Solutions Manual to Walter Rudin's *Principles of Mathematical Analysis*

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Chapter 7

Sequences and Series of Functions

Exercise 7.1 Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Solution. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a uniformly convergent sequence of bounded functions, say $|f_n(x)| \leq M_n$ for all x and all n. Since the sequence converges uniformly, it is a uniformly Cauchy sequence. Hence there exists N such that $|f_m(x) - f_n(x)| < 1$ for all $m, n \geq N$. In particular if $m \geq N$, we have $|f_m(x)| \leq |f_N(x)| + |f_m(x) - f_N(x)| \leq M_N + 1$, and therefore if $M = 1 + \max(M_1, \ldots, M_N)$ we have $|f_n(x)| \leq M$ for all n and x.

Exercise 7.2 If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

Solution. Let f and g denote the limits of the two sequences. Let $\varepsilon > 0$. There exist N_1 and N_2 such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all x if $n > N_1$ and $|g_n(x) - g(x)| < \frac{\varepsilon}{2}$ for all x if $n > N_2$. Let $N = \max(N_1, N_2)$. Then for $n > N$ we have, for all x,

$$|(f_n + g_n)(x) - (f + g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon.$$

Hence $\{f_n + g_n\}$ converges uniformly.

Suppose now that each of the functions f_n and g_n is bounded. By the previous problem, both sequences are uniformly bounded. Hence there exists M such that $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$ for all n and all x. It follows that $|g(x)| \leq M$ also. Then, given $\varepsilon > 0$, choose N_1 and N_2 such that $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2M}$ for all x and all $n > N_1$ and $|g_n(x) - g(x)| < \frac{\varepsilon}{2M}$ for all x and $n > N_2$.

Again let $N = \max(N_1, N_2)$. We then have, for all x and all $n > N$,

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)g_n(x) - f_n(x)g(x)| + \\ &\qquad +|f_n(x)g(x) - f(x)g(x)| \\ &\leq M|g_n(x) - g(x)| + M|f_n(x) - f(x)| \\ &< M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} \\ &= \varepsilon. \end{aligned}$$

Exercise 7.3 Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

Solution. Let $f_n(x) = x$ for all x and all n, and let $g_n(x) = \frac{1}{n}$ for all x and all n. Then $f_n(x)$ converges uniformly to x, and $g_n(x)$ converges uniformly to 0. Therefore $f_n(x)g_n(x)$ converges to 0, but not uniformly. In fact for every n there is an x, namely $x = n$, such that $f_n(x)g_n(x) = 1$. Hence, no matter how large n is taken, the inequality $|f_n(x)g_n(x)| < 1$ will never hold for all x.

Exercise 7.4 Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

Solution. The series converges for all x except 0 and $x = \frac{-1}{n^2}$, $n = 1, 2, \ldots$. For $x = 0$ all the terms of the series are defined, but the terms do not tend to 0. For $x = \frac{-1}{n^2}$ the nth term is not defined. For all other values of x the series converges. By Theorem 7.10 (the Weierstrass M-test) the series converges uniformly on the interval $[\delta, \infty)$ if $\delta > 0$, since on that interval

$$\frac{1}{1 + n^2 x} \leq \frac{1}{n^2 \delta}.$$

Likewise, the series converges uniformly on $(-\infty, -\delta]$ except at the points $x = -\frac{1}{n^2}$, since for $n \geq \sqrt{\frac{2}{\delta}}$ we have

$$\left|\frac{1}{1 + n^2 x}\right| \leq \frac{1}{n^2} \cdot \frac{1}{\delta - \frac{1}{n^2}} \leq \frac{2}{\delta n^2}.$$

The series does not converge uniformly on any interval having 0 as an endpoint. This is easy to see in the case when 0 is the left-hand endpoint. For each of the terms of the series is a bounded function on $[0,\infty)$. If the series converged uniformly, the limit would be bounded by Problem 1 above. But we have

$$f\Big(\frac{1}{m^2}\Big) \ge \sum_{n=1}^{m} \frac{1}{1+\frac{n^2}{m^2}} \ge \frac{m}{2}.$$

Likewise the series cannot be a uniformly Cauchy series (i.e., the sequence of partial sums cannot be a uniformly Cauchy sequence) on any interval $(-\delta, 0)$, since, no matter how large n is taken, there is a point x in this interval, namely $x = -\frac{1}{2n^2}$, at which the nth term has the value 2. Hence, if S_n denotes the sum of the first n terms, then $|S_n(x) - S_{n-1}(x)| = 2$.

The uniform convergence shows that the limiting function $f(x)$ is continuous wherever it is defined on $(-\infty, \delta] \cup [\delta, +\infty)$. Since δ is arbitrary, $f(x)$ is continuous wherever it is defined. The argument given above shows that $f(x)$ is not bounded.

Exercise 7.5 Let

$$f_n(x) = \begin{cases} 0 & (x < \frac{1}{n+1}), \\ \sin^2 \frac{\pi}{x} & (\frac{1}{n+1} \le x \le \frac{1}{n}), \\ 0 & (\frac{1}{n} < x). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x does not imply uniform convergence.

Solution. The limit of $f_n(x)$ is zero. If $x \le 0$ or $x \ge 1$, then $f_n(x) = 0$ for all n, and so this assertion is obvious. If $0 < x < 1$, then $f_n(x) = 0$ for all $n \ge \frac{1}{x}$, and so once again the assertion is obvious.

The convergence is not uniform, since, no matter how large n is taken, there is a point x, namely $x = \frac{1}{2n+\frac{1}{2}}$, for which $f_n(x) = 1$.

The series $\sum f_n(x)$ converges to 0 for $x \le 0$ and $x \ge 1$, and to $\sin^2 \frac{\pi}{x}$ for $0 < x < 1$. Since the terms are nonnegative, the series obviously converges absolutely. Since the sum is not continuous at 0, the series does not converge uniformly on any interval containing 0.

