Solutions Manual to Walter Rudin's *Principles of Mathematical Analysis*

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Chapter 4

Continuity

Exercise 4.1 Suppose f is a real function defined on R^1 which satisfies

$$\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$$

for every $x \in R^1$. Does this imply that f is continuous?

Solution. No. In fact even the stronger statement

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{h^n} = 0$$

for every $x \in R^1$, where n is an arbitrary positive number, does not imply that f is continuous, since this property is possessed by the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer,} \\ 0 & \text{if } x \text{ is not an integer.} \end{cases}$$

(If x is an integer, then $f(x+h) - f(x-h) \equiv 0$ for all h; while if x is not an integer, $f(x+h) - f(x-h) = 0$ for $|h| < \min(x - [x], 1 + [x] - x)$.

Exercise 4.2 If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. ($\overline{E}$ denotes the closure of E.) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Solution. Let $x \in \overline{E}$. We need to show that $f(x) \in \overline{f(E)}$. To this end, let O be any neighborhood of $f(x)$. Since f is continuous, $f^{-1}(O)$ contains (is) a neighborhood of x. Since $x \in \overline{E}$, there is a point u of E in $f^{-1}(O)$. Hence $f(u) \in O \cap f(E)$. Since O was any neighborhood of $f(x)$, it follows that $f(x) \in \overline{f(E)}$.

Consider $f : R^1 \to R^1$ given by $f(x) = \dfrac{x}{1+x^2}$, and let $E = \overline{E} = [1, \infty)$, so that $f(E) = f(\overline{E}) = (0, \frac{1}{2}]$, yet $\overline{f(E)} = [0, \frac{1}{2}]$.

Exercise 4.3 Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Solution. $Z(f) = f^{-1}(\{0\})$, which is the inverse image of a closed set. Hence $Z(f)$ is closed.

Exercise 4.4 Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

Solution. To prove that $f(E)$ is dense in $f(X)$, simply use Exercise 2 above: $f(X) = f(\overline{E}) \subseteq \overline{f(E)}$.

The function $\varphi : X \to R^1$ given by

$$\varphi(p) = d_Y(f(p), g(p))$$

is continuous, since

$$|d_Y(f(p), g(p)) - d_Y(f(q), g(q))| \leq d_Y(f(p), f(q)) + d_Y(g(p), g(q)).$$

(This inequality follows from the triangle inequality, since

$$d_Y(f(p), g(p)) \leq d_Y(f(p), f(q)) + d_Y(f(q), g(q)) + d_Y(g(q), g(p)),$$

and the same inequality holds with p and q interchanged. The absolute value $|d_Y(f(p), g(p)) - d_Y(f(q), g(q))|$ must be either $d_Y(f(p), g(p)) - d_Y(f(q), g(q))$ or $d_Y(f(q), g(q)) - d_Y(f(p), g(p))$, and the triangle inequality shows that both of these numbers are at most $d_Y(f(p), f(q)) + d_Y(g(p), g(q))$.)

By the previous problem, the zero set of φ is closed. But by definition

$$Z(\varphi) = \{p : f(p) = g(p)\}.$$

Hence the set of p for which $f(p) = g(p)$ is closed. Since by hypothesis it is dense, it must be X.

Exercise 4.5 If f is a real continuous function defined on a closed set $E \subset R^1$, prove that there exist continuous real functions g on R^1 such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called *continuous extensions* of f from E to R^1.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector-valued functions. *Hint:* Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 29, Chap. 2). The result remains true if R^1 is replaced by any metric space, but the proof is not so simple.

Solution. Following the hint, let the complement of E consist of a countable collection of finite open intervals (a_k, b_k) together with possibly one or both of the the semi-infinite intervals $(b, +\infty)$ and $(-\infty, a)$. The function $f(x)$ is already defined at a_k and b_k, as well as at a and b (if these last two points exist). Define $g(x)$ to be $f(b)$ for $x > b$ and $f(a)$ for $x < a$ if a and b exist. On the interval (a_k, b_k) let

$$g(x) = f(a_k) + \frac{x - a_k}{b_k - a_k}(f(b_k) - f(a_k)).$$

Of course we let $g(x) = f(x)$ for $x \in E$. It is now fairly clear that $g(x)$ is continuous. A rigorous proof proceeds as follows. Let $\varepsilon > 0$. To choose $\delta > 0$ such that $|x - u| < \delta$ implies $|g(x) - g(u)| < \varepsilon$, we consider three cases.

i. If $x > b$, let $\delta = x - b$. Then if $|x - u| < \delta$, it follows that $u > b$ also, so that $g(u) = f(b) = g(x)$, and $|g(u) - g(x)| = 0 < \varepsilon$. Similarly if $x < a$, let $\delta = a - x$.

