

1 Smooth Convex Optimization

We are interested in unconstrained minimization of convex and smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}$ given first order oracle

$$\text{minimize}_{x \in \mathbb{R}^n} f(x)$$

We may impose additional assumption on f , i.e. L -lipschitz, μ -strongly convex

1.1 Gradient Descent

Gradient descent achieves sublinear convergence $\mathcal{O}(\frac{1}{\epsilon})$ for $f \in \mathcal{F}_L^1$ and $\mathcal{O}(\log \frac{1}{\epsilon})$ for $f \in \mathcal{S}_{L,\mu}^1$.

$$x^{k+1} = x^k - \alpha_k \nabla f(x^k)$$

for some stepsize $\alpha_k \geq 0$. Note $\alpha_k = \frac{1}{L}$ is the optimal stepsize.

1.2 Gradient Descent with Barzilai & Borwein Stepsize

Barzilai & Borwein stepsize relaxes the constraint on monotonic descent [1]. The idea is to choose α_k such that $\alpha_k g^k$ approximates the Newton update.

$$\alpha_k = \frac{\langle u^k, v^k \rangle}{\|v^k\|^2} \quad \text{or} \quad \alpha_k = \frac{\|u^k\|^2}{\langle u^k, v^k \rangle}$$

where

$$u^k = x^k - x^{k-1} \quad v^k = \nabla f(x^k) - \nabla f(x^{k-1})$$

This algorithm enjoys fast empirical convergence.

1.3 Nesterov's Accelerated Gradient

Nesterov's accelerated gradient achieves lower bound for minimization of function $f \in \mathcal{S}_{L,\mu}^1$ and improves the rate for gradient descent from $\mathcal{O}(\kappa \log \frac{1}{\epsilon})$ to $\mathcal{O}(\sqrt{\kappa} \log \frac{1}{\epsilon})$. Similarly, acceleration improves convergence rate for function $f \in \mathcal{F}_L^1$ from $\mathcal{O}(\frac{1}{\epsilon})$ to $\mathcal{O}(\frac{1}{\sqrt{\epsilon}})$.

1.3.1 Intuition

The following comes from Nesterov's book [2] and [lecture note](#).

Definition. A pair of sequences $(\{\phi_k(x)\}_{k=0}^\infty, \{\lambda_k\}_{k=0}^\infty)$ where $\lambda_k \geq 0$ are called the estimating sequences of the function $f(\cdot)$ if

1. $\lambda_k \rightarrow 0$ and
2. (**lower bound**) for any $x \in \mathbb{R}^n$ and for all $k \geq 0$, $\phi_k(x) \leq (1 - \lambda_k)f(x) + \lambda_k \phi_0(x)$

In addition, If we can find some sequence of points $\{x^k\}_{k=0}^\infty$ such that

3. (**upper bound**) for any $x \in \mathbb{R}^n$, $f(x^k) \leq \phi_k(x)$

then the rate of convergence can be derived from convergence rate of λ_k , i.e.

$$f(x^k) - f^* \leq \lambda_k \{\phi_0^* - f^*\} \rightarrow 0$$

where $\phi_k^* = \min_{x \in \mathbb{R}^n} \phi_k(x)$. Intuitively, $\phi_k(\cdot)$ are approximations for $f(\cdot)$, providing tighter and tighter bound on the optimality gap $f(x^k) - f^*$ as $\lambda_k \rightarrow 0$. In addition, from (2) and (3), we have that the sequence $\{x^k\}$ converges to the minimizer of f .

