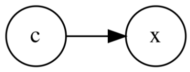
1 InfoGAN

InfoGAN extends the GAN objective to include a new term which encourages high mutual information between generated data and a subset of latent codes [1]. Let (c, z) be latent variable, where c are latent codes capturing semantic features of the data distribution and z are source of incompressible noise.

1.1 Probabilistic Interpretation



A simpler view of method presented in the paper is to consider the above generative model. The joint density can be factorized as follows

$$p_{\mathsf{c},\mathsf{x}} = p_{\mathsf{c}}(c)p_{\mathsf{x}|\mathsf{c}}(x|c) = \prod_{l=1}^{L} p_{\mathsf{c}_l}(c_l)p_{\mathsf{x}|\mathsf{c}}(x|c)$$

The paper implicitly model $p_{\mathsf{x}|\mathsf{c}}$ by using a combination of 1) a deterministic generator $G : \mathcal{C} \times \mathcal{Z} \to \mathcal{X}$ and 2) a stochastic noise sampler $\mathsf{z} \sim p_{\mathsf{z}}$. In particular, $f : \mathcal{C} \to \mathcal{X}; c \mapsto G(c, z)$ for some $z \sim p_{\mathsf{z}}$ is trained to sample from $p_{\mathsf{x}|\mathsf{c}}(\cdot|c)$ using the adversarial loss [2].

1.2 Variational Maximization of Mutual Information

The paper is motivated to construct latent code in such a way such that when given a sample, we would be quite certain what the latent codes are. In other words, we are interested in the following optimization problem

$$\min_{G} H(\mathsf{c}|\mathsf{x}) \qquad \text{where} \qquad \mathsf{x} = G(\mathsf{c}, \mathsf{z}) \tag{1}$$

If we know the parametric family of distribution c is in, this is equivalent to maximizing mutual information between latent codes and generated sample. Given $H(\mathsf{c}|\mathsf{x}) = H(\mathsf{c}) - I(\mathsf{c}; \mathsf{x})$, we can rewrite (2) as

$$\max_{G} I(\mathsf{c}; \mathsf{x}) = \mathbb{E}_{\mathsf{c},\mathsf{x}}\left[\log \frac{p_{\mathsf{c},\mathsf{x}}(c, x)}{p_{\mathsf{c}}(c)p_{\mathsf{x}}(x)}\right]$$

which is intractable, since we do not know the implicit likelihood $p_{\mathsf{x}|\mathsf{c}}$ nor the posterior $p_{\mathsf{c}|\mathsf{x}}$. Instead we approximate $p_{\mathsf{c}|\mathsf{x}}$ with using $q_{\mathsf{c}|\mathsf{x}}$, parameterize by a neural network, and derive a lower bound for the objective [3, 4],

$$\begin{aligned}
I(\mathsf{c}; \mathsf{x}) &= H(\mathsf{c}) - H(\mathsf{c}|\mathsf{x}) \\
&= \sum_x p_{\mathsf{x}}(x) \sum_c p_{\mathsf{c}|\mathsf{x}}(c|x) \log p_{\mathsf{c}|\mathsf{x}}(c|x) + H(\mathsf{c}) \\
&= \sum_x p_{\mathsf{x}}(x) \sum_c p_{\mathsf{c}|\mathsf{x}}(c|x) \log \frac{p_{\mathsf{c}|\mathsf{x}}(c|x)}{q_{\mathsf{c}|\mathsf{x}}(c|x)} + \sum_x p_{\mathsf{x}}(x) \sum_c p_{\mathsf{c}|\mathsf{x}}(c|x) \log q_{\mathsf{c}|\mathsf{x}}(c|x) + H(\mathsf{c}) \\
&= \mathbb{E}_{\mathsf{x}}\left[KL(p_{\mathsf{c}|\mathsf{x}}(c|x)||q_{\mathsf{c}|\mathsf{x}}(c|x))\right] + \mathbb{E}_{\mathsf{c},\mathsf{x}}\left[\log q_{\mathsf{c}|\mathsf{x}}(c|x)\right] + H(\mathsf{c}) \\
&\geq \mathbb{E}_{\mathsf{c},\mathsf{x}}\left[\log q_{\mathsf{c}|\mathsf{x}}(c|x)\right] + H(\mathsf{c}) && (KL \geq 0) \\
&= \mathbb{E}_{\mathsf{c},\mathsf{z}}\left[\log q_{\mathsf{c}|\mathsf{x}}(c|G(c, z))\right] + H(\mathsf{c}) && \text{(LOTUS)}
\end{aligned}$$