Exercise 7.6 Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Solution. The series is the sum of two series:

$$x^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

The first of these converges both uniformly and absolutely on any bounded interval $[a,b]$ by the M-test (with $M_n=\frac{M^2}{n^2}$, where $M=\max(|a|,|b|)$). The second is independent of x and converges, hence it converges uniformly in x. By Exercise 2 above, the sum of the two series converges uniformly.

The series does not converge absolutely since the absolute value of each term is at least $\frac{1}{n}$ for any x.

Exercise 7.7 For $n=1,2,3,\ldots$, x real, put

$$f_n(x)=\frac{x}{1+nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x)=\lim_{n\to\infty}f_n'(x)$$

is correct if $x\neq 0$, but false if $x=0$.

Solution. The Schwarz inequality, which implies that $|f_n(x)|\leq\frac{|x|}{2\sqrt{n}|x|}=\frac{1}{2\sqrt{n}}$ for $x\neq 0$, shows that $f_n(x)$ tends uniformly to 0. Now $f_n'(x)=\frac{1-nx^2}{(1+nx^2)^2}$, which tends to 0 if $x\neq 0$, though $f_n'(0)=1$ for all n.

Exercise 7.8 If

$$I(x)=\begin{cases}0 & (x\leq 0),\\ 1 & (x>0),\end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a,b), and if $\sum|c_n|$ converges, prove that the series

$$f(x)=\sum_{n=1}^{\infty}c_nI(x-x_n)\quad(a\leq x\leq b)$$

converges uniformly, and that f is continuous for every $x\neq x_n$.

Solution. The uniform convergence is a consequence of the M-test with $M_n=|c_n|$. Hence f is continuous wherever each of the individual terms is continuous, in particular, at least for $x\neq x_n$.

Exercise 7.9 Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

Solution. Let $\varepsilon > 0$. Choose N_1 so large that $|f_m(x) - f(x)| < \frac{\varepsilon}{2}$ for all $m > N_1$. Then, since f is continuous at x, choose $\delta > 0$ so small that $|f(y) - f(x)| < \frac{\varepsilon}{2}$ if $|y - x| < \delta$. Finally, choose N_2 so large that $|x_n - x| < \delta$ if $n > N_2$. Then if $n > \max(N_1, N_2)$ we have

$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon.$$

The converse is not true in general. For example let $f_n(x)$ be given by $f_n(x) = \sin^2 \pi x$ for $n \le |x| \le n+1$ and $f_n(x) = 0$ for $|x| \le n$ or $|x| \ge n+1$. Thus $f_n(x)$ tends to zero, since $f_n(x) = 0$ if $n \ge |x|$, but $f_n(x)$ does not converge uniformly, since $f_n(n + \frac{1}{2}) = 1$ Then for any convergent sequence, say $x_n \to x$, let $N \ge \max(|x|, |x_1|, |x_2|, \ldots, |x_n|, \ldots)$. We then have $f_n(x_n) = 0$ for all $n \ge N$, and so $f_n(x_n) \to f(x)$.

This condition does guarantee uniform convergence on any *compact* set, however. For if $\{f_n(x)\}$ is *not* a uniformly Cauchy sequence, then for some $\varepsilon_0 > 0$ there is a sequence of integers $n_1 < n_2 < \cdots$ and a sequence of points $x_1, x_2, \ldots$ such that

$$|f_{n_{2k-1}}(x_k) - f_{n_{2k}}(x_k)| \ge \varepsilon_0$$

for $k = 1, 2, \ldots$. Since K is compact, some subsequence of $\{x_k\}$ converges, say $x_{k_r} \to x$ as $r \to \infty$. Now define $y_n = x$ for all $n \ne n_{2k_r}$, $n \ne n_{2k_r - 1}$, and let $y_{n_{2k_r-1}} = y_{n_{2k}} = x_{k_r}$, so that so that $y_n \to x$. Then the sequence $\{z_n\} = \{f_n(y_n)\}$ is not a Cauchy sequence, since $|z_{n_{2k_r}} - z_{n_{2k_r-1}}| \ge \varepsilon_0$.

Exercise 7.10 Let $(x) = x - n$, where n is the unique integer such that $n \le x < n+1$. Prove that

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$$

is discontinuous at a dense set of points.

Solution. We shall prove that $f(x)$ is discontinuous at every rational number. Since $f(x)$ has period 1, it suffices to prove this for $0 \le x < 1$. To that end, let $x = \dfrac{p}{q}$ where p and q are relatively prime integers, $0 \le p < q$. We "stratify" the sum that defines $f(x)$ by grouping all the indices n that are congruent modulo q, i.e., we let $n = kq + r$, where $1 \le r \le q$:

$$f(x) = \sum_{k=0}^{\infty} \sum_{r=1}^{q} \frac{((kq+r)x)}{(kq+r)^2}.$$

Reversing the order of summation, we find

$$f(x) = f_1(x) + f_2(x) + \cdots + f_{q-1}(x) + f_q(x),$$

where

$$f_r(x) = \sum_{k=0}^{\infty} \frac{((kq+r)x)}{(kq+r)^2}.$$

Now it is easy to see that $f_1(x), \ldots, f_{q-1}(x)$ are continuous at $x = \frac{p}{q}$. For if $1 \leq r < q$, then $((kq+r)x)$ is continuous at that point, since (x) is continuous at the point $x = (kq+r)\frac{p}{q} = kp + \frac{rp}{q}$. (This point is not an integer, since p and q are relatively prime.) Since the series defining $f_r(x)$ converges uniformly, its limit is continuous at each point where all of the terms are continuous. In particular $f_r(x)$ is continuous at $x = \frac{p}{q}$, for $1 \leq r < q$.

