

# Contents



# Chapter 1

## Real Analysis

(1.5)

*OrderTrichotomy* $[<, S] := \forall_{x,y \in S} (x < y \vee x = y \vee y < x)$

*OrderTransitivity* $[<, S] := \forall_{x,y,z \in S} ((x < y \wedge y < z) \implies x < z)$

*Order* $[<, S] := (\textcolor{teal}{OrderTrichotomy}[<, S]) \wedge (\textcolor{teal}{OrderTransitivity}[<, S])$

(1.7)

*Bounded Above* $[E, S, <] := (\textcolor{teal}{Order}[<, S]) \wedge (E \subset S) \wedge (\exists_{\beta \in S} \forall_{x \in E} (x \leq \beta))$

*Bounded Below* $[E, S, <] := (\textcolor{teal}{Order}[<, S]) \wedge (E \subset S) \wedge (\exists_{\beta \in S} \forall_{x \in E} (\beta \leq x))$

*Upper Bound* $[\beta, E, S, <] := (\textcolor{teal}{Order}[<, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E} (x \leq \beta))$

*Lower Bound* $[\beta, E, S, <] := (\textcolor{teal}{Order}[<, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E} (\beta \leq x))$

(1.8)

*LU B* $[\alpha, E, S, <] := (\textcolor{teal}{Upper Bound}[\alpha, E, S, <]) \wedge (\forall_{\gamma} (\gamma < \alpha \implies \neg \textcolor{teal}{Upper Bound}[\gamma, E, S, <]))$

*GLB* $[\alpha, E, S, <] := (\textcolor{teal}{Lower Bound}[\alpha, E, S, <]) \wedge (\forall_{\beta} (\alpha < \beta \implies \neg \textcolor{teal}{Lower Bound}[\beta, E, S, <]))$

(1.10)

*LU B Property* $[S, <] := \forall_E (((\emptyset \neq E \subset S) \wedge (\textcolor{teal}{Bounded Above}[E, S, <]) \implies \exists_{\alpha \in S} (\textcolor{teal}{LU B}[\alpha, E, S, <])))$

*GLB Property* $[S, <] := \forall_E (((\emptyset \neq E \subset S) \wedge (\textcolor{teal}{Bounded Below}[E, S, <]) \implies \exists_{\alpha \in S} (\textcolor{teal}{GLB}[\alpha, E, S, <])))$

(1.11)

*LU B Property Implies GLB Property*  $:= \textcolor{teal}{LU B Property}[S, <] \implies \textcolor{teal}{GLB Property}[S, <]$

(1)  $\textcolor{teal}{LU B Property}[S, <] \implies \dots$

wts: 2

(1.1)  $(\emptyset \neq B \subset S \wedge \textcolor{teal}{Bounded Below}[B, S, <]) \implies \dots$

wts: 1.2

from: *Bounded Below*, 1.1

(1.1.1)  $\textcolor{teal}{Order}[<, S] \wedge \exists_{\delta' \in S} (\textcolor{teal}{Lower Bound}[\delta', B, S, <])$

(1.1.2)  $|B| = 1 \implies \dots$

wts: 1.1.3

from: 1.1.2

(1.1.2.1)  $\exists_{u'} (u' \in B) \blacksquare u := \textit{choice}(\{u' : u' \in B\}) \blacksquare B = \{u\}$

(1.1.2.2)  $\textcolor{teal}{GLB}[u, B, S, <] \blacksquare \exists_{\epsilon_0 \in S} (\textcolor{teal}{GLB}[\epsilon_0, B, S, <])$

(1.1.3)  $|B| = 1 \implies \exists_{\epsilon_0 \in S} (\textcolor{teal}{GLB}[\epsilon_0, B, S, <])$

(1.1.4)  $|B| \neq 1 \implies \dots$

wts: 1.1.5

from: *LU B Property*, 1

(1.1.4.1)  $\forall_E (((\emptyset \neq E \subset S \wedge \textcolor{teal}{Bounded Above}[E, S, <]) \implies \exists_{\alpha \in S} (\textcolor{teal}{LU B}[\alpha, E, S, <])))$

(1.1.4.2)  $L := \{s \in S : \textcolor{teal}{Lower Bound}[s, B, S, <]\}$

(1.1.4.3)  $|B| > 1 \wedge \textcolor{teal}{OrderTrichotomy}[<, S] \blacksquare \exists_{b_1' \in B} \exists_{b_0' \in B} (b_0' < b_1')$

from: *Order*, 1.1.1  
wts: 1.1.4.7

(1.1.4.4)  $b_1 := \textit{choice}(\{b_1' \in B : \exists_{b_0' \in B} (b_0' < b_1')\}) \blacksquare \neg \textcolor{teal}{Lower Bound}[b_1, B, S, <]$

from: 1.1.4.2

(1.1.4.5)  $b_1 \notin L \blacksquare L \subset S$

(1.1.4.6)  $\delta := \textit{choice}(\{\delta' \in S : \textcolor{teal}{Lower Bound}[\delta', B, S, <]\}) \blacksquare \delta \in L \blacksquare \emptyset \neq L$

from: 1.1.1

(1.1.4.7)  $\emptyset \neq L \subset S$

from: 1.1.4.5, 1.1.4.6

(1.1.4.8)  $\forall_{y \in L} (\textcolor{teal}{Lower Bound}[y_0, B, S, <]) \blacksquare \forall_{y \in L} \forall_{x \in B} (y_0 \leq x)$

from: *Lower Bound*, 1.1.4.2  
wts: 1.1.4.10

(1.1.4.9)  $\forall_{x \in B} (x \in S \wedge \forall_{y \in L} (y_0 \leq x)) \blacksquare \forall_{x \in B} (\textcolor{teal}{Upper Bound}[x, L, S, <])$

from: *Upper Bound*

(1.1.4.10)  $\exists_{x \in S} (\textcolor{teal}{Upper Bound}[x, L, S, <]) \blacksquare \textcolor{teal}{Bounded Above}[L, S, <]$

(1.1.4.11)	$\emptyset \neq L \subset S \wedge \text{Bounded Above}[L, S, <]$	from: 1.1.4.7, 1.1.4.10
(1.1.4.12)	$\exists_{\alpha' \in S}(\text{LUB}[\alpha', L, S, <]) \blacksquare \alpha := \text{choice}(\{\alpha' \in S : (\text{LUB}[\alpha', L, S, <])\})$	from: 1.1.4.1 wts: 1.1.4.21
(1.1.4.13)	$\forall_x(x \in B \implies \text{Upper Bound}[x, L, S, <])$	from: 1.1.4.9 wts: 1.1.4.17
(1.1.4.14)	$\forall_x(\neg \text{Upper Bound}[x, L, S, <] \implies x \notin B)$	
(1.1.4.15)	$\gamma < \alpha \implies \dots$	wts: 1.1.4.16
(1.1.4.15.1)	$\neg \text{Upper Bound}[\gamma, L, S, <] \blacksquare \gamma \notin B$	from: LUB, 1.1.4.12, 1.1.4.14
(1.1.4.16)	$\gamma < \alpha \implies \gamma \notin B \blacksquare \gamma \in B \implies \gamma \geq \alpha$	
(1.1.4.17)	$\forall_{\gamma \in B}(\alpha \leq \gamma) \blacksquare \text{Lower Bound}[\alpha, B, S, <]$	from: Lower Bound
(1.1.4.18)	$\alpha < \beta \implies \dots$	wts: 1.1.4.19
(1.1.4.18.1)	$\forall_{y \in L}(y_0 \leq \alpha < \beta) \blacksquare \forall_{y \in L}(y_0 \neq \beta)$	from: LUB, 1.1.4.12, 1.1.4.18
(1.1.4.18.2)	$\beta \notin L \blacksquare \neg \text{Lower Bound}[\beta, B, S, <]$	from: 1.1.4.2
(1.1.4.19)	$\alpha < \beta \implies \neg \text{Lower Bound}[\beta, B, S, <] \blacksquare \forall_{\beta \in S}(\alpha < \beta \implies \neg \text{Lower Bound}[\beta, B, S, <])$	
(1.1.4.20)	$\text{Lower Bound}[\alpha, B, S, <] \wedge \forall_{\beta \in S}(\alpha < \beta \implies \neg \text{Lower Bound}[\beta, B, S, <])$	from: 1.1.4.17, 1.1.4.19
(1.1.4.21)	$\text{GLB}[\alpha, B, S, <] \blacksquare \exists_{\epsilon_1 \in S}(\text{GLB}[\epsilon_1, B, S, <])$	
(1.1.5)	$ B  \neq 1 \implies \exists_{\epsilon_1 \in S}(\text{GLB}[\epsilon_1, B, S, <])$	
(1.1.6)	$( B  = 1 \implies \exists_{\epsilon_0 \in S}(\text{GLB}[\epsilon_0, B, S, <])) \wedge ( B  \neq 1 \implies \exists_{\epsilon_1 \in S}(\text{GLB}[\epsilon_1, B, S, <]))$	from: 1.1.3, 1.1.5
(1.1.7)	$( B  = 1 \vee  B  \neq 1) \implies \exists_{\epsilon \in S}(\text{GLB}[\epsilon, B, S, <]) \blacksquare \exists_{\epsilon \in S}(\text{GLB}[\epsilon, B, S, <])$	
(1.2)	$(\emptyset \neq B \subset S \wedge \text{Bounded Below}[B, S, <]) \implies \exists_{\epsilon \in S}(\text{GLB}[\epsilon, B, S, <])$	
(1.3)	$\forall_B((\emptyset \neq B \subset S \wedge \text{Bounded Below}[B, S, <]) \implies \exists_{\epsilon \in S}(\text{GLB}[\epsilon, B, S, <]))$	
(1.4)	$\text{GLBProperty}[S, <]$	
(2)	$\text{LUBProperty}[S, <] \implies \text{GLBProperty}[S, <]$	

(1.12)

$$\text{Field}[F, +, *] := \exists_{0, 1 \in F} \forall_{x, y, z \in F} \left( \begin{array}{llll} x + y \in F & \wedge & x * y \in F & \wedge \\ x + y = y + x & \wedge & x * y = y * x & \wedge \\ (x + y) + z = x + (y + z) & \wedge & (x * y) * z = x * (y * z) & \wedge \\ 1 \neq 0 & \wedge & x * (y + z) = (x * y) + (x * z) & \wedge \\ 0 + x = x & \wedge & 1 * x = x & \wedge \\ \exists_{-x \in F}(x + (-x) = 0) \wedge (x \neq 0 \implies \exists_{1/x \in F}(x * (1/x) = 1)) \end{array} \right)$$

\*\*\*\*\* (Field[F, +, \*]  $\wedge$   $x, y, z \in F$ )  $\implies$  ... \*\*\*\*\*

(1.14)

$$\text{AdditiveCancellation} := (x + y = x + z) \implies y = z$$

(1)	$y = 0 + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \dots$	from: Field
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(2)	$(-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = 0 + z = z$	from: Field
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$$\text{AdditiveIdentityUniqueness} := (x + y = x) \implies y = 0$$

(1)	$x + y = x = 0 + x = x + 0$	from: Field
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(2)	$y = 0$	from: AdditiveCancellation
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$$\text{AdditiveInverseUniqueness} := (x + y = 0) \implies y = -x$$

(1)	$x + y = 0 = x + (-x)$	from: Field
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(2)	$y = -x$	from: AdditiveCancellation
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$$\text{DoubleNegative} := x = -(-x)$$

(1)	$0 = x + (-x) = (-x) + x \blacksquare 0 = (-x) + x$	from: Field
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(2)	$x = -(-x)$	from: AdditiveInverseUniqueness
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(1.15)

