

# Contents



# Chapter 1

## Real Analysis

(1.5)

*OrderTrichotomy* $[\prec, S] := \forall_{x,y \in S} (x \prec y \vee x = y \vee y \prec x)$

*OrderTransitivity* $[\prec, S] := \forall_{x,y,z \in S} ((x \prec y \wedge y \prec z) \implies x \prec z)$

*Order* $[\prec, S] := (\textcolor{teal}{OrderTrichotomy}[\prec, S]) \wedge (\textcolor{teal}{OrderTransitivity}[\prec, S])$

(1.7)

*Bounded Above* $[E, S, \prec] := (\textcolor{teal}{Order}[\prec, S]) \wedge (E \subset S) \wedge (\exists_{\beta \in S} \forall_{x \in E} (x \leq \beta))$

*Bounded Below* $[E, S, \prec] := (\textcolor{teal}{Order}[\prec, S]) \wedge (E \subset S) \wedge (\exists_{\beta \in S} \forall_{x \in E} (\beta \leq x))$

*Upper Bound* $[\beta, E, S, \prec] := (\textcolor{teal}{Order}[\prec, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E} (x \leq \beta))$

*Lower Bound* $[\beta, E, S, \prec] := (\textcolor{teal}{Order}[\prec, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E} (\beta \leq x))$

(1.8)

*LU B* $[\alpha, E, S, \prec] := (\textcolor{teal}{Upper Bound}[\alpha, E, S, \prec]) \wedge (\forall_{\gamma} (\gamma < \alpha \implies \neg \textcolor{teal}{Upper Bound}[\gamma, E, S, \prec]))$

*GLB* $[\alpha, E, S, \prec] := (\textcolor{teal}{Lower Bound}[\alpha, E, S, \prec]) \wedge (\forall_{\beta} (\alpha < \beta \implies \neg \textcolor{teal}{Lower Bound}[\beta, E, S, \prec]))$

(1.10)

*LU BProperty* $[S, \prec] := \forall_E (((\emptyset \neq E \subset S) \wedge (\textcolor{teal}{Bounded Above}[E, S, \prec]) \implies \exists_{\alpha \in S} (\textcolor{teal}{LU B}[\alpha, E, S, \prec])))$

*GLBProperty* $[S, \prec] := \forall_E (((\emptyset \neq E \subset S) \wedge (\textcolor{teal}{Bounded Below}[E, S, \prec]) \implies \exists_{\alpha \in S} (\textcolor{teal}{GLB}[\alpha, E, S, \prec])))$

(1.11)

*LU BPropertyImpliesGLBProperty*  $:= \textcolor{teal}{LU BProperty}[S, \prec] \implies \textcolor{teal}{GLBProperty}[S, \prec]$

(1)  $\textcolor{teal}{LU BProperty}[S, \prec] \implies \dots$

wts: 2

(1.1)  $(\emptyset \neq B \subset S \wedge \textcolor{teal}{Bounded Below}[B, S, \prec]) \implies \dots$

wts: 1.2

from: *Bounded Below*, 1.1

(1.1.1)  $\textcolor{teal}{Order}[\prec, S] \wedge \exists_{\delta' \in S} (\textcolor{teal}{Lower Bound}[\delta', B, S, \prec])$

(1.1.2)  $|B| = 1 \implies \dots$

wts: 1.1.3

from: 1.1.2

(1.1.2.1)  $\exists_{u'} (u' \in B) \blacksquare u := \textit{choice}(\{u' : u' \in B\}) \blacksquare B = \{u\}$

(1.1.2.2)  $\textcolor{teal}{GLB}[u, B, S, \prec] \blacksquare \exists_{\epsilon_0 \in S} (\textcolor{teal}{GLB}[\epsilon_0, B, S, \prec])$

(1.1.3)  $|B| = 1 \implies \exists_{\epsilon_0 \in S} (\textcolor{teal}{GLB}[\epsilon_0, B, S, \prec])$

(1.1.4)  $|B| \neq 1 \implies \dots$

wts: 1.1.5

from: *LU BProperty*, 1

(1.1.4.1)  $\forall_E ((\emptyset \neq E \subset S \wedge \textcolor{teal}{Bounded Above}[E, S, \prec]) \implies \exists_{\alpha \in S} (\textcolor{teal}{LU B}[\alpha, E, S, \prec]))$

(1.1.4.2)  $L := \{s \in S : \textcolor{teal}{Lower Bound}[s, B, S, \prec]\}$

(1.1.4.3)  $|B| > 1 \wedge \textcolor{teal}{OrderTrichotomy}[\prec, S] \blacksquare \exists_{b_1' \in B} \exists_{b_0' \in B} (b_0' < b_1')$

from: *Order*, 1.1.1  
wts: 1.1.4.7

(1.1.4.4)  $b_1 := \textit{choice}(\{b_1' \in B : \exists_{b_0' \in B} (b_0' < b_1')\}) \blacksquare \neg \textcolor{teal}{Lower Bound}[b_1, B, S, \prec]$

from: 1.1.4.2

(1.1.4.5)  $b_1 \notin L \blacksquare L \subset S$

(1.1.4.6)  $\delta := \textit{choice}(\{\delta' \in S : \textcolor{teal}{Lower Bound}[\delta', B, S, \prec]\}) \blacksquare \delta \in L \blacksquare \emptyset \neq L$

from: 1.1.1

(1.1.4.7)  $\emptyset \neq L \subset S$

from: 1.1.4.5, 1.1.4.6

(1.1.4.8)  $\forall_{y \in L} (\textcolor{teal}{Lower Bound}[y_0, B, S, \prec]) \blacksquare \forall_{y \in L} \forall_{x \in B} (y_0 \leq x)$

from: *Lower Bound*, 1.1.4.2  
wts: 1.1.4.10

(1.1.4.9)  $\forall_{x \in B} (x \in S \wedge \forall_{y \in L} (y_0 \leq x)) \blacksquare \forall_{x \in B} (\textcolor{teal}{Upper Bound}[x, L, S, \prec])$

from: *Upper Bound*

(1.1.4.10)  $\exists_{x \in S} (\textcolor{teal}{Upper Bound}[x, L, S, \prec]) \blacksquare \textcolor{teal}{Bounded Above}[L, S, \prec]$

(1.1.4.11)	$\emptyset \neq L \subset S \wedge \text{BoundedAbove}[L, S, <]$	from: 1.1.4.7, 1.1.4.10
(1.1.4.12)	$\exists_{\alpha' \in S}(\text{LUB}[\alpha', L, S, <]) \blacksquare \alpha := \text{choice}(\{\alpha' \in S : (\text{LUB}[\alpha', L, S, <])\})$	from: 1.1.4.1 wts: 1.1.4.21
(1.1.4.13)	$\forall_x(x \in B \implies \text{UpperBound}[x, L, S, <])$	from: 1.1.4.9 wts: 1.1.4.17
(1.1.4.14)	$\forall_x(\neg \text{UpperBound}[x, L, S, <] \implies x \notin B)$	
(1.1.4.15)	$\gamma < \alpha \implies \dots$	wts: 1.1.4.16
(1.1.4.15.1)	$\neg \text{UpperBound}[\gamma, L, S, <] \blacksquare \gamma \notin B$	from: LUB, 1.1.4.12, 1.1.4.14
(1.1.4.16)	$\gamma < \alpha \implies \gamma \notin B \blacksquare \gamma \in B \implies \gamma \geq \alpha$	
(1.1.4.17)	$\forall_{\gamma \in B}(\alpha \leq \gamma) \blacksquare \text{LowerBound}[\alpha, B, S, <]$	from: LowerBound
(1.1.4.18)	$\alpha < \beta \implies \dots$	wts: 1.1.4.19
(1.1.4.18.1)	$\forall_{y \in L}(y_0 \leq \alpha < \beta) \blacksquare \forall_{y \in L}(y_0 \neq \beta)$	from: LUB, 1.1.4.12, 1.1.4.18
(1.1.4.18.2)	$\beta \notin L \blacksquare \neg \text{LowerBound}[\beta, B, S, <]$	from: 1.1.4.2
(1.1.4.19)	$\alpha < \beta \implies \neg \text{LowerBound}[\beta, B, S, <] \blacksquare \forall_{\beta \in S}(\alpha < \beta \implies \neg \text{LowerBound}[\beta, B, S, <])$	
(1.1.4.20)	$\text{LowerBound}[\alpha, B, S, <] \wedge \forall_{\beta \in S}(\alpha < \beta \implies \neg \text{LowerBound}[\beta, B, S, <])$	from: 1.1.4.17, 1.1.4.19
(1.1.4.21)	$\text{GLB}[\alpha, B, S, <] \blacksquare \exists_{\epsilon_1 \in S}(\text{GLB}[\epsilon_1, B, S, <])$	
(1.1.5)	$ B  \neq 1 \implies \exists_{\epsilon_1 \in S}(\text{GLB}[\epsilon_1, B, S, <])$	
(1.1.6)	$( B  = 1 \implies \exists_{\epsilon_0 \in S}(\text{GLB}[\epsilon_0, B, S, <])) \wedge ( B  \neq 1 \implies \exists_{\epsilon_1 \in S}(\text{GLB}[\epsilon_1, B, S, <]))$	from: 1.1.3, 1.1.5
(1.1.7)	$( B  = 1 \vee  B  \neq 1) \implies \exists_{\epsilon \in S}(\text{GLB}[\epsilon, B, S, <]) \blacksquare \exists_{\epsilon \in S}(\text{GLB}[\epsilon, B, S, <])$	
(1.2)	$(\emptyset \neq B \subset S \wedge \text{BoundedBelow}[B, S, <]) \implies \exists_{\epsilon \in S}(\text{GLB}[\epsilon, B, S, <])$	
(1.3)	$\forall_B((\emptyset \neq B \subset S \wedge \text{BoundedBelow}[B, S, <]) \implies \exists_{\epsilon \in S}(\text{GLB}[\epsilon, B, S, <]))$	
(1.4)	$\text{GLBProperty}[S, <]$	
(2)	$\text{LUBProperty}[S, <] \implies \text{GLBProperty}[S, <]$	

(1.12)

$$\text{Field}[F, +, *] := \exists_{0, 1 \in F} \forall_{x, y, z \in F} \left( \begin{array}{l} x + y \in F \quad \wedge \quad x * y \in F \quad \wedge \\ x + y = y + x \quad \wedge \quad x * y = y * x \quad \wedge \\ (x + y) + z = x + (y + z) \quad \wedge \quad (x * y) * z = x * (y * z) \quad \wedge \\ 1 \neq 0 \quad \wedge \quad x * (y + z) = (x * y) + (x * z) \quad \wedge \\ 0 + x = x \quad \wedge \quad 1 * x = x \quad \wedge \\ \exists_{-x \in F}(x + (-x) = 0) \wedge (x \neq 0 \implies \exists_{1/x \in F}(x * (1/x) = 1)) \end{array} \right)$$

\*\*\*\*\* (Field[F, +, \*]  $\wedge$   $x, y, z \in F$ )  $\implies$  ... \*\*\*\*\*

(1.14)

$$\text{AdditiveCancellation} := (x + y = x + z) \implies y = z$$

(1)	$y = 0 + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \dots$	from: Field
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(2)	$(-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = 0 + z = z$	from: Field
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$$\text{AdditiveIdentityUniqueness} := (x + y = x) \implies y = 0$$

(1)	$x + y = x = 0 + x = x + 0$	from: Field
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(2)	$y = 0$	from: AdditiveCancellation
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$$\text{AdditiveInverseUniqueness} := (x + y = 0) \implies y = -x$$

(1)	$x + y = 0 = x + (-x)$	from: Field
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(2)	$y = -x$	from: AdditiveCancellation
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$$\text{DoubleNegative} := x = -(-x)$$

(1)	$0 = x + (-x) = (-x) + x \blacksquare 0 = (-x) + x$	from: Field
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(2)	$x = -(-x)$	from: AdditiveInverseUniqueness
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(1.15)

$$\textcolor{red}{\textit{MultiplicativeCancellation}} := (x \neq 0 \wedge x * y = x * z) \implies y = z \quad \text{---}$$

$$\textcolor{red}{\textit{MultiplicativeIdentityUniqueness}} := (x \neq 0 \wedge x * y = x) \implies y = 1 \quad \text{---}$$

$$\textcolor{red}{\textit{MultiplicativeInverseUniqueness}} := (x \neq 0 \wedge x * y = 1) \implies y = 1/x \quad \text{---}$$

$$\textcolor{red}{\textit{DoubleReciprocal}} := (x \neq 0) \implies x = 1/(1/x) \quad \text{---}$$

(1.16)

$$\textcolor{red}{\textit{Domination}} := 0 * x = 0$$

$$(1) \quad 0 * x = (0 + 0) * x = 0 * x + 0 * x \quad \blacksquare \quad 0 * x = 0 * x + 0 * x$$

from: [Field](#)

$$(2) \quad 0 * x = 0$$

from: [AdditiveIdentityUniqueness](#)

$$\textcolor{red}{\textit{NonDomination}} := (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$$

$$(1) \quad (x \neq 0 \wedge y \neq 0) \implies \dots$$

$$(1.1) \quad (x * y = 0) \implies \dots$$

$$(1.1.1) \quad 1 = 1 * 1 = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = 0 * ((1/x) * (1/y)) = 0$$

from: [Field](#), [Domination](#), 1, 1.1

$$(1.1.2) \quad 1 = 0 \wedge 1 \neq 0 \quad \blacksquare \quad \perp$$

from: [Field](#)

$$(1.2) \quad (x * y = 0) \implies \perp \quad \blacksquare \quad x * y \neq 0$$

$$(2) \quad (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$$

$$\textcolor{red}{\textit{NegationCommutativity}} := (-x) * y = -(x * y) = x * (-y)$$

$$(1) \quad x * y + (-x) * y = (x + -x) * y = 0 * y = 0 \quad \blacksquare \quad x * y + (-x) * y = 0$$

from: [Field](#), [Domination](#)  
wts: 2

$$(2) \quad (-x) * y = -(x * y)$$

from: [AdditiveInverseUniqueness](#)

$$(3) \quad x * y + x * (-y) = x * (y_0 + -y) = x * 0 = 0 \quad \blacksquare \quad x * y + x * (-y) = 0$$

from: [Field](#), [Domination](#)  
wts: 4

$$(4) \quad x * (-y) = -(x * y)$$

from: [AdditiveInverseUniqueness](#)

$$(5) \quad (-x) * y = -(x * y) = x * (-y)$$

from: 2, 4

$$\textcolor{red}{\textit{NegativeMultiplication}} := (-x) * (-y) = x * y$$

$$(1) \quad (-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$$

from: [NegationCommutativity](#), [DoubleNegative](#)

(1.17)

$$\textcolor{red}{\textit{OrderedField}}[F, +, *, <] := \left( \begin{array}{l} \textcolor{blue}{\textit{Field}}[F, +, *] \quad \wedge \quad \textcolor{blue}{\textit{Order}}[<, F] \quad \wedge \\ \forall_{x,y,z \in F} (y_0 < z \implies x + y < x + z) \quad \wedge \\ \forall_{x,y \in F} ((x > 0 \wedge y > 0) \implies x * y > 0) \end{array} \right)$$

$$***** \textcolor{blue}{(\textit{OrderedField}[F, +, *, <] \wedge x, y, z \in F)} \implies \dots *****$$

(1.18)

$$\textcolor{red}{\textit{NegationOnOrder}} := x > 0 \iff -x < 0$$

$$(1) \quad x > 0 \implies \dots$$

$$(1.1) \quad 0 = (-x) + x > (-x) + 0 = -x \quad \blacksquare \quad 0 > -x \quad \blacksquare \quad -x < 0$$

from: [OrderedField](#)

$$(2) \quad x > 0 \implies -x < 0$$

$$(3) \quad -x < 0 \implies \dots$$

$$(3.1) \quad 0 = x + (-x) < x + 0 = x \quad \blacksquare \quad 0 < x \quad \blacksquare \quad x > 0$$

from: [OrderedField](#)

$$(4) \quad -x < 0 \implies x > 0$$

$$(5) \quad x > 0 \implies -x < 0 \wedge -x < 0 \implies x > 0 \quad \blacksquare \quad x > 0 \iff -x < 0$$

from: 2, 4

$$\textcolor{red}{\textit{PositiveFactorPreservesOrder}} := (x > 0 \wedge y < z) \implies x * y < x * z$$

$$(1) \quad (x > 0 \wedge y < z) \implies \dots$$

$$(1.1) \quad (-y) + z > (-y) + y = 0 \quad \blacksquare \quad z + (-y) = 0$$

from: [OrderedField](#)

$$(1.2) \quad x * (z + (-y)) > 0 \quad \blacksquare \quad x * z + x * (-y) > 0$$

from: [OrderedField](#)

$$(1.3) \quad x * z = 0 + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \dots$$

from: [Field](#), [NegationCommutativity](#)

$$(1.4) \quad x * y + (x * z + x * (-y)) > x * y + 0 = x * y$$

from: [Field](#), 1.2

$$(1.5) \quad x * z > x * y$$

from: 1.3, 1.4

$$(2) \quad (x > 0 \wedge y < z) \implies x * z > x * y$$

$$\textcolor{red}{NegativeFactorFlipsOrder} := (x < 0 \wedge y < z) \implies x * y > x * z$$

$$(1) \quad (x < 0 \wedge y < z) \implies \dots$$

$$(1.1) \quad -x > 0$$

from: [NegationOnOrder](#)

$$(1.2) \quad (-x) * y < (-x) * z \quad \blacksquare \quad 0 = x * y + (-x) * y < x * y + (-x) * z \quad \blacksquare \quad 0 < x * y + (-x) * z$$

from: [PositiveFactorPreservesOrder](#)

$$(1.3) \quad 0 < (-x) * (-y + z) \quad \blacksquare \quad 0 > x * (-y + z) \quad \blacksquare \quad 0 > -(x * y) + x * z$$

from: [NegationOnOrder](#)

$$(1.4) \quad x * y > x * z$$

$$(2) \quad (x < 0 \wedge y < z) \implies x * y > x * z$$

$$\textcolor{red}{SquareIsPositive} := (x \neq 0) \implies x * x > 0$$

$$(1) \quad (x > 0) \implies x * x > 0$$

from: [OrderedField](#)

$$(2) \quad (x < 0) \implies \dots$$

$$(2.1) \quad -x > 0 \quad \blacksquare \quad x * x = (-x) * (-x) > 0 \quad \blacksquare \quad x * x > 0$$

from: [NegationOnOrder](#), [OrderedField](#), [NegativeMultiplication](#)

$$(3) \quad (x < 0) \implies x * x > 0$$

$$(4) \quad x \neq 0 \implies (x > 0 \vee x < 0) \implies x * x > 0 \quad \blacksquare \quad x \neq 0 \implies x * x > 0$$

from: [OrderTrichotomy](#), 1, 3

$$\textcolor{red}{OneIsPositive} := 1 > 0$$

$$(1) \quad 1 \neq 0 \quad \blacksquare \quad 1 = 1 * 1 > 0$$

from: [Field](#), [SquareIsPositive](#)

$$\textcolor{red}{ReciprocationOnOrder} := (0 < x < y) \implies 0 < 1/y < 1/x$$

$$(1) \quad (0 < x < y) \implies \dots$$

$$(1.1) \quad x * (1/x) = 1 > 0 \quad \blacksquare \quad x * (1/x) > 0$$

from: [Field](#), [OneIsPositive](#)

$$(1.2) \quad 1/x < 0 \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp \quad \blacksquare \quad 1/x > 0$$

from: [NegativeFactorFlipsOrder](#), 1

$$(1.3) \quad y * (1/y) = 1 > 0 \quad \blacksquare \quad y * (1/y) > 0$$

from: [Field](#), [OneIsPositive](#)

$$(1.4) \quad 1/y < 0 \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp \quad \blacksquare \quad 1/y > 0$$

from: [NegativeFactorFlipsOrder](#), 1

$$(1.5) \quad (1/x) * (1/y) > 0$$

from: [OrderedField](#)

$$(1.6) \quad 0 < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$$

from: [OrderedField](#), 1, 1.4, 1.5

$$(1.19)$$

$$\textcolor{red}{OrderedFieldQ} := \textcolor{blue}{OrderedField}[\mathbb{Q}, +, *, <] \quad \text{---}$$

$$\textcolor{blue}{Subfield}[K, F, +, *] := \textcolor{blue}{Field}[F, +, *] \wedge K \subset F \wedge \textcolor{blue}{Field}[K, +, *]$$

$$\textcolor{blue}{OrderedSubfield}[K, F, +, *, <] := \textcolor{blue}{OrderedField}[F, +, *, <] \wedge K \subset F \wedge \textcolor{blue}{OrderedField}[K, +, *, <]$$

$$\textcolor{blue}{CutI}[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$$

$$\textcolor{blue}{CutII}[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$$

$$\textcolor{blue}{CutIII}[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$$

$$\mathbb{R} := \{\alpha \in \mathbb{Q} : \textcolor{blue}{CutI}[\alpha] \wedge \textcolor{blue}{CutII}[\alpha] \wedge \textcolor{blue}{CutIII}[\alpha]\}$$

$$\textcolor{red}{CutCorollaryI} := (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$$

$$(1) \quad (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \dots$$

$$(1.1) \quad \forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$$

from: [CutII](#), 1

$$(1.2) \quad q < p \implies q \in \alpha \quad \blacksquare \quad q \notin \alpha \implies q \geq p$$

from: 1

$$(1.3) \quad (q \notin \alpha) \implies \dots$$

$$(1.3.1) \quad q \geq p$$

from: 1.2

$$(1.3.2) \quad (q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp \quad \blacksquare \quad q \neq p$$

from: 1, 1.3

$$(1.3.3) \quad q \geq p \wedge q \neq p \quad \blacksquare \quad p < q$$

$$(1.4) \quad q \notin \alpha \implies p < q \quad \blacksquare \quad p < q$$

from: 1

$$(2) \quad (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$$

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**CutCorollaryI1**  $:= (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