$$f(x^k) \leq \phi_k(x^*) \leq f(x^*)$$

In [2], Nesterov showed that for $f \in \mathcal{S}_{\mu, L}^1$, we can construct estimating sequences for f recursively

$$\begin{aligned} \lambda_{k+1} &= (1 - \alpha_k) \lambda_k \\ \phi_{k+1}(x) &= (1 - \alpha_k) \phi_k(x) + \alpha_k L_k(x) \\ \text{where } L_k(x) &= f(y^k) + \langle \nabla f(y^k), x - y^k \rangle + \frac{\mu}{2} \|x - y^k\|^2 \end{aligned}$$

where $\{y^k\}_{k=0}^\infty$ is an arbitrary sequence of points, coefficients $\{\alpha_k\}_{k=0}^\infty$ satisfy $\alpha_k \in (0, 1)$ and $\sum_k \alpha_k = \infty$ with $\lambda_0 = 1$ and that $\phi_0(\cdot)$ is an arbitrary convex function. Note that ϕ_k is simply a convex combination of the previous approximate ϕ_{k-1} and a quadratic lower bound L_{k-1} on f , at some carefully chosen point y^{k-1} . If we let $\phi_0(x) = \phi_0^* + \frac{\gamma_0}{2} \|x - v_0\|^2$ be a quadratic function, then $\phi_k(\cdot)$ has a convenient closed form expression

$$\phi_k(x) = \phi_k^* + \frac{\gamma_k}{2} \|x - v_k\|^2$$

where $\{\gamma_k\}, \{v_k\}, \{\phi_k^*\}$ follow certain recurrence relation detailed in [2]. Additional constraint needs to be satisfied to ensure (3) holds.

1. For (3) to hold, it must be that $f(y^k) - \frac{1}{2L} \|\nabla f(y^k)\|^2 \geq f(x^{k+1})$, which can be achieved if we obtain x^{k+1} by taking a gradient step $x^{k+1} = y^k - \frac{1}{L} \nabla f(y^k)$ at y^k and apply descent lemma.
2. To apply the previous, we need the coefficient before $\|\nabla f(y^k)\|^2$ to agree, i.e. want α_k such that $L\alpha_k^2 = (a - \alpha_k)\gamma_k + \alpha_k\mu$.
3. Choose y^k accordingly to ensure (3) holds

By making these constraints invariant to iterative updates, we arrive at the accelerated gradient methods. In addition to the algebra tricks, there are efforts that tries to interpret what Nesterov's method is doing under the hood. For example, [3] interpreted Nesterov's accelerated method as a linear coupling of gradient descent and mirror descent. [4] showed that in the limit of small stepsizes (when taking the gradient step to obtain x^{k+1}) is equivalent to the dynamics of some continuous second-order ODE.

1.3.2 The Algorithm

There are several equivalent algorithm for Nesterov's Accelerated Gradient Method. The following came from the original paper by Nesterov in 1983 [5] and later adapted to LASSO

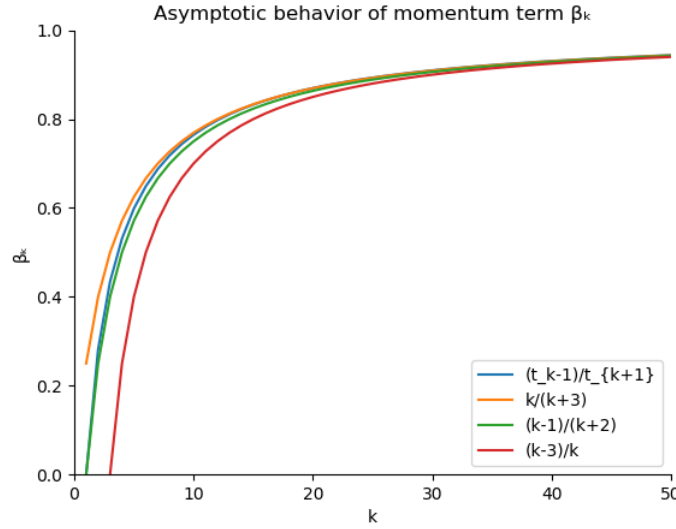
[6]. Assume $f \in \mathcal{F}_L^1$. Given $t_1 = 1$ and $y_1 = x_0$, accelerated gradient updates according to

$$\begin{aligned} x^{k+1} &= y^k - \frac{1}{L} \nabla f(y^k) \\ t_{k+1} &= \frac{1 + \sqrt{1 + 4t_k^2}}{2} \\ y^{k+1} &= x^{k+1} + \frac{t_k - 1}{t_{k+1}} (x^{k+1} - x^k) \end{aligned}$$

