Contents

Chapter 1

Real Analysis

(1.5)

$OrderTrichotomy[<, S] := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$

$OrderTransitivity[<, S] := \forall_{x,y,z \in S}\big((x < y \wedge y < z) \implies x < z\big)$

$Order[<, S] := (OrderTrichotomy[<, S]) \wedge (OrderTransitivity[<, S])$

(1.7)

$BoundedAbove[E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge \Big(\exists_{\beta \in S}\forall_{x \in E}(x \le \beta)\Big)$

$BoundedBelow[E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge \Big(\exists_{\beta \in S}\forall_{x \in E}(\beta \le x)\Big)$

$UpperBound[\beta, E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge \big(\beta \in S \wedge \forall_{x \in E}(x \le \beta)\big)$

$LowerBound[\beta, E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge \big(\beta \in S \wedge \forall_{x \in E}(\beta \le x)\big)$

(1.8)

$LUB[\alpha, E, S, <] := (UpperBound[\alpha, E, S, <]) \wedge \Big(\forall_{\gamma}(\gamma < \alpha \implies \neg UpperBound[\gamma, E, S, <])\Big)$

$GLB[\alpha, E, S, <] := (LowerBound[\alpha, E, S, <]) \wedge \Big(\forall_{\beta}(\alpha < \beta \implies \neg LowerBound[\beta, E, S, <])\Big)$

(1.10)

$LUBProperty[S, <] := \forall_E\Big(\big((\emptyset \ne E \subset S) \wedge (BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <])\big)\Big)$

$GLBProperty[S, <] := \forall_E\Big(\big((\emptyset \ne E \subset S) \wedge (BoundedBelow[E, S, <]) \implies \exists_{\alpha \in S}(GLB[\alpha, E, S, <])\big)\Big)$

(1.11)

$LUBPropertyImpliesGLBProperty := LUBProperty[S, <] \implies GLBProperty[S, <]$

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(1) $LUBProperty[S, <] \implies \ldots$ — wts: 2

(1.1) $(\emptyset \ne B \subset S \wedge BoundedBelow[B, S, <]) \implies \ldots$ — wts: 1.2

(1.1.1) $Order[<, S] \wedge \exists_{\delta' \in S}(LowerBound[\delta', B, S, <])$ — from: $BoundedBelow$, 1.1

(1.1.2) $|B| = 1 \implies \ldots$ — wts: 1.1.3

(1.1.2.1) $\exists_{u'}(u' \in B) \blacksquare u := choice(\{u' : u' \in B\}) \blacksquare B = \{u\}$ — from: 1.1.2

(1.1.2.2) $GLB[u, B, S, <] \blacksquare \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$

(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$

(1.1.4) $|B| \ne 1 \implies \ldots$ — wts: 1.1.5

(1.1.4.1) $\forall_E\big((\emptyset \ne E \subset S \wedge BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <])\big)$ — from: $LUBProperty$, 1

(1.1.4.2) $L := \{s \in S : LowerBound[s, B, S, <]\}$

(1.1.4.3) $|B| > 1 \wedge OrderTrichotomy[<, S] \blacksquare \exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$ — from: $Order$, 1.1.1; wts: 1.1.4.7

(1.1.4.4) $b_1 := choice\Big(\{b_1' \in B : \exists_{b_0' \in B}(b_0' < b_1')\}\Big) \blacksquare \neg LowerBound[b_1, B, S, <]$ — from: 1.1.4.2

(1.1.4.5) $b_1 \notin L \blacksquare L \subset S$

(1.1.4.6) $\delta := choice(\{\delta' \in S : LowerBound[\delta', B, S, <]\}) \blacksquare \delta \in L \blacksquare \emptyset \ne L$ — from: 1.1.1

(1.1.4.7) $\emptyset \ne L \subset S$ — from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y \in L}(LowerBound[y_0, B, S, <]) \blacksquare \forall_{y \in L}\forall_{x \in B}(y_0 \le x)$ — from: $LowerBound$, 1.1.4.2; wts: 1.1.4.10

(1.1.4.9) $\forall_{x\in B}\Big(x \in S \wedge \forall_{y\in L}(y_0 \le x)\Big)$ ▌ $\forall_{x\in B}(UpperBound[x, L, S, <])$ from: UpperBound

(1.1.4.10) $\exists_{x\in S}(UpperBound[x, L, S, <])$ ▌ $BoundedAbove[L, S, <]$

(1.1.4.11) $\emptyset \ne L \subset S \wedge BoundedAbove[L, S, <]$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha'\in S}(LUB[\alpha', L, S, <])$ ▌ $\alpha := choice\Big(\{\alpha' \in S : (LUB[\alpha', L, S, <])\}\Big)$ from: 1.1.4.1 wts: 1.1.4.21

(1.1.4.13) $\forall_x(x \in B \implies UpperBound[x, L, S, <])$ from: 1.1.4.9 wts: 1.1.4.17

(1.1.4.14) $\forall_x(\neg UpperBound[x, L, S, <] \implies x \notin B)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$ wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound[\gamma, L, S, <]$ ▌ $\gamma \notin B$ from: LUB, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B$ ▌ $\gamma \in B \implies \gamma \ge \alpha$

(1.1.4.17) $\forall_{\gamma\in B}(\alpha \le \gamma)$ ▌ $LowerBound[\alpha, B, S, <]$ from: LowerBound

(1.1.4.18) $\alpha < \beta \implies \ldots$ wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y\in L}(y_0 \le \alpha < \beta)$ ▌ $\forall_{y\in L}(y_0 \ne \beta)$ from: LUB, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta \notin L$ ▌ $\neg LowerBound[\beta, B, S, <]$ from: 1.1.4.2

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound[\beta, B, S, <]$ ▌ $\forall_{\beta\in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.20) $LowerBound[\alpha, B, S, <] \wedge \forall_{\beta\in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB[\alpha, B, S, <]$ ▌ $\exists_{\epsilon_1\in S}(GLB[\epsilon_1, B, S, <])$

(1.1.5) $|B| \ne 1 \implies \exists_{\epsilon_1\in S}(GLB[\epsilon_1, B, S, <])$

(1.1.6) $\Big(|B| = 1 \implies \exists_{\epsilon_0\in S}(GLB[\epsilon_0, B, S, <])\Big) \wedge \Big(|B| \ne 1 \implies \exists_{\epsilon_1\in S}(GLB[\epsilon_1, B, S, <])\Big)$ from: 1.1.3, 1.1.5

(1.1.7) $(|B| = 1 \vee |B| \ne 1) \implies \exists_{\epsilon\in S}(GLB[\epsilon, B, S, <])$ ▌ $\exists_{\epsilon\in S}(GLB[\epsilon, B, S, <])$

(1.2) $(\emptyset \ne B \subset S \wedge BoundedBelow[B, S, <]) \implies \exists_{\epsilon\in S}(GLB[\epsilon, B, S, <])$

(1.3) $\forall_B\Big((\emptyset \ne B \subset S \wedge BoundedBelow[B, S, <]) \implies \exists_{\epsilon\in S}(GLB[\epsilon, B, S, <])\Big)$

(1.4) $GLBProperty[S, <]$

(2) $LUBProperty[S, <] \implies GLBProperty[S, <]$

(1.12)

$$Field[F, +, *] := \exists_{\mathbb{0},\mathbb{1}\in F}\forall_{x,y,z\in F}\left(\begin{array}{l} x + y \in F \quad\wedge\quad x * y \in F \quad\wedge \\ x + y = y + x \quad\wedge\quad x * y = y * x \quad\wedge \\ (x + y) + z = x + (y_0 + z) \quad\wedge\quad (x * y) * z = x * (y_0 * z) \quad\wedge \\ \mathbb{1} \ne \mathbb{0} \quad\wedge\quad x * (y_0 + z) = (x * y) + (x * z) \quad\wedge \\ \mathbb{0} + x = x \quad\wedge\quad \mathbb{1} * x = x \quad\wedge \\ \exists_{-x\in F}\Big(x + (-x) = \mathbb{0}\Big) \wedge \Big(x \ne \mathbb{0} \implies \exists_{1/x\in F}\Big(x * (1/x) = \mathbb{1}\Big)\Big) \end{array}\right)$$

************************ $(Field[F, +, *] \wedge x, y, z \in F) \implies \ldots$ ************************

(1.14)

$AdditiveCancellation := (x + y = x + z) \implies y = z$

(1) $y = \mathbb{0} + y = \Big(x + (-x)\Big) + y = \Big((-x) + x\Big) + y = (-x) + (x + y) = \ldots$ from: Field

(2) $(-x) + (x + z) = \Big((-x) + x\Big) + z = \Big(x + (-x)\Big) + z = \mathbb{0} + z = z$ from: Field

$AdditiveIdentityUniqueness := (x + y = x) \implies y = \mathbb{0}$

(1) $x + y = x = \mathbb{0} + x = x + \mathbb{0}$ from: Field

(2) $y = 0$ from: AdditiveCancellation

$AdditiveInverseUniqueness := (x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$ from: Field

(2) $y = -x$ from: AdditiveCancellation

$DoubleNegative := x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x$ ▌ $\mathbb{0} = (-x) + x$ from: Field

(2) $x = -(-x)$ from: *AdditiveInverseUniqueness*

(1.15)
$\textit{MultiplicativeCancellation} := (x \neq 0 \land x * y = x * z) \implies y = z$ —
$\textit{MultiplicativeIdentityUniqueness} := (x \neq 0 \land x * y = x) \implies y = \mathbb{1}$ —
$\textit{MultiplicativeInverseUniqueness} := (x \neq 0 \land x * y = \mathbb{1}) \implies y = 1/x$ —
$\textit{DoubleReciprocal} := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)
$\textit{Domination} := \mathbb{0} * x = \mathbb{0}$

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x \;\blacksquare\; \mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$ from: *Field*

(2) $\mathbb{0} * x = \mathbb{0}$ from: *AdditiveIdentityUniqueness*

$\textit{NonDomination} := (x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies \ldots$

(1.1) $(x * y = 0) \implies \ldots$

(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = \big(x * (1/x)\big) * \big(y * (1/y)\big) = (x * y) * \big((1/x) * (1/y)\big) = \mathbb{0} * \big((1/x) * (1/y)\big) = \mathbb{0}$ from: *Field*, *Domination*, 1, 1.1

(1.1.2) $\mathbb{1} = \mathbb{0} \land \mathbb{1} \neq \mathbb{0} \;\blacksquare\; \bot$ from: *Field*

(1.2) $(x * y = 0) \implies \bot \;\blacksquare\; x * y \neq 0$

(2) $(x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq 0$

$\textit{NegationCommutativity} := (-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0} \;\blacksquare\; x * y + (-x) * y = \mathbb{0}$ from: *Field*, *Domination* wts: 2

(2) $(-x) * y = -(x * y)$ from: *AdditiveInverseUniqueness*

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * \mathbb{0} = \mathbb{0} \;\blacksquare\; x * y + x * (-y) = \mathbb{0}$ from: *Field*, *Domination* wts: 4

(4) $x * (-y) = -(x * y)$ from: *AdditiveInverseUniqueness*

(5) $(-x) * y = -(x * y) = x * (-y)$ from: 2, 4

$\textit{NegativeMultiplication} := (-x) * (-y) = x * y$

(1) $(-x) * (-y) = -\big(x * (-y)\big) = -\big(-(x * y)\big) = x * y$ from: *NegationCommutativity*, *DoubleNegative*

(1.17)

$$\textit{OrderedField}[F, +, *, <] := \left(\begin{array}{l} \textit{Field}[F, +, *] \quad \land \quad \textit{Order}[<, F] \quad \land \\ \forall_{x,y,z \in F} (y_0 < z \implies x + y < x + z) \;\land \\ \forall_{x,y \in F} \big((x > \mathbb{0} \land y > \mathbb{0}) \implies x * y > \mathbb{0}\big) \end{array} \right)$$

*********************** $(\textit{OrderedField}[F, +, *, <] \land x, y, z \in F) \implies \ldots$ ***********************

(1.18)
$\textit{NegationOnOrder} := x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \ldots$

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x \;\blacksquare\; \mathbb{0} > -x \;\blacksquare\; -x < \mathbb{0}$ from: *OrderedField*