ii. If $a_k < x < b_k$ and $f(a_k) = f(b_k)$, let $\delta = \min(x - a_k, b_k - x)$. Since $|x - u| < \delta$ implies $a_k < u < b_k$, so that $g(u) = f(a_k) = f(b_k) = g(x)$, we again have $|g(x) - g(u)| = 0 < \varepsilon$. If $a_k < x < b_k$ and $f(a_k) \neq f(b_k)$, let $\delta = \min\left(x - a_k, b_k - x, \dfrac{(b_k - a_k)\varepsilon}{|f(b_k) - f(a_k)|}\right)$. Then if $|x - u| < \delta$, we again have $a_k < u < b_k$ and so

$$|g(x) - g(u)| = \frac{|x - u|}{b_k - a_k}|f(b_k) - f(a_k)| < \varepsilon.$$

iii. If $x \in E$, let δ_1 be such that $|f(u) - f(x)| < \varepsilon$ if $u \in E$ and $|x - u| < \delta_1$. (Subcase a). If there are points $x_1 \in E \cap (x - \delta_1, x)$ and $x_2 \in E \cap (x, x + \delta_1)$, let $\delta = \min(x - x_1, x_2 - x)$. If $|u - x| < \delta$ and $u \in E$, then $|f(u) - f(x)| < \varepsilon$ by definition of δ_1. if $u \notin E$, then, since x_1, x, and x_2 are all in E, it follows that $u \in (a_k, b_k)$, where $a_k \in E$, $b_k \in E$, and $|a_k - x| < \delta$ and $|b_k - x| < \delta$, so that $|f(a_k) - f(x)| < \varepsilon$ and $|f(b_k) - f(x)| < \varepsilon$. If $f(a_k) = f(b_k)$, then $f(u) = f(a_k)$ also, and we have $|f(u) - f(x)| < \varepsilon$. If $f(a_k) \neq f(b_k)$, then

$$\begin{aligned}
|f(u) - f(x)| &= \left|f(a_k) - f(x) + \frac{u - a_k}{b_k - a_k}(f(b_k) - f(a_k))\right| \\
&= \left|\frac{b_k - u}{b_k - a_k}(f(a_k) - f(x)) + \frac{u - a_k}{b_k - a_k}(f(b_k) - f(x))\right| \\
&< \frac{b_k - u}{b_k - a_k}\varepsilon + \frac{u - a_k}{b_k - a_k}\varepsilon \\
&= \varepsilon
\end{aligned}$$

(Subcase b). Suppose x_2 does not exist, i.e., either $x = a_k$ or $x = a_k$ and $b_k > a_k + \delta_1$. Let us consider the second of these cases and show how to get $|f(u) - f(x)| < \varepsilon$ for $x < u < x + \delta$. If $f(a_k) = f(b_k)$, let $\delta = \delta_1$. If $u > x$ we have $a_k < u < b_k$ and $f(u) = f(a_k) = f(x)$. If $f(a_k) \neq f(b_k)$, let $\delta = \min\left(\delta_1, \dfrac{(b_k - a_k)\varepsilon}{|f(b_k) - f(a_k)|}\right)$. Then, just as in Subcase a, we have $|f(u) - f(x)| < \varepsilon$.

The case when $x = b_k$ for some k and $a_k < x - \delta_1$ is handled in exactly the same way.

If $x = b$, let $\delta = \delta_1$. If $u > x$ we have $f(x) - f(u)$; and if $u < x$ and $u \notin E$, we use the same argument as in Subcases a and b.

The case $x = a$ is handled similarly.

The extension of this result to vector-valued functions is immediate: Simply extend each component of the function. A vector-valued function is continuous if and only if each of its components is continuous.

Exercise 4.6 If f is defined on E, the *graph* of f is the set of points $(x, f(x))$ for $x \in E$. In particular, if E is the set of real numbers and f is real-valued, the graph of f is a subset of the plane.

Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

Solution. Let Y be the co-domain of the function f. We invent a new metric space $E \times Y$ as the set of pairs of points (x, y), $x \in E$, $y \in Y$, with the metric $\rho((x_1, y_1), (x_2, y_2)) = d_E(x_1, x_2) + d_Y(y_1, y_2)$. The function $\varphi(x) = (x, f(x))$ is then a mapping of E into $E \times Y$.

We claim that the mapping φ is continuous if f is continuous. Indeed, let $x \in X$ and $\varepsilon > 0$ be given. Choose $\eta > 0$ so that $d_Y(f(x), f(u)) < \dfrac{\varepsilon}{2}$ if $d_E(x, y) < \eta$. Then let $\delta = \min\left(\eta, \dfrac{\varepsilon}{2}\right)$. It is easy to see that $\rho(\varphi(x), \varphi(u)) < \varepsilon$ if $d_E(x, u) < \delta$. Conversely if φ is continuous, it is obvious from the inequality $\rho(\varphi(x), \varphi(u)) \geq d_Y(f(x), f(u))$ that f is continuous.

From these facts we deduce immediately that the graph of a continuous function f on a compact set E is compact, being the image of E under the continuous mapping φ. Conversely, if f is not continuous at some point x, there is a sequence of points x_n converging to x such that $f(x_n)$ does not converge to $f(x)$. If no subsequence of $f(x_n)$ converges, then the sequence $\{(x_n, f(x_n))\}_{n=1}^{\infty}$ has no convergent subsequence, and so the graph is not compact. If some subsequence of $f(x_n)$ converges, say $f(x_{n_k}) \to z$, but $z \neq f(x)$, then the graph of f fails to contain the limit point (x, z), and hence is not closed. A fortiori it is not compact.

Exercise 4.7 If $E \subset X$ and if f is a function defined on X, the *restriction* of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on R^2 by $f(0,0) = g(0,0) = 0$, $f(x,y) = xy^2/(x^2 + y^4)$, $g(x,y) = xy^2/(x^2 + y^6)$ if $(x,y) \neq (0,0)$. Prove that f is bounded on R^2, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in R^2 are continuous!