We can simplify the expression by noting that (slides)

$$\frac{t_k - 1}{t_{k+1}} = 1 - \frac{3}{k} + o\left(\frac{1}{k}\right) = \frac{k-3}{k} + o\left(\frac{1}{k}\right)$$

The momentum coefficient is asymptotically equivalent to $\frac{k-1}{k+2}$ ($\frac{t_1-1}{t_2} = 0$)



And updates is now given by

$$\begin{aligned} x^{k+1} &= y^k - \frac{1}{L} \nabla f(y^k) \\ y^{k+1} &= x^{k+1} + \frac{k-1}{k+2} (x^{k+1} - x^k) \end{aligned}$$

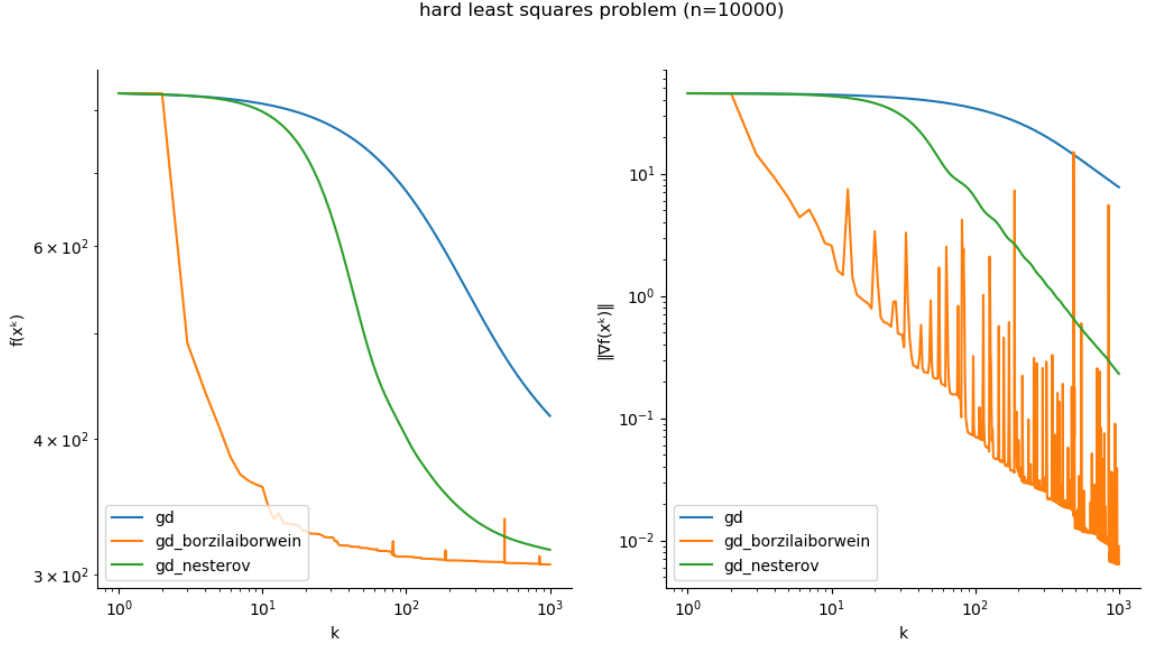
Another formulation of the algorithm comes from Nesterov's textbook [2]. If we take a constant step, i.e. $\frac{1}{L}$, to find the x^{k+1} , and that we pick $\alpha_0 = \sqrt{\frac{\mu}{L}} = 1/\sqrt{\kappa}$, which is the interpolating coefficient for recursive construction of the estimating sequence. Then we have the following updates

$$\begin{aligned} x^{k+1} &= y^k - \frac{1}{L} \nabla f(y^k) \\ y^{k+1} &= x^{k+1} + \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} (x^{k+1} - x^k) \end{aligned}$$

However, in practice the condition number κ is hard to compute.

