Counting Independent sets in Graphs

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UCL

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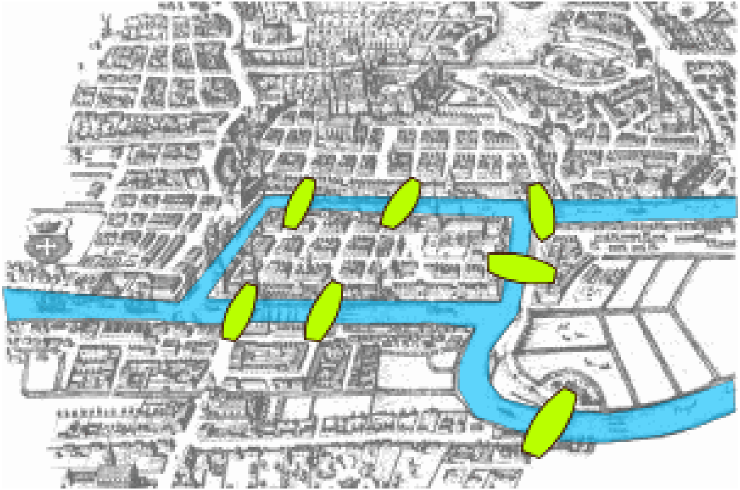
Why are we interested in graphs?



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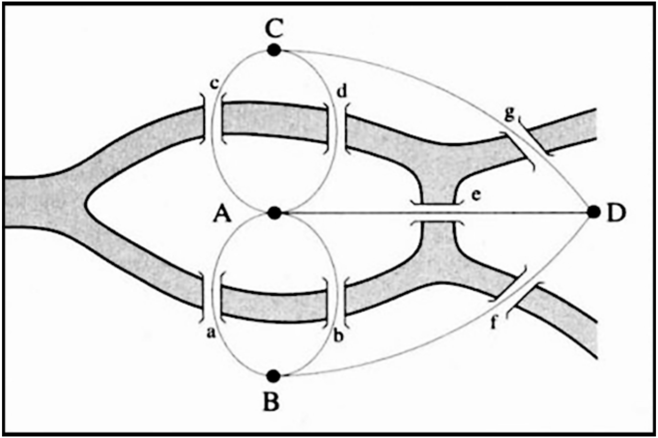

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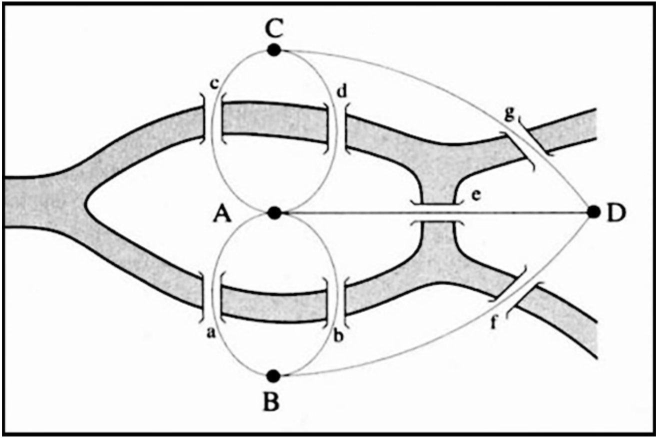

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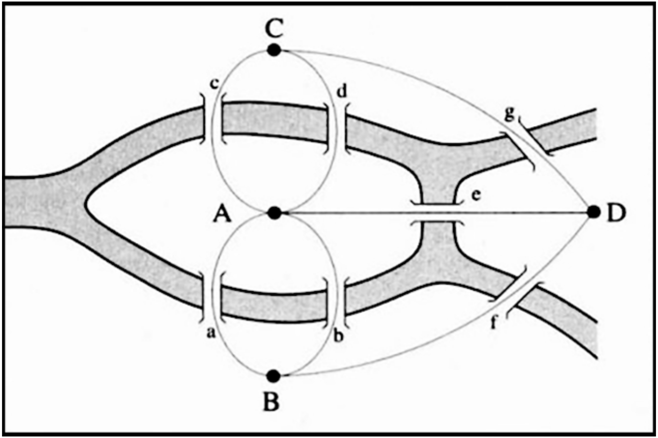