Solution. The fact that $|f(x,y)| \le \frac{1}{2}$ is an easy consequence of the inequality $(x-y^2)^2 \ge 0$. The fact that $\lim_{y\to 0} g(y^3,y) = \lim_{y\to 0} \frac{y^5}{2y^6} = \lim_{y\to 0} \frac{1}{2y} = \infty$ shows that g is unbounded on every neighborhood of infinity. The fact that $\lim_{y\to 0} f(y^2,y) = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$ shows that f is not continuous at $(0,0)$.

Since f and g are continuous except at $(0,0)$, it is obvious that their restrictions to any line that does not pass through $(0,0)$ are continuous. Now a line that *does* pass through $(0,0)$ has an equation that is either $x = 0$ or $y = ax$ for some a. Both f and g are constantly 0 on the first of these, and on the second we have $f(x,ax) = a^2x^3/(x^2+a^4x^4) = a^2x/(1+a^4x^2)$, while $g(x,ax) = a^2x^3/(x^2+a^6x^6) = a^2x/(1+a^6x^4)$. Both of the latter are obviously continuous functions.

Exercise 4.8 Let f be a real uniformly continuous function on the bounded set E in R^1. Prove that f is bounded on E.

Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Let $a = \inf E$ and $b = \sup E$, and let $\delta > 0$ be such that $|f(x) - f(y)| < 1$ if $x, y \in E$ and $|x-y| < \delta$. Now choose a positive integer N larger than $(b-a)/\delta$, and consider the N intervals $I_k = \left[a + \frac{k-1}{b-a}, a + \frac{k}{b-a}\right]$, $k = 1, 2, \ldots, N$. For each k such that $I_k \cap E \ne \varnothing$ let $x_k \in E \cap I_k$. Then let $M = 1 + \max\{|f(x_k)|\}$. If $x \in E$, we have $|x - x_k| < \delta$ for some k, and hence $|f(x)| < M$.

The function $f(x) = x$ is uniformly continuous on the entire line, but not bounded.

Exercise 4.9 Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\operatorname{diam} f(E) < \varepsilon$ for all $E \subset X$ with $\operatorname{diam} E < \delta$.

Solution. Suppose f is uniformly continuous and $\varepsilon > 0$ is given. Choose any positive number α smaller than ε. Then there exists $\delta > 0$ such that $d_Y(f(x), f(u)) < \alpha$ if $d_X(x,u) < \delta$. Hence if E is any set of diameter less than δ and x and u are any two points in E we have $d_Y(f(x), f(u)) < \alpha$, so that $\operatorname{diam} f(E) \le \alpha < \varepsilon$.

Conversely if f satisfies the condition stated in the problem, it is obvious that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x), f(u)) < \varepsilon$ whenever $d_X(x,u) < \delta$. (Choose $\delta > 0$ corresponding to ε in the condition of the problem and then let E be the two-point set $\{x, u\}$.)

Exercise 4.10 Complete the details of the following alternate proof of Theorem 4.19: If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \longrightarrow 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$. Use Theorem 2.37 to obtain a contradiction.

Solution. Theorem 4.19 asserts that a continuous function on a compact set is uniformly continuous. By Theorem 2.37 there are subsequences $\{p_{n_k}\}$ and $\{q_{n_k}\}$ that converge to points p and q respectively. Since $d_X(p_n, q_n) \longrightarrow 0$, it follows that $p = q$. However, since f is continuous, it follows from Theorem 4.2 that $f(p_{n_k})$ and $f(q_{n_k})$ converge to $f(p)$, which, since $d_Y(f(p_{n_k}), f(q_{n_k}) \le d_Y(f(p_{n_k}), f(p)) + d_Y(f(p), f(q_{n_k}))$, implies that $d_Y(f(p_{n_k}), f(q_{n_k})) \longrightarrow 0$, contradicting the inequality $d_Y(f(p_{n_k}), f(q_{n_k})) > \varepsilon$.

Exercise 4.11 Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof of the theorem stated in Exercise 13.

Solution. Suppose $\{x_n\}$ is a Cauchy sequence in X. Let $\varepsilon > 0$ be given. Let $\delta > 0$ be such that $d_Y(f(x), f(u)) < \varepsilon$ if $d_X(x, u) < \delta$. Then choose N so that $d_X(x_n, x_m) < \delta$ if $n, m > N$. Obviously $d_Y(f(x_n), f(x_m)) < \varepsilon$ if $m, n > N$, showing that $\{f(x_n)\}$ is a Cauchy sequence.

Now let f be a uniformly continuous function defined on a dense subset E of X, mapping E into a *complete* metric space Y (for example, Y could be the real numbers). To prove that f has a unique continuous extension to all of X, proceed as follows. For each $x \in X \setminus E$ let $\{x_n\}$ be a sequence of points in E converging to x. Define $f(x)$ to be the limit of the Cauchy sequence $\{f(x_n)\}$. This definition is unambiguous; for if $\{u_n\}$ also converges to x, then the sequence $\{y_n\}$ defined by

$$y_n = \begin{cases} x_{n/2} & \text{if } n \text{ is even,} \\ u_{(n+1)/2} & \text{if } n \text{ is odd,} \end{cases}$$

also converges to x. Hence $\{f(y_n)\}$ is a Cauchy sequence in Y, and so all subsequences of $\{f(y_n)\}$ converge to the same limit. In particular $\{f(x_n)\}$ and $\{f(u_n)\}$ both converge to the same value.

