1 Time Variable LQR

- System: $\boldsymbol{x}_{[t+1]} = \boldsymbol{A}_{[t]}\boldsymbol{x}_{[t]} + \boldsymbol{B}_{[t]}\boldsymbol{u}_{[t]}$
- Cost:

$$J = \sum_{t=1}^{T} g(\boldsymbol{x}_{[t]}, \boldsymbol{u}_{[t]}) = \sum_{t=1}^{T} \boldsymbol{x}_{[t]}^T \boldsymbol{Q}_{[t]}\boldsymbol{x}_{[t]} + \boldsymbol{u}_{[t]}^T \boldsymbol{R}_{[t]}\boldsymbol{u}_{[t]}$$

- Value function:

$$V(\boldsymbol{x}_{[t]}, t) = \min_{\boldsymbol{u}_{[t]}} \left[g(\boldsymbol{x}_{[t]}, \boldsymbol{u}_{[t]}) + V\left(f(\boldsymbol{x}_{[t]}, \boldsymbol{u}_{[t]}, t) \right) \right] \tag{1}$$

we assume that the value function takes the following form:

$$V(\boldsymbol{x}_{[t]}, t) = \boldsymbol{x}_{[t]}^T \boldsymbol{S}_{[t]}\boldsymbol{x}_{[t]}$$

- solving Eq. 1:

$$\begin{aligned} \boldsymbol{x}_{[t]}^T \boldsymbol{S}_{[t]}\boldsymbol{x}_{[t]} &= \min_{\boldsymbol{u}_{[t]}} \boldsymbol{x}_{[t]}^T \boldsymbol{Q}_{[t]}\boldsymbol{x}_{[t]} + \boldsymbol{u}_{[t]}^T \boldsymbol{R}_{[t]}\boldsymbol{u}_{[t]} + f\left(\boldsymbol{x}_{[t]}, \boldsymbol{u}_{[t]}, t\right)^T \boldsymbol{S}_{[t+1]} f\left(\boldsymbol{x}_{[t]}, \boldsymbol{u}_{[t]}, t\right) \qquad (2) \\ &= \min_{\boldsymbol{u}_{[t]}} \boldsymbol{x}_{[t]}^T \boldsymbol{Q}_{[t]}\boldsymbol{x}_{[t]} + \boldsymbol{u}_{[t]}^T \boldsymbol{R}_{[t]}\boldsymbol{u}_{[t]} + \left((\boldsymbol{A}_{[t]}\boldsymbol{x}_{[t]})^T + (\boldsymbol{B}_{[t]}\boldsymbol{u}_{[t]})^T \right) \boldsymbol{S}_{[t+1]} \left(\boldsymbol{A}_{[t]}\boldsymbol{x}_{[t]} + \boldsymbol{B}_{[t]}\boldsymbol{u}_{[t]} \right) \end{aligned}$$

dropping time notation, assume everything depends on t, while $\hat{\boldsymbol{S}} = \boldsymbol{S}_{[t+1]}$

$$Vt = \boldsymbol{x}^T \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{R}\boldsymbol{u} + (\boldsymbol{A}\boldsymbol{x})^T \hat{\boldsymbol{S}}(\boldsymbol{A}\boldsymbol{x}) + 2(\boldsymbol{A}\boldsymbol{x})^T \hat{\boldsymbol{S}}(\boldsymbol{B}\boldsymbol{u}) + (\boldsymbol{B}\boldsymbol{u})^T \hat{\boldsymbol{S}}(\boldsymbol{B}\boldsymbol{u})$$

solving the minimization problem:

$$\begin{aligned} &\frac{\partial Vt}{\partial \boldsymbol{u}} = 2\boldsymbol{R}\boldsymbol{u} + 2\boldsymbol{B}^T \hat{\boldsymbol{S}}\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{B}^T \hat{\boldsymbol{S}}\boldsymbol{B}\boldsymbol{u} = 0 \\ &\underbrace{(\boldsymbol{R} + \boldsymbol{B}^T \hat{\boldsymbol{S}}\boldsymbol{B})}_{\boldsymbol{M}}\boldsymbol{u} = -\underbrace{\boldsymbol{B}^T \hat{\boldsymbol{S}}\boldsymbol{A}}_{\boldsymbol{C}}\boldsymbol{x} \\ &\boldsymbol{u} = -\boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{x} \qquad (3) \end{aligned}$$

Now we need to solve for $\hat{\boldsymbol{S}}$ by replacing 3 into 2:

$$\begin{aligned} \boldsymbol{x}^T \boldsymbol{S}\boldsymbol{x} &= \boldsymbol{x}^T \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{A}^T \hat{\boldsymbol{S}}\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{x}^T \underbrace{\left(\boldsymbol{A}^T \hat{\boldsymbol{S}}\boldsymbol{B}\right)}_{\boldsymbol{C}^T} \boldsymbol{u} + \boldsymbol{u}^T \underbrace{\left(\boldsymbol{R} + \boldsymbol{B}^T \hat{\boldsymbol{S}}\boldsymbol{B}\right)}_{\boldsymbol{M}} \boldsymbol{u} \\ &= \boldsymbol{x}^T \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{A}^T \hat{\boldsymbol{S}}\boldsymbol{A}\boldsymbol{x} - 2\boldsymbol{x}^T \boldsymbol{C}^T \boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{C}^T \boldsymbol{M}^{-T}\boldsymbol{M}\boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{x} \\ &= \boldsymbol{x}^T \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{x}^T \boldsymbol{A}^T \hat{\boldsymbol{S}}\boldsymbol{A}\boldsymbol{x} - \boldsymbol{x}^T \boldsymbol{C}^T \boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{x} \\ &= \boldsymbol{x}^T \left(\boldsymbol{Q} + \boldsymbol{A}^T \hat{\boldsymbol{S}}\boldsymbol{A} - \boldsymbol{C}^T \boldsymbol{M}^{-1}\boldsymbol{C}\right) \boldsymbol{x} \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \boldsymbol{S} &= \boldsymbol{Q} + \boldsymbol{A}^T \hat{\boldsymbol{S}}\boldsymbol{A} - \boldsymbol{C}^T \boldsymbol{M}^{-1}\boldsymbol{C} \\ &= \boldsymbol{Q} + \boldsymbol{A}^T \hat{\boldsymbol{S}}\boldsymbol{A} - \boldsymbol{A}^T \hat{\boldsymbol{S}}\boldsymbol{B}\left(\boldsymbol{R} + \boldsymbol{B}^T \hat{\boldsymbol{S}}\boldsymbol{B}\right)^{-1} \boldsymbol{B}^T \hat{\boldsymbol{S}}\boldsymbol{A} \end{aligned}$$