# Contents

# Chapter 1

# Real Analysis

(1.5)

$\mathit{OrderTrichotomy}[<, S] := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$

$\mathit{OrderTransitivity}[<, S] := \forall_{x,y,z \in S}\big((x < y \land y < z) \implies x < z\big)$

$\mathit{Order}[<, S] := (\mathit{OrderTrichotomy}[<, S]) \land (\mathit{OrderTransitivity}[<, S])$

(1.7)

$\mathit{BoundedAbove}[E, S, <] := (\mathit{Order}[<, S]) \land (E \subset S) \land \Big(\exists_{\beta \in S}\forall_{x \in E}(x \leq \beta)\Big)$

$\mathit{BoundedBelow}[E, S, <] := (\mathit{Order}[<, S]) \land (E \subset S) \land \Big(\exists_{\beta \in S}\forall_{x \in E}(\beta \leq x)\Big)$

$\mathit{UpperBound}[\beta, E, S, <] := (\mathit{Order}[<, S]) \land (E \subset S) \land \big(\beta \in S \land \forall_{x \in E}(x \leq \beta)\big)$

$\mathit{LowerBound}[\beta, E, S, <] := (\mathit{Order}[<, S]) \land (E \subset S) \land \big(\beta \in S \land \forall_{x \in E}(\beta \leq x)\big)$

(1.8)

$\mathit{LUB}[\alpha, E, S, <] := (\mathit{UpperBound}[\alpha, E, S, <]) \land \Big(\forall_{\gamma}(\gamma < \alpha \implies \neg \mathit{UpperBound}[\gamma, E, S, <])\Big)$

$\mathit{GLB}[\alpha, E, S, <] := (\mathit{LowerBound}[\alpha, E, S, <]) \land \Big(\forall_{\beta}(\alpha < \beta \implies \neg \mathit{LowerBound}[\beta, E, S, <])\Big)$

(1.10)

$\mathit{LUBProperty}[S, <] := \forall_{E}\Big(\big((\emptyset \neq E \subset S) \land (\mathit{BoundedAbove}[E, S, <]) \implies \exists_{\alpha \in S}(\mathit{LUB}[\alpha, E, S, <])\big)\Big)$

$\mathit{GLBProperty}[S, <] := \forall_{E}\Big(\big((\emptyset \neq E \subset S) \land (\mathit{BoundedBelow}[E, S, <]) \implies \exists_{\alpha \in S}(\mathit{GLB}[\alpha, E, S, <])\big)\Big)$

(1.11)

$\mathit{LUBPropertyImpliesGLBProperty} := \mathit{LUBProperty}[S, <] \implies \mathit{GLBProperty}[S, <]$

(1) $\mathit{LUBProperty}[S, <] \implies \ldots$ — wts: 2

(1.1) $(\emptyset \neq B \subset S \land \mathit{BoundedBelow}[B, S, <]) \implies \ldots$ — wts: 1.2

(1.1.1) $\mathit{Order}[<, S] \land \exists_{\delta' \in S}(\mathit{LowerBound}[\delta', B, S, <])$ — from: *BoundedBelow*, 1.1

(1.1.2) $|B| = 1 \implies \ldots$ — wts: 1.1.3

(1.1.2.1) $\exists_{u'}(u' \in B) \blacksquare u := \mathit{choice}(\{u' : u' \in B\}) \blacksquare B = \{u\}$ — from: 1.1.2

(1.1.2.2) $\mathit{GLB}[u, B, S, <] \blacksquare \exists_{\epsilon_0 \in S}(\mathit{GLB}[\epsilon_0, B, S, <])$

(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S}(\mathit{GLB}[\epsilon_0, B, S, <])$

(1.1.4) $|B| \neq 1 \implies \ldots$ — wts: 1.1.5

(1.1.4.1) $\forall_{E}\big((\emptyset \neq E \subset S \land \mathit{BoundedAbove}[E, S, <]) \implies \exists_{\alpha \in S}(\mathit{LUB}[\alpha, E, S, <])\big)$ — from: *LUBProperty*, 1

