Project Proposal

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I am planning to implement diffeomorphic registration with optimal transport cost [1] and possibly extend it to probabilistic settings, referencing [2].

Large deformation metric mapping is a registration algorithm that handles large deformation of objects and ensures the transformations are smooth and invertible [3]. It solves for a time-varying velocity vector field $v_t : \Omega \to \mathbb{R}^n$ for $t \in [0,1]$ dictating the dynamic of a time-varying transformation $\phi_t : \Omega \to \Omega$,

$$\frac{d}{dt}\phi_t = v_t(\phi_t) \qquad \phi_0 = \mathrm{Id} \tag{1}$$

where Ω is ambient space, e.g. $\mathbb{R}^2$ when registering 2d images. The desired transformation is the end point of the above ODE problem, $\varphi = \phi_1 = \phi_0 + \int_0^1 v_t(\phi_t)\, dt$. It has been shown that if the velocities are sufficiently smooth, then φ is a diffeomorphic map. Assume that $\mu = \sum_{i \in I} p_i \delta_{x_i}, \nu = \sum_{j \in J} q_j \delta_j$ is some discrete representation of shape X, Y. We are interested in finding a diffeomorphic transformation φ s.t. the pushforward $\varphi_{\#}\mu$ is similar to the target ν, captured by the following energy

$$\mathcal{E}(\varphi) = \mathcal{R}(\varphi) + \mathcal{L}(\varphi_{\#}\mu, \nu) \tag{2}$$

where $\mathcal{R}$ regularizes φ be smooth and that $\mathcal{L}$ is a data fidelity term. [1] proposes to use a regularized unbalanced optimal transport cost between two discrete measures, i.e. $\mathcal{L} := W_{\epsilon,\rho}$ where

$$W_{\epsilon,\rho}(\mu,\nu) = \min_{\gamma \in \mathbb{R}_+^{I \times J}} \sum_{i,j} c(x_i, x_j)\gamma_{i,j} - \epsilon H(\gamma) + \rho \mathrm{KL}(\gamma \mathbf{1} \mid p) + \rho \mathrm{KL}(\gamma^T \mathbf{1} \mid q) \tag{3}$$

[1] then went on to compute $W_{\epsilon,\rho}$ with generalized Sinkhorn algorithm, and supply the gradient $\nabla W_{\epsilon,\rho}$ with respect to a discrete parameterization of $\varphi_{\#}\mu$ to minimize the energy $\mathcal{E}$.

One possible extension is to extend lddmm+ot to the Bayesian setup. [2] uses variational method to compute an approximate posterior distribution $q(\varphi)$ that is close to the true posterior,

$$\mathrm{KL}(q(\varphi) \| p(\varphi \mid X, Y)) = \mathrm{KL}(q(\varphi) \| p(\varphi)) - \langle \log p(X, Y \mid \varphi) \rangle_q + \log p(X, Y) \tag{4}$$

where φ is assumed to be a Gaussian process $p(\varphi) = \mathcal{GP}(\mu_\varphi, k_\varphi)$. The paper made some derivations and proposed to minimize a lower bound of the form

$$\mathcal{E}'(\varphi) = \mathcal{R}(\varphi) + \langle \mathcal{L}(\varphi_{\#}\mu, \nu) \rangle_{q(\varphi)} \tag{5}$$

We can think of the second term as measuring the average data fidelity of a discrete random variable μ transformed by a random mapping $\varphi \sim q$ to another discrete random variable ν. In case $\mathcal{L} = W_{\epsilon,\rho}$, the question is if there is some simplification of the second term when we are taking expectation with respect to $q(\varphi)$ and/or if semi-discrete transport distance can be used instead somehow. I probably also need to parameterize velocity/transformation instead of optimizing for free variables of the pushforward $\varphi_{\#}\mu$ directly as [1] did.

References

[1] Jean Feydy et al. “Optimal Transport for Diffeomorphic Registration”. In: *arXiv:1706.05218 [math]* (June 16, 2017). arXiv: 1706.05218. URL: http://arxiv.org/abs/1706.05218 (visited on 11/05/2020).

[2] Demian Wassermann et al. “Probabilistic Diffeomorphic Registration: Representing Uncertainty”. In: *arXiv:1701.03266 [cs]* 8545 (2014), pp. 72–82. DOI: 10.1007/978-3-319-08554-8_8. arXiv: 1701.03266. URL: http://arxiv.org/abs/1701.03266 (visited on 03/12/2021).

[3] M. Faisal Beg et al. “Computing Large Deformation Metric Mappings via Geodesic Flows of Diffeomorphisms”. In: *International Journal of Computer Vision* 61.2 (Feb. 1, 2005), pp. 139–157. ISSN: 1573-1405. DOI: 10.1023/B:VISI.0000043755.93987.aa. URL: https://doi.org/10.1023/B:VISI.0000043755.93987.aa (visited on 10/19/2019).