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(1)  $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \dots$

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(1.1)  $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$

---

from: [CutI1](#), 1

(1.2)  $s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)) \blacksquare s \in \alpha \implies r \in \alpha$

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from: 1, 1.1

(1.3)  $r \notin \alpha \implies s \notin \alpha \blacksquare s \notin \alpha$

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from: 1, 1.2

(2)  $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

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$<_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha < \beta$

**OrderTrichotomyOfR**  $:= \text{OrderTrichotomy}[\mathbb{R}, <_{\mathbb{R}}]$

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(1)  $(\alpha, \beta \in \mathbb{R}) \implies \dots$

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(1.1)  $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \dots$

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(1.1.1)  $\alpha \not\subset \beta \wedge \alpha \neq \beta$

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from:  $<_{\mathbb{R}}$ , 1.1

(1.1.2)  $\exists_{p'} (p' \in \alpha \wedge p' \notin \beta) \blacksquare p := \text{choice}(\{p' : p' \in \alpha \wedge p' \notin \beta\})$

---

(1.1.3)  $q \in \beta \implies \dots$

---

(1.1.3.1)  $p, q \in \mathbb{Q}$

---

(1.1.3.2)  $q < p$

---

from: [CutCorollaryI](#)

(1.1.3.3)  $q \in \alpha$

---

from: [CutI1](#)

(1.1.4)  $q \in \beta \implies q \in \alpha$

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(1.1.5)  $\forall_{q \in \beta} (q \in \alpha) \blacksquare \beta \subseteq \alpha$

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(1.1.6)  $\beta \subset \alpha \blacksquare \beta <_{\mathbb{R}} \alpha$

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(1.2)  $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

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(1.3)  $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \blacksquare (\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

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(1.4)  $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

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(1.5)  $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

---

(1.6)  $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

---

(1.7)  $\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta$

---

(2)  $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

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(3)  $\forall_{\alpha, \beta \in \mathbb{R}} (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

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(4) [OrderTrichotomy](#) $[\mathbb{R}, <_{\mathbb{R}}]$

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**OrderTransitivityOfR**  $:= \text{OrderTransitivity}[\mathbb{R}, <_{\mathbb{R}}]$

---

(1)  $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$

---

(1.1)  $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \dots$

---

(1.1.1)  $\alpha \subset \beta \wedge \beta \subset \gamma$

---

(1.1.2)  $\forall_{a \in \alpha} (a \in \beta) \wedge \forall_{b \in \beta} (b \in \gamma)$

---

(1.1.3)  $\forall_{a \in \alpha} (\alpha \in \gamma) \blacksquare \alpha \subset \gamma \blacksquare \alpha <_{\mathbb{R}} \gamma$

---

(1.2)  $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

---

(2)  $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

---

(3)  $\forall_{\alpha, \beta, \gamma \in \mathbb{R}} ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

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(4) [OrderTransitivity](#) $[\mathbb{R}, <_{\mathbb{R}}]$

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**OrderOfR**  $:= \text{Order}[<_{\mathbb{R}}, \mathbb{R}]$

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from: [OrderTrichotomyR](#), [OrderTransitivityR](#)  
wts:

**LUBPropertyOfR**  $:= \text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}]$

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(1)  $(\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \dots$

---

(1.1)  $\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A} (p \in \alpha)\}$

---

(1.2)  $A \neq \emptyset \blacksquare \exists_{\alpha} (\alpha \in A) \blacksquare \alpha_0 := \text{choice}(\{\alpha : \alpha \in A\})$

---

(1.3)  $\alpha_0 \neq \emptyset \blacksquare \exists_a (a \in \alpha_0) \blacksquare a_0 := \text{choice}(\{a : a \in \alpha_0\}) \blacksquare a_0 \in \gamma \blacksquare \gamma \neq \emptyset$

---

(1.4) [BoundedAbove](#) $[A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare \exists_{\beta} (\text{UpperBound}[\beta, A, \mathbb{R}, <_{\mathbb{R}}])$

---

(1.5)	$\beta_0 := \text{choice}(\{\beta : \text{UpperBound}[\beta, A, \mathbb{R}, <_{\mathbb{R}}]\})$
(1.6)	$\text{UpperBound}[\beta_0, A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \beta_0) \blacksquare \forall_{\alpha \in A}(\alpha \subseteq \beta_0) \blacksquare \forall_{\alpha \in A} \forall_{a \in \alpha}(a \in \beta_0)$
(1.7)	$(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \blacksquare \forall_{a \in \gamma}(a \in \beta_0) \blacksquare \gamma \subseteq \beta_0$
(1.8)	$\beta_0 \subset \mathbb{Q} \blacksquare \gamma \subseteq \beta_0 \subset \mathbb{Q} \blacksquare \gamma \subset \mathbb{Q}$
(1.9)	$\emptyset \neq \gamma \subset \mathbb{Q} \blacksquare \text{CutI}[\gamma]$
(1.10)	$(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
(1.10.1)	$p \in \gamma \blacksquare \exists_{\alpha \in A}(p \in \alpha) \blacksquare \alpha_1 := \text{choice}(\{\alpha \in A : p \in \alpha\})$
(1.10.2)	$p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \blacksquare q \in \alpha_1 \blacksquare q \in \gamma$
(1.11)	$(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \blacksquare \forall_{p \in \gamma} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma) \blacksquare \text{CutII}[\gamma]$
(1.12)	$p \in \gamma \implies \dots$
(1.12.1)	$\exists_{\alpha \in A}(p \in \alpha) \blacksquare \alpha_2 := \text{choice}(\{\alpha \in A : p \in \alpha\})$
(1.12.2)	$\alpha_2 \in \mathbb{R} \blacksquare \text{CutII}[\alpha_2] \blacksquare \exists_{r \in \alpha_2}(p < r) \blacksquare r_0 := \text{choice}(\{r \in \alpha_2 : p < r\})$
(1.12.3)	$r_0 \in \alpha_2 \blacksquare r_0 \in \gamma$
(1.12.4)	$p < r_0 \blacksquare p < r_0 \wedge r_0 \in \gamma \blacksquare \exists_{r \in \gamma}(p < r)$
(1.13)	$p \in \gamma \implies \exists_{r \in \gamma}(p < r) \blacksquare \forall_{p \in \gamma} \exists_{r \in \gamma}(p < r) \blacksquare \text{CutIII}[\gamma]$
(1.14)	$\text{CutI}[\gamma] \wedge \text{CutII}[\gamma] \wedge \text{CutIII}[\gamma] \blacksquare \gamma \in \mathbb{R}$
(1.15)	$\forall_{\alpha \in A}(\alpha \subseteq \gamma) \blacksquare \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma)$
(1.16)	$\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R} \blacksquare \text{UpperBound}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$
(1.17)	$\delta <_{\mathbb{R}} \gamma \implies \dots$
(1.17.1)	$\delta \subset \gamma \blacksquare \exists_s(s \in \gamma \wedge s \notin \delta) \blacksquare s_0 := \text{choice}(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$
(1.17.2)	$s_0 \in \gamma \blacksquare \exists_{\alpha \in A}(s_0 \in \alpha) \blacksquare \alpha_3 := \text{choice}(\{\alpha \in A : s_0 \in \alpha\})$
(1.17.3)	$s_0 \in \alpha_3 \wedge s_0 \notin \delta \blacksquare \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$
(1.17.4)	$\delta \geq_{\mathbb{R}} \alpha_3 \implies \dots$
(1.17.4.1)	$\alpha_3 \subseteq \delta \blacksquare \forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \blacksquare \neg \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$
(1.17.4.2)	$\neg \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \blacksquare \perp$
(1.17.5)	$\delta \geq_{\mathbb{R}} \alpha_3 \implies \perp \blacksquare \delta <_{\mathbb{R}} \alpha_3 \blacksquare \exists_{\alpha \in A}(\delta <_{\mathbb{R}} \alpha) \blacksquare \exists_{\alpha \in A}(\neg(\alpha \leq_{\mathbb{R}} \delta))$
(1.17.6)	$\neg \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \delta) \blacksquare \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$
(1.18)	$\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare \forall_{\delta}(\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$
(1.19)	$\text{UpperBound}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \wedge \forall_{\delta}(\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$
(1.20)	$\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$
(2)	$(\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$
(3)	$\forall_A((\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])) \blacksquare \text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}]$

$$+_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$$

$$0_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 0\}$$

$$\text{ZeroInR} := 0_{\mathbb{R}} \in \mathbb{R}$$

(1)	$-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}} \blacksquare \emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q} \blacksquare \text{CutI}[0_{\mathbb{R}}]$
(2)	$(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}} \blacksquare \forall_{x \in 0_{\mathbb{R}}} \forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}}) \blacksquare \text{CutII}[0_{\mathbb{R}}]$
(3)	$y := x/2 \blacksquare (x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y) \blacksquare \forall_{x \in 0_{\mathbb{R}}} \exists_{y \in 0_{\mathbb{R}}}(x < y) \blacksquare \text{CutIII}[0_{\mathbb{R}}]$
(4)	$\text{CutI}[0_{\mathbb{R}}] \wedge \text{CutII}[0_{\mathbb{R}}] \wedge \text{CutIII}[0_{\mathbb{R}}] \blacksquare 0_{\mathbb{R}} \in \mathbb{R}$

$$\text{FieldAdditionClosureOfR} := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$$

(1)	$(\alpha, \beta \in \mathbb{R}) \implies \dots$
(1.1)	$(\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$
(1.2)	$\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$
(1.3)	$\exists_a(a \in \alpha) ; \exists_b(b \in \beta) \blacksquare a_0 := \text{choice}(\{a : a \in \alpha\}) ; b_0 := \text{choice}(\{b : b \in \beta\}) \blacksquare a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$
(1.4)	$\exists_x(x \notin \alpha) ; \exists_y(y_0 \notin \beta) \blacksquare x_0 := \text{choice}(\{x : x \notin \alpha\}) ; y_0 := \text{choice}(\{y : y \notin \beta\})$
(1.5)	$\forall_{r \in \alpha}(r < x_0) ; \forall_{s \in \beta}(s < y_0) \blacksquare \forall_{r \in \alpha} \forall_{s \in \beta}(r + s < x_0 + y_0) \blacksquare x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$
(1.6)	$\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q} \blacksquare \text{CutI}[\alpha +_{\mathbb{R}} \beta]$



(1.7)	$(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
(1.7.1)	$\exists_{r \in \alpha} \exists_{s \in \beta} (p = r + s) \blacksquare (r_0, s_0) := \text{choice}((r, s) \in \alpha \times \beta : p = r + s)$
(1.7.2)	$q < p = r_0 + s_0 \blacksquare (q - s_0) < r_0 \blacksquare (q - s_0) \in \alpha$
(1.7.3)	$s_0 \in \beta \blacksquare q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta \blacksquare q \in \alpha +_{\mathbb{R}} \beta$
(1.8)	$(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} \beta} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha +_{\mathbb{R}} \beta) \blacksquare \text{CutII}[\alpha +_{\mathbb{R}} \beta]$
(1.9)	$p \in \alpha \implies \dots$
(1.9.1)	$\exists_{r \in \alpha} \exists_{s \in \beta} (p = r + s) \blacksquare (r_1, s_1) := \text{choice}(\{(r, s) \in \alpha \times \beta : p = r + s\})$
(1.9.2)	$r_1 \in \alpha \blacksquare \exists_{t \in \alpha} (r_1 < t) \blacksquare t_0 := \text{choice}(\{t \in \alpha : r_1 < t\})$
(1.9.3)	$s_1 \in \beta \blacksquare t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1 \blacksquare \exists_{r \in \alpha +_{\mathbb{R}} \beta} (p < r)$
(1.10)	$p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta} (p < r) \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} \beta} \exists_{r \in \alpha +_{\mathbb{R}} \beta} (p < r) \blacksquare \text{CutIII}[\alpha +_{\mathbb{R}} \beta]$
(1.11)	$\text{CutI}[\alpha +_{\mathbb{R}} \beta] \wedge \text{CutII}[\alpha +_{\mathbb{R}} \beta] \wedge \text{CutIII}[\alpha +_{\mathbb{R}} \beta] \blacksquare \alpha +_{\mathbb{R}} \beta \in \mathbb{R}$
(2)	$(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

**FieldAdditionCommutativityOfR** :=  $(\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1)	$\alpha +_{\mathbb{R}} \beta = \{r + s : r \in \alpha \wedge s \in \beta\} = \{s + r : s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$
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**FieldAdditionAssociativityOfR** :=  $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma))$

(1)	$(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$
(1.1)	$(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \dots$
(1.2)	$\{a + (b + c) : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$
(2)	$(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

**FieldAdditionIdentityOfR** :=  $(\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1)	$\alpha \in \mathbb{R} \implies \dots$
(1.1)	$(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \dots$
(1.1.1)	$s < 0 \blacksquare r + s < r + 0 = r \blacksquare r + s < r \blacksquare r + s \in \alpha$
(1.2)	$(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha \blacksquare \forall_{r \in \alpha} \forall_{s \in 0_{\mathbb{R}}} (r + s \in \alpha)$
(1.3)	$(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}} (p \in \alpha) \blacksquare \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$
(1.4)	$p \in \alpha \implies \dots$
(1.4.1)	$\exists_{r \in \alpha} (p < r) \blacksquare r_2 := \text{choice}(\{r \in \alpha : p < r\})$
(1.4.2)	$p < r_2 \blacksquare p - r_2 < r_2 - r_2 = 0 \blacksquare (p - r_2) < 0 \blacksquare (p - r_2) \in 0_{\mathbb{R}}$
(1.4.3)	$r_2 \in \alpha \blacksquare p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$
(1.5)	$p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare \forall_{p \in \alpha} (p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$
(1.6)	$\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$
(2)	$\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

**FieldAdditionInverseOfR** :=  $(\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}} (\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(1)	$\alpha \in \mathbb{R} \implies \dots$
(1.1)	$\beta := \{p \in \mathbb{Q} : \exists_{r > 0} (-p - r \notin \alpha)\}$
(1.2)	$\alpha \subset \mathbb{Q} \blacksquare \exists_{s \in \mathbb{Q}} (s \notin \alpha) \blacksquare s_0 := \text{choice}(\{s : s \notin \alpha\}) \blacksquare p_0 := -s_0 - 1$
(1.3)	$-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha \blacksquare -p_0 - 1 \notin \alpha \blacksquare \exists_{r > 0} (-p_0 - r \notin \alpha) \blacksquare p_0 \in \beta$
(1.4)	$\emptyset \neq \alpha \blacksquare \exists_{q \in \alpha} \blacksquare q_0 := \text{choice}(\{q \in \mathbb{Q} : q \in \alpha\})$
(1.5)	$r > 0 \implies \dots$
(1.5.1)	$q_0 \in \alpha \blacksquare -(-q_0) - r = q_0 - r < q_0 \blacksquare -(-q_0) - r < q_0 \blacksquare -(-q_0) - r \in \alpha$
(1.6)	$\forall_{r > 0} (-(-q_0) - r \in \alpha) \blacksquare \neg \exists_{r > 0} (-(-q_0) - r \notin \alpha) \blacksquare -q_0 \notin \beta$
(1.7)	$\emptyset \neq \beta \subset \mathbb{Q} \blacksquare \text{CutI}[\beta]$
(1.8)	$(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
(1.8.1)	$p \in \beta \blacksquare \exists_{r > 0} (-p - r \notin \alpha) \blacksquare r_0 := \text{choice}(\{r > 0 : -p - r \notin \alpha\})$
(1.8.2)	$q < p \blacksquare -p - r < -q - r$
(1.8.3)	$-q - r \notin \alpha \blacksquare q \in \beta$
(1.9)	$(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta \blacksquare \forall_{p \in \beta} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \beta) \blacksquare \text{CutII}[\beta]$

(1.10)	$p \in \beta \implies \dots$	
(1.10.1)	$p \in \beta \implies \exists_{r>0}(-p-r \notin \alpha) \blacksquare r_1 := \text{choice}(\{r > 0 : -p-r \notin \alpha\})$	
(1.10.2)	$t_0 := p + (r_1/2)$	
(1.10.3)	$r_1 > 0 \blacksquare r_1/2 > 0$	
(1.10.4)	$t_0 > t_0 - (r_1/2) = p \blacksquare t_0 > p$	
(1.10.5)	$-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$	
(1.10.6)	$-p - r_1 \notin \alpha \blacksquare -t_0 - (r_1/2) \notin \alpha \blacksquare \exists_{r>0}(-t_0 - r \notin \alpha) \blacksquare t_0 \in \beta$	
(1.10.7)	$t_0 > p \wedge t_0 \in \beta \blacksquare \exists_{t \in \beta}(p < t)$	
(1.11)	$p \in \beta \implies \exists_{t \in \beta}(p < t) \blacksquare \forall_{p \in \beta} \exists_{t \in \beta}(p < t) \blacksquare \text{CutIII}[\beta]$	
(1.12)	$\text{CutI}[\beta] \wedge \text{CutII}[\beta] \wedge \text{CutIII}[\beta] \blacksquare \beta \in \mathbb{R}$	
(1.13)	$(r \in \alpha \wedge s \in \beta) \implies \dots$	
(1.13.1)	$s \in \beta \blacksquare \exists_{t>0}(-s-t \notin \alpha) \blacksquare t_1 := \text{choice}(\{t > 0 : -s-t \notin \alpha\}) \blacksquare -s-t_1 < -s$	
(1.13.2)	$\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s-t_1 < -s \wedge -s-t_1 \notin \alpha \blacksquare -s \notin \alpha$	
(1.13.3)	$\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha \blacksquare r < -s \blacksquare r+s < 0 \blacksquare r+s \in 0_{\mathbb{R}}$	
(1.14)	$(r \in \alpha \wedge s \in \beta) \implies r+s \in 0_{\mathbb{R}} \blacksquare \forall_{(r,s) \in \alpha \times \beta}(r+s \in 0_{\mathbb{R}}) \blacksquare \alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$	
(1.15)	$v \in 0_{\mathbb{R}} \implies \dots$	
(1.15.1)	$v < 0 \blacksquare w_0 := -v/2 \blacksquare w > 0$	
(1.15.2)	$\exists_{n \in \mathbb{Z}}(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha) \blacksquare n_0 := \text{choice}(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$	from: ARCHIMEDEANPROPERTYOFQ + LUB???
(1.15.3)	$p_0 := -(n_0+2)w_0 \blacksquare -p_0 - w_0 = (n_0+2)w_0 - w_0 = (n_0+1)w_0 \notin \alpha \blacksquare -p_0 - w_0 \notin \alpha \blacksquare p_0 \in \beta$	
(1.15.4)	$n_0 w_0 \in \alpha \wedge p_0 \in \beta \blacksquare n_0 w_0 + p_0 = n_0(-v/2) + -(n_0+2)w_0 = v \in \alpha +_{\mathbb{R}} \beta$	
(1.16)	$v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{v \in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta) \blacksquare 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$	
(1.17)	$\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta \blacksquare \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$	
(1.18)	$\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}} \blacksquare \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$	
(2)	$\alpha \in \mathbb{R} \implies \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$	