(1.1.4.2) $L := \{s \in S : \mathit{LowerBound}[s, B, S, <]\}$

(1.1.4.3) $|B| > 1 \land \mathit{OrderTrichotomy}[<, S] \blacksquare \exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$ — from: *Order*, 1.1.1; wts: 1.1.4.7

(1.1.4.4) $b_1 := \mathit{choice}\Big(\{b_1' \in B : \exists_{b_0' \in B}(b_0' < b_1')\}\Big) \blacksquare \neg \mathit{LowerBound}[b_1, B, S, <]$ — from: 1.1.4.2

(1.1.4.5) $b_1 \notin L \blacksquare L \subset S$

(1.1.4.6) $\delta := \mathit{choice}(\{\delta' \in S : \mathit{LowerBound}[\delta', B, S, <]\}) \blacksquare \delta \in L \blacksquare \emptyset \neq L$ — from: 1.1.1

(1.1.4.7) $\emptyset \neq L \subset S$ — from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y \in L}(\mathit{LowerBound}[y_0, B, S, <]) \blacksquare \forall_{y \in L}\forall_{x \in B}(y_0 \leq x)$ — from: *LowerBound*, 1.1.4.2; wts: 1.1.4.10

(1.1.4.9) $\forall_{x\in B}\Big(x \in S \land \forall_{y\in L}(y_0 \le x)\Big) \blacksquare \forall_{x\in B}(UpperBound[x, L, S, <])$ from: $UpperBound$

(1.1.4.10) $\exists_{x\in S}(UpperBound[x, L, S, <]) \blacksquare BoundedAbove[L, S, <]$

(1.1.4.11) $\emptyset \neq L \subset S \land BoundedAbove[L, S, <]$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha'\in S}(LUB[\alpha', L, S, <]) \blacksquare \alpha := choice\Big(\{\alpha' \in S : (LUB[\alpha', L, S, <])\}\Big)$ from: 1.1.4.1 wts: 1.1.4.21

(1.1.4.13) $\forall_x(x \in B \implies UpperBound[x, L, S, <])$ from: 1.1.4.9 wts: 1.1.4.17

(1.1.4.14) $\forall_x(\neg UpperBound[x, L, S, <] \implies x \notin B)$

(1.1.4.15) $\gamma < \alpha \implies \dots$ wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound[\gamma, L, S, <] \blacksquare \gamma \notin B$ from: $LUB$, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B \blacksquare \gamma \in B \implies \gamma \ge \alpha$

(1.1.4.17) $\forall_{\gamma\in B}(\alpha \le \gamma) \blacksquare LowerBound[\alpha, B, S, <]$ from: $LowerBound$

(1.1.4.18) $\alpha < \beta \implies \dots$ wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y\in L}(y_0 \le \alpha < \beta) \blacksquare \forall_{y\in L}(y_0 \neq \beta)$ from: $LUB$, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta \notin L \blacksquare \neg LowerBound[\beta, B, S, <]$ from: 1.1.4.2

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound[\beta, B, S, <] \blacksquare \forall_{\beta\in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.20) $LowerBound[\alpha, B, S, <] \land \forall_{\beta\in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB[\alpha, B, S, <] \blacksquare \exists_{\epsilon_1\in S}(GLB[\epsilon_1, B, S, <])$

(1.1.5) $|B| \neq 1 \implies \exists_{\epsilon_1\in S}(GLB[\epsilon_1, B, S, <])$

(1.1.6) $\Big(|B| = 1 \implies \exists_{\epsilon_0\in S}(GLB[\epsilon_0, B, S, <])\Big) \land \Big(|B| \neq 1 \implies \exists_{\epsilon_1\in S}(GLB[\epsilon_1, B, S, <])\Big)$ from: 1.1.3, 1.1.5

(1.1.7) $(|B| = 1 \lor |B| \neq 1) \implies \exists_{\epsilon\in S}(GLB[\epsilon, B, S, <]) \blacksquare \exists_{\epsilon\in S}(GLB[\epsilon, B, S, <])$