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \ldots$

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x \;\blacksquare\; \mathbb{0} < x \;\blacksquare\; x > \mathbb{0}$ from: *OrderedField*

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \land -x < \mathbb{0} \implies x > \mathbb{0} \;\blacksquare\; x > \mathbb{0} \iff -x < \mathbb{0}$ from: 2, 4

$\textit{PositiveFactorPreservesOrder} := (x > \mathbb{0} \land y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \land y < z) \implies \ldots$

(1.1) $(-y) + z > (-y) + y = \mathbb{0} \;\blacksquare\; z + (-y) = \mathbb{0}$ from: *OrderedField*

(1.2) $x * \big(z + (-y)\big) > \mathbb{0} \;\blacksquare\; x * z + x * (-y) > \mathbb{0}$ from: *OrderedField*

(1.3) $x * z = \mathbb{0} + x * z = \big(x * y + -(x * y)\big) + x * z = \big(x * y + x * (-y)\big) + x * z = \dots$ from: *Field*, *NegationCommutativity*

(1.4) $x * y + \big(x * z + x * (-y)\big) > x * y + \mathbb{0} = x * y$ from: *Field*, 1.2

(1.5) $x * z > x * y$ from: 1.3, 1.4

(2) $(x > \mathbb{0} \wedge y < z) \implies x * z > x * y$

NegativeFactorFlipsOrder $:= (x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \wedge y < z) \implies \dots$

(1.1) $-x > \mathbb{0}$ from: *NegationOnOrder*

(1.2) $(-x) * y < (-x) * z \;\blacksquare\; \mathbb{0} = x * y + (-x) * y < x * y + (-x) * z \;\blacksquare\; \mathbb{0} < x * y + (-x) * z$ from: *PositiveFactorPreservesOrder*

(1.3) $\mathbb{0} < (-x) * (-y + z) \;\blacksquare\; \mathbb{0} > x * (-y + z) \;\blacksquare\; \mathbb{0} > -(x * y) + x * z$ from: *NegationOnOrder*

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

SquareIsPositive $:= (x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ from: *OrderedField*

(2) $(x < \mathbb{0}) \implies \dots$

(2.1) $-x > \mathbb{0} \;\blacksquare\; x * x = (-x) * (-x) > \mathbb{0} \;\blacksquare\; x * x > \mathbb{0}$ from: *NegationOnOrder*, *OrderedField*, *NegativeMultiplication*

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \vee x < \mathbb{0}) \implies x * x > \mathbb{0} \;\blacksquare\; x \neq \mathbb{0} \implies x * x > \mathbb{0}$ from: *OrderTrichotomy*, 1, 3

OneIsPositive $:= \mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0} \;\blacksquare\; \mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ from: *Field*, *SquareIsPositive*

ReciprocationOnOrder $:= (0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \dots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0} \;\blacksquare\; x * (1/x) > 0$ from: *Field*, *OneIsPositive*

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp \;\blacksquare\; 1/x > \mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0} \;\blacksquare\; y * (1/y) > 0$ from: *Field*, *OneIsPositive*

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp \;\blacksquare\; 1/y > \mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

(1.5) $(1/x) * (1/y) > \mathbb{0}$ from: *OrderedField*

(1.6) $\mathbb{0} < 1/y = \big((1/x) * (1/y)\big) * x < \big((1/x) * (1/y)\big) * y = 1/x$ from: *OrderedField*, 1, 1.4, 1.5

(1.19)
OrderedFieldQ $:=$ *OrderedField*$[\mathbb{Q}, +, *, <]$ —

Subfield$[K, F, +, *] :=$ *Field*$[F, +, *] \wedge K \subset F \wedge$ *Field*$[K, +, *]$
OrderedSubfield$[K, F, +, *, <] :=$ *OrderedField*$[F, +, *, <] \wedge K \subset F \wedge$ *OrderedField*$[K, +, *, <]$
CutI$[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$
CutII$[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
CutIII$[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} :$ *CutI*$[\alpha] \wedge$ *CutII*$[\alpha] \wedge$ *CutIII*$[\alpha]\}$

CutCorollaryI $:= (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \dots$

(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ from: *CutII*, 1

(1.2) $q < p \implies q \in \alpha \;\blacksquare\; q \notin \alpha \implies q \geq p$ from: 1

(1.3) $(q \notin \alpha) \implies \dots$

(1.3.1) $q \geq p$ from: 1.2

(1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp \;\blacksquare\; q \neq p$ from: 1, 1.3

(1.3.3) $q \geq p \wedge q \neq p \;\blacksquare\; p < q$

(1.4) $q \notin \alpha \implies p < q \;\blacksquare\; p < q$ from: 1

(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

$CutCorollaryII := (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$

(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ from: $CutII$, 1

(1.2) $s \in \alpha \implies \big(r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)\big) \;\blacksquare\; s \in \alpha \implies r \in \alpha$ from: 1, 1.1

(1.3) $r \notin \alpha \implies s \notin \alpha \;\blacksquare\; s \notin \alpha$ from: 1, 1.2

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$OrderTrichotomyOfR := OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$

(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$ from: $<_{\mathbb{R}}$, 1.1

(1.1.2) $\exists_{p'} (p' \in \alpha \wedge p' \notin \beta) \;\blacksquare\; p := choice(\{p' : p' \in \alpha \wedge p' \notin \beta\})$

(1.1.3) $q \in \beta \implies \ldots$

(1.1.3.1) $p, q \in \mathbb{Q}$

(1.1.3.2) $q < p$ from: $CutCorollaryI$

(1.1.3.3) $q \in \alpha$ from: $CutII$

(1.1.4) $q \in \beta \implies q \in \alpha$

(1.1.5) $\forall_{q \in \beta} (q \in \alpha) \;\blacksquare\; \beta \subseteq \alpha$

(1.1.6) $\beta \subset \alpha \;\blacksquare\; \beta <_{\mathbb{R}} \alpha$

(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \;\blacksquare\; (\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}} (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

$OrderTransitivityOfR := OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a \in \alpha} (a \in \beta) \wedge \forall_{b \in \beta} (b \in \gamma)$

(1.1.3) $\forall_{a \in \alpha} (\alpha \in \gamma) \;\blacksquare\; \alpha \subset \gamma \;\blacksquare\; \alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}} \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(4) $OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

$OrderOfR := Order[<_{\mathbb{R}}, \mathbb{R}]$ from: $OrderTrichotomyR$, $OrderTransitivityR$ wts:

$LUBPropertyOfR := LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A} (p \in \alpha)\}$

(1.2) $A \neq \emptyset$ ▮ $\exists_\alpha(\alpha \in A)$ ▮ $\alpha_0 := choice(\{\alpha : \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset$ ▮ $\exists_a(a \in \alpha_0)$ ▮ $a_0 := choice(\{a : a \in \alpha_0\})$ ▮ $a_0 \in \gamma$ ▮ $\gamma \neq \emptyset$

(1.4) $\mathit{BoundedAbove}[A, \mathbb{R}, <_\mathbb{R}]$ ▮ $\exists_\beta(\mathit{UpperBound}[\beta, A, \mathbb{R}, <_\mathbb{R}])$

(1.5) $\beta_0 := choice(\{\beta : \mathit{UpperBound}[\beta, A, \mathbb{R}, <_\mathbb{R}]\})$

(1.6) $\mathit{UpperBound}[\beta_0, A, \mathbb{R}, <_\mathbb{R}]$ ▮ $\forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \beta_0)$ ▮ $\forall_{\alpha\in A}(\alpha \subseteq \beta_0)$ ▮ $\forall_{\alpha\in A}\forall_{a\in\alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma$ ▮ $\forall_{a\in\gamma}(a \in \beta_0)$ ▮ $\gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q}$ ▮ $\gamma \subseteq \beta_0 \subset \mathbb{Q}$ ▮ $\gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q}$ ▮ $\mathit{CutI}[\gamma]$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.10.1) $p \in \gamma$ ▮ $\exists_{\alpha\in A}(p \in \alpha)$ ▮ $\alpha_1 := choice(\{\alpha \in A : p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p$ ▮ $q \in \alpha_1$ ▮ $q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma$ ▮ $\forall_{p\in\gamma}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \gamma)$ ▮ $\mathit{CutII}[\gamma]$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha\in A}(p \in \alpha)$ ▮ $\alpha_2 := choice(\{\alpha \in A : p \in \alpha\})$

(1.12.2) $\alpha_2 \in \mathbb{R}$ ▮ $\mathit{CutII}[\alpha_2]$ ▮ $\exists_{r\in\alpha_2}(p < r)$ ▮ $r_0 := choice(\{r \in \alpha_2 : p < r\})$

(1.12.3) $r_0 \in \alpha_2$ ▮ $r_0 \in \gamma$

(1.12.4) $p < r_0$ ▮ $p < r_0 \wedge r_0 \in \gamma$ ▮ $\exists_{r\in\gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r\in\gamma}(p < r)$ ▮ $\forall_{p\in\gamma}\exists_{r\in\gamma}(p < r)$ ▮ $\mathit{CutIII}[\gamma]$

(1.14) $\mathit{CutI}[\gamma] \wedge \mathit{CutII}[\gamma] \wedge \mathit{CutIII}[\gamma]$ ▮ $\gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha\in A}(\alpha \subseteq \gamma)$ ▮ $\forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \gamma)$

(1.16) $\forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \gamma) \wedge \gamma \in \mathbb{R}$ ▮ $\mathit{UpperBound}[\gamma, A, \mathbb{R}, <_\mathbb{R}]$

(1.17) $\delta <_\mathbb{R} \gamma \implies \ldots$

(1.17.1) $\delta \subset \gamma$ ▮ $\exists_s(s \in \gamma \wedge s \notin \delta)$ ▮ $s_0 := choice(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$

(1.17.2) $s_0 \in \gamma$ ▮ $\exists_{\alpha\in A}(s_0 \in \alpha)$ ▮ $\alpha_3 := choice(\{\alpha \in A : s_0 \in \alpha\})$

(1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta$ ▮ $\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4) $\delta \geq_\mathbb{R} \alpha_3 \implies \ldots$

(1.17.4.1) $\alpha_3 \subseteq \delta$ ▮ $\forall_{s\in\mathbb{Q}}(s \in \alpha_3 \implies s \in \delta)$ ▮ $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4.2) $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$ ▮ $\bot$

(1.17.5) $\delta \geq_\mathbb{R} \alpha_3 \implies \bot$ ▮ $\delta <_\mathbb{R} \alpha_3$ ▮ $\exists_{\alpha\in A}(\delta <_\mathbb{R} \alpha)$ ▮ $\exists_{\alpha\in A}\big(\neg(\alpha \leq_\mathbb{R} \delta)\big)$

(1.17.6) $\neg\forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \delta)$ ▮ $\neg\mathit{UpperBound}[\delta, A, \mathbb{R}, <_\mathbb{R}]$

(1.18) $\delta <_\mathbb{R} \gamma \implies \neg\mathit{UpperBound}[\delta, A, \mathbb{R}, <_\mathbb{R}])$ ▮ $\forall_\delta(\delta <_\mathbb{R} \gamma \implies \neg\mathit{UpperBound}[\delta, A, \mathbb{R}, <_\mathbb{R}])$

(1.19) $\mathit{UpperBound}[\gamma, A, \mathbb{R}, <_\mathbb{R}] \wedge \forall_\delta(\delta <_\mathbb{R} \gamma \implies \neg\mathit{UpperBound}[\delta, A, \mathbb{R}, <_\mathbb{R}])$

(1.20) $\mathit{LUB}[\gamma, A, \mathbb{R}, <_\mathbb{R}]$ ▮ $\exists_{\gamma\in S}(\mathit{LUB}[\gamma, A, \mathbb{R}, <_\mathbb{R}])$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge \mathit{BoundedAbove}[A, \mathbb{R}, <_\mathbb{R}]) \implies \exists_{\gamma\in S}(\mathit{LUB}[\gamma, A, \mathbb{R}, <_\mathbb{R}])$