The extended function is also uniformly continuous. For if $\varepsilon > 0$, let $\delta > 0$ be such that $d_Y(f(x), f(u)) < \dfrac{\varepsilon}{3}$ if $x, u \in E$ and $d_X(x, u) < \delta$. Then if $x \in E$, $u \in X \setminus E$, and $d_X(x, u) < \delta$, choose $v \in E$ with $d_X(v, u) < \delta - d_X(x, u)$ and $d_Y(f(v), f(u)) < \dfrac{\varepsilon}{3}$ (this is possible because of the definition of $f(u)$). We then have $d_X(x, v) \le d_X(x, u) + d_X(u, v) < \delta$, and so

$$d_Y(f(x), f(u)) \le d_Y(f(x), f(v)) + d_Y(f(v), f(u)) < \frac{2\varepsilon}{3} < \varepsilon.$$

Similarly if $x \in X \setminus E$, $u \in X \setminus E$, and $d_X(x, u) < \delta$, choose $v, w \in E$ with $d_X(v, u) < \dfrac{1}{2}(\delta - d_X(x, u))$, $d_X(x, w) < \dfrac{1}{2}(\delta - d_X(x, u))$, $d_Y(f(v), f(u)) < \dfrac{\varepsilon}{3}$, and $d_Y(f(w), f(x)) < \frac{\varepsilon}{3}$. We then have

$$d_X(v,w) \le d_X(v,u) + d_X(u,x) + d_X(x,w) < \delta$$

and hence

$$d_Y(f(x), f(u)) \le d_Y(f(x), f(w)) + d_Y(f(w), f(v)) + d_Y(f(v), f(u)) < \varepsilon.$$

The uniqueness of this extension follows from Exercise 4 above.

Exercise 4.12 A uniformly continuous function of a uniformly continuous function is uniformly continuous.

State this more precisely and prove it.

Solution. Let $f : X \to Y$ and $g : Y \to Z$ be uniformly continuous. Then $g \circ f : X \to Z$ is uniformly continuous, where $g \circ f(x) = g\big(f(x)\big)$ for all $x \in X$.

To prove this fact, let $\varepsilon > 0$ be given. Then, since g is uniformly continuous, there exists $\eta > 0$ such that $d_Z(g(u), g(v)) < \varepsilon$ if $d_Y(u,v) < \eta$. Since f is uniformly continuous, there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \eta$ if $d_X(x,y) < \delta$.

It is then obvious that $d_Z(g(f(x)), g(f(y))) < \varepsilon$ if $d_X(x,y) < \delta$, so that $g \circ f$ is uniformly continuous.

Exercise 4.13 Let E be a dense subset of a metric space X, and let f be a uniformly continuous *real* function defined on E. Prove that f has a continuous extension from E to X (see Exercise 5 for terminology). (Uniqueness follows from Exercise 4.) *Hint:* For each $p \in X$ and each positive ineger n, let $V_n(p)$ be the set of all $q \in E$ with $d(p,q) < 1/n$. Use Exercise 9 to show that the intersection of the closures of the sets $f(V_1(p)), f(V_2(p)), \ldots$, consists of a single point, say $g(p)$, of R^1. Prove that the function g so defined is the desired extension of f.

Could the range space R^1 be replaced by R^n. By any compact metric space? By any complete metric space? By any metric space?

Solution. We shall carry out the proof in the context of any complete metric space, showing that the range space could be R^n or any compact metric space.

The diameter of the closure of $f(V_i(p))$ is the same as the diameter of $f(V_i(p))$ itself. Hence by Exercise 9 above these diameters tend to zero. Since they form a nested sequence of nonempty closed sets, their intersection must consist of a single point, which can be defined to be $g(p)$. If $p \in E$, the intersection of these sets is just $f(p)$ (since $f(p)$ is in all the sets, and only one point belongs to all of them), so that g coincides with f on E. It remains to show that g is continuous. This proof is identical to the proof given in Exercise 11 above, which depends only on the fact that for $u \in X \setminus E$ and $\varepsilon > 0$, $\delta > 0$ there is a point $v \in E$

with $d_X(v, u) < \delta$ and $d_Y(f(v), f(u)) < \varepsilon$. This condition clearly holds in the present context as well.

In general this theorem fails on an incomplete metric space. For example, take X to be the real numbers, Y and E the rational numbers, and let $f : E \to Y$ be given by $f(x) = x$. There is no possible extension of f to a mapping from X into Y. (There is a unique extension of f to a mapping from X into X, but its range is not contained in Y. If there were an extension of f to a mapping from X into Y, there would be two extensions of f to mappings from X into X, contradicting the uniqueness of the extension.)

Exercise 4.14 Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Solution. If $f(0) = 0$ or $f(1) = 1$, we are done. If not, then $0 < f(0)$ and $f(1) < 1$. Hence the continuous function $g(x) = x - f(x)$ satisfies $g(0) < 0 < g(1)$. By the intermediate value theorem, there must be a point $x \in (0, 1)$ where $g(x) = 0$.

Exercise 4.15 Call a mapping from X into Y *open* if $f(V)$ is an open set in Y whenever V is an open set in X.

Prove that every continuous open mapping of R^1 into R^1 is monotonic.

