Temporal Information Processing

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Máster en Ciencia de Datos

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Part I.
Introduction

1. Time Series

A time series, also known as discrete time signal, is a sequence of observations taken periodically in time. We can use time series to perform many tasks such as predictions of future values, behaviour analysis or information extraction. Examples of time series are audio signals, industrial instrument measures or diary finantial activity.

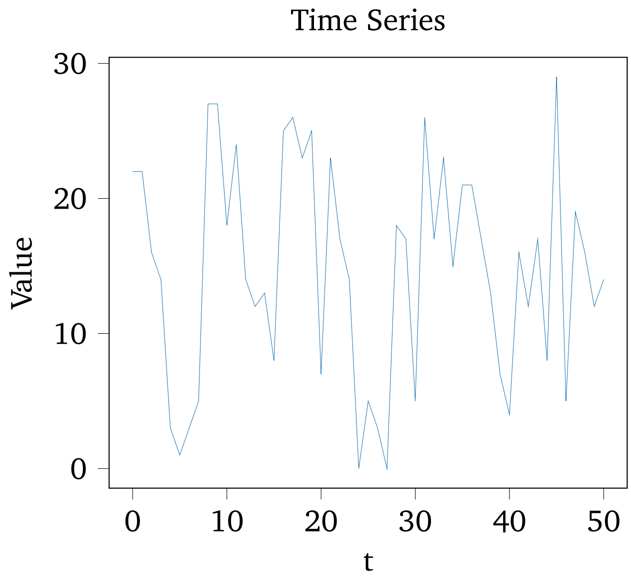


Figure 1: Example of a random time series.

A system can be determined comparing the input and the output. We call the system a filter if it is linear and time invariant. Considering the dynamic system as a black box, we can estimate the transference function or the impulse response to taht filter.

We can also consider **multivariate** time series, where some values of the time series have an influence on the other values in different or the same time instant. We can **classify** the time series in two wide types:

- Determinist: based in dynamic systems, they exploit the phisics of the generation algorithm of the time series.
- Stochastic: where the series are realizations of a stochastic process, which can be modelated.

In this subject, we will focus on stochastic models.

1.1. Stochastic Models

We can make three big considerations on the stochastic models.

- Stationary models.

  Definition 1.1. Let $\{X_t\}$ be a stochastic process and let $F_X\left(x_{t_1+\tau}, \ldots, x_{t_n+\tau}\right)$ represent the CDF of the **unconditional** joint distribution of $\{X_t\}$ at times $t_1 + \tau, \ldots, t_n + \tau$. Then $\{X_t\}$ is strictly stationary if

  $$F_X\left(x_{t_1+\tau}, \ldots, x_{t_n+\tau}\right) = F_X\left(X_{t_1}, \ldots, x_{t_n}\right)$$

  However, we will use the case of **weak stationarity**, where we assume that the expectation of the stochastic process and the covariance at times $t, t + \tau$ are constant.

  Example 1.1. AR, MA, ARMA

- Non stationary models, where we do not make the assumption that the average of the process is constant in time and that there is seasonality

  Example 1.2. ARIMA, SARIMA

- Influenced by exogenous(extern) variables. In this cases, the exogenous variable affects the model, but the model does not affect this variable.

  Example 1.3. SARIMAX

Let us introduce some **notation** for the following explanations

Definition 1.2. Let z_t be the value of the time series at instant t.

- The **backward shift** operator is $z_{t-m} = B^m z_t$
- The **forward shift** operator is $z_{t+m} = F^m z_t = B^{-m} z_t$
- The difference or discrete gradient operator is $\nabla z_t = z_t - z_{t-1} = (1 - B) z_t$

Recall that, having a time series we can consider its **Z-transform**, that converts the discrete-time signal into a complex frequency-domain representation. In the Z-transform representation, the previously introduced notation is:

- The backward shift is $z_{t-m} = B^m z_t = Z^{-m} z_t$
- The forward shift is $z_{t+m} = B^{-m} z_t = Z^m z_t$
- The difference or discrete gradient is $\nabla z_t = (1 - Z^{-1}) z_t$

2. Linear filter based models

The stochastic models we use are based on time series z_t in which sucessive values are highly dependent. In these cases, we can see that the time series is generated from a series of independependent "shocks".

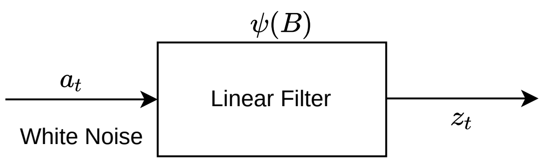
Definition 2.1. Let $a_t \sim \mathcal{N}\left(0, \sigma_a^2\right)$ be *white noise* (where each *shock* is related to a_t) which is not observed. Consider a linear filter that transforms the unobserved a_t to a observed time series z_t. We say that a **linear filter model is**

$$z_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots = \mu + \psi(B) a_t,$$

where

$$\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \dots$$

is called the **transfer function** of the filter.



As we can see, we are expressing the filter in terms of a infinite sum of the coefficients ψ_i. If there are finite coefficients of the sum is *absolutely summable*, that is: $\sum_{j=0}^{\infty} \left|\psi_j\right| < \infty$ or the vector of coefficients has finite ℓ^1 norm, we say that the filter is **stable** and the process z_t is **stationary**.

In the case where the ℓ^1 norm is not finite, our filter are non-stable and produce non-stationary series.

2.1. Autoregressive Models (AR)

Let us firstly consider the simplest case of linear filter. An **autoregressive model** is a linear filter where the current value of the process $\tilde{z}_t$ is expressed as a finite sum of the previous values and a random shock a_t.

Definition 2.2. Let us denote the values of a process af equally spaced times $t, t-1, \dots$ by $z_t, z_{t-1}, \dots$. Consider that the values are centered, that is $\tilde{z}_t = z_t - \mu$. Then, the **autoregressive (AR) process** of **order p** is

$$\tilde{z}_t = \phi_1 \tilde{z}_{t-1} + \phi_2 \tilde{z}_{t-2} + \cdots + \phi_p \tilde{z}_{t-p} + a_t \tag{1}$$

Now, if we define the **autoregressive operator of order p** using the backward shift operator B as:

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p,$$

we can economically write the autoregressive model in (1) as

$$\phi(B)\tilde{z}_t = a_t \tag{2}$$