Deep Learning, Neural Networks and Kernel Machines

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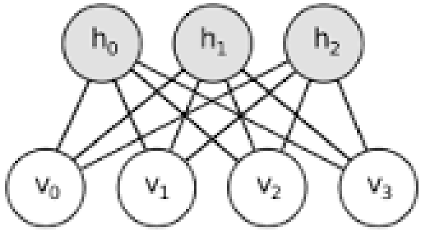
Part III: Deep RKM and future perspectives

- RBM and Deep BM: "sandwich" interpretation
- From RKM to Deep RKM
- Deep RKM example: LS-SVM + KPCA + KPCA
  Connection with Deep BM
- Deep RKM example: KPCA + KPCA + LS-SVM
  Connection with stacked autoencoders
- Learning algorithms
- Numerical examples

RBM and deep learning

Restricted Boltzmann Machines (RBM)



- Markov random field, bipartite graph, stochastic binary units
  Layer of <u>visible units</u> v and layer of <u>hidden units</u> h
  No hidden-to-hidden connections
- Energy:

$$E(v,h;\theta) = -v^T W h - c^T v - a^T h \quad \text{with} \quad \theta = \{W, c, a\}$$

Joint distribution:

$$P(v,h;\theta) = \frac{1}{Z(\theta)} \exp(-E(v,h;\theta))$$

with partition function $Z(\theta) = \sum_v \sum_h \exp(-E(v,h;\theta))$

[Hinton, Osindero, Teh, Neural Computation 2006]

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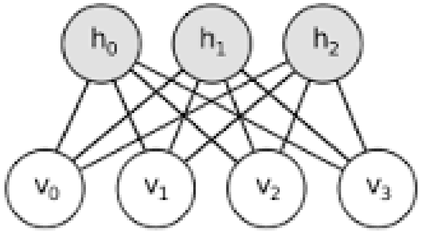

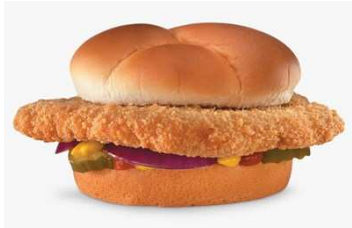

- Markov random field, bipartite graph, stochastic binary units
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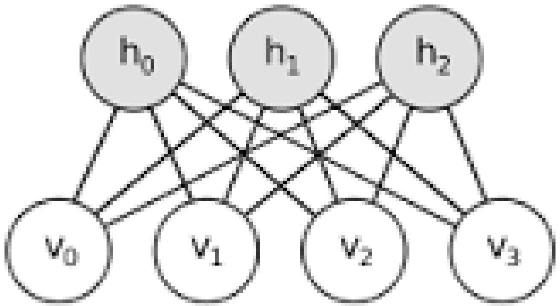
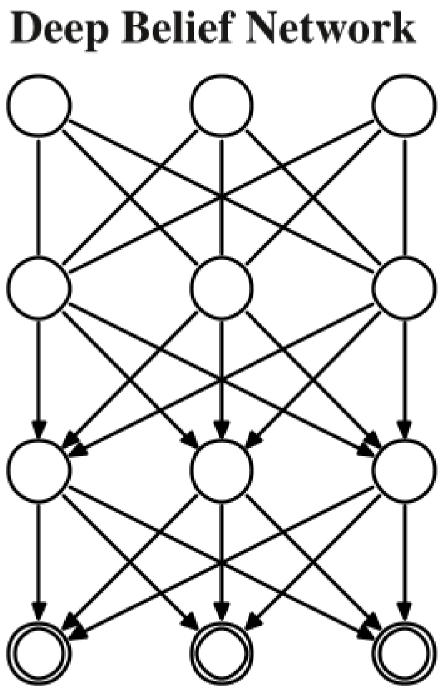
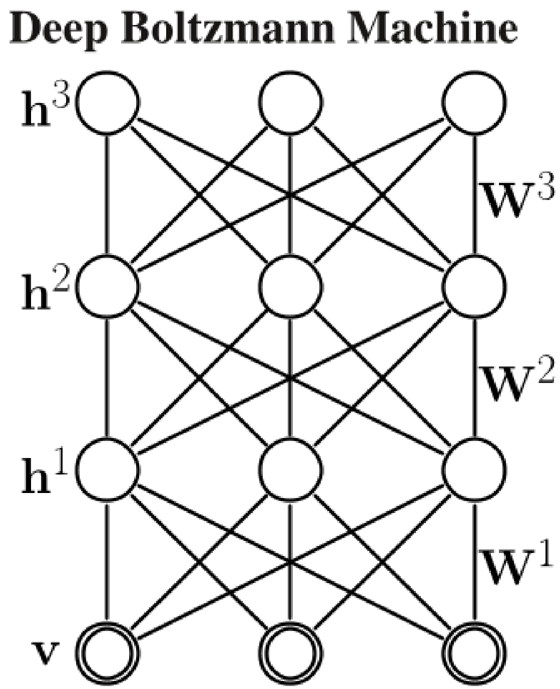
[Hinton, Osindero, Teh, Neural Computation 2006]

RBM and deep learning







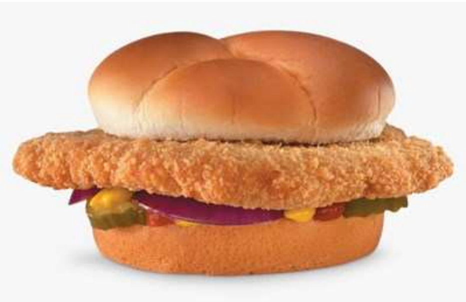
$p(v, h)$

$p(v, h^1, h^2, h^3, ...)$

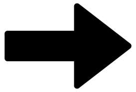
[Hinton et al., 2006; Salakhutdinov, 2015]

in other words ...

"sandwich"



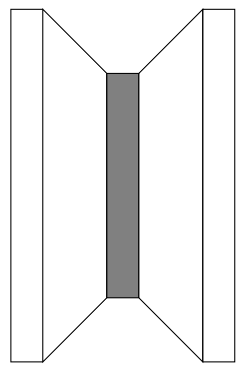
$$E = -v^T W h$$



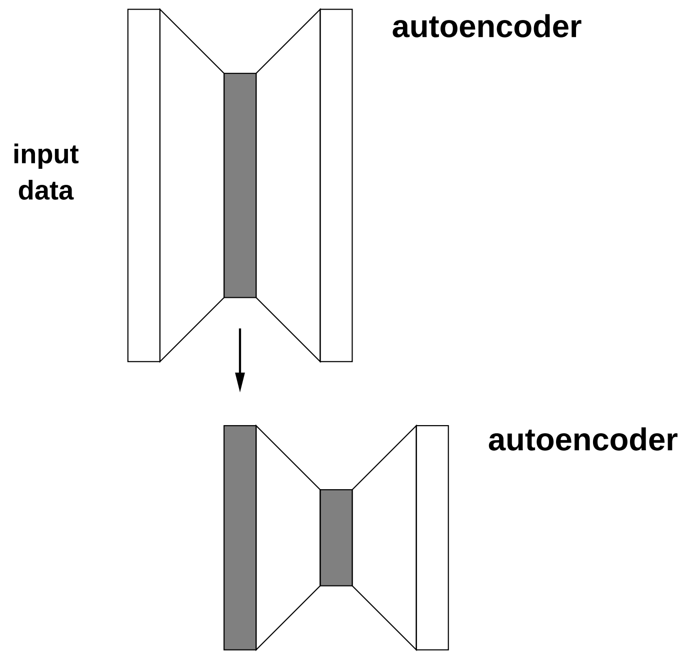
"deep sandwich"