$*_{\mathbb{R}}[\alpha, \beta] := \text{---}$

$1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$11sNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}} \text{ ---}$

$11nR := 1_{\mathbb{R}} \in \mathbb{R} \text{ ---}$

$\text{FieldMultiplicationClosureOf } \mathbb{R} := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R}) \text{ ---}$

$\text{FieldMultiplicationCommutativityOf } \mathbb{R} := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha) \text{ ---}$

$\text{FieldMultiplicationAssociativityOf } \mathbb{R} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)) \text{ ---}$

$\text{FieldMultiplicationIdentityOf } \mathbb{R} := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha \text{ ---}$

$\text{FieldMultiplicationInverseOf } \mathbb{R} := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha \in \mathbb{R}}(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}) \text{ ---}$

$\text{FieldDistributivityOf } \mathbb{R} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta \text{ ---}$

$\text{FieldWith } \mathbb{R} := \text{Field}[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}] \text{ ---}$

$\text{OrderedFieldWith } \mathbb{R} := \text{OrderedField}[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}] \text{ ---}$

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$

$\text{QROrderedSubfieldOf } \mathbb{R} := \text{OrderedSubfield}[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}] \text{ ---}$

$\text{QIsomorphicToQR} := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q} \text{ ---}$

$\text{CompletenessOf } \mathbb{R} := \exists_{\mathbb{R}}(\text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}] \wedge \text{OrderedSubfield}[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]) \text{ ---}$

(1.20)

$\text{ArchimedeanPropertyOf } \mathbb{R} := \forall_{x,y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

(1)  $(x, y \in \mathbb{R} \wedge x > 0) \implies \dots$

(1.1)  $A := \{nx : n \in \mathbb{N}^+\} \blacksquare (\emptyset \neq A \subset \mathbb{R}) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$

(1.2)  $\neg \exists_{n \in \mathbb{N}^+}(nx > y) \implies \dots$

(1.2.1)  $\neg \exists_{n \in \mathbb{N}^+}(nx > y) \blacksquare \forall_{n \in \mathbb{N}^+}(nx \leq y) \blacksquare \text{UpperBound}[y_0, A, \mathbb{R}, <] \blacksquare \text{BoundedAbove}[A, \mathbb{R}, <]$

(1.2.2)  $\text{CompletenessOf } \mathbb{R} \blacksquare \text{LUBProperty}[\mathbb{R}, <]$

(1.2.3)  $(\text{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (\text{BoundedAbove}[A, \mathbb{R}, <]) \blacksquare \exists_{\alpha \in \mathbb{R}}(\text{LUB}[\alpha, A, \mathbb{R}, <]) \dots$

(1.2.4)	$\dots \alpha_0 := \text{choice}(\{\alpha \in \mathbb{R} : \textcolor{teal}{LUB}[\alpha, A, \mathbb{R}, <]\}) \quad \blacksquare \quad \textcolor{teal}{LUB}[\alpha_0, A, \mathbb{R}, <]$
(1.2.5)	$x > 0 \quad \blacksquare \quad \alpha_0 - x < \alpha_0$
(1.2.6)	$(\alpha_0 - x < \alpha_0) \wedge (\textcolor{teal}{LUB}[\alpha_0, A, \mathbb{R}, <]) \quad \blacksquare \quad \neg \textcolor{teal}{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <]$
(1.2.7)	$\neg \textcolor{teal}{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <] \quad \blacksquare \quad \exists_{c \in A}(\alpha_0 - x < c) \quad \dots$
(1.2.8)	$\dots c_0 := \text{choice}(\{c \in A : \alpha_0 - x < c\}) \quad \blacksquare \quad (c_0 \in A) \wedge (\alpha_0 - x < c_0)$
(1.2.9)	$(c_0 \in A) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)) \quad \blacksquare \quad \exists_{m \in \mathbb{N}^+}(mx = c_0) \quad \dots$
(1.2.10)	$\dots m_0 := \text{choice}(\{m \in \mathbb{N}^+ : mx = c_0\}) \quad \blacksquare \quad (m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$
(1.2.11)	$(\alpha_0 - x < c_0) \wedge (m_0 x = c_0) \quad \blacksquare \quad \alpha_0 - x < c_0 = m_0 x \quad \blacksquare \quad \alpha_0 < m_0 x + x \quad \blacksquare \quad \alpha_0 < (m_0 + 1)x$
(1.2.12)	$m_0 \in \mathbb{N}^+ \quad \blacksquare \quad m_0 + 1 \in \mathbb{N}^+$
(1.2.13)	$(m_0 + 1 \in \mathbb{N}^+) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)) \quad \blacksquare \quad (m_0 + 1)x \in A$
(1.2.14)	$(\alpha_0 < (m_0 + 1)x) \wedge ((m_0 + 1)x \in A) \quad \blacksquare \quad \exists_{c \in A}(\alpha_0 < c)$
(1.2.15)	$\textcolor{teal}{LUB}[\alpha_0, A, \mathbb{R}, <] \quad \blacksquare \quad \textcolor{teal}{UpperBound}[\alpha_0, A, \mathbb{R}, <] \quad \blacksquare \quad \forall_{c \in A}(c \leq \alpha_0) \quad \blacksquare \quad \neg \exists_{c \in A}(c > \alpha_0) \quad \blacksquare \quad \neg \exists_{c \in A}(\alpha_0 < c)$
(1.2.16)	$(\exists_{c \in A}(\alpha_0 < c)) \wedge (\neg \exists_{c \in A}(\alpha_0 < c)) \quad \blacksquare \quad \perp$
(1.3)	$\neg \exists_{n \in \mathbb{N}^+}(nx > y) \implies \perp \quad \blacksquare \quad \exists_{n \in \mathbb{N}^+}(nx > y)$
(2)	$(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y) \quad \blacksquare \quad \forall_{x, y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

***QDenseInR*** :=  $\forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1)	$(x, y \in \mathbb{R} \wedge x < y) \implies \dots$
(1.1)	$x < y \quad \blacksquare \quad (0 < y - x) \wedge (y - x \in \mathbb{R})$
(1.2)	$\textcolor{teal}{ArchimedeanPropertyOfR} \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R}) \quad \blacksquare \quad \exists_{n \in \mathbb{N}^+}(n(y - x) > 1) \quad \dots$
(1.3)	$\dots n_0 := \text{choice}(\{n \in \mathbb{N}^+ : n(y - x) > 1\}) \quad \blacksquare \quad (n_0 \in \mathbb{N}^+) \wedge (n_0(y - x) > 1)$
(1.4)	$(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R}) \quad \blacksquare \quad n_0 x, -n_0 x \in \mathbb{R}$
(1.5)	$\textcolor{teal}{ArchimedeanPropertyOfR} \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R}) \quad \blacksquare \quad \exists_{m \in \mathbb{N}^+}(m(1) > n_0 x) \quad \dots$
(1.6)	$\dots m_1 := \text{choice}(\{m \in \mathbb{N}^+ : m(1) > n_0 x\}) \quad \blacksquare \quad (m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$
(1.7)	$\textcolor{teal}{ArchimedeanPropertyOfR} \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R}) \quad \blacksquare \quad \exists_{m \in \mathbb{N}^+}(m(1) > -n_0 x) \quad \dots$
(1.8)	$\dots m_2 := \text{choice}(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\}) \quad \blacksquare \quad (m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$
(1.9)	$(m_1 > n_0 x) \wedge (m_2 > -n_0 x) \quad \blacksquare \quad -m_2 < n_0 x < m_1$
(1.10)	$m_1, m_2 \in \mathbb{N}^+ \quad \blacksquare \quad  m_1 - (-m_2)  \geq 2$
(1.11)	$(-m_2 < n_0 x < m_1) \wedge ( m_1 - (-m_2)  \geq 2) \quad \blacksquare \quad \exists_{m \in \mathbb{Z}}((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)) \quad \dots$
(1.12)	$\dots m_0 := \text{choice}(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\}) \quad \blacksquare \quad (-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$
(1.13)	$(n_0(y - x) > 1) \wedge (m_0 - 1 \leq n_0 x < m_0) \quad \blacksquare \quad n_0 x < m_0 \leq 1 + n_0 x < n_0 y \quad \blacksquare \quad n_0 x < m_0 < n_0 y$
(1.14)	$(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y) \quad \blacksquare \quad x < m_0/n_0 < y$
(1.15)	$m_0, n_0 \in \mathbb{Z} \quad \blacksquare \quad m_0/n_0 \in \mathbb{Q}$
(1.16)	$(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y) \quad \blacksquare \quad \exists_{p \in \mathbb{Q}}(x < p < y)$
(2)	$(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y) \quad \blacksquare \quad \forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1.21)

***Root Lemma*** :=  $(0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

(1)	$(0 < a < b) \implies \dots$
(1.1)	$b^n - a^n = (b - a) \sum_{i=1}^n (b^{n-i} a^{i-1})$
(1.2)	$0 < a < b \quad \blacksquare \quad b/a > 1$
(1.3)	$b/a > 1 \quad \blacksquare \quad \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq \sum_{i=1}^n (b^{n-i} a^{i-1} (b/a)^{i-1}) = \sum_{i=1}^n (b^{n-1}) = nb^{n-1} \quad \blacksquare \quad \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq \sum_{i=1}^n (b^{n-1}) = nb^{n-1}$
(1.4)	$b^n - a^n = (b - a) \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq (b - a)nb^{n-1} \quad \blacksquare \quad b^n - a^n \leq (b - a)nb^{n-1}$
(2)	$(0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

***Root Existence In R*** :=  $\forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}} (y_0^n = x)$

(1)	$(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \dots$
(1.1)	$E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\} \quad \blacksquare \quad t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$
(1.2)	$t_0 := x/(1 + x) \quad \blacksquare \quad (t_0 = x/(1 + x)) \wedge (t_0 \in \mathbb{R})$
(1.3)	$0 < x \quad \blacksquare \quad 0 < x < 1 + x \quad \blacksquare \quad t_0 = x/(1 + x) > 0 \quad \blacksquare \quad t_0 > 0$

$$(1.4) \quad 1 = (1+x)/(1+x) > x/(1+x) = t_0 \quad \blacksquare \quad 1 > t_0$$

$$(1.5) \quad (t_0 > 0) \wedge (1 > t_0) \quad \blacksquare \quad 0 < t_0 < 1$$

$$(1.6) \quad (0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1) \quad \blacksquare \quad t_0^n \leq t_0$$

$$(1.7) \quad 0 < x \quad \blacksquare \quad x > x/(1+x) = t_0 \quad \blacksquare \quad x > t_0$$

$$(1.8) \quad (t_0^n \leq t_0) \wedge (x > t_0) \quad \blacksquare \quad t_0^n < x$$

$$(1.9) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge (t_0^n < x) \quad \blacksquare \quad t_0 \in E \quad \blacksquare \quad \emptyset \neq E$$

$$(1.10) \quad t_1 := \text{choice}(\{t \in \mathbb{R} : t > 1+x\}) \quad \blacksquare \quad (t_1 \in \mathbb{R}) \wedge (t_1 > 1+x)$$

$$(1.11) \quad x > 0 \quad \blacksquare \quad t_1 > 1+x > 1 \quad \blacksquare \quad t_1 > 1 \quad \blacksquare \quad t_1^n \geq t_1$$

$$(1.12) \quad (t_1^n \geq t_1) \wedge (t_1 > 1+x) \wedge (1 > 0) \quad \blacksquare \quad t_1^n \geq t_1 > 1+x > x \quad \blacksquare \quad t_1^n > x$$

$$(1.13) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_1^n > x) \quad \blacksquare \quad t_1 \notin E \quad \blacksquare \quad E \subset \mathbb{R}$$

$$(1.14) \quad (\emptyset \neq E) \wedge (E \subset \mathbb{R}) \quad \blacksquare \quad \emptyset \neq E \subset \mathbb{R}$$

$$(1.15) \quad t \in E \implies \dots$$

$$(1.15.1) \quad (t \in E) \wedge (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \quad \blacksquare \quad t^n < x$$

$$(1.15.2) \quad (t_1^n > x) \wedge (t^n < x) \quad \blacksquare \quad t^n < x < t_1^n \quad \blacksquare \quad t < t_1$$

$$(1.16) \quad t \in E \implies t < t_1 \quad \blacksquare \quad \forall_{t \in E} (t \leq t_1) \quad \blacksquare \quad \text{UpperBound}[t_1, E, \mathbb{R}, <] \quad \blacksquare \quad \text{BoundedAbove}[E, \mathbb{R}, <]$$

$$(1.17) \quad \text{CompletenessOfR} \quad \blacksquare \quad \text{LUBProperty}[\mathbb{R}, <]$$

$$(1.18) \quad (\text{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (\text{BoundedAbove}[E, \mathbb{R}, <]) \quad \blacksquare \quad \exists_{y \in \mathbb{R}} (\text{LUB}[y, E, \mathbb{R}, <]) \quad \dots$$

$$(1.19) \quad \dots y_0 := \text{choice}(\{y \in \mathbb{R} : \text{LUB}[y, E, \mathbb{R}, <]\}) \quad \blacksquare \quad \text{LUB}[y_0, E, \mathbb{R}, <]$$

$$(1.20) \quad (\text{LUB}[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0) \quad \blacksquare \quad 0 < t_0 \leq y_0 \in \mathbb{R} \quad \blacksquare \quad 0 < y_0 \in \mathbb{R}$$

$$(1.21) \quad y_0^n < x \implies \dots$$

$$(1.21.1) \quad k_0 := \frac{x-y_0^n}{n(y_0+1)^{n-1}} \quad \blacksquare \quad k_0 \in \mathbb{R}$$

$$(1.21.2) \quad y_0^n < x \quad \blacksquare \quad 0 < x - y_0^n$$

$$(1.21.3) \quad (n > 0) \wedge (y_0 > 0) \quad \blacksquare \quad 0 < n(y_0 + 1)^{n-1}$$

$$(1.21.4) \quad (0 < x - y_0^n) \wedge (0 < n(y_0 + 1)^{n-1}) \quad \blacksquare \quad 0 < \frac{x-y_0^n}{n(y_0+1)^{n-1}} = k_0 \quad \blacksquare \quad 0 < k_0$$

$$(1.21.5) \quad (0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R}) \quad \blacksquare \quad 0 < \min(1, k_0) \in \mathbb{R}$$

$$(1.21.6) \quad \text{QDenseInR} \wedge (0, \min(1, k_0) \in \mathbb{R}) \wedge (0 < \min(1, k_0)) \quad \blacksquare \quad \exists_{h \in \mathbb{Q}} (0 < h < \min(1, k_0)) \quad \dots$$

$$(1.21.7) \quad \dots h_0 := \text{choice}(\{h \in \mathbb{Q} : 0 < h < \min(1, k_0)\}) \quad \blacksquare \quad (0 < h_0 < 1) \wedge (h_0 < k_0 = \frac{x-y_0^n}{n(y_0+1)^{n-1}})$$

$$(1.21.8) \quad (y_0 > 0) \wedge (h_0 > 0) \quad \blacksquare \quad 0 < y_0 < y_0 + h_0$$

$$(1.21.9) \quad \text{RootLemma} \wedge (0 < y_0 < y_0 + h_0) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < h_0 n (y_0 + h_0)^{n-1}$$

$$(1.21.10) \quad h_0 < 1 \quad \blacksquare \quad h_0 n (y_0 + h_0)^{n-1} < h_0 n (y_0 + 1)^{n-1}$$

$$(1.21.11) \quad ((y_0 + h_0)^n - y_0^n < h_0 n (y_0 + h_0)^{n-1}) \wedge (h_0 n (y_0 + h_0)^{n-1} < h_0 n (y_0 + 1)^{n-1}) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < h_0 n (y_0 + 1)^{n-1}$$

$$(1.21.12) \quad (0 < n(y_0 + 1)^{n-1}) \wedge (h_0 < k_0 = \frac{x-y_0^n}{n(y_0+1)^{n-1}}) \quad \blacksquare \quad h_0 n (y_0 + 1)^{n-1} < x - y_0^n$$

$$(1.21.13) \quad ((y_0 + h_0)^n - y_0^n < h_0 n (y_0 + 1)^{n-1}) \wedge (h_0 n (y_0 + 1)^{n-1} < x - y_0^n) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < x - y_0^n \quad \blacksquare \quad (y_0 + h_0)^n < x$$

$$(1.21.14) \quad (y_0 + h_0)^n - y_0^n < x - y_0^n \quad \blacksquare \quad (y_0 + h_0)^n < x$$

$$(1.21.15) \quad (0 < y_0 \in \mathbb{R}) \wedge (0 < h_0 \in \mathbb{R}) \quad \blacksquare \quad 0 < y_0 < y_0 + h_0 \in \mathbb{R}$$

$$(1.21.16) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ((y_0 + h_0)^n < x) \wedge (0 < y_0 + h_0 \in \mathbb{R}) \quad \blacksquare \quad (y_0 + h_0)^n \in E$$

$$(1.21.17) \quad ((y_0 + h_0)^n \in E) \wedge (y_0 < y_0 + h_0) \quad \blacksquare \quad \exists_{e \in E} (y_0 < e)$$

$$(1.21.18) \quad \text{LUB}[y_0, E, \mathbb{R}, <] \quad \blacksquare \quad \text{UpperBound}[y_0, E, \mathbb{R}, <] \quad \blacksquare \quad \forall_{e \in E} (e \leq y_0) \quad \blacksquare \quad \neg \exists_{e \in E} (e > y_0)$$

$$(1.21.19) \quad (\exists_{e \in E} (e > y_0)) \wedge (\neg \exists_{e \in E} (e > y_0)) \quad \blacksquare \quad \perp$$

$$(1.22) \quad y_0^n < x \implies \perp \quad \blacksquare \quad y_0^n \geq x$$

$$(1.23) \quad y_0^n > x \implies \dots$$

$$(1.23.1) \quad k_1 := \frac{y_0^n - x}{n y_0^{n-1}} \quad \blacksquare \quad (k_1 \in \mathbb{R}) \wedge (k_1 n y_0^{n-1} = y_0^n - x)$$

$$(1.23.2) \quad (0 < x) \wedge (0 < n \in \mathbb{Z}) \quad \blacksquare \quad y_0^n - x < y_0^n \leq n y_0^n \quad \blacksquare \quad y_0^n - x < n y_0^n$$

$$(1.23.3) \quad y_0^n - x < n y_0^n \quad \blacksquare \quad k_1 = \frac{y_0^n - x}{n y_0^{n-1}} < \frac{n y_0^n}{n y_0^{n-1}} = y_0 \quad \blacksquare \quad k_1 < y_0$$

$$(1.23.4) \quad y_0^n > x \quad \blacksquare \quad 0 < y_0^n - x$$

$$(1.23.5) \quad (n > 0) \wedge (y_0 > 0) \quad \blacksquare \quad 0 < n y_0^{n-1}$$

$$(1.23.6) \quad (0 < y_0^n - x) \wedge 0 < (n y_0^{n-1}) \quad \blacksquare \quad 0 < \frac{y_0^n - x}{n y_0^{n-1}} = k_1 \quad \blacksquare \quad 0 < k_1$$