We shall now show that $f_q(x)$ is discontinuous at $x = \frac{p}{q}$. It will then follow that $f(x)$ is discontinuous at that point. Observe that

$$f_q(x) = \frac{1}{q^2} \sum_{k=0}^{\infty} \frac{((k+1)qx)}{(k+1)^2} = \frac{1}{q^2} \sum_{k=1}^{\infty} \frac{(kqx)}{k^2},$$

so that

$$f_q\left(\frac{p}{q}\right) = \frac{1}{q^2} \sum_{k=1}^{1} \frac{(kp)}{k^2} = 0.$$

We shall prove that $\lim_{x\omega\, \frac{p}{q}+} f_q(x) > 0$, and this will show that $f_q(x)$ is discontinuous at $x = \frac{p}{q}$. Since all the terms of the series for f_q are nonnegative, it suffices to show that the limit of the first term is positive. To that end, let $\delta = \frac{1}{2q}$. If $\frac{p}{q} - \delta < x < \frac{p}{q}$, then $p - \frac{1}{2} < qx < p$, and hence $(qx) > \frac{1}{2}$, from which it follows that $f_q(x) \geq \frac{1}{2q^2}$. Therefore the lower left-hand limit of $f_q(x)$ at $x = \frac{p}{q}$ is at least $\frac{1}{2q^2}$.

Since, by the M-test with $M_n = \frac{1}{n^2}$, this series converges uniformly and each of its terms is Riemann-integrable, it follows from Theorem 7.16 that the sum of the series is Riemann-integrable.

Exercise 7.11 Suppose $\{f_n\}$, $\{g_n\}$ are defined on E and

(*a*) $\sum f_n$ has uniformly bounded partial sums;

(*b*) $g_n \to 0$ uniformly on E;

(c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E. *Hint:* Compare with Theorem 3.42.

Solution. Following the hint, we let $S_N(x) = \sum_{n=1}^{N} f_n(x)g_n(x)$ and $F_N(x) = \sum_{n=1}^{N} f_n(x)$ ($F_0(x) = 0$), so that $|F_N(x)| \leq B$ for all x. Then if $N > M$, we have

$$\begin{aligned}
|S_N(x) - S_M(x)| &= \left| \sum_{n=M+1}^{N} [F_n(x) - F_{n-1}(x)]g_n(x) \right| \\
&= \Bigg| F_N(x)g_N(x) - F_M(x)g_{M+1}(x) + \\
&\quad + \sum_{n=M+1}^{N-1} F_n(x)[g_n(x) - g_{n+1}(x)] \Bigg| \\
&\leq B\Big\{ |g_N(x)| + |g_{M+1}(x)| + \sum_{n=M+1}^{N-1} [g_n(x) - g_{n+1}(x)] \Big\} \\
&= B\big[|g_N(x)| + |g_{M+1}(x)| + g_{M+1}(x) - g_N(x) \big],
\end{aligned}$$

and this last expression can be made uniformly small by choosing M sufficiently large by hypothesis (b). Hypothesis (c) was used in moving the summation sign outside the absolute value.

Exercise 7.12 Suppose g and f_n ($n = 1, 2, 3, \ldots$) are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^\infty g(x)\, dx < \infty.$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx = \int_0^\infty f(x)\, dx.$$

(See Exercises 7 and 8 of Chap. 6 for the relevant definitions.)

This is a rather weak form of Lebesgue's dominated convergence theorem (Theorem 11.32). Even in the context of the Riemann integral, uniform convergence can be replaced by pointwise convergence if it is assumed that $f \in \mathcal{R}$. (See the articles by F. Cunningham in *Math. Mag.*, vol. 40, 1967, pp. 179–186, and by H. Kestelman in *Amer. Math. Monthly*, vol. 77, 1970, pp. 182–187.)

Solution. We shall prove that $\int_0^\infty f_n(x)\, dx$ converges for each n, that the limit $\lim_{n\to\infty} \int_0^\infty f_n(x)\, dx$ exists, that $\int_0^\infty f(x)\, dx$ converges and that these last two quantities are equal.

Since we obviously have $|f(x)| \le g(x)$ also, it follows that for any interval $[r,s] \subset (0,\infty)$ we have

$$\left|\int_r^s f_n(x)\,dx\right| \le \int_r^s g(x)\,dx,$$
$$\left|\int_r^s f(x)\,dx\right| \le \int_r^s g(x)\,dx,$$
$$\left|\int_r^s f_n(x) - f(x)\,dx\right| \le 2\int_r^s g(x)\,dx.$$

Now let $\varepsilon > 0$. Choose a and b with $0 < a < b < \infty$ so that if $0 < c < a < b < d < \infty$, then

$$\left|\int_c^d g(x)\,dx - \int_0^\infty g(x)\,dx\right| < \frac{\varepsilon}{2}.$$

It follows in particular that if $d > e > b$ we have

$$\begin{aligned}
\int_e^d g(x)\,dx &= \int_{\frac{a}{2}}^d g(x)\,dx - \int_{\frac{a}{2}}^e g(x)\,dx \\
&\le \left|\int_0^\infty g(x)\,dx - \int_{\frac{a}{2}}^d g(x)\,dx)\right| \\
&\quad + \left|\int_0^\infty g(x)\,dx - \int_{\frac{a}{2}}^e g(x)\,dx\right| \\
&< \varepsilon.
\end{aligned}$$

Then for any $d > e > b > r$ and any n we certainly have

$$\left|\int_r^d f_n(x)\,dx - \int_r^e f_n(x)\,dx\right| = \left|\int_d^e f_n(x)\,dx\right| \le \int_d^e g(x)\,dx < \varepsilon.$$

Thus by the Cauchy criterion $\lim\limits_{d\to\infty} \displaystyle\int_r^d f_n(x)\,dx$ exists. A similar argument shows that all the improper integrals in question converge. Moreover the argument shows that

$$\left|\int_c^d \varphi(x)\,dx - \int_0^\infty \varphi(x)\,dx\right| < \varepsilon$$

when $0 < c < a < b < d$, whether $\varphi(x) = f_n(x)$, $\varphi(x) = f(x)$, or $\varphi(x) = g(x)$.

