Lecture Notes[1] in MTH601M/D (Term 1 AY 2020-2021)
§1 Fundamentals of Logic, §2 Arguments, and §3 Quantifiers
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1 Fundamentals of Logic

1.1 Propositions & Logical Operators

A *proposition* P is a sentence which is either true or false but not both. If P is true, P is assigned the *truth value* 1. If P is false, P is assigned the *truth value* 0.

Exercise 1. Which of the following is a proposition?

1) $5 + 6 - 10$
2) $5 + 6 < 10$
3) $2n$ is an even integer
4) $2n$ is even given that n is an integer

Definition 2. The *negation* of a proposition P, denoted by $\neg P$ (read as "not P"), is the proposition whose truth value depends on P as shown below.

Truth Table

P	$\neg P$
1	0
0	1

Definition 3. Let P and Q be propositions. The *conjunction* of P and Q, denoted by $P \wedge Q$ (read as "*P and Q*") is the proposition whose truth value depends on P and Q as shown below. We call each of P and Q a *conjunct*.

Truth Table

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

Definition 4. Let P and Q be propositions. The *disjunction* of P and Q, denoted by $P \vee Q$ (read as " *P or Q* ") is the proposition whose truth value depends on P and Q as shown below. We call each of P and Q a *disjunct*.

Truth Table

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

[1]References: Irving M. Copi, *Symnbolic Logic* 5th ed. Pearson (1979); and Severino D. Diesto, *Lecture Notes in Discrete Mathematics* (unpublished), De La Salle University (1989).

Definition 5. Let P and Q be propositions. The *conditional statement* with *premise* P and *conclusion* Q, denoted by $P \Rightarrow Q$, is the proposition whose truth value depends on P and Q as shown below. The proposition $P \Rightarrow Q$ is read "if P then Q".

Truth Table

P	Q	$P \Rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

Definition 6. Let P and Q be propositions. The biconditional statement, denoted by $P \Leftrightarrow Q$, is the proposition whose truth value depends on P and Q as shown below. The proposition $P \Leftrightarrow Q$ is read "P if and only if Q".

Truth Table

P	Q	$P \Leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

Definition 7. A finite sequence of propositions and logical operators which results in a proposition is called a *compound proposition.*

Exercise 8. Construct a truth table for each compound proposition.

1) $(P \vee Q) \Rightarrow P$ 2) $(P \wedge Q) \Rightarrow P$ 3) $[P \Rightarrow (Q \Rightarrow R)] \iff [(P \Rightarrow Q) \Rightarrow R]$

1.2 Tautology and Contradiction

Definition 9. A compound proposition that is always true is called a *tautology.* A compound proposition that is always false is called an *absurdity* or *contradiction.* A compound proposition that is neither a tautology nor a contradiction is called a *contingency.*

Example 10. Classify each proposition as a tautology, contradiction or contingency.

1) $P \vee \neg P$ 2) $P \wedge \neg P$ 3) $P \Rightarrow \neg P$

Exercise 11. A. Determine if each conditional statement is a tautology.

1) $(P \vee Q) \Rightarrow P$
2) $(P \wedge Q) \Rightarrow P$
3) $[(P \Rightarrow Q) \wedge P] \Rightarrow Q$
4) $[(P \Rightarrow Q) \wedge Q] \Rightarrow P$

B. Determine if each biconditional is a tautology.

1) $(P \wedge Q) \Leftrightarrow (Q \wedge P)$
2) $(P \vee Q) \Leftrightarrow (Q \vee P)$
3) $(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$
4) $(P \Leftrightarrow Q) \Leftrightarrow (Q \Leftrightarrow P)$

1.3 Tautological Implication and Tautological Equivalence

Definition 12. A conditional statement $P \Rightarrow Q$ which is a tautology is called a *logical implication* or a *tautological implication*.

Definition 13. If a biconditional $P \Leftrightarrow Q$ is a tautology we say that P and Q are *logically equivalent* and we write $P \equiv Q$.

We will write $P \nRightarrow Q$ to mean that P does not imply Q, and $P \nLeftrightarrow Q$ to mean that P and Q are not equivalent.

The Rule of Replacement: Any two logically equivalent propositions can replace each other wherever they occur. The following are called *rules of replacement*.