$\textcolor{red}{MultiplicativeCancellation} := (x \neq 0 \wedge x * y = x * z) \implies y = z \quad \text{---}$   
 $\textcolor{red}{MultiplicativeIdentityUniqueness} := (x \neq 0 \wedge x * y = x) \implies y = 1 \quad \text{---}$   
 $\textcolor{red}{MultiplicativeInverseUniqueness} := (x \neq 0 \wedge x * y = 1) \implies y = 1/x \quad \text{---}$   
 $\textcolor{red}{DoubleReciprocal} := (x \neq 0) \implies x = 1/(1/x) \quad \text{---}$

(1.16)

 $\textcolor{red}{Domination} := 0 * x = 0$ 
 $(1) \quad 0 * x = (0 + 0) * x = 0 * x + 0 * x \quad \blacksquare \quad 0 * x = 0 * x + 0 * x$ 
from: *Field*
 $(2) \quad 0 * x = 0$ 
from: *AdditiveIdentityUniqueness*
 $\textcolor{red}{NonDomination} := (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$ 
 $(1) \quad (x \neq 0 \wedge y \neq 0) \implies \dots$ 
 $(1.1) \quad (x * y = 0) \implies \dots$ 
 $(1.1.1) \quad 1 = 1 * 1 = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = 0 * ((1/x) * (1/y)) = 0$ 
from: *Field, Domination*, 1, 1.1
 $(1.1.2) \quad 1 = 0 \wedge 1 \neq 0 \quad \blacksquare \quad \perp$ 
from: *Field*
 $(1.2) \quad (x * y = 0) \implies \perp \quad \blacksquare \quad x * y \neq 0$ 
 $(2) \quad (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$ 
 $\textcolor{red}{NegationCommutativity} := (-x) * y = -(x * y) = x * (-y)$ 
 $(1) \quad x * y + (-x) * y = (x + -x) * y = 0 * y = 0 \quad \blacksquare \quad x * y + (-x) * y = 0$ 
from: *Field, Domination*  
wts: 2
 $(2) \quad (-x) * y = -(x * y)$ 
from: *AdditiveInverseUniqueness*
 $(3) \quad x * y + x * (-y) = x * (y_0 + -y) = x * 0 = 0 \quad \blacksquare \quad x * y + x * (-y) = 0$ 
from: *Field, Domination*  
wts: 4
 $(4) \quad x * (-y) = -(x * y)$ 
from: *AdditiveInverseUniqueness*
 $(5) \quad (-x) * y = -(x * y) = x * (-y)$ 

from: 2, 4

 $\textcolor{red}{NegativeMultiplication} := (-x) * (-y) = x * y$ 
 $(1) \quad (-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$ 
from: *NegationCommutativity, DoubleNegative*

(1.17)

 $\textcolor{red}{OrderedField}[F, +, *, <] := \left( \begin{array}{l} \textcolor{blue}{Field}[F, +, *] \quad \wedge \quad \textcolor{blue}{Order}[<, F] \quad \wedge \\ \forall_{x,y,z \in F} (y_0 < z \implies x + y < x + z) \quad \wedge \\ \forall_{x,y \in F} ((x > 0 \wedge y > 0) \implies x * y > 0) \end{array} \right)$ 
 $***** (\textcolor{blue}{OrderedField}[F, +, *, <] \wedge x, y, z \in F) \implies \dots *****$ 

(1.18)

 $\textcolor{red}{NegationOnOrder} := x > 0 \iff -x < 0$ 
 $(1) \quad x > 0 \implies \dots$ 
 $(1.1) \quad 0 = (-x) + x > (-x) + 0 = -x \quad \blacksquare \quad 0 > -x \quad \blacksquare \quad -x < 0$ 
from: *OrderedField*
 $(2) \quad x > 0 \implies -x < 0$ 
 $(3) \quad -x < 0 \implies \dots$ 
 $(3.1) \quad 0 = x + (-x) < x + 0 = x \quad \blacksquare \quad 0 < x \quad \blacksquare \quad x > 0$ 
from: *OrderedField*
 $(4) \quad -x < 0 \implies x > 0$ 
 $(5) \quad x > 0 \implies -x < 0 \wedge -x < 0 \implies x > 0 \quad \blacksquare \quad x > 0 \iff -x < 0$ 

from: 2, 4

 $\textcolor{red}{PositiveFactorPreservesOrder} := (x > 0 \wedge y < z) \implies x * y < x * z$ 
 $(1) \quad (x > 0 \wedge y < z) \implies \dots$ 
 $(1.1) \quad (-y) + z > (-y) + y = 0 \quad \blacksquare \quad z + (-y) = 0$ 
from: *OrderedField*
 $(1.2) \quad x * (z + (-y)) > 0 \quad \blacksquare \quad x * z + x * (-y) > 0$ 
from: *OrderedField*
 $(1.3) \quad x * z = 0 + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \dots$ 
from: *Field, NegationCommutativity*
 $(1.4) \quad x * y + (x * z + x * (-y)) > x * y + 0 = x * y$ 
from: *Field*, 1.2
 $(1.5) \quad x * z > x * y$ 

from: 1.3, 1.4

$$(2) \quad (x > 0 \wedge y < z) \implies x * z > x * y$$

$$\textcolor{red}{NegativeFactorFlipsOrder} := (x < 0 \wedge y < z) \implies x * y > x * z$$

$$(1) \quad (x < 0 \wedge y < z) \implies \dots$$

$$(1.1) \quad -x > 0$$

from: [NegationOnOrder](#)

$$(1.2) \quad (-x) * y < (-x) * z \quad \blacksquare \quad 0 = x * y + (-x) * y < x * y + (-x) * z \quad \blacksquare \quad 0 < x * y + (-x) * z$$

from: [PositiveFactorPreservesOrder](#)

$$(1.3) \quad 0 < (-x) * (-y + z) \quad \blacksquare \quad 0 > x * (-y + z) \quad \blacksquare \quad 0 > -(x * y) + x * z$$

from: [NegationOnOrder](#)

$$(1.4) \quad x * y > x * z$$

$$(2) \quad (x < 0 \wedge y < z) \implies x * y > x * z$$

$$\textcolor{red}{SquareIsPositive} := (x \neq 0) \implies x * x > 0$$

$$(1) \quad (x > 0) \implies x * x > 0$$

from: [OrderedField](#)

$$(2) \quad (x < 0) \implies \dots$$

$$(2.1) \quad -x > 0 \quad \blacksquare \quad x * x = (-x) * (-x) > 0 \quad \blacksquare \quad x * x > 0$$

from: [NegationOnOrder](#), [OrderedField](#), [NegativeMultiplication](#)

$$(3) \quad (x < 0) \implies x * x > 0$$

$$(4) \quad x \neq 0 \implies (x > 0 \vee x < 0) \implies x * x > 0 \quad \blacksquare \quad x \neq 0 \implies x * x > 0$$

from: [OrderTrichotomy](#), 1, 3

$$\textcolor{red}{OneIsPositive} := 1 > 0$$

$$(1) \quad 1 \neq 0 \quad \blacksquare \quad 1 = 1 * 1 > 0$$

from: [Field](#), [SquareIsPositive](#)

$$\textcolor{red}{ReciprocationOnOrder} := (0 < x < y) \implies 0 < 1/y < 1/x$$

$$(1) \quad (0 < x < y) \implies \dots$$

$$(1.1) \quad x * (1/x) = 1 > 0 \quad \blacksquare \quad x * (1/x) > 0$$

from: [Field](#), [OneIsPositive](#)

$$(1.2) \quad 1/x < 0 \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp \quad \blacksquare \quad 1/x > 0$$

from: [NegativeFactorFlipsOrder](#), 1

$$(1.3) \quad y * (1/y) = 1 > 0 \quad \blacksquare \quad y * (1/y) > 0$$

from: [Field](#), [OneIsPositive](#)

$$(1.4) \quad 1/y < 0 \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp \quad \blacksquare \quad 1/y > 0$$

from: [NegativeFactorFlipsOrder](#), 1

$$(1.5) \quad (1/x) * (1/y) > 0$$

from: [OrderedField](#)

$$(1.6) \quad 0 < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$$

from: [OrderedField](#), 1, 1.4, 1.5

$$(1.19)$$

$$\textcolor{red}{OrderedFieldQ} := \textcolor{blue}{OrderedField}[\mathbb{Q}, +, *, <] \quad \text{---}$$

$$\textcolor{blue}{Subfield}[K, F, +, *] := \textcolor{blue}{Field}[F, +, *] \wedge K \subset F \wedge \textcolor{blue}{Field}[K, +, *]$$

$$\textcolor{blue}{OrderedSubfield}[K, F, +, *, <] := \textcolor{blue}{OrderedField}[F, +, *, <] \wedge K \subset F \wedge \textcolor{blue}{OrderedField}[K, +, *, <]$$

$$\textcolor{blue}{CutI}[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$$

$$\textcolor{blue}{CutII}[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$$

$$\textcolor{blue}{CutIII}[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$$

$$\mathbb{R} := \{\alpha \in \mathbb{Q} : \textcolor{blue}{CutI}[\alpha] \wedge \textcolor{blue}{CutII}[\alpha] \wedge \textcolor{blue}{CutIII}[\alpha]\}$$

$$\textcolor{red}{CutCorollaryI} := (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$$

$$(1) \quad (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \dots$$

$$(1.1) \quad \forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$$

from: [CutII](#), 1

$$(1.2) \quad q < p \implies q \in \alpha \quad \blacksquare \quad q \notin \alpha \implies q \geq p$$

from: 1

$$(1.3) \quad (q \notin \alpha) \implies \dots$$

$$(1.3.1) \quad q \geq p$$

from: 1.2

$$(1.3.2) \quad (q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp \quad \blacksquare \quad q \neq p$$

from: 1, 1.3

$$(1.3.3) \quad q \geq p \wedge q \neq p \quad \blacksquare \quad p < q$$

$$(1.4) \quad q \notin \alpha \implies p < q \quad \blacksquare \quad p < q$$

from: 1

$$(2) \quad (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$$

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**CutCorollaryI1**  $:= (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

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(1)  $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \dots$

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(1.1)  $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$

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from: [CutI1](#), 1

(1.2)  $s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)) \blacksquare s \in \alpha \implies r \in \alpha$

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from: 1, 1.1

(1.3)  $r \notin \alpha \implies s \notin \alpha \blacksquare s \notin \alpha$

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from: 1, 1.2

(2)  $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

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$<_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha < \beta$

**OrderTrichotomyOfR**  $:= \text{OrderTrichotomy}[\mathbb{R}, <_{\mathbb{R}}]$

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(1)  $(\alpha, \beta \in \mathbb{R}) \implies \dots$

---

(1.1)  $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \dots$

---

(1.1.1)  $\alpha \not\subset \beta \wedge \alpha \neq \beta$

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from:  $<_{\mathbb{R}}$ , 1.1

(1.1.2)  $\exists_{p'} (p' \in \alpha \wedge p' \notin \beta) \blacksquare p := \text{choice}(\{p' : p' \in \alpha \wedge p' \notin \beta\})$

---

(1.1.3)  $q \in \beta \implies \dots$

---

(1.1.3.1)  $p, q \in \mathbb{Q}$

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(1.1.3.2)  $q < p$

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from: [CutCorollaryI](#)

(1.1.3.3)  $q \in \alpha$

---

from: [CutI1](#)

(1.1.4)  $q \in \beta \implies q \in \alpha$

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(1.1.5)  $\forall_{q \in \beta} (q \in \alpha) \blacksquare \beta \subseteq \alpha$

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(1.1.6)  $\beta \subset \alpha \blacksquare \beta <_{\mathbb{R}} \alpha$

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(1.2)  $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

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(1.3)  $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \blacksquare (\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

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(1.4)  $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

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(1.5)  $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

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(1.6)  $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

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(1.7)  $\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta$