$$E = -v^T W^1 h^1 - {h^1}^T W^2 h^2 - {h^2}^T W^3 h^3$$

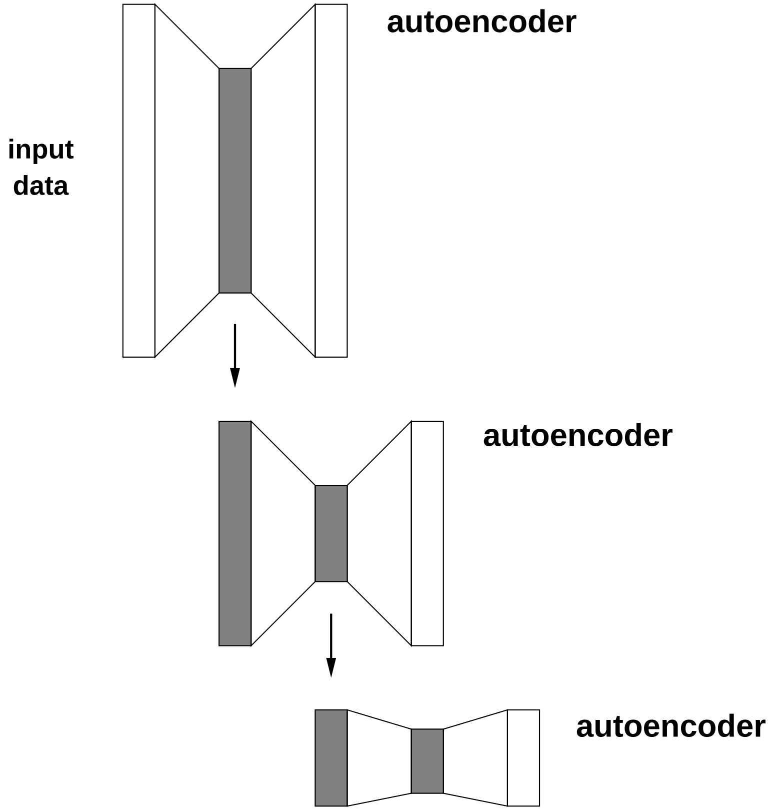
Stacked autoencoders



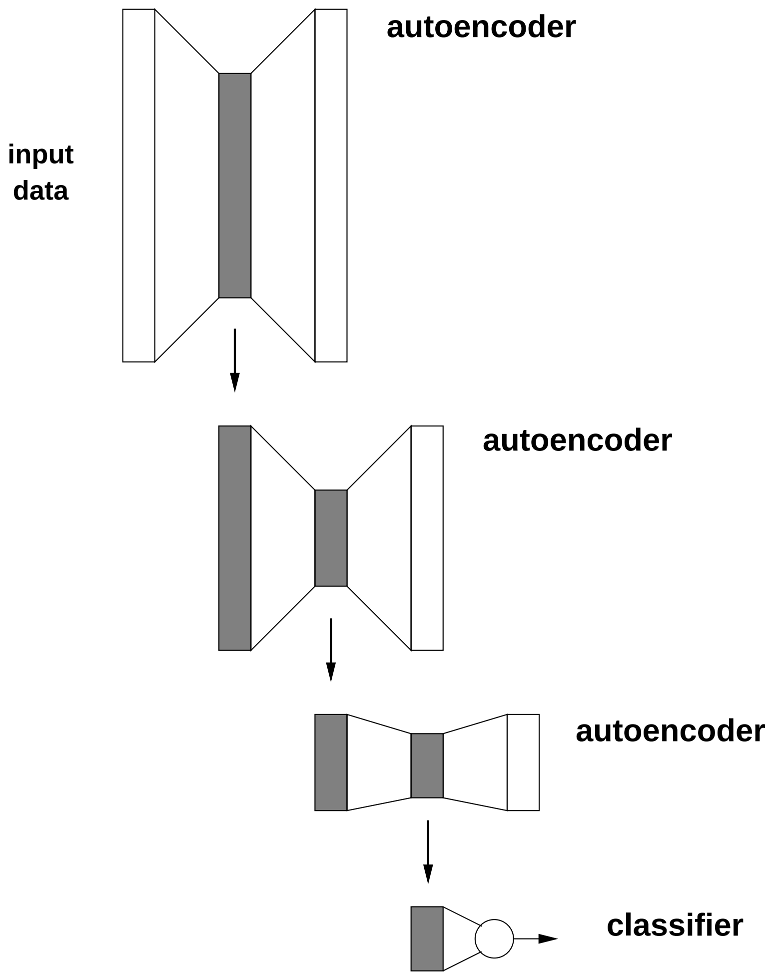
Stacked autoencoders



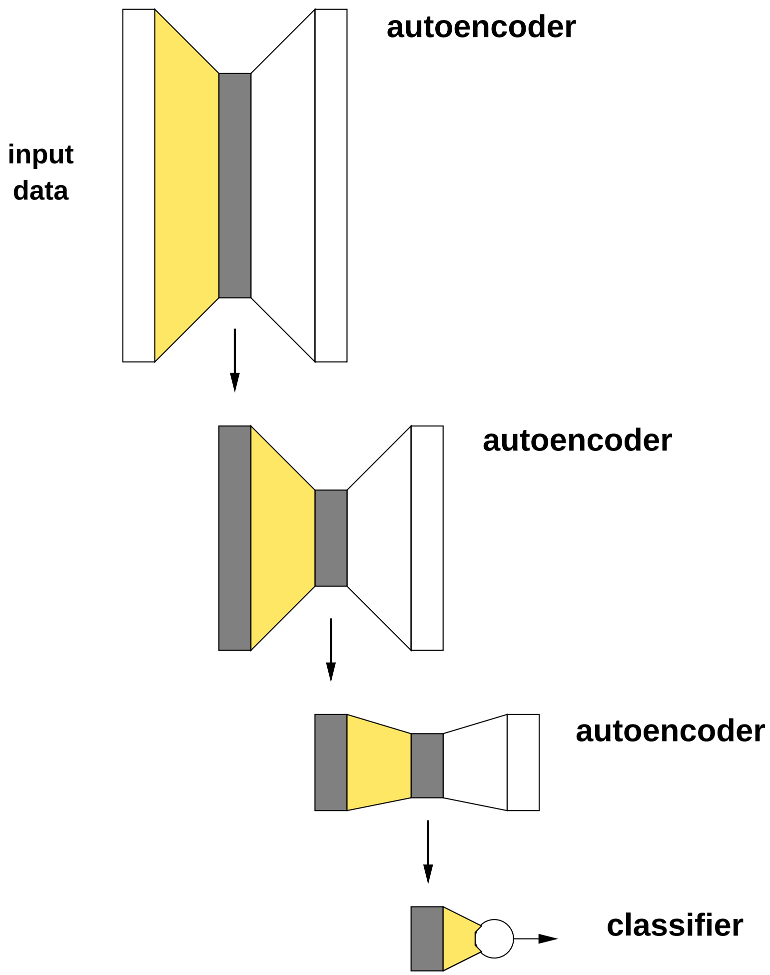
Stacked autoencoders



Stacked autoencoders



Stacked autoencoders



pretraining phase

& finetuning phase

[Bengio, 2009]

Restricted kernel machine

Restricted Kernel Machines (RKM)

Main characteristics:

- Kernel machine interpretations in terms of **visible and hidden units** (similar to Restricted Boltzmann Machines (**RBM**))

- Restricted Kernel Machine (**RKM**) representations for
  - LS-SVM regression/classification
  - Kernel PCA
  - Matrix SVD
  - Parzen-type models
  - other

- Based on principle of **conjugate feature duality** (with hidden features corresponding to dual variables)

Model: living in two worlds ...