Solution. Suppose f is continuous and not monotonic, say there exist points $a < b < c$ with $f(a) < f(b)$, and $f(c) < f(b)$. Then the maximum value of f on the closed interval $[a, c]$ is assumed at a point u in the open interval (a, c). If there is also a point v in the open interval (a, c) where f assumes its minimum value on $[a, c]$, then $f(a, c) = [f(v), f(u)]$. If no such point v exists, then $f(a, c) = (d, f(u)]$, where $d = \min(f(a), f(c))$. In either case, the image of (a, c) is not open.

Exercise 4.16 Let $[x]$ denote the largest integer contained in x, that is $[x]$ is the integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the functions $[x]$ and (x) have?

Solution. The two functions have the same discontinuities, since each can be written as the difference of the continuous function $f(x) = x$ and the other function. Now the function $[x]$ is constant on each open interval $(k, k + 1)$; hence its only possible discontinuities are the integers. These are of course real discontinuities, since if $\varepsilon = 1$, there is no $\delta > 0$ such that $|[x] - [k]| < \varepsilon$ whenever $|x - k| < \delta$. (For if any δ is given, let $\eta = \min(1, \delta)$. Then $[k] - [k - \frac{\eta}{2}] = 1$.)

Exercise 4.17 Let f be a real function defined on (a, b). Prove that the set of points at which f has a simple discontinuity is at most countable. *Hint:* Let E be the set on which $f(x-) < f(x+)$. With each point x of E associate a triple (p, q, r) of rational numbers such that
(*a*) $f(x-) < p < f(x+)$,
(*b*) $a < q < t < x$ implies $f(t) < p$,
(*c*) $x < t < r < b$ implies $f(t) > p$.

The set of such triples is countable. Show that each triple is associated with at most one point of E. Deal similarly with the other possible types of simple discontinuities.

Solution. The existence of three such rational numbers (p, q, r) for each simple discontinuity of this type follows from the assumption $f(x-) < f(x+)$, and the definition of $f(x-)$ and $f(x+)$. We need to show that a given triple (p, q, r) cannot be associated with any other discontinuity of this type. To that end, suppose $y > x$ and $f(y-) < f(y+)$. If we do not have $f(y-) < p < f(y+)$, then the triple chosen for y will differ from (p, q, r) in its first element. Hence suppose $f(y-) < p < f(y+)$. In this case we definitely cannot have $r > y$, since there are points $t \in (x, y)$ such that $f(t) < p$ (if there weren't, we would have $f(y-) \geq p$).

We have thus shown that the set of points $x \in (a, b)$ at which $f(x-) < f(x+)$ is at most countable. The proof that the set of points at which $f(x-) > f(x+)$ is at most countable is, of course, nearly identical.

Now consider the set of points x at which $\lim_{t \to x} f(t)$ exists, but is not equal to $f(x)$. For each point $x \in (a, b)$ such that $\lim_{t \to x} f(t) < f(x)$, we take a triple (p, q, r) of rational numbers such that
(*a*) $\lim_{t \to x} f(t) < p < f(x)$,
(*b*) $a < q < t < x$ or $x < t < r < b$ implies $f(t) < p$.

As before, if $y > x$ and $\lim_{t \to y} f(t) < f(y)$, the triple associated with y will be different from that associated with x. For even if $\lim_{t \to y} f(t) < p < f(y)$, we cannot have $r > y$, since $f(y) > p$ and $x < y$.

The proof that the set of points $x \in (a, b)$ at which $\lim_{t \to x} f(t) > f(x)$ is countable is nearly identical.

Hence, the number of points in $[a, b]$ at which f has a discontinuity of first kind is countable.

Exercise 4.18 Every rational x can be written in the form $x = m/n$, where $n > 0$ and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on R^1 by

$$f(x) = \begin{cases} 0 & (x \text{ irrational}), \\ \dfrac{1}{n} & \left(x = \dfrac{m}{n}\right). \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Solution. We shall show that $\lim_{t \to x} f(t) = 0$ for every t. Both assertions follow immediately from this fact. To this end, let $\varepsilon > 0$ be given, and let x be any real number. Let N be the unique positive integer such that $N \leq 1/\varepsilon < N+1$, and for each positive integer $n = 1, 2, \ldots, N$, let k_n be the unique integer such that

$$\frac{k_n}{n} \leq x < \frac{k_n + 1}{n}$$

Then for each such n let $\delta_n = \frac{1}{n}$ if $x = \frac{k_n}{n}$, otherwise let $\delta_n = \min\left(x - \frac{k}{n}, \frac{k_n + 1}{n} - x\right)$. Finally let $\delta = \min(\delta_1, \ldots, \delta_N)$. We claim that $|f(t)| < \varepsilon$ if $0 < |x - t| < \delta$. This is obvious if t is irrational, while if t is rational and $t = \frac{m}{n}$, we necessarily have $n > N$ by the choice of the numbers δ_n for $n \leq N$. Hence if t is rational, then $f(t) \leq \frac{1}{N+1} < \varepsilon$. The proof is now complete.

Exercise 4.19 Suppose f is a real function with domain R^1 which has the intermediate-value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b.

Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed.

Prove that f is continuous.

Hint: If $x_n \to x_0$ but $f(x_n) > r > f(x_0)$ for some r and all n, then $f(t_n) = r$ for some t_n between x_0 and x_n; thus $t_n \to x_0$. Find a contradiction. (N. M. Fine, *Amer. Math. Monthly*, vol. 73, 1966, p. 782.)

