

# 1 Gaussian Process

[1] introduces Gaussian Process regression as an alternative view to Bayesian regression.

## 1.1 Bayesian Regression

In linear regression setup, we assume output  $y \in \mathbb{R}$  is a linear function of inputs  $x \in \mathbb{R}^d$ , corrupted with additive iid normal noise,

$$y = f(x) + \epsilon \quad \text{where} \quad f(x) = w^T x \quad \text{and} \quad \epsilon \sim \mathcal{N}(0, \sigma_n^2) \quad (1)$$

The Bayesian setup considers  $w \in \mathbb{R}^d$  as a random variable, endowed with prior  $w \sim \mathcal{N}(0, \Sigma_p)$ . Using Bayes rule, we can find the posterior of weights given data, which is again a normal random variable  $p(w | X, y) = \mathcal{N}(w; A^{-1}b, A^{-1})$  where  $A = \frac{1}{\sigma_n^2} X^T X + \Sigma_p^{-1}$  and  $b = \frac{1}{\sigma_n^2} X^T y$  and  $X \in \mathbb{R}^{n \times d}$ ,  $y \in \mathbb{R}^{n \times 1}$  are design matrices. For test point  $x_*$ , the predictive distribution of  $f_* = f(x_*)$  is the average likelihood of  $f_*$  under model  $f(x; w)$  with respect to posterior of  $w$ .

$$p(f_* | x_*, X, y) = \int p(f_* | x_*, w) p(w | X, y) dw \quad (2)$$

We can think of the predictive distribution as a linear function  $f_* = x_*^T w$  of weights, a normal random variable, and therefore is normal. Therefore,  $f_* | x_*, X, y \sim \mathcal{N}(x_*^T A^{-1} b, x_*^T A^{-1} x_*)$ . The natural extension to Bayesian linear regression is to kernelize it, for example assume a linear model in some feature space  $f(x) = \phi(x)^T w$  for some feature map  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^D$ . We can write predictive distribution as

$$f_* | x_*, X, y \sim \mathcal{N}(k(x_*, X)(k(X, X) + \sigma_n^2 I)^{-1} y, \quad (3)$$

$$k(x_*, x_*) - k(x_*, X)(k(X, X) + \sigma_n^2 I)^{-1} k(X, x_*)) \quad (4)$$

where  $k(X, X') = \Phi \Sigma_p \Phi^T \in \mathbb{R}^{n \times n'}$  and  $\Phi \in \mathbb{R}^{n \times D}$  are the feature vectors.

## 1.2 Gaussian Process

**Definition 1.** (Gaussian Process) A Gaussian process is a stochastic process  $\{X_t\}_{t \in T}$  such that, for every finite subset of indices  $t_1, \dots, t_k \in T$ ,  $(X_{t_1}, \dots, X_{t_k})$  is multivariate normal.

A Gaussian process  $f \sim \mathcal{GP}(m, k)$  over  $\mathcal{X}$  is fully specified by the mean function  $m : \mathcal{X} \rightarrow \mathbb{R}$  and covariance function  $k : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$  where

$$m(x) = \mathbb{E}[f(x)] \quad k(x, x') = \mathbb{E}[(f(x) - m(x))(f(x') - m(x')))] \quad (5)$$

The covariance function determines function's behavior, for example its stationarity, smoothness, and periodicity etc. For example, the squared exponential covariance function  $k_{SE}(x, x') = \exp(-\frac{1}{2\ell^2} \|x - x'\|_2^2)$  enforces the prior knowledge that functions are smooth, i.e. inputs are close in the Euclidean sense will have similar outputs. See Figure (1) for some examples of samples from Gaussian process.

**Example 1.** (Intuition about covariance matrix for  $\mathcal{GP}$ ) First consider  $(y_1, y_2) \sim \mathcal{N}(\mathbf{0}, \Sigma)$  where  $\Sigma_{11} = \Sigma_{22} = 1$  and  $\Sigma_{12} = \Sigma_{21} = \rho_{12}$ . We know  $y_1 | y_2 = a \sim \mathcal{N}(\rho_{12}a, 1 - \rho_{12}^2)$ . When  $\text{Cov}(y_1, y_2) = \rho_{12} \uparrow 1$ ,  $y_1$ 's samples conditioned on  $y_2 = a$  fall close to  $a$  with high probability. When  $\rho_{12} \downarrow 0$ ,  $y_1 | y_2 = a$  will distribute like a unit normal, regardless of values that  $y_2$  take. Extend this intuition to gaussian process: whenever  $k(x_i, x_j)$  is large,  $y_i, y_j$  are correlated and so observing  $y_i = a$  provides a strong prior on how  $y_j$  will behave, or in other words, reduce the uncertainty of values that  $y_j$  can take dramatically.