Identities (ID)

$$\begin{aligned} \neg 1 &\equiv 0 \\ \neg 0 &\equiv 1 \end{aligned}$$

$$\begin{aligned} (P \wedge 1) &\equiv P \\ (P \vee 1) &\equiv 1 \\ (P \vee \neg P) &\equiv 1 \end{aligned}$$

$$\begin{aligned} (P \wedge 0) &\equiv 0 \\ (P \vee 0) &\equiv P \\ (P \wedge \neg P) &\equiv 0 \end{aligned}$$

Double Negation (DN)

$$\neg(\neg P) \equiv P$$

Idempotence (IDEM)

$$\begin{aligned} (P \wedge P) &\equiv P \\ (P \vee P) &\equiv P \end{aligned}$$

Commutativity (COMM)

$$\begin{aligned} (P \wedge Q) &\equiv (Q \wedge P) \\ (P \vee Q) &\equiv (Q \vee P) \end{aligned}$$

Associativity (ASSO)

$$\begin{aligned} (P \wedge Q) \wedge R &\equiv P \wedge (Q \wedge R) \\ (P \vee Q) \vee R &\equiv P \vee (Q \vee R) \end{aligned}$$

Distributivity (DIST)

$$\begin{aligned} P \wedge (Q \vee R) &\equiv (P \wedge Q) \vee (P \wedge R) \\ P \vee (Q \wedge R) &\equiv (P \vee Q) \wedge (P \vee R) \end{aligned}$$

De Morgan's Laws (DeM)

$$\begin{aligned} \neg(P \wedge Q) &\equiv (\neg P \vee \neg Q) \\ \neg(P \vee Q) &\equiv (\neg P \wedge \neg Q) \end{aligned}$$

Material Implication (MI)

$$(P \Rightarrow Q) \equiv (\neg P \vee Q)$$

Material Equivalence (ME)

$$(P \Leftrightarrow Q) \quad \equiv \quad (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Exportation (EXP)

$$(P \wedge Q) \Rightarrow R \quad \equiv \quad P \Rightarrow (Q \Rightarrow R)$$

Contrapositive (CONTRA)

$$(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$$

Exercise 14. A. Use the Rules of Replacement to prove the following logical equivalences.

1. $(P \Rightarrow Q) \quad \equiv \quad (P \wedge \neg Q) \Rightarrow 0$
2. $(P \vee Q) \Rightarrow R \quad \equiv \quad (P \Rightarrow R) \wedge (Q \Rightarrow R)$.
3. $P \vee (P \wedge Q) \quad \equiv \quad P$
4. $P \wedge (P \vee Q) \quad \equiv \quad P$

The last two equivalences are called the Absorption Laws (ABS).

B. Classify each propositional form as a tautology, contradiction or contingency.

1) $P \vee (Q \vee \neg P)$ 2) $P \wedge \neg(Q \vee \neg Q)$ 3) $P \vee \neg(Q \vee \neg Q)$

Definition 15. Given a conditional statement $P \Rightarrow Q$. The conditional statements $Q \Rightarrow P$, $\neg P \Rightarrow \neg Q$ $\neg Q \Rightarrow \neg P$ are called the *converse*, *inverse* and *contrapositive* of $P \Rightarrow Q$ respectively.

Exercise 16. 1. Write the converse, inverse and contrapositive of the conditional statement "If $\lim_{n \Rightarrow \infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges."

2. Complete the truth table below of $P \Rightarrow Q$, its inverse, converse and contrapositive. Which of the three is/are equivalent to $P \Rightarrow Q$?

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$\neg P \Rightarrow \neg Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$
1	1						
1	0						
0	1						
0	0						

We end this section with the following note. In Mathematics, theorems (statements that have been established as true) are often stated as implications. A theorem of the form $P \Rightarrow Q$ is read in many different ways, namely:

- ♡ *P implies Q,*
- ♡ *P only if Q,*
- ♡ *P is a sufficient condition for Q,*
- ♣ *Q if P,*
- ♣ *Q whenever P,*
- ♣ *Q is a necessary condition for P.*

2 Arguments

2.1 Inductive and Deductive Arguments

Definition 17. An *argument*

$$\begin{array}{c} P_1 \\ P_2 \\ . \\ . \\ . \\ P_n \\ \hline \therefore\ Q \end{array}$$

is a collection of propositions $P_1, P_2, \ldots, P_n, Q$ where it is claimed that Q follows from $P_1, P_2, \ldots, P_n$. We call Q the *conclusion* and we call $P_1, P_2, \ldots, P_n$ the *premises* of the argument. We shall also use the notation

$$\{P_1, P_2, \ldots, P_n\} / \therefore Q$$

to denote the above argument.