We now merely observe that

$$\begin{aligned}
\left|\int_0^\infty f_n(x)\,dx - \int_0^\infty f(x)\,dx\right| \le& \left|\int_0^\infty f_n(x)\,dx - \int_c^d f_n(x)\,dx\right| + \\
&+\left|\int_c^d [f_n(x) - f(x)]\,dx\right| + \\
&+\left|\int_c^d f(x)\,dx - \int_0^\infty f(x)\,dx\right|.
\end{aligned}$$

Given $\varepsilon > 0$ we can choose c and d so that the first and last terms on the right are less than $\frac{\varepsilon}{3}$ (for all n, in the case of the first term). Then, since $f_n(x) \longrightarrow f(x)$ uniformly on c, d, we can choose n_0 so large that the middle term is less than $\frac{\varepsilon}{3}$ if $n > n_0$.

Exercise 7.13 Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on R^1 with $0 \le f_n(x) \le 1$ for all x and all n.

(a) Prove that there is a function f and a sequence $\{n_k\}$ such that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

for every $x \in R^1$. (The existence of such a pointwise convergent subsequence is usually called *Helly's selection theorem.*)

(b) If, moreover f is continuous, prove that $f_{n_k} \longrightarrow f$ uniformly on R^1.

Hint: (i) Some subsequence $\{f_{n_i}\}$ converges at all rational points r, say to $f(r)$. (ii) Define $f(x)$ for any $x \in R^1$ to be $\sup f(r)$, the sup being taken over all $r \le x$. (iii) Show that $f_{n_i}(x) \longrightarrow f(x)$ at every x at which f is continuous. (This where monotonicity is strongly used.) (iv) A subsequence of $\{f_{n_i}\}$ converges at every point of discontinuity of f since there are at most countably many such points. This proves (a). To prove (b), modify your proof of (iii) appropriately.

Solution. (a) Following the hint, we enumerate the rational numbers (or any countable dense set) as $\{r_n\}$ and use the well-known diagonal procedure to get first a subsequence that converges at r_1, then a further subsequence that converges at r_2, etc. The sequence formed by taking the nth term of the nth subsequence is itself a subsequence and converges at each r_n. (Note that we have not used the fact that $0 \le f_n(x) \le 1$ for all x and n, only the much weaker fact that for each x there is an $M(x)$ such that $|f_n(x)| \le M(x)$ for all x and n.) Let the function $f(x)$ be defined as $f(r_k) = \lim_{n_i} f_{n_i}(r_k)$ and $f(x) = \sup\{f(r_k) : r_k \le x\}$ for all other x. The second definition could be taken as the general one if we wished, since it is consistent with the definition already given at the points $x = r_k$.

Since each of the functions is nondecreasing, it is clear that the function $f(x)$ is nondecreasing. By its definition it is continuous from the left. Suppose $f(x)$ is continuous at x_0. Let $\varepsilon > 0$ be given. Choose rational numbers r and s with $r \le x_0 \le s$, $f(x_0) - \frac{\varepsilon}{4} \le f(r) \le f(x_0) \le f(s) \le f(x_0) + \frac{\varepsilon}{4}$. Then choose i_0 so large that $|f_{n_i}(t) - f(t)| < \frac{\varepsilon}{4}$ for all $i > i_0$, $t = r$ or $t = s$. We then have

$$f(x_0) - \frac{\varepsilon}{2} \le f(r) - \frac{\varepsilon}{4} < f_{n_i}(r) \le f_{n_i}(x_0) \le f_{n_i}(s) < f(s) + \frac{\varepsilon}{4} \le f(x_0) + \frac{\varepsilon}{2}.$$

Hence $|f(x_0) - f_{n_i}(x_0)| < \varepsilon$ if $i > i_0$, which proves the convergence at points of continuity. One more application of the diagonal procedure now allows us to assure that some subsequence converges at every point (since the set of of discontinuities of a nondecreasing function is countable). We can then modify the definition of $f(x)$ at these points.

The claim that the convergence is uniform if the limit is continuous is not true. Let

$$f_n(x) = \begin{cases} \frac{1}{3} + \frac{x}{1+3|x|}, & \text{if } x \leq n, \\ 1, & \text{if } x > n. \end{cases}$$

It is clear that $\lim_{n\to\infty} f_n(x) = \dfrac{1}{3} + \dfrac{x}{1+3|x|}$ for each x, yet $f_n(y) - f(y) \geq \frac{1}{3}$ if $y > n$, so that the convergence is not uniform. Here the functions $f_n(x)$ are not continuous, but they could easily be made so without violating the conditions of the problem.

To get uniform convergence we must assume in addition that $f(x) \to 1$ as $x \to \infty$ and $f(x) \to 0$ as $x \to -\infty$. Let us grant these relations and assume that $f(x)$ is continuous at all points x. To simplify the notation we shall write f_k instead of f_{n_k}. Given $\varepsilon > 0$, choose an interval $[a, b]$ such that $f(x) < \dfrac{\varepsilon}{2}$ if $x \leq a$ and $f(x) > 1 - \dfrac{\varepsilon}{2}$ if $x > b$. Then, since $f(x)$ is uniformly continuous on $[a, b]$, let $a = t_0 < t_1 < \cdots < t_n = b$ be such that $f(t_i) - f(t_{i-1}) < \frac{\varepsilon}{5}$. Choose k so large that $|f_l(t_i) - f(t_i)| < \frac{\varepsilon}{5}$ for all $i = 1, \ldots, n$ and all $l > k$.Then for all $y \geq b = t_n$ we have

$$1 \geq f_l(y) \geq f_l(t_k) > 1 - \frac{4\varepsilon}{5}$$

and

$$1 \geq f(y) > 1 - \frac{\varepsilon}{2} > 1 - \frac{4\varepsilon}{5}.$$

Hence certainly

$$|f_l(y) - f(y)| \leq 1 - \left(1 - \frac{4\varepsilon}{5}\right) < \varepsilon$$

for all $l > k$ and all $y \geq b$.

A similar argument shows that f_l converges uniformly to f on $(-\infty, a]$. The argument that f_l converges uniformly to f on $[t_{i-1}, t_i]$ is identical to that given above.