Original model:

$$\boxed{\hat{y} = W^T \varphi(x) + b,\ e = y - \hat{y}}$$

objective J
$=$ regularization term $\mathrm{Tr}(W^T W)$
$+$ $(\frac{1}{\lambda})$ error term $\sum_i e_i^T e_i$

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$$\downarrow \quad \frac{1}{2\lambda} e^T e \geq e^T h - \frac{\lambda}{2} h^T h$$

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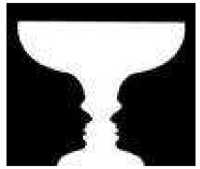

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$$\downarrow \quad \frac{1}{2\lambda} e^T e \geq e^T h - \frac{\lambda}{2} h^T h$$

New representation:

$$\hat{y} = \sum_j h_j K(x_j, x) + b$$

obtain $J \geq \underline{J}(h_i, W, b)$
solution from stationary points of $\underline{J}$:
$\frac{\partial \underline{J}}{\partial h_i} = 0$, $\frac{\partial \underline{J}}{\partial W} = 0$, $\frac{\partial \underline{J}}{\partial b} = 0$

where $\underline{J} = \sum_{i=1}^N (y_i^T - x_i^T W - b^T) h_i - \frac{\lambda}{2} \sum_{i=1}^N h_i^T h_i + \frac{\eta}{2} \mathrm{Tr}(W^T W)$

RKM regression problem

- Stationary points of $\underline{J}(h_i, W, b)$ (nonlinear case, feature map $\varphi(\cdot)$)

$$\left\{ \begin{array}{lcl} \frac{\partial \underline{J}}{\partial h_i} = 0 & \Rightarrow & y_i = W^T \varphi(x_i) + b + \lambda h_i, \ \forall i \\ \frac{\partial \underline{J}}{\partial W} = 0 & \Rightarrow & W = \frac{1}{\eta} \sum_i \varphi(x_i) h_i^T \\ \frac{\partial \underline{J}}{\partial b} = 0 & \Rightarrow & \sum_i h_i = 0. \end{array} \right.$$

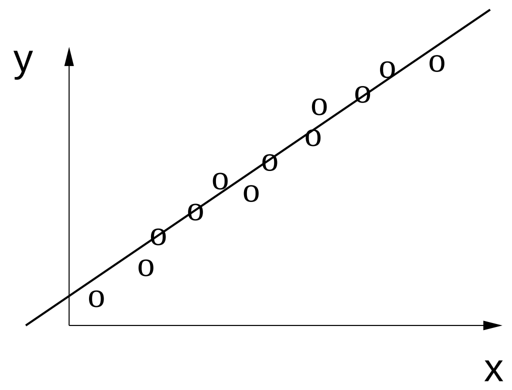
- Solution in h_i and b with positive definite kernel $K(x_i, x_j) = \varphi(x_i)^T \varphi(x_j)$

$$\left[\begin{array}{c|c} \frac{1}{\eta} K + \lambda I_N & 1_N \\ \hline 1_N^T & 0 \end{array} \right] \left[\begin{array}{c} H^T \\ \hline b^T \end{array} \right] = \left[\begin{array}{c} Y^T \\ \hline 0 \end{array} \right]$$

with $K = [K(x_i, x_j)]$, $H = [h_1 ... h_N]$, $Y = [y_1 ... y_N]$.

Simple example: line fitting

Given data: $\{(x_i, y_i)\}_{i=1}^N$, $x_i, y_i \in \mathbb{R}$

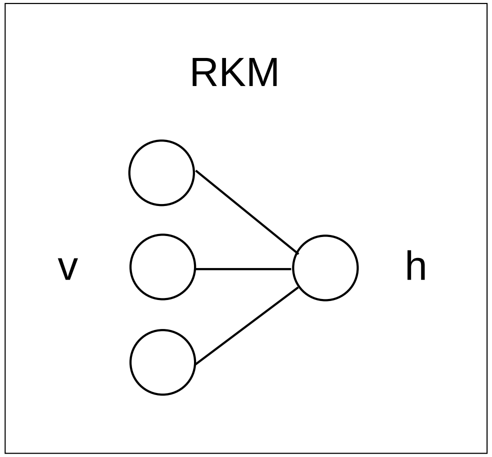


Linear model:

$$\hat{y} = wx + b, \;\; e = y - \hat{y}$$

RKM representation:

$$\hat{y} = \sum_i h_i x_i x + b$$

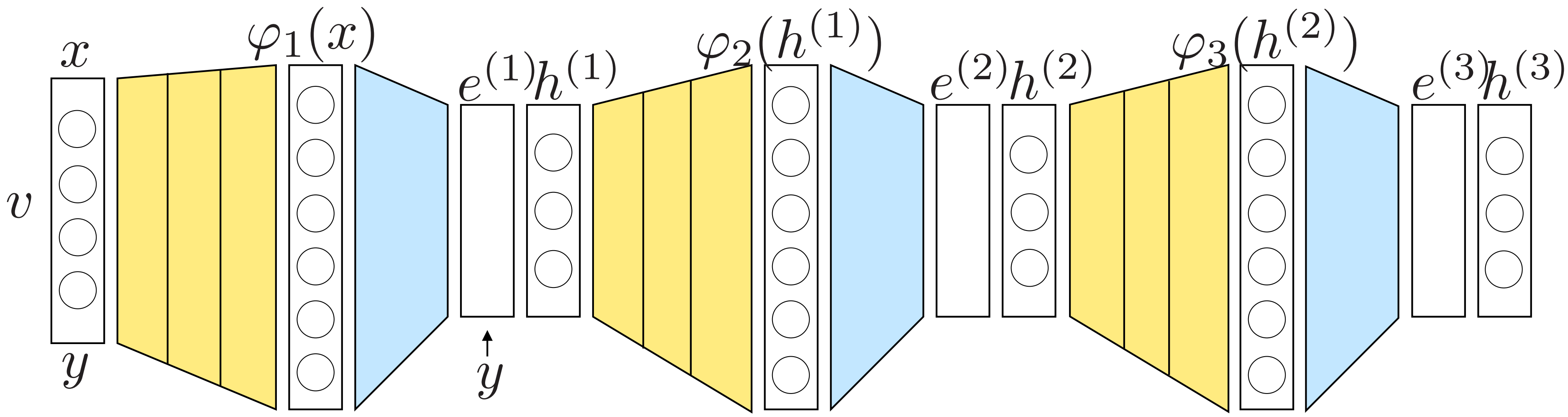


3 visible units: $\quad v = [x; 1; -y]$

1 hidden unit: $\quad h \in \mathbb{R}$

Deep Restricted Kernel Machines

Deep RKM: Example



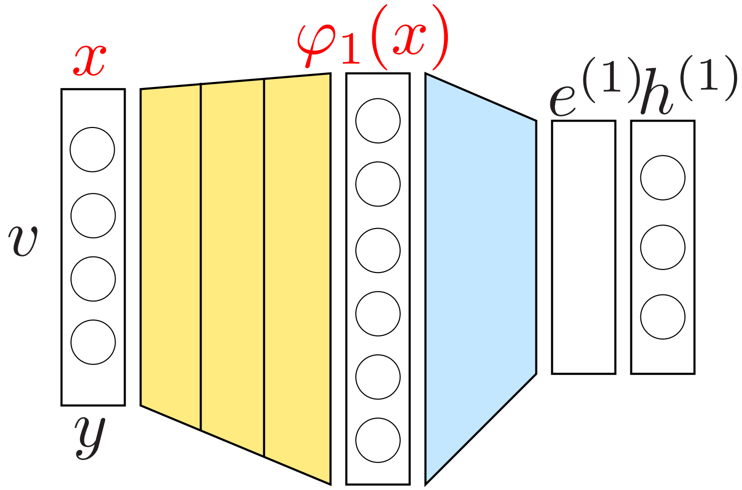
Deep RKM: LSSVM + KPCA + KPCA

Coupling of RKMs by taking sum of the objectives

$$J_{\text{deep}} = \underline{J}_1 + \overline{J}_2 + \overline{J}_3$$

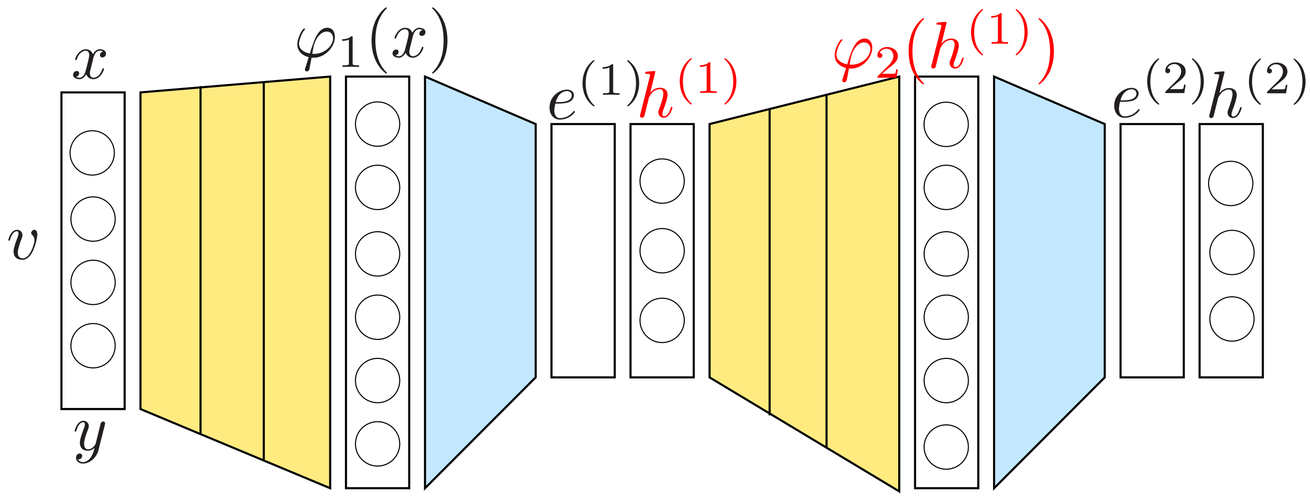
Multiple *levels* and multiple *layers* per level.

in more detail ...



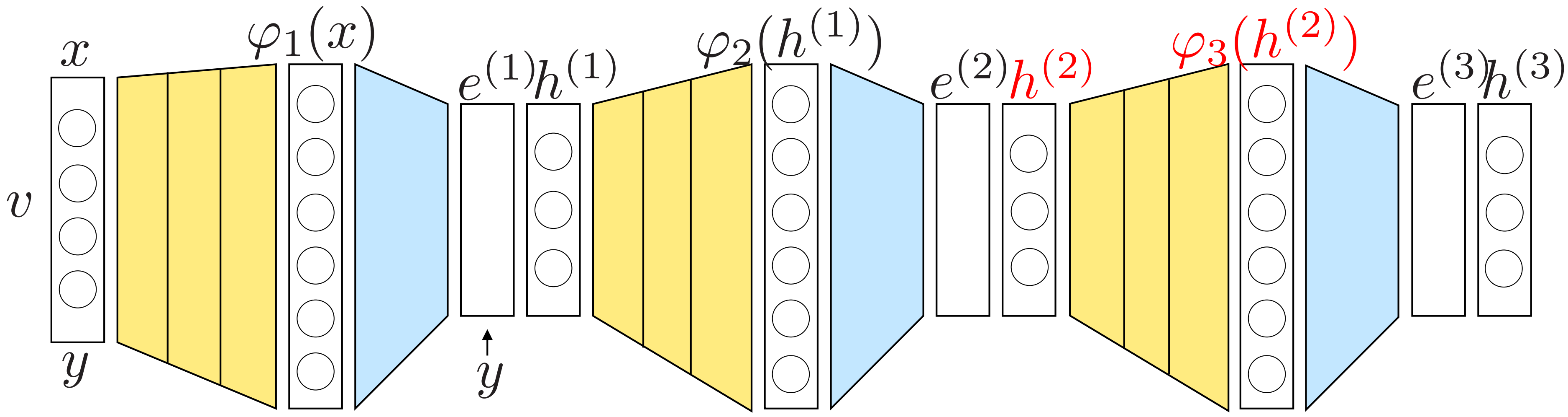
$$J_{\text{deep}} = \sum_{i=1}^{N} (y_i^T - \varphi_1(x_i)^T W_1 - b^T) h_i^{(1)} - \frac{\lambda_1}{2} \sum_{i=1}^{N} {h_i^{(1)}}^T h_i^{(1)} + \frac{\eta_1}{2} \mathrm{Tr}(W_1^T W_1)$$

in more detail ...



$$
\begin{aligned}
J_{\text{deep}} \quad = \quad & \sum_{i=1}^{N} (y_i^T - \varphi_1(x_i)^T W_1 - b^T) h_i^{(1)} - \frac{\lambda_1}{2} \sum_{i=1}^{N} {h_i^{(1)}}^T h_i^{(1)} + \frac{\eta_1}{2} \mathrm{Tr}(W_1^T W_1) \\
& - \sum_{i=1}^{N} \varphi_2(h_i^{(1)})^T W_2 h_i^{(2)} + \frac{\lambda_2}{2} \sum_{i=1}^{N} {h_i^{(2)}}^T h_i^{(2)} + \frac{\eta_2}{2} \mathrm{Tr}(W_2^T W_2)
\end{aligned}
$$

in more detail ...



$$
\begin{aligned}
J_{\mathrm{deep}} \quad = \quad & \sum_{i=1}^{N} (y_i^T - \varphi_1(x_i)^T W_1 - b^T) h_i^{(1)} - \frac{\lambda_1}{2} \sum_{i=1}^{N} {h_i^{(1)}}^T h_i^{(1)} + \frac{\eta_1}{2} \mathrm{Tr}(W_1^T W_1) \\
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& - \sum_{i=1}^{N} \varphi_3(h_i^{(2)})^T W_3 h_i^{(3)} + \frac{\lambda_3}{2} \sum_{i=1}^{N} {h_i^{(3)}}^T h_i^{(3)} + \frac{\eta_3}{2} \mathrm{Tr}(W_3^T W_3)
\end{aligned}
$$