(1.23.7)	$(k_1 < y_0) \wedge (0 < k_1) \quad \blacksquare \quad (0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$	
(1.23.8)	$t \geq y_0 - k_1 \implies \dots$	
(1.23.8.1)	$t \geq y_0 - k_1 \quad \blacksquare \quad t^n \geq (y_0 - k_1)^n \quad \blacksquare \quad -t^n \leq -(y_0 - k_1)^n \quad \blacksquare \quad y_0^n - t^n \leq y_0^n - (y_0 - k_1)^n$	
(1.23.8.2)	$\textcolor{blue}{RootLemma} \wedge (0 < y_0 - k_1 < y_0) \quad \blacksquare \quad y_0^n - (y_0 - k_1)^n < k_1 n y_0^{n-1}$	
(1.23.8.3)	$(y_0^n - t^n \leq y_0^n - (y_0 - k_1)^n) \wedge (y_0^n - t^n < k_1 n y_0^{n-1}) \quad \blacksquare \quad y_0^n - t^n < k_1 n y_0^{n-1}$	
(1.23.8.4)	$(k_1 n y_0^{n-1} = y_0^n - x) \wedge (y_0^n - t^n < k_1 n y_0^{n-1}) \quad \blacksquare \quad y_0^n - t^n < y_0^n - x \quad \blacksquare \quad -t^n < -x \quad \blacksquare \quad t^n > x$	
(1.23.8.5)	$(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t^n > x) \quad \blacksquare \quad t \notin E$	
(1.23.9)	$t \geq y_0 - k_1 \implies t \notin E \quad \blacksquare \quad t \in E \implies t < y_0 - k_1 \quad \blacksquare \quad \forall_{t \in E} (t \leq y_0 - k_1) \quad \blacksquare \quad \textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$	
(1.23.10)	$(\textcolor{blue}{LUB}[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)) \quad \blacksquare \quad \neg \textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$	
(1.23.11)	$(\textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg \textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \quad \blacksquare \quad \perp$	
(1.24)	$y_0^n > x \implies \perp \quad \blacksquare \quad y_0^n \leq x$	
(1.25)	$\textcolor{blue}{Order}[\mathbb{R}, <] \quad \blacksquare \quad \textcolor{blue}{OrderTrichotomy}[\mathbb{R}, <]$	
(1.26)	$(\textcolor{blue}{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_0^n \geq x) \wedge (y_0^n \leq x) \quad \blacksquare \quad y_0^n = x$	
(1.27)	$(y_0^n = x) \wedge (y_0 \in \mathbb{R}) \quad \blacksquare \quad \exists_{y \in \mathbb{R}} (y^n = x)$	
(1.28)	$y_1, y_2 := \textit{choice}(\{y \in \mathbb{R} : y^n = x\})$	
(1.29)	$y_1 \neq y_2 \implies \dots$	
(1.29.1)	$(\textcolor{blue}{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_1 \neq y_2) \quad \blacksquare \quad (y_1 < y_2) \vee (y_2 < y_1) \quad \dots$	
(1.29.2)	$\dots (x = y_1^n < y_2^n = x) \vee (x = y_2^n < y_1^n = x) \quad \blacksquare \quad (x < x) \vee (x > x) \quad \blacksquare \quad \perp \vee \perp \quad \blacksquare \quad \perp$	
(1.30)	$y_1 \neq y_2 \implies \perp \quad \blacksquare \quad y_1 = y_2 \quad \blacksquare \quad \forall_{a,b \in \mathbb{R}} ((a^n = x \wedge b^n = x) \implies a = b)$	
(1.31)	$(\exists_{y \in \mathbb{R}} (y^n = x)) \wedge (\forall_{a,b \in \mathbb{R}} ((a^n = x \wedge b^n = x) \implies a = b)) \quad \blacksquare \quad \exists!_{y \in \mathbb{R}} (y^n = x)$	
(2)	$(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}} (y^n = x) \quad \blacksquare \quad \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}} (y_0^n = x)$	

$$\textcolor{red}{RootExistenceInRCorollary} := \forall_{0 < a \in \mathbb{R}} \forall_{0 < b \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} ((ab)^{1/n} = a^{1/n} b^{1/n}) \quad \text{---}$$

$$\textcolor{red}{ExtendedRealSystem}[\bar{\mathbb{R}}, +, *, <] := \left( \begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \quad \wedge \quad x - \infty = -\infty \quad \wedge \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \quad \wedge \\ (x > 0) \implies (x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty) \wedge \\ (x < 0) \implies (x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty) \end{array} \right)$$

$$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$$

$$+_C[\langle a, b \rangle, \langle c, d \rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d \rangle$$

$$*_C[\langle a, b \rangle, \langle c, d \rangle] := \langle a *_R c - b *_R d, a *_R d + b *_R c \rangle$$

$$\textcolor{red}{FieldC} := \textcolor{blue}{Field}[\mathbb{C}, +_C, *_C] \quad \text{---}$$

$$\textcolor{red}{RSubfieldC} := \textcolor{blue}{Subfield}[\mathbb{R}, \mathbb{C}, +, *] \quad \text{---}$$

$$i := \langle 0, 1 \rangle \in \mathbb{C}$$

$$\textcolor{red}{iProperty} := i^2 = -1 \quad \text{---}$$

$$\textcolor{red}{CProperty} := (a, b \in \mathbb{R}) \implies (\langle a, b \rangle = a + bi) \quad \text{---}$$

$$\textcolor{red}{Conjugate}[\overline{a + bi}] := a - bi$$

$$\textcolor{red}{ConjugateProperties} := (w, z \in \mathbb{C}) \implies \dots \quad \text{---}$$

$$(1) \quad \overline{z + w} = \bar{z} + \bar{w}$$

$$(2) \quad \overline{z * w} = \bar{z} * \bar{w}$$

$$(3) \quad \textit{Re}(z) = (1/2)(z + \bar{z}) \wedge \textit{Im}(z) = (1/2)(z - \bar{z})$$

$$(4) \quad 0 \leq z * \bar{z} \in \mathbb{R}$$

$$\textcolor{red}{AbsoluteValueC}[|z|] = (z * \bar{z})^{1/2}$$

$$\textcolor{red}{AbsoluteValueProperties} := (z, w \in \mathbb{C}) \implies \dots \quad \text{---}$$

$$(1) \quad 123123$$



# Chapter 2

## Abstract Algebra

$Relation(f, X) := f \subseteq X$   
 $Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y} ((x, y) \in f)$

$(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \dots$

(1)  $Domain(f) := X; Codomain(f) := Y$

(2)  $Image(f, A) := \{f(a) : a \in A\}; Preimage(f, B) := \{a : f(a) \in B\}$

(3)  $Range(f) := Image(Domain(f))$

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X} (x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$

$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X} (y = f(x))$

$Bijjective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$

**Surjective Equivalent**  $:= (Range(f) = Codomain(f)) \implies Surjective(f)$

$(Function(f, X, Y) \wedge Function(g, Y, Z)) \implies (f \circ g)(x) := f(g(x)); Function(f \circ g, X, Z)$

**Properties of Functions**  $:= (Function(f, A, B) \wedge Function(g, B, C) \wedge Function(h, C, D)) \implies \dots$

(1)  $h \circ (g \circ f) = (h \circ g) \circ f$

(2)  $(Injective(f) \wedge Injective(g)) \implies Injective(g \circ f)$

(3)  $(Surjective(f) \wedge Surjective(g)) \implies Surjective(g \circ f)$

(4)  $(Bijjective(f, A, B)) \implies \exists_{f^{-1}} (Function(f^{-1}, B, A) \wedge \forall_{a \in A} (f^{-1}(f(a)) = a) \wedge \forall_{b \in B} (f(f^{-1}(b)) = b))$

$|(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}} (b = ac)$

**Divisibility Theorems**  $:= (a, b, c, m, x, y \in \mathbb{Z}) \implies \dots$

(1)  $(a|b) \implies a|bc$

(2)  $(a|b \wedge b|c) \implies a|c$

(3)  $(a|b \wedge b|c) \implies a|(bx + cy)$

(4)  $(a|b \wedge b|a) \implies a = \pm b$

(5)  $(a|b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$

(6)  $(a|b) \iff (m \neq 0 \wedge ma|mb)$

**Division Algorithm**  $:= (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}} (b = aq + r)$

**CD**  $(a, b, c) := a, b, c \in \mathbb{Z} \wedge a : b \wedge a : c$

**GCD**  $(a, b, c) := CD(a, b, c) \wedge \forall_d ((d : b \wedge d : c) \implies d : a)$

**GCD Equivalent**  $:= 123123$





# Chapter 3

## Linear Algebra

### 3.1 Matrix Operations and Special Matrices

$Matrix[A, m, n] := [a_{i,j}]_{m \times n} := m \text{ rows, } n \text{ columns of real numbers}$

$\mathcal{M}_{m,n} := \{A : Matrix[A, m, n]\}$

$O_{m,n} := (Matrix[O, m, n]) \wedge (a_{i,j} = 0)$

$Square[A, n] := Matrix[A, n, n]$

$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$

$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$

$Diagonal[A, n] := (Square[A, n]) \wedge (i \neq j \implies a_{i,j} = 0)$

$Scalar[A, n, k] := (Diagonal[A, n]) \wedge (a_{i,i} = k)$

$I_n := Scalar[I, n, 1]$

$+(A, B) := ((Matrix[A, m, n]) \wedge (Matrix[B, m, n])) \implies (A + B = [a_{i,j} + b_{i,j}]_{m \times n})$

$*(r, A) := ((r \in \mathbb{R}) \wedge (Matrix[A, m, n])) \implies (r * A = [ra_{i,j}]_{m \times n})$

$*(A, B) := ((Matrix[A, m, p]) \wedge (Matrix[B, p, n])) \implies (A * B = \left[ \sum_{k=1}^p (a_{i,k} b_{k,j}) \right]_{m \times n})$

$^T[A] := (Matrix[A, m, n]) \implies (A^T = [a_{j,i}]_{n \times m})$

$AddCom := \forall_{A,B \in \mathcal{M}} (A + B = B + A)$

---

(1)  $A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

---

$AddAssoc := \forall_{A,B,C \in \mathcal{M}} ((A + B) + C = A + (B + C))$

---

(1)  $(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

---

$AddId := \forall_{A \in \mathcal{M}} \exists!_{O \in \mathcal{M}} (A + O = A = O + A)$

---

(1)  $A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$

---

(2)  $A + O_1 = A = A + O_2 \implies O_1 = O_2$

---

$AddInv := \forall_{A \in \mathcal{M}} \exists!_{(-A) \in \mathcal{M}} (A + (-A) = O = (-A) + A)$

---

(1)  $A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$

---

(2)  $A + (-A_1) = O = A + (-A_2) \implies -A_1 = -A_2 \implies A_1 = A_2$

---

$MulAssoc := \forall_{A,B,C \in \mathcal{M}} ((A * B) * C = A * (B * C))$

---

(1)  $(A * B) * C = \left[ \sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,j}) \right] * C = \left[ \sum_{k_2=1}^{p_2} (\sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,k_2}) c_{k_2,j}) \right] = \left[ \sum_{k_2=1}^{p_2} \sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,k_2} c_{k_2,j}) \right] = \dots$

---

(2)  $\dots \left[ \sum_{k_1=1}^{p_1} \sum_{k_2=1}^{p_2} (a_{i,k_1} b_{k_1,k_2} c_{k_2,j}) \right] = \left[ \sum_{k_1=1}^{p_1} (a_{i,k_1} \sum_{k_2=1}^{p_2} (b_{k_1,k_2} c_{k_2,j})) \right] = \dots = A * (B * C)$

---

$MulId := \forall_{A: Square[A,n]} (A * I_n = A = I_n * A)$

---

$$(1) \quad A * I_n = \left[ \sum_{k=1}^n a_{i,k} \begin{pmatrix} 1 & k=j \\ 0 & k \neq j \end{pmatrix} \right] = [a_{i,j}] = A$$

$$(2) \quad \text{TODO} = A$$

$$\text{ScalAssoc} := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} (r(sA) = (rs)A = s(rA))$$

$$(1) \quad r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$$

$$(2) \quad (rs)A = [rsa_{i,j}]$$

$$(3) \quad s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$$

$$\text{TransCancel} := \forall_{A \in \mathcal{M}} (A = (A^T)^T)$$

$$(1) \quad A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$$

$$\text{ScalMulCom} := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} ((rA) * B = r(A * B) = A * (rB))$$

$$(1) \quad (rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[ \sum_{k=1}^p (ra_{i,k} b_{k,j}) \right] = r(A * B)$$

$$(2) \quad A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[ \sum_{k=1}^p (a_{i,k} rb_{k,j}) \right] = \left[ \sum_{k=1}^p (ra_{i,k} b_{k,j}) \right] = r(A * B)$$

$$\text{ScalDistLeft} := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} ((r+s)A = rA + sA)$$

$$(1) \quad \text{TODO}$$

$$\text{ScalDistRight} := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} (r(A+B) = rA + rB)$$

$$(1) \quad \text{TODO}$$

$$\text{MulDistRight} := \forall_{A,B,C \in \mathcal{M}} ((A+B) * C = A * C + B * C)$$

$$(1) \quad (A+B) * C = [a_{i,j} + b_{i,j}] * C = \left[ \sum_{k=1}^p ((a_{i,k} + b_{i,k}) c_{k,j}) \right] = \dots$$

$$(2) \quad \dots \left[ \sum_{k=1}^p (a_{i,k} c_{k,j} + b_{i,k} c_{k,j}) \right] = \left[ \sum_{k=1}^p (a_{i,k} c_{k,j}) \right] + \left[ \sum_{k=1}^p (b_{i,k} c_{k,j}) \right] = A * C + B * C$$

$$\text{MulDistLeft} := \forall_{A,B,C \in \mathcal{M}} (C * (A+B) = C * A + C * B)$$

$$(1) \quad \text{TODO}$$

$$\text{TransAddDist} := \forall_{A,B \in \mathcal{M}} ((A+B)^T = A^T + B^T)$$

$$(1) \quad \text{TODO}$$

$$\text{TransMulDist} := \forall_{A,B \in \mathcal{M}} ((A * B)^T = B^T * A^T)$$

$$(1) \quad (A * B)^T = \left[ \sum_{k=1}^p (a_{i,k} b_{k,j}) \right]^T = \left[ \sum_{k=1}^p (a_{j,k} b_{k,i}) \right] = \left[ \sum_{k=1}^p (b_{k,i} a_{j,k}) \right] = \left[ \sum_{k=1}^p (b_{i,k}^T a_{k,j}^T) \right] = B^T * A^T$$

$$\text{Sym}[A] := A = A^T$$

$$\text{SkewSym}[A] := A = -A^T$$

$$\text{Invertible}[A] := (\text{Square}[A, n]) \wedge (\exists_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A))$$

$$\text{SymGen} := \forall_{A \in \mathcal{M}} (\text{Sym}[A + A^T])$$

$$(1) \quad (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

$$\text{SkewSymGen} := \forall_{A \in \mathcal{M}} (\text{SkewSym}[A - A^T])$$

$$(1) \quad -(A - A^T)^T = -(A^T - (A^T)^T) = -(A^T - A) = (A - A^T)$$

$$SymDecomp := \forall_{A \in \mathcal{M}} \exists!_{B: Sym[B]} \exists!_{C: SkewSym[C]} (A = B + C)$$

- (1)  $B := (1/2) * (A + A^T) ; C := (1/2) * (A - A^T)$
- (2)  $SymGen[B] \wedge SkewSymGen[C]$
- (3)  $A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$
- (4)  $(1/2) * (A_1 + A_1^T) = (1/2) * (A_2 + A_2^T) \quad \blacksquare \quad A_1 = A_2$
- (5)  $(1/2) * (A_3 - A_3^T) = (1/2) * (A_4 - A_4^T) \quad \blacksquare \quad A_3 = A_4$

$$InvId := \forall_{A: Invertible[A]} (\exists!_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A))$$

- (1)  $A^{-1}_1 = A^{-1}_1 * I_n = A^{-1}_1 * (A * A^{-1}_2) = (A^{-1}_1 * A) * A^{-1}_2 = I_n * A^{-1}_2 = A^{-1}_2$

$$InvCancel := \forall_{A: Invertible[A]} ((A^{-1})^{-1} = A)$$

- (1)  $(A * A^{-1})^{-1} = I_n^{-1} = I_n$
- (2)  $(A^{-1})^{-1} * A^{-1} = I_n \quad \blacksquare \quad (A^{-1})^{-1} = I_n * A = A$

$$InvDist := \forall_{A: Invertible[A]} \forall_{B: Invertible[B]} ((A * B)^{-1} = B^{-1} * A^{-1})$$

- (1)  $(A * B) * (A * B)^{-1} = I \quad \blacksquare \quad B * (A * B)^{-1} = A^{-1} \quad \blacksquare \quad (A * B)^{-1} = B^{-1} * A^{-1}$

$$InvTrans := \forall_{A: Invertible[A]} ((A^T)^{-1} = (A^{-1})^T) \quad \blacksquare \quad \Leftarrow$$

- (1)  $A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I \quad \blacksquare \quad (A^{-1})^T = (A^T)^{-1}$

## 3.2 Elementary Matrices on Invertibility and Systems of Linear Equations

$$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$$

$$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$$

$$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X (Sol[X, A, B])$$

$$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$$

$$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$$

$$HomoSysProps := (Sys[A, O]) \implies \dots$$

- (1)  $u_0 := O ; u_1 := choice(\{X \in \mathcal{M} | X \neq O\}) ; k := choice(\mathbb{R})$
- (2)  $TrivSol[u_0, A]$
- (3)  $(NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$
- (4)  $(TrivSol[\vec{X}, A]) \implies (TrivSol[LC(\vec{X}), A])$

$$ElemMat[E] := (E = Swap(I_n, i, j)) \vee (Scale_*(I_n, i, c)) \vee (Combine_*(I_n, i, c, j))$$

$$ElemMatProd[E^*] := \exists_{\langle E \rangle} (\forall_{E_i \in E^*} (ElemMat[E_i]) \wedge (E^* = \prod_{E_i \in E^*} (E_i)))$$

$$RowEquiv[A, B] := \exists_{E^*} ((ElemMatProd[E^*]) \wedge (B = E^* * A))$$

$$ElemMatInv := \forall_{E \in \mathcal{M}} ((ElemMat[E]) \implies (Invertible[E]))$$

- (1)  $E - RowSwap[E] \implies TODO ; E - RowScale_*(E) \implies TODO ; E - RowCombine_*(E) \implies TODO$

$$ElemMatProdInv := \forall_{E^*} ((ElemMatProd[E^*]) \implies (Invertible[E^*]))$$

- (1)  $TODO$

$$RowEquivSys := \forall_{A, B, C, D, X \in \mathcal{M}} (((Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]])) \implies (Sol[X, A, B] \iff Sol[X, C, D]))$$