---

(2)  $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

---

(3)  $\forall_{\alpha, \beta \in \mathbb{R}} (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

---

(4) [OrderTrichotomy](#) $[\mathbb{R}, <_{\mathbb{R}}]$

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**OrderTransitivityOfR**  $:= \text{OrderTransitivity}[\mathbb{R}, <_{\mathbb{R}}]$

---

(1)  $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$

---

(1.1)  $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \dots$

---

(1.1.1)  $\alpha \subset \beta \wedge \beta \subset \gamma$

---

(1.1.2)  $\forall_{a \in \alpha} (a \in \beta) \wedge \forall_{b \in \beta} (b \in \gamma)$

---

(1.1.3)  $\forall_{a \in \alpha} (\alpha \in \gamma) \blacksquare \alpha \subset \gamma \blacksquare \alpha <_{\mathbb{R}} \gamma$

---

(1.2)  $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

---

(2)  $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

---

(3)  $\forall_{\alpha, \beta, \gamma \in \mathbb{R}} ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

---

(4) [OrderTransitivity](#) $[\mathbb{R}, <_{\mathbb{R}}]$

---

**OrderOfR**  $:= \text{Order}[<_{\mathbb{R}}, \mathbb{R}]$

---

from: [OrderTrichotomyR](#), [OrderTransitivityR](#)  
wts:

**LUBPropertyOfR**  $:= \text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}]$

---

(1)  $(\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \dots$

---

(1.1)  $\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A} (p \in \alpha)\}$

---

(1.2)  $A \neq \emptyset \blacksquare \exists_{\alpha} (\alpha \in A) \blacksquare \alpha_0 := \text{choice}(\{\alpha : \alpha \in A\})$

---

(1.3)  $\alpha_0 \neq \emptyset \blacksquare \exists_a (a \in \alpha_0) \blacksquare a_0 := \text{choice}(\{a : a \in \alpha_0\}) \blacksquare a_0 \in \gamma \blacksquare \gamma \neq \emptyset$

---

(1.4) [BoundedAbove](#) $[A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare \exists_{\beta} (\text{UpperBound}[\beta, A, \mathbb{R}, <_{\mathbb{R}}])$

---

(1.5)	$\beta_0 := \text{choice}(\{\beta : \text{UpperBound}[\beta, A, \mathbb{R}, <_{\mathbb{R}}]\})$
(1.6)	$\text{UpperBound}[\beta_0, A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \beta_0) \blacksquare \forall_{\alpha \in A}(\alpha \subseteq \beta_0) \blacksquare \forall_{\alpha \in A} \forall_{a \in \alpha}(a \in \beta_0)$
(1.7)	$(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \blacksquare \forall_{a \in \gamma}(a \in \beta_0) \blacksquare \gamma \subseteq \beta_0$
(1.8)	$\beta_0 \subset \mathbb{Q} \blacksquare \gamma \subseteq \beta_0 \subset \mathbb{Q} \blacksquare \gamma \subset \mathbb{Q}$
(1.9)	$\emptyset \neq \gamma \subset \mathbb{Q} \blacksquare \text{CutI}[\gamma]$
(1.10)	$(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
(1.10.1)	$p \in \gamma \blacksquare \exists_{\alpha \in A}(p \in \alpha) \blacksquare \alpha_1 := \text{choice}(\{\alpha \in A : p \in \alpha\})$
(1.10.2)	$p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \blacksquare q \in \alpha_1 \blacksquare q \in \gamma$
(1.11)	$(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \blacksquare \forall_{p \in \gamma} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma) \blacksquare \text{CutII}[\gamma]$
(1.12)	$p \in \gamma \implies \dots$
(1.12.1)	$\exists_{\alpha \in A}(p \in \alpha) \blacksquare \alpha_2 := \text{choice}(\{\alpha \in A : p \in \alpha\})$
(1.12.2)	$\alpha_2 \in \mathbb{R} \blacksquare \text{CutII}[\alpha_2] \blacksquare \exists_{r \in \alpha_2}(p < r) \blacksquare r_0 := \text{choice}(\{r \in \alpha_2 : p < r\})$
(1.12.3)	$r_0 \in \alpha_2 \blacksquare r_0 \in \gamma$
(1.12.4)	$p < r_0 \blacksquare p < r_0 \wedge r_0 \in \gamma \blacksquare \exists_{r \in \gamma}(p < r)$
(1.13)	$p \in \gamma \implies \exists_{r \in \gamma}(p < r) \blacksquare \forall_{p \in \gamma} \exists_{r \in \gamma}(p < r) \blacksquare \text{CutIII}[\gamma]$
(1.14)	$\text{CutI}[\gamma] \wedge \text{CutII}[\gamma] \wedge \text{CutIII}[\gamma] \blacksquare \gamma \in \mathbb{R}$
(1.15)	$\forall_{\alpha \in A}(\alpha \subseteq \gamma) \blacksquare \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma)$
(1.16)	$\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R} \blacksquare \text{UpperBound}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$
(1.17)	$\delta <_{\mathbb{R}} \gamma \implies \dots$
(1.17.1)	$\delta \subset \gamma \blacksquare \exists_s(s \in \gamma \wedge s \notin \delta) \blacksquare s_0 := \text{choice}(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$
(1.17.2)	$s_0 \in \gamma \blacksquare \exists_{\alpha \in A}(s_0 \in \alpha) \blacksquare \alpha_3 := \text{choice}(\{\alpha \in A : s_0 \in \alpha\})$
(1.17.3)	$s_0 \in \alpha_3 \wedge s_0 \notin \delta \blacksquare \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$
(1.17.4)	$\delta \geq_{\mathbb{R}} \alpha_3 \implies \dots$
(1.17.4.1)	$\alpha_3 \subseteq \delta \blacksquare \forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \blacksquare \neg \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$
(1.17.4.2)	$\neg \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \blacksquare \perp$
(1.17.5)	$\delta \geq_{\mathbb{R}} \alpha_3 \implies \perp \blacksquare \delta <_{\mathbb{R}} \alpha_3 \blacksquare \exists_{\alpha \in A}(\delta <_{\mathbb{R}} \alpha) \blacksquare \exists_{\alpha \in A}(\neg(\alpha \leq_{\mathbb{R}} \delta))$
(1.17.6)	$\neg \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \delta) \blacksquare \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$
(1.18)	$\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare \forall_{\delta}(\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$
(1.19)	$\text{UpperBound}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \wedge \forall_{\delta}(\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$
(1.20)	$\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$
(2)	$(\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$
(3)	$\forall_A((\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])) \blacksquare \text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}]$

$$+_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$$

$$0_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 0\}$$

$$0InR := 0_{\mathbb{R}} \in \mathbb{R}$$

(1)	$-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}} \blacksquare \emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q} \blacksquare \text{CutI}[0_{\mathbb{R}}]$
(2)	$(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}} \blacksquare \forall_{x \in 0_{\mathbb{R}}} \forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}}) \blacksquare \text{CutII}[0_{\mathbb{R}}]$
(3)	$y := x/2 \blacksquare (x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y) \blacksquare \forall_{x \in 0_{\mathbb{R}}} \exists_{y \in 0_{\mathbb{R}}}(x < y) \blacksquare \text{CutIII}[0_{\mathbb{R}}]$
(4)	$\text{CutI}[0_{\mathbb{R}}] \wedge \text{CutII}[0_{\mathbb{R}}] \wedge \text{CutIII}[0_{\mathbb{R}}] \blacksquare 0_{\mathbb{R}} \in \mathbb{R}$

$$\text{FieldAdditionClosureOf } R := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$$

(1)	$(\alpha, \beta \in \mathbb{R}) \implies \dots$
(1.1)	$(\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$
(1.2)	$\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$
(1.3)	$\exists_a(a \in \alpha) ; \exists_b(b \in \beta) \blacksquare a_0 := \text{choice}(\{a : a \in \alpha\}) ; b_0 := \text{choice}(\{b : b \in \beta\}) \blacksquare a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$
(1.4)	$\exists_x(x \notin \alpha) ; \exists_y(y_0 \notin \beta) \blacksquare x_0 := \text{choice}(\{x : x \notin \alpha\}) ; y_0 := \text{choice}(\{y : y \notin \beta\})$
(1.5)	$\forall_{r \in \alpha}(r < x_0) ; \forall_{s \in \beta}(s < y_0) \blacksquare \forall_{r \in \alpha} \forall_{s \in \beta}(r + s < x_0 + y_0) \blacksquare x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$
(1.6)	$\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q} \blacksquare \text{CutI}[\alpha +_{\mathbb{R}} \beta]$



- (1.7)  $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
- (1.7.1)  $\exists_{r \in \alpha} \exists_{s \in \beta} (p = r + s) \blacksquare (r_0, s_0) := \text{choice}((r, s) \in \alpha \times \beta : p = r + s)$
- (1.7.2)  $q < p = r_0 + s_0 \blacksquare (q - s_0) < r_0 \blacksquare (q - s_0) \in \alpha$
- (1.7.3)  $s_0 \in \beta \blacksquare q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta \blacksquare q \in \alpha +_{\mathbb{R}} \beta$
- (1.8)  $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} \beta} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha +_{\mathbb{R}} \beta) \blacksquare \text{CutII}[\alpha +_{\mathbb{R}} \beta]$
- (1.9)  $p \in \alpha \implies \dots$
- (1.9.1)  $\exists_{r \in \alpha} \exists_{s \in \beta} (p = r + s) \blacksquare (r_1, s_1) := \text{choice}(\{(r, s) \in \alpha \times \beta : p = r + s\})$
- (1.9.2)  $r_1 \in \alpha \blacksquare \exists_{t \in \alpha} (r_1 < t) \blacksquare t_0 := \text{choice}(\{t \in \alpha : r_1 < t\})$
- (1.9.3)  $s_1 \in \beta \blacksquare t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1 \blacksquare \exists_{r \in \alpha +_{\mathbb{R}} \beta} (p < r)$
- (1.10)  $p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta} (p < r) \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} \beta} \exists_{r \in \alpha +_{\mathbb{R}} \beta} (p < r) \blacksquare \text{CutIII}[\alpha +_{\mathbb{R}} \beta]$
- (1.11)  $\text{CutI}[\alpha +_{\mathbb{R}} \beta] \wedge \text{CutII}[\alpha +_{\mathbb{R}} \beta] \wedge \text{CutIII}[\alpha +_{\mathbb{R}} \beta] \blacksquare \alpha +_{\mathbb{R}} \beta \in \mathbb{R}$
- (2)  $(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

**FieldAdditionCommutativityOfR**  $:= (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

- (1)  $\alpha +_{\mathbb{R}} \beta = \{r + s : r \in \alpha \wedge s \in \beta\} = \{s + r : s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

**FieldAdditionAssociativityOfR**  $:= (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma))$

- (1)  $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$
- (1.1)  $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \dots$
- (1.2)  $\{a + (b + c) : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$
- (2)  $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

**FieldAdditionIdentityOfR**  $:= (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

- (1)  $\alpha \in \mathbb{R} \implies \dots$
- (1.1)  $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \dots$
- (1.1.1)  $s < 0 \blacksquare r + s < r + 0 = r \blacksquare r + s < r \blacksquare r + s \in \alpha$
- (1.2)  $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha \blacksquare \forall_{r \in \alpha} \forall_{s \in 0_{\mathbb{R}}} (r + s \in \alpha)$
- (1.3)  $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}} (p \in \alpha) \blacksquare \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$
- (1.4)  $p \in \alpha \implies \dots$
- (1.4.1)  $\exists_{r \in \alpha} (p < r) \blacksquare r_2 := \text{choice}(\{r \in \alpha : p < r\})$
- (1.4.2)  $p < r_2 \blacksquare p - r_2 < r_2 - r_2 = 0 \blacksquare (p - r_2) < 0 \blacksquare (p - r_2) \in 0_{\mathbb{R}}$
- (1.4.3)  $r_2 \in \alpha \blacksquare p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$
- (1.5)  $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare \forall_{p \in \alpha} (p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$
- (1.6)  $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$
- (2)  $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