Exercise 7.14 Let f be a continuous real function on R^1 with the following properties: $0 \leq f(t) \leq 1$, $f(t+2) = f(t)$ for every t, and

$$f(t) = \begin{cases} 0 & (0 \leq t \leq \frac{1}{3}) \\ 1 & (\frac{2}{3} \leq t \leq 1). \end{cases}$$

Put $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \quad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

Prove that Φ is *continuous* and that Φ maps $I = [0,1]$ *onto* the unit square $I^2 \subset R^2$. In fact, show that Φ maps the Cantor set onto I^2.

Hint: Each $(x_0, y_0) \in I^2$ has the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \quad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n},$$

where each a_i is 0 or 1. If

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i),$$

show that $f(3^k t_0) = a_k$, and hence that $x(t_0) = x_0$, $y(t_0) = y_0$.

(This simple example of a so-called "space-filling curve" is due to I. J. Schoenberg, *Bull. A.M.S.*, vol. 44, 1938, p. 519.)

Solution. We note that $3^k t_0$ is the sum of the even integer $2(3^{k-2}a_1 + \cdots + 3a_{k-2} + a_{k-1})$ and a fractional part $\sum_{i=k}^{\infty} \frac{2a_i}{3^{i-k+1}}$. This fractional part lies in $\left[\frac{2}{3}, 1\right]$ if $a_k = 1$, while if $a_k = 0$ it is at least 0 and at most $\frac{2}{9}$. Thus it lies in the interval $\left[0, \frac{1}{3}\right]$ if $a_k = 0$. In either case $f(3^k t_0) = a_k$, as claimed. We therefore have

$$x(t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1} = x_0, \quad y(t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n} = y_0,$$

as asserted.

Exercise 7.15 Suppose f is a real continuous function on R^1, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$, and $\{f_n\}$ is equicontinuous on $[0,1]$. What conclusion can you draw about f?

Solution. The function $f(t)$ must be constant on $[0, \infty)$. For if $f(x) \neq f(y)$ and $0 \le x < y < \infty$, say $|f(x) - f(y)| = \varepsilon > 0$, it follows that $|f_n(\frac{x}{n}) - f_n(\frac{y}{n})| = \varepsilon$ for all n. Since $\frac{x-y}{n} \to 0$, it follows that the family $\{f_n\}$ cannot be equicontinuous on $[0,1]$, or, indeed, on any neighborhood of 0.

Exercise 7.16 Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

Solution. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$ for all $n \neq m$ if $x, y \in K$ and $|x - y| < \delta$. Choose a finite number of points $x_1, \ldots, x_N$ such that for every $x \in K$ there exists j with $|x - x_j| < \delta$. (Such a finite set exists; otherwise we could inductively select a sequence $\{x_n\}$ such that $|x_m - x_n| \ge \delta$

for all n, and this sequence would have no Cauchy subsequence, contradicting the compactness of K.) Then choose n_0 so large that $|f_m(x_j) - f_n(x_j)| < \frac{\varepsilon}{3}$ for all $m, n > n_0$ and all $j = 1, 2, \ldots, N$. Then for any point $x \in K$, fix j so that $|x - x_j| < \delta$. If $m, n > n_0$ we have

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_m(x_j)| + |f_m(x_j) - f_n(x_j)| + |f_n(x_j) - f_n(x)|.$$

The first and last terms are smaller than $\frac{\varepsilon}{3}$ because $|x - x_j| < \delta$; the middle term is smaller than $\frac{\varepsilon}{3}$ since $m, n > n_0$. Thus the sequence is a uniformly Cauchy sequence.

Exercise 7.17 Define the notions of uniform convergence and equicontinuity for mappings into any metric space. Show that Theorems 7.9 and 7.12 are valid for mappings into any metric space, that Theorems 7.8 and 7.11 are valid for mappings into any complete metric space, and that Theorems 7.10, 7.16, 7.17, 7.24, and 7.25 hold for vector-valued functions, that is, for mappings into any R^n.

Solution. Let X and Y be any metric spaces. The sequence $\{f_n\}$, where $f_n : X \to Y$, *converges uniformly* to $f : X \to Y$ if for every $\varepsilon > 0$ there exists N such that $d_Y(f_n(x), f(x)) < \varepsilon$ for all $x \in X$ and all $n > N$. A family of functions $\mathcal{F}$ is *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x_1), f(x_2)) < \varepsilon$ for all $f \in \mathcal{F}$ whenever $d_X(x_1, x_2) < \delta$.

An immediate consequence of this definition is that $\{f_n\}$ converges uniformly to f if and only if $M_n \to 0$, where $M_n = \sup\limits_{x \in X} d_Y(f_n(x), f(x))$ (Theorem 7.9).

The same $\frac{\varepsilon}{3}$ argument that proves Theorem 7.12 shows that the uniform limit of a sequence of continuous functions is continuous.

The Cauchy convergence criterion accepts the additional word *uniformly* without any change, provided Y is complete. Suppose for every $\varepsilon > 0$ there exists N such that $d_Y(f_m(x), f_n(x)) < \varepsilon$ for all $m, n > N$ and all x. Then, in particular, for each $x \in X$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, this sequence converges to a value that we shall call $f(x)$. We now claim that $\{f_n\}$ converges uniformly to f. Indeed, given $\varepsilon > 0$ choose N so that $d_Y(f_m(x), f_n(x)) < \frac{\varepsilon}{2}$ if $m, n > N$. Since a metric is a continuous function, it follows that $d_Y(f(x), f_n(x)) \leq \frac{\varepsilon}{2} < \varepsilon$ if $n > N$, that is $\{f_n\}$ converges uniformly to f. This is Theorem 7.8.