- (1)  $\exists_{E^* : ElemMatProd[E^*]} ([CD] = E^* * [AB])$
- (2)  $(E^* * A = C) \wedge (E^* * B = D)$
- (3)  $Sol[Y, A, B] \implies \dots$

$$(3.1) \quad A * Y = B$$

$$(3.2) \quad C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D \quad \blacksquare \text{Sol}[Y, C, D]$$

$$(4) \quad \text{Sol}[Y, A, B] \implies \text{Sol}[Y, C, D]$$

$$(5) \quad (A = (E^*)^{-1} * C) \wedge (B = (E^*)^{-1} * D)$$

$$(6) \quad \text{Sol}[Z, C, D] \implies \dots$$

$$(6.1) \quad C * Z = D$$

$$(6.2) \quad A * Z = ((E^*)^{-1} * C) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$$

$$(7) \quad \text{Sol}[Z, C, D] \implies \text{Sol}[Z, A, B]$$

$$(8) \quad \text{Sol}[X, A, B] \iff \text{Sol}[X, C, D]$$

$$\text{RowEquivHomoSysSol} := \forall_{A, C, X \in \mathcal{M}} ((\text{RowEquiv}[A, C]) \implies ((\text{Sol}[X, A, O]) \iff (\text{Sol}[X, C, O])))$$

$$(1) \quad \text{Set } B = D = O$$

$$\text{RREF}[A] := (A \in \mathcal{M}) \wedge \left( \begin{array}{l} \text{All zero rows are at the bottom of the matrix.} \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} \\ \text{The leading entry in any nonzero row is 1.} \\ \text{All entries in the column above and below a leading 1 are zero.} \end{array} \right)$$

$$\text{GaussJordanElim} := \forall_{A \in \mathcal{M}} \exists!_{B \in \mathcal{M}} ((\text{RREF}[B]) \wedge (\text{RowEquiv}[A, B]))$$

$$(1) \quad \text{Hit } A \text{ with } \text{ElemMat}'\text{s until it becomes } B$$

$$(2) \quad (B = E^* * A) \wedge (\text{RREF}[B])$$

$$\text{HasZero}[A] := (\text{Matrix}(A, m, n)) \wedge (\exists_{i \leq m} (A_{i,:} = O))$$

$$\text{HasZeroNonInvertible} := \forall_{A \in \mathcal{M}} ((\text{HasZero}[A]) \implies (\neg \text{Invertible}[A]))$$

$$(1) \quad i := \text{choice}(\{i \leq m \mid A_{i,:} = O\})$$

$$(2) \quad (B \in \mathcal{M}) \implies \dots$$

$$(2.1) \quad (A * B)_{i,:} = O \neq I_{n_{i,:}} \quad \blacksquare \quad A * B \neq I_n$$

$$(3) \quad (B \in \mathcal{M}) \implies (A * B \neq I_n) \quad \blacksquare \quad \forall_{B \in \mathcal{M}} (A * B \neq I_n) \quad \blacksquare \quad \neg \text{Invertible}[A]$$

$$\text{InvIf f RowEquivI} := \forall_{A \in \mathcal{M}} ((\text{Invertible}[A]) \iff (\text{RowEquiv}[A, I_n]))$$

$$(1) \quad (\text{Invertible}[A]) \implies \dots$$

$$(1.1) \quad (\text{RREF}[B]) \wedge (\text{RowEquiv}[A, B])$$

$$(1.2) \quad B = E^* * A$$

$$(1.3) \quad (\text{Invertible}[E^*]) \wedge (\text{Invertible}[A]) \quad \blacksquare \quad \text{Invertible}[B]$$

$$(1.4) \quad \text{Invertible}[B] \quad \blacksquare \quad \neg \text{HasZero}[B]$$

$$(1.5) \quad (\text{RREF}[B]) \wedge (\neg \text{HasZero}[B]) \quad \blacksquare \quad B = I_n$$

$$(1.6) \quad \text{RowEquiv}[A, I_n]$$

$$(2) \quad (\text{Invertible}[A]) \implies (\text{RowEquiv}[A, I_n])$$

$$(3) \quad (\text{RowEquiv}[A, I_n]) \implies \dots$$

$$(3.1) \quad I_n = E^* * A \quad \blacksquare \quad (E^*)^{-1} = A$$

$$(3.2) \quad A^{-1} = E^*_{\text{DescSort}} \quad \blacksquare \quad \text{Invertible}[A]$$

$$(4) \quad (\text{RowEquiv}[A, I_n]) \implies (\text{Invertible}[A])$$

$$(5) \quad (\text{Invertible}[A]) \iff (\text{RowEquiv}[A, I_n])$$

$$\text{RowEquivIf f TrivSol} := \forall_{A \in \mathcal{M}} ((\text{RowEquiv}[A, I_n]) \iff (\forall_X ((X = O) \iff (\text{Sol}[X, A, O])))$$

$$(1) \quad (\text{RowEquiv}[A, I_n]) \implies \dots$$

$$(1.1) \quad \text{RowEquiv}[A, I_n] \quad \blacksquare \quad \text{Invertible}[A]$$

$$(1.2) \quad (\text{Sol}[X, A, O]) \implies \dots$$

$$(1.2.1) \quad A * X = O \quad \blacksquare \quad X = A^{-1} * O = O \quad \blacksquare \quad X = O$$

$$(1.3) \quad (\text{Sol}[X, A, O]) \implies (X = O)$$

$$(1.4) \quad (X = O) \implies (\text{Sol}[X, A, O])$$

$$(1.5) \quad (X = O) \iff (Sol[X, A, O]) \quad \blacksquare \quad \forall_X((X = O) \iff (Sol[X, A, O]))$$

$$(2) \quad (RowEquiv[A, I_n]) \implies (\forall_X((X = O) \iff (Sol[X, A, O])))$$

$$(3) \quad (\forall_X((X = O) \iff (Sol[X, A, O]))) \implies \dots$$

$$(3.1) \quad (RREF[B]) \wedge (RowEquiv[A, B])$$

$$(3.2) \quad Sol[X, B, O]$$

$$(3.3) \quad (B \neq I_n) \implies \dots$$

$$(3.3.1) \quad (\exists_{Y \neq X}(Sol[Y, B, O]))$$

$$(3.3.2) \quad Sol[Y, A, O] \quad \blacksquare \quad Y = X$$

$$(3.3.3) \quad (Y \neq X) \wedge (Y = X) \quad \blacksquare \quad \perp$$

$$(3.4) \quad (B \neq I_n) \implies \perp \quad \blacksquare \quad B = I_n$$

$$(3.5) \quad (RowEquiv[A, B]) \wedge (B = I_n) \quad \blacksquare \quad RowEquiv[A, I_n]$$

$$(4) \quad (\forall_X((X = O) \iff (Sol[X, A, O]))) \implies (RowEquiv[A, I_n])$$

$$(5) \quad (RowEquiv[A, I_n]) \iff (\forall_X((X = O) \iff (Sol[X, A, O])))$$

$$InvIf fUniqSol := \forall_{A \in \mathcal{M}}((Invertible[A]) \iff (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])))$$

$$(1) \quad (Invertible[A] \wedge B \in \mathcal{M}) \implies \dots$$

$$(1.1) \quad (Invertible[A]) \wedge (Sys[A, B])$$

$$(1.2) \quad (X = A^{-1} * B) \iff (Sol[X, A, B]) \quad \blacksquare \quad \exists!_{X \in \mathcal{M}}(Sol[X, A, B])$$

$$(2) \quad (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies \dots$$

$$(2.1) \quad X_i := choice(\{X_i | Sol[X_i, A, I_{n:,i}]\})$$

$$(2.2) \quad A * [X_1 \dots X_n] = [(A * X_1) \dots (A * X_n)] = [I_{n:,1} \dots I_{n:,n}] = I_n$$

$$(2.3) \quad A^{-1} = [X_1 \dots X_n]$$

$$(3) \quad (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies (Invertible[A])$$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}} \left( \begin{array}{ccc} (Invertible[A]) & \iff & \\ (RowEquiv[A, I_n]) & \iff & \\ (\forall_X((X = O) \iff (Sol[X, A, O]))) & \iff & \\ (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) & & \end{array} \right)$$

### 3.3 Vector Spaces

$$VectorSpace[V, +, *] := \exists_{O \in V} \forall_{\alpha, \beta \in \mathbb{R}} \forall_{u, v, w \in V} \left( \begin{array}{l} (u + v \in V) \wedge (u + v = v + u) \wedge ((u + v) + w = u + (v + w)) \wedge \\ (u + O = u) \wedge (\exists_{-u \in V}(u + (-u) = O)) \wedge \\ (\alpha * u \in V) \wedge (\alpha * (\beta * u) = (\alpha\beta) * u) \wedge (1 * u = u) \wedge \\ (\alpha * (u + v) = (\alpha * u) + (\alpha * v)) \wedge ((\alpha + \beta) * u = (\alpha * u) + (\beta * u)) \end{array} \right)$$

$$ZeroVectorUniq := \forall_{O', v \in V}((v + O' = v) \implies (O' = O))$$

$$(1) \quad O' = O' + O = O + O' = O \quad \blacksquare \quad O' = O$$

$$AddInvUniq := \forall_{-v', v \in V}((v + -v' = O) \implies (-v' = -v))$$

$$(1) \quad -v' = -v' + O = -v' + (v + -v) = (-v' + v) + -v = (v + -v') + -v = O + -v = -v \quad \blacksquare \quad -v' = -v$$

$$AddInvGen := \forall_{v \in V}((-1) * v = -v)$$

$$(1) \quad v + (-1) * v = (1 - 1) * v = 0 * v = O \quad \blacksquare \quad (-1) * v = -v$$

$$ZeroVectorGenLeft := \forall_{v \in V}(0 * v = O)$$

$$(1) \quad 0 * v = (0 + 0) * v = (0 * v) + (0 * v) \quad \blacksquare \quad O = 0 * v$$

$$ZeroVectorGenRight := \forall_{r \in \mathbb{R}}(r * O = O)$$

$$(1) \quad r * O = r * (O + O) = (r * O) + (r * O) \quad \blacksquare \quad O = r * O$$

$$ZeroVectorEquiv := \forall_{r \in \mathbb{R}} \forall_{v \in V} ((r * v = O) \iff ((v = O) \vee (r = 0)))$$

$$(1) \quad (ZeroVectorGenLeft) \wedge (ZeroVectorGenRight) \quad \blacksquare \quad ((v = O) \vee (r = 0)) \implies (r * v = O)$$

$$(2) \quad (r * v = O) \implies \dots$$

$$(2.1) \quad (r \neq 0) \implies \dots$$

$$(2.1.1) \quad r \neq 0 \quad \blacksquare \quad r^{-1} \in \mathbb{R}$$

$$(2.1.2) \quad ZeroVectorGenRight \quad \blacksquare \quad O = r^{-1} * O = r^{-1} * (r * v) = (r^{-1}r) * v = 1 * v = v \quad \blacksquare \quad O = v$$

$$(2.2) \quad (r \neq 0) \implies (v = O) \quad \blacksquare \quad (r = 0) \vee (v = O)$$

$$(3) \quad (r * v = O) \implies ((r = 0) \vee (v = O))$$

$$(4) \quad (r * v = O) \iff ((r = 0) \vee (v = O))$$

### 3.4 Subspaces and Special Subspaces

$$Subspace[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (S \subseteq V) \wedge (VectorSpace[S, +, *])$$

$$SubspaceEquiv := \forall_{V, S} \left( \begin{array}{l} (VectorSpace[V, +, *]) \\ ((Subspace[S, V, +, *]) \iff ((\emptyset \neq S \subseteq V) \wedge (\forall_{r, s \in S} (r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S} (\alpha * s \in S)))) \end{array} \implies \right)$$

$$(1) \quad (Subspace[S, V, +, *]) \implies \dots$$

$$(1.1) \quad Subspace[S, V, +, *] \quad \blacksquare \quad S \subseteq V$$

$$(1.2) \quad VectorSpace[S, V, +, *] \quad \blacksquare \quad \exists_{O \in V} \forall_{v \in V} (v + O = v) \quad \blacksquare \quad O \in S \quad \blacksquare \quad \emptyset \neq S$$

$$(1.3) \quad (\emptyset \neq S) \wedge (S \subseteq V) \quad \blacksquare \quad \emptyset \neq S \subseteq V$$

$$(1.4) \quad VectorSpace[S, V, +, *] \quad \blacksquare \quad (\forall_{r, s \in S} (r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S} (\alpha * s \in S))$$

$$(1.5) \quad (\emptyset \neq S \subseteq V) \wedge (\forall_{r, s \in S} (r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S} (\alpha * s \in S))$$

$$(2) \quad (Subspace[S, V, +, *]) \implies ((\emptyset \neq S \subseteq V) \wedge (\forall_{r, s \in S} (r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S} (\alpha * s \in S)))$$

$$(3) \quad ((\emptyset \neq S \subseteq V) \wedge (\forall_{r, s \in S} (r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S} (\alpha * s \in S))) \implies \dots$$

$$(3.1) \quad ((\emptyset \neq S) \wedge (\alpha, \beta \in \mathbb{R}) \wedge (u, v, w \in S)) \implies \dots$$

$$(3.1.1) \quad \emptyset \neq S \quad \blacksquare \quad \exists_x (x \in V)$$

$$(3.1.2) \quad (ZeroVectorGenLeft) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S} (\alpha * s \in S)) \wedge (x \in V) \quad \blacksquare \quad O = 0 * x \in S \quad \blacksquare \quad O \in S$$

$$(3.1.3) \quad u, v \in V \quad \blacksquare \quad u + v = v + u$$

$$(3.1.4) \quad u, v, w \in V \quad \blacksquare \quad (u + v) + w = u + (v + w)$$

$$(3.1.5) \quad u \in V \quad \blacksquare \quad u + O = u$$

$$(3.1.6) \quad (AddInvGen) \wedge (u \in S) \quad \blacksquare \quad (-1) * u = -u \in S$$

$$(3.1.7) \quad u \in V \quad \blacksquare \quad \alpha * (\beta * u) = (\alpha\beta) * u$$

$$(3.1.8) \quad u \in V \quad \blacksquare \quad 1 * u = u$$

$$(3.1.9) \quad u, v \in V \quad \blacksquare \quad \alpha * (u + v) = (\alpha * u) + (\alpha * v)$$

$$(3.1.10) \quad u \in V \quad \blacksquare \quad (\alpha + \beta) * u = (\alpha * u) + (\beta * u)$$

$$(4) \quad ((\emptyset \neq S) \wedge (\forall_{r, s \in S} (r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S} (\alpha * s \in S))) \implies (Subspace[S, V, +, *])$$

$$(5) \quad (Subspace[S, V, +, *]) \iff ((\emptyset \neq S) \wedge (\forall_{r, s \in S} (r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S} (\alpha * s \in S)))$$

$$SetSum[A + B, A, B, V, +, *] := (VectorSpace[V, +, *]) \wedge (A, B \subseteq V) \wedge (A + B = \{a + b | (a \in A) \wedge (b \in B)\})$$

$$SumSubContains := \forall_{A, B, V} \left( \begin{array}{l} ((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])) \\ ((Subspace[A + B, V, +, *]) \wedge (A, B \subseteq A + B)) \end{array} \implies \right)$$

$$(1) \quad (Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \quad \blacksquare \quad (O \in A) \wedge (O \in B)$$

$$(2) \quad (SetSum[A + B, A, B, V, +, *]) \wedge (O \in A) \wedge (O \in B) \quad \blacksquare \quad O = O + O \in A + B \quad \blacksquare \quad \emptyset \neq A + B$$

$$(3) \quad (v \in A + B) \implies \dots$$

$$(3.1) \quad \exists_{a \in A} \exists_{b \in B} (v = a + b)$$

$$(3.2) \quad (A \subseteq V) \wedge (B \subseteq V) \quad \blacksquare \quad a, b \in V$$

$$(3.3) \quad VectorSpace[V, +, *] \quad \blacksquare \quad v = a + b \in V$$

- (4)  $(v \in A + B) \implies (v \in V) \blacksquare A + B \subseteq V$
- (5)  $(\emptyset \neq A + B) \wedge (A + B \subseteq V) \blacksquare \emptyset \neq A + B \subseteq V$
- (6)  $(u, v \in A + B) \implies \dots$
- (6.1)  $(\exists_{a_1 \in A} \exists_{b_1 \in B} (u = a_1 + b_1)) \wedge (\exists_{a_2 \in A} \exists_{b_2 \in B} (v = a_2 + b_2))$
- (6.2)  $u + v = (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$
- (6.3)  $(a_1 + a_2 \in A) \wedge (b_1 + b_2 \in B) \blacksquare u + v \in A + B$
- (7)  $(u, v \in A + B) \implies (u + v \in A + B) \blacksquare \forall_{u, v \in A + B} (u + v \in A + B)$
- (8)  $((r \in \mathbb{R}) \wedge (v \in A + B)) \implies \dots$
- (8.1)  $\exists_{a \in A} \exists_{b \in B} (v = a + b)$
- (8.2)  $r * v = r * (a + b) = r * a + r * b$
- (8.3)  $(r * a \in A) \wedge (r * b \in B) \blacksquare r * v \in A + B$
- (9)  $((r \in \mathbb{R}) \wedge (v \in A + B)) \implies (r * v \in A + B) \blacksquare \forall_{r \in \mathbb{R}} \forall_{v \in A + B} (r * v \in A + B)$
- (10)  $(SubspaceEquiv) \wedge (\emptyset \neq A + B \subseteq V) \wedge (\forall_{u, v \in A + B} (u + v \in A + B)) \wedge (\forall_{r \in \mathbb{R}} \forall_{v \in A + B} (r * v \in A + B)) \blacksquare Subspace[A + B, V, +, *]$
- (11)  $(O \in B) \wedge (\forall_{a \in A} (a + O) = a) \blacksquare A \subseteq A + B$
- (12)  $(O \in A) \wedge (\forall_{b \in B} (b + O) = b) \blacksquare B \subseteq A + B$
- (13)  $(A \subseteq A + B) \wedge (B \subseteq A + B) \blacksquare A, B \subseteq A + B$
- (14)  $(Subspace[A + B, V, +, *]) \wedge (A, B \subseteq A + B)$

$$SumSubMinContains := \forall_{A, B, V} \left( ((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])) \implies \right. \\ \left. (\forall_C ((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies (A + B \subseteq C)) \right)$$

- (1)  $SumSub \blacksquare (A, B \subseteq A + B) \wedge (Subspace[A + B, V, +, *])$
- (2)  $((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies \dots$
- (2.1)  $(s \in A + B) \implies \dots$
- (2.1.1)  $\exists_{a \in A} \exists_{b \in B} (s = a + b)$
- (2.1.2)  $(A, B \subseteq C) \blacksquare a, b \in C$
- (2.1.3)  $(VectorSpace[C, V, +, *]) \wedge (a, b \in C) \blacksquare s = a + b \in C$
- (2.2)  $(s \in A + B) \implies (s \in C) \blacksquare A + B \subseteq C$
- (3)  $((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies (A + B \subseteq C)$

$$DirSum[A \oplus B, A, B, V, +, *] := \left( (Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge \right. \\ \left. (SetSum[A + B, A, B, V, +, *]) \wedge (\forall_{s \in A + B} \exists!_{\langle a, b \rangle \in A \times B} (s = a + b)) \right)$$

$$DirSumEquiv := \forall_{A, B, V} \left( ((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])) \implies \right. \\ \left. ((DirSum[A \oplus B, A, B, V, +, *]) \iff (\exists!_{\langle a, b \rangle \in A \times B} (O = a + b))) \right)$$