**FieldAdditionInverseOfR**  $:= (\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}} (\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

- (1)  $\alpha \in \mathbb{R} \implies \dots$
- (1.1)  $\beta := \{p \in \mathbb{Q} : \exists_{r > 0} (-p - r \notin \alpha)\}$
- (1.2)  $\alpha \subset \mathbb{Q} \blacksquare \exists_{s \in \mathbb{Q}} (s \notin \alpha) \blacksquare s_0 := \text{choice}(\{s : s \notin \alpha\}) \blacksquare p_0 := -s_0 - 1$
- (1.3)  $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha \blacksquare -p_0 - 1 \notin \alpha \blacksquare \exists_{r > 0} (-p_0 - r \notin \alpha) \blacksquare p_0 \in \beta$
- (1.4)  $\emptyset \neq \alpha \blacksquare \exists_{q \in \alpha} \blacksquare q_0 := \text{choice}(\{q \in \mathbb{Q} : q \in \alpha\})$
- (1.5)  $r > 0 \implies \dots$
- (1.5.1)  $q_0 \in \alpha \blacksquare -(-q_0) - r = q_0 - r < q_0 \blacksquare -(-q_0) - r < q_0 \blacksquare -(-q_0) - r \in \alpha$
- (1.6)  $\forall_{r > 0} (-(-q_0) - r \in \alpha) \blacksquare \neg \exists_{r > 0} (-(-q_0) - r \notin \alpha) \blacksquare -q_0 \notin \beta$
- (1.7)  $\emptyset \neq \beta \subset \mathbb{Q} \blacksquare \text{CutI}[\beta]$
- (1.8)  $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
- (1.8.1)  $p \in \beta \blacksquare \exists_{r > 0} (-p - r \notin \alpha) \blacksquare r_0 := \text{choice}(\{r > 0 : -p - r \notin \alpha\})$
- (1.8.2)  $q < p \blacksquare -p - r < -q - r$
- (1.8.3)  $-q - r \notin \alpha \blacksquare q \in \beta$
- (1.9)  $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta \blacksquare \forall_{p \in \beta} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \beta) \blacksquare \text{CutII}[\beta]$

(1.10)	$p \in \beta \implies \dots$	
(1.10.1)	$p \in \beta \implies \exists_{r>0}(-p-r \notin \alpha) \blacksquare r_1 := \text{choice}(\{r > 0 : -p-r \notin \alpha\})$	
(1.10.2)	$t_0 := p + (r_1/2)$	
(1.10.3)	$r_1 > 0 \blacksquare r_1/2 > 0$	
(1.10.4)	$t_0 > t_0 - (r_1/2) = p \blacksquare t_0 > p$	
(1.10.5)	$-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$	
(1.10.6)	$-p - r_1 \notin \alpha \blacksquare -t_0 - (r_1/2) \notin \alpha \blacksquare \exists_{r>0}(-t_0 - r \notin \alpha) \blacksquare t_0 \in \beta$	
(1.10.7)	$t_0 > p \wedge t_0 \in \beta \blacksquare \exists_{t \in \beta}(p < t)$	
(1.11)	$p \in \beta \implies \exists_{t \in \beta}(p < t) \blacksquare \forall_{p \in \beta} \exists_{t \in \beta}(p < t) \blacksquare \text{CutIII}[\beta]$	
(1.12)	$\text{CutI}[\beta] \wedge \text{CutII}[\beta] \wedge \text{CutIII}[\beta] \blacksquare \beta \in \mathbb{R}$	
(1.13)	$(r \in \alpha \wedge s \in \beta) \implies \dots$	
(1.13.1)	$s \in \beta \blacksquare \exists_{t>0}(-s-t \notin \alpha) \blacksquare t_1 := \text{choice}(\{t > 0 : -s-t \notin \alpha\}) \blacksquare -s-t_1 < -s$	
(1.13.2)	$\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s-t_1 < -s \wedge -s-t_1 \notin \alpha \blacksquare -s \notin \alpha$	
(1.13.3)	$\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha \blacksquare r < -s \blacksquare r+s < 0 \blacksquare r+s \in 0_{\mathbb{R}}$	
(1.14)	$(r \in \alpha \wedge s \in \beta) \implies r+s \in 0_{\mathbb{R}} \blacksquare \forall_{(r,s) \in \alpha \times \beta}(r+s \in 0_{\mathbb{R}}) \blacksquare \alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$	
(1.15)	$v \in 0_{\mathbb{R}} \implies \dots$	
(1.15.1)	$v < 0 \blacksquare w_0 := -v/2 \blacksquare w > 0$	
(1.15.2)	$\exists_{n \in \mathbb{Z}}(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha) \blacksquare n_0 := \text{choice}(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$	from: ARCHIMEDEANPROPERTYOFQ + LUB???
(1.15.3)	$p_0 := -(n_0+2)w_0 \blacksquare -p_0 - w_0 = (n_0+2)w_0 - w_0 = (n_0+1)w_0 \notin \alpha \blacksquare -p_0 - w_0 \notin \alpha \blacksquare p_0 \in \beta$	
(1.15.4)	$n_0 w_0 \in \alpha \wedge p_0 \in \beta \blacksquare n_0 w_0 + p_0 = n_0(-v/2) + -(n_0+2)w_0 = v \in \alpha +_{\mathbb{R}} \beta$	
(1.16)	$v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{v \in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta) \blacksquare 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$	
(1.17)	$\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta \blacksquare \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$	
(1.18)	$\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}} \blacksquare \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$	
(2)	$\alpha \in \mathbb{R} \implies \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$	

$*_{\mathbb{R}}[\alpha, \beta] := \text{---}$

$1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$11sNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}} \text{ ---}$

$11nR := 1_{\mathbb{R}} \in \mathbb{R} \text{ ---}$

$\text{FieldMultiplicationClosureOf } \mathbb{R} := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R}) \text{ ---}$

$\text{FieldMultiplicationCommutativityOf } \mathbb{R} := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha) \text{ ---}$

$\text{FieldMultiplicationAssociativityOf } \mathbb{R} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)) \text{ ---}$

$\text{FieldMultiplicationIdentityOf } \mathbb{R} := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha \text{ ---}$

$\text{FieldMultiplicationInverseOf } \mathbb{R} := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha \in \mathbb{R}}(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}) \text{ ---}$

$\text{FieldDistributivityOf } \mathbb{R} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta \text{ ---}$

$\text{FieldWith } \mathbb{R} := \text{Field}[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}] \text{ ---}$

$\text{OrderedFieldWith } \mathbb{R} := \text{OrderedField}[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}] \text{ ---}$

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$

$\text{QROrderedSubfieldOf } \mathbb{R} := \text{OrderedSubfield}[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}] \text{ ---}$

$\text{QIsomorphicToQR} := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q} \text{ ---}$

$\text{CompletenessOf } \mathbb{R} := \exists_{\mathbb{R}}(\text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}] \wedge \text{OrderedSubfield}[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]) \text{ ---}$

(1.20)

$\text{ArchimedeanPropertyOf } \mathbb{R} := \forall_{x,y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

(1)  $(x, y \in \mathbb{R} \wedge x > 0) \implies \dots$

(1.1)  $A := \{nx : n \in \mathbb{N}^+\} \blacksquare (\emptyset \neq A \subset \mathbb{R}) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$

(1.2)  $\neg \exists_{n \in \mathbb{N}^+}(nx > y) \implies \dots$

(1.2.1)  $\neg \exists_{n \in \mathbb{N}^+}(nx > y) \blacksquare \forall_{n \in \mathbb{N}^+}(nx \leq y) \blacksquare \text{UpperBound}[y_0, A, \mathbb{R}, <] \blacksquare \text{BoundedAbove}[A, \mathbb{R}, <]$

(1.2.2)  $\text{CompletenessOf } \mathbb{R} \blacksquare \text{LUBProperty}[\mathbb{R}, <]$

(1.2.3)  $(\text{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (\text{BoundedAbove}[A, \mathbb{R}, <]) \blacksquare \exists_{\alpha \in \mathbb{R}}(\text{LUB}[\alpha, A, \mathbb{R}, <]) \dots$

(1.2.4)	$\dots \alpha_0 := \text{choice}(\{\alpha \in \mathbb{R} : \textcolor{teal}{LUB}[\alpha, A, \mathbb{R}, <]\}) \quad \blacksquare \quad \textcolor{teal}{LUB}[\alpha_0, A, \mathbb{R}, <]$
(1.2.5)	$x > 0 \quad \blacksquare \quad \alpha_0 - x < \alpha_0$
(1.2.6)	$(\alpha_0 - x < \alpha_0) \wedge (\textcolor{teal}{LUB}[\alpha_0, A, \mathbb{R}, <]) \quad \blacksquare \quad \neg \textcolor{teal}{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <]$
(1.2.7)	$\neg \textcolor{teal}{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <] \quad \blacksquare \quad \exists_{c \in A}(\alpha_0 - x < c) \quad \dots$
(1.2.8)	$\dots c_0 := \text{choice}(\{c \in A : \alpha_0 - x < c\}) \quad \blacksquare \quad (c_0 \in A) \wedge (\alpha_0 - x < c_0)$
(1.2.9)	$(c_0 \in A) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)) \quad \blacksquare \quad \exists_{m \in \mathbb{N}^+}(mx = c_0) \quad \dots$
(1.2.10)	$\dots m_0 := \text{choice}(\{m \in \mathbb{N}^+ : mx = c_0\}) \quad \blacksquare \quad (m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$
(1.2.11)	$(\alpha_0 - x < c_0) \wedge (m_0 x = c_0) \quad \blacksquare \quad \alpha_0 - x < c_0 = m_0 x \quad \blacksquare \quad \alpha_0 < m_0 x + x \quad \blacksquare \quad \alpha_0 < (m_0 + 1)x$
(1.2.12)	$m_0 \in \mathbb{N}^+ \quad \blacksquare \quad m_0 + 1 \in \mathbb{N}^+$
(1.2.13)	$(m_0 + 1 \in \mathbb{N}^+) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)) \quad \blacksquare \quad (m_0 + 1)x \in A$
(1.2.14)	$(\alpha_0 < (m_0 + 1)x) \wedge ((m_0 + 1)x \in A) \quad \blacksquare \quad \exists_{c \in A}(\alpha_0 < c)$
(1.2.15)	$\textcolor{teal}{LUB}[\alpha_0, A, \mathbb{R}, <] \quad \blacksquare \quad \textcolor{teal}{UpperBound}[\alpha_0, A, \mathbb{R}, <] \quad \blacksquare \quad \forall_{c \in A}(c \leq \alpha_0) \quad \blacksquare \quad \neg \exists_{c \in A}(c > \alpha_0) \quad \blacksquare \quad \neg \exists_{c \in A}(\alpha_0 < c)$
(1.2.16)	$(\exists_{c \in A}(\alpha_0 < c)) \wedge (\neg \exists_{c \in A}(\alpha_0 < c)) \quad \blacksquare \quad \perp$
(1.3)	$\neg \exists_{n \in \mathbb{N}^+}(nx > y) \implies \perp \quad \blacksquare \quad \exists_{n \in \mathbb{N}^+}(nx > y)$
(2)	$(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y) \quad \blacksquare \quad \forall_{x, y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