(3) $\forall_A\Big((\emptyset \neq A \subset \mathbb{R} \wedge \mathit{BoundedAbove}[A, \mathbb{R}, <_\mathbb{R}]) \implies \exists_{\gamma\in S}(\mathit{LUB}[\gamma, A, \mathbb{R}, <_\mathbb{R}])\Big)$ ▮ $\mathit{LUBProperty}[\mathbb{R}, <_\mathbb{R}]$

$+_\mathbb{R}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_\mathbb{R} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

$0_\mathbb{R} := \{x \in \mathbb{Q} : x < 0\}$

$\mathit{ZeroInR} := 0_\mathbb{R} \in \mathbb{R}$

(1) $-1 \in 0_\mathbb{R} \wedge 1 \notin 0_\mathbb{R}$ ▮ $\emptyset \neq 0_\mathbb{R} \subseteq \mathbb{Q}$ ▮ $\mathit{CutI}[0_\mathbb{R}]$

(2) $(x \in 0_\mathbb{R} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_\mathbb{R}$ ▮ $\forall_{x\in 0_\mathbb{R}}\forall_{y\in\mathbb{Q}}(y_0 < x \implies y \in 0_\mathbb{R})$ ▮ $\mathit{CutII}[0_\mathbb{R}]$

(3) $y := x/2$ ▮ $(x \in 0_\mathbb{R}) \implies (x < y < 0) \implies \exists_{y\in 0_\mathbb{R}}(x < y)$ ▮ $\forall_{x\in 0_\mathbb{R}}\exists_{y\in 0_\mathbb{R}}(x < y)$ ▮ $\mathit{CutIII}[0_\mathbb{R}]$

(4) $\mathit{CutI}[0_\mathbb{R}] \wedge \mathit{CutII}[0_\mathbb{R}] \wedge \mathit{CutIII}[0_\mathbb{R}]$ ▮ $0_\mathbb{R} \in \mathbb{R}$

$\mathit{FieldAdditionClosureOfR} := (\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_\mathbb{R} \beta) \in \mathbb{R}\big)$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_\mathbb{R} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha) ; \exists_b(b \in \beta)$ ▮ $a_0 := choice(\{a : a \in \alpha\}) ; b_0 := choice(\{b : b \in \beta\})$ ▮ $a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha) ; \exists_y(y_0 \notin \beta)$ ▮ $x_0 := choice(\{x : x \notin \alpha\}) ; y_0 := choice(\{y : y \notin \beta\})$

(1.5) $\forall_{r\in\alpha}(r < x_0) ; \forall_{s\in\beta}(s < y_0)$ ▮ $\forall_{r\in\alpha}\forall_{s\in\beta}(r + s < x_0 + y_0)$ ▮ $x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q}$ ▮ $\mathit{CutI}[\alpha +_{\mathbb{R}} \beta]$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \land q \in \mathbb{Q} \land q < p) \implies \ldots$

(1.7.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s)$ ▮ $(r_0, s_0) := choice\big((r, s) \in \alpha \times \beta : p = r + s\big)$

(1.7.2) $q < p = r_0 + s_0$ ▮ $(q - s_0) < r_0$ ▮ $(q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta$ ▮ $q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta$ ▮ $q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \land q \in \mathbb{Q} \land q < p) \implies q \in \alpha +_{\mathbb{R}} \beta$ ▮ $\forall_{p\in\alpha+_{\mathbb{R}}\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta)$ ▮ $\mathit{CutII}[\alpha +_{\mathbb{R}} \beta]$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s)$ ▮ $(r_1, s_1) := choice\big(\{(r, s) \in \alpha \times \beta : p = r + s\}\big)$

(1.9.2) $r_1 \in \alpha$ ▮ $\exists_{t\in\alpha}(r_1 < t)$ ▮ $t_0 := choice(\{t \in \alpha : r_1 < t\})$

(1.9.3) $s_1 \in \beta$ ▮ $t + s_1 \in \alpha +_{\mathbb{R}} \beta \land p = r_1 + s_1 < t + s_1$ ▮ $\exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$ ▮ $\forall_{p\in\alpha+_{\mathbb{R}}\beta}\exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$ ▮ $\mathit{CutIII}[\alpha +_{\mathbb{R}} \beta]$

(1.11) $\mathit{CutI}[\alpha +_{\mathbb{R}} \beta] \land \mathit{CutII}[\alpha +_{\mathbb{R}} \beta] \land \mathit{CutIII}[\alpha +_{\mathbb{R}} \beta]$ ▮ $\alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

$\mathit{FieldAdditionCommutativityOfR} := (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s : r \in \alpha \land s \in \beta\} = \{s + r : s \in \beta \land r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

$\mathit{FieldAdditionAssociativityOfR} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)\big)$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c : a \in \alpha \land b \in \beta \land c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) : a \in \alpha \land b \in \beta \land c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

$\mathit{FieldAdditionIdentityOfR} := (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \implies \ldots$

(1.1.1) $s < 0$ ▮ $r + s < r + 0 = r$ ▮ $r + s < r$ ▮ $r + s \in \alpha$

(1.2) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \implies r + s \in \alpha$ ▮ $\forall_{r\in\alpha}\forall_{s\in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▮ $\forall_{p\in\alpha+_{\mathbb{R}}0_{\mathbb{R}}}(p \in \alpha)$ ▮ $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r\in\alpha}(p < r)$ ▮ $r_2 := choice(\{r \in \alpha : p < r\})$

(1.4.2) $p < r_2$ ▮ $p - r_2 < r_2 - r_2 = 0$ ▮ $(p - r_2) < 0$ ▮ $(p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha$ ▮ $p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▮ $p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▮ $\forall_{p\in\alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▮ $\alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \land \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▮ $0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

$\mathit{FieldAdditionInverseOfR} := (\alpha \in \mathbb{R}) \implies \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} : \exists_{r>0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q}$ ▮ $\exists_{s\in\mathbb{Q}}(s \notin \alpha)$ ▮ $s_0 := choice(\{s : s \notin \alpha\})$ ▮ $p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha$ ▮ $-p_0 - 1 \notin \alpha$ ▮ $\exists_{r>0}(-p_0 - r \notin \alpha)$ ▮ $p_0 \in \beta$

(1.4) $\emptyset \neq \alpha$ ▮ $\exists_{q\in\alpha}$ ▮ $q_0 := choice(\{q \in \mathbb{Q} : q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha$ ▮ $-(-q_0) - r = q_0 - r < q_0$ ▮ $-(-q_0) - r < q_0$ ▮ $-(-q_0) - r \in \alpha$

(1.6) $\forall_{r>0}\big(-(-q_0) - r \in \alpha\big)$ ▮ $\neg\exists_{r>0}\big(-(-q_0) - r \notin \alpha\big)$ ▮ $-q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q}$ ▮ $\mathit{CutI}[\beta]$

(1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.8.1) $p \in \beta \blacksquare \exists_{r>0}(-p - r \notin \alpha) \blacksquare r_0 := choice(\{r > 0 : -p - r \notin \alpha\})$

(1.8.2) $q < p \blacksquare -p - r < -q - r$

(1.8.3) $-q - r \notin \alpha \blacksquare q \in \beta$

(1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta \blacksquare \forall_{p\in\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \beta) \blacksquare CutII[\beta]$

(1.10) $p \in \beta \implies \ldots$

(1.10.1) $p \in \beta \blacksquare \exists_{r>0}(-p - r \notin \alpha) \blacksquare r_1 := choice(\{r > 0 : -p - r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0 \blacksquare r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p \blacksquare t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -\big(p + (r_1/2)\big) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha \blacksquare -t_0 - (r_1/2) \notin \alpha \blacksquare \exists_{r>0}(-t_0 - r \notin \alpha) \blacksquare t_0 \in \beta$

(1.10.7) $t_0 > p \wedge t_0 \in \beta \blacksquare \exists_{t\in\beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t\in\beta}(p < t) \blacksquare \forall_{p\in\beta}\exists_{t\in\beta}(p < t) \blacksquare CutIII[\beta]$

(1.12) $CutI[\beta] \wedge CutII[\beta] \wedge CutIII[\beta] \blacksquare \beta \in \mathbb{R}$

(1.13) $(r \in \alpha \wedge s \in \beta) \implies \ldots$

(1.13.1) $s \in \beta \blacksquare \exists_{t>0}(-s - t \notin \alpha) \blacksquare t_1 := choice(\{t > 0 : -s - t \notin \alpha\}) \blacksquare -s - t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha \blacksquare -s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha \blacksquare r < -s \blacksquare r + s < 0 \blacksquare r + s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}} \blacksquare \forall_{(r,s)\in\alpha\times\beta}(r + s \in 0_{\mathbb{R}}) \blacksquare \alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \ldots$

(1.15.1) $v < 0 \blacksquare w_0 := -v/2 \blacksquare w > 0$

(1.15.2) $\exists_{n\in\mathbb{Z}}\big(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\big) \blacksquare n_0 := choice\big(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\}\big)$ from: $ARCHIMEDEANPROPERTYOFQ + LUB$???

(1.15.3) $p_0 := -(n_0 + 2)w_0 \blacksquare -p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha \blacksquare -p_0 - w_0 \notin \alpha \blacksquare p_0 \in \beta$

(1.15.4) $n_0 w_0 \in \alpha \wedge p_0 \in \beta \blacksquare n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{v\in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta) \blacksquare 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta \blacksquare \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}} \blacksquare \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

$*_{\mathbb{R}}[\alpha, \beta] :=$ —
$1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$1IsNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ —
$1InR := 1_{\mathbb{R}} \in \mathbb{R}$ —
$FieldMultiplicationClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R}\big)$ —
$FieldMultiplicationCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ —
$FieldMultiplicationAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)\big)$ —
$FieldMultiplicationIdentityOfR := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ —
$FieldMultiplicationInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha\in\mathbb{R}}\big(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}\big)$ —

$FieldDistributativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ —

$FieldWithR := Field[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}]$ —
$OrderedFieldWithR := OrderedField[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$
$QROrderedSubfieldOfR := OrderedSubfield[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —
$QIsomorphicToQR := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ —
$CompletenessOfR := \exists_{\mathbb{R}}(LUBProperty[\mathbb{R}, <_{\mathbb{R}}] \wedge OrderedSubfield[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}])$ —

(1.20)
$ArchimedeanPropertyOfR := \forall_{x,y\in\mathbb{R}}\big(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y)\big)$

(1) $(x, y \in \mathbb{R} \land x > 0) \implies \ldots$

(1.1) $A := \{nx : n \in \mathbb{N}^+\}$ ▌ $(\emptyset \neq A \subset \mathbb{R}) \land \big(a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)\big)$

(1.2) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n \in \mathbb{N}^+}(nx > y)$ ▌ $\forall_{n \in \mathbb{N}^+}(nx \leq y)$ ▌ $UpperBound[y_0, A, \mathbb{R}, <]$ ▌ $BoundedAbove[A, \mathbb{R}, <]$

(1.2.2) $CompletenessOfR$ ▌ $LUBProperty[\mathbb{R}, <]$

(1.2.3) $(LUBProperty[\mathbb{R}, <]) \land (\emptyset \neq A \subset \mathbb{R}) \land (BoundedAbove[A, \mathbb{R}, <])$ ▌ $\exists_{\alpha \in \mathbb{R}}(LUB[\alpha, A, \mathbb{R}, <])\ \ldots$

(1.2.4) $\ldots \alpha_0 := choice(\{\alpha \in \mathbb{R} : LUB[\alpha, A, \mathbb{R}, <]\})$ ▌ $LUB[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0$ ▌ $\alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \land (LUB[\alpha_0, A, \mathbb{R}, <])$ ▌ $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$ ▌ $\exists_{c \in A}(\alpha_0 - x < c)\ \ldots$

(1.2.8) $\ldots c_0 := choice(\{c \in A : \alpha_0 - x < c\})$ ▌ $(c_0 \in A) \land (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \land \big(a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)\big)$ ▌ $\exists_{m \in \mathbb{N}^+}(mx = c_0)\ \ldots$

(1.2.10) $\ldots m_0 := choice(\{m \in \mathbb{N}^+ : mx = c_0\})$ ▌ $(m_0 \in \mathbb{N}^+) \land (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \land (m_0 x = c_0)$ ▌ $\alpha_0 - x < c_0 = m_0 x$ ▌ $\alpha_0 < m_0 x + x$ ▌ $\alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+$ ▌ $m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \land \big(a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)\big)$ ▌ $(m_0 + 1)x \in A$