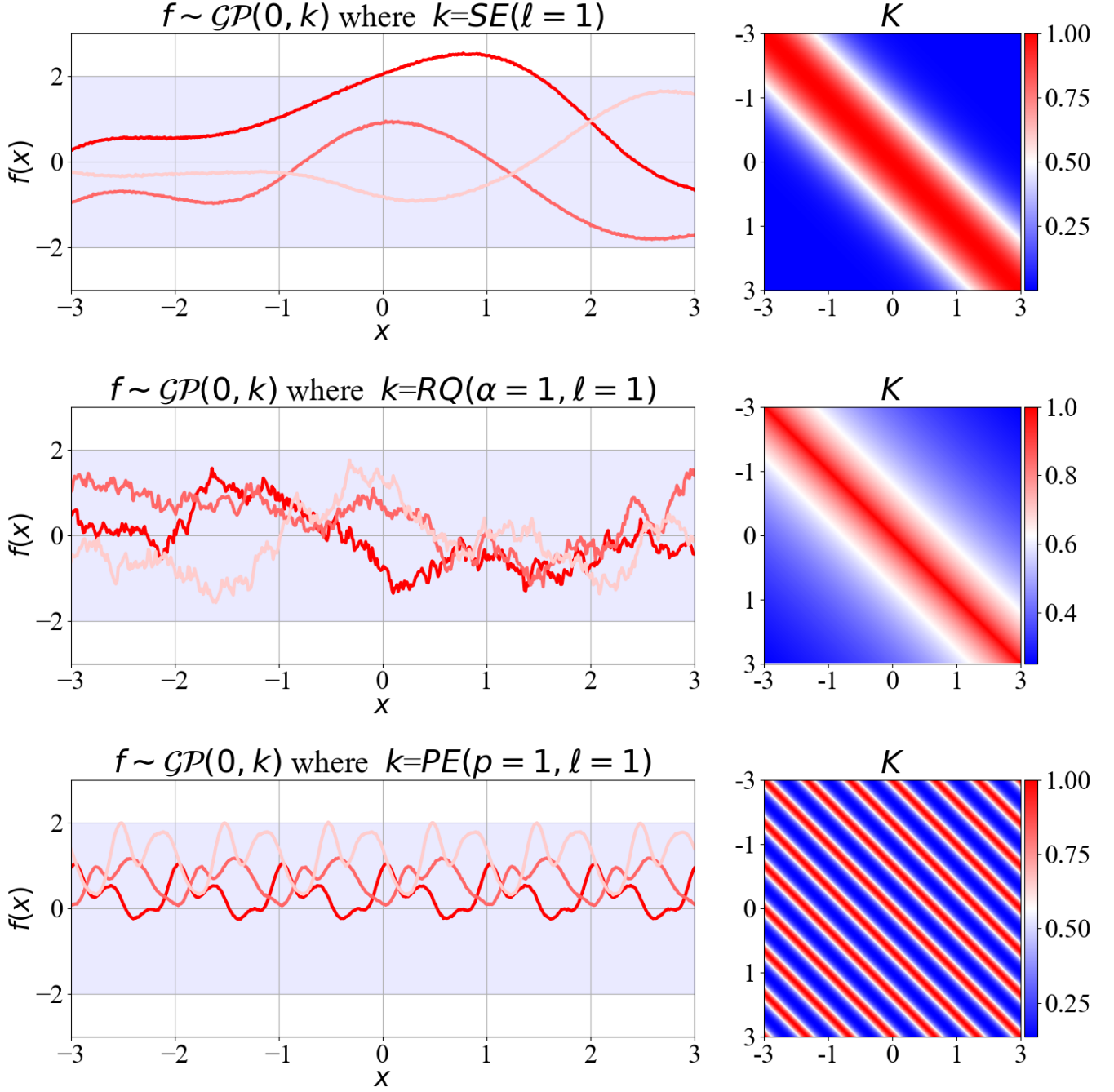


Figure 1: (Left) Samples from Gaussian process prior and (Right) covariance matrix at test locations.