1.3 Gradient Estimator

This lower bound can be optimized using stochastic gradient via Monte Carlo estimation,

$$\begin{aligned}
\nabla_\theta \left\{ \mathbb{E}_{\mathsf{c},\mathsf{z}} \left[\log q_{\mathsf{c}|\mathsf{x}}(c|G(c,z)) \right] + H(\mathsf{c}) \right\} &= \mathbb{E}_{\mathsf{c},\mathsf{z}} \left[\nabla_\theta \log q_{\mathsf{c}|\mathsf{x}}(c|G(c,z)) \right] \\
&\approx \sum_{i=1}^{N} \nabla_\theta \log q_{\mathsf{c}|\mathsf{x}}(c^{(i)}|G(c^{(i)}, z^{(i)})) \\
&\quad \text{where} \quad c^{(i)} \sim p_{\mathsf{c}} \quad z^{(i)} \sim p_{\mathsf{z}} \qquad i = 1, \cdots, N
\end{aligned}$$

We could also interpret the idea of randomizing the generator using a noise sampler as performing the reparameterization trick [5]. We avoid taking gradient of expectation with respect to $p_{\mathsf{x}|\mathsf{c}}$; Instead, we take sample from a known distribution $z \sim p_{\mathsf{z}}$ and then compute the desired sample $x = G(c,z)$ via a deterministic function.

1.4 Optimization

Note, Bernoulli distributed $p_{\mathsf{y}|\mathsf{x}}$ is approximated with the discriminator network D, parameterized by θ_D. Similarly, $q_{\mathsf{c}|\mathsf{x}}$ is approximated by a neural netowrk Q, parameterized by θ_Q. We assume $q_{\mathsf{c}|\mathsf{x}}$ to be factored, i.e. $q_{\mathsf{c}|\mathsf{x}} = \prod_i q_{\mathsf{c}_i|\mathsf{x}}$. For each i, $Q(c_i|x)$ outputs the parameters for distributions of c_i, e.g. class probabilities for categorical c_i and mean and variance for Gaussian c_i.

$$\begin{aligned}
p_{\mathsf{y}|\mathsf{x}}(y|x;\theta_D) &= p_1(x;\theta_D)^{\mathbb{1}_{y=1}} (1-p_1)(x;\theta_D)^{\mathbb{1}_{y=0}} & p_1(x;\theta_D) &\leftarrow D(x;\theta_D) \\
q_{\mathsf{c}_i|\mathsf{x}}(c_i|x;\theta_Q) &= \prod_{i=1}^{K} p_k(x;\theta_Q)^{\mathbb{1}_{c_i=k}} & \{p_k(x;\theta_Q)\}_{k=1}^{K} &\leftarrow Q(x;\theta_Q) \\
q_{\mathsf{c}_i|\mathsf{x}}(c_i|x;\theta_Q) &= \mathcal{N}(c_i; \mu(x;\theta_Q), \sigma^2(x;\theta_Q)) & (\mu(x;\theta_Q), \sigma^2(x;\theta_Q)) &\leftarrow Q(x;\theta_Q)
\end{aligned}$$

Let θ_G be parameters for the generator. Following convention in section (2), we can write

$$\begin{aligned}
\mathcal{L}_{GAN}(\theta_D, \theta_G) &= \mathbb{E}_{\mathsf{x}} \left[-\log p_{\mathsf{y}|\mathsf{x}}(1|x;\theta_D) \right] + \mathbb{E}_{\mathsf{c},\mathsf{z}} \left[-\log \left(1 - p_{\mathsf{y}|\mathsf{x}}(1|G(c,z;\theta_G);\theta_D) \right) \right] \\
\mathcal{L}_I(\theta_D, \theta_Q) &= \mathbb{E}_{\mathsf{c},\mathsf{z}} \left[\log q_{\mathsf{c}|\mathsf{x}}(c|G(c,z;\theta_G);\theta_Q) \right]
\end{aligned}$$