(1.2.14) $\big(\alpha_0 < (m_0 + 1)x\big) \land \big((m_0 + 1)x \in A\big)$ ▌ $\exists_{c \in A}(\alpha_0 < c)$

(1.2.15) $LUB[\alpha_0, A, \mathbb{R}, <]$ ▌ $UpperBound[\alpha_0, A, \mathbb{R}, <]$ ▌ $\forall_{c \in A}(c \leq \alpha_0)$ ▌ $\neg\exists_{c \in A}(c > \alpha_0)$ ▌ $\neg\exists_{c \in A}(\alpha_0 < c)$

(1.2.16) $\big(\exists_{c \in A}(\alpha_0 < c)\big) \land \big(\neg\exists_{c \in A}(\alpha_0 < c)\big)$ ▌ $\bot$

(1.3) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \bot$ ▌ $\exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \land x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y)$ ▌ $\forall_{x, y \in \mathbb{R}}\big(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y)\big)$

$QDenseInR := \forall_{x, y \in \mathbb{R}}\Big(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y)\Big)$

(1) $(x, y \in \mathbb{R} \land x < y) \implies \ldots$

(1.1) $x < y$ ▌ $(0 < y - x) \land (y - x \in \mathbb{R})$

(1.2) $ArchimedeanPropertyOfR \land (0 < y - x) \land (y - x, 1 \in \mathbb{R})$ ▌ $\exists_{n \in \mathbb{N}^+}\big(n(y - x) > 1\big)\ \ldots$

(1.3) $\ldots n_0 := choice\big(\{n \in \mathbb{N}^+ : n(y - x) > 1\}\big)$ ▌ $(n_0 \in \mathbb{N}^+) \land \big(n_0(y - x) > 1\big)$

(1.4) $(n_0 \in \mathbb{N}^+) \land (x \in \mathbb{R})$ ▌ $n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $ArchimedeanPropertyOfR \land (1 > 0) \land (n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m \in \mathbb{N}^+}\big(m(1) > n_0 x\big)\ \ldots$

(1.6) $\ldots m_1 := choice\big(\{m \in \mathbb{N}^+ : m(1) > n_0 x\}\big)$ ▌ $(m_1 \in \mathbb{N}^+) \land (m_1 > n_0 x)$

(1.7) $ArchimedeanPropertyOfR \land (1 > 0) \land (-n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m \in \mathbb{N}^+}\big(m(1) > -n_0 x\big)\ \ldots$

(1.8) $\ldots m_2 := choice\big(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\}\big)$ ▌ $(m_2 \in \mathbb{N}^+) \land (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \land (m_2 > -n_0 x)$ ▌ $-m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+$ ▌ $|m_1 - (-m_2)| \geq 2$

(1.11) $(-m_2 < n_0 x < m_1) \land \big(|m_1 - (-m_2)| \geq 2\big)$ ▌ $\exists_{m \in \mathbb{Z}}\big((-m_2 < m < m_1) \land (m - 1 \leq n_0 x < m)\big)\ \ldots$

(1.12) $\ldots m_0 := choice\big(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \land (m - 1 \leq n_0 x < m)\}\big)$ ▌ $(-m_2 < m_0 < m_1) \land (m_0 - 1 \leq n_0 x < m_0)$

(1.13) $\big(n_0(y - x) > 1\big) \land (m_0 - 1 \leq n_0 x < m_0)$ ▌ $n_0 x < m_0 \leq 1 + n_0 x < n_0 y$ ▌ $n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \land (n_0 x < m_0 < n_0 y)$ ▌ $x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z}$ ▌ $m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \land (x < m_0/n_0 < y)$ ▌ $\exists_{p \in \mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \land x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y)$ ▌ $\forall_{x, y \in \mathbb{R}}\Big(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y)\Big)$

(1.21)

$RootLemma := (0 < a < b) \implies \Big(b^n - a^n \leq (b - a)nb^{n-1}\Big)$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b$ ▌ $b/a > 1$

(1.3) $b/a > 1$ ▮ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq \sum_{i=1}^{n}\left(b^{n-i}a^{i-1}(b/a)^{i-1}\right) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$ ▮ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b-a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq (b-a)nb^{n-1}$ ▮ $b^n - a^n \leq (b-a)nb^{n-1}$

(2) $(0 < a < b) \implies \left(b^n - a^n \leq (b-a)nb^{n-1}\right)$

$RootExistenceInR := \forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}({y_0}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\}$ ▮ $t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$

(1.2) $t_0 := x/(1+x)$ ▮ $\left(t_0 = x/(1+x)\right) \wedge (t_0 \in \mathbb{R})$

(1.3) $0 < x$ ▮ $0 < x < 1 + x$ ▮ $t_0 = x/(1+x) > 0$ ▮ $t_0 > 0$

(1.4) $1 = (1+x)/(1+x) > x/(1+x) = t_0$ ▮ $1 > t_0$

(1.5) $(t_0 > 0) \wedge (1 > t_0)$ ▮ $0 < t_0 < 1$

(1.6) $(0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1)$ ▮ ${t_0}^n \leq t_0$

(1.7) $0 < x$ ▮ $x > x/(1+x) = t_0$ ▮ $x > t_0$

(1.8) $({t_0}^n \leq t_0) \wedge (x > t_0)$ ▮ ${t_0}^n < x$

(1.9) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge ({t_0}^n < x)$ ▮ $t_0 \in E$ ▮ $\emptyset \neq E$

(1.10) $t_1 := choice(\{t \in \mathbb{R} : t > 1 + x\})$ ▮ $(t_1 \in \mathbb{R}) \wedge (t_1 > 1 + x)$

(1.11) $x > 0$ ▮ $t_1 > 1 + x > 1$ ▮ $t_1 > 1$ ▮ ${t_1}^n \geq t_1$

(1.12) $({t_1}^n \geq t_1) \wedge (t_1 > 1 + x) \wedge (1 > 0)$ ▮ ${t_1}^n \geq t_1 > 1 + x > x$ ▮ ${t_1}^n > x$

(1.13) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge ({t_1}^n > x)$ ▮ $t_1 \notin E$ ▮ $E \subset \mathbb{R}$

(1.14) $(\emptyset \neq E) \wedge (E \subset \mathbb{R})$ ▮ $\emptyset \neq E \subset \mathbb{R}$

(1.15) $t \in E \implies \ldots$

(1.15.1) $(t \in E) \wedge \left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right)$ ▮ $t^n < x$

(1.15.2) $({t_1}^n > x) \wedge (t^n < x)$ ▮ $t^n < x < {t_1}^n$ ▮ $t < t_1$

(1.16) $t \in E \implies t < t_1$ ▮ $\forall_{t\in E}(t \leq t_1)$ ▮ $UpperBound[t_1, E, \mathbb{R}, <]$ ▮ $BoundedAbove[E, \mathbb{R}, <]$

(1.17) $CompletenessOfR$ ▮ $LUBProperty[\mathbb{R}, <]$

(1.18) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (BoundedAbove[E, \mathbb{R}, <])$ ▮ $\exists_{y\in\mathbb{R}}(LUB[y, E, \mathbb{R}, <])$ ...

(1.19) ... $y_0 := choice(\{y \in \mathbb{R} : LUB[y, E, \mathbb{R}, <]\})$ ▮ $LUB[y_0, E, \mathbb{R}, <]$

(1.20) $(LUB[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0)$ ▮ $0 < t_0 \leq y_0 \in \mathbb{R}$ ▮ $0 < y_0 \in \mathbb{R}$

(1.21) ${y_0}^n < x \implies \ldots$

(1.21.1) $k_0 := \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}$ ▮ $k_0 \in \mathbb{R}$

(1.21.2) ${y_0}^n < x$ ▮ $0 < x - {y_0}^n$

(1.21.3) $(n > 0) \wedge (y_0 > 0)$ ▮ $0 < n(y_0 + 1)^{n-1}$

(1.21.4) $(0 < x - {y_0}^n) \wedge \left(0 < n(y_0 + 1)^{n-1}\right)$ ▮ $0 < \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} = k_0$ ▮ $0 < k_0$

(1.21.5) $(0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R})$ ▮ $0 < min(1, k_0) \in \mathbb{R}$

(1.21.6) $QDenseInR \wedge \left(0, min(1, k_0) \in \mathbb{R}\right) \wedge \left(0 < min(1, k_0)\right)$ ▮ $\exists_{h\in\mathbb{Q}}\left(0 < h < min(1, k_0)\right)$ ...

(1.21.7) ... $h_0 := choice\left(\{h \in \mathbb{Q} : 0 < h < min(1, k_0)\}\right)$ ▮ $(0 < h_0 < 1) \wedge \left(h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}\right)$

(1.21.8) $(y_0 > 0) \wedge (h_0 > 0)$ ▮ $0 < y_0 < y_0 + h_0$

(1.21.9) $RootLemma \wedge (0 < y_0 < y_0 + h_0)$ ▮ $(y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}$

(1.21.10) $h_0 < 1$ ▮ $h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$

(1.21.11) $\left((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}\right) \wedge \left(h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}\right)$ ▮ $(y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}$

(1.21.12) $\left(0 < n(y_0 + 1)^{n-1}\right) \wedge \left(h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}\right)$ ▮ $h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n$

(1.21.13) $\left((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}\right) \wedge \left(h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n\right)$ ▮ $(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n$ ▮ $(y_0 + h_0)^n < x$

(1.21.14) $(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n$ ▮ $(y_0 + h_0)^n < x$

(1.21.15) $(0 < y_0\mathbb{R}) \wedge (0 < h_0 < \mathbb{R})$ ▮ $0 < y_0 < y_0 + h_0 \in \mathbb{R}$

(1.21.16) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge \left((y_0 + h_0)^n < x\right) \wedge (0 < y_0 + h_0 \in \mathbb{R})$ ▮ $(y_0 + h_0)^n \in E$

(1.21.17) $\big((y_0+h_0)^n \in E\big) \wedge (y_0 < y_0 + h_0) \ \blacksquare\ \exists_{e\in E}(y_0 < e)$

(1.21.18) $LUB[y_0, E, \mathbb{R}, <] \ \blacksquare\ UpperBound[y_0, E, \mathbb{R}, <] \ \blacksquare\ \forall_{e\in E}(e \leq y_0) \ \blacksquare\ \neg\exists_{e\in E}(e > y_0)$

(1.21.19) $\big(\exists_{e\in E}(e > y_0)\big) \wedge \big(\neg\exists_{e\in E}(e > y_0)\big) \ \blacksquare\ \bot$

(1.22) ${y_0}^n < x \implies \bot \ \blacksquare\ {y_0}^n \geq x$

(1.23) ${y_0}^n > x \implies \ldots$

(1.23.1) $k_1 := \frac{{y_0}^n - x}{n{y_0}^{n-1}} \ \blacksquare\ (k_1 \in \mathbb{R}) \wedge (k_1 n {y_0}^{n-1} = {y_0}^n - x)$

(1.23.2) $(0 < x) \wedge (0 < n \in \mathbb{Z}) \ \blacksquare\ {y_0}^n - x < {y_0}^n \leq n{y_0}^n \ \blacksquare\ {y_0}^n - x < n{y_0}^n$

(1.23.3) ${y_0}^n - x < n{y_0}^n \ \blacksquare\ k_1 = \frac{{y_0}^n - x}{n{y_0}^{n-1}} < \frac{n{y_0}^n}{n{y_0}^{n-1}} = y_0 \ \blacksquare\ k_1 < y_0$

(1.23.4) ${y_0}^n > x \ \blacksquare\ 0 < {y_0}^n - x$

(1.23.5) $(n > 0) \wedge (y_0 > 0) \ \blacksquare\ 0 < n{y_0}^{n-1}$

(1.23.6) $(0 < {y_0}^n - x) \wedge 0 < (n{y_0}^{n-1}) \ \blacksquare\ 0 < \frac{{y_0}^n - x}{n{y_0}^{n-1}} = k_1 \ \blacksquare\ 0 < k_1$