### 1.3 Gaussian Process Regression

Instead of placing a prior over weights  $p(w)$  to quantify randomness in function  $f(x) = \phi(x)^T w$ , we model function directly as a Gaussian process,  $f \sim \mathcal{GP}(0, k)$ . There is a one-to-one correspondence between the two views. For example,  $f(x) = \phi(x)^T w$  with prior  $w \sim \mathcal{N}(0, \Sigma_p)$  used in kernel Bayesian regression has

$$\mathbb{E}[f(x)] = \phi(x)^T \mathbb{E}[w] = 0 \quad \mathbb{E}[f(x)f(x')] = \phi(x)^T \Sigma_p \phi(x') \quad (6)$$

Therefore,  $f \sim \mathcal{GP}(0, k)$  where  $k(x, x') = \phi(x)^T \Sigma_p \phi(x')$ . Note  $k$  is in fact a valid kernel. (Since  $\Sigma_p$  is psd,  $\Sigma_p = UDU^T$  by SVD. We can write  $k(x, x') = \langle \psi(x), \psi(x') \rangle$  where  $\psi(x) = \Sigma_p^{1/2} \phi(x)$  and  $\Sigma_p^{1/2} = UD^{1/2}U^T$ ). Conversely, any valid kernel used in kernel Bayesian regression can be used to parameterize the covariance function of the Gaussian process model over  $f$ .

Since  $y = f(x) + \epsilon$ , we have  $\mathbf{y} = \mathbf{f} + \sigma_n^2 I \sim \mathcal{N}(0, k(X, X) + \sigma_n^2 I)$  where  $\mathbf{y}, \mathbf{f} \in \mathbb{R}^{n \times 1}$ . We can write the joint distribution of observed values and function values at some test locations  $X_*$  as

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{f}_* \end{bmatrix} \sim \mathcal{N} \left( \mathbf{0}, \begin{bmatrix} k(X, X) + \sigma_n^2 I & k(X, X_*) \\ k(X_*, X) & k(X_*, X_*) \end{bmatrix} \right) \quad (7)$$

We can derive the predictive distribution for  $\mathbf{f}_* | \mathbf{y}$  by simply apply conditional distribution formula

$$\mathbf{f}_* | X, \mathbf{y}, X_* \sim \mathcal{N}(k(X_*, X)(k(X, X) + \sigma_n^2 I)^{-1} \mathbf{y} \quad (8)$$

$$k(X_*, X_*) - k(X_*, X)(k(X, X) + \sigma_n^2 I)^{-1} k(X, X_*)) \quad (9)$$

which has exact form compared to (4). More compactly, for a single test point  $x_*$ ,  $\mu_{\mathbf{f}_*} = k_*^T (K + \sigma_n^2 I)^{-1} \mathbf{y}$  and  $\text{Var}(\mathbf{f}_*) = k(x_*, x_*) - k_*^T (K + \sigma_n^2 I)^{-1} k_*$  where  $k_* = k(X, x_*) \in \mathbb{R}^{n \times 1}$ . See Figure (2) for examples of Gaussian process regression with varying data size and hyperparameters.