Solution. The contradiction is evidently that x_0 is a limit point of the set of t such that $f(t) = r$, yet, x_0 does not belong to this set. This contradicts the hypothesis that the set is closed.

Exercise 4.20 If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.
(b) Prove that ρ_E is a uniformly continuous function on X by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

for all $x \in X$ and $y \in X$.

Hint: $\rho_E(x) \leq d(x,z) \leq d(x,y) + d(y,z)$, so that

$$\rho_E(x) \leq d(x,y) + \rho_E(y).$$

Solution. (a) For each positive integer n, let $z_n \in E$ be such that $\rho_E(x) \leq d(x,z_n) < \rho_E(x) + \frac{1}{n}$. It follows that $d(x,z_n) \longrightarrow \rho_E(x)$. If $\rho_E(x) = 0$, this means $z_n \longrightarrow x$, i.e., $x \in \overline{E}$. Conversely, if $x \in \overline{E}$, there exists a sequence $\{z_n\}_{n=1}^{\infty} \subseteq E$ such that $z_n \longrightarrow x$, and this means $d(z_n, x) \longrightarrow 0$, so that $\rho_E(x) = 0$.

(b) The last inequality given in the hint follows form the first by taking the infimum over z on the right-hand side. This inequality immediately implies that

$$\rho_E(x) - \rho_E(y) \leq d(x,y).$$

By interchanging x and y, we also obtain

$$\rho_E(y) - \rho_E(x) \leq d(y,x) = d(x,y).$$

Since $|\rho_E(x) - \rho_E(y)|$ must be either $\rho_E(x) - \rho_E(y)$ or $\rho_E(y) - \rho_E(x)$, it follows that

$$|\rho_E(x) - \rho_E(y)| \leq d(x,y).$$

Exercise 4.21 Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p,q) > \delta$ if $p \in K$, $q \in F$. *Hint:* ρ_F is a continuous positive function on K.

Show that the conclusion may fail for two disjoint closed sets if neither is compact.

Solution. Following the hint, we observe that $\rho_F(x)$ must attain its minimum value on K, i.e., there is some point $r \in K$ such that

$$\rho_F(r) = \min_{q \in K} \rho_F(q).$$

Since F is closed and $r \notin F$, it follows from Exercise 4.20 that $\rho_F(r) > 0$. Let δ be any positive number smaller than $\rho_F(r)$. Then for any $p \in F$, $q \in K$, we have

$$d(p,q) \geq \rho_F(q) \geq \rho_F(r) > \delta.$$

This proves the positive assertion.

As for closed sets in general, one could let $F = \{1, 2, 3, \ldots\}$ and $K = \{1 + \frac{1}{2}, 2 + \frac{1}{3}, 3 + \frac{1}{4} \ldots\}$ in R^1, or one could let $F = \{(x,y) : y = 0\}$ and $K = \{(x,y) : y = \frac{1}{1+x^2}\}$ in R^2. In both cases there are sequences of points $p_n \in F$, $q_n \in K$ such that $d(p_n, q_n) \longrightarrow 0$.

Exercise 4.22 Let A and B be disjoint nonempty closed sets in a metric space X, and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)} \quad (p \in X).$$

Show that f is a continuous function on X whose range lies in $[0,1]$, that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B. This establishes a converse of Exercise 3: Every closed set $A \subset X$ is $Z(f)$ for some continuous real f on X. Setting

$$V = f^{-1}\left(\left[0, \frac{1}{2}\right)\right), \quad W = f^{-1}\left(\left(\frac{1}{2}, 1\right]\right),$$

show that V and W are open and disjoint, and that $A \subset V$, $B \subset W$. (Thus pairs of disjoint closed sets in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called *normality*.)

Solution. The continuity of f follows from the fact that the quotient of two continuous real-valued continuous functions is continuous wherever the denominator is non-zero. Now the denominator of the fraction that defines f cannot be zero, since the first term is zero only on A and the second is zero only on B, while A and B are disjoint. The fact that $f(p) = 0$ if and only if $p \in A$ follows from Exercise 20 and the fact that A is closed. Likewise the fact that $f(p) = 1$ if and only if $p \in B$ follows from Exercise 20 and the fact that B is closed. The assertion about V and W is immediate, since V and W are the inverse images of disjoint open sets containing 0 and 1 respectively.

Exercise 4.23 A real-valued function f defined in (a,b) is said to be *convex* if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f.)

If f is convex in (a,b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \le \frac{f(u) - f(s)}{u - s} \le \frac{f(u) - f(t)}{u - t}.$$