- (1)  $(DirSum[A \oplus B, A, B, V, +, *]) \implies \dots$
- (1.1)  $(Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \blacksquare (O \in A) \wedge (O \in B)$
- (1.2)  $(SubSum[A \oplus B, A, B, V, +, *]) \wedge (O \in A) \wedge (O \in B) \blacksquare O = O + O \in A \oplus B$
- (1.3)  $(DirSum[A \oplus B, A, B, V, +, *]) \wedge (O \in A \oplus B) \blacksquare \exists!_{\langle a, b \rangle \in A \times B} (O = a + b)$
- (2)  $(DirSum[A \oplus B, A, B, V, +, *]) \implies (\exists!_{\langle a, b \rangle \in A \times B} (O = a + b))$
- (3)  $(\exists!_{\langle a, b \rangle \in A \times B} (O = a + b)) \implies \dots$
- (3.1)  $(s \in A \oplus B) \implies \dots$
- (3.1.1)  $(\exists_{\langle a, b \rangle \in A \times B} (s = a + b))$
- (3.1.2)  $((s = a_1 + b_1) \wedge (s = a_2 + b_2)) \implies \dots$
- (3.1.2.1)  $O = s - s = (a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2)$
- (3.1.2.2)  $(Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \blacksquare (a_1 - a_2 \in A) \wedge (b_1 - b_2 \in B)$
- (3.1.2.3)  $((a_1 - a_2 \neq O) \vee (b_1 - b_2 \neq O)) \implies (\neg \exists!_{\langle a, b \rangle \in A \times B} (O = a + b)) \implies \perp$
- (3.1.2.4)  $(a_1 - a_2 = O) \wedge (b_1 - b_2 = O) \blacksquare \langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$
- (3.1.3)  $((s = a_1 + b_1) \wedge (s = a_2 + b_2)) \implies \langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$
- (3.1.4)  $\forall_{\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B} (((s = a_1 + b_1) \wedge (s = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle))$
- (3.1.5)  $\exists_{\langle a, b \rangle \in A \times B} (s = a + b) \wedge \forall_{\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B} (((s = a_1 + b_1) \wedge (s = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle)) \blacksquare \exists!_{\langle a, b \rangle \in A \times B} (s = a + b)$
- (3.2)  $(s \in A + B) \implies \exists!_{\langle a, b \rangle \in A \times B} (s = a + b) \blacksquare \forall_{s \in A + B} \exists!_{\langle a, b \rangle \in A \times B} (s = a + b) \blacksquare DirSum[A \oplus B, A, B, V, +, *]$

$$(4) \quad (\exists!_{\langle a,b \rangle \in A \times B} (O = a + b)) \implies (DirSum[A \oplus B, A, B, V, +, *])$$

$$(5) \quad (DirSum[A \oplus B, A, B, V, +, *]) \iff (\exists!_{\langle a,b \rangle \in A \times B} (O = a + b))$$

$$DirSumSubspace := \forall_{A,B,V} \left( ((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])) \implies ((DirSum[A \oplus B, A, B, V, +, *]) \iff (A \cap B = \{O\})) \right)$$

$$(1) \quad (DirSum[A \oplus B, A, B, V, +, *]) \implies \dots$$

$$(1.1) \quad (v \in A \cap B) \implies \dots$$

$$(1.1.1) \quad (v \in A \cap B) \wedge (VectorSpace[B, +, *]) \blacksquare (v \in A) \wedge (v \in B) \blacksquare (v \in A) \wedge (-v \in B)$$

$$(1.1.2) \quad (v \in A) \wedge (-v \in B) \blacksquare v + (-v) = O \in A + B$$

$$(1.1.3) \quad DirSum[A \oplus B, A, B, V, +, *] \blacksquare \exists!_{\langle a,b \rangle \in A \times B} (O = a + b)$$

$$(1.1.4) \quad (v \neq O) \implies (\neg \exists!_{\langle a,b \rangle \in A \times B} (O = a + b)) \implies \perp \blacksquare v = O$$

$$(1.2) \quad (v \in A \cap B) \implies (v = O) \blacksquare A + B \subseteq \{O\}$$

$$(1.3) \quad (v = O) \implies \dots$$

$$(1.3.1) \quad (Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \blacksquare (O \in A) \wedge (O \in B) \blacksquare v = O \in A \cup B$$

$$(1.4) \quad (v = O) \implies (v \in A \cap B) \blacksquare \{O\} \subseteq A \cap B$$

$$(1.5) \quad (A + B \subseteq \{O\}) \wedge (\{O\} \subseteq A \cap B) \blacksquare A \cap B = \{O\}$$

$$(2) \quad (DirSum[A \oplus B, A, B, V, +, *]) \implies (A \cap B = \{O\})$$

$$(3) \quad (A \cap B = \{O\}) \implies \dots$$

$$(3.1) \quad (O \in A) \wedge (O \in B) \wedge (O = O + O \in A + B) \blacksquare \exists_{\langle a,b \rangle \in A \times B} (O = a + b)$$

$$(3.2) \quad (((\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle) \in A \times B) \wedge (O = a_1 + b_1) \wedge (O = a_2 + b_2)) \implies \dots$$

$$(3.2.1) \quad (O = a_1 + b_1) \wedge (O = a_2 + b_2) \blacksquare (a_1 = -b_1) \wedge (a_2 = -b_2)$$

$$(3.2.2) \quad VectorSpace[B, +, *] \blacksquare -b_1, -b_2 \in B$$

$$(3.2.3) \quad (a_1 \in A) \wedge (a_1 = -b_1 \in B) \blacksquare a_1 \in A \cap B \blacksquare a_1 = O \blacksquare a_1 = b_1 = O$$

$$(3.2.4) \quad (a_2 \in A) \wedge (a_2 = -b_2 \in B) \blacksquare a_2 \in A \cap B \blacksquare a_2 = O \blacksquare a_2 = b_2 = O$$

$$(3.2.5) \quad \langle a_1, b_1 \rangle = \langle O, O \rangle = \langle a_2, b_2 \rangle$$

$$(3.3) \quad (((\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle) \in A \times B) \wedge (O = a_1 + b_1) \wedge (O = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle)$$

$$(3.4) \quad \forall_{\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B} (((O = a_1 + b_1) \wedge (O = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle))$$

$$(3.5) \quad (\exists_{\langle a,b \rangle \in A \times B} (O = a + b)) \wedge (\forall_{\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B} (((O = a_1 + b_1) \wedge (O = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle)))$$

$$(3.6) \quad (\exists!_{\langle a,b \rangle \in A \times B} (O = a + b)) \wedge (DirSumEquiv) \blacksquare DirSum[A \oplus B, A, B, V, +, *]$$

$$(4) \quad (A \cap B = \{O\}) \implies (DirSum[A \oplus B, A, B, V, +, *])$$

$$(5) \quad (DirSum[A \oplus B, A, B, V, +, *]) \iff (A \cap B = \{O\})$$

$$NullSpace[N, A, m, n] := (Matrix[A, m, n]) \wedge (N = \{x \in \mathbb{R}^n | A * x = O\})$$

$$RowSpace[R, A, m, n] := (Matrix[A, m, n]) \wedge (R = \{x^T * A \in \mathbb{R}^n | x \in \mathbb{R}^m\})$$

$$ColSpace[C, A, m, n] := (Matrix[A, m, n]) \wedge (C = \{A * x \in \mathbb{R}^m | x \in \mathbb{R}^n\})$$

$$NullSubspace := (NullSpace[N, A, m, n]) \implies (Subspace[N, \mathbb{R}^n, +, *])$$

$$(1) \quad \text{TODO}$$

$$RowSubspace := (RowSpace[R, A, m, n]) \implies (Subspace[R, \mathbb{R}^n, +, *])$$

$$(1) \quad \text{TODO}$$

$$ColSubspace := (ColSpace[C, A, m, n]) \implies (Subspace[C, \mathbb{R}^m, +, *])$$

$$(1) \quad \text{TODO}$$

### 3.5 Linear Combination, Linear Span, Linear Independence

$$LinComb[c, U, K, V, +, *] := (VectorSpace[V, +, *]) \wedge (n \in \mathbb{N}) \wedge (U \in V^n) \wedge (K \in \mathbb{R}^n) \wedge (c = \sum_{i=1}^n (k_i * u_i))$$

$$LinSpan[S', S, V, +, *] := \left( (VectorSpace[V, +, *]) \wedge (S \in V^n) \wedge ((S = \emptyset) \implies (S' = \{O\})) \wedge ((S \neq \emptyset) \implies (S' = \{c \in V | (K \in \mathbb{R}^n) \wedge (LinComb[c, S, K, V, +, *])\})) \right)$$



$$\text{LinSpanSubContains} := \forall_{S', S, V} ((\text{LinSpan}[S', S, V, +, *]) \implies ((\text{Subspace}[S', V, +, *]) \wedge (S \subseteq S')))$$

(1)  $(S = \emptyset) \implies \dots$

(1.1)  $\text{LinSpan}[S', S, V, +, *] \blacksquare S' = \{O\}$

(1.2)  $\text{Subspace}[\{O\}, V, +, *] \blacksquare \text{Subspace}[S', V, +, *]$

(1.3)  $S = \emptyset \subseteq \{O\} = S' \blacksquare S \subseteq S'$

(1.4)  $(\text{Subspace}[S', V, +, *]) \wedge (S \subseteq S')$

(2)  $(S = \emptyset) \implies ((\text{Subspace}[S', V, +, *]) \wedge (S \subseteq S'))$

(3)  $(S \neq \emptyset) \implies \dots$

(3.1)  $\text{LinSpan}[S', S, V, +, *] \blacksquare S' = \{c \in V \mid (K \in \mathbb{R}^n) \wedge (\text{LinComb}[c, S, K, V, +, *])\} \blacksquare S' \subseteq V$

(3.2)  $(\{0\}^n \subseteq \mathbb{R}^n) \wedge (\text{LinComb}[O, S, \{0\}^n, V, +, *]) \blacksquare O \in S' \blacksquare \emptyset \neq S'$

(3.3)  $(S' \subseteq V) \wedge (\emptyset \neq S') \blacksquare \emptyset \neq S' \subseteq V$

(3.4)  $(a, b \in S') \implies \dots$

(3.4.1)  $(\exists_{K_a \in \mathbb{R}^n} (\text{LinComb}[a, S, K_a, V, +, *])) \wedge (\exists_{K_b \in \mathbb{R}^n} (\text{LinComb}[b, S, K_b, V, +, *])) \blacksquare (a = \sum_{i=1}^n (k_{ai} * s_i)) \wedge (b = \sum_{i=1}^n (k_{bi} * s_i))$

(3.4.2)  $a + b = \sum_{i=1}^n (k_{ai} * s_i) + \sum_{i=1}^n (k_{bi} * s_i) = \sum_{i=1}^n ((k_{ai} + k_{bi}) * s_i) \blacksquare a + b = \sum_{i=1}^n ((k_{ai} + k_{bi}) * s_i)$

(3.4.3)  $\langle k_{ai} + k_{bi} \mid i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n$

(3.4.4)  $(a + b = \sum_{i=1}^n ((k_{ai} + k_{bi}) * s_i)) \wedge (\langle k_{ai} + k_{bi} \mid i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n) \dots$

(3.4.5)  $\dots \exists_{M \in \mathbb{N}^n} (a + b = \sum_{i=1}^n (m_i * s_i)) \blacksquare \exists_{M \in \mathbb{N}^n} (\text{LinComb}[a + b, S, M, V, +, *]) \blacksquare a + b \in S'$

(3.5)  $(a, b \in S') \implies (a + b \in S') \blacksquare \forall_{a, b \in S'} (a + b \in S')$

(3.6)  $((r \in \mathbb{R}) \wedge (u \in S')) \implies \dots$

(3.6.1)  $\exists_{K \in \mathbb{R}^n} (\text{LinComb}[u, S, K, V, +, *]) \blacksquare u = \sum_{i=1}^n (k_i * s_i)$

(3.6.2)  $r * u = r * \sum_{i=1}^n (k_i * s_i) = \sum_{i=1}^n (r * (k_i * s_i)) = \sum_{i=1}^n (rk_i) * s_i \blacksquare r * u = \sum_{i=1}^n (rk_i) * s_i$

(3.6.3)  $\langle rk_i \in \mathbb{R} \mid i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n$

(3.6.4)  $(r * u = \sum_{i=1}^n (rk_i) * s_i) \wedge (\langle rk_i \in \mathbb{R} \mid i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n) \blacksquare \exists_{M \in \mathbb{R}^n} (r * u = \sum_{i=1}^n (m_i * s_i))$

(3.6.5)  $\exists_{M \in \mathbb{R}^n} (\text{LinComb}[r * u, S, M, V, +, *]) \blacksquare r * u \in S'$

(3.7)  $((r \in \mathbb{R}) \wedge (u \in S')) \implies (r * u \in S') \blacksquare \forall_{r \in \mathbb{R}} \forall_{u \in S'} (r * u \in S')$

(3.8)  $(\text{SubspaceEquiv}) \wedge (\emptyset \neq S' \subseteq V) \wedge (\forall_{a, b \in S'} (a + b \in S')) \wedge (\forall_{r \in \mathbb{R}} \forall_{u \in S'} (r * u \in S')) \blacksquare \text{Subspace}[S', V, +, *]$

(3.9)  $(s_i \in S) \implies \dots$

(3.9.1)  $K_s := \left\langle \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \mid j \in \mathbb{N}_{1,n} \right\rangle \blacksquare (K_s \in \mathbb{R}^n) \wedge (\sum_{j=1}^n (k_{sj} * s_j) = s_i)$

(3.9.2)  $\exists_{K \in \mathbb{R}^n} (\text{LinComb}[s_j, S, K, V, +, *]) \blacksquare s_j \in S'$

(3.10)  $(s_i \in S) \implies (s_i \in S') \blacksquare S \subseteq S'$

(3.11)  $(\text{Subspace}[S', V, +, *]) \wedge (S \subseteq S')$

(4)  $(S \neq \emptyset) \implies ((\text{Subspace}[S', V, +, *]) \wedge (S \subseteq S'))$

(5)  $((S = \emptyset) \implies ((\text{Subspace}[S', V, +, *]) \wedge (S \subseteq S'))) \wedge ((S \neq \emptyset) \implies ((\text{Subspace}[S', V, +, *]) \wedge (S \subseteq S'))) \dots$

(6)  $\dots (\text{Subspace}[S', V, +, *]) \wedge (S \subseteq S')$

$$\text{LinSpanSubMinContains} := \forall_{S', S, V, +, *} ((\text{LinSpan}[S', S, V, +, *]) \implies (\forall_{W} (((\text{Subspace}[W, V, +, *]) \wedge (S \subseteq W)) \implies (S' \subseteq W)))$$

(1)  $(s' \in S') \implies \dots$

(1.1)  $\exists_{K \in \mathbb{R}^n} (\text{LinComb}[s', S, K, V, +, *]) \blacksquare s' = \sum_{i=1}^n (k_i * s_i)$

(1.2)  $(S \subseteq W) \wedge (\text{VectorSpace}[W, V, +, *]) \blacksquare s' = \sum_{i=1}^n (k_i * s_i) \in W \blacksquare s' \in W$

(2)  $(s' \in S') \implies (s' \in W) \blacksquare S' \subseteq W$

$$\text{Spans}[S, V, +, *] := \text{LinSpan}[V, S, V, +, *]$$

$$\text{FinDim}[V, +, *] := \exists_{S \in V^n} (\text{Spans}[S, V, +, *])$$

$$\text{LinInd}[S, V, +, *] := (\text{VectorSpace}[V, +, *]) \wedge (S \in V^n) \wedge ((S \neq \emptyset) \implies (\forall_{K \in \mathbb{R}^n} ((\text{LinComb}[O, S, K, V, +, *]) \implies (K = \{0\}^n))))$$

$$\text{ZeroDependent} := (O \in S) \implies (\neg \text{LinInd}[S, V, +, *])$$

$$(1) \quad O \in S \quad \blacksquare \quad \exists_{u_i \in S} (u_i = O) \quad \blacksquare \quad K := \left\langle \begin{pmatrix} 1 & u_i = O \\ 0 & u_i \neq O \end{pmatrix} \middle| i \in \mathbb{N}_{1,n} \right\rangle \quad \blacksquare \quad \{O\}^n \neq K \in \mathbb{R}^n$$

$$(2) \quad O = \sum_{i=1}^n (k_i * s_i) \quad \blacksquare \quad \text{LinComb}[O, S, K, V, +, *]$$

$$(3) \quad (\text{LinComb}[O, S, K, V, +, *]) \wedge (\{O\}^n \neq K \in \mathbb{R}^n) \quad \blacksquare \quad \exists_{K \in \mathbb{R}^n} ((\text{LinComb}[O, S, K, V, +, *]) \wedge (K \neq \{O\}^n)) \quad \blacksquare \quad \neg \text{LinInd}[S, V, +, *]$$

$$\text{SingletonNonZeroIndependent} := (v \neq O) \implies (\text{LinInd}[\langle v \rangle, V, +, *])$$

$$(1) \quad ((\langle r \rangle \in \mathbb{R}^1) \wedge (\text{LinComb}[O, \langle v \rangle, \langle r \rangle, V, +, *])) \implies \dots$$

$$(1.1) \quad (\text{ZeroVectorEquiv}) \wedge (r * v = O) \quad \blacksquare \quad (r * v = O) \iff ((r = 0) \vee (v \neq O))$$

$$(1.2) \quad v \neq O \quad \blacksquare \quad r = 0$$

$$(2) \quad ((\langle r \rangle \in \mathbb{R}^1) \wedge (\text{LinComb}[O, \langle v \rangle, \langle r \rangle, V, +, *])) \implies (r = 0) \quad \blacksquare \quad \forall_{r \in \mathbb{R}} ((\text{LinComb}[O, \langle v \rangle, \langle r \rangle, V, +, *]) \implies (r = 0))$$

$$(3) \quad \text{LinInd}[\langle v \rangle, V, +, *]$$

$$\text{SubIndependent} := \forall_{V,A,B} \left( ((\text{VectorSpace}[V, +, *]) \wedge (A \subseteq B) \wedge (A \in V^n) \wedge (B \in V^m)) \implies \right. \\ \left. ((\text{LinInd}[B, V, +, *]) \implies (\text{LinInd}[A, V, +, *])) \right)$$

$$(1) \quad ((K \in \mathbb{R}^n) \wedge (\text{LinComb}[O, A, K, V, +, *])) \implies \dots$$

$$(1.1) \quad n \leq m \quad \blacksquare \quad L := \left\langle \begin{pmatrix} k_j & j \leq n \\ 0 & j > n \end{pmatrix} \middle| j \in \mathbb{N}_{1,m} \right\rangle \quad \blacksquare \quad L \in \mathbb{R}^m$$

$$(1.2) \quad A \subseteq B \quad \blacksquare \quad \forall_{j \in \mathbb{N}_{1,n}} (a_j = b_j) \quad \blacksquare \quad \sum_{i=1}^n (k_i * a_i) = \sum_{j=1}^m (l_j * b_j)$$

$$(1.3) \quad \text{LinComb}[O, A, K, V, +, *] \quad \blacksquare \quad O = \sum_{i=1}^n (k_i * a_i) = \sum_{j=1}^m (l_j * b_j) \quad \blacksquare \quad \text{LinComb}[O, B, L, V, +, *]$$

$$(1.4) \quad (\text{LinInd}[B, V, +, *]) \wedge (\text{LinComb}[O, B, L, V, +, *]) \quad \blacksquare \quad L = \{0\}^m \quad \blacksquare \quad K = \{0\}^n$$

$$(2) \quad ((K \in \mathbb{R}^n) \wedge (\text{LinComb}[O, A, K, V, +, *])) \implies (K = \{0\}^n) \quad \blacksquare \quad \text{LinInd}[A, V, +, *]$$

$$\text{SuperDependent} := \forall_{V,A,B} (((\text{VectorSpace}[V, +, *]) \wedge (A \subseteq B \subseteq V)) \implies ((\neg \text{LinInd}[A, V, +, *]) \implies (\neg \text{LinInd}[B, V, +, *])))$$

$$(1) \quad \neg \text{LinInd}[A, V, +, *] \quad \blacksquare \quad \exists_K ((\text{LinComb}[O, A, K, V, +, *]) \wedge (K \neq \{0\}^n))$$

$$(2) \quad n \leq m \quad \blacksquare \quad L := \left\langle \begin{pmatrix} k_j & j \leq n \\ 0 & j > n \end{pmatrix} \middle| j \in \mathbb{N}_{1,m} \right\rangle \quad \blacksquare \quad L \in \mathbb{R}^m$$

$$(3) \quad A \subseteq B \quad \blacksquare \quad \forall_{j \in \mathbb{N}_{1,n}} (a_j = b_j) \quad \blacksquare \quad \sum_{i=1}^n (k_i * a_i) = \sum_{j=1}^m (l_j * b_j)$$

$$(4) \quad \text{LinComb}[O, A, K, V, +, *] \quad \blacksquare \quad \text{LinComb}[O, B, L, V, +, *]$$

$$(5) \quad K \neq \{0\}^n \quad \blacksquare \quad L \neq \{0\}^m$$

$$(6) \quad \exists_L ((\text{LinComb}[O, B, L, V, +, *]) \wedge (L \neq \{0\}^m)) \quad \blacksquare \quad \neg \text{LinInd}[B, V, +, *]$$

$$\text{LinDepProp} := \forall_{S,V} ((\neg \text{LinInd}[S, V, +, *]) \implies (\exists_{s_j \in S} \exists_{K \in \mathbb{R}^{n-1}} (\text{LinComb}[s_j, S \setminus \{s_j\}, K, V, +, *])))$$

$$(1) \quad \neg \text{LinInd}[S, V, +, *] \quad \blacksquare \quad \exists_{K \in \mathbb{R}^n} ((\text{LinComb}[O, S, K, V, +, *]) \wedge (K \neq \{0\}^n))$$

$$(2) \quad K \neq \{0\}^n \quad \blacksquare \quad \exists_{j \in \mathbb{N}_{1,n}} ((k_j \neq 0) \wedge (\forall_{i \in \mathbb{N}_{j+1,n}} (k_i = 0))) \quad \dots$$

$$(3) \quad \dots \sum_{i=1}^n (k_i * s_i) = \sum_{i=1}^j (k_i * s_i) = \sum_{i=1}^{j-1} (k_i * s_i) + k_j * s_j \quad \blacksquare \quad \sum_{i=1}^n (k_i * s_i) = \sum_{i=1}^{j-1} (k_i * s_i) + k_j * s_j$$

$$(4) \quad (\text{LinComb}[O, S, K, V, +, *]) \wedge (\sum_{i=1}^n (k_i * s_i) = \sum_{i=1}^{j-1} (k_i * s_i) + k_j * s_j) \quad \blacksquare \quad O = \sum_{i=1}^n (k_i * s_i) = \sum_{i=1}^{j-1} (k_i * s_i) + k_j * s_j$$

$$(5) \quad s_j = (-1/k_j) \sum_{i=1}^{j-1} (k_i * s_i) = \sum_{i=1}^{j-1} ((-k_i/k_j) * s_i) \quad \blacksquare \quad s_j = \sum_{i=1}^{j-1} ((-k_i/k_j) * s_i)$$