An *inductive argument* is one where it is claimed that within a certain probability of error the conclusion follows from the premises. A *deductive argument* is one where it is claimed that the conclusion follows absolutely from the premises.

From now on, by an argument we shall mean a deductive argument.

Example 18. The following are examples of arguments.

1) If there is peace then there is progress.
   There is progress.
   Therefore, there is peace.

2) All lizards are mammals.
   All mammals have wings.
   Therefore, all lizards have wings.

3) If a sequence is bounded and monotonic then the sequence is convergent. If a sequence is convergent and monotonic then the sequence is bounded. Therefore, if a sequence is monotonic then the sequence is bounded if and only if it is convergent.

We will classify an argument as valid or invalid.

2.2 Valid and Invalid Arguments

Definition 19. An argument is said to be *valid* if whenever the premises are all true the conclusion is also true. If an argument is not valid, we say that it is *invalid*.

Remark 20. It follows from Definition 19 that an argument

$$\begin{array}{c} P_1 \\ P_2 \\ . \\ . \\ . \\ P_n \\ \hline \therefore\ Q \end{array} \qquad \text{is valid}$$

if and only if

$$(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \Longrightarrow Q.$$

Exercise 21.

State if each argument form below is valid or invalid.

1) $\begin{array}{c} P \Rightarrow Q \\ P \\ \hline \therefore Q \end{array}$

2) $\begin{array}{c} P \Rightarrow Q \\ Q \\ \hline \therefore P \end{array}$

3) $\begin{array}{c} P \Rightarrow Q \\ \neg P \\ \hline \therefore \neg Q \end{array}$

4) $\begin{array}{c} P \Rightarrow Q \\ \neg Q \\ \hline \therefore \neg P \end{array}$

2.3 Proof of Invalidity

Remark 22. To prove that an argument is invalid we exhibit a truth value assignment to the component propositions that will make the premises of the argument true and the conclusion false.

Exercise 23. Prove that the following argument forms are invalid by the method of assigning truth values to the component propositions.

1) $\begin{array}{c} A \Rightarrow B \\ C \Rightarrow D \\ B \vee C \\ \hline \therefore A \vee D \end{array}$

2) $\begin{array}{c} E \Rightarrow (F \vee G) \\ G \Rightarrow (H \wedge J) \\ \neg H \\ \hline \therefore E \Rightarrow J \end{array}$

2.4 Rules of Inference

Definition 24. The following basic valid argument forms are called *Rules of Inference*.

i) Addition (ADDN)

$$\begin{array}{c} P \\ \hline \therefore P \vee Q \end{array}$$

ii) Simplification (SIMP)

$$\begin{array}{c} P \wedge Q \\ \hline \therefore P \end{array}$$

iii) Conjunction (CONJ)

$$\begin{array}{c} P \\ \underline{Q} \\ \therefore P \wedge Q \end{array}$$

iv) Modus Ponens (MP)

$$\begin{array}{c} P \Rightarrow Q \\ \underline{P} \\ \therefore Q \end{array}$$

v) Modus Tollens (MT)

$$\begin{array}{c} P \Rightarrow Q \\ \underline{\neg Q} \\ \therefore \neg P \end{array}$$

vi) Disjunctive Syllogism (DS)

$$\begin{array}{c} P \vee Q \\ \underline{\neg P} \\ \therefore Q \end{array}$$

vii) Hypothetical Syllogism (HS)

$$\begin{array}{c} P \Rightarrow Q \\ \underline{Q \Rightarrow R} \\ \therefore P \Rightarrow R \end{array}$$

viii) Constructive Dilemma (CD)

$$\begin{array}{c} (P \Rightarrow Q) \wedge (R \Rightarrow S) \\ \underline{P \vee R} \\ \therefore Q \vee S \end{array}$$

ix) Destructive Dilemma (DD)

$$\begin{array}{c} (P \Rightarrow Q) \wedge (R \Rightarrow S) \\ \underline{\neg Q \vee \neg S} \\ \therefore \neg P \vee \neg R \end{array}$$

The validity of each of the above argument forms is proved using a truth table. The rules of inference and the rules of replacement are then used to establish the validity of other arguments as shown in the following example.