***QDenseInR*** :=  $\forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1)	$(x, y \in \mathbb{R} \wedge x < y) \implies \dots$
(1.1)	$x < y \quad \blacksquare \quad (0 < y - x) \wedge (y - x \in \mathbb{R})$
(1.2)	$\textcolor{teal}{ArchimedeanPropertyOfR} \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R}) \quad \blacksquare \quad \exists_{n \in \mathbb{N}^+}(n(y - x) > 1) \quad \dots$
(1.3)	$\dots n_0 := \text{choice}(\{n \in \mathbb{N}^+ : n(y - x) > 1\}) \quad \blacksquare \quad (n_0 \in \mathbb{N}^+) \wedge (n_0(y - x) > 1)$
(1.4)	$(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R}) \quad \blacksquare \quad n_0 x, -n_0 x \in \mathbb{R}$
(1.5)	$\textcolor{teal}{ArchimedeanPropertyOfR} \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R}) \quad \blacksquare \quad \exists_{m \in \mathbb{N}^+}(m(1) > n_0 x) \quad \dots$
(1.6)	$\dots m_1 := \text{choice}(\{m \in \mathbb{N}^+ : m(1) > n_0 x\}) \quad \blacksquare \quad (m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$
(1.7)	$\textcolor{teal}{ArchimedeanPropertyOfR} \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R}) \quad \blacksquare \quad \exists_{m \in \mathbb{N}^+}(m(1) > -n_0 x) \quad \dots$
(1.8)	$\dots m_2 := \text{choice}(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\}) \quad \blacksquare \quad (m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$
(1.9)	$(m_1 > n_0 x) \wedge (m_2 > -n_0 x) \quad \blacksquare \quad -m_2 < n_0 x < m_1$
(1.10)	$m_1, m_2 \in \mathbb{N}^+ \quad \blacksquare \quad  m_1 - (-m_2)  \geq 2$
(1.11)	$(-m_2 < n_0 x < m_1) \wedge ( m_1 - (-m_2)  \geq 2) \quad \blacksquare \quad \exists_{m \in \mathbb{Z}}((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)) \quad \dots$
(1.12)	$\dots m_0 := \text{choice}(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\}) \quad \blacksquare \quad (-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$
(1.13)	$(n_0(y - x) > 1) \wedge (m_0 - 1 \leq n_0 x < m_0) \quad \blacksquare \quad n_0 x < m_0 \leq 1 + n_0 x < n_0 y \quad \blacksquare \quad n_0 x < m_0 < n_0 y$
(1.14)	$(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y) \quad \blacksquare \quad x < m_0/n_0 < y$
(1.15)	$m_0, n_0 \in \mathbb{Z} \quad \blacksquare \quad m_0/n_0 \in \mathbb{Q}$
(1.16)	$(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y) \quad \blacksquare \quad \exists_{p \in \mathbb{Q}}(x < p < y)$
(2)	$(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y) \quad \blacksquare \quad \forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1.21)

***Root Lemma*** :=  $(0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

(1)	$(0 < a < b) \implies \dots$
(1.1)	$b^n - a^n = (b - a) \sum_{i=1}^n (b^{n-i} a^{i-1})$
(1.2)	$0 < a < b \quad \blacksquare \quad b/a > 1$
(1.3)	$b/a > 1 \quad \blacksquare \quad \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq \sum_{i=1}^n (b^{n-i} a^{i-1} (b/a)^{i-1}) = \sum_{i=1}^n (b^{n-1}) = nb^{n-1} \quad \blacksquare \quad \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq \sum_{i=1}^n (b^{n-1}) = nb^{n-1}$
(1.4)	$b^n - a^n = (b - a) \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq (b - a)nb^{n-1} \quad \blacksquare \quad b^n - a^n \leq (b - a)nb^{n-1}$
(2)	$(0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

***Root Existence In R*** :=  $\forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}} (y_0^n = x)$

(1)	$(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \dots$
(1.1)	$E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\} \quad \blacksquare \quad t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$
(1.2)	$t_0 := x/(1 + x) \quad \blacksquare \quad (t_0 = x/(1 + x)) \wedge (t_0 \in \mathbb{R})$
(1.3)	$0 < x \quad \blacksquare \quad 0 < x < 1 + x \quad \blacksquare \quad t_0 = x/(1 + x) > 0 \quad \blacksquare \quad t_0 > 0$

$$(1.4) \quad 1 = (1+x)/(1+x) > x/(1+x) = t_0 \quad \blacksquare \quad 1 > t_0$$

$$(1.5) \quad (t_0 > 0) \wedge (1 > t_0) \quad \blacksquare \quad 0 < t_0 < 1$$

$$(1.6) \quad (0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1) \quad \blacksquare \quad t_0^n \leq t_0$$

$$(1.7) \quad 0 < x \quad \blacksquare \quad x > x/(1+x) = t_0 \quad \blacksquare \quad x > t_0$$

$$(1.8) \quad (t_0^n \leq t_0) \wedge (x > t_0) \quad \blacksquare \quad t_0^n < x$$

$$(1.9) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge (t_0^n < x) \quad \blacksquare \quad t_0 \in E \quad \blacksquare \quad \emptyset \neq E$$

$$(1.10) \quad t_1 := \text{choice}(\{t \in \mathbb{R} : t > 1+x\}) \quad \blacksquare \quad (t_1 \in \mathbb{R}) \wedge (t_1 > 1+x)$$

$$(1.11) \quad x > 0 \quad \blacksquare \quad t_1 > 1+x > 1 \quad \blacksquare \quad t_1 > 1 \quad \blacksquare \quad t_1^n \geq t_1$$

$$(1.12) \quad (t_1^n \geq t_1) \wedge (t_1 > 1+x) \wedge (1 > 0) \quad \blacksquare \quad t_1^n \geq t_1 > 1+x > x \quad \blacksquare \quad t_1^n > x$$

$$(1.13) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_1^n > x) \quad \blacksquare \quad t_1 \notin E \quad \blacksquare \quad E \subset \mathbb{R}$$

$$(1.14) \quad (\emptyset \neq E) \wedge (E \subset \mathbb{R}) \quad \blacksquare \quad \emptyset \neq E \subset \mathbb{R}$$

$$(1.15) \quad t \in E \implies \dots$$

$$(1.15.1) \quad (t \in E) \wedge (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \quad \blacksquare \quad t^n < x$$

$$(1.15.2) \quad (t_1^n > x) \wedge (t^n < x) \quad \blacksquare \quad t^n < x < t_1^n \quad \blacksquare \quad t < t_1$$

$$(1.16) \quad t \in E \implies t < t_1 \quad \blacksquare \quad \forall_{t \in E} (t \leq t_1) \quad \blacksquare \quad \text{UpperBound}[t_1, E, \mathbb{R}, <] \quad \blacksquare \quad \text{BoundedAbove}[E, \mathbb{R}, <]$$

$$(1.17) \quad \text{CompletenessOfR} \quad \blacksquare \quad \text{LUBProperty}[\mathbb{R}, <]$$

$$(1.18) \quad (\text{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (\text{BoundedAbove}[E, \mathbb{R}, <]) \quad \blacksquare \quad \exists_{y \in \mathbb{R}} (\text{LUB}[y, E, \mathbb{R}, <]) \quad \dots$$

$$(1.19) \quad \dots y_0 := \text{choice}(\{y \in \mathbb{R} : \text{LUB}[y, E, \mathbb{R}, <]\}) \quad \blacksquare \quad \text{LUB}[y_0, E, \mathbb{R}, <]$$

$$(1.20) \quad (\text{LUB}[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0) \quad \blacksquare \quad 0 < t_0 \leq y_0 \in \mathbb{R} \quad \blacksquare \quad 0 < y_0 \in \mathbb{R}$$

$$(1.21) \quad y_0^n < x \implies \dots$$

$$(1.21.1) \quad k_0 := \frac{x-y_0^n}{n(y_0+1)^{n-1}} \quad \blacksquare \quad k_0 \in \mathbb{R}$$

$$(1.21.2) \quad y_0^n < x \quad \blacksquare \quad 0 < x - y_0^n$$

$$(1.21.3) \quad (n > 0) \wedge (y_0 > 0) \quad \blacksquare \quad 0 < n(y_0 + 1)^{n-1}$$

$$(1.21.4) \quad (0 < x - y_0^n) \wedge (0 < n(y_0 + 1)^{n-1}) \quad \blacksquare \quad 0 < \frac{x-y_0^n}{n(y_0+1)^{n-1}} = k_0 \quad \blacksquare \quad 0 < k_0$$

$$(1.21.5) \quad (0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R}) \quad \blacksquare \quad 0 < \min(1, k_0) \in \mathbb{R}$$

$$(1.21.6) \quad \text{QDenseInR} \wedge (0, \min(1, k_0) \in \mathbb{R}) \wedge (0 < \min(1, k_0)) \quad \blacksquare \quad \exists_{h \in \mathbb{Q}} (0 < h < \min(1, k_0)) \quad \dots$$

$$(1.21.7) \quad \dots h_0 := \text{choice}(\{h \in \mathbb{Q} : 0 < h < \min(1, k_0)\}) \quad \blacksquare \quad (0 < h_0 < 1) \wedge (h_0 < k_0 = \frac{x-y_0^n}{n(y_0+1)^{n-1}})$$

$$(1.21.8) \quad (y_0 > 0) \wedge (h_0 > 0) \quad \blacksquare \quad 0 < y_0 < y_0 + h_0$$

$$(1.21.9) \quad \text{RootLemma} \wedge (0 < y_0 < y_0 + h_0) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < h_0 n (y_0 + h_0)^{n-1}$$

$$(1.21.10) \quad h_0 < 1 \quad \blacksquare \quad h_0 n (y_0 + h_0)^{n-1} < h_0 n (y_0 + 1)^{n-1}$$

$$(1.21.11) \quad ((y_0 + h_0)^n - y_0^n < h_0 n (y_0 + h_0)^{n-1}) \wedge (h_0 n (y_0 + h_0)^{n-1} < h_0 n (y_0 + 1)^{n-1}) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < h_0 n (y_0 + 1)^{n-1}$$

$$(1.21.12) \quad (0 < n(y_0 + 1)^{n-1}) \wedge (h_0 < k_0 = \frac{x-y_0^n}{n(y_0+1)^{n-1}}) \quad \blacksquare \quad h_0 n (y_0 + 1)^{n-1} < x - y_0^n$$

$$(1.21.13) \quad ((y_0 + h_0)^n - y_0^n < h_0 n (y_0 + 1)^{n-1}) \wedge (h_0 n (y_0 + 1)^{n-1} < x - y_0^n) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < x - y_0^n \quad \blacksquare \quad (y_0 + h_0)^n < x$$

$$(1.21.14) \quad (y_0 + h_0)^n - y_0^n < x - y_0^n \quad \blacksquare \quad (y_0 + h_0)^n < x$$

$$(1.21.15) \quad (0 < y_0 \in \mathbb{R}) \wedge (0 < h_0 \in \mathbb{R}) \quad \blacksquare \quad 0 < y_0 < y_0 + h_0 \in \mathbb{R}$$

$$(1.21.16) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ((y_0 + h_0)^n < x) \wedge (0 < y_0 + h_0 \in \mathbb{R}) \quad \blacksquare \quad (y_0 + h_0)^n \in E$$

$$(1.21.17) \quad ((y_0 + h_0)^n \in E) \wedge (y_0 < y_0 + h_0) \quad \blacksquare \quad \exists_{e \in E} (y_0 < e)$$

$$(1.21.18) \quad \text{LUB}[y_0, E, \mathbb{R}, <] \quad \blacksquare \quad \text{UpperBound}[y_0, E, \mathbb{R}, <] \quad \blacksquare \quad \forall_{e \in E} (e \leq y_0) \quad \blacksquare \quad \neg \exists_{e \in E} (e > y_0)$$

$$(1.21.19) \quad (\exists_{e \in E} (e > y_0)) \wedge (\neg \exists_{e \in E} (e > y_0)) \quad \blacksquare \quad \perp$$

$$(1.22) \quad y_0^n < x \implies \perp \quad \blacksquare \quad y_0^n \geq x$$

$$(1.23) \quad y_0^n > x \implies \dots$$

$$(1.23.1) \quad k_1 := \frac{y_0^n - x}{n y_0^{n-1}} \quad \blacksquare \quad (k_1 \in \mathbb{R}) \wedge (k_1 n y_0^{n-1} = y_0^n - x)$$

$$(1.23.2) \quad (0 < x) \wedge (0 < n \in \mathbb{Z}) \quad \blacksquare \quad y_0^n - x < y_0^n \leq n y_0^n \quad \blacksquare \quad y_0^n - x < n y_0^n$$

$$(1.23.3) \quad y_0^n - x < n y_0^n \quad \blacksquare \quad k_1 = \frac{y_0^n - x}{n y_0^{n-1}} < \frac{n y_0^n}{n y_0^{n-1}} = y_0 \quad \blacksquare \quad k_1 < y_0$$

$$(1.23.4) \quad y_0^n > x \quad \blacksquare \quad 0 < y_0^n - x$$

$$(1.23.5) \quad (n > 0) \wedge (y_0 > 0) \quad \blacksquare \quad 0 < n y_0^{n-1}$$

$$(1.23.6) \quad (0 < y_0^n - x) \wedge 0 < (n y_0^{n-1}) \quad \blacksquare \quad 0 < \frac{y_0^n - x}{n y_0^{n-1}} = k_1 \quad \blacksquare \quad 0 < k_1$$