Stationary points

Stationary points of $J_{\text{deep}}(h_i^{(1)}, W_1, b, h_i^{(2)}, W_2, h_i^{(3)}, W_3)$ are given by

$$\left\{\begin{array}{lll}
\frac{\partial J_{\text{deep}}}{\partial h_i^{(1)}} = 0 & \Rightarrow & y_i - W_1^T \varphi_1(x_i) - b = \lambda_1 h_i^{(1)} + \frac{\partial}{\partial h_i^{(1)}}[\varphi_2(h_i^{(1)})^T W_2 h_i^{(2)}], \ \forall i \\
\frac{\partial J_{\text{deep}}}{\partial W_1} = 0 & \Rightarrow & W_1 = \frac{1}{\eta_1} \sum_i \varphi_1(x_i) {h_i^{(1)}}^T \\
\frac{\partial J_{\text{deep}}}{\partial b} = 0 & \Rightarrow & \sum_i h_i^{(1)} = 0 \\
\frac{\partial J_{\text{deep}}}{\partial h_i^{(2)}} = 0 & \Rightarrow & W_2^T \varphi_2(h_i^{(1)}) = \lambda_2 h_i^{(2)} - \frac{\partial}{\partial h_i^{(2)}}[\varphi_3(h_i^{(2)})^T W_3 h_i^{(3)}], \ \forall i \\
\frac{\partial J_{\text{deep}}}{\partial W_2} = 0 & \Rightarrow & W_2 = \frac{1}{\eta_2} \sum_i \varphi_2(h_i^{(1)}) {h_i^{(2)}}^T \\
\frac{\partial J_{\text{deep}}}{\partial h_i^{(3)}} = 0 & \Rightarrow & W_3^T \varphi_3(h_i^{(2)}) = \lambda_3 h_i^{(3)}, \ \forall i \\
\frac{\partial J_{\text{deep}}}{\partial W_3} = 0 & \Rightarrow & W_3 = \frac{1}{\eta_3} \sum_i \varphi_3(h_i^{(2)}) {h_i^{(3)}}^T
\end{array}\right.$$

Kernel trick

Elimination of W_1, W_2, W_3 and application kernel trick to each level 1,2,3 $\rightarrow$ set of nonlinear equations:

$$\left\{ \begin{array}{l} y_i = \frac{1}{\eta_1} \sum_j h_j^{(1)} K_1(x_j, x_i) + b + \lambda_1 h_i^{(1)} + \frac{1}{\eta_2} \sum_j \frac{\partial K_2(h_i^{(1)}, h_j^{(1)})}{\partial h_i^{(1)}} {h_j^{(2)}}^T h_i^{(2)}, \ \forall i \\ \sum_i h_i^{(1)} = 0 \\ \frac{1}{\eta_2} \sum_j h_j^{(2)} K_2(h_j^{(1)}, h_i^{(1)}) = \lambda_2 h_i^{(2)} - \frac{1}{\eta_3} \sum_j \frac{\partial K_3(h_i^{(2)}, h_j^{(2)})}{\partial h_i^{(2)}} {h_j^{(3)}}^T h_i^{(3)}, \ \forall i \\ \frac{1}{\eta_3} \sum_j h_j^{(3)} K_3(h_j^{(2)}, h_i^{(2)}) = \lambda_3 h_i^{(3)}, \ \forall i \end{array} \right.$$

$$(P)_{\mathrm{DeepRKM}}: \quad \hat{y} = W_1^T \varphi_1(x) + b$$

$$\mathcal{M} \quad \nearrow \quad \searrow$$

$$(D)_{\mathrm{DeepRKM}}: \quad \hat{y} = \frac{1}{\eta_1} \sum_j h_j^{(1)} K_1(x_j, x) + b$$

Data fusion interpretation

- Special case of linear kernel PCA levels

$$
\left\{
\begin{array}{ll}
\text{Level 1}: & \left[\begin{array}{c|c} \frac{1}{\eta_1}[K_1(x_j,x_i)] + \frac{1}{\eta_2}[K_{\rm lin}(h_j^{(2)},h_i^{(2)})] + \lambda_1 I_N & 1_N \\ \hline 1_N^T & 0 \end{array}\right] \left[\begin{array}{c} H_1^T \\ \hline b^T \end{array}\right] = \left[\begin{array}{c} Y^T \\ \hline 0 \end{array}\right] \\
\text{Level 2}: & \left(\frac{1}{\eta_2}[K_{2,{\rm lin}}(h_j^{(1)},h_i^{(1)})] + \frac{1}{\eta_3}[K_{\rm lin}(h_j^{(3)},h_i^{(3)})]\right) H_2^T = H_2^T \Lambda_2 \\
\text{Level 3}: & \left(\frac{1}{\eta_3}[K_{3,{\rm lin}}(h_j^{(2)},h_i^{(2)})]\right) H_3^T = H_3^T \Lambda_3
\end{array}
\right.
$$

with $H_l = [h_1^{(l)} ... h_N^{(l)}]$, $l = 1, 2, 3$.

- Data fusion:
  - Level 1: between K_1 and $K_{\rm lin}$
  - Level 2: between $K_{2,{\rm lin}}$ and $K_{\rm lin}$

Heuristic algorithm

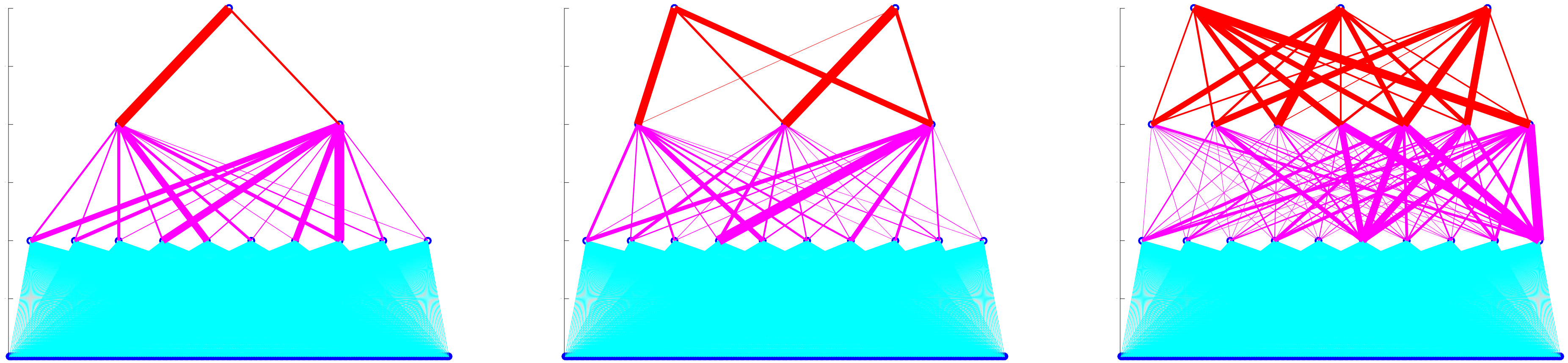
Forward phase (level 1 $\rightarrow$ level 3)

$$\begin{array}{ll} & H_2, H_3 \text{ initialization} \\ \text{Level 1}: & H_1 := f_1(X, Y, H_2) \\ \text{Level 2}: & H_2 := f_2(H_1, H_3) \\ \text{Level 3}: & H_3 := f_3(H_2) \end{array}$$

Backward phase (level 3 $\rightarrow$ level 1)

$$\begin{array}{ll} \text{Level 2}: & H_2 := f_2(H_1, H_3) \\ \text{Level 1}: & H_1 := f_1(X, Y, H_2) \end{array}$$

Deep RKM - Example USPS data



USPS (10 classes): Deep RKM: LSSVM (K_{rbf}) + KPCA (K_{lin}) + KPCA (K_{lin})