We are interested in the following optimization problem

$$\begin{aligned}
&\min_{\theta_G,\theta_Q} \max_{\theta_D} \mathcal{L}_{GAN}(\theta_D,\theta_G) + \mathcal{L}_I(\theta_D,\theta_Q) \\
&\min_{\theta_G,\theta_Q} \max_{\theta_D} \mathbb{E}_{\mathsf{x}} \left[\log p_{\mathsf{y}|\mathsf{x}}(1;x;\theta_D) \right] + \mathbb{E}_{\mathsf{x}'} \left[\log(1 - p_{\mathsf{y}|\mathsf{x}}(1|x';\theta_D)) - \lambda \log q_{\mathsf{c}|\mathsf{x}}(c|x';\theta_Q) \right] \\
&\qquad\qquad \text{where} \qquad x' = G(c,z;\theta_G)
\end{aligned}$$

Similar to equation (3), we can optimizing in an alternating fashion

$$\begin{aligned}
&\min_{\theta_D} \mathbb{E}_{\mathsf{x}} \left[-\log p_{\mathsf{y}|\mathsf{x}}(1|x;\theta_D) \right] + \mathbb{E}_{\mathsf{x}'} \left[-\log \left(1 - p_{\mathsf{y}|\mathsf{x}}(1|x';\theta_D) \right) \right] \\
&\min_{\theta_G} \mathbb{E}_{\mathsf{c},\mathsf{z}} \left[\log \left(1 - p_{\mathsf{y}|\mathsf{x}}(1|G(c,z;\theta_G)) - \lambda \log q_{\mathsf{c}|\mathsf{x}}(c|G(c,z;\theta_G)) \right) \right] \\
&\min_{\theta_Q} \lambda \mathbb{E}_{\mathsf{x}'} \left[-\log q_{\mathsf{c}|\mathsf{x}}(c|x';\theta_Q) \right]
\end{aligned}$$

2 Clarification on GAN's loss

Formulation of GAN loss bears assemblance to the idea of *learning by comparison* in the noise contrastive estimation (NCE) paper [6]. It turns out the connection between hypothesis testing and learning implicit generative models is quite extensively studied [7]. Here is a reproduction of a subset of ideas in these two papers, in addition to a brief comparison between NCE and GAN.

2.1 Learning by Comparison

The goal of both GAN and NCE is to approximate the true data distribution $p_d(\cdot)$ with a parameterized model $p_m(\cdot)$, where learning is driven by classification of which data distribution the sample come from. We formulate this idea below. Let $\mathcal{X}_d = \{x_1, \cdots, x_N\}$ be the training dataset and $\mathcal{X}_g = \{x'_1, \cdots, x'_N\}$ be the generated dataset. Let u be a random variable that takes value on $\mathcal{U} = \mathcal{X}_d \cup \mathcal{X}_g$. We can assign all data points in $\mathcal{U}$ binary class labels $\mathcal{Y} = \{y_i \mid y_i = \mathbb{1}_{u_i \in \mathcal{X}_d}\}$, i.e. assign value of 1 to real data point and 0 to generated data point. We can think of each label following a Bernoulli distribution $\mathsf{y}_i \sim \mathrm{Bern}(p)$. We want to build a *max a posterior* classifier to classify y given u,

$$\hat{y}(u) = \arg\max_{y \in \{0,1\}} p_{\mathsf{y}|\mathsf{u}}(y|u)$$

Equivalently, we can arrive at an equivalent decision rule based on (log) density ratio and notice its similarity to logistic regression. Let $\mathsf{y} \sim \mathrm{Bern}(1/2)$ and use Bayes rule,

$$\begin{aligned}
p_{\mathsf{y}|\mathsf{u}}(1|u) &= \frac{p_{\mathsf{u}|\mathsf{y}}(u|1)p_{\mathsf{y}}(1)}{p_{\mathsf{u}|\mathsf{y}}(u|1)p_{\mathsf{y}}(1) + p_{\mathsf{u}|\mathsf{y}}(u|0)p_{\mathsf{y}}(0)} = \sigma\left(\log \frac{p_{\mathsf{u}|\mathsf{y}}(u|1)}{p_{\mathsf{u}|\mathsf{y}}(u|0)}\right) \\
p_{\mathsf{y}|\mathsf{u}}(0|u) &= 1 - p_{\mathsf{y}|\mathsf{u}}(1|u)
\end{aligned}$$