(1.23.7)	$(k_1 < y_0) \wedge (0 < k_1) \quad \blacksquare \quad (0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$	
(1.23.8)	$t \geq y_0 - k_1 \implies \dots$	
(1.23.8.1)	$t \geq y_0 - k_1 \quad \blacksquare \quad t^n \geq (y_0 - k_1)^n \quad \blacksquare \quad -t^n \leq -(y_0 - k_1)^n \quad \blacksquare \quad y_0^n - t^n \leq y_0^n - (y_0 - k_1)^n$	
(1.23.8.2)	$\textcolor{blue}{RootLemma} \wedge (0 < y_0 - k_1 < y_0) \quad \blacksquare \quad y_0^n - (y_0 - k_1)^n < k_1 n y_0^{n-1}$	
(1.23.8.3)	$(y_0^n - t^n \leq y_0^n - (y_0 - k_1)^n) \wedge (y_0^n - t^n < k_1 n y_0^{n-1}) \quad \blacksquare \quad y_0^n - t^n < k_1 n y_0^{n-1}$	
(1.23.8.4)	$(k_1 n y_0^{n-1} = y_0^n - x) \wedge (y_0^n - t^n < k_1 n y_0^{n-1}) \quad \blacksquare \quad y_0^n - t^n < y_0^n - x \quad \blacksquare \quad -t^n < -x \quad \blacksquare \quad t^n > x$	
(1.23.8.5)	$(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t^n > x) \quad \blacksquare \quad t \notin E$	
(1.23.9)	$t \geq y_0 - k_1 \implies t \notin E \quad \blacksquare \quad t \in E \implies t < y_0 - k_1 \quad \blacksquare \quad \forall_{t \in E} (t \leq y_0 - k_1) \quad \blacksquare \quad \textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$	
(1.23.10)	$(\textcolor{blue}{LUB}[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)) \quad \blacksquare \quad \neg \textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$	
(1.23.11)	$(\textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg \textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \quad \blacksquare \quad \perp$	
(1.24)	$y_0^n > x \implies \perp \quad \blacksquare \quad y_0^n \leq x$	
(1.25)	$\textcolor{blue}{Order}[\mathbb{R}, <] \quad \blacksquare \quad \textcolor{blue}{OrderTrichotomy}[\mathbb{R}, <]$	
(1.26)	$(\textcolor{blue}{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_0^n \geq x) \wedge (y_0^n \leq x) \quad \blacksquare \quad y_0^n = x$	
(1.27)	$(y_0^n = x) \wedge (y_0 \in \mathbb{R}) \quad \blacksquare \quad \exists_{y \in \mathbb{R}} (y^n = x)$	
(1.28)	$y_1, y_2 := \textit{choice}(\{y \in \mathbb{R} : y^n = x\})$	
(1.29)	$y_1 \neq y_2 \implies \dots$	
(1.29.1)	$(\textcolor{blue}{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_1 \neq y_2) \quad \blacksquare \quad (y_1 < y_2) \vee (y_2 < y_1) \quad \dots$	
(1.29.2)	$\dots (x = y_1^n < y_2^n = x) \vee (x = y_2^n < y_1^n = x) \quad \blacksquare \quad (x < x) \vee (x > x) \quad \blacksquare \quad \perp \vee \perp \quad \blacksquare \quad \perp$	
(1.30)	$y_1 \neq y_2 \implies \perp \quad \blacksquare \quad y_1 = y_2 \quad \blacksquare \quad \forall_{a,b \in \mathbb{R}} ((a^n = x \wedge b^n = x) \implies a = b)$	
(1.31)	$(\exists_{y \in \mathbb{R}} (y^n = x)) \wedge (\forall_{a,b \in \mathbb{R}} ((a^n = x \wedge b^n = x) \implies a = b)) \quad \blacksquare \quad \exists!_{y \in \mathbb{R}} (y^n = x)$	
(2)	$(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}} (y^n = x) \quad \blacksquare \quad \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}} (y_0^n = x)$	

$$\textcolor{red}{RootExistenceInRCorollary} := \forall_{0 < a \in \mathbb{R}} \forall_{0 < b \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} ((ab)^{1/n} = a^{1/n} b^{1/n}) \quad \text{---}$$

$$\textcolor{red}{ExtendedRealSystem}[\bar{\mathbb{R}}, +, *, <] := \left( \begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \quad \wedge \quad x - \infty = -\infty \quad \wedge \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \quad \wedge \\ (x > 0) \implies (x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty) \wedge \\ (x < 0) \implies (x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty) \end{array} \right)$$

$$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$$

$$+_C[\langle a, b \rangle, \langle c, d \rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d \rangle$$

$$*_C[\langle a, b \rangle, \langle c, d \rangle] := \langle a *_R c - b *_R d, a *_R d + b *_R c \rangle$$

$$\textcolor{red}{FieldC} := \textcolor{blue}{Field}[\mathbb{C}, +_C, *_C] \quad \text{---}$$

$$\textcolor{red}{RSubfieldC} := \textcolor{blue}{Subfield}[\mathbb{R}, \mathbb{C}, +, *] \quad \text{---}$$

$$i := \langle 0, 1 \rangle \in \mathbb{C}$$

$$\textcolor{red}{iProperty} := i^2 = -1 \quad \text{---}$$

$$\textcolor{red}{CProperty} := (a, b \in \mathbb{R}) \implies (\langle a, b \rangle = a + bi) \quad \text{---}$$

$$\textcolor{red}{Conjugate}[\overline{a + bi}] := a - bi$$

$$\textcolor{red}{ConjugateProperties} := (w, z \in \mathbb{C}) \implies \dots \quad \text{---}$$

$$(1) \quad \overline{z + w} = \bar{z} + \bar{w}$$

$$(2) \quad \overline{z * w} = \bar{z} * \bar{w}$$

$$(3) \quad \textit{Re}(z) = (1/2)(z + \bar{z}) \wedge \textit{Im}(z) = (1/2)(z - \bar{z})$$

$$(4) \quad 0 \leq z * \bar{z} \in \mathbb{R}$$

$$\textcolor{red}{AbsoluteValueC}[|z|] = (z * \bar{z})^{1/2}$$

$$\textcolor{red}{AbsoluteValueProperties} := (z, w \in \mathbb{C}) \implies \dots \quad \text{---}$$

$$(1) \quad 123123$$



# Chapter 2

## Abstract Algebra

$Relation(f, X) := f \subseteq X$   
 $Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y} ((x, y) \in f)$

$(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \dots$

- |     |   |
|-----|---|
| (1) | $Domain(f) := X; Codomain(f) := Y$  |
| (2) | $Image(f, A) := \{f(a) : a \in A\}; Preimage(f, B) := \{a : f(a) \in B\}$ |
| (3) | $Range(f) := Image(Domain(f))$  |

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X} (x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$   
 $Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X} (y = f(x))$   
 $Bijjective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$   
 $SurjectiveEquivalent := (Range(f) = Codomain(f)) \implies Surjective(f)$

$(Function(f, X, Y) \wedge Function(g, Y, Z)) \implies (f \circ g)(x) := f(g(x)); Function(f \circ g, X, Z)$

$PropertiesofFunctions := (Function(f, A, B) \wedge Function(g, B, C) \wedge Function(h, C, D)) \implies \dots$

- |     |  |
|-----|--|
| (1) | $h \circ (g \circ f) = (h \circ g) \circ f$  |
| (2) | $(Injective(f) \wedge Injective(g)) \implies Injective(g \circ f)$   |
| (3) | $(Surjective(f) \wedge Surjective(g)) \implies Surjective(g \circ f)$  |
| (4) | $(Bijjective(f, A, B)) \implies \exists_{f^{-1}} (Function(f^{-1}, B, A) \wedge \forall_{a \in A} (f^{-1}(f(a)) = a) \wedge \forall_{b \in B} (f(f^{-1}(b)) = b))$ |

$|(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}} (b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \dots$

- |     |   |
|-----|---|
| (1) | $(a b) \implies a bc$                                 |
| (2) | $(a b \wedge b c) \implies a c$                       |
| (3) | $(a b \wedge b c) \implies a (bx + cy)$               |
| (4) | $(a b \wedge b a) \implies a = \pm b$                 |
| (5) | $(a b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$ |
| (6) | $(a b) \iff (m \neq 0 \wedge ma mb)$                  |

$DivisionAlgorithm := (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}} (b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \wedge a : b \wedge a : c$   
 $GCD(a, b, c) := CD(a, b, c) \wedge \forall_d ((d : b \wedge d : c) \implies d : a)$   
 $GCDEquivalent := 123123$





# Chapter 3

## Linear Algebra

$Matrix[A, m, n] := [a_{i,j}]_{m \times n} := m \text{ rows, } n \text{ columns of real numbers}$

$\mathcal{M}_{m,n} := \{A : Matrix[A, m, n]\}$

$O_{m,n} := (Matrix[O, m, n]) \wedge (a_{i,j} = 0)$

$Square[A, n] := Matrix[A, n, n]$

$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$

$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$

$Diagonal[A, n] := (Square[A, n]) \wedge (i \neq j \implies a_{i,j} = 0)$

$Scalar[A, n, k] := (Diagonal[A, n]) \wedge (a_{i,i} = k)$

$I_n := Scalar[I, n, 1]$

$+(A, B) := ((Matrix[A, m, n]) \wedge (Matrix[B, m, n])) \implies (A + B = [a_{i,j} + b_{i,j}]_{m \times n})$

$*(r, A) := ((r \in \mathbb{R}) \wedge (Matrix[A, m, n])) \implies (r * A = [ra_{i,j}]_{m \times n})$

$*(A, B) := ((Matrix[A, m, p]) \wedge (Matrix[B, p, n])) \implies (A * B = \left[ \sum_{k=1}^p (a_{i,k} b_{k,j}) \right]_{m \times n})$

$^T[A] := (Matrix[A, m, n]) \implies (A^T = [a_{j,i}]_{n \times m})$

$AddCom := \forall_{A,B \in \mathcal{M}} (A + B = B + A)$

(1)  $A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C \in \mathcal{M}} ((A + B) + C = A + (B + C))$