Training algorithm: forward & backward phases, kernel fusion between levels

$N = 2000$: test error 3.26% (basic) - 3.18% (deep) ($N_{\mathrm{test}} = 5000$)

$N = 4000$: test error 2.14% (basic) - 2.12% (deep) ($N_{\mathrm{test}} = 5000$)

Deep RKM - Example MNIST data

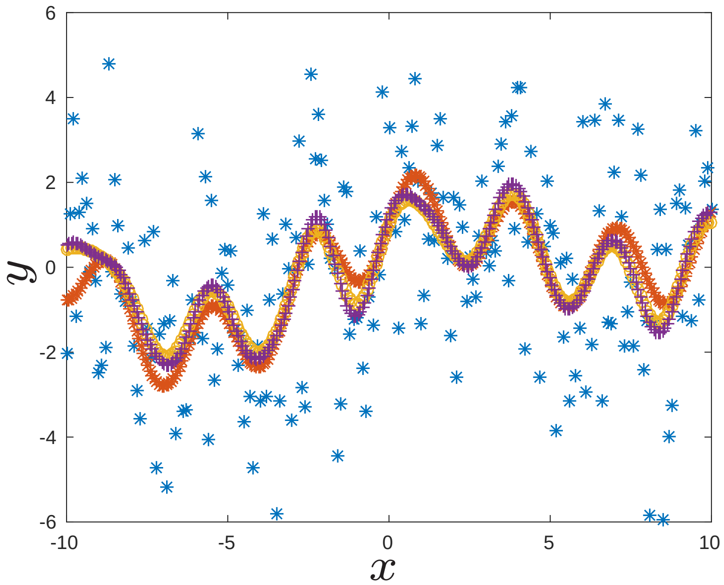
$d = 784$ (images of size 28x28 for each of the 10 classes), $N = 50000$
partitioned into non-overlapping subsets of size 50 (i.e. 5 data points/class)

Deep RKMs on subsets: LSSVM (K_{rbf}) + KPCA (K_{lin}) + KPCA (K_{lin})
forward-backward phases
extra 50000 noise corrupted training data
linearly combined submodels ($\tanh$ applied to their outputs)

Test error: 1.28%

Deep belief networks	(1.2%)
Deep Boltzmann machines	$(0.95, 1.01\%)$
SVM with Gaussian kernel	(1.4%)

Deep RKM - Example nonlinear regression

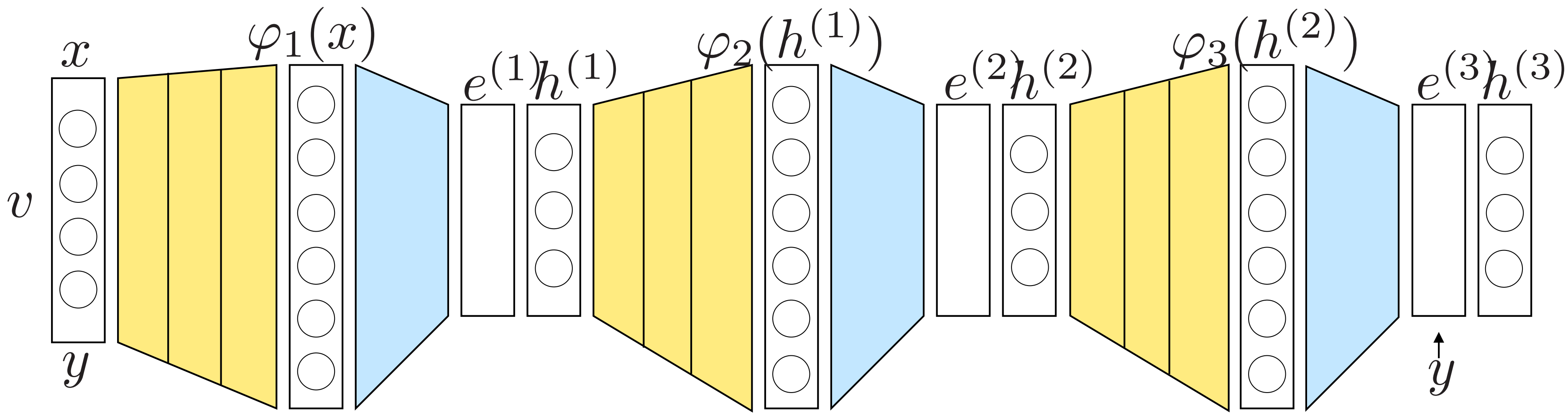


$$f(x) = \sin(0.3x) + \cos(0.5x) + \sin(2x)$$

with zero mean Gaussian noise with standard deviation 0.1, 0.5, 1, 2

noise	LS-SVM	deep (1+1)	deep (7+2)
0.1	$0.0019 \pm 4.3\,10^{-4}$	$0.0018 \pm 4.2\,10^{-4}$	$0.0019 \pm 4.4\,10^{-4}$
0.5	0.0403 ± 0.0098	0.0374 ± 0.0403	0.0397 ± 0.0089
1	0.1037 ± 0.0289	0.0934 ± 0.0269	0.0994 ± 0.0301
2	0.3368 ± 0.0992	0.2902 ± 0.0875	0.3080 ± 0.0954

Deep RKM: Other example



Deep RKM: KPCA + KPCA + LSSVM

Coupling of RKMs by taking sum of the objectives

$$J_{\text{deep}} = \overline{J}_1 + \overline{J}_2 + \underline{J}_3$$

Primal and dual model representations

$$\mathcal{M} \begin{array}{l} \nearrow \quad (P)_{\mathrm{DeepRKM}}: \quad \begin{array}{l} \hat{e}^{(1)} = W_1^T \varphi_1(x) \\ \hat{e}^{(2)} = W_2^T \varphi_2(\Lambda_1^{-1} \hat{e}^{(1)}) \\ \hat{y} = W_3^T \varphi_3(\Lambda_2^{-1} \hat{e}^{(2)}) + b \end{array} \\ \\ \searrow \quad (D)_{\mathrm{DeepRKM}}: \quad \begin{array}{l} \hat{e}^{(1)} = \frac{1}{\eta_1} \sum_j h_j^{(1)} K_1(x_j, x) \\ \hat{e}^{(2)} = \frac{1}{\eta_2} \sum_j h_j^{(2)} K_2(h_j^{(1)}, \Lambda_1^{-1} \hat{e}^{(1)}) \\ \hat{y} = \frac{1}{\eta_3} \sum_j h_j^{(3)} K_3(h_j^{(2)}, \Lambda_2^{-1} \hat{e}^{(2)}) + b \end{array} \end{array}$$

Deep reduced set kernel-based models (1)

Subset of training set $\{\tilde{x}_j\}_{j=1}^{M} \subset \{x_i\}_{i=1}^{N}$ with $M \ll N$:

$$
\begin{array}{rcl}
W_1 & = & \frac{1}{\eta_1} \sum_{i=1}^{N} \varphi_1(x_i) {h_i^{(1)}}^T \simeq \tilde{W}_1 = \frac{1}{\eta_1} \sum_{j=1}^{M} \varphi_1(\tilde{x}_j) \tilde{h}_j^{(1)T} \\
\tilde{W}_2 & = & \frac{1}{\eta_2} \sum_{j=1}^{M} \varphi_2(\tilde{h}_j^{(1)}) \tilde{h}_j^{(2)T} \\
\tilde{W}_3 & = & \frac{1}{\eta_3} \sum_{j=1}^{M} \varphi_2(\tilde{h}_j^{(2)}) \tilde{h}_j^{(3)T}
\end{array}
$$