(1.23.7) $(k_1 < y_0) \wedge (0 < k_1) \ \blacksquare\ (0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$

(1.23.8) $t \geq y_0 - k_1 \implies \ldots$

(1.23.8.1) $t \geq y_0 - k_1 \ \blacksquare\ t^n \geq (y_0 - k_1)^n \ \blacksquare\ -t^n \leq -(y_0 - k_1)^n \ \blacksquare\ {y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n$

(1.23.8.2) $RootLemma \wedge (0 < y_0 - k_1 < y_0) \ \blacksquare\ {y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}$

(1.23.8.3) $\big({y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n\big) \wedge \Big({y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}\Big) \ \blacksquare\ {y_0}^n - t^n < k_1 n {y_0}^{n-1}$

(1.23.8.4) $(k_1 n {y_0}^{n-1} = {y_0}^n - x) \wedge ({y_0}^n - t^n < k_1 n {y_0}^{n-1}) \ \blacksquare\ {y_0}^n - t^n < {y_0}^n - x \ \blacksquare\ -t^n < -x \ \blacksquare\ t^n > x$

(1.23.8.5) $\big(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\big) \wedge (t^n > x) \ \blacksquare\ t \notin E$

(1.23.9) $t \geq y_0 - k_1 \implies t \notin E \ \blacksquare\ t \in E \implies t < y_0 - k_1 \ \blacksquare\ \forall_{t\in E}(t \leq y_0 - k_1) \ \blacksquare\ UpperBound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.10) $\big(LUB[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)\big) \ \blacksquare\ \neg UpperBound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.11) $(UpperBound[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg UpperBound[y_0 - k_1, E, \mathbb{R}, <]) \ \blacksquare\ \bot$

(1.24) ${y_0}^n > x \implies \bot \ \blacksquare\ {y_0}^n \leq x$

(1.25) $Order[\mathbb{R}, <] \ \blacksquare\ OrderTrichotomy[\mathbb{R}, <]$

(1.26) $(OrderTrichotomy[\mathbb{R}, <]) \wedge ({y_0}^n \geq x) \wedge ({y_0}^n \leq x) \ \blacksquare\ {y_0}^n = x$

(1.27) $({y_0}^n = x) \wedge (y_0 \in \mathbb{R}) \ \blacksquare\ \exists_{y\in\mathbb{R}}(y^n = x)$

(1.28) $y_1, y_2 := choice(\{y \in \mathbb{R} : y^n = x\})$

(1.29) $y_1 \neq y_2 \implies \ldots$

(1.29.1) $(OrderTrichotomy[\mathbb{R}, <]) \wedge (y_1 \neq y_2) \ \blacksquare\ (y_1 < y_2) \veebar (y_2 < y_1) \ \ldots$

(1.29.2) $\ldots (x = {y_1}^n < {y_2}^n = x) \veebar (x = {y_2}^n < {y_1}^n = x) \ \blacksquare\ (x < x) \veebar (x > x) \ \blacksquare\ \bot \veebar \bot \ \blacksquare\ \bot$

(1.30) $y_1 \neq y_2 \implies \bot \ \blacksquare\ y_1 = y_2 \ \blacksquare\ \forall_{a,b\in\mathbb{R}}\big((a^n = x \wedge b^n = x) \implies a = b\big)$

(1.31) $\Big(\exists_{y\in\mathbb{R}}(y^n = x)\Big) \wedge \Big(\forall_{a,b\in\mathbb{R}}\big((a^n = x \wedge b^n = x) \implies a = b\big)\Big) \ \blacksquare\ \exists!_{y\in\mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y\in\mathbb{R}}(y^n = x) \ \blacksquare\ \forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}({y_0}^n = x)$

$RootExistenceInRCorollary := \forall_{0<a\in\mathbb{R}}\forall_{0<b\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\Big((ab)^{1/n} = a^{1/n}b^{1/n}\Big)$ —

$$ExtendedRealSystem[\bar{\mathbb{R}}, +, *, <] := \left(\begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \ \wedge \ x - \infty = -\infty \ \wedge \ \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \ \wedge \\ (x > 0) \implies \big(x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty\big) \wedge \\ (x < 0) \implies \big(x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty\big) \end{array}\right)$$

$\mathbb{C} := \{\langle a, b\rangle \in \mathbb{R} \times \mathbb{R}\}$
$+_{\mathbb{C}}[\langle a, b\rangle, \langle c, d\rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d\rangle$
$*_{\mathbb{C}}[\langle a, b\rangle, \langle c, d\rangle] := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c\rangle$

$FieldC := Field[\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}}]$ —
$RSubfieldC := Subfield[\mathbb{R}, \mathbb{C}, +, *]$ —

$i := \langle 0, 1\rangle \in \mathbb{C}$

$iProperty := i^2 = -1$ —
$CProperty := (a, b \in \mathbb{R}) \implies (\langle a, b\rangle = a + bi)$ —

$Conjugate[\overline{a+bi}] := a - bi$

$ConjugateProperties := (w, z \in \mathbb{C}) \implies \ldots$ —

(1)	$\overline{z+w} = \overline{z} + \overline{w}$
(2)	$\overline{z*w} = \overline{z} * \overline{w}$
(3)	$Re(z) = (1/2)(z+\overline{z}) \wedge Im(z) = (1/2)(z-\overline{z})$
(4)	$0 \leq z * \overline{z} \in \mathbb{R}$

$AbsoluteValueC[|z|] = (z * \overline{z})^{1/2}$
$AbsoluteValueProperties := (z, w \in \mathbb{C}) \implies \ldots$ —

(1)	123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists !_{y \in Y} \big((x, y) \in f\big)$

$\big(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y\big) \implies \ldots$

(1)	$Domain(f) := X; Codomain(f) := Y$
(2)	$Image(f, A) := \{f(a) : a \in A\}; Preimage(f, B) := \{a : f(a) \in B\}$
(3)	$Range(f) := Image\big(Domain(f)\big)$

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X}\big(x_1 \neq x_2 \implies f(x_1) \neq f(x_2)\big)$
$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X}\big(y_0 = f(x)\big)$
$Bijective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$
$SurjectiveEquivalent := \big(Range(f) = Codomain(f)\big) \implies Surjective(f)$

$\big(Function(f, X, Y) \wedge Function(g, Y, Z)\big) \implies (f \circ g)(x) := f\big(g(x)\big); Function(f \circ g, X, Z)$

$Properties of Functions := \big(Function(f, A, B) \wedge Function(g, B, C) \wedge Function(h, C, D)\big) \implies \ldots$

(1)	$h \circ (g \circ f) = (h \circ g) \circ f$
(2)	$\big(Injective(f) \wedge Injective(g)\big) \implies Injective(g \circ f)$
(3)	$\big(Surjective(f) \wedge Surjective(g)\big) \implies Surjective(g \circ f)$
(4)	$\big(Bijective(f, A, B)\big) \implies \exists_{f^{-1}}\Bigg(Function(f^{-1}, B, A) \wedge \forall_{a \in A}\Big(f^{-1}\big(f(a)\big) = a\Big) \wedge \forall_{b \in B}\Big(f\Big(f^{-1}(b)\Big) = b\Big)\Bigg)$

$|(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}}(b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \ldots$

(1)	$(a \mid b) \implies a \mid bc$
(2)	$(a \mid b \wedge b \mid c) \implies a \mid c$
(3)	$(a \mid b \wedge b \mid c) \implies a \mid (bx + cy)$
(4)	$(a \mid b \wedge b \mid a) \implies a = \pm b$
(5)	$(a \mid b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$
(6)	$(a \mid b) \iff (m \neq 0 \wedge ma \mid mb)$

$DivisionAlgorithm := (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists !_{q, r \in \mathbb{Z}}(b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \wedge a : b \wedge a : c$
$GCD(a, b, c) := CD(a, b, c) \wedge \forall_d\big((d : b \wedge d : c) \implies d : a\big)$
$GCDEquivalent := 123123$

Chapter 3

Linear Algebra

$Matrix[A, m, n] := [a_{i,j}]_{m \times n}$:= m rows, n columns of real numbers
$\mathcal{M}_{m,n} := \{A : Matrix[A, m, n]\}$
$O_{m,n} := (Matrix[O, m, n]) \wedge (a_{i,j} = 0)$
$Square[A, n] := Matrix[A, n, n]$
$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$
$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$
$Diagonal[A, n] := (Square[A, n]) \wedge (i \neq j \implies a_{i,j} = 0)$
$Scalar[A, n, k] := (Diagonal[A, n]) \wedge (a_{i,i} = k)$
$I_n := Scalar[I, n, 1]$

$+(A, B) := \big((Matrix[A, m, n]) \wedge (Matrix[B, m, n])\big) \implies (A + B = [a_{i,j} + b_{i,j}]_{m \times n})$
$*\,(r, A) := \big((r \in \mathbb{R}) \wedge (Matrix[A, m, n])\big) \implies (r * A = [ra_{i,j}]_{m \times n})$
$*\,(A, B) := \big((Matrix[A, m, p]) \wedge (Matrix[B, p, n])\big) \implies \left(A * B = \left[\sum_{k=1}^{p} (a_{i,k} b_{k,j})\right]_{m \times n}\right)$
${}^{T}[A] := (Matrix[A, m, n]) \implies (A^T = [a_{j,i}]_{n \times m})$

$AddCom := \forall_{A,B \in \mathcal{M}} (A + B = B + A)$

(1) $A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C \in \mathcal{M}} \big((A + B) + C = A + (B + C)\big)$

(1) $(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A \in \mathcal{M}} \exists!_{O \in \mathcal{M}} (A + O = A = O + A)$

(1) $A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$

(2) $A + O_1 = A = A + O_2$ ∎ $O_1 = O_2$

$AddInv := \forall_{A \in \mathcal{M}} \exists!_{(-A) \in \mathcal{M}} \big(A + (-A) = O = (-A) + A\big)$

(1) $A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$

(2) $A + (-A_1) = O = A + (-A_2)$ ∎ $-A_1 = -A_2$ ∎ $A_1 = A_2$

$MulAssoc := \forall_{A,B,C \in \mathcal{M}} \big((A * B) * C = A * (B * C)\big)$

(1) $(A * B) * C = \left[\sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,j})\right] * C = \left[\sum_{k_2=1}^{p_2} \left(\sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,k_2}) c_{k_2,j}\right)\right] = \left[\sum_{k_2=1}^{p_2} \sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,k_2} c_{k_2,j})\right] = \ ...$

(2) $...\ \left[\sum_{k_1=1}^{p_1} \sum_{k_2=1}^{p_2} (a_{i,k_1} b_{k_1,k_2} c_{k_2,j})\right] = \left[\sum_{k_1=1}^{p_1} \left(a_{i,k_1} \sum_{k_2=1}^{p_2} (b_{k_1,k_2} c_{k_2,j})\right)\right] = ... = A * (B * C)$

$MulId := \forall_{A : Square[A,n]} (A * I_n = A = I_n * A)$

(1) $A * I_n = \left[\sum_{k=1}^{n}\left(a_{i,k}\left(\begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}\right)\right)\right] = [a_{i,j}] = A$

(2) $TODO = A$

$ScalAssoc := \forall_{r,s\in\mathbb{R}}\forall_{A\in\mathcal{M}}\big(r(sA) = (rs)A = s(rA)\big)$

(1) $r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$

(2) $(rs)A = [rsa_{i,j}]$

(3) $s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$

$TransCancel := \forall_{A\in\mathcal{M}}\big(A = (A^T)^T\big)$

(1) $A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$

$ScalMulCom := \forall_{r\in\mathbb{R}}\forall_{A,B\in\mathcal{M}}\big((rA) * B = r(A * B) = A * (rB)\big)$

(1) $(rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[\sum_{k=1}^{p}(ra_{i,k}b_{k,j})\right] = r(A * B)$

(2) $A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[\sum_{k=1}^{p}(a_{i,k}rb_{k,j})\right] = \left[\sum_{k=1}^{p}(ra_{i,k}b_{k,j})\right] = r(A * B)$

$ScalDistLeft := \forall_{r,s\in\mathbb{R}}\forall_{A\in\mathcal{M}}\big((r + s)A = rA + sA\big)$