Example 25. Use the Rules of Inference or Replacement to show the validity of the argument below.

$$\begin{array}{c} (P \Rightarrow Q) \wedge (R \Rightarrow S) \\ \underline{P \wedge R} \\ \therefore Q \wedge S \end{array}$$

A formal proof of validity of an argument is presented in two-column form. The first column contains the statements and the second column, the reason for each statement that can be inferred from the premises. Notice the numbers appearing on the second column. For example, opposite statement 5 we write "3,4 MP" as reason. This means that statement 5 is obtained by applying Modus Ponens to statements 3 and 4.

	Statement		Reason
1.	$(P \Rightarrow Q) \wedge (R \Rightarrow S)$		
2.	$P \wedge R$	$/ \therefore Q \wedge S$	
3.	$P \Rightarrow Q$		1, SIMP
4.	P		2, SIMP
5.	Q		3,4 MP
6.	$(R \Rightarrow S) \wedge (P \Rightarrow Q)$		2, COMM
7.	$R \Rightarrow S$		6, SIMP
8.	$R \wedge P$		2, COMM
9.	R		8, SIMP
10.	S		7,9 MP
11.	$Q \wedge S$		5,10 CONJ

Exercise 26. Construct a formal proof of validity for each argument.

1) $$\begin{array}{c} (A \vee B) \Rightarrow (C \wedge D) \\ A \\ \hline \therefore C \end{array}$$

2) $$\begin{array}{c} (P \vee Q) \Rightarrow R \\ \neg R \\ \hline \therefore \neg P \end{array}$$

3) $$\begin{array}{c} A \Rightarrow B \\ A \vee (B \vee \neg C) \\ \neg B \\ \hline \therefore \neg C \end{array}$$

4) $$\begin{array}{c} (A \vee B) \Rightarrow (C \wedge D) \\ \neg C \\ \hline \therefore \neg B \end{array}$$

5) $$\begin{array}{c} A \Rightarrow B \\ C \Rightarrow D \\ A \vee (C \wedge E) \\ \hline \therefore (B \vee D) \end{array}$$

6) $$\begin{array}{c} Q \Rightarrow (R \wedge S) \\ P \Rightarrow Q \\ P \\ \hline \therefore S \vee (R \Rightarrow P) \end{array}$$

7) $$\begin{array}{c} \neg B \Rightarrow (C \vee \neg D) \\ B \Rightarrow \neg A \\ A \\ \hline \therefore D \Rightarrow C \end{array}$$

8) $$\begin{array}{c} (P \wedge Q) \Rightarrow [P \Rightarrow (R \wedge S)] \\ P \wedge (Q \wedge T) \\ \hline \therefore S \end{array}$$

9) $$\begin{array}{c} \neg(S \Rightarrow P) \Rightarrow N \\ P \Rightarrow Q \\ \neg N \\ \hline \therefore S \Rightarrow Q \end{array}$$

10) $$\begin{array}{c} K \Rightarrow L \\ \hline \therefore K \Rightarrow (L \vee M) \end{array}$$

2.5 The Rule of Conditional Proof and the Rule of Indirect Proof

Remark 27. The following are equivalent.

i) The argument $\{P_1, P_2, \ldots, P_n\} / \therefore (P \Rightarrow Q)$ is valid.

ii) $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \Longrightarrow (P \Rightarrow Q)$.

iii) $(P_1 \wedge P_2 \wedge \cdots \wedge P_n \wedge P) \Longrightarrow Q$.

iv) The argument $\{P_1, P_2, \ldots, P_n, P\} / \therefore Q$ is valid.

Definition 28. By the equivalence of (i) and (iv) in Remark 2.5.1, to prove $P \Rightarrow Q$ we may include P in the premises and prove Q instead. This is called the *Rule of Conditional Proof.*

Exercise 29. Give a conditional proof of validity for each argument.

1) $$\frac{A}{\therefore B \Rightarrow (A \wedge B)}$$

2) $$\frac{P \Rightarrow Q}{\therefore\ P \Rightarrow (P \wedge Q)}$$

3) $$\frac{\begin{array}{c}(J \Rightarrow \neg L) \wedge (\neg J \Rightarrow \neg K)\\ L \vee M\end{array}}{\therefore\ K \Rightarrow M}$$

4) $$\frac{\begin{array}{l}(W \Rightarrow X) \wedge (Y \Rightarrow Z)\\ (X \vee Z) \Rightarrow (W \wedge Y)\end{array}}{\therefore\ W \Leftrightarrow Y}$$

5) $$\frac{A \Leftrightarrow (B \wedge C)}{\therefore\ C \Rightarrow (A \Leftrightarrow B)}$$

Remark 30. The following are equivalent.

i) The argument $\{P_1, P_2, \dots, P_n\}/ \therefore Q$ is valid.

ii) $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \Longrightarrow Q$.

iii) $(P_1 \wedge P_2 \wedge \cdots \wedge P_n \wedge \neg Q) \Longrightarrow 0$.

iv) The argument $\{P_1, P_2, \dots, P_n, \neg Q\}/ \therefore 0$ is valid.