1.4 Numerical Experiments

We are given a hard least squares problem of minimizing $f(x) = \frac{1}{2} \|D^T x - b\|_2^2$ where $D \in \mathbb{R}^{n \times (n+1)}$ is the differencing matrix, with all -1 on the main diagonal and all 1 on the superdiagonal. The gradient is given by $\nabla f(x) = D(D^T x - b)$. We compare gradient descent with either constant stepsize or using barzilai borwein stepsize, and nesterov's accelerated gradient descent.



We see that the barzilai borwein stepsize is the fastest method, followed by nesterov's accelerated gradient, then the naive gradient descent method.

2 Nonsmooth Convex Optimization

We are interested in unconstrained minimization of convex, possibly nondifferentiable, $f : \mathbb{R}^n \rightarrow \mathbb{R}$

$$\text{minimize}_{x \in \mathbb{R}^n} f(x)$$

given first order oracle

2.1 Subgradient Method

Given bounded subgradient $\|g^k\| \leq G$ and bounded domain $\|x^0 - x^*\| \leq R$, subgradient method is in a sense optimal as it achieves the lower bound $\mathcal{O}(\frac{1}{\epsilon^2})$ for this problem class. Subgradient method iteratively updates as follows

$$x^{k+1} = x^k - \alpha_k g^k$$

where $g^k \in \partial f(x^k)$ is *any* subgradient of f . First order optimality condition is now $0 \in \partial f(x^*)$, which is impossible to test for nontrivial function f . Therefore, using $\|g^k\| \leq \epsilon$ is not informative and subgradient method does not really have a stopping criterion.

2.1.1 Solving Support Vector Machine w/ Subgradient Method

We are given data $\mathcal{D} = \{(x_i, y_i) \mid x_i \in \mathbb{R}^n, y_i \in \{\pm 1\}\}$, support vector machine is supervised learning model that tries to find $w \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that the empirical risk and regularizer on w is minimized

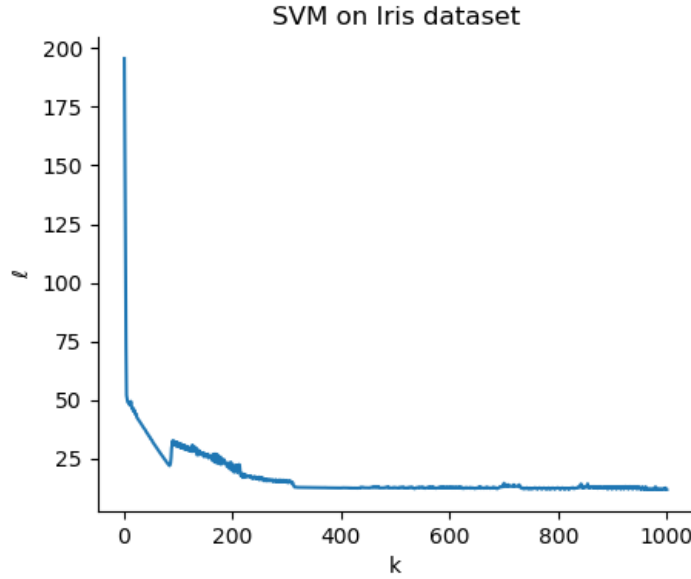
$$\text{minimize}_{w,b} \quad \frac{1}{2} \|w\|_2^2 + \lambda \sum_{i=1}^m \max[0, 1 - y_i(w^T x_i + b)] \quad (:= f(w, b))$$

Support vector machines can be solved using subgradient method. We first find a subgradient of f

$$g_w^k = w^k - \lambda \sum_{i \in [m]: y_i(w^T x_i + b) < 1} y_i x_i$$

$$g_b = -\lambda \sum_{i \in [m]: y_i(w^T x_i + b) < 1} y_i$$

where we have picked $0 \in \partial(\max 0, 1 - y_i(w^T x_i + b))$ when $y_i(w^T x_i + b) = 1$, the only case where the *max term* is non-differentiable. When tested on the Iris dataset, subgradient method worked!



