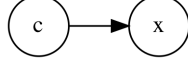# 1 InfoGAN

InfoGAN extends the GAN objective to include a new term which encourages high mutual information between generated data and a subset of latent codes [1]. Let (c, z) be latent variable, where c are latent codes capturing semantic features of the data distribution and z are source of incompressible noise.

1.1 Probabilistic Interpretation

A simpler view of method presented in the paper is to consider the above generative model. The joint density can be factorized as follows

$$p_{c,x} = p_c(c) p_{x|c}(x|c) = \prod_{l=1}^{L} p_{c_l}(c_l) p_{x|c}(x|c)$$

The paper implicitly model $p_{x|c}$ by using a combination of 1) a deterministic generator $G : \mathcal{C} \times \mathcal{Z} \to \mathcal{X}$ and 2) a stochastic noise sampler $z \sim p_z$. In particular, $f : \mathcal{C} \to \mathcal{X}; c \mapsto G(c, z)$ for some $z \sim p_z$ is trained to sample from $p_{x|c}(\cdot|c)$ using the adversarial loss [2].

1.2 Variational Maximization of Mutual Information

The paper is motivated to construct latent code in such a way such that when given a sample, we would be quite certain what the latent codes are. In other words, we are interested in the following optimization problem

$$\min_{G} H(c|x) \qquad \text{where} \qquad x = G(c, z) \tag{1}$$

If we know the parametric family of distribution c is in, this is equivalent to maximizing mutual information between latent codes and generated sample. Given $H(c|x) = H(c) - I(c; x)$, we can rewrite (2) as

$$\max_{G} I(c; x) = \mathbb{E}_{c,x} \left[\log \frac{p_{c,x}(c, x)}{p_c(c) p_x(x)} \right]$$

which is intractable, since we do not know the implicit likelihood $p_{x|c}$ nor the posterior $p_{c|x}$. Instead we approximate $p_{c|x}$ with using $q_{c|x}$, parameterize by a neural network, and derive a lower bound for the objective [3, 4],

$$
\begin{aligned}
I(c; x) &= H(c) - H(c|x) \\
&= \sum_x p_x(x) \sum_c p_{c|x}(c|x) \log p_{c|x}(c|x) + H(c) \\
&= \sum_x p_x(x) \sum_c p_{c|x}(c|x) \log \frac{p_{c|x}(c|x)}{q_{c|x}(c|x)} + \sum_x p_x(x) \sum_c p_{c|x}(c|x) \log q_{c|x}(c|x) + H(c) \\
&= \mathbb{E}_x \left[KL(p_{c|x}(c|x) || q_{c|x}(c|x)) \right] + \mathbb{E}_{c,x} \left[\log q_{c|x}(c|x) \right] + H(c) \\
&\geq \mathbb{E}_{c,x} \left[\log q_{c|x}(c|x) \right] + H(c) && (KL \geq 0) \\
&= \mathbb{E}_{c,z} \left[\log q_{c|x}(c|G(c, z)) \right] + H(c) && (\text{LOTUS})
\end{aligned}
$$

1.3 Gradient Estimator

This lower bound can be optimized using stochastic gradient via Monte Carlo estimation,

$$\nabla_\theta \left\{ \mathbb{E}_{\mathsf{c},\mathsf{z}} \left[\log q_{\mathsf{c}|\mathsf{x}}(c|G(c,z)) \right] + H(\mathsf{c}) \right\} = \mathbb{E}_{\mathsf{c},\mathsf{z}} \left[\nabla_\theta \log q_{\mathsf{c}|\mathsf{x}}(c|G(c,z)) \right]$$

$$\approx \sum_{i=1}^{N} \nabla_\theta \log q_{\mathsf{c}|\mathsf{x}}(c^{(i)}|G(c^{(i)}, z^{(i)}))$$

$$\text{where} \quad c^{(i)} \sim p_{\mathsf{c}} \quad z^{(i)} \sim p_{\mathsf{z}} \quad i = 1, \cdots, N$$

We could also interpret the idea of randomizing the generator using a noise sampler as performing the reparameterization trick [5]. We avoid taking gradient of expectation with respect to $p_{\mathsf{x}|\mathsf{c}}$; Instead, we take sample from a known distribution $z \sim p_{\mathsf{z}}$ and then compute the desired sample $x = G(c,z)$ via a deterministic function.