Definition 31. By the equivalence of (i) and (iv) in Remark 2.5.3, to prove Q we may include $\neg Q$ in the premises and derive a contradiction. This is called the *Rule of Indirect Proof* or *reductio ad absurdum.*

Exercise 32. Give an indirect proof of validity for each argument.

1) $$\frac{\begin{array}{l}A \vee (B \wedge C)\\ A \Rightarrow C\end{array}}{\therefore C}$$

2) $$\frac{\begin{array}{l}(L \vee M) \Rightarrow Q\\ \neg L \Rightarrow \neg E\\ \neg Q\end{array}}{\therefore \neg E}$$

3) $$\frac{\begin{array}{l}(D \vee E) \Rightarrow (F \Rightarrow G)\\ (\neg G \vee H) \Rightarrow (D \wedge F)\end{array}}{\therefore\ G}$$

4) $$\frac{\begin{array}{l}(P \Rightarrow Q) \wedge (R \Rightarrow S)\\ (Q \vee S) \Rightarrow T\\ \neg T\end{array}}{\therefore\ \neg(P \vee R)}$$

5) $$\frac{\begin{array}{l}(A \vee B) \Rightarrow (C \wedge D)\\ (C \vee E) \Rightarrow (\neg F \wedge G)\\ (F \vee H) \Rightarrow (A \wedge J)\end{array}}{\therefore \neg F}$$

6) $$\frac{\begin{array}{c}A \Rightarrow B\\ [A \Rightarrow (A \wedge B)] \Rightarrow C\end{array}}{\therefore C}$$

3 Quantification

The logical techniques for proving validity of an argument that we have so far do not apply to arguments such as the following.

All fruits are nourishing,
Banana is a fruit,
Therefore, banana is nourishing.

In this section, we develop methods for symbolizing and analyzing the structure of these arguments.

3.1 Universal and Existential Quantifiers

Consider the sentence "x is an even integer". This sentence is not a proposition but it becomes a proposition when the variable x is replaced with a specific number. We shall call a sentence like this a *propositional function*. Let $E(x)$ denote the propositional function

$$E(x) : x \text{ is an even integer.}$$

The variable x is used as place-marker to indicate where a number may be substituted. Thus, by $E(5)$ and $E(100)$ we mean

$$\begin{aligned} E(5) &\ :\ \text{5 is an even integer} \\ E(100) &\ :\ \text{100 is an even integer} \end{aligned}$$

Clearly, $E(5)$ is false while $E(100)$ is true. Since x is used simply as a place-marker in the propositional function $E(x)$, changing x to y gives

$$E(y) : y \text{ is an even integer,}$$

which is the same propositional function as $E(x)$.

Given a propositional function $P(x)$. A *universe of discourse* (or simply, *universe*) for the variable x is a specified set from which an object is taken to substitute for x. For example, for the propositional function *"x is an even integer"*, the universe could be the set of all integers.

Notation $\mathcal{U}$: universe of discourse

Definition 33. The phrase "For all x" is called a *universal quantifier* and is denoted by $\forall x$.

Other ways of reading $\forall x$:

- "For every x"
- "For each x"

Definition 34. The statement "For some x" is called an *existential quantifier* and is denoted by $\exists x$.

Other ways of reading $\exists x$:

- "There exists an x"
- "There is at least one x"

Remark 35. Let $P(x)$ denote a propositional function and let $\mathcal{U}$ be the universe for x. The general proposition $\forall x P(x)$ is true if and only if $P(c)$ is true for all objects c in $\mathcal{U}$.

Truth Table

	$\forall x P(x)$
$P(c)$ is true for every object c in $\mathcal{U}$	1
$P(c)$ is false for at least one object c in $\mathcal{U}$	0

Remark 36. Let $P(x)$ denote a propositional function and let $\mathcal{U}$ be the universe for x. The general proposition $\exists x P(x)$ is true if and only if $P(c)$ is true for at least one object c in $\mathcal{U}$.