Solution. Fix any points c, d with $a < c < d < b$, let $\eta > 0$ be any fixed positive number with $\eta < \dfrac{d-c}{2}$ and consider any two points x, y satisfying $c + \eta \le x < y \le d - \eta$. The inequality in the definition implies that $f(t)$ is bounded above on $[c,d]$. Indeed, if $c < t < d$, taking $\lambda = \dfrac{t-c}{d-c}$, we have $t = (1-\lambda)c + \lambda d$, and so, if $M = \max(f(c), f(d))$, we have

$$f(t) \le (1-\lambda)f(c) + \lambda f(d) \le (1-\lambda)M + \lambda M = M.$$

It is less obvious that f is also bounded below on $[c,d]$. In fact if $\frac{c+d}{2} < t < d$, we have

$$\frac{c+d}{2} = (1-\lambda)c + \lambda t,$$

where $\lambda = \frac{d-c}{2(t-c)}$, so that

$$f\Big(\frac{c+d}{2}\Big) \leq \Big(\frac{2t-(c+d)}{2(t-c)}\Big) f(c) + \Big(\frac{d-c}{2(t-c)}\Big) f(t),$$

which implies

$$f(t) \geq \Big(\frac{2(t-c)}{d-c}\Big) f\Big(\frac{c+d}{2}\Big) - \frac{2t-(c+d)}{d-c}\Big) f(c) \geq -2\Big|f\Big(\frac{c+d}{2}\Big)\Big| - |f(c)|.$$

The proof that f is bounded below on $\left[c, \frac{c+d}{2}\right]$ is similar. Hence there exists M such that $|f(t)| \leq M$ for all $t \in [c,d]$.

We can also write

$$x = (1-\lambda)c + \lambda y,$$

where $\lambda = \frac{x-c}{y-c} \in (0,1)$. Accordingly we have

$$\begin{aligned} f(x) - f(y) &\leq (1-\lambda)\big(f(c) - f(y)\big) = \\ &= \frac{y-x}{y-c}\big(f(c) - f(y)\big) \leq \frac{y-x}{\eta}|f(c) - f(y)|. \end{aligned}$$

Thus

$$f(x) - f(y) \leq \frac{2M}{\eta}(y-x).$$

Similarly, writing $y = \lambda x + (1-\lambda)d$, where $\lambda = \frac{d-y}{d-x} \in (0,1)$, we find

$$\begin{aligned} f(y) - f(x) &\leq (1-\lambda)\big(f(d) - f(x)\big) = \\ &= \frac{y-x}{d-x}\big(f(d) - f(x)\big) \leq \frac{y-x}{\eta}|f(d) - f(x)|. \end{aligned}$$

Hence we also have

$$f(y) - f(x) \leq \frac{2M}{\eta}(y-x).$$

Therefore

$$|f(y) - f(x)| \leq \frac{2M}{\eta}|y-x|$$

for all $x, y \in [c+\eta, d-\eta]$. Since c, d, and η are arbitrary, it follows that f is continuous on (a,b).

If $f(x)$ is convex on (a,b), and $g(x)$ is an increasing convex function on $f((a,b))$, we have

$$g(f(\lambda x+(1-\lambda)y)) \leq g(\lambda f(x)+(1-\lambda)f(y)) \leq \lambda g(f(x))+(1-\lambda)g(f(y)).$$

The inequality

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s}$$

can be rewritten as

$$f(t) \leq \frac{t-s}{u-s}f(u)+\Big(1-\frac{t-s}{u-s}\Big)f(s),$$

which is precisely the definition of convexity if we note that

$$t=\lambda u+(1-\lambda)s$$

when $\lambda = \dfrac{t-s}{u-s}$.

The other inequality is proved in exactly the same way.

Exercise 4.24 Assume that f is a continuous real function defined in (a,b) such that

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{f(x)+f(y)}{2}$$

for all $x,y \in (a,b)$. Prove that f is convex.

Solution. We shall prove that

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$$

for all "dyadic rational" numbers, i.e., all numbers of the form $\lambda = \dfrac{k}{2^n}$, where k is a nonnegative integer not larger than 2^n. We do this by induction on n. The case $n=0$ is trivial (since $\lambda=0$ or $\lambda=1$). In the case $n=1$ we have $\lambda=0$ or $\lambda=1$ or $\lambda=\frac{1}{2}$. The first two cases are again trivial, and the third is precisely the hypothesis of the theorem. Suppose the result is proved for $n \leq r$, and consider $\lambda = \dfrac{k}{2^{r+1}}$. If k is even, say $k=2l$, then $\dfrac{k}{2^{r+1}} = \dfrac{l}{2^r}$, and we can appeal to the induction hypothesis. Now suppose k is odd. Then $1 \leq k \leq 2^{r+1}-1$, and so the numbers $l = \dfrac{k-1}{2}$ and $m = \dfrac{k+1}{2}$ are integers with $0 \leq l < m \leq 2^r$. We can now write

$$\lambda = \frac{s+t}{2},$$

where $s = \dfrac{k-1}{2^{r+1}} = \dfrac{l}{2^r}$ and $t = \dfrac{k+1}{2^{r+1}} = \dfrac{m}{2^r}$. We then have

$$\lambda x+(1-\lambda)y = \frac{[sx+(1-s)y]+[tx+(1-t)y]}{2}.$$

Hence by the hypothesis of the theorem and the induction hypothesis we have

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\leq \frac{f(sx+(1-s)y) + f(tx+(1-t)y)}{2} \\ &\leq \frac{sf(x) + (1-s)f(y) + tf(x) + (1-t)f(y)}{2} \\ &= \Big(\frac{s+t}{2}\Big)f(x) + \Big(1 - \frac{s+t}{2}\Big)f(y) \\ &= \lambda f(x) + (1-\lambda)f(y). \end{aligned}$$

This completes the induction.

Now for each fixed x and y both sides of the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

are continuous functions of λ. Hence the set on which this inequality holds (the inverse image of the closed set $[0,\infty)$ under the mapping $\lambda \mapsto \lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y)$) is a closed set. Since it contains all the points $\frac{k}{2^n}$, $0 \leq k \leq n$, $n = 1, 2, \ldots$, it must contain the closure of this set of points, i.e., it must contain all of $[0,1]$. Thus f is convex.