Suppose now $\{f_n\}$ converges uniformly to f, Y is complete, $x_0 \in X$, and $\lim\limits_{x \to x_0} f_n(x) = A_n$ for $n = 1, 2, \ldots$. Then $\{A_n\}$ converges, and $\lim\limits_{x \to x_0} f(x) = \lim\limits_{n \to \infty} A_n$. (This is Theorem 7.11.) The proof is as follows. Given $\varepsilon > 0$ choose N so that $d_Y(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for all x if $n \geq N$. Let $n > N$ be fixed. Choose $\delta > 0$ (depending on n and ε in general) such that $d_Y(f_n(x), A_n) < \frac{\varepsilon}{3}$ if $0 < d_X(x, x_0) < \delta$. (The fact that $\lim_{x \to x_0} f_n(x) = A_n$ implies that there must exist x satisfying these inequalities, i.e., that x_0 is an accumulation point of X.) We then have, for $m, n > N$,

$$d_Y(A_m, A_n) \leq d_Y(A_m, f_m(x)) + d_Y(f_m(x), f_n(x)) + d_Y(f_n(x), f(x))$$

The middle term is less than $\frac{\varepsilon}{3}$ for all $m, n > N$ and all $x \in X$. If m and n are then fixed integers larger than N, the first and last terms can be made smaller than $\frac{\varepsilon}{3}$ by choosing x sufficiently close to x_0. Hence we have $d_Y(A_m, A_n) < \varepsilon$ if $m, n > N$. Since Y is complete, the sequence $\{A_n\}$ converges, say to A. Now observe that

$$d_Y(f(x), A) \leq d_Y(f(x), f_n(x)) + d_Y(f_n(x), A_n) + d_Y(A_n, A).$$

If N is chosen sufficiently large, the first and last terms on the right-hand side will be less than $\frac{\varepsilon}{3}$ (for all x, in the case of the first term). For a fixed n satisfying these conditions, if $\delta > 0$ is sufficiently small, the second term will be less than $\frac{\varepsilon}{3}$ whenever $0 < d_X(x, x_0) < \delta$, and hence $d_Y(f(x), A) < \varepsilon$ if $0 < d_X(x, x_0) < \delta$.

The proof of the stated theorems for vector-valued functions is a consequence of the obvious facts that a vector-valued function $\mathbf{f}$ is integrable, differentiable or continuous if and only if each of its components has the corresponding property, and that a series of vector-valued functions $\{\mathbf{f}_n\}$ is Cauchy, bounded, convergent, uniformly convergent, majorized by a convergent sequence, equicontinuous, etc., if and only if each component has those properties. A typical proof proceeds as follows (Theorem 7.25). Suppose $\{\mathbf{f}_n\}$ is a bounded equicontinuous sequence of vector-valued functions on a compact set K. Let $\|\mathbf{f}_n(x)\| \leq M$ for all $x \in K$ and all n, and given $\varepsilon > 0$ choose $\delta > 0$ such that $\|\mathbf{f}_n(x) - \mathbf{f}_n(y)\| < \varepsilon$ whenever $d(x, y) < \delta$. Then for each component f_n^i of $\mathbf{f}_n$ we have $|f_n^i(x)| \leq \|\mathbf{f}_n\| \leq M$ and $|f_n^i(x) - f_n^i(y)| \leq \|\mathbf{f}_n(x) - \mathbf{f}_n(y)\| < \varepsilon$ whenever $d(x, y) < \delta$. Hence each sequence of components $\{f_n^i\}_n$, $i = 1, \ldots, k$, is bounded and equicontinuous. Therefore for each i there is a subsequence $\{n_r\}$ such that $f_{n_r}^i$ converges uniformly. By refining to subsubsequences, we can obtain a single subsequence $\{n_r\}$ such that $\{f_{n_r}^i\}$ converges uniformly for all i, say to $f^i(x)$. Then, given $\varepsilon > 0$, choose r_0 so large that $|f_{n_r}^i(x) - f^i(x)| < \frac{\varepsilon}{k}$ for $i = 1, 2, \ldots, k$ and $r > r_0$. It then follows that $\|\mathbf{f}_{n_r}(x) - \mathbf{f}(x)\| < \varepsilon$ if $r > r_0$. The proofs of the other results all follow this model argument.

Exercise 7.18 Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t)\, dt \quad (a \leq x \leq b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

Solution. Let M be such that $|f_n(x)| \leq M$ for all n and x. Then clearly $|F_n(x)| \leq M(b - a)$ for all n, so that $\{F_n\}$ is uniformly bounded. Also, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{M}$. Then if $x < y$ and $|x - y| < \delta$, we have

$$|F_n(y) - F_n(x)| = \left| \int_x^y f_n(t)\, dt \right| < M|x - y| < \varepsilon.$$

Hence $\{F_n\}$ is also uniformly equicontinuous. Therefore by Ascoli's Theorem (Theorem 7.25), there exists a uniformly convergent subsequence of $\{F_n\}$.

Exercise 7.19 Let K be a compact metric space, let S be a subset of $\mathcal{C}(K)$. Prove that S is compact (with respect to the metric defined in Section 7.14) if and only if S is uniformly closed, pointwise bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no sequence that converges uniformly on K.)

Solution. First suppose S is compact. Then we know that S has to be closed and bounded (in this metric *bounded* means the same thing as *uniformly bounded*). If S is not equicontinuous, then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist x, y and $g \in S$ such that $d(x,y) < \delta$ and $|g(x) - g(y)| \geq \varepsilon$. Let $x_n, y_n \in K$ and $g_n \in S$ be such that $d(x_n, y_n) < \frac{1}{n}$ and $|g_n(x_n) - g_n(y_n)| \geq \varepsilon$. Then no subsequence of $\{g_n\}$ can be equicontinuous, since $|g_{n_k}(x_{n_k} - g_{n_k}(y_{n_k}) \geq \varepsilon$. Hence by Theorem 7.24 no subsequence of $\{g_n\}$ can converge in $\mathcal{C}(K)$, and so S cannot be compact. We conclude, then, that if S is compact, then S is closed, bounded, and equicontinuous.

Conversely, if S is closed, pointwise bounded, and equicontinuous, then every sequence $\{g_n\}$ contains a subsequence that converges uniformly, hence converges in the metric of $\mathcal{C}(K)$ (by Ascoli's theorem). Since S is closed, the limit belongs to S, and so S is compact by Exercise 26 of Chapter 2.