Truth Table

	$\exists x P(x)$
$P(c)$ is true for at least one object c in $\mathcal{U}$	1
$P(c)$ is false for every object c in $\mathcal{U}$	0

Exercise 37. Let $\mathcal{U} = \{1,2,4,6,8\}$ and let $E(x), P(x), S(x)$ and $M(x)$ denote the propositional functions

$$\begin{array}{lcl} E(x) & : & x \text{ is even,} \\ P(x) & : & x \text{ is positive,} \\ S(x) & : & x+1=4, \\ M(x) & : & 2x=2. \end{array}$$

Determine if each proposition is true or false.

1) $E(1)$
2) $S(2) \wedge M(1)$
3) $E(1) \vee \neg P(1)$
4) $P(1) \Rightarrow M(4)$
5) $\forall x P(x)$
6) $\forall x E(x)$
7) $\exists x S(x)$
8) $\exists x M(x)$
9) $\exists x \neg E(x)$
10) $\neg \exists x E(x)$
11) $\forall x \neg E(x)$
12) $\neg \forall x E(x)$
13) $\forall x E(x) \vee \forall x M(x)$
14) $\forall x (E(x) \vee M(x))$
15) $\exists x E(x) \wedge \exists x M(x)$
16) $\exists x (E(x) \wedge M(x))$
17) $\forall x (E(x) \Rightarrow M(x))$
18) $\forall x E(x) \Rightarrow \forall x M(x)$
19) $\exists x (E(x) \Rightarrow S(x))$
20) $\exists x E(x) \Rightarrow \exists x S(x)$

We may view a propositional function $P(x)$ as a sentence that ascribes a certain property P to an object x in the universe. For example, in the propositional function "$P(x)$: x is a prime number" we may view P as the property of "being prime" which is attributed to x. In the propositional function "$D(y)$: y is divisible by 2" we may think of D as the property of "being divisible by 2" which is attributed to y .

3.2 Quantification Negation

Let $P(x)$ denote a propositional function and let $\mathcal{U}$ denote the universe for the variable x. The logical equivalences below are referred to as the Quantification Negation Rules (QN):

$$\begin{aligned} \neg \forall x P(x) &\Leftrightarrow \exists x \neg P(x), \\ \neg \exists x P(x) &\Leftrightarrow \forall x \neg P(x). \end{aligned}$$

Exercise 38. State the negation of each general proposition.

a) All integers are real numbers.

b) Some prime numbers are even.

c) No divisors of 15 are even.

d) All integers are either even or odd.

e) Some functions are both injective and surjective.

There are propositional functions that require two or more variables as place-marker for its objects. The propositional function "x likes y" expresses a relation between two objects x and y. Such propositions are called *binary relations.* Other propositional functions may relate three or more objects. For example, "x introduced y to z" is an example of a *ternary relation*, while "w, x, y and z are friends" is an example of a *quaternary* relation. In general, a propositional function that expresses a relation between n objects is called an *n-ary relation.*

Exercise 39. A. Determine the truth value of the proposition obtained when the propositional function in each column is preceded by the quantifiers in each row. Assume that the universe for both variables is the set of real numbers.

	$xy = yx$	$x + y = 0$	$xy = 0$	$xy = 1$
$\forall x \forall y$				
$\forall x \exists y$				
$\exists x \forall y$				
$\exists x \exists y$				

B. State the negation of each proposition.

1) In any Nash equilibrium of the Bertrand model with $j > 2$ firms, all sales take place at a price equal to cost.
2) For all distinct real numbers x and y, $f(x) \neq f(y)$.
3) There is a real number y such that for every real number x, $f(x) \leq y$.
4) For every real number x there exists an integer n such that $n > x$.
5) For every integer $n > 1$, there exists a prime number p such that $n < p < 2n$.
6) For each real number r, there are integers p and q with $q \neq 0$ such that $r = \frac{p}{q}$.
7) There is a real number $M > 0$ such that for all elements x in a set S, $|x| < M$.
8) For each positive integer n, there is an integer $M > 0$ such that $f(x) > n$ for all numbers x with $x > M$.
9) For every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for all real numbers y with $|x - y| \leq \delta$, $|f(x) - f(y)| \leq \varepsilon$.
10) For every real number $\varepsilon > 0$ there exists a positive integer N such that $|a_n - L| < \varepsilon$ for all integers $n > N$.
11) Every efficient production y in a convex set Y is a profit-maximizing production for some nonzero price vector $p \geq 0$.
12) There is $p \in \mathbb{R}^N$ with $p \neq 0$, and a value $c \in \mathbb{R}$, such that $p \cdot x \geq c$ for every $x \in A$ and $p \cdot y \leq c$ for every $y \in B$.