1.4 Optimization

Note, Bernoulli distributed $p_{\mathsf{y}|\mathsf{x}}$ is approximated with the discriminator network D, parameterized by θ_D. Similarly, $q_{\mathsf{c}|\mathsf{x}}$ is approximated by a neural netowrk Q, parameterized by θ_Q. We assume $q_{\mathsf{c}|\mathsf{x}}$ to be factored, i.e. $q_{\mathsf{c}|\mathsf{x}} = \prod_i q_{\mathsf{c}_i|\mathsf{x}}$. For each i, $Q(c_i|x)$ outputs the parameters for distributions of c_i, e.g. class probabilities for categorical c_i and mean and variance for Gaussian c_i.

$$p_{\mathsf{y}|\mathsf{x}}(y|x;\theta_D) = p(x;\theta_D)^{\mathbb{1}_{y=1}}(1 - p(x;\theta_D))^{\mathbb{1}_{y=0}} \qquad\qquad p(x;\theta_D) \leftarrow D(x;\theta_D)$$

$$q_{\mathsf{c}_i|\mathsf{x}}(c_i|x;\theta_Q) = \prod_{i=1}^{K} p_k(x;\theta_Q)^{\mathbb{1}_{c_i=k}} \qquad\qquad \{p_k(x;\theta_Q)\}_{k=1}^{K} \leftarrow Q(x;\theta_Q)$$

$$q_{\mathsf{c}_i|\mathsf{x}}(c_i|x;\theta_Q) = \mathcal{N}(c_i; \mu(x;\theta_Q), \sigma^2(x;\theta_Q)) \qquad\qquad (\mu(x;\theta_Q), \sigma^2(x;\theta_Q)) \leftarrow Q(x;\theta_Q)$$

Let θ_G be parameters for the generator. Following convention in section (2), we can write

$$\mathcal{L}_{GAN}(\theta_D, \theta_G) = \mathbb{E}_{\mathsf{x}} \left[-\log p_{\mathsf{y}|\mathsf{x}}(1|x;\theta_D) \right] + \mathbb{E}_{\mathsf{c},\mathsf{z}} \left[-\log \left(1 - p_{\mathsf{y}|\mathsf{x}}(1|G(c,z;\theta_G);\theta_D)\right) \right]$$
$$\mathcal{L}_I(\theta_D, \theta_Q) = \mathbb{E}_{\mathsf{c},\mathsf{z}} \left[\log q_{\mathsf{c}|\mathsf{x}}(c|G(c,z;\theta_G);\theta_Q) \right]$$

We are interested in the following optimization problem

$$\min_{\theta_G, \theta_Q} \max_{\theta_D} \mathcal{L}_{GAN}(\theta_D, \theta_G) + \mathcal{L}_I(\theta_D, \theta_Q)$$
$$\min_{\theta_G, \theta_Q} \max_{\theta_D} \mathbb{E}_{\mathsf{x}} \left[\log p_{\mathsf{y}|\mathsf{x}}(1; x;\theta_D) \right] + \mathbb{E}_{\mathsf{x}'} \left[\log(1 - p_{\mathsf{y}|\mathsf{x}}(1|x';\theta_D)) - \lambda \log q_{\mathsf{c}|\mathsf{x}}(c|x';\theta_Q) \right]$$
$$\text{where} \qquad x' = G(c,z;\theta_G)$$

Similar to equation (3), we can optimizing in an alternating fashion

$$\min_{\theta_D} \mathbb{E}_{\mathsf{x}} \left[-\log p_{\mathsf{y}|\mathsf{x}}(1|x;\theta_D) \right] + \mathbb{E}_{\mathsf{x}'} \left[-\log \left(1 - p_{\mathsf{y}|\mathsf{x}}(1|x';\theta_D)\right) \right]$$
$$\min_{\theta_G} \mathbb{E}_{\mathsf{c},\mathsf{z}} \left[\log \left(1 - p_{\mathsf{y}|\mathsf{x}}(1|G(c,z;\theta_G)) - \lambda \log q_{\mathsf{c}|\mathsf{x}}(c|G(c,z;\theta_G))\right) \right]$$
$$\min_{\theta_Q} \lambda \mathbb{E}_{\mathsf{x}'} \left[-\log q_{\mathsf{c}|\mathsf{x}}(c|x';\theta_Q) \right]$$