Exercise 7.20 If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)x^n\,dx = 0 \quad (n = 0, 1, 2 \ldots),$$

prove that $f(x) = 0$ on $[0,1]$. *Hint:* The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x)\,dx = 0$.

Solution. There exists a sequence of polynomials $p_n(x)$ such that $p_n(x)$ converges uniformly to $f(x)$. Since f is bounded, $\{p_n\}$ is uniformly bounded, and hence $p_n f$ converges uniformly to f^2. Then by Theorem 7.16

$$\int_0^1 f^2(x)\,dx = \lim_{n\to\infty} \int_0^1 p_n(x)f(x)\,dx = 0.$$

But we know already (Exercise 2 of Chapter 6) that this implies $f^2(x) \equiv 0$.

Exercise 7.21 Let K be the unit circle in the complex plane (i.e., the set of all z with $|z| = 1$), and let $\mathcal{A}$ be the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta} \quad (\theta \text{ real}).$$

The $\mathcal{A}$ separates points on K, and $\mathcal{A}$ vanishes at no point of K, but nevertheless there are continuous functions on K which are not in the uniform closure of $\mathcal{A}$.
Hint: For every $f \in \mathcal{A}$

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\, d\theta = 0,$$

and this is also true for every f in the closure of $\mathcal{A}$.

Solution. The function $f(z) = z \in \mathcal{A}$ separates points on K and never vanishes. The equality given in the hint is a straighforward computation. It implies that the continuous function $\frac{1}{z}$, which is $e^{-i\theta}$, is not in the uniform closure of $\mathcal{A}$, since

$$\int_0^{2\pi} e^{-i\theta}e^{i\theta}\, d\theta = 2\pi.$$

Exercise 7.22 Assume $f \in \mathcal{R}(\alpha)$ on $[a, b]$, and prove that there are polynomials P_n such that

$$\lim_{n\to\infty} \int_a^b |f - P_n|^2\, d\alpha = 0.$$

(Compare with Exercise 12, Chap. 6.)

Solution. The parenthetical remark refers to the proof that there is a sequence of *continuous* functions $\{f_n\}$ such that

$$\lim_{n\to\infty} \int_a^b |f - f_n|^2\, d\alpha = 0.$$

All that is now needed is to note that one can find polynomials P_n such that $|f_n(x) - P_n(x)| < \frac{1}{n}$ for all $x \in [a, b]$ and all n.

Exercise 7.23 Put $P_n = 0$, and define, for $n = 0, 1, 2, \ldots$,

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}$$

Prove that

$$\lim_{n\to\infty} P_n(x) = |x|,$$

uniformly on $[-1, 1]$.

(This makes it possible to prove the Stone-Weierstrass theorem without first proving Theorem 7.26.

Hint: Use the identity

$$|x| - P_{n+1}(x) = [|x| - P_n(x)]\left[1 - \frac{|x| + P_n(x)}{2}\right]$$

to prove that $0 \le P_n(x) \le P_{n+1}(x) \le |x|$ if $|x| \le 1$, and that

$$|x| - P_n(x) \le |x|\left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1}$$

if $|x| \le 1$.

Solution. The identity given in the hint is a trivial consequence of the identity $x^2 - P_n^2(x) = [|x| - P_n(x)][|x| + P_n(x)]$. Then, granting that $0 \le P_n(x) \le |x|$, we conclude that $0 \le 1 - \dfrac{|x| + P_n(x)}{2} < 1$ for $|x| \le 1$, and hence that $0 \le |x| - P_{n+1}(x) \le |x| - P_n(x)$, which gives all of the desired inequalities. An immediate corollary of the same identity (obtained by replacing $P_n(x)$ by 0 in the second factor on the right-hand side) is

$$|x| - P_{n+1}(x) \le [|x| - P_n(x)]\left(1 - \frac{|x|}{2}\right),$$

and this inequality makes it possible to obtain the inequality

$$|x| - P_n(x) \le |x|\left(1 - \frac{|x|}{2}\right)^n$$

by induction on n. Finally, by symmetry, the maximum of $|x|(1 - \frac{|x|}{2})^n$ on $[-1, 1]$ is its maximum on $[0, 1]$, and this can be found by simple calculus to occur at $x = \frac{2}{n+1}$. Since this function is always less than $|x|$, the final inequality now follows.

Exercise 7.24 Let X be a metric space, with metric d. Fix a point $a \in X$. Assign to each $p \in X$ the function f_p defined by

$$f_p(x) = d(x, p) - d(x, a) \quad (x \in X).$$

Prove that $|f_p(x)| \le d(a, p)$ for all $x \in X$, and therefore $f_p \in \mathcal{C}(X)$.
Prove that

$$\|f_p - f_q\| = d(p, q)$$

for all $p, q \in X$.

If $\Phi(p) = f_p$, it follows that Φ is an *isometry* (a distance-preserving mapping) of X onto $\Phi(X) \subset \mathcal{C}(X)$.

Let Y be the closure of $\Phi(X)$ in $\mathcal{C}(X)$. Show that Y is complete.

Conclusion: X is isometric to a dense subset of a complete metric space Y. (Exercise 24, Chap. 3 contains a different proof of this.)

Solution. The inequality $|f_p(x)| \le d(a, p)$ is well-known, i.e., the fact that

$$|d(x, p) - d(x, a)| \le d(a, p)$$

and follows from the triangle inequality by merely transposing a term. (The left-hand side is either $d(x,p) - d(x,a)$ or $d(x,a) - d(x,p)$. Whichever is the case, if the subtracted term is moved to the other side, we have the ordinary triangle inequality.)

As for the isometry, we certainly have, for all x,

$$|f_q(x) - f_p(x)| = |d(x,q) - d(x,p)| \leq d(p,q)$$

and equality holds here if $x = q$ or $x = p$. Hence the supremum over all x is exactly $d(p,q)$.

As for the closure Y of $\Phi(X)$ being complete, it is a closed subset of a complete metric space, hence necessarily complete. By definition of closure, $\Phi(X)$ is dense in Y.