3.3 Proof of Validity

The Rules of Replacement and the Rules of Inference have limited applicability when it comes to propositions containing quantifiers. For example, the validity of the argument

$$\frac{\forall x P(x) \wedge \forall x Q(x)}{\therefore \forall x P(x)}$$

follows from Simplification. However, Simplification alone cannot be used to establish the validity of the argument

$$\frac{\forall x (P(x) \wedge Q(x))}{\therefore \forall x P(x).}$$

Remark 40. Let $P(x)$ denote a propositional function in x and let $\mathcal{U}$ be the universe for x. The following valid argument forms are called Quantification Rules.

i) *Universal Instantiation (UI)*

$$\frac{\forall x P(x)}{\therefore\ P(a)} \quad \text{where } a \text{ is any object in } \mathcal{U}.$$

Given that $\forall x P(x)$ is true, we can infer the truth of $P(a)$ for any object a in $\mathcal{U}$.

ii) *Existential Instantiation (EI)*

$$\frac{\exists x P(x)}{\therefore\ P(a)} \quad \text{where } a \text{ is a particular (specified) object in } \mathcal{U}.$$

Given that $\exists x P(x)$ is true, we can infer that there is at least one object a in $\mathcal{U}$ such that $P(a)$ is true.

iii) *Universal Generalization (UG)*

$$\frac{P(a)}{\therefore\ \forall x P(x)} \quad \text{where } a \text{ is an arbitrary (unspecified) object in } \mathcal{U}.$$

Given that the proposition $P(a)$ is true for any arbitrary object a in $\mathcal{U}$, we can conclude that $\forall x P(x)$ is true.

iv) *Existential Generalization (EG)*

$$\frac{P(a)}{\therefore\ \exists x P(x)} \quad \text{where } a \text{ is an object in } \mathcal{U}.$$

Given that $P(a)$ is true for some object a in $\mathcal{U}$, we can infer that $\exists x P(x)$ is true.

Exercise 41. State the Quantification Rule illustrated in each argument form, if any.

1) $\dfrac{\forall x(P(x) \Rightarrow Q(x))}{\therefore\ P(a) \Rightarrow Q(a)}$

2) $\dfrac{\forall x(P(x) \Rightarrow Q(x))}{\therefore\ P(a) \Rightarrow Q(b)}$

3) $\dfrac{\neg Q(b)}{\therefore\ \exists x \neg Q(x)}$

4) $\dfrac{P(a) \wedge Q(a)}{\therefore\ \exists x(P(x) \wedge Q(x))}$

5) $\dfrac{P(a) \wedge Q(b)}{\therefore\ \exists x(P(x) \wedge Q(x))}$

6) $\dfrac{\forall x P(x) \Rightarrow \forall x Q(x)}{\therefore\ P(a) \Rightarrow Q(a)}$

7) $\dfrac{\forall x(P(x) \Rightarrow \exists y Q(y))}{\therefore\ P(a) \Rightarrow Q(a)}$

8) $\dfrac{\forall x(P(x) \Rightarrow \exists y Q(y))}{\therefore\ P(a) \Rightarrow \exists y Q(y)}$

Exercise 42. A. Find the mistake in the following 'proof of validity' of the argument

Some cats are animals.
Some dogs are animals.
Therefore, some cats are dogs.

'Proof.'

1) $\exists x(C(x) \wedge A(x))$	
2) $\exists x(D(x) \wedge A(x)) / \therefore \exists x(C(x) \wedge D(x)$	
3) $C(w) \wedge A(w)$	1, EI
4) $D(w) \wedge A(w)$	2, EI
5) $C(w)$	3, Simp
6) $D(w)$	4, Simp
7) $C(w) \wedge D(w)$	5,6 Conj
8) $\exists x(C(x) \wedge D(x))$	7, EG

B. Construct a formal proof of validity for each of the following arguments.

1) $\forall x(F(x) \Rightarrow H(x))$
$H(a) \Rightarrow P$
$\underline{F(a)}$
$\therefore\ P$

2) $\exists x(P(x) \vee Q(x))$
$\underline{\neg\exists x Q(x)}$
$\therefore\ \exists x P(x)$