$$(6) \quad \exists_{K \in \mathbb{R}^{n-1}} (\text{LinComb}[s_j, S \setminus \{s_j\}, K, V, +, *])$$

$$\text{LinDepPropCorollary} := \forall_{P,S,V} (((\neg \text{LinInd}[S, V, +, *]) \wedge (\text{LinSpan}[P, S, V, +, *])) \implies (\exists_{s_j \in S} (\text{LinSpan}[P, S \setminus \{s_j\}, V, +, *])))$$

$$(1) \quad \text{LinDepProp} \quad \blacksquare \quad \exists_{s_j \in S} \exists_{K \in \mathbb{R}^{n-1}} (\text{LinComb}[s_j, S \setminus \{s_j\}, K, V, +, *])$$

$$(2) \quad \forall_{u \in P} ((\exists_{K_1} (\text{LinComb}[u, S, K_1, V, +, *])) \implies (\exists_{K_2} (\text{LinComb}[u, S \setminus \{s_j\}, K_2, V, +, *]))) \quad \blacksquare \quad \text{LinSpan}[P, S \setminus \{s_j\}, V, +, *]$$

$$\text{LinIndEquiv} := \forall_{S,V} ((\text{LinInd}[S, V, +, *]) \iff (\forall_{s_j \in S} \forall_{K \in \mathbb{R}^{n-1}} (\neg \text{LinComb}[s_j, S \setminus \{s_j\}, K, V, +, *])))$$

$$(1) \text{ LinDepProp} \blacksquare (\neg \text{LinInd}[S, V, +, *]) \implies (\exists_{s_j \in S} \exists_{K \in \mathbb{R}^{n-1}} (\text{LinComb}[s_j, S \setminus \{s_j\}, K, V, +, *])) \dots$$

$$(2) \dots (\forall_{s_j \in S} \forall_{K \in \mathbb{R}^{n-1}} (\neg \text{LinComb}[s_j, S \setminus \{s_j\}, K, V, +, *])) \implies (\text{LinInd}[S, V, +, *])$$

$$(3) (\exists_{s_j \in S} \exists_{K \in \mathbb{R}^{n-1}} (\text{LinComb}[s_j, S \setminus \{s_j\}, K, V, +, *])) \implies \dots$$

$$(3.1) L := \left\langle \left\{ \begin{array}{cc} k_i & i \neq j \\ -1 & i = j \end{array} \right\} \middle| i \in \mathbb{N}_{1,n} \right\rangle \blacksquare (L \in \mathbb{R}^n) \wedge (L \neq \{0\}^n)$$

$$(3.2) \text{LinComb}[s_j, S \setminus \{s_j\}, K, V, +, *] \blacksquare \dots \blacksquare \sum_{i=1}^{j-1} (k_i * s_i) + k_j * s_j = \sum_{i=1}^{j-1} (k_i * s_i) + - \sum_{i=1}^{j-1} (k_i * s_i) = O \dots$$

$$(3.3) \dots \text{LinComb}[O, S, L, V, +, *]$$

$$(3.4) (\text{LinComb}[O, S, L, V, +, *]) \wedge (L \neq \{0\}^n) \blacksquare \exists_{L \in \mathbb{R}^n} ((\text{LinComb}[O, S, L, V, +, *]) \wedge (L \neq \{0\}^n)) \blacksquare (\neg \text{LinInd}[S, V, +, *])$$

$$(4) (\exists_{s_j \in S} \exists_{K \in \mathbb{R}^{n-1}} (\text{LinComb}[s_j, S \setminus \{s_j\}, K, V, +, *])) \implies (\neg \text{LinInd}[S, V, +, *])$$

$$(5) (\text{LinInd}[S, V, +, *]) \implies (\forall_{s_j \in S} \forall_{K \in \mathbb{R}^{n-1}} (\neg \text{LinComb}[s_j, S \setminus \{s_j\}, K, V, +, *]))$$

$$(6) (\text{LinInd}[S, V, +, *]) \iff (\forall_{s_j \in S} \forall_{K \in \mathbb{R}^{n-1}} (\neg \text{LinComb}[s_j, S \setminus \{s_j\}, K, V, +, *]))$$

$$\text{LinIndSuperspace} := \forall_{U,V} ((\text{Subspace}[U, V]) \implies (\forall_W ((\text{LinInd}[W, U, +, *]) \implies (\text{LinInd}[W, V, +, *])))$$

$$(1) (\neg \text{LinInd}[W, V, +, *]) \implies \dots$$

$$(1.1) \exists_{j \in W} (\text{LinComb}[j, W \setminus \{j\}, +, *]) \blacksquare \neg \text{LinInd}[W, U, +, *]$$

$$(1.2) (\text{LinInd}[W, U, +, *]) \wedge (\neg \text{LinInd}[W, U, +, *]) \blacksquare \perp$$

$$(2) (\neg \text{LinInd}[W, V, +, *]) \implies \perp \blacksquare \text{LinInd}[W, V, +, *]$$

### 3.6 Bases and Dimensions

$$\text{Basis}[S, V, +, *] := (\text{Spans}[S, V, +, *]) \wedge (\text{LinInd}[S, V, +, *])$$

$$\text{BasisEquiv} := \forall_{S,V} ((\text{Basis}[S, V, +, *]) \iff (\forall_{v \in V} \exists!_{K \in \mathbb{R}^n} (\text{LinComb}[v, S, K, V, +, *])))$$

$$(1) (\text{Basis}[S, V, +, *]) \implies \dots$$

$$(1.1) (v \in V) \implies \dots$$

$$(1.1.1) \text{Basis}[S, V, +, *] \blacksquare \text{Spans}[V, S, +, *] \blacksquare \exists_{K \in \mathbb{R}^n} (\text{LinComb}[v, S, K, V, +, *])$$

$$(1.1.2) ((K_1, K_2 \in \mathbb{R}^n) \wedge (\text{LinComb}[v, S, K_1, V, +, *]) \wedge (\text{LinComb}[v, S, K_2, V, +, *])) \implies \dots$$

$$(1.1.2.1) (v = \sum (k_{1i} * s_i)) \wedge (v = \sum (k_{2i} * s_i))$$

$$(1.1.2.2) O = v - v = \sum (k_{1i} * s_i) - \sum (k_{2i} * s_i) = \sum ((k_{1i} - k_{2i}) * s_i)$$

$$(1.1.2.3) L := \langle k_{1i} - k_{2i} | i \in \mathbb{N}_{i=1}^n \rangle \in \mathbb{R}^n$$

$$(1.1.2.4) (\text{LinInd}[S, V, +, *]) \wedge (\text{LinComb}[O, S, L, V, +, *]) \blacksquare L = \{0\}^n \blacksquare K_2 = K_1$$

$$(1.1.3) ((K_1, K_2 \in \mathbb{R}^n) \wedge (\text{LinComb}[v, S, K_1, V, +, *]) \wedge (\text{LinComb}[v, S, K_2, V, +, *])) \implies (K_1 = K_2)$$

$$(1.1.4) \forall_{K_1, K_2 \in \mathbb{R}^n} ((\text{LinComb}[v, S, K_1, V, +, *]) \wedge (\text{LinComb}[v, S, K_2, V, +, *]) \implies (K_1 = K_2))$$

$$(1.1.5) \exists!_{K \in \mathbb{R}^n} (\text{LinComb}[v, S, K, V, +, *])$$

$$(1.2) (v \in V) \implies (\exists!_{K \in \mathbb{R}^n} (\text{LinComb}[v, S, K, V, +, *]))$$

$$(2) (\text{Basis}[S, V, +, *]) \implies (\forall_{v \in V} \exists!_{K \in \mathbb{R}^n} (\text{LinComb}[v, S, K, V, +, *]))$$

$$(3) (\forall_{v \in V} \exists!_{K \in \mathbb{R}^n} (\text{LinComb}[v, S, K, V, +, *])) \implies \dots$$

$$(3.1) \forall_{v \in V} \exists!_{K \in \mathbb{R}^n} (\text{LinComb}[v, S, K, V, +, *]) \blacksquare \forall_{v \in V} \exists_{K \in \mathbb{R}^n} (\text{LinComb}[v, S, K, V, +, *]) \blacksquare \text{Spans}[S, V, +, *]$$

$$(3.2) O \in V \blacksquare \exists!_{K \in \mathbb{R}^n} (\text{LinComb}[O, S, K, V, +, *])$$

$$(3.3) (K \neq \{0\}^n) \implies (\neg \exists!_{K \in \mathbb{R}^n} (\text{LinComb}[O, S, K, V, +, *])) \implies \perp \blacksquare K = \{0\}^n$$

$$(3.4) (\exists!_{K \in \mathbb{R}^n} (\text{LinComb}[O, S, K, V, +, *])) \wedge (K = \{0\}^n) \blacksquare \text{LinInd}[S, V, +, *]$$

$$(3.5) (\text{Spans}[S, V, +, *]) \wedge (\text{LinInd}[S, V, +, *]) \blacksquare \text{Basis}[S, V, +, *]$$

$$(4) (\forall_{v \in V} \exists!_{K \in \mathbb{R}^n} (\text{LinComb}[v, S, K, V, +, *])) \implies (\text{Basis}[S, V, +, *])$$

$$\text{SpanReduceBasis} := \forall_{S,V} ((\text{Spans}[S, V, +, *]) \implies (\exists_B ((B \subseteq S) \wedge (\text{Basis}[B, V, +, *]))))$$

$$(1) \text{LinDepPropCorollary} \blacksquare \exists_B ((B \subseteq S) \wedge (\text{LinInd}[B, V, +, *]) \wedge (\text{Spans}[B, V, +, *])) \blacksquare \exists_B ((B \subseteq S) \wedge (\text{Basis}[B, V, +, *]))$$

$$(2) \text{TODO - formalize removing latter entries first}$$

$$FinDimBasis := \forall_V((FinDim[V, +, *]) \implies (\exists_B(Basis[B, V, +, *])))$$

$$(1) \quad FinDim[V, +, *] \quad \blacksquare \quad \exists_{S \in V^n}(Spans[S, V, +, *])$$

$$(2) \quad (SpanReduceBasis) \wedge (Spans[S, V, +, *]) \quad \blacksquare \quad \exists_B(Basis[B, V, +, *])$$

$$LinIndExpandBasis := \forall_{L,V}((LinInd[L, V, +, *]) \implies (\exists_B((L \subseteq B) \wedge (Basis[B, V, +, *])))$$

$$(1) \quad FinDimBasis \quad \blacksquare \quad \exists_C(Basis[C, V, +, *])$$

$$(2) \quad S := L \cup C$$

$$(3) \quad Basis[C, V, +, *] \quad \blacksquare \quad Spans[C, V, +, *] \quad \blacksquare \quad Spans[S, V, +, *]$$

$$(4) \quad SpanReduceBasis \quad \blacksquare \quad (\exists_B((B \subseteq S) \wedge (Basis[B, V, +, *]))) \wedge (L \subseteq B)$$

$$SpanLinIndLength := \forall_{S,T,V}(((Span[S, V, +, *]) \wedge (LinInd[T, V, +, *])) \implies (|T| \leq |S|))$$

$$(1) \quad ((Span[S, V, +, *]) \wedge (|T| > |S|)) \implies \dots$$

$$(1.1) \quad Span[S, V, +, *] \quad \blacksquare \quad \forall_{i \in \mathbb{N}_{1,|H|}} \exists_{K_i \in \mathbb{R}^{|S|}}(LinComb[t_i, S, K_i V, +, *])$$

$$(1.2) \quad |H| > |S| \quad \blacksquare \quad \exists_{L \in \mathbb{R}^{|H|-1}}(LinComb[t_{|H|}, T \setminus \{t_{|H|}\}, L, V, +, *])$$

$$(1.3) \quad L = -1 * K \quad \blacksquare \quad (\sum(K + L) = 0) \wedge (K + L \neq \{0\}^{|T|}) \quad \blacksquare \quad \neg LinInd[T, V, +, *]$$

$$(1.4) \quad \text{TODO - tidy up}$$

$$(2) \quad ((Span[S, V, +, *]) \wedge (|T| > |S|)) \implies (\neg LinInd[T, V, +, *]) \quad \blacksquare \quad ((Span[S, V, +, *]) \wedge (LinInd[T, V, +, *])) \implies (|T| \leq |S|)$$

$$BasisLength := \forall_{S,T,V}(((Basis[S, V, +, *]) \wedge (Basis[T, V, +, *])) \implies (|T| = |S|))$$

$$(1) \quad (Span[T, V, +, *]) \wedge (LinInd[S, V, +, *]) \quad \blacksquare \quad |S| \leq |T|$$

$$(2) \quad (Span[S, V, +, *]) \wedge (LinInd[T, V, +, *]) \quad \blacksquare \quad |T| \leq |S|$$

$$(3) \quad (|S| \leq |T|) \wedge (|T| \leq |S|) \quad \blacksquare \quad |T| = |S|$$

$$Dim[d, V, +, *] := ((V = \{O\}) \implies (d = 0)) \wedge ((V \neq \{O\}) \implies ((\exists_B(Basis[B, V, +, *])) \wedge (d = |B|)))$$

$$LinIndLengthDim := \forall_{U,V}(((LinInd[U, V, +, *]) \wedge (Dim[|U|, V, +, *])) \implies (Basis[U, V, +, *]))$$

$$(1) \quad (LinIndExpandBasis) \wedge (LinInd[U, V, +, *]) \quad \blacksquare \quad \exists_B((U \subseteq B) \wedge (Basis[B, V, +, *]))$$

$$(2) \quad (BasisLength) \wedge (Dim[|U|, V, +, *]) \wedge (Basis[B, V, +, *]) \quad \blacksquare \quad |B| = |U| \quad \blacksquare \quad B = U \quad \blacksquare \quad Basis[U, V, +, *]$$

$$SpanLengthDim := \forall_{U,V}(((Spans[U, V, +, *]) \wedge (Dim[|U|, V, +, *])) \implies (Basis[U, V, +, *]))$$

$$(1) \quad (SpanReduceBasis) \wedge (Spans[U, V, +, *]) \quad \blacksquare \quad \exists_B((B \subseteq U) \wedge (Basis[B, V, +, *]))$$

$$(2) \quad (BasisLength) \wedge (Dim[|U|, V, +, *]) \wedge (Basis[B, V, +, *]) \quad \blacksquare \quad |B| = |U| \quad \blacksquare \quad B = U \quad \blacksquare \quad Basis[U, V, +, *]$$

$$LinDepLengthDim := \forall_{U,V}(((U \subseteq V) \wedge (|U| > Dim[V])) \implies (\neg LinInd[U, V, +, *]))$$

$$(1) \quad \text{Contrapositive of } BasisLinearIndCard$$

$$(2) \quad \text{TODO - cleanup}$$

$$NonSpanLengthDim := \forall_{U,V}(((U \subseteq V) \wedge (|U| < Dim[V])) \implies (\neg Spans[U, V, +, *]))$$

$$(1) \quad \text{Suppose } Spans[U, V, +, *], B = SpanReduceBasis[U] \text{ to form a basis, } (|B| \leq |U| < Dim[V]) \wedge |B| = Dim[V] \quad \blacksquare \quad \perp$$

$$(2) \quad \neg Spans[U, V, +, *]$$

$$(3) \quad \text{TODO - cleanup}$$

## 3.7 Rank

$$Nullity[n, A] := (NullSpace[N, A]) \wedge (Dim[n, N, +, *])$$

$$Rank[r, A, m, n] := (Matrix[A, m, n]) \wedge (RowSpace[R, A, m, n]) \wedge (Dim[r, R, A, +, *])$$

$$RowRankEqColRank := \forall_A( TODO )$$

---

(1) TODO

---

$$\text{RankNullity} := \forall_A((\text{Matrix}[A, m, n]) \implies (\text{Rank}[A] + \text{Nullity}[A] = n))$$

---

(1) TODO

---

$$\text{RankInv} := \forall_A((\text{Matrix}[A, m, n]) \implies ((\text{Rank}[A] = n) \iff (\text{Inv}[A])))$$

---

(1) TODO

---

$$\text{RankNonTrivialSol} := (\exists_X((A * X = O) \wedge (X \neq O))) \iff (\text{Rank}[A] < n)$$

---

(1) TODO

---

$$\text{RankUniqueSol} := (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(\text{Sol}[X, A, B])) \iff (\text{Rank}[A] = n)$$

---

(1) TODO

---

$$\text{SquareTheorems}_8 := \forall_{A \in \mathcal{M}} \left( \begin{array}{l} (\text{Invertible}[A]) \iff \\ (\text{RowEquiv}[A, I_n]) \iff \\ (\forall_X((X = O) \iff (\text{Sol}[X, A, O]))) \iff \\ (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(\text{Sol}[X, A, B])) \iff \\ (\text{Rank}[A] = n) \iff \\ (\text{Nullity}[A] = 0) \iff \\ (\text{The rows form a linearly independent set of vectors (to get full rank)}) \iff \\ (\text{The columns form a linearly independent set of vectors (to get full rank)}) \iff \end{array} \right)$$

### 3.8 Linear Transformations

$$\text{LinTrans}[L, V, +_v, *_v, W, +_w, *_w] := \left( \begin{array}{l} (\text{Function}[f, V, W]) \wedge (\text{VectorSpace}[V, +_v, *_v]) \wedge (\text{VectorSpace}[W, +_w, *_w]) \wedge \\ (\forall_{\alpha, \beta \in V} (L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))) \quad \wedge \quad (\forall_{r \in \mathbb{R}} \forall_{\alpha \in V} (L(r *_v \alpha) = r *_w L(\alpha))) \end{array} \right)$$

$$\text{LinOp}[L, V, +_v, *_v] := \text{LinTrans}[L, V, +_v, *_v, V, +_v, *_v]$$

$$\mathcal{L}[V, W] := \{L \mid \text{LinTrans}[L, V, +_v, *_v, W, +_w, *_w]\}$$

$$\text{ZeroMapsToZero} := \forall_{L, V, W}((\text{LinTrans}[L, V, +_v, *_v, W, +_w, *_w]) \implies (L(O_v) = O_w))$$

---

(1)  $L(O_v) = L(O_v +_v O_v) = L(O_v) +_w L(O_v)$

---

(2)  $O_w = L(O_v) - L(O_v) = L(O_v)$

---

$$\text{SplitAddInv} := \forall_{L, V, W}((\text{LinTrans}[L, V, +_v, *_v, W, +_w, *_w]) \implies (\forall_{\alpha, \beta \in V} (L(\alpha -_v \beta) = L(\alpha) -_w L(\beta))))$$

---

(1)  $L(\alpha - \beta) = L(\alpha + (-\beta)) = L(\alpha) + L(-\beta) = L(\alpha) + (-1) * L(\beta) = L(\alpha) - L(\beta)$

---

$$\text{BasisDomainInduceLinTrans} := \forall_{V, W} \left( \begin{array}{l} ((\text{Basis}[A, V, +_v, *_v]) \wedge (B \subseteq W) \wedge (n = |B| = |A|) \wedge (\text{VectorSpace}[W, +_w, *_w])) \implies \\ (\exists!_T((\text{LinTrans}[T, V, +_v, *_v, W, +_w, *_w]) \wedge (\forall_{i \in \mathbb{N}_{1,n}} (T(a_i) = b_i)))) \end{array} \right)$$

---

(1)  $T(\sum_{i=1}^n (k_i * a_i)) := \sum_{i=1}^n (k_i * b_i)$

---

(2)  $(i \in \mathbb{N}_{1,n}) \implies \dots$

---

$$(2.1) \quad L := \left\langle \left\{ \begin{array}{ll} 1 & j = i \\ 0 & j \neq i \end{array} \right\} \mid j \in \mathbb{N}_{1,n} \right\rangle \quad \blacksquare \quad L \in \mathbb{R}^n$$

$$(2.2) \quad T(a_i) = T(\sum_{i=1}^n (l_i * a_i)) = \sum_{i=1}^n (l_i * b_i) = b_i \quad \blacksquare \quad T(a_i) = b_i$$

---

(3)  $(i \in \mathbb{N}_{1,n}) \implies (T(a_i) = b_i) \quad \blacksquare \quad \forall_{i \in \mathbb{N}_{1,n}} (T(a_i) = b_i)$