Let ϕ be parameters for our classifier $\hat{y}(\cdot)$. We want our data $\{(u_i, y_i)\}_{i=1}^{2N}$ to be likely under the result of classification. This is equivalent to maximizing log likelihood of parameters ϕ

$$\begin{aligned}
\ell(\phi) &= \frac{1}{2N} \log \prod_{i=1}^{2N} p_{\mathsf{y}|\mathsf{u}}(y_i|u_i;\phi) \\
&= \frac{1}{2N} \left(\sum_{i=1}^{2N} y \log p_{\mathsf{y}|\mathsf{u}}(1|u_i;\phi) + (1-y) \log p_{\mathsf{y}|\mathsf{u}}(0|u_i;\phi) \right) \\
&= \frac{1}{2N} \sum_{i=1}^{N} \left(\log p_{\mathsf{y}|\mathsf{u}}(1|x_i;\phi) + \log\left(1 - p_{\mathsf{y}|\mathsf{u}}(1|x'_i;\phi)\right)\right) \\
&= \frac{1}{2N} \left(\mathbb{E}_{p_d}\left[\log p_{\mathsf{y}|\mathsf{u}}(1|x;\phi)\right] + \mathbb{E}_{p_g}\left[\log\left(1 - p_{\mathsf{y}|\mathsf{u}}(1|x;\phi)\right)\right]\right) \qquad (2)
\end{aligned}$$

2.2 NCE

In NCE, we model the class conditional likelihood with parametric distributions,

$$\begin{aligned}
p_{\mathsf{u}|\mathsf{y}}(u|1) &= p_m(u;\phi) && \text{(model distribution)} \\
p_{\mathsf{u}|\mathsf{y}}(u|0) &= p_n(u) && \text{(fixed noise distribution)}
\end{aligned}$$

In essence, we estimate parameters for the data distribution by learning the parameters for the classifier, ϕ by maximizing the objective function (2).

2.3 GAN

In GAN, we model the posterior directly with a discriminator network $D : \mathcal{U} \to [0, 1]$

$$p_{\mathsf{y}|\mathsf{u}}(1|u) = D(u; \phi)$$
$$p_{\mathsf{y}|\mathsf{u}}(1|u) = 1 - D(u; \phi)$$

We can interpret the score that the discriminator network computes as an approximation for the log likelihood ratio. In addition to classification, GAN uses a generator network $D : \mathcal{Z} \to \mathcal{X}$, which takes a sample from a latent distribution $z \sim p_{\mathsf{z}}$ to generate samples for an implicit model data distribution $p_m(\cdot)$. GAN's loss can be derived from (2)

$$\begin{aligned}\mathcal{L} &= \mathbb{E}_{p_d}\left[-\log D(x; \phi)\right] + \mathbb{E}_{p_g}\left[-\log\left(1 - D(x; \phi)\right)\right] \\ &= \mathbb{E}_{p_d}\left[-\log D(x; \phi)\right] + \mathbb{E}_{p_z}\left[-\log\left(1 - D(G(z; \theta); \phi)\right)\right] \qquad \text{(LOTUS)}\end{aligned}$$

We form a minimax game where the discriminator tries to identify counterfakes and the generator tries to generate realistic samples.

$$\min_G \max_D \; \mathbb{E}_{p_d}\left[-\log D(x; \phi)\right] + \mathbb{E}_{p_z}\left[-\log\left(1 - D(G(z; \theta); \phi)\right)\right]$$

Note $\mathcal{L}$ is separable with respect to ϕ, θ, so we can do alternating optimization,

$$\begin{aligned}&\min_\phi \; \mathbb{E}_{p_d}\left[-\log D(x; \phi)\right] + \mathbb{E}_{p_g}\left[-\log\left(1 - D(x; \phi)\right)\right] \\ &\min_\theta \; \mathbb{E}_{p_z}\left[\log\left(1 - D(G(z; \theta))\right)\right] \qquad (3)\end{aligned}$$

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