(1) TODO

$ScalDistRight := \forall_{r\in\mathbb{R}}\forall_{A,B\in\mathcal{M}}\big(r(A + B) = rA + rB\big)$

(1) TODO

$MulDistRight := \forall_{A,B,C\in\mathcal{M}}\big((A + B) * C = A * C + B * C\big)$

(1) $(A + B) * C = [a_{i,j} + b_{i,j}] * C = \left[\sum_{k=1}^{p}\Big((a_{i,k} + b_{i,k})c_{k,j}\Big)\right] = \ldots$

(2) $\ldots \left[\sum_{k=1}^{p}(a_{i,k}c_{k,j} + b_{i,k}c_{k,j})\right] = \left[\sum_{k=1}^{p}(a_{i,k}c_{k,j})\right] + \left[\sum_{k=1}^{p}(b_{i,k}c_{k,j})\right] = A * C + B * C$

$MulDistLeft := \forall_{A,B,C\in\mathcal{M}}\big(C * (A + B) = C * A + C * B\big)$

(1) TODO

$TransAddDist := \forall_{A,B\in\mathcal{M}}\big((A + B)^T = A^T + B^T\big)$

(1) TODO

$TransMulDist := \forall_{A,B\in\mathcal{M}}\big((A * B)^T = B^T * A^T\big)$

(1) $(A * B)^T = \left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]^T = \left[\sum_{k=1}^{p}(a_{j,k}b_{k,i})\right] = \left[\sum_{k=1}^{p}(b_{k,i}a_{j,k})\right] = \left[\sum_{k=1}^{p}(b^T_{i,k}a^T_{k,j})\right] = B^T * A^T$

$Sym[A] := A = A^T$

$SkewSym[A] := A = -A^T$

$Invertible[A] := (Square[A, n]) \wedge \Big(\exists_{A^{-1}\in\mathcal{M}}(A * A^{-1} = I_n = A^{-1} * A)\Big)$

$SymGen := \forall_{A\in\mathcal{M}}(Sym[A + A^T])$

(1) $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$SkewSymGen := \forall_{A\in\mathcal{M}}(SkewSym[A - A^T])$

(1) $-(A - A^T)^T = -\big(A^T - (A^T)^T\big) = -(A^T - A) = (A - A^T)$

$SymDecomp := \forall_{A \in \mathcal{M}} \exists!_{B:Sym[B]} \exists!_{C:SkewSym[C]} (A = B + C)$

(1) $B := (1/2) * (A + A^T)$; $C := (1/2) * (A - A^T)$

(2) $SymGen[B] \wedge SkewSymGen[C]$

(3) $A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$

(4) $(1/2) * (A_1 + {A_1}^T) = (1/2) * (A_2 + {A_2}^T)$ ▮ $A_1 = A_2$

(5) $(1/2) * (A_3 - {A_3}^T) = (1/2) * (A_4 - {A_4}^T)$ ▮ $A_3 = A_4$

$InvId := \forall_{A:Invertible[A]} \left(\exists!_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A) \right)$

(1) ${A^{-1}}_1 = {A^{-1}}_1 * I_n = {A^{-1}}_1 * (A * {A^{-1}}_2) = ({A^{-1}}_1 * A) * {A^{-1}}_2 = I_n * {A^{-1}}_2 = {A^{-1}}_2$

$InvCancel := \forall_{A:Invertible[A]} \left((A^{-1})^{-1} = A \right)$

(1) $(A * A^{-1})^{-1} = {I_n}^{-1} = I_n$

(2) $(A^{-1})^{-1} * A^{-1} = I_n$ ▮ $A^{-1})^{-1} = I_n * A = A$

$InvDist := \forall_{A:Invertible[A]} \forall_{B:Invertible[B]} \left((A * B)^{-1} = B^{-1} * A^{-1} \right)$

(1) $(A * B) * (A * B)^{-1} = I$ ▮ $B * (A * B)^{-1} = A^{-1}$ ▮ $(A * B)^{-1} = B^{-1} * A^{-1}$

$InvTrans := \forall_{A:Invertible[A]} \left((A^T)^{-1} = (A^{-1})^T \right)$ ▮ $\Leftarrow$

(1) $A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I$ ▮ $(A^{-1})^T = (A^T)^{-1}$

$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$
$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$
$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X (Sol[X, A, B])$
$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$
$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$

$HomoSysProps := (Sys[A, O]) \implies \ldots$

(1) $u_0 := O$; $u_1 := choice(\{X \in \mathcal{M} | X \neq O\})$; $k := choice(\mathbb{R})$

(2) $TrivSol[u_0, A]$

(3) $(NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$

(4) $(TrivSol[\overrightarrow{X}, A]) \implies \left(TrivSol[LC(\overrightarrow{X}), A] \right)$

$ElemMat[E] := (E = Swap[I_n, i, j]) \vee \left(Scale_*(I_n, i, c) \right) \vee \left(Combine_*(I_n, i, c, j) \right)$
$ElemMatProd[E^*] := \exists_{\langle E \rangle} \left(\forall_{E_i \in E^*} (ElemMat[E_i]) \wedge \left(E^* = \Pi_{E_i \in E^*} (E_i) \right) \right)$
$RowEquiv[A, B] := \exists_{E^*} \left((ElemMatProd[E^*]) \wedge (B = E^* * A) \right)$

$ElemMatInv := \forall_{E \in \mathcal{M}} \left((ElemMat[E]) \implies (Invertible[E]) \right)$

(1) $E - RowSwap[E] \implies TODO$; $E - RowScale_*(E) \implies TODO$; $E - RowCombine_*(E) \implies TODO$

$ElemMatProdInv := \forall_{E^*} \left((ElemMatProd[E^*]) \implies (Invertible[E^*]) \right)$

(1) TODO

$RowEquivSys := \forall_{A,B,C,D,X \in \mathcal{M}} \left(\left((Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]]) \right) \implies (Sol[X, A, B] \iff Sol[X, C, D]) \right)$

(1) $\exists_{E^*:ElemMatProd[E^*]} ([CD] = E^* * [AB])$

(2) $(E^* * A = C) \wedge (E^* * B = D)$

(3) $Sol[Y, A, B] \implies \ldots$

(3.1) $A * Y = B$

(3.2) $C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D$ ▌ $Sol[Y, C, D]$

(4) $Sol[Y, A, B] \implies Sol[Y, C, D]$

(5) $\left(A = (E^*)^{-1} * C\right) \wedge \left(B = (E^*)^{-1} * D\right)$

(6) $Sol[Z, C, D] \implies \ldots$

(6.1) $C * Z = D$

(6.2) $A * Z = \left((E^*)^{-1} * C\right) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$

(7) $Sol[Z, C, D] \implies Sol[Z, A, B]$

(8) $Sol[X, A, B] \iff Sol[X, C, D]$

$RowEquivHomoSysSol := \forall_{A,C,X \in \mathcal{M}}\Big((RowEquiv[A, C]) \implies \big((Sol[X, A, O]) \iff (Sol[X, C, O])\big)\Big)$

(1) Set $B = D = O$

$$RREF[A] := (A \in \mathcal{M}) \wedge \left(\begin{array}{l} \text{All zero rows are at the bottom of the matrix.} \wedge \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} \wedge \\ \text{The leading entry in any nonzero row is 1.} \wedge \\ \text{All entries in the column above and below a leading 1 are zero.} \wedge \end{array}\right)$$

$GaussJordanElim := \forall_{A \in \mathcal{M}} \exists!_{B \in \mathcal{M}}\big((RREF[B]) \wedge (RowEquiv[A, B])\big)$

(1) Hit A with $ElemMat$'s until it becomes B

(2) $(B = E^* * A) \wedge (RREF[B])$

$HasZero[A] := \big(Matrix(A, m, n)\big) \wedge \big(\exists_{i \le m}(A_{i,:} = O)\big)$

$HasZeroNonInvertible := \forall_{A \in \mathcal{M}}\big((HasZero[A]) \implies (\neg Invertible[A])\big)$

(1) $i := choice(\{i \le m | A_{i,:} = O\})$

(2) $(B \in \mathcal{M}) \implies \ldots$

(2.1) $(A * B)_{i,:} = O \neq I_{ni,:}$ ▌ $A * B \neq I_n$

(3) $(B \in \mathcal{M}) \implies (A * B \neq I_n)$ ▌ $\forall_{B \in \mathcal{M}}(A * B \neq I_n)$ ▌ $\neg Invertible[A]$

$InvIffRowEquivI := \forall_{A \in \mathcal{M}}\big((Invertible[A]) \iff (RowEquiv[A, I_n])\big)$

(1) $(Invertible[A]) \implies \ldots$

(1.1) $(RREF[B]) \wedge (RowEquiv[A, B])$

(1.2) $B = E^* * A$

(1.3) $(Invertible[E^*]) \wedge (Invertible[A])$ ▌ $Invertible[B]$

(1.4) $Invertible[B]$ ▌ $\neg HasZero[B]$

(1.5) $(RREF[B]) \wedge (\neg HasZero[B])$ ▌ $B = I_n$

(1.6) $RowEquiv[A, I_n]$

(2) $(Invertible[A]) \implies (RowEquiv[A, I_n])$

(3) $(RowEquiv[A, I_n]) \implies \ldots$

(3.1) $I_n = E^* * A$ ▌ $(E^*)^{-1} = A$

(3.2) $A^{-1} = E^*_{DescSort}$ ▌ $Invertible[A]$

(4) $(RowEquiv[A, I_n]) \implies (Invertible[A])$

(5) $(Invertible[A]) \iff (RowEquiv[A, I_n])$

$RowEquivIIffTrivSol := \forall_{A \in \mathcal{M}}\Big((RowEquiv[A, I_n]) \iff \Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big)\Big)$

(1) $(RowEquiv[A, I_n]) \implies \ldots$

(1.1) $RowEquiv[A, I_n]$ ▌ $Invertible[A]$

(1.2) $(Sol[X, A, O]) \implies \ldots$

(1.2.1) $A * X = O$ ▌ $X = A^{-1} * O = O$ ▌ $X = O$

(1.3) $(Sol[X, A, O]) \implies (X = O)$

(1.4) $(X = O) \implies (Sol[X, A, O])$

(1.5) $(X = O) \iff (Sol[X, A, O]) \blacksquare \forall_X\big((X = O) \iff (Sol[X, A, O])\big)$

(2) $(RowEquiv[A, I_n]) \implies \Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big)$

(3) $\Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big) \implies \ldots$

(3.1) $(RREF[B]) \wedge (RowEquiv[A, B])$

(3.2) $Sol[X, B, O]$

(3.3) $(B \neq I_n) \implies \ldots$

(3.3.1) $\Big(\exists_{Y \neq X}(Sol[Y, B, O])\Big)$

(3.3.2) $Sol[Y, A, O] \blacksquare Y = X$

(3.3.3) $(Y \neq X) \wedge (Y = X) \blacksquare \perp$

(3.4) $(B \neq I_n) \implies \perp \blacksquare B = I_n$

(3.5) $(RowEquiv[A, B]) \wedge (B = I_n) \blacksquare RowEquiv[A, I_n]$

(4) $\Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big) \implies (RowEquiv[A, I_n])$

(5) $(RowEquiv[A, I_n]) \iff \Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big)$

$InvIffUniqSol := \forall_{A \in \mathcal{M}}\Big((Invertible[A]) \iff \big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\big)\Big)$

(1) $(Invertible[A] \wedge B \in \mathcal{M}) \implies \ldots$

(1.1) $(Invertible[A]) \wedge (Sys[A, B])$

(1.2) $(X = A^{-1} * B) \iff (Sol[X, A, B]) \blacksquare \exists!_{X \in \mathcal{M}}(Sol[X, A, B])$

(2) $\big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\big) \implies \ldots$

(2.1) $X_i := choice(\{X_i | Sol[X_i, A, I_{n:,i}]\})$

(2.2) $A * [X_1 \ldots X_n] = [(A * X_1) \ldots (A * X_n)] = [I_{n:,1} \ldots I_{n:,n}] = I_n$