2 Clarification on GAN's loss

Formulation of GAN loss bears assemblance to the idea of *learning by comparison* in the noise contrastive estimation (NCE) paper [6]. It turns out the connection between hypothesis testing and learning implicit generative models is quite extensively studied [7]. Here is a reproduction of a subset of ideas in these two papers, in addition to a brief comparison between NCE and GAN.

2.1 Learning by Comparison

The goal of both GAN and NCE is to approximate the true data distribution $p_d(\cdot)$ with a parameterized model $p_m(\cdot)$, where learning is driven by classification of which data distribution the sample come from. We formulate this idea below. Let $\mathcal{X}_d = \{x_1, \cdots, x_N\}$ be the training dataset and $\mathcal{X}_g = \{x'_1, \cdots, x'_N\}$ be the generated dataset. Let u be a random variable that takes value on $\mathcal{U} = \mathcal{X}_d \cup \mathcal{X}_g$. We can assign all data points in $\mathcal{U}$ binary class labels $\mathcal{Y} = \{y_i \mid y_i = \mathbb{1}_{u_i \in \mathcal{X}_d}\}$, i.e. assign value of 1 to real data point and 0 to generated data point. We can think of each label following a Bernoulli distribution $\mathsf{y}_i \sim \text{Bern}(p)$. We want to build a *max a posterior* classifier to classify y given u,

$$\hat{y}(u) = \arg \max_{y \in \{0,1\}} p_{\mathsf{y}|\mathsf{u}}(y|u)$$

Equivalently, we can arrive at an equivalent decision rule based on (log) density ratio. Let $\mathsf{y} \sim \text{Bern}(1/2)$ and use Bayes rule,

$$p_{\mathsf{y}|\mathsf{u}}(1|u) = \frac{p_{\mathsf{u}|\mathsf{y}}(u|1)p_{\mathsf{y}}(1)}{p_{\mathsf{u}|\mathsf{y}}(u|1)p_{\mathsf{y}}(1) + p_{\mathsf{u}|\mathsf{y}}(u|0)p_{\mathsf{y}}(0)} = \sigma \left(\log \frac{p_{\mathsf{u}|\mathsf{y}}(u|1)}{p_{\mathsf{u}|\mathsf{y}}(u|0)} \right)$$

$$p_{\mathsf{y}|\mathsf{u}}(0|u) = 1 - p_{\mathsf{y}|\mathsf{u}}(1|u)$$

Let ϕ be parameters for our classifier $\hat{y}(\cdot)$. We want our data $\{(u_i, y_i)\}_{i=1}^{2N}$ to be likely under the result of classification. This is equivalent to maximizing log likelihood of parameters ϕ

$$
\begin{aligned}
\ell(\phi) &= \frac{1}{2N} \log \prod_{i=1}^{2N} p_{\mathsf{y}|\mathsf{u}}(y_i|u_i; \phi) \\
&= \frac{1}{2N} \sum_{i=1}^{2N} \left(y \log p_{\mathsf{y}|\mathsf{u}}(1|u_i; \phi) + (1-y) \log p_{\mathsf{y}|\mathsf{u}}(0|u_i; \phi) \right) \\
&= \frac{1}{2N} \sum_{i=1}^{N} \left(\log p_{\mathsf{y}|\mathsf{u}}(1|x_i; \phi) + \log \left(1 - p_{\mathsf{y}|\mathsf{u}}(1|x'_i; \phi) \right) \right) \\
&= \frac{1}{2N} \left(\mathbb{E}_{p_d} \left[\log p_{\mathsf{y}|\mathsf{u}}(1|x; \phi) \right] + \mathbb{E}_{p_g} \left[\log \left(1 - p_{\mathsf{y}|\mathsf{u}}(1|x; \phi) \right) \right] \right)
\end{aligned}
\tag{2}
$$

2.2 NCE

In NCE, we model the class conditional likelihood with parametric distributions,

$$p_{\mathsf{u}|\mathsf{y}}(u|1) = p_m(u; \phi) \qquad \text{(model distribution)}$$
$$p_{\mathsf{u}|\mathsf{y}}(u|0) = p_n(u) \qquad \text{(fixed noise distribution)}$$

In essence, we estimate parameters for the data distribution by learning the parameters for the classifier, ϕ by maximizing the objective function (2).