(1.2) $(\emptyset \neq B \subset S \land BoundedBelow[B, S, <]) \implies \exists_{\epsilon\in S}(GLB[\epsilon, B, S, <])$

(1.3) $\forall_B\Big((\emptyset \neq B \subset S \land BoundedBelow[B, S, <]) \implies \exists_{\epsilon\in S}(GLB[\epsilon, B, S, <])\Big)$

(1.4) $GLBProperty[S, <]$

(2) $LUBProperty[S, <] \implies GLBProperty[S, <]$

(1.12)

$$Field[F, +, *] := \exists_{\mathbb{0},\mathbb{1}\in F}\forall_{x,y,z\in F}\left(\begin{array}{l} x + y \in F \quad \land \quad x * y \in F \quad \land \\ x + y = y + x \quad \land \quad x * y = y * x \quad \land \\ (x + y) + z = x + (y_0 + z) \quad \land \quad (x * y) * z = x * (y_0 * z) \quad \land \\ \mathbb{1} \neq \mathbb{0} \quad \land \quad x * (y_0 + z) = (x * y) + (x * z) \quad \land \\ \mathbb{0} + x = x \quad \land \quad \mathbb{1} * x = x \quad \land \\ \exists_{-x\in F}\Big(x + (-x) = \mathbb{0}\Big) \land \Big(x \neq \mathbb{0} \implies \exists_{1/x\in F}\Big(x * (1/x) = \mathbb{1}\Big)\Big) \end{array}\right)$$

************************ $(Field[F, +, *] \land x, y, z \in F) \implies \dots$ ************************

(1.14)

$AdditiveCancellation := (x + y = x + z) \implies y = z$

(1) $y = \mathbb{0} + y = \Big(x + (-x)\Big) + y = \Big((-x) + x\Big) + y = (-x) + (x + y) = \dots$ from: $Field$

(2) $(-x) + (x + z) = \Big((-x) + x\Big) + z = \Big(x + (-x)\Big) + z = \mathbb{0} + z = z$ from: $Field$

$AdditiveIdentityUniqueness := (x + y = x) \implies y = \mathbb{0}$

(1) $x + y = x = \mathbb{0} + x = x + \mathbb{0}$ from: $Field$

(2) $y = 0$ from: $AdditiveCancellation$

$AdditiveInverseUniqueness := (x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$ from: $Field$

(2) $y = -x$ from: $AdditiveCancellation$

$DoubleNegative := x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x \blacksquare \mathbb{0} = (-x) + x$ from: $Field$

(2) $x = -(-x)$ from: *AdditiveInverseUniqueness*

(1.15)

$MultiplicativeCancellation := (x \neq 0 \land x * y = x * z) \implies y = z$ —

$MultiplicativeIdentityUniqueness := (x \neq 0 \land x * y = x) \implies y = \mathbb{1}$ —

$MultiplicativeInverseUniqueness := (x \neq 0 \land x * y = \mathbb{1}) \implies y = 1/x$ —

$DoubleReciprocal := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)

$Domination := \mathbb{0} * x = \mathbb{0}$

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x \;\blacksquare\; \mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$ from: *Field*

(2) $\mathbb{0} * x = \mathbb{0}$ from: *AdditiveIdentityUniqueness*

$NonDomination := (x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies \ldots$

(1.1) $(x * y = 0) \implies \ldots$

(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = \big(x * (1/x)\big) * \big(y * (1/y)\big) = (x * y) * \big((1/x) * (1/y)\big) = \mathbb{0} * \big((1/x) * (1/y)\big) = \mathbb{0}$ from: *Field*, *Domination*, 1, 1.1

(1.1.2) $\mathbb{1} = \mathbb{0} \land \mathbb{1} \neq \mathbb{0} \;\blacksquare\; \bot$ from: *Field*

(1.2) $(x * y = 0) \implies \bot \;\blacksquare\; x * y \neq 0$

(2) $(x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq 0$

$NegationCommutativity := (-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0} \;\blacksquare\; x * y + (-x) * y = \mathbb{0}$ from: *Field*, *Domination* wts: 2