Predictive model:

$$
\begin{array}{rcl}
\hat{e}^{(1)} & = & \frac{1}{\eta_1} \sum_{j=1}^{M} \tilde{h}_j^{(1)} K_1(\tilde{x}_j, x) \\
\hat{e}^{(2)} & = & \frac{1}{\eta_2} \sum_{j=1}^{M} \tilde{h}_j^{(2)} K_2(\tilde{h}_j^{(1)}, \Lambda_1^{-1} \hat{e}^{(1)}) \\
\hat{y} & = & \frac{1}{\eta_3} \sum_{j=1}^{M} \tilde{h}_j^{(3)} K_3(\tilde{h}_j^{(2)}, \Lambda_2^{-1} \hat{e}^{(2)}) + b
\end{array}
$$

Deep reduced set kernel-based models (2)

Training in primal:

$$\begin{aligned}\min_{\tilde{h}_j^{(1)},\tilde{h}_j^{(2)},\tilde{h}_j^{(3)},b,\Lambda_1,\Lambda_2} J_{\mathrm{deep},\mathrm{P_{stab}}} = \quad & -\frac{1}{2}\sum_{j=1}^{M} {e_j^{(1)}}^T \Lambda_1^{-1} e_j^{(1)} + \frac{\eta_1}{2}\mathrm{Tr}(\tilde{W}_1^T \tilde{W}_1) \\ & -\frac{1}{2}\sum_{j=1}^{M} {e_j^{(2)}}^T \Lambda_2^{-1} e_j^{(2)} + \frac{\eta_2}{2}\mathrm{Tr}(\tilde{W}_2^T \tilde{W}_2) \\ & +\frac{1}{2\lambda_3}\sum_{j=1}^{M} {e_j^{(3)}}^T e_j^{(3)} + \frac{\eta_3}{2}\mathrm{Tr}(\tilde{W}_3^T \tilde{W}_3) \\ & +\frac{1}{2}\mathbf{c}_{\mathrm{stab}}\Big(-\frac{1}{2}\sum_{j=1}^{M} {e_j^{(1)}}^T \Lambda_1^{-1} e_j^{(1)} + \frac{\eta_1}{2}\mathrm{Tr}(\tilde{W}_1^T \tilde{W}_1)\Big)^{\mathbf{2}} \\ & +\frac{1}{2}\mathbf{c}_{\mathrm{stab}}\Big(-\frac{1}{2}\sum_{j=1}^{M} {e_j^{(2)}}^T \Lambda_2^{-1} e_j^{(2)} + \frac{\eta_2}{2}\mathrm{Tr}(\tilde{W}_2^T \tilde{W}_2)\Big)^{\mathbf{2}}\end{aligned}$$

with stabilization terms.

Deep feedforward neural networks in the primal

Model

$$\begin{aligned}\hat{e}^{(1)} &= W_1^T \sigma(U_1 x + \beta_1) \\ \hat{e}^{(2)} &= W_2^T \sigma(U_2 \Lambda_1^{-1} \hat{e}^{(1)} + \beta_2) \\ \hat{y} &= W_3^T \sigma(U_3 \Lambda_2^{-1} \hat{e}^{(2)} + \beta_3) + b\end{aligned}$$

Training objective:

$$\begin{aligned}\min_{W_{1,2,3}, U_{1,2,3}, \beta_{1,2,3}, b, \Lambda_1, \Lambda_2} J_{\text{deep},\text{P}_{\text{stab}}} = \; & -\tfrac{1}{2} \textstyle\sum_{j=1}^M {e_j^{(1)}}^T \Lambda_1^{-1} e_j^{(1)} + \tfrac{\eta_1}{2} \mathrm{Tr}(W_1^T W_1) \\ & -\tfrac{1}{2} \textstyle\sum_{j=1}^M {e_j^{(2)}}^T \Lambda_2^{-1} e_j^{(2)} + \tfrac{\eta_2}{2} \mathrm{Tr}(W_2^T W_2) \\ & +\tfrac{1}{2\lambda_3} \textstyle\sum_{j=1}^M {e_j^{(3)}}^T e_j^{(3)} + \tfrac{\eta_3}{2} \mathrm{Tr}(W_3^T W_3) \\ & +\tfrac{1}{2} \mathbf{c}_{\text{stab}} \Big(-\tfrac{1}{2} \textstyle\sum_{j=1}^M {e_j^{(1)}}^T \Lambda_1^{-1} e_j^{(1)} + \tfrac{\eta_1}{2} \mathrm{Tr}(W_1^T W_1) \Big)^{\mathbf{2}} \\ & +\tfrac{1}{2} \mathbf{c}_{\text{stab}} \Big(-\tfrac{1}{2} \textstyle\sum_{j=1}^M {e_j^{(2)}}^T \Lambda_2^{-1} e_j^{(2)} + \tfrac{\eta_2}{2} \mathrm{Tr}(W_2^T W_2) \Big)^{\mathbf{2}}\end{aligned}$$

Toeplitz matrices (convolutional)

Taking matrices $U \in \mathbb{R}^{n_1 \times n_2}$ as Toeplitz matrices:

$$\begin{bmatrix} a & b & c & d & \\ e & a & b & c & \\ f & e & a & b & \\ g & f & e & a & \\ & & & & \ddots \end{bmatrix}$$

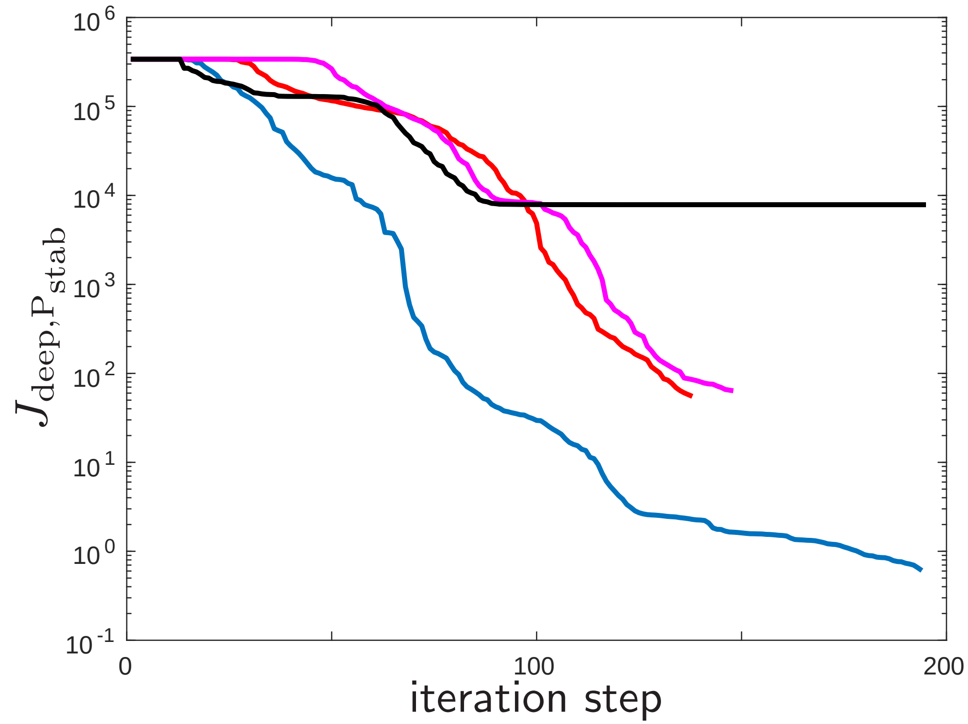
$\rightarrow$ reduces to $n_1 + n_2 - 1$ instead of $n_1 n_2$ unknowns