2.3 GAN

In GAN, we model the posterior directly with a discriminator network $D : \mathcal{U} \to [0, 1]$

$$p_{\mathsf{y}|\mathsf{u}}(1|u) = D(u; \phi)$$
$$p_{\mathsf{y}|\mathsf{u}}(1|u) = 1 - D(u; \phi)$$

We can interpret the score that the discriminator network computes as an approximation for the log likelihood ratio. In addition to classification, GAN uses a generator network $D : \mathcal{Z} \to \mathcal{X}$, which takes a sample from a latent distribution $z \sim p_{\mathsf{z}}$ to generate samples for an implicit model data distribution $p_m(\cdot)$. GAN's loss can be derived from (2)

$$\mathcal{L} = \mathbb{E}_{p_d} \left[-\log D(x; \phi) \right] + \mathbb{E}_{p_g} \left[-\log \left(1 - D(x; \phi) \right) \right]$$
$$= \mathbb{E}_{p_d} \left[-\log D(x; \phi) \right] + \mathbb{E}_{p_z} \left[-\log \left(1 - D(G(z; \theta); \phi) \right) \right] \qquad \text{(LOTUS)}$$

We form a minimax game where the discriminator tries to identify counterfeits and the generator tries to generate realistic samples.

$$\min_G \max_D \; \mathbb{E}_{p_d} \left[-\log D(x; \phi) \right] + \mathbb{E}_{p_z} \left[-\log \left(1 - D(G(z; \theta); \phi) \right) \right]$$

Note $\mathcal{L}$ is separable with respect to ϕ, θ, so we can do alternating optimization,

$$\min_\phi \; \mathbb{E}_{p_d} \left[-\log D(x; \phi) \right] + \mathbb{E}_{p_g} \left[-\log \left(1 - D(x; \phi) \right) \right]$$
$$\min_\theta \; \mathbb{E}_{p_z} \left[\log \left(1 - D(G(z; \theta)) \right) \right] \qquad\qquad (3)$$

References

[1] Xi Chen et al. "InfoGAN: Interpretable Representation Learning by Information Maximizing Generative Adversarial Nets". In: *arXiv:1606.03657 [cs, stat]* (June 11, 2016). arXiv: 1606.03657. URL: http://arxiv.org/abs/1606.03657 (visited on 12/04/2019).

[2] Ian J. Goodfellow et al. "Generative Adversarial Networks". In: *arXiv:1406.2661 [cs, stat]* (June 10, 2014). arXiv: 1406.2661. URL: http://arxiv.org/abs/1406.2661 (visited on 12/12/2019).

[3] David Barber and Felix Agakov. "The IM Algorithm: A Variational Approach to Information Maximization." In: Jan. 1, 2003.

[4] Ben Poole and Sherjil Ozair. "On Variational Bounds of Mutual Information". In: (2019), p. 10.

[5] Diederik P. Kingma and Max Welling. "Auto-Encoding Variational Bayes". In: *arXiv:1312.6114 [cs, stat]* (May 1, 2014). arXiv: 1312.6114. URL: http://arxiv.org/abs/1312.6114 (visited on 11/13/2019).

[6] Michael Gutmann and Aapo Hyvärinen. "Noise-contrastive estimation: A new estimation principle for unnormalized statistical models". In: *Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics*. Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics. Mar. 31, 2010, pp. 297–304. URL: http://proceedings.mlr.press/v9/gutmann10a.html (visited on 12/10/2019).

[7] Shakir Mohamed and Balaji Lakshminarayanan. "Learning in Implicit Generative Models". In: *arXiv:1610.03483 [cs, stat]* (Feb. 27, 2017). arXiv: 1610.03483. URL: http://arxiv.org/abs/1610.03483 (visited on 01/27/2020).