Exercise 4.25 If $A \subset R^k$ and $B \subset R^k$, define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$, $\mathbf{y} \in B$.

(a) If K is compact and C is closed in R^k, prove that $K + C$ is closed.

Hint: Take $\mathbf{z} \notin K + C$, put $F = \mathbf{z} - C$, the set of all $\mathbf{z} - \mathbf{y}$ with $\mathbf{y} \in C$. Then K and F are disjoint. Choose δ as in Exercise 21. Show that the open ball with center $\mathbf{z}$ and radius δ does not intersect $K + C$.

(b) Let α be an irrational number. Let C_1 be the set of all integers. Let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of R^1 whose sum $C_1 + C_2$ is not closed, by showing that $C_1 + C_2$ is a countable dense subset of R^1.

Solution. (a) It is clear that the set F defined in the hint is a closed set. It is disjoint from K, since $\mathbf{z} \notin K + C$. Let δ be such that $|\mathbf{p} - \mathbf{q}| > \delta$ if $\mathbf{p} \in F$ and $\mathbf{q} \in K$. We claim that there is no point of $K + C$ inside the ball of radius δ about $\mathbf{z}$. For suppose $\mathbf{w}$ were such a point. By definition we would have $\mathbf{w} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in K$ and $\mathbf{v} \in C$. But then we would have

$$|\mathbf{u} - (\mathbf{z} - \mathbf{v})| = |\mathbf{w} - \mathbf{z}| < \delta,$$

which is a contradiction, since $\mathbf{u} \in K$ and $\mathbf{z} - \mathbf{v} \in F$. Thus $K + C$ is closed.

(b) Neither of the sets C_1 and C_2 has any limit points; hence both are closed sets. For each fixed integer $N \geq 2$, consider the fractional parts $\beta_1 = \alpha - [\alpha]$, $\beta_2 = 2\alpha - [2\alpha], \ldots, \beta_N = N\alpha - [N\alpha]$. There must be some half-open interval

$\left[\frac{k-1}{N-1}, \frac{k}{N-1}\right)$, $k = 1, 2, \ldots, N-1$ containing two of the numbers $\beta_1, \ldots, \beta_N$, since there are N numbers and only $N-1$ intervals. (Note: No two of these numbers are equal, since $\beta_i = \beta_j$, $i \neq j$, would imply

$$\alpha = \frac{[i\alpha] - [j\alpha]}{i - j},$$

i.e., α would be a rational number.) Now the inequalities

$$0 < (i\alpha - [i\alpha]) - (j\alpha - [j\alpha]) < \frac{1}{N-1}$$

say that $(i-j)\alpha + ([j\alpha] - [i\alpha]) \in \left(0, \frac{1}{N-1}\right)$, that is, there is certainly a point of $C_1 + C_2$ in $\left(0, \frac{1}{N-1}\right)$ for any $N \geq 2$. We shall now prove that there is a point of $C_1 + C_2$ in $\left(\frac{k}{n}, \frac{k+1}{n}\right)$ for any integer k and any positive integer n. To do so, fix the integer q such that $qn \leq k < (q+1)n$, and choose $y \in C_1 + C_2$ such that $0 < y < \frac{1}{n}$. Then $x = ny \in C_1 + C_2$ and $0 < x < 1$. Hence there is a positive integer p such that $k < px + qn < k + 1$. This says precisely that

$$\frac{k}{n} < py + q < \frac{k+1}{n},$$

and certainly $py + q \in C_1 + C_2$. Now let O be any nonempty open subset of R^1. Then O contains an interval (a, b). If $n > \frac{2}{b-a}$, there is an integer k such that $\left(\frac{k}{n}, \frac{k+1}{n}\right) \subset (a, b)$. This interval, as just shown, contains a point of $C_1 + C_2$, and hence O contains such a point. Therefore $C_1 + C_2$ is dense in R^1. Since it is a countable set, it is not all of R^1, and hence not closed.

Exercise 4.26 Suppose X, Y, Z are metric spaces and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$.

Prove that f is uniformly continuous if h is uniformly continuous.

Hint: g^{-1} has compact domain $g(Y)$, and $f(x) = g^{-1}(h(x))$.

Prove also that f is continuous if h is continuous.

Show (by modifying Example 4.21, or by finding a different example) that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

Solution. Theorem 4.17 asserts that g^{-1} is continuous, and since its domain is compact, it is uniformly continuous. Exercise 12 above then implies that f is uniformly continuous. The same argument, with the word "uniformly" omitted, shows that f is continuous if h is continuous.

To get a counterexample when Y is not compact, let $X = [0,1] = Z$, $Y = \{0\} \cup [1, \infty)$, and let $f : X \to Y$ and $g : Y \to Z$ be given by

$$f(x) = \begin{cases} \frac{1}{x}, & 0 < x \leq 1, \\ 0, & x = 0, \end{cases}$$

$$g(y) = \begin{cases} \frac{1}{y}, & 1 \leq y < \infty, \\ 0, & y = 0. \end{cases}$$

Then $h(x) = g(f(x)) = x$, so that h is uniformly continuous, and g is continuous and one-to-one, yet f is not even continuous.