Experimental results

	pid	bld	ion	adu
$\mathcal{M}_{1,a}$	19.53 [20.02(1.53)]	26.09 [30.96(3.34)]	**0** [0.68(1.60)]	16.99 [17.46(0.65)]
$\mathcal{M}_{1,b}$	**18.75** [19.39(0.89)]	**25.22** [31.48(4.11)]	**0** [5.38(12.0)]	17.08 [17.48(0.56)]
$\mathcal{M}_{1,c}$	21.88 [24.73(5.91)]	28.69 [32.39(3.48)]	**0** [8.21(6.07)]	17.83 [21.21(4.78)]
$\mathcal{M}_{2,a}$	21.09 [20.20(1.51)]	27.83 [28.86(2.83)]	1.71 [5.68(2.22)]	15.07 [15.15(0.15)]
$\mathcal{M}_{2,b}$	**18.75** [20.33(2.75)]	28.69 [28.38(2.80)]	10.23 [6.92(3.69)]	14.91 [15.08 (0.15)]
$\mathcal{M}_{2,b,T}$	19.03 [19.16(1.10)]	26.08 [27.74(9.40)]	6.83 [6.50(8.31)]	15.71 [15.97(0.07)]
$\mathcal{M}_{2,c}$	24.61 [22.34(1.95)]	32.17 [27.61(3.69)]	3.42 [9.66(6.74)]	15.21 [15.19(0.08)]
bestbmark	22.7(2.2)	29.6(3.7)	4.0(2.1)	14.4(0.3)

$\mathcal{M}_1$: *Deep reduced set kernel-based models* (with RBF kernel)
$\mathcal{M}_{1,a}$: additional term $-c_0(\mathrm{Tr}(\Lambda_1) + \mathrm{Tr}(\Lambda_2))$ and $\mathrm{Tr}(\tilde{H}^{(l)} \tilde{H}^{(l)T})$ regularization
$\mathcal{M}_{1,b}$: without term $-c_0(\mathrm{Tr}(\Lambda_1) + \mathrm{Tr}(\Lambda_2))$
$\mathcal{M}_{1,c}$: with objective function $\frac{1}{2\lambda_3} \sum_{j=1}^{M} {e_j^{(3)}}^T e_j^{(3)} + \frac{\eta_3}{2} \mathrm{Tr}(W_3^T W_3)$
$\mathcal{M}_2$: *Deep feedforward neural networks*
$\mathcal{M}_{2,b,T}$: with Toeplitz matrices for U matrices

Training process

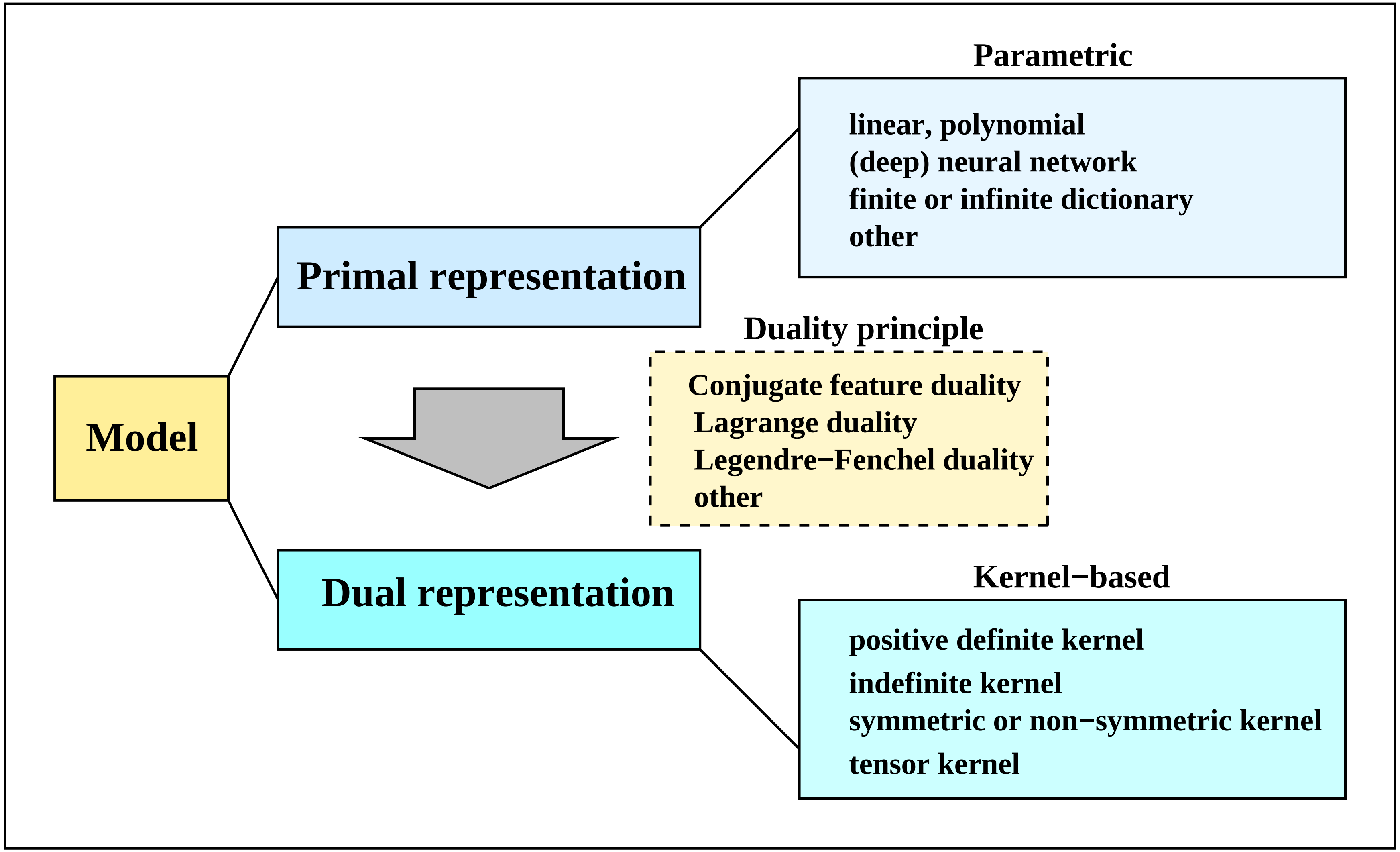


Evolution of the objective function (logarithmic scale) during training on the ion data set. Shown are training curves for the model $\mathcal{M}_{2,a}$ for different choices of c_{stab} (equal to 1, 10, 100 in blue, red, magenta color, respectively) in comparison with $\mathcal{M}_{2,c}$ (level 3 objective only, in black color), for the same initialization, with quasi-Newton method.

Future challenges

- efficient algorithms and implementations for large data
- extension to other loss functions and regularization schemes
- multimodal data, tensor models, coupling schemes
- models for deep clustering and semi-supervised learning
- combination with convolutional layers
- choice kernel functions, invariances
- deep generative models
- optimal transport

Towards a unifying picture



[Suykens 2017]

Conclusions

- From RBM to deep BM
- From RKM to deep RKM
- RKM and RBM representation: visible and hidden units
- Deep RKM: new framework for deep kernel machines and deep feedforward neural networks
- Primal and dual models representations for deep learning

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Exploring duality for future data-driven modelling

Thank you