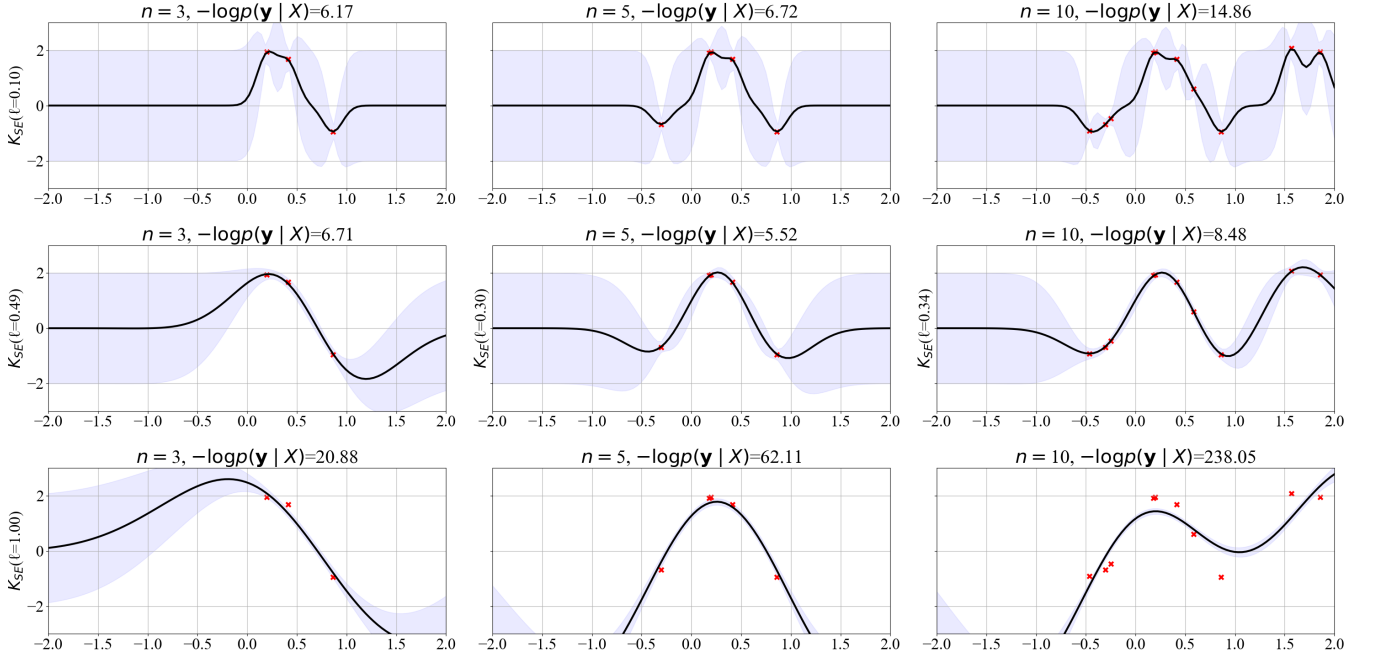


Figure 2: This plot shows mean (black) and 95% confidence interval (light blue) for predictive distribution  $p(\mathbf{f}_* | X, \mathbf{y}, X_*)$  fit using Gaussian process regression assuming a SE prior over  $f \sim \mathcal{GP}(0, k_{SE})$  of varying lengthscale  $\ell$  and observed sample sizes  $n$ . The middle row's kernel hyperparameter is taken to be the empirical Bayes estimate  $\ell = \arg \max p(\mathbf{y} | X, \ell)$  where  $p(\mathbf{y} | X, \ell)$  given by Equation (11), optimized via gradient descent.

## 1.4 Model Selection

Model selection for Gaussian process regression involves picking the form and the hyperparameters of the covariance function. Given data  $(X, \mathbf{y})$ , marginal likelihood  $p(\mathbf{y} | X)$  quantifies how likely data is observed

under our additive noise model  $\mathbf{y} \mid \mathbf{f} \sim \mathcal{N}(\mathbf{f}, \sigma_n^2 I)$  on average with respect to latent function values (or parameters of our model)  $\mathbf{f} \mid X \sim \mathcal{N}(0, K)$  (due to assumption of Gaussian process prior over  $\mathbf{f}$ ),

$$p(\mathbf{y} \mid X) = \int p(\mathbf{y} \mid \mathbf{f}, X) p(\mathbf{f} \mid X) d\mathbf{f} \quad (10)$$

This formulation is analogous to that in Bayesian linear regression where the marginal likelihood marginalizes over weights  $p(\mathbf{y} \mid X) = \int p(\mathbf{y} \mid X, w) p(w) dw$ . We can obtain a closed form expression by reading off (7), i.e.  $\mathbf{y} \sim \mathcal{N}(\mathbf{0}, K + \sigma_n^2 I)$ .

$$\log p(\mathbf{y} \mid X) = -\frac{1}{2} \mathbf{y}^T (K + \sigma_n^2 I)^{-1} \mathbf{y} - \frac{1}{2} \log |K + \sigma_n^2 I| - \frac{n}{2} \log 2\pi \quad (11)$$

In Bayesian model selection, model hyperparameters for Gaussian process regression  $\theta = \{\sigma_n, \ell\}$  can be found by maximizing the marginal likelihood or the type II likelihood  $\theta^* = \arg \max_{\theta} \log p(\mathbf{y} \mid X, \theta)$ .

## References

- [1] Carl Edward Rasmussen and Williams Christopher. *Gaussian Process for Machine Learning*. MIT Press, 2006.