(2) $(-x) * y = -(x * y)$ from: *AdditiveInverseUniqueness*

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * \mathbb{0} = \mathbb{0} \;\blacksquare\; x * y + x * (-y) = \mathbb{0}$ from: *Field*, *Domination* wts: 4

(4) $x * (-y) = -(x * y)$ from: *AdditiveInverseUniqueness*

(5) $(-x) * y = -(x * y) = x * (-y)$ from: 2, 4

$NegativeMultiplication := (-x) * (-y) = x * y$

(1) $(-x) * (-y) = -\big(x * (-y)\big) = -\big(-(x * y)\big) = x * y$ from: *NegationCommutativity*, *DoubleNegative*

(1.17)

$$OrderedField[F, +, *, <] := \left( \begin{array}{l} Field[F, +, *] \quad \land \quad Order[<, F] \quad \land \\ \forall_{x,y,z \in F}(y_0 < z \implies x + y < x + z) \;\land \\ \forall_{x,y \in F}\big((x > \mathbb{0} \land y > \mathbb{0}) \implies x * y > \mathbb{0}\big) \end{array} \right)$$

$*********************** \; (OrderedField[F, +, *, <] \land x, y, z \in F) \implies \ldots \; ***********************$

(1.18)

$NegationOnOrder := x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \ldots$

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x \;\blacksquare\; \mathbb{0} > -x \;\blacksquare\; -x < \mathbb{0}$ from: *OrderedField*

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \ldots$

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x \;\blacksquare\; \mathbb{0} < x \;\blacksquare\; x > \mathbb{0}$ from: *OrderedField*

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \land -x < \mathbb{0} \implies x > \mathbb{0} \;\blacksquare\; x > \mathbb{0} \iff -x < \mathbb{0}$ from: 2, 4

$PositiveFactorPreservesOrder := (x > \mathbb{0} \land y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \land y < z) \implies \ldots$

(1.1) $(-y) + z > (-y) + y = \mathbb{0} \;\blacksquare\; z + (-y) = \mathbb{0}$ from: *OrderedField*

(1.2) $x * \big(z + (-y)\big) > \mathbb{0} \;\blacksquare\; x * z + x * (-y) > \mathbb{0}$ from: *OrderedField*

(1.3) $x * z = \mathbb{0} + x * z = \big(x * y + -(x * y)\big) + x * z = \big(x * y + x * (-y)\big) + x * z = \ldots$ — from: *Field*, *NegationCommutativity*

(1.4) $x * y + \big(x * z + x * (-y)\big) > x * y + \mathbb{0} = x * y$ — from: *Field*, 1.2

(1.5) $x * z > x * y$ — from: 1.3, 1.4

(2) $(x > \mathbb{0} \wedge y < z) \implies x * z > x * y$

*NegativeFactorFlipsOrder* $:= (x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \wedge y < z) \implies \ldots$

(1.1) $-x > \mathbb{0}$ — from: *NegationOnOrder*

(1.2) $(-x) * y < (-x) * z \;\blacksquare\; \mathbb{0} = x * y + (-x) * y < x * y + (-x) * z \;\blacksquare\; \mathbb{0} < x * y + (-x) * z$ — from: *PositiveFactorPreservesOrder*

(1.3) $\mathbb{0} < (-x) * (-y + z) \;\blacksquare\; \mathbb{0} > x * (-y + z) \;\blacksquare\; \mathbb{0} > -(x * y) + x * z$ — from: *NegationOnOrder*

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

*SquareIsPositive* $:= (x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ — from: *OrderedField*

(2) $(x < \mathbb{0}) \implies \ldots$

(2.1) $-x > \mathbb{0} \;\blacksquare\; x * x = (-x) * (-x) > \mathbb{0} \;\blacksquare\; x * x > \mathbb{0}$ — from: *NegationOnOrder*, *OrderedField*, *NegativeMultiplication*

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \vee x < \mathbb{0}) \implies x * x > \mathbb{0} \;\blacksquare\; x \neq \mathbb{0} \implies x * x > \mathbb{0}$ — from: *OrderTrichotomy*, 1, 3