3) $\forall x(F(x) \Rightarrow \neg G(x))$
$\underline{\forall x(G(x) \wedge \neg H(x))}$
$\therefore\ \neg\exists x F(x)$

4) $\exists y K(y) \vee \exists x J(x)$
$\underline{\forall x(J(x) \Rightarrow K(x))}$
$\therefore\ \exists y K(y)$

5) $\exists x A(x) \Rightarrow \forall y(B(y) \Rightarrow C(y))$
$\underline{\exists x D(x) \rightarrow \exists y B(y)}$
$\therefore \exists x(A(x) \wedge D(x)) \Rightarrow \exists y C(y)$

6) $\underline{\exists x L(x) \Rightarrow \forall y M(y)}$
$\therefore\ \forall x(L(x) \Rightarrow \forall y M(y))$

7) $\underline{\forall x(P(x) \wedge Q(x))}$
$\therefore\ \forall x P(x) \wedge \forall x Q(x)$

8) $\underline{\forall x P(x) \wedge \forall x Q(x)}$
$\therefore\ \forall x(P(x) \wedge Q(x))$

9) $\underline{\exists x(P(x) \vee Q(x))}$
$\therefore\ \exists x P(x) \vee \exists x Q(x)$

10) $\underline{\exists x P(x) \vee \exists x Q(x)}$
$\therefore\ \exists x(P(x) \vee Q(x))$

11) $\underline{\exists x(P(x) \wedge Q(x))}$
$\therefore\ \exists x P(x) \wedge \exists x Q(x)$

12) $\underline{\forall x P(x) \vee \forall x Q(x)}$
$\therefore\ \forall x(P(x) \vee Q(x))$

13) $\forall x(A(x) \Rightarrow B(x))$
$\underline{\exists x(A(x) \vee B(x))}$
$\therefore\ \exists x B(x)$

14) $\forall x(F(x) \Rightarrow \neg G(x))$
$\underline{\forall x(\neg H(x) \wedge G(x))}$
$\therefore\ \neg\exists x F(x)$

15) $\exists x A(x) \Rightarrow \forall y(B(y) \Rightarrow C(y))$
$\underline{\exists x D(x) \Rightarrow \exists y B(y)}$
$\therefore\ \exists x(A(x) \wedge D(x)) \Rightarrow \exists y C(y)$

Quantification Identities

The logical equivalences below are called Quantification Identities (QI). In the following, Px and Qx are propositional functions in x, and R is a proposition free of x.

$$
\begin{aligned}
\exists x(P(x) \vee Q(x)) &\equiv (\exists x P(x) \vee \exists x Q(x)) \\
\forall x(P(x) \wedge Q(x)) &\equiv (\forall x P(x) \wedge \forall x Q(x))
\end{aligned}
$$

$$
\begin{aligned}
\exists x(R \wedge Q(x)) &\equiv (R \wedge \exists x Q(x)) \\
\forall x(R \vee Q(x)) &\equiv (R \vee \forall x Q(x))
\end{aligned}
$$

$$
\begin{aligned}
(\forall x P(x) \Rightarrow R) &\equiv \exists x(P(x) \Rightarrow R) \\
(\exists x P(x) \Rightarrow R) &\equiv \forall x(P(x) \Rightarrow R)
\end{aligned}
$$

$$
\begin{aligned}
(R \Rightarrow \forall x P(x)) &\equiv \forall x(R \Rightarrow P(x)) \\
(R \Rightarrow \exists x P(x)) &\equiv \exists x(R \Rightarrow P(x))
\end{aligned}
$$

Exercise 43. 1. Express each quantification identity in argument form then construct a formal proof of validity for each argument.

2. Investigate whether or not each of the following identities is true.

   a) $\exists x(P(x) \wedge Q(x)) \equiv (\exists x P(x) \wedge \exists x Q(x))$

   b) $\forall x(P(x) \vee Q(x)) \equiv (\forall x P(x) \vee \forall x Q(x))$

3. Construct a formal proof of validity for the argument

$$
\begin{array}{c}
\exists x[V(x) \wedge \forall y(Wy \Rightarrow S(y))] \\
\forall x\{V(x) \Rightarrow [\exists y(T(y) \wedge S(y)) \Rightarrow Z(x)]\} \\
\exists y(T(y) \wedge W(y)) \\
\hline
\therefore\ \exists x Z(x)
\end{array}
$$