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3 Second Order Methods

For unconstrained problem $\min_{x \in \mathbb{R}^n} f(x)$, the standard Newton scheme updates according to

$$x^{k+1} = x^k - \left[\nabla^2 f(x^k) \right]^{-1} \nabla f(x^k) \quad (1)$$

where $f \in C_L^{2,2}(\mathbb{R}^n)$. The method has quadratic local convergence rate when initial iterate is close to the optimum of f [2]. Cubic regularized Newton's Method converges globally to

second order stationary points ($\nabla f(x) = 0$ and $\nabla^2 f(x) \succeq 0$) assuming $f \in C_L^{2,2}$, i.e. twice continuously differentiable with lipschitz continuous hessian [7, 8]. The idea is to iteratively minimize a global upper bound of the objective,

$$x^{k+1} = \arg \min_{y \in \mathbb{R}^n} \tilde{f}_{x^k} y \quad (2)$$

where $\tilde{f}(y)$ is a cubic regularized quadratic model of the objective,

$$\tilde{f}_x(y) = f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2} \langle \nabla^2 f(x)(y - x), y - x \rangle + \frac{L}{6} \|y - x\|^3 \quad (3)$$

This modified Newton step ensures that function values of iterates are monotonic non-increasing.

4 Cheat Sheet

4.1 Lipschitz Continuous

Definition 1. $f \in C_L^{k,p}(Q)$ is k times continuously differentiable on Q if for all $x, y \in Q$,

$$\|\nabla^p f(y) - \nabla^p f(x)\| \leq L \|y - x\| \quad (4)$$

$f \in C_L^{1,1}(\mathbb{R}^n)$ is continuously differentiable on \mathbb{R}^n if for all $x, y \in \mathbb{R}^n$

$$\|\nabla f(y) - \nabla f(x)\| \leq L \|y - x\| \quad (5)$$

Definition 2. $f \in C_L^{2,1}(\mathbb{R}^n) \subset C_L^{1,1}(\mathbb{R}^n)$ if for all $x \in \mathbb{R}^n$, either condition is satisfied

$$\|\nabla^2 f(x)\| \leq L \quad (6)$$

$$-L\mathbf{I} \preceq \nabla^2 f(x) \preceq L\mathbf{I} \quad (7)$$

Property for $f \in C_L^{1,1}(\mathbb{R}^n)$,

$$|f(y) - f(x) - \langle \nabla f(x), y - x \rangle| \leq \frac{L}{2} \|y - x\|^2 \quad (8)$$

$$f(x - \alpha \nabla f(x)) \leq f(x) - \alpha \left(1 - \frac{\alpha}{2} L\right) \|\nabla f(x)\|^2 \quad (9)$$

$$f\left(x - \frac{1}{L} \nabla f(x)\right) \leq f(x) - \frac{1}{2L} \|\nabla f(x)\|^2 \quad (10)$$

Note (8) implies that quadratic functions $\phi_-(\cdot), \phi_+(\cdot)$ are global lower/upper bound of $f(\cdot)$ respectively, i.e. $\phi_-(y) \leq f(y) \leq \phi_+(y)$ for any $x \in \mathbb{R}^n$, where

$$\phi_-(y) = f(x) + \langle \nabla f(x), y - x \rangle - \frac{L}{2} \|y - x\|^2$$

$$\phi_+(y) = f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2$$

Note (10) is important in proving convergence of descent methods, where we see progress made in reducing function value of iterates by stepping in $-\frac{1}{L} \nabla f(x)$ is at least some constant times the gradient norm.