---

(4)  $(\text{BasisEquiv}) \wedge (\text{Basis}[A, V, +_v, *_v]) \quad \blacksquare \quad \forall_{v \in V} \exists!_{K \in \mathbb{R}^n} (\text{LinComb}[v, A, K, V, +, *]) \quad \dots$

---

(5)  $\dots \forall_{v_1, v_2 \in V} ((v_1 = v_2) \implies (T(v_1) = T(v_2))) \quad \blacksquare \quad \text{Function}[T, V, W]$

---

(6)  $(\alpha, \beta \in V) \implies \dots$

---

$$(6.1) \quad (\exists_{K_\alpha} (\text{LinComb}[\alpha, A, K_\alpha, V, +_v, *_v]) \wedge (\exists_{K_\beta} (\text{LinComb}[\beta, A, K_\beta, V, +_v, *_v]))) \quad \blacksquare \quad (\alpha = \sum_{i=1}^n (k_{\alpha i} * a_i)) \wedge (\beta = \sum_{i=1}^n (k_{\beta i} * a_i))$$

$$(6.2) \quad T(\alpha + \beta) = T(\sum_{i=1}^n (k_{\alpha i} * a_i) + \sum_{i=1}^n (k_{\beta i} * a_i)) = T(\sum_{i=1}^n ((k_{\alpha i} + k_{\beta i}) * a_i)) = \sum_{i=1}^n ((k_{\alpha i} + k_{\beta i}) * b_i) = \dots$$

(6.3)	$\dots \sum_{i=1}^n (k_{\alpha_i} * b_i) + \sum_{i=1}^n (k_{\beta_i} * b_i) = T(\sum_{i=1}^n (k_{\alpha_i} * a_i)) + T(\sum_{i=1}^n (k_{\beta_i} * a_i)) = T(\alpha) + T(\beta)$
(7)	$(\alpha, \beta \in V) \implies (L(\alpha +_v \beta) = L(\alpha) +_w L(\beta)) \blacksquare \forall_{\alpha, \beta \in V} (L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))$
(8)	$((r \in \mathbb{R}) \wedge (\alpha \in V)) \implies \dots$
(8.1)	$\exists_K (LinComb[\alpha, A, K, V, +_v, *_v]) \blacksquare \alpha = \sum_{i=1}^n (k_i * a_i)$
(8.2)	$L(r *_v \alpha) = L(r *_v \sum_{i=1}^n (k_i *_v a_i)) = L(\sum_{i=1}^n ((rk_i) *_v a_i)) = \dots$
(8.3)	$\dots \sum_{i=1}^n ((rk_i) *_w b_i) = r *_w \sum_{i=1}^n (k_i *_w b_i) = r *_w L(\sum_{i=1}^n (k_i *_v a_i)) = r *_w L(\alpha)$
(9)	$((r \in \mathbb{R}) \wedge (\alpha \in V)) \implies (L(r *_v \alpha) = r *_w L(\alpha)) \blacksquare \forall_{r \in \mathbb{R}} \forall_{\alpha \in V} (L(r *_v \alpha) = r *_w L(\alpha))$
(10)	$(\forall_{i \in \mathbb{N}_{1,n}} (T(a_i) = b_i)) \wedge (Function[T, V, W]) \wedge (\forall_{\alpha, \beta \in V} (L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))) \wedge (\forall_{r \in \mathbb{R}} \forall_{\alpha \in V} (L(r *_v \alpha) = r *_w L(\alpha))) \wedge \dots$
(11)	$\dots (VectorSpace[V, +_v, *_v]) \wedge (VectorSpace[W, +_w, *_w]) \blacksquare (\forall_{i \in \mathbb{N}_{1,n}} (T(a_i) = b_i)) \wedge (LinTrans[T, V, +_v, *_v, W, +_w, *_w])$
(12)	$(\forall_{i \in \mathbb{N}_{1,n}} (T_2(a_i) = b_i)) \wedge (LinTrans[T_2, V, +_v, *_v, W, +_w, *_w]) \implies \dots$
(12.1)	$\forall_{i \in \mathbb{N}_{1,n}} (T_2(a_i) = b_i) \blacksquare \forall_{i \in \mathbb{N}_{1,n}} (T_2(c_i * a_i) = c_i * b_i) \blacksquare T_2(\sum_{i=1}^n (c_i * a_i)) = \sum_{i=1}^n (c_i * b_i) \blacksquare T_2 = T$
(13)	$(\forall_{i \in \mathbb{N}_{1,n}} (T_2(a_i) = b_i)) \wedge (LinTrans[T_2, V, +_v, *_v, W, +_w, *_w]) \implies (T_2 = T)$

$+_{\mathcal{L}}[S + T, S, T] := (S + T)(v) = S(v) + T(v)$   
 $*_{\mathcal{L}}[r * T, r, T] := (r * T)(v) = r * (T(v))$   
 $LTVectorSpace := \forall_{V, W} (VectorSpace[\mathcal{L}[V, W], +_{\mathcal{L}}, *__{\mathcal{L}}])$

(1) TODO

$*_{\mathcal{L}}[S * T, S, T] := (S * T)(v) = S(T(v))$   
 $LTProdProperties := (associativity) \wedge (identity) \wedge (distributive)$

(1) TODO

$Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (ker_L = \{\alpha \in V \mid L(\alpha) = O_w\})$

$KerSubspace := \forall_{L, V, W} ((Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w]) \implies (Subspace[ker_L, V, +_v, *_v]))$

(1)  $ZeroMapsToZero \blacksquare L(O_v) = O_w \blacksquare O_v \in ker_L \blacksquare \emptyset \neq ker_L \blacksquare \emptyset \neq ker_L \subseteq V$

(2)  $(\alpha, \beta \in ker_L) \implies \dots$

(2.1)  $(L(\alpha) = O_w) \wedge (L(\beta) = O_w)$

(2.2)  $L(\alpha + \beta) = L(\alpha) + L(\beta) = O_w + O_w = O_w \blacksquare L(\alpha + \beta) \in ker_L$

(3)  $(\alpha, \beta \in ker_L) \implies (\alpha + \beta \in ker_L) \blacksquare \forall_{\alpha, \beta \in ker_L} (\alpha + \beta \in ker_L)$

(4)  $((r \in \mathbb{R}) \wedge (\alpha \in ker_L)) \implies \dots$

(4.1)  $L(\alpha) = O_w \blacksquare L(r * \alpha) = r * L(\alpha) = r * O_w = O_w \blacksquare r * \alpha \in ker_L$

(5)  $((r \in \mathbb{R}) \wedge (\alpha \in ker_L)) \implies (r * \alpha \in ker_L) \blacksquare \forall_{r \in \mathbb{R}} \forall_{\alpha \in ker_L} (r * \alpha \in ker_L)$

(6)  $(SubspaceEquiv) \wedge (\emptyset \neq ker_L \subseteq V) \wedge (\forall_{\alpha, \beta \in ker_L} (\alpha + \beta \in ker_L)) \wedge (\forall_{r \in \mathbb{R}} \forall_{\alpha \in ker_L} (r * \alpha \in ker_L)) \blacksquare Subspace[ker_L, V, +_v, *_v]$

$Rng[rng_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (rng_L = \{\beta \in W \mid \exists_{\alpha \in V} (\beta = L(\alpha))\})$

$RangeSubspace := \forall_{L, V, W} ((Ran[rng_L, L, V, +_v, *_v, W, +_w, *_w]) \implies (Subspace[rng_L, W, +_w, *_w]))$

(1)  $ZeroMapsToZero \blacksquare O_w = L(O_v) \blacksquare \exists_{\alpha \in V} (O_w = L(\alpha)) \blacksquare O_w \in rng_L \blacksquare \emptyset \neq rng_L \blacksquare \emptyset \neq rng_L \subseteq W$

(2)  $(\alpha, \beta \in rng_L) \implies \dots$

(2.1)  $(\exists_{u \in V} (\alpha = L(u))) \wedge (\exists_{v \in V} (\beta = L(v)))$

(2.2)  $\alpha + \beta = L(u) + L(v) = L(u + v) \blacksquare \exists_{w \in V} (\alpha + \beta = L(w)) \blacksquare \alpha + \beta \in rng_L$

(3)  $(\alpha, \beta \in rng_L) \implies (\alpha + \beta \in rng_L) \blacksquare \forall_{\alpha, \beta \in rng_L} (\alpha + \beta \in rng_L)$

(4)  $((r \in \mathbb{R}) \wedge (\alpha \in rng_L)) \implies \dots$

(4.1)  $\exists_{v \in V} (\alpha = L(v)) \blacksquare L(r * v) = r * L(v) = r * \alpha \blacksquare \exists_{w \in V} (r * \alpha = L(w)) \blacksquare r * \alpha \in rng_L$

(5)  $((r \in \mathbb{R}) \wedge (\alpha \in rng_L)) \implies (r * \alpha \in rng_L) \blacksquare \forall_{r \in \mathbb{R}} \forall_{\alpha \in rng_L} (r * \alpha \in rng_L)$

(6)  $(SubspaceEquiv) \wedge (\emptyset \neq rng_L \subseteq W) \wedge (\forall_{\alpha, \beta \in rng_L} (\alpha + \beta \in rng_L)) \wedge (\forall_{r \in \mathbb{R}} \forall_{\alpha \in rng_L} (r * \alpha \in rng_L)) \blacksquare Subspace[rng_L, W, +_w, *_w]$

$$KerInjective := \forall_{L,V,W}((Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w]) \implies ((Injective[L, V, W]) \iff (ker_L = \{O_v\})))$$

$$(1) \quad (Injective[L, V, W]) \implies \dots$$

$$(1.1) \quad ZeroMapsToZero \quad \blacksquare \quad L(O_v) = O_w$$

$$(1.2) \quad O_v \in ker_L \quad \blacksquare \quad \{O_v\} \subseteq ker_L$$

$$(1.3) \quad (v \in ker_L) \implies \dots$$

$$(1.3.1) \quad L(v) = O_w$$

$$(1.3.2) \quad (Injective[L, V, W]) \wedge (L(O_v) = O_w) \quad \blacksquare \quad O_v = v$$

$$(1.4) \quad (v \in ker_L) \implies (v = O_v) \quad \blacksquare \quad ker_L \subseteq \{O_v\}$$

$$(1.5) \quad (\{O_v\} \subseteq ker_L) \wedge (ker_L \subseteq \{O_v\}) \quad \blacksquare \quad ker_L = \{O_v\}$$

$$(2) \quad (Injective[L, V, W]) \implies (ker_L = \{O_v\})$$

$$(3) \quad (ker_L = \{O_v\}) \implies \dots$$

$$(3.1) \quad ((u, v \in V) \wedge (L(u) = L(v))) \implies \dots$$

$$(3.1.1) \quad O_w = L(u) - L(v) = L(u - v) \quad \blacksquare \quad u - v \in ker_L$$

$$(3.1.2) \quad ker_L = \{O_v\} \quad \blacksquare \quad u - v = O_v \quad \blacksquare \quad u = v$$

$$(3.2) \quad ((u, v \in V) \wedge (L(u) = L(v))) \implies (u = v) \quad \blacksquare \quad \forall_{u,v \in V}((L(u) = L(v)) \implies (u = v)) \quad \blacksquare \quad Injective[L, V, W]$$

$$(4) \quad (ker_L = \{O_v\}) \implies (Injective[L, V, W])$$

$$(5) \quad (Injective[L, V, W]) \iff (ker_L = \{O_v\})$$

$$RankNullity := \forall_{L,V,W}((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies (Dim[V] = Dim[ker_L] + Dim[rng_L]))$$

$$(1) \quad KerSubspace \quad \blacksquare \quad (\exists_U(Basis[U, ker_L, +_v, *_v]) \wedge (Dim[ker_L] = |U|))$$

$$(2) \quad (LinIndSuperspace) \wedge (LinInd[U, ker_L, +_v, *_v]) \quad \blacksquare \quad LinInd[U, V, +_v, *_v]$$

$$(3) \quad (LinIndExpandBasis) \wedge (LinInd[U, V, +_v, *_v]) \quad \blacksquare \quad (\exists_B((U \subseteq B) \wedge (Basis[B, V, +_v, *_v]))) \wedge (Dim[V] = |B|)$$

$$(4) \quad U \subseteq B \quad \blacksquare \quad \exists_T(B = U \cup T)$$

$$(5) \quad (w \in rng_L) \implies \dots$$

$$(5.1) \quad \exists_{v \in V}(w = L(v))$$

$$(5.2) \quad (Basis[B, V, +_v, *_v]) \wedge (B = U \cup T) \quad \blacksquare \quad \exists_{K \in \mathbb{R}^{|B|}}(w = \sum_{i=1}^{|B|}(k_i * b_i) = \sum_{i=1}^{|U|}(k_i * u_i) + \sum_{i=1}^{|T|}(k_{|U|+i} * t_i))$$

$$(5.3) \quad w = L(v) = L(\sum_{i=1}^{|U|}(k_i * u_i) + \sum_{i=1}^{|T|}(k_{|U|+i} * t_i)) = L(\sum_{i=1}^{|U|}(k_i * u_i)) + L(\sum_{i=1}^{|T|}(k_{|U|+i} * t_i)) = \dots$$

$$(5.4) \quad O + L(\sum_{i=1}^{|T|}(k_{|U|+i} * t_i)) = \sum_{i=1}^{|T|}(L(k_{|U|+i} * t_i)) = \sum_{i=1}^{|T|}(k_{|U|+i} * L(t_i)) \quad \blacksquare \quad \exists_K(LinComb[w, L(T), K, W, +, *])$$

$$(6) \quad (w \in rng_L) \implies (\exists_L(LinComb[w, L(T), L, W, +, *])) \quad \blacksquare \quad Spans[L(T), rng_L, W, +, *]$$

$$(7) \quad ((K \in \mathbb{R}^n) \wedge (LinComb[O_w, L(T), K, W, +_w, *_w])) \implies \dots$$

$$(7.1) \quad O_w = \sum_{i=1}^n(k_i * L(t_i)) = L(\sum_{i=1}^n(k_i * t_i)) \quad \blacksquare \quad \sum_{i=1}^n(k_i * t_i) \in ker_L$$

$$(7.2) \quad (Basis[U, ker_L, +_v, *_v]) \wedge (\sum_{i=1}^n(k_i * t_i) \in ker_L) \quad \blacksquare \quad \exists_{D \in \mathbb{R}^m}(\sum_{i=1}^n(k_i * t_i) = \sum_{i=1}^m(d_i * u_i))$$

$$(7.3) \quad Basis[B] \quad \blacksquare \quad LinInd[B] \quad \blacksquare \quad LinInd[U \cup T] \quad \blacksquare \quad \forall_{s_j \in U \cup T} \forall_{K \in \mathbb{R}^{n-1}}(\neg LinComb[s_j, U \cup T \setminus \{s_j\}, K, V, +, *])$$

$$(7.4) \quad (\sum_{i=1}^n(k_i * t_i) = \sum_{i=1}^m(d_i * u_i)) \wedge (\forall_{s_j \in U \cup T} \forall_{K \in \mathbb{R}^{n-1}}(\neg LinComb[s_j, U \cup T \setminus \{s_j\}, K, V, +, *])) \quad \blacksquare \quad (D = \{O\}) \wedge (K = \{O\})$$

$$(8) \quad ((K \in \mathbb{R}^n) \wedge (LinComb[O_w, L(T), K, W, +_w, *_w])) \implies (K = \{O\}) \quad \blacksquare \quad LinInd[L(T), W, +_w, *_w]$$

$$(9) \quad (SubIndependent) \wedge (LinInd[L(T), W, +_w, *_w]) \quad \blacksquare \quad LinInd[L(T), rng_L, +_w, *_w]$$

$$(10) \quad (Spans[L(T), rng_L, W, +, *]) \wedge (LinInd[L(T), rng_L, +_w, *_w]) \quad \blacksquare \quad Basis[L(T), rng_L, +_w, *_w] \quad \blacksquare \quad Dim[rng_L] = |L(T)| = |T|$$

$$(11) \quad B = U \cup T \quad \blacksquare \quad |B| = |U| + |T| \quad \blacksquare \quad Dim[V] = Dim[ker_L] + Dim[rng_L]$$

$$SmallerMapNotInjective := \forall_{T,V,W}(((LinTrans[T, V, +_v, *_v, W, +_w, *_w]) \wedge (Dim[V] > Dim[W])) \implies (\neg Injective[T, V, W]))$$

$$(1) \quad (RankNullity) \wedge (Dim[W] \geq Dim[rng_T]) \quad \blacksquare \quad Dim[ker_T] = Dim[V] - Dim[rng_T] \geq Dim[V] - Dim[W] > 0 \quad \blacksquare \quad Dim[ker_T] \neq 0$$

$$(2) \quad (KerInjective) \wedge (Dim[ker_T] \neq 0) \quad \blacksquare \quad \neg Injective[T, V, W]$$

$$LargerMapNotSurjective := \forall_{T,V,W}(((LinTrans[T, V, +_v, *_v, W, +_w, *_w]) \wedge (Dim[V] < Dim[W])) \implies (\neg Surjective[T, V, W]))$$

$$(1) \quad RankNullity \quad \blacksquare \quad Dim[rng_T] = Dim[V] - Dim[ker_T] \leq Dim[V] < Dim[W]$$

$$(2) \quad Dim[rng_T] < Dim[W] \quad \blacksquare \quad Dim[rng_T] \neq Dim[W] \quad \blacksquare \quad \neg Surjective[T, V, W]$$

A linear transformation  $L : V \rightarrow W$  is one-to-one if and only if the image of every linearly independent set of vectors in  $V$  is linearly independent set of vectors in  $W$ .

(1) TODO

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

(1) TODO

An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

(1) TODO

### 3.9 Invertibility of Linear Transformations

$$LTInv[L^{-1}, L, V, +_v, *_v, W, +_w, *_w] := \left( \begin{array}{c} (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (LinTrans[L^{-1}, W, +_w, *_w, V, +_v, *_v]) \wedge \\ (L^{-1} \circ L = 1_v) \qquad \qquad \qquad \wedge \qquad \qquad \qquad (L \circ L^{-1} = 1_w) \end{array} \right)$$

$$LTInvUniq := \forall_{L_1^{-1}, L_2^{-1}} (((LTInv[L_1^{-1}, L, V, +_v, *_v, W, +_w, *_w]) \wedge (LTInv[L_2^{-1}, L, V, +_v, *_v, W, +_w, *_w])) \implies (L_1^{-1} = L_2^{-1}))$$

$$(1) \quad L_1^{-1} = L_1^{-1} \circ 1_w = L_1^{-1} \circ (L \circ L_2^{-1}) = (L_1^{-1} \circ L) \circ L_2^{-1} = 1_v \circ L_2^{-1} = L_2^{-1} \quad \blacksquare \quad L_1^{-1} = L_2^{-1}$$

$$LTInvertible[L, V, +_v, *_v, W, +_w, *_w] := \exists_{L^{-1}} (LTInv[L^{-1}, L, V, +_v, *_v, W, +_w, *_w])$$

$$InvertibleBijectiveEquiv := \forall_L ((LTInvertible[L, V, +_v, *_v, W, +_w, *_w]) \iff ((Injective[L, V, W]) \wedge (Surjective[L, V, W])))$$

$$(1) \quad (LTInvertible[L, V, +_v, *_v, W, +_w, *_w]) \implies \dots$$

(1.1) TODO

$$(2) \quad (LTInvertible[L, V, +_v, *_v, W, +_w, *_w]) \implies ((Injective[L, V, W]) \wedge (Surjective[L, V, W]))$$

$$(3) \quad ((Injective[L, V, W]) \wedge (Surjective[L, V, W])) \implies \dots$$

(3.1) TODO

$$(4) \quad ((Injective[L, V, W]) \wedge (Surjective[L, V, W])) \implies (LTInvertible[L, V, +_v, *_v, W, +_w, *_w])$$

$$(5) \quad (LTInvertible[L, V, +_v, *_v, W, +_w, *_w]) \iff ((Injective[L, V, W]) \wedge (Surjective[L, V, W]))$$

3.69 87 Injectivity is equivalent to surjectivity in finite dimensions Suppose  $V$  is finite-dimensional and  $T \in L(V)$ . Then the following are equivalent: (a)  $T$  is invertible; (b)  $T$  is injective; (c)  $T$  is surjective.

### 3.10 Matrix of a Linear Transform

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