(2.3) $A^{-1} = [X_1 \ldots X_n]$

(3) $\big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\big) \implies (Invertible[A])$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}} \left(\begin{array}{l} (Invertible[A]) \iff \\ (RowEquiv[A, I_n]) \iff \\ \Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big) \iff \\ \big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\big) \end{array} \right)$$

$$VectorSpace[V, +, *] := \forall_{\alpha,\beta \in \mathbb{R}} \forall_{u,v,w \in V} \exists_{O \in V} \left(\begin{array}{l} (u + v \in V) \wedge (u + v = v + u) \wedge \big((u + v) + w = u + (v + w)\big) \wedge \\ (u + O = u) \wedge \Big(\exists_{-u \in V}\big(u + (-u) = O\big)\Big) \wedge \\ (\alpha * u \in V) \wedge \big(\alpha * (\beta * u) = (\alpha\beta) * u\big) \wedge (1 * u = u) \wedge \\ \big(\alpha * (u + v) = (\alpha * u) + (\alpha * v)\big) \wedge \big((\alpha + \beta) * u = (\alpha * u) + (\beta * u)\big) \end{array} \right)$$

$ZeroVectorUniq := \forall_{u,v \in V}\big((v + u = v) \implies (u = O)\big)$

(1) $(v + u = v) \blacksquare -v + v + u = -v + v \blacksquare u = O$

$ZeroVectorGen := \forall_{\alpha \in \mathbb{R}} \forall_{u \in V}\Big((\alpha * u = O) \iff \big((u = O) \vee (\alpha = 0)\big)\Big)$

(1) $(u = O) \implies \ldots$

(1.1) $\alpha * u = \alpha * O = \alpha * (O + O) = (\alpha * O) + (\alpha * O)$

(1.2) $\alpha * O = (\alpha * O) + (\alpha * O) \blacksquare \alpha * O = O \blacksquare \alpha * u = O$

(2) $(u = O) \implies (\alpha * u = O)$

(3) $(\alpha = 0) \implies \ldots$

(3.1) $\alpha * u = 0 * u = (0 + 0) * u = (0 * u) + (0 * u)$

(3.2) $0 * u = (0 * u) + (0 * u) \blacksquare 0 * u = O \blacksquare \alpha * u = O$

(4) $(\alpha = 0) \implies (\alpha * u = O)$

(5) $\big((u = O) \implies (\alpha * u = O)\big) \wedge \big((\alpha = 0) \implies (\alpha * u = O)\big) \blacksquare \big((u = O) \vee (\alpha = 0)\big) \implies (\alpha * u = O))$

(6) $(\alpha * u = O) \implies \ldots$

(6.1) $(\alpha \neq 0) \implies \ldots$

(6.1.1) $\alpha^{-1} \in \mathbb{R}$

(6.1.2) $O = \alpha^{-1} * O = \alpha^{-1} * (\alpha * u) = (\alpha^{-1} * \alpha) * u = 1 * u = u \;\blacksquare\; u = O$

(6.2) $(\alpha \neq 0) \implies (u = O)$

(6.3) $(\alpha = 0) \lor (\alpha \neq 0) \;\blacksquare\; (\alpha = 0) \lor (u = O)$

(7) $(\alpha * u = O) \implies \big((\alpha = 0) \lor (u = O)\big)$

(8) $(\alpha * u = O) \iff \big((\alpha = 0) \lor (u = O)\big)$

$$NegVectorGen := \forall_{u \in V}\big((-1) * u = -u\big)$$

(1) $O = 0 * u = \big(1 + (-1)\big) * u = (1 * u) + \big((-1) * u\big) = u + \big((-1) * u\big) \;\blacksquare\; O = u + \big((-1) * u\big) \;\blacksquare\; -u = (-1) * u$

$$Subspace[S, V, +, *] := (VectorSpace[V, +, *]) \land (\emptyset \neq S \subseteq V) \land (VectorSpace[S, +, *])$$

$$SubspaceEquiv := \forall_{V,S}\left(\begin{array}{l} \big((VectorSpace[V, +, *]) \land (\emptyset \neq S \subseteq V)\big) \qquad \implies \\ \Big((Subspace[S, V, +, *]) \iff \Big(\big(\forall_{r,s \in S}(r + s \in S)\big) \land \big(\forall_{\alpha \in \mathbb{R}}\forall_{s \in S}(\alpha * s \in S)\big)\Big)\Big) \end{array} \right)$$

(1) $(Subspace[S, V, +, *]) \implies \ldots$

(1.1) $VectorSpace[S, +, *] \;\blacksquare\; \big(\forall_{r,s \in S}(r + s \in S)\big) \land \big(\forall_{\alpha \in \mathbb{R}}\forall_{s \in S}(\alpha * s \in S)\big)$

(2) $(Subspace[S, V, +, *]) \implies \Big(\big(\forall_{r,s \in S}(r + s \in S)\big) \land \big(\forall_{\alpha \in \mathbb{R}}\forall_{s \in S}(\alpha * s \in S)\big)\Big)$

(3) $\Big(\big(\forall_{r,s \in S}(r + s \in S)\big) \land \big(\forall_{\alpha \in \mathbb{R}}\forall_{s \in S}(\alpha * s \in S)\big)\Big) \implies \ldots$

(3.1) $\big((\alpha, \beta \in \mathbb{R}) \land (u, v, w \in S)\big) \implies \ldots$

(3.1.1) $u, v \in V \;\blacksquare\; u + v = v + u$

(3.1.2) $u, v, w \in V \;\blacksquare\; (u + v) + w = u + (v + w)$

(3.1.3) $(ZeroVectorGen) \land (u \in S) \;\blacksquare\; 0 * u = O \in S$

(3.1.4) $u \in V \;\blacksquare\; u + O = u$

(3.1.5) $(NegVectorGen) \land (u \in S) \;\blacksquare\; (-1) * u = -u \in S$

(3.1.6) $u \in V \;\blacksquare\; \alpha * (\beta * u) = (\alpha\beta) * u$

(3.1.7) $u \in V \;\blacksquare\; 1 * u = u$

(3.1.8) $u, v \in V \;\blacksquare\; \alpha * (u + v) = (\alpha * u) + (\alpha * v)$

(3.1.9) $u \in V \;\blacksquare\; (\alpha + \beta) * u = (\alpha * u) + (\beta * u)$

(4) $\Big(\big(\forall_{r,s \in S}(r + s \in S)\big) \land \big(\forall_{\alpha \in \mathbb{R}}\forall_{s \in S}(\alpha * s \in S)\big)\Big) \implies (Subspace[S, V, +, *])$

(5) $(Subspace[S, V, +, *]) \iff \Big(\big(\forall_{r,s \in S}(r + s \in S)\big) \land \big(\forall_{\alpha \in \mathbb{R}}\forall_{s \in S}(\alpha * s \in S)\big)\Big)$

$$LinComb[c, U, K, V, +, *] := (VectorSpace[V, +, *]) \land (U \subseteq V) \land (K \subseteq \mathbb{R}) \land (n = |U| = |K| \in \mathbb{N}) \land \big(c = \textstyle\sum_{i=1}^{n}(k_i * u_i)\big)$$

$$LinSpan[S', S, V, +, *] := \left(\begin{array}{l} (VectorSpace[V, +, *]) \land (S \subseteq V) \land \big((S = \emptyset) \implies (S' = \{O\})\big) \land \\ \Big((S \neq \emptyset) \implies \big(S' = \{c \in V | \exists_{K \subseteq \mathbb{R}}(LinComb[c, S, K, V, +, *])\}\big)\Big) \end{array} \right)$$

$$Spans[S, V, +, *] := LinSpan[V, S, V, +, *]$$

$$LSSubspaceContaining := \forall_{S',S,V}\Big((LinSpan[S', S, V, +, *]) \implies \big((Subspace[S', V, +, *]) \land (S \subseteq S')\big)\Big)$$

(1) $LinSpan[S', S, V, +, *] \;\blacksquare\; O \in S' \;\blacksquare\; \emptyset \neq S'$

(2) $(\emptyset \neq S') \land (S' \subseteq V) \;\blacksquare\; \emptyset \neq S' \subseteq V$

(3) $(u, v \in S') \implies \ldots$

(3.1) $u \in S' \;\blacksquare\; \exists_{K \subseteq \mathbb{R}}(LinComb[u, S, K, V, +, *]) \;\blacksquare\; \exists_{K \subseteq \mathbb{R}}\big(u = \sum_{i=1}^{n}(k_i * s_i)\big)$

(3.2) $v \in S' \;\blacksquare\; \exists_{L \subseteq \mathbb{R}}(LinComb[v, S, L, V, +, *]) \;\blacksquare\; \exists_{L \subseteq \mathbb{R}}\big(v = \sum_{i=1}^{n}(l_i * s_i)\big)$

(3.3) $u + v = \sum_{i=1}^{n}(k_i * s_i) + \sum_{i=1}^{n}(l_i * s_i) = \sum_{i=1}^{n}\big((k_i + l_i) * s_i\big)$

(3.4) $M := \{k_i + l_i \in \mathbb{R} | (1 \leq i \leq n) \land (i \in \mathbb{N})\} \;\blacksquare\; M \subseteq \mathbb{R}$

(3.5) $\exists_{M \subseteq \mathbb{R}}\big(u + v = \sum_{i=1}^{n}(m_i * s_i)\big) \;\blacksquare\; \exists_{M \subseteq \mathbb{R}}(LinComb[u + v, S, M, V, +, *]) \;\blacksquare\; u + v \in S'$

(4) $(u, v \in S') \implies (u + v \in S') \blacksquare \forall_{u,v \in S'}(u + v \in S')$

(5) $\big((r \in \mathbb{R}) \land (u \in S')\big) \implies \ldots$

(5.1) $u \in S' \blacksquare \exists_{K \subseteq \mathbb{R}}(LinComb[u, S, K, V, +, *]) \blacksquare \exists_{K \subseteq \mathbb{R}}\big(u = \sum_{i=1}^{n}(k_i * s_i)\big)$

(5.2) $r * u = r * \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{n}\big(r * (k_i * s_i)\big) = \sum_{i=1}^{n}(rk_i) * s_i)$

(5.3) $M := \{rk_i \in \mathbb{R} | (1 \leq i \leq n) \land (i \in \mathbb{N})\} \blacksquare M \subseteq \mathbb{R}$

(5.4) $\exists_{M \subseteq \mathbb{R}}\big(r * u = \sum_{i=1}^{n}(m_i * s_i)\big) \blacksquare \exists_{M \subseteq \mathbb{R}}(LinComb[r * u, S, M, V, +, *]) \blacksquare r * u \in S'$

(6) $\big((r \in \mathbb{R}) \land (u \in S')\big) \implies (r * u \in S') \blacksquare \forall_{r \in \mathbb{R}} \forall_{u \in S'}(r * u \in S')$

(7) $SubspaceEquiv \blacksquare Subspace[S', V, +, *]$

(8) $(s_j \in S) \implies \ldots$

(8.1) $K := \{\left(\begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}\right) | (1 \leq i \leq n) \land (i \in \mathbb{N})\} \blacksquare K \subseteq \mathbb{R}$

(8.2) $\ldots \blacksquare \exists_{K \subseteq \mathbb{R}}(LinComb[s_j, S, K, V, +, *]) \blacksquare s_j \in S'$

(9) $(s_j \in S) \implies (s_j \in S') \blacksquare \forall_{x \in S}(x \in S') \blacksquare S \subseteq S'$

(10) $(Subspace[S', V, +, *]) \land (S \subseteq S')$

$LSSubspaceIdentity := (LinSpan[W', W, V, +, *]) \implies \big((W' = W) \iff (Subspace[W, V, +, *])\big)$

(1) $(W' = W) \implies \ldots$

(1.1) $LSSubspaceContaining \blacksquare Subspace[W', V, +, *] \blacksquare Subspace[W, V, +, *]$

(2) $(W' = W) \implies (Subspace[W, V, +, *])$

(3) $(Subspace[W, V, +, *]) \implies \ldots$

(3.1) $SubspaceEquiv \blacksquare \big(\forall_{r,s \in W}(r + s \in W)\big) \land \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in W}(\alpha * s \in W)\big) \blacksquare \forall_{w \in W}(LinComb[w, W, K, V, +, *])$

(3.2) $(w \in W) \iff (LinComb[w, W, K, V, +, *]) \iff (w \in W') \blacksquare W = W' \blacksquare W' = W$