(1)  $(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A \in \mathcal{M}} \exists!_{O \in \mathcal{M}} (A + O = A = O + A)$

(1)  $A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$

(2)  $A + O_1 = A = A + O_2 \quad \blacksquare \quad O_1 = O_2$

$AddInv := \forall_{A \in \mathcal{M}} \exists!_{(-A) \in \mathcal{M}} (A + (-A) = O = (-A) + A)$

(1)  $A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$

(2)  $A + (-A_1) = O = A + (-A_2) \quad \blacksquare \quad -A_1 = -A_2 \quad \blacksquare \quad A_1 = A_2$

$MulAssoc := \forall_{A,B,C \in \mathcal{M}} ((A * B) * C = A * (B * C))$

(1)  $(A * B) * C = \left[ \sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,j}) \right] * C = \left[ \sum_{k_2=1}^{p_2} (\sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,k_2}) c_{k_2,j}) \right] = \left[ \sum_{k_2=1}^{p_2} \sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,k_2} c_{k_2,j}) \right] = \dots$

(2)  $\dots \left[ \sum_{k_1=1}^{p_1} \sum_{k_2=1}^{p_2} (a_{i,k_1} b_{k_1,k_2} c_{k_2,j}) \right] = \left[ \sum_{k_1=1}^{p_1} (a_{i,k_1} \sum_{k_2=1}^{p_2} (b_{k_1,k_2} c_{k_2,j})) \right] = \dots = A * (B * C)$

$MulId := \forall_{A: Square[A,n]} (A * I_n = A = I_n * A)$

(1)  $A * I_n = \left[ \sum_{k=1}^n \left( a_{i,k} \left( \begin{cases} 1 & k=j \\ 0 & k \neq j \end{cases} \right) \right) \right] = [a_{i,j}] = A$

(2)  $TODO = A$

---


$$\text{ScalAssoc} := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} (r(sA) = (rs)A = s(rA))$$


---

$$(1) \quad r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$$


---

$$(2) \quad (rs)A = [rsa_{i,j}]$$


---

$$(3) \quad s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$$


---

---


$$\text{TransCancel} := \forall_{A \in \mathcal{M}} (A = (A^T)^T)$$


---

$$(1) \quad A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$$


---

---


$$\text{ScalMulCom} := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} ((rA) * B = r(A * B) = A * (rB))$$


---

$$(1) \quad (rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[ \sum_{k=1}^p (ra_{i,k} b_{k,j}) \right] = r(A * B)$$


---

$$(2) \quad A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[ \sum_{k=1}^p (a_{i,k} rb_{k,j}) \right] = \left[ \sum_{k=1}^p (ra_{i,k} b_{k,j}) \right] = r(A * B)$$


---

---


$$\text{ScalDistLeft} := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} ((r+s)A = rA + sA)$$


---

$$(1) \quad \text{TODO}$$


---

---


$$\text{ScalDistRight} := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} (r(A+B) = rA + rB)$$


---

$$(1) \quad \text{TODO}$$


---

---


$$\text{MulDistRight} := \forall_{A,B,C \in \mathcal{M}} ((A+B) * C = A * C + B * C)$$


---

$$(1) \quad (A+B) * C = [a_{i,j} + b_{i,j}] * C = \left[ \sum_{k=1}^p ((a_{i,k} + b_{i,k}) c_{k,j}) \right] = \dots$$


---

$$(2) \quad \dots \left[ \sum_{k=1}^p (a_{i,k} c_{k,j} + b_{i,k} c_{k,j}) \right] = \left[ \sum_{k=1}^p (a_{i,k} c_{k,j}) \right] + \left[ \sum_{k=1}^p (b_{i,k} c_{k,j}) \right] = A * C + B * C$$


---

---


$$\text{MulDistLeft} := \forall_{A,B,C \in \mathcal{M}} (C * (A+B) = C * A + C * B)$$


---

$$(1) \quad \text{TODO}$$


---

---


$$\text{TransAddDist} := \forall_{A,B \in \mathcal{M}} ((A+B)^T = A^T + B^T)$$


---

$$(1) \quad \text{TODO}$$


---

---


$$\text{TransMulDist} := \forall_{A,B \in \mathcal{M}} ((A * B)^T = B^T * A^T)$$


---

$$(1) \quad (A * B)^T = \left[ \sum_{k=1}^p (a_{i,k} b_{k,j}) \right]^T = \left[ \sum_{k=1}^p (a_{j,k} b_{k,i}) \right] = \left[ \sum_{k=1}^p (b_{k,i} a_{j,k}) \right] = \left[ \sum_{k=1}^p (b_{i,k}^T a_{k,j}^T) \right] = B^T * A^T$$


---

---


$$\text{Sym}[A] := A = A^T$$

$$\text{SkewSym}[A] := A = -A^T$$

$$\text{Invertible}[A] := (\text{Square}[A, n]) \wedge (\exists_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A))$$

---


$$\text{SymGen} := \forall_{A \in \mathcal{M}} (\text{Sym}[A + A^T])$$


---

$$(1) \quad (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$


---

---


$$\text{SkewSymGen} := \forall_{A \in \mathcal{M}} (\text{SkewSym}[A - A^T])$$


---

$$(1) \quad -(A - A^T)^T = -(A^T - (A^T)^T) = -(A^T - A) = (A - A^T)$$


---

---


$$\text{SymDecomp} := \forall_{A \in \mathcal{M}} \exists!_{B: \text{Sym}[B]} \exists!_{C: \text{SkewSym}[C]} (A = B + C)$$


---

$$(1) \quad B := (1/2) * (A + A^T) ; C := (1/2) * (A - A^T)$$


---

$$(2) \quad \text{SymGen}[B] \wedge \text{SkewSymGen}[C]$$


---

$$(3) \quad A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$$


---

---


$$(4) \quad (1/2) * (A_1 + A_1^T) = (1/2) * (A_2 + A_2^T) \quad \blacksquare \quad A_1 = A_2$$


---

$$(5) \quad (1/2) * (A_3 - A_3^T) = (1/2) * (A_4 - A_4^T) \quad \blacksquare \quad A_3 = A_4$$


---

---


$$InvId := \forall_{A: Invertible[A]} (\exists!_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A))$$


---

$$(1) \quad A^{-1}_1 = A^{-1}_1 * I_n = A^{-1}_1 * (A * A^{-1}_2) = (A^{-1}_1 * A) * A^{-1}_2 = I_n * A^{-1}_2 = A^{-1}_2$$


---

---


$$InvCancel := \forall_{A: Invertible[A]} ((A^{-1})^{-1} = A)$$


---

$$(1) \quad (A * A^{-1})^{-1} = I_n^{-1} = I_n$$


---

$$(2) \quad (A^{-1})^{-1} * A^{-1} = I_n \quad \blacksquare \quad A^{-1})^{-1} = I_n * A = A$$


---

---


$$InvDist := \forall_{A: Invertible[A]} \forall_{B: Invertible[B]} ((A * B)^{-1} = B^{-1} * A^{-1})$$


---

$$(1) \quad (A * B) * (A * B)^{-1} = I \quad \blacksquare \quad B * (A * B)^{-1} = A^{-1} \quad \blacksquare \quad (A * B)^{-1} = B^{-1} * A^{-1}$$


---

---


$$InvTrans := \forall_{A: Invertible[A]} ((A^T)^{-1} = (A^{-1})^T) \quad \blacksquare \quad \Leftarrow$$


---

$$(1) \quad A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I \quad \blacksquare \quad (A^{-1})^T = (A^T)^{-1}$$


---

$$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$$

$$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$$

$$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X (Sol[X, A, B])$$

$$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$$

$$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$$

---


$$HomoSysProps := (Sys[A, O]) \implies \dots$$


---

$$(1) \quad u_0 := O ; u_1 := choice(\{X \in \mathcal{M} | X \neq O\}) ; k := choice(\mathbb{R})$$


---

$$(2) \quad TrivSol[u_0, A]$$


---

$$(3) \quad (NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$$


---

$$(4) \quad (TrivSol[\vec{X}, A]) \implies (TrivSol[LC(\vec{X}), A])$$


---

$$ElemMat[E] := (E = Swap[I_n, i, j]) \vee (Scale_*(I_n, i, c)) \vee (Combine_*(I_n, i, c, j))$$

$$ElemMatProd[E^*] := \exists_{\langle E \rangle} (\forall_{E_i \in E^*} (ElemMat[E_i]) \wedge (E^* = \Pi_{E_i \in E^*} (E_i)))$$

$$RowEquiv[A, B] := \exists_{E^*} ((ElemMatProd[E^*]) \wedge (B = E^* * A))$$

---


$$ElemMatInv := \forall_{E \in \mathcal{M}} ((ElemMat[E]) \implies (Invertible[E]))$$


---

$$(1) \quad E - RowSwap[E] \implies TODO ; E - RowScale_*(E) \implies TODO ; E - RowCombine_*(E) \implies TODO$$


---

---


$$ElemMatProdInv := \forall_{E^*} ((ElemMatProd[E^*]) \implies (Invertible[E^*]))$$


---

$$(1) \quad TODO$$


---

---


$$RowEquivSys := \forall_{A, B, C, D, X \in \mathcal{M}} (((Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]])) \implies (Sol[X, A, B] \iff Sol[X, C, D]))$$


---

$$(1) \quad \exists_{E^*: ElemMatProd[E^*]} ([CD] = E^* * [AB])$$


---

$$(2) \quad (E^* * A = C) \wedge (E^* * B = D)$$


---

$$(3) \quad Sol[Y, A, B] \implies \dots$$


---

$$(3.1) \quad A * Y = B$$


---

$$(3.2) \quad C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D \quad \blacksquare \quad Sol[Y, C, D]$$


---

$$(4) \quad Sol[Y, A, B] \implies Sol[Y, C, D]$$


---

$$(5) \quad (A = (E^*)^{-1} * C) \wedge (B = (E^*)^{-1} * D)$$


---

$$(6) \quad Sol[Z, C, D] \implies \dots$$


---

$$(6.1) \quad C * Z = D$$


---

$$(6.2) \quad A * Z = ((E^*)^{-1} * C) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$$