Exercise 7.25 Suppose ϕ is a continuous bounded real function in the strip defined by $0 \leq x \leq 1$, $-\infty < y < \infty$. Prove that the initial-value problem

$$y' = \phi(x,y), \quad y(0) = c$$

has a solution. (Note that the hypotheses of this existence theorem are less strigent than those of the corresponding uniqueness theorem; see Exercise 27, Chap. 5.)

Hint: Fix n. For $i = 0, \ldots, n$; put $x_i = i/n$. Let f_n be a continuous function on $[0,1]$ such that $f_n(0) = c$,

$$f_n'(t) = \phi(x_i, f_n(x_i)) \quad \text{if} \quad x_i < t < x_{i+1},$$

and put

$$\Delta_n(t) = f_n'(t) - \phi(t, f_n(t)),$$

except at the points x_i, where $\Delta_n(t) = 0$. Then

$$f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]\, dt.$$

Choose $M < \infty$ so that $|\phi| \leq M$. Verify the following assertions.

(*a*) $|f_n'| \leq M$, $|\Delta_n| \leq 2M$, $\Delta_n \in \mathcal{R}$, and $|f_n| \leq |c| + M = M_1$, say, on $[0,1]$ for all n.

(*b*) $\{f_n\}$ is equicontinuous on $[0,1]$, since $|f_n'| \leq M$.

(*c*) Some $\{f_{n_k}\}$ converges to some f, uniformly on $[0,1]$.

(*d*) Since ϕ is uniformly continuous on the rectangle $0 \leq x \leq 1$, $|y| \leq M_1$,

$$\phi(t, f_{n_k}(t)) \to \phi(t, f(t))$$

uniformly on $[0,1]$.

(e) $\Delta_n(t) \to 0$ uniformly on $[0,1]$, since

$$\Delta_n(t) = \phi(x_i, f_n(x_i)) - \phi(t, f_n(t))$$

in (x_i, x_{i+1}).
(f) Hence

$$f(x) = c + \int_0^x \phi(t, f(t))\,dt.$$

This f is a solution of the given problem.

Solution. It will save trouble if we assume that ϕ is a bounded continuous mapping from $[0,1] \times R^k$ into R^k and that c is a vector in R^k. That way we can do Exercise 26 simultaneously with this one. Since we are defining the functions $f_n(t)$ to be piecewise-linear, there is no difficulty in doing this with vector-valued functions. We simply define $f_n(t) = c + t\phi(0,c)$ for $0 \le t \le x_1$, and then, by induction on i,

$$f_n(t) = f_n(x_i) + (t - x_i)\phi(x_i, f_n(x_i))$$

for $x_i < t \le x_{i+1}$.

Then, if $\Delta_n(t)$ is defined as indicated, we have $f_n'(t) = \Delta_n(t) + \phi(t, f_n(t))$ except at a finite set of points, and therefore

$$f_n(x) = f_n(0) + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]\,dt.$$

(a) The assertions $|f_n'| \le M$ and $|\Delta_n| \le 2M$ are immediate consequences of the definitions of these two functions and the fact that $|\phi(x,y)| \le M$ for all x and y (here in general $y \in R^k$). Since $\Delta_n(t)$ is bounded and continuous except at x_i, it is Riemann-integrable. The inequality $|f_n| \le |c| + M = M_1$ is then immediate.

(b) $|f_n(x) - f_n(y)| \le \int_x^y |f_n'(t)|\,dt \le M|x-y|$.

(c) This is Ascoli's Theorem (Theorem 7.25).

(d) Given $\varepsilon > 0$ let $\delta > 0$ be such that $|\phi(t,y) - \phi(t,z)| < \varepsilon$ if $|y - z| < \delta$, for all $t \in [0,1]$, and $y, z \in R^k$. Then if $|f_{n_k}(t) - f(t)| < \delta$ for all t (which is the case if k is large), we have $|\phi(t, f_{n_k}(t)) - \phi(t, f(t))| < \varepsilon$ for all t.

(e) For each t and n let $i(n)$ be chosen so that $t \in [x_{i(n)}, x_{i(n)+1}]$, so that $|t - x_{i(n)}| \le \frac{1}{n}$. Since $f_{n_k}(t)$ converges uniformly to $f(t)$ and $x_{i(n)} \to t$, it follows that $\phi(x_{i(n)}, f_n(x_{i(n)})) - \phi(t, f_n(t)) \to 0$.

(f) We now invoke Theorem 7.16 to get

$$f(x) = c + \int_0^x \phi(t, f(t))\,dt.$$

Clearly $f(0) = c$, and since the right-hand side has a continuous derivative, so does the left-hand side, and $f'(x) = \phi(x, f(x))$.

Exercise 7.26 Prove an analogous existence theorem for the initial-value problem

$$\mathbf{y}' = \Phi(x, \mathbf{y}), \quad \mathbf{y}(0) = \mathbf{c},$$

where now $\mathbf{c} \in R^k$, $\mathbf{y} \in R^k$, and Φ is a continuous bounded mapping of the part of R^{k+1} defined by $0 \leq x \leq 1$, $\mathbf{y} \in R^k$ into R^k. (Compare Exercise 28, Chap. 5.) *Hint:* Use the vector-valued version of Theorem 7.25.

Solution. Since we were foresightful enough to make all the necessary notes in the solution of the previous problem, there is nothing to be done. Observe that an k-th order initial-value problem

$$y^{(k)} = \phi(x, y, y', y'', \ldots, y^{(k-1)})$$

with $y(0) = c_0$, $y'(0) = c_1, \ldots$, $y^{(k-1)}(0) = c_{k-1}$ falls under this theorem if we let

$$\Phi(x, y_1, y_2, \ldots, y_k) = (y_2, y_3, \ldots, y_k, \phi(x, y_1, \ldots, y_{k-1})),$$

$\mathbf{y}(0) = (c_0, \ldots, c_{k-1})$ Any solution of this problem provides a solution of the k-th order equation (namely y_1 if $\mathbf{y} = (y_1, \ldots, y_k)$).