(4) $(Subspace[W, V, +, *]) \implies (W' = W)$

(5) $(W' = W) \iff (Subspace[W, V, +, *])$

$LSSubspaceSubset := \big((LinSpan[S', S, V, +, *]) \land (Subspace[W, V, +, *]) \land (S \subseteq W)\big) \implies (Subspace[S', W, +, *])$

(1) $(LinSpan[S', S, V, +, *]) \land (S \subseteq W) \blacksquare (LinSpan[S', S, W, +, *])$

(2) $(LSSubspaceContaining) \land (LinSpan[S', S, W, +, *]) \blacksquare Subspace[S', W, +, *]$

$SmallestSubspaceContaining := \forall_{W,S',S,V}\Big(\big((LinSpan[S', S, V, +, *]) \land (Subspace[W, V, +, *]) \land (S \subseteq W)\big) \implies (S' \subseteq W)\Big)$

(1) $\big((Subspace[W, V, +, *]) \land (S \subseteq W)\big) \implies \ldots$

(1.1) $LSSubspaceSubset \blacksquare Subspace[S', W, +, *] \blacksquare S' \subseteq W$

(2) $\big((Subspace[W, V, +, *]) \land (S \subseteq W)\big) \implies (S' \subseteq W)$

$NullSpace[N, A, m, n] := (Matrix[A, m, n]) \land (N = \{x \in \mathbb{R}^n | A * x = O\})$
$RowSpace[R, A, m, n] := (Matrix[A, m, n]) \land (R = \{x^T * A \in \mathbb{R}^n | x \in \mathbb{R}^m\})$
$ColSpace[C, A, m, n] := (Matrix[A, m, n]) \land (C = \{A * x \in \mathbb{R}^m | x \in \mathbb{R}^n\})$

$NullSubspace := (NullSpace[N, A, m, n]) \implies (Subspace[N, \mathbb{R}^n, +, *])$

(1) TODO

$RowSubspace := (RowSpace[R, A, m, n]) \implies (Subspace[R, \mathbb{R}^n, +, *])$

(1) TODO

$ColSubspace := (ColSpace[C, A, m, n]) \implies (Subspace[C, \mathbb{R}^m, +, *])$

(1) TODO

$LinInd[S, V, +, *] := (VectorSpace[V, +, *]) \land (\emptyset \neq S \subseteq V) \land \Big(\forall_{K \subseteq \mathbb{R}}\big((LinComb[O, S, K, V, +, *]) \implies (K = \{0\})\big)\Big)$

TODO, SAY K is a sequence instead of a set $K \subseteq \mathbb{R}^n$ instead ??

$$LinInd[U,V,+,*] := \left(\begin{array}{l} (VectorSpace[V,+,*]) \wedge (U \subseteq V) \wedge (n = |U| \in \mathbb{N}) \wedge \\ \left(\forall_{\Gamma \in \mathbb{R}^n} \left(\left(\sum_{i=1}^{n} (\gamma_i * u_i) = O \right) \implies (\Gamma = \{0\}^n) \right) \right) \end{array} \right)$$

$SingletonZeroDependent := \neg LinInd[\{O\},V,+,*]$

(1) $(1 * O = O) \wedge (1 \neq 0)$

$SingletonNonZeroIndependent := (\alpha \neq O) \implies (LinInd[\{\alpha\},V,+,*])$

(1) $(r * \alpha = O) \iff \big((r = 0) \vee (\alpha \neq O)\big)$

(2) $\alpha \neq O \ \blacksquare \ r = 0$

(3) $\forall_{r \in \mathbb{R}^1} \big((r * \alpha = O) \implies (r = 0)\big)$

$$SubIndependent := \forall_{V,A,B} \left(\begin{array}{l} \big((VectorSpace[V,+,*]) \wedge (A \subseteq B \subseteq V) \wedge (|B| \in \mathbb{N})\big) \implies \\ \big((LinInd[B,V,+,*]) \implies (LinInd[A,V,+,*])\big) \end{array} \right)$$

(1) $U := B \setminus A$

(2) $\big(\sum(\gamma_i * a_i = O)\big) \implies \ldots$

(2.1) $\sum(0 * u_i) = O$

(2.2) $\sum(\gamma'_i = b_i) = \sum(\gamma_i * a_i) + \sum(0 * u_i) = O + O = O$

(2.3) $LinInd[B,V,+,*] \ \blacksquare \ \Gamma' = \{0\}^{|B|}$

(2.4) $\Gamma' = \{0\}^{|B|} = \Gamma \times \{0\}^{|U|} \ \blacksquare \ \Gamma = \{0\}^{|A|}$

(3) $\big(\sum(\gamma_i * a_i = O)\big) \implies (\Gamma = \{0\}^{|A|}) \ \blacksquare \ LinInd[A,V,+,*]$

$$SuperDependent := \forall_{V,A,B} \left(\begin{array}{l} \big((VectorSpace[V,+,*]) \wedge (A \subseteq B \subseteq V) \wedge (|B| \in \mathbb{N})\big) \implies \\ \big((\neg LinInd[A,V,+,*]) \implies (\neg LinInd[B,V,+,*])\big) \end{array} \right)$$

(1) $U := B \setminus A$

(2) $\sum(0 * u_i) = O$

(3) $\left(\exists_{\Gamma \in \mathbb{R}^{|A|}} \left(\left(\sum(\gamma_i * a_i) = O \right) \wedge (\Gamma \neq \{0\}^{|A|}) \right) \right)$

(4) $\left(\sum(\gamma_i * b_i) = \sum(\gamma_i * a_i) + O = O + O = O \right) \wedge (\Gamma' = \Gamma \times \{0\}^{|U|} \neq \{0\}^{|}B|)$

$$LinIndEquiv := \forall_{U,V} \left((LinInd[U,V,+,*]) \iff \left(\forall_{j \in U} (\neg LinComb[j, U \setminus \{j\}, +, *]) \right) \right)$$

(1) $\Gamma' = \Gamma \setminus \{j\}$

(2) $(\neg LinInd[U,V,+,*]) \implies \ldots$

(2.1) $\left(\exists_{\Gamma \in \mathbb{R}^{|U|}} \left(\left(\sum(\gamma_i * u_i) = O \right) \wedge (\Gamma \neq \{0\}^{|U|}) \right) \right)$

(2.2) $\exists_{\gamma_k \in \Gamma} (\gamma_k \neq 0)$

(2.3) $\sum(\gamma'_i * u_i) = \sum(\gamma_i * u_i) - \gamma_k * u_k = -\gamma_k * u_j$

(2.4) $u_k = (-1/\gamma_k)\left(\sum(\gamma'_i * u_i) \right) = \sum\left((-\gamma'_i/\gamma_k) * u_i \right) \ \blacksquare \ \exists_{j \in U} (LinComb[j, U \setminus \{j\}, +, *])$

(3) $(\neg LinInd[U,V,+,*]) \implies \left(\exists_{j \in U} (LinComb[j, U \setminus \{j\}, +, *]) \right)$

(4) $\left(\forall_{j \in U} (\neg LinComb[j, U \setminus \{j\}, +, *]) \right) \implies (LinInd[U,V,+,*])$

(5) $\left(\exists_{j \in U} (LinComb[j, U \setminus \{j\}, +, *]) \right) \implies \ldots$

(5.1) $\exists_{j \in U} \left(j = \sum(\gamma'_i * u_i) \right)$

(5.2) $\Gamma := \Gamma' \cup \{-1\}$

(5.3) $\left(\sum(\gamma_i * u_i) = \sum(\gamma'_i * u_i) + (-1) * \gamma_j = O \right) \wedge (\Gamma \neq \{0\}^n) \ \blacksquare \ \neg LinInd[U,V,+,*]$

(6) $\left(\exists_{j \in U} (LinComb[j, U \setminus \{j\}, +, *]) \right) \implies (\neg LinInd[U,V,+,*])$

(7) $(LinInd[U,V,+,*]) \implies \left(\forall_{j\in U}(\neg LinComb[j,U\setminus\{j\},+,*])\right)$

(8) $(LinInd[U,V,+,*]) \iff \left(\forall_{j\in U}(\neg LinComb[j,U\setminus\{j\},+,*])\right)$

$$Basis[S,V,+,*] := (Spans[S,V,+,*]) \wedge (LinInd[S,V,+,*])$$

$$UniqueLinComb := \forall_{S,V}\left((Basis[S,V]) \implies \left(\forall_{v,\Gamma,\Delta}\left(\left(\left(v=\sum(\gamma_i * u_i)\right) \wedge \left(v=\sum(\delta_i * u_i)\right)\right) \implies (\Gamma=\Delta)\right)\right)\right)$$

(1) $(v \in V) \implies \ldots$

(1.1) $Spans[V,S,+,*] \;\blacksquare\; \left(\exists_{\Gamma\in\mathbb{R}^n}\left(v=\sum(\gamma_i * s_i)\right)\right) \wedge \left(\exists_{\Gamma\in\mathbb{R}^n}\left(v=\sum(\gamma_i * s_i)\right)\right)$

(1.2) $O = v - v = \sum(\gamma_i * s_i) - \sum(\delta_i * s_i) = \sum\left((\gamma_i-\delta_i) * s_i\right) \;\blacksquare\; \sum\left((\gamma_i-\delta_i) * s_i\right) = O$

(1.3) $(LinInd[S,V,+,*]) \wedge \left(\sum\left((\gamma_i-\delta_i) * s_i\right) = O\right) \;\blacksquare\; \{\gamma_i-\delta_i\} = \{0\}^n \;\blacksquare\; \{\gamma_i\} = \{\delta_i\} \;\blacksquare\; \Gamma = \Delta$

(2) $\Gamma = \Delta$

$$BasisSubSpan := \forall_{S,V}\left((Spans[S,V,+,*]) \implies \left(\exists_{B\subseteq S}(Basis[B,V,+,*])\right)\right)$$

(1) $A = B$

(2) While $\neg LinInd(A,V,+,*)$, $\exists_{j\in A}(LinearCombination[j,A\setminus\{j\},+,*])$, $A' = A\setminus\{j\}$

(3) $Spans[A',S,+,*]$, until $(LinInd[A',V,+,*]) \wedge (Spans[A',V,+,*])$

$$BasisLinearIndCard := \forall_{S,T,V}\left(\left((Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*])\right) \implies (|T| \leq |S|)\right)$$

(1) $(Basis[S,V,+,*]) \implies \ldots$

(1.1) $(|T| > |S|) \implies \ldots$

(1.1.1) $(Spans[S,V,+,*]) \wedge (T \subseteq V) \;\blacksquare\; t_{1\ldots t_j} = \sum(\gamma_i * s * i)\ \ldots$

(1.1.2) $\ldots t_j = \sum(\gamma_i' * t_i) \;\blacksquare\; \neg LinInd[T,V,+,*]$

(1.2) $(|T| > |S|) \implies (\neg LinInd[T,V,+,*]) \;\blacksquare\; (LinInd[T,V,+,*]) \implies (|T| \leq |S|)$

(2) $\left((Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*])\right) \implies (|T| \leq |S|)$

$$BasisCardProp := \forall_{S,T,V}\left(\left((Basis[S,V,+,*]) \wedge (Basis[T,V,+,*])\right) \implies (|T| = |S|)\right)$$

(1) $Basis[S,V,+,*] \;\blacksquare\; LinInd[S,V,+,*]$

(2) $(Basis[T,V,+,*]) \wedge (LinInd[S,V,+,*]) \;\blacksquare\; |S| \leq |T|$

(3) $Basis[T,V,+,*] \;\blacksquare\; LinInd[T,V,+,*]$

(4) $(Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*]) \;\blacksquare\; |T| \leq |S|$

(5) $(|S| \leq |T|) \wedge (|T| \leq |S|) \;\blacksquare\; |T| = |S|$

$$Dim[d,V,+,*] := \left(\exists_B(Basis[B,V,+,*])\right) \wedge \left((V=\{O\}) \implies (d=0)\right) \wedge \left((V \neq \{O\}) \implies (d=|B|)\right)$$

$$Nullity[n,A] := (NullSpace[N,A]) \wedge (Dim[n,N,+,*])$$

??????????????????????????????