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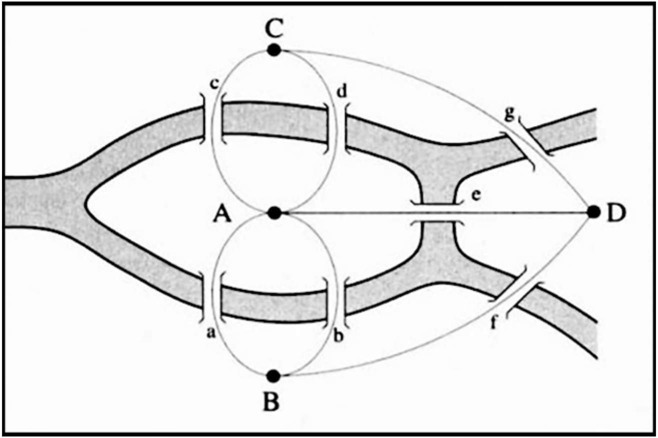

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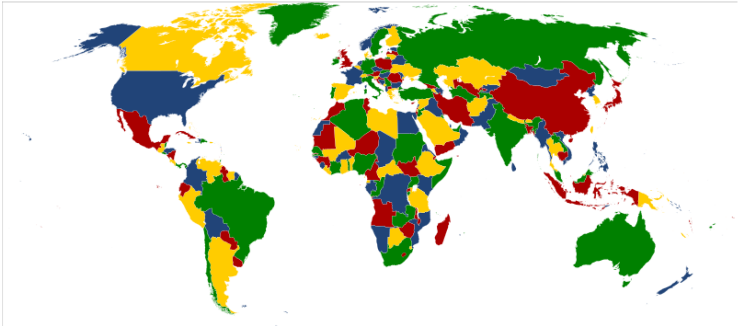

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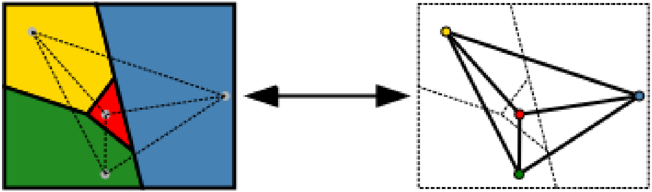

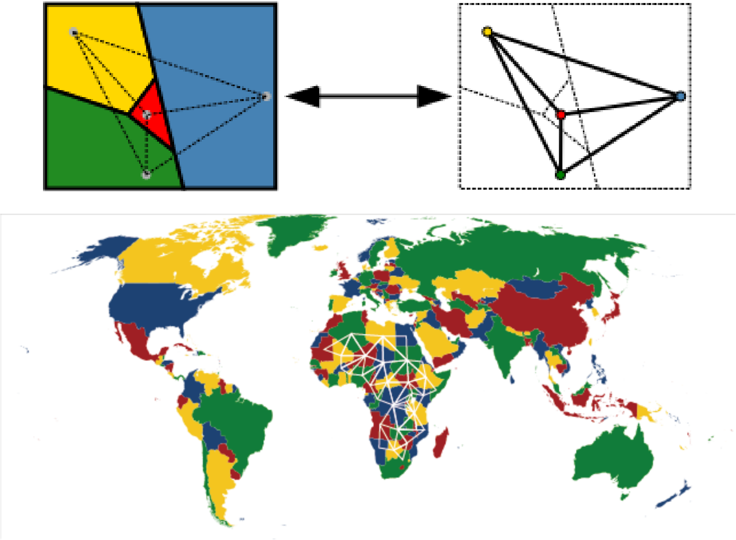

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Next: a slightly weaker bound by Alon, which can be proved using graphs.

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Sum over possible combinations of small subsets.

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- Other interesting applications of the KW-algorithm.

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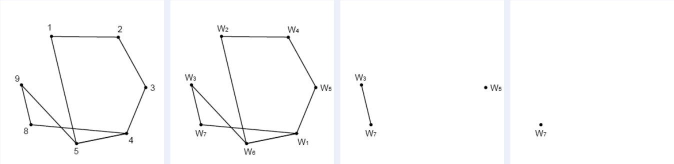

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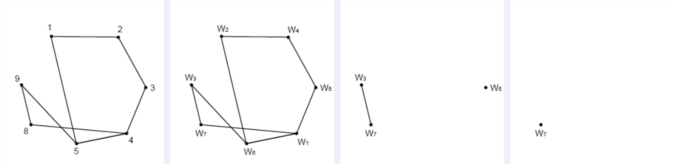