Property for $f \in C_M^{2,2}(\mathbb{R}^n)$,

$$\|\nabla f(y) - \nabla f(x) - \nabla^2 f(x)(y - x)\| \leq \frac{M}{2} \|y - x\|^2 \quad (11)$$

$$\nabla^2 f(x) - C \preceq \nabla^2 f(y) \preceq \nabla^2 f(x) + C \quad \text{where } c = M \|y - x\| \mathbf{I} \quad (12)$$

4.2 Convex

Definition 3. The following are equivalent

1. A continuously differentiable function f is convex on convex set Q ($f \in \mathcal{F}^1(Q)$)
2. For all $x, y \in Q$

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle \quad (13)$$

3. For all $x, y \in Q$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (14)$$

4. For all $x, y \in Q$,

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle > 0 \quad (15)$$

Definition 4. A twice differentiable function f belongs to $\mathcal{F}^2(Q)$ if for any $x \in Q$,

$$\nabla^2 f(x) \succeq 0 \quad (16)$$

4.3 Smooth & Convex

Definition 5. $f \in \mathcal{F}_L^{1,1}(Q, \|\cdot\|)$ if f is convex with Lipschitz continuous gradient, i.e. for all $x, y \in Q$,

$$\|\nabla f(x) - \nabla f(y)\|_* \leq L \|x - y\| \quad (17)$$

Property of $f \in \mathcal{F}^{1,1}(\mathbb{R}^n, \|\cdot\|)$. Let $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$

$$0 \leq f(y) - f(x) - \langle \nabla f(x), y - x \rangle \leq \frac{L}{2} \|x - y\|^2 \quad (18)$$

$$f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L} \|\nabla f(x) - \nabla f(y)\|_*^2 \leq f(y) \quad (19)$$

$$\frac{1}{L} \|\nabla f(x) - \nabla f(y)\|_*^2 \leq \langle \nabla f(x) - \nabla f(y), x - y \rangle \quad (20)$$

$$0 \leq \langle \nabla f(x) - \nabla f(y), x - y \rangle \leq L \|x - y\|^2 \quad (21)$$

Note (18) implies a tighter (linear) lower bound to $f(\cdot)$ if we assume convexity. In fact, the lower bound can be improved further to a upward quadratic by (19).

4.4 Strongly Convex

Definition 6. A continuously differentiable function f is strongly convex on \mathbb{R}^n ($f \in \mathcal{S}_\mu^1(Q, \|\cdot\|)$) if there exists a convexity parameter $\mu > 0$ such that for all $x, y \in Q$,

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2 \quad (22)$$

Property for $f \in \mathcal{S}_\mu^1(\mathbb{R}^n)$. Let $x, y \in Q$ and $\lambda \in [0, 1]$,

$$f(x) \geq f(x^*) + \frac{\mu}{2} \|x - x^*\|^2 \quad \text{where } \nabla f(x^*) = 0 \quad (23)$$

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \|x - y\|^2 \quad (24)$$

$$\lambda f(x) + (1 - \lambda)f(y) \geq f(\lambda x + (1 - \lambda)y) + \lambda(1 - \lambda)\frac{\mu}{2} \|x - y\|^2 \quad (25)$$

4.5 Smooth & Strongly Convex

Definition 7. A continuously differentiable function f that is strongly convex with L -lipschitz continuous gradients ($f \in \mathcal{S}_{L,\mu}^{1,1}(\mathbb{R}^n)$). Note $\kappa = L/\mu \geq 1$ is the condition number of f .

Property for $f \in \mathcal{S}_{\mu,L}^{1,1}(\mathbb{R}^n)$. For any $x, y \in \mathbb{R}^n$

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{\mu L}{\mu + L} \|x - y\|^2 + \frac{1}{\mu + L} \|\nabla f(x) - \nabla f(y)\|^2 \quad (26)$$

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