---

$$(7) \quad Sol[Z, C, D] \implies Sol[Z, A, B]$$


---

$$(8) \quad Sol[X, A, B] \iff Sol[X, C, D]$$


---

$$\text{RowEquivHomoSysSol} := \forall_{A,C,X \in \mathcal{M}} ((\text{RowEquiv}[A, C]) \implies ((\text{Sol}[X, A, O]) \iff (\text{Sol}[X, C, O])))$$

$$(1) \quad \text{Set } B = D = O$$

$$\text{RREF}[A] := (A \in \mathcal{M}) \wedge \left( \begin{array}{l} \text{All zero rows are at the bottom of the matrix.} \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} \\ \text{The leading entry in any nonzero row is 1.} \\ \text{All entries in the column above and below a leading 1 are zero.} \end{array} \right)$$

$$\text{GaussJordanElim} := \forall_{A \in \mathcal{M}} \exists!_{B \in \mathcal{M}} ((\text{RREF}[B]) \wedge (\text{RowEquiv}[A, B]))$$

$$(1) \quad \text{Hit } A \text{ with } \text{ElemMat}'\text{'s until it becomes } B$$

$$(2) \quad (B = E^* * A) \wedge (\text{RREF}[B])$$

$$\text{HasZero}[A] := (\text{Matrix}(A, m, n)) \wedge (\exists_{i \leq m} (A_{i,:} = O))$$

$$\text{HasZeroNonInvertible} := \forall_{A \in \mathcal{M}} ((\text{HasZero}[A]) \implies (\neg \text{Invertible}[A]))$$

$$(1) \quad i := \text{choice}(\{i \leq m \mid A_{i,:} = O\})$$

$$(2) \quad (B \in \mathcal{M}) \implies \dots$$

$$(2.1) \quad (A * B)_{i,:} = O \neq I_{ni,:} \quad \blacksquare \quad A * B \neq I_n$$

$$(3) \quad (B \in \mathcal{M}) \implies (A * B \neq I_n) \quad \blacksquare \quad \forall_{B \in \mathcal{M}} (A * B \neq I_n) \quad \blacksquare \quad \neg \text{Invertible}[A]$$

$$\text{InvIf f RowEquivI} := \forall_{A \in \mathcal{M}} ((\text{Invertible}[A]) \iff (\text{RowEquiv}[A, I_n]))$$

$$(1) \quad (\text{Invertible}[A]) \implies \dots$$

$$(1.1) \quad (\text{RREF}[B]) \wedge (\text{RowEquiv}[A, B])$$

$$(1.2) \quad B = E^* * A$$

$$(1.3) \quad (\text{Invertible}[E^*]) \wedge (\text{Invertible}[A]) \quad \blacksquare \quad \text{Invertible}[B]$$

$$(1.4) \quad \text{Invertible}[B] \quad \blacksquare \quad \neg \text{HasZero}[B]$$

$$(1.5) \quad (\text{RREF}[B]) \wedge (\neg \text{HasZero}[B]) \quad \blacksquare \quad B = I_n$$

$$(1.6) \quad \text{RowEquiv}[A, I_n]$$

$$(2) \quad (\text{Invertible}[A]) \implies (\text{RowEquiv}[A, I_n])$$

$$(3) \quad (\text{RowEquiv}[A, I_n]) \implies \dots$$

$$(3.1) \quad I_n = E^* * A \quad \blacksquare \quad (E^*)^{-1} = A$$

$$(3.2) \quad A^{-1} = E^*_{\text{DescSort}} \quad \blacksquare \quad \text{Invertible}[A]$$

$$(4) \quad (\text{RowEquiv}[A, I_n]) \implies (\text{Invertible}[A])$$

$$(5) \quad (\text{Invertible}[A]) \iff (\text{RowEquiv}[A, I_n])$$

$$\text{RowEquivIf f TrivSol} := \forall_{A \in \mathcal{M}} ((\text{RowEquiv}[A, I_n]) \iff (\forall_X ((X = O) \iff (\text{Sol}[X, A, O])))$$

$$(1) \quad (\text{RowEquiv}[A, I_n]) \implies \dots$$

$$(1.1) \quad \text{RowEquiv}[A, I_n] \quad \blacksquare \quad \text{Invertible}[A]$$

$$(1.2) \quad (\text{Sol}[X, A, O]) \implies \dots$$

$$(1.2.1) \quad A * X = O \quad \blacksquare \quad X = A^{-1} * O = O \quad \blacksquare \quad X = O$$

$$(1.3) \quad (\text{Sol}[X, A, O]) \implies (X = O)$$

$$(1.4) \quad (X = O) \implies (\text{Sol}[X, A, O])$$

$$(1.5) \quad (X = O) \iff (\text{Sol}[X, A, O]) \quad \blacksquare \quad \forall_X ((X = O) \iff (\text{Sol}[X, A, O]))$$

$$(2) \quad (\text{RowEquiv}[A, I_n]) \implies (\forall_X ((X = O) \iff (\text{Sol}[X, A, O])))$$

$$(3) \quad (\forall_X ((X = O) \iff (\text{Sol}[X, A, O]))) \implies \dots$$

$$(3.1) \quad (\text{RREF}[B]) \wedge (\text{RowEquiv}[A, B])$$

$$(3.2) \quad \text{Sol}[X, B, O]$$

$$(3.3) \quad (B \neq I_n) \implies \dots$$

$$(3.3.1) \quad (\exists_{Y \neq X} (\text{Sol}[Y, B, O]))$$

$$(3.3.2) \quad \text{Sol}[Y, A, O] \quad \blacksquare \quad Y = X$$

$$(3.3.3) \quad (Y \neq X) \wedge (Y = X) \quad \blacksquare \quad \perp$$

$$(3.4) \quad (B \neq I_n) \implies \perp \quad \blacksquare \quad B = I_n$$

$$(3.5) \quad (RowEquiv[A, B]) \wedge (B = I_n) \quad \blacksquare \quad RowEquiv[A, I_n]$$

$$(4) \quad (\forall_X((X = O) \iff (Sol[X, A, O]))) \implies (RowEquiv[A, I_n])$$

$$(5) \quad (RowEquiv[A, I_n]) \iff (\forall_X((X = O) \iff (Sol[X, A, O])))$$

$$InvIf fUniqSol := \forall_{A \in \mathcal{M}}((Invertible[A]) \iff (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])))$$

$$(1) \quad (Invertible[A] \wedge B \in \mathcal{M}) \implies \dots$$

$$(1.1) \quad (Invertible[A]) \wedge (Sys[A, B])$$

$$(1.2) \quad (X = A^{-1} * B) \iff (Sol[X, A, B]) \quad \blacksquare \quad \exists!_{X \in \mathcal{M}}(Sol[X, A, B])$$

$$(2) \quad (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies \dots$$

$$(2.1) \quad X_i := choice(\{X_i | Sol[X_i, A, I_{n:,i}]\})$$

$$(2.2) \quad A * [X_1 \dots X_n] = [(A * X_1) \dots (A * X_n)] = [I_{n:,1} \dots I_{n:,n}] = I_n$$

$$(2.3) \quad A^{-1} = [X_1 \dots X_n]$$

$$(3) \quad (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies (Invertible[A])$$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}} \left( \begin{array}{c} (Invertible[A]) \iff \\ (RowEquiv[A, I_n]) \iff \\ (\forall_X((X = O) \iff (Sol[X, A, O]))) \iff \\ (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \end{array} \right)$$

$$VectorSpace[V, +, *] := \forall_{\alpha, \beta \in \mathbb{R}} \forall_{u, v, w \in V} \exists_{O \in V} \left( \begin{array}{c} (u + v \in V) \wedge (u + v = v + u) \wedge ((u + v) + w = u + (v + w)) \wedge \\ (u + O = u) \wedge (\exists_{-u \in V}(u + (-u) = O)) \wedge \\ (\alpha * u \in V) \wedge (\alpha * (\beta * u) = (\alpha\beta) * u) \wedge (1 * u = u) \wedge \\ (\alpha * (u + v) = (\alpha * u) + (\alpha * v)) \wedge ((\alpha + \beta) * u = (\alpha * u) + (\beta * u)) \end{array} \right)$$

$$ZeroVectorUniq := \forall_{u, v \in V}((v + u = v) \implies (u = O))$$

$$(1) \quad (v + u = v) \quad \blacksquare \quad -v + v + u = -v + v \quad \blacksquare \quad u = O$$

$$ZeroVectorGen := \forall_{\alpha \in \mathbb{R}} \forall_{u \in V}((\alpha * u = O) \iff ((u = O) \vee (\alpha = 0)))$$

$$(1) \quad (u = O) \implies \dots$$

$$(1.1) \quad \alpha * u = \alpha * O = \alpha * (O + O) = (\alpha * O) + (\alpha * O)$$

$$(1.2) \quad \alpha * O = (\alpha * O) + (\alpha * O) \quad \blacksquare \quad \alpha * O = O \quad \blacksquare \quad \alpha * u = O$$

$$(2) \quad (u = O) \implies (\alpha * u = O)$$

$$(3) \quad (\alpha = 0) \implies \dots$$

$$(3.1) \quad \alpha * u = 0 * u = (0 + 0) * u = (0 * u) + (0 * u)$$

$$(3.2) \quad 0 * u = (0 * u) + (0 * u) \quad \blacksquare \quad 0 * u = O \quad \blacksquare \quad \alpha * u = O$$

$$(4) \quad (\alpha = 0) \implies (\alpha * u = O)$$

$$(5) \quad ((u = O) \implies (\alpha * u = O)) \wedge ((\alpha = 0) \implies (\alpha * u = O)) \quad \blacksquare \quad ((u = O) \vee (\alpha = 0)) \implies (\alpha * u = O)$$

$$(6) \quad (\alpha * u = O) \implies \dots$$

$$(6.1) \quad (\alpha \neq 0) \implies \dots$$

$$(6.1.1) \quad \alpha^{-1} \in \mathbb{R}$$

$$(6.1.2) \quad O = \alpha^{-1} * O = \alpha^{-1} * (\alpha * u) = (\alpha^{-1} * \alpha) * u = 1 * u = u \quad \blacksquare \quad u = O$$

$$(6.2) \quad (\alpha \neq 0) \implies (u = O)$$

$$(6.3) \quad (\alpha = 0) \vee (\alpha \neq 0) \quad \blacksquare \quad (\alpha = 0) \vee (u = O)$$

$$(7) \quad (\alpha * u = O) \implies ((\alpha = 0) \vee (u = O))$$

$$(8) \quad (\alpha * u = O) \iff ((\alpha = 0) \vee (u = O))$$

$$NegVectorGen := \forall_{u \in V}((-1) * u = -u)$$

$$(1) \quad O = 0 * u = (1 + (-1)) * u = (1 * u) + ((-1) * u) = u + ((-1) * u) \quad \blacksquare \quad O = u + ((-1) * u) \quad \blacksquare \quad -u = (-1) * u$$

$$Subspace[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (\emptyset \neq S \subseteq V) \wedge (VectorSpace[S, +, *])$$

$$SubspaceEquiv := \forall_{V: VectorSpace[V, +, *]} \forall_{S: \emptyset \neq S \subseteq V}((Subspace[S, V, +, *]) \iff ((\forall_{r, s \in S}(r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S))))$$

(1)	$(Subspace[S, V, +, *]) \implies \dots$
(1.1)	$VectorSpace[S, +, *] \blacksquare (\forall_{r,s \in S}(r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S))$
(2)	$(Subspace[S, V, +, *]) \implies ((\forall_{r,s \in S}(r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)))$
(3)	$((\forall_{r,s \in S}(r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S))) \implies \dots$
(3.1)	$((\alpha, \beta \in \mathbb{R}) \wedge (u, v, w \in S)) \implies \dots$
(3.1.1)	$u, v \in V \blacksquare u + v = v + u$
(3.1.2)	$u, v, w \in V \blacksquare (u + v) + w = u + (v + w)$
(3.1.3)	$u \in S \blacksquare 0 * u = O \in S$
(3.1.4)	$u \in V \blacksquare u + O = u$
(3.1.5)	$u \in S \blacksquare (-1) * u = -u \in S$
(3.1.6)	$u \in V \blacksquare \alpha * (\beta * u) = (\alpha\beta) * u$
(3.1.7)	$u \in V \blacksquare 1 * u = u$
(3.1.8)	$u, v \in V \blacksquare \alpha * (u + v) = (\alpha * u) + (\alpha * v)$
(3.1.9)	$u \in V \blacksquare (\alpha + \beta) * u = (\alpha * u) + (\beta * u)$
(3.2)	$((\alpha, \beta \in \mathbb{R}) \wedge (u, v, w \in S)) \implies \dots$
(3.3)	$(VectorSpace[S, +, *]) \wedge (\emptyset \neq S \subseteq V) \wedge (VectorSpace[V, S, +, *]) \blacksquare Subspace[S, V, +, *]$
(4)	$((\forall_{r,s \in S}(r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S))) \implies (Subspace[S, V, +, *])$
(5)	$(Subspace[S, V, +, *]) \iff ((\forall_{r,s \in S}(r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)))$

123123  $LinearCombination[c, U, V, +, *] := (VectorSpace[V, +, *]) \wedge (\emptyset \neq U \subseteq V) \wedge (\exists_{\Gamma \subseteq \mathbb{R}}(c = \sum_{u_i \in U}(\gamma_i * u_i)))$

TRASH CAN