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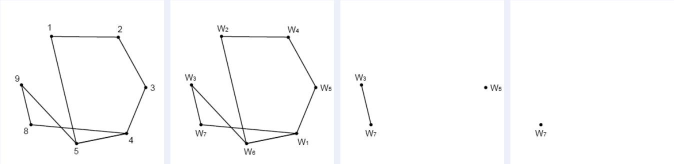

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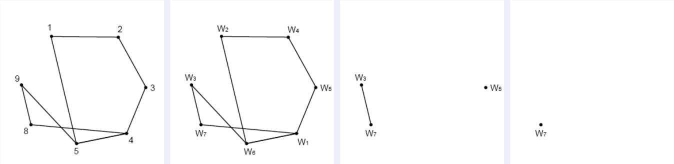

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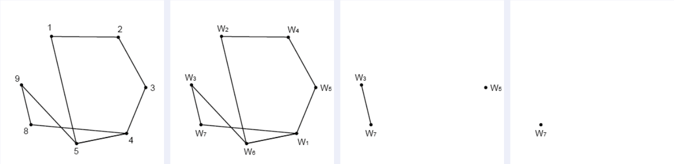

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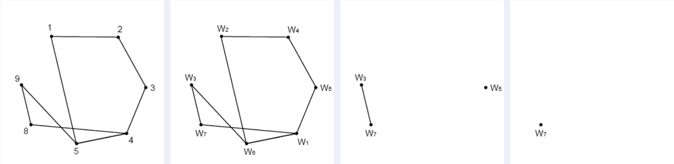

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We have: $i(G, m) \le \sum_{(j_s)} i\left(G\left[A(j_1, j_2, ..., j_q), m - q\right]\right)$

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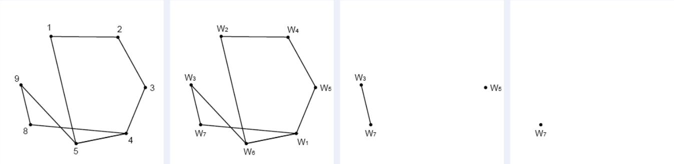
Given a graph G and an independent set I in G, fix an integer $q \leq |I|$.
Idea: reorder vertices in I, find the 'first q' vertices in I.
Outputs: $(j_1, j_2, ..., j_q)$ and $A(j_1, j_2, ..., j_q) \cap I$.

Example (Output)

Give graph G and choose independent set $I = \{1, 3, 8\}$.



$q = 1$: $(j_1) = (2)$ and $A \cap I = \{w_5, w_7\}$. Here $A(j_1) = \{w_3, w_5, w_7\}$.
$q = 2$: $(j_1, j_2) = (2, 2)$ and $A \cap I = \{w_7\}$. Here $A(j_1, j_2) = \{w_7\}$.

We have: $i(G, m) \leq \sum_{(j_s)} i\left(G\left[A(j_1, j_2, ..., j_q), m-q\right]\right) \leq \sum_{(j_s)} \binom{|A(j_1, j_2, \ldots, j_q)|}{m-q}$.

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Theorem (Kahn, 2001; Zhao, 2010)

For each n-vertex d-regular G: $i(G) \leq i(K_{d,d})^{n/2d} = (2^{d+1} - 1)^{n/2d}$.

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Theorem (Sapozhenko, 2001)

There is a constant C so that all n-vertex d-regular graph G satisfy
$i(G) \leq 2^{\left(1+c\sqrt{\log(d)/d}\right)n/2}$.

More Applications

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Theorem (Alon, Balogh, Morris and Samotij, 2014)

Let G be an n-vertex d-regular graph, λ denotes the second eigenvalue of its adjacency matrix. For all positive ϵ, there is a constant C so that if $m \geq Cn/d$ then $i(G, m) \leq \binom{\left(\frac{\lambda}{d+\lambda}+\epsilon\right)n}{m}$.

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Theorem (Kövári, Sós, and Turán; Brown and Erdös, Rényi, and Sós; Kleitman and Winston)

There is a positive constant C s.t. $\left(\frac{1}{2} + o(1)\right) n^{3/2} \leq \log_2 f_n(C_4) \leq Cn^{3/2}$, where $f_n(C_4)$ denotes the number of C_4 free graphs based on n vertices.

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Theorem (Kohayakawa, Luczak, and Rödl, 1996)

For each positive δ, there is a C such that if $C\sqrt{n} \leq m \leq n$, then P(random m-element subset of $[n]$ is δ-Roth)$\to 1$ as $n \to \infty$.

Thank You!