*OneIsPositive* $:= \mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0} \;\blacksquare\; \mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ — from: *Field*, *SquareIsPositive*

*ReciprocationOnOrder* $:= (0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \ldots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0} \;\blacksquare\; x * (1/x) > 0$ — from: *Field*, *OneIsPositive*

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp \;\blacksquare\; 1/x > \mathbb{0}$ — from: *NegativeFactorFlipsOrder*, 1

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0} \;\blacksquare\; y * (1/y) > 0$ — from: *Field*, *OneIsPositive*

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp \;\blacksquare\; 1/y > \mathbb{0}$ — from: *NegativeFactorFlipsOrder*, 1

(1.5) $(1/x) * (1/y) > \mathbb{0}$ — from: *OrderedField*

(1.6) $\mathbb{0} < 1/y = \big((1/x) * (1/y)\big) * x < \big((1/x) * (1/y)\big) * y = 1/x$ — from: *OrderedField*, 1, 1.4, 1.5

(1.19)
*OrderedFieldQ* $:=$ *OrderedField*$[\mathbb{Q}, +, *, <]$ —

*Subfield*$[K, F, +, *] :=$ *Field*$[F, +, *] \wedge K \subset F \wedge$ *Field*$[K, +, *]$
*OrderedSubfield*$[K, F, +, *, <] :=$ *OrderedField*$[F, +, *, <] \wedge K \subset F \wedge$ *OrderedField*$[K, +, *, <]$
*CutI*$[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$
*CutII*$[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
*CutIII*$[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} :$ *CutI*$[\alpha] \wedge$ *CutII*$[\alpha] \wedge$ *CutIII*$[\alpha]\}$

*CutCorollaryI* $:= (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \ldots$

(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ — from: *CutII*, 1

(1.2) $q < p \implies q \in \alpha \;\blacksquare\; q \notin \alpha \implies q \geq p$ — from: 1

(1.3) $(q \notin \alpha) \implies \ldots$

(1.3.1) $q \geq p$ — from: 1.2

(1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp \;\blacksquare\; q \neq p$ — from: 1, 1.3

(1.3.3) $q \geq p \wedge q \neq p$ ▮ $p < q$

(1.4) $q \notin \alpha \implies p < q$ ▮ $p < q$ — from: 1

(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

$CutCorollaryII := (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$

(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ — from: $CutII$, 1

(1.2) $s \in \alpha \implies \big(r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)\big)$ ▮ $s \in \alpha \implies r \in \alpha$ — from: 1, 1.1

(1.3) $r \notin \alpha \implies s \notin \alpha$ ▮ $s \notin \alpha$ — from: 1, 1.2

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$OrderTrichotomyOfR := OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$

(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$ — from: $<_{\mathbb{R}}$, 1.1

(1.1.2) $\exists_{p'}(p' \in \alpha \wedge p' \notin \beta)$ ▮ $p := choice(\{p' : p' \in \alpha \wedge p' \notin \beta\})$

(1.1.3) $q \in \beta \implies \ldots$

(1.1.3.1) $p, q \in \mathbb{Q}$

(1.1.3.2) $q < p$ — from: $CutCorollaryI$

(1.1.3.3) $q \in \alpha$ — from: $CutII$

(1.1.4) $q \in \beta \implies q \in \alpha$

(1.1.5) $\forall_{q \in \beta}(q \in \alpha)$ ▮ $\beta \subseteq \alpha$

(1.1.6) $\beta \subset \alpha$ ▮ $\beta <_{\mathbb{R}} \alpha$

(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$ ▮ $(\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

$OrderTransitivityOfR := OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a \in \alpha}(a \in \beta) \wedge \forall_{b \in \beta}(b \in \gamma)$

(1.1.3) $\forall_{a \in \alpha}(\alpha \in \gamma)$ ▮ $\alpha \subset \gamma$ ▮ $\alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}}\big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(4) $OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

$OrderOfR := Order[<_{\mathbb{R}}, \mathbb{R}]$ — from: $OrderTrichotomyR$, $OrderTransitivityR$

wts:

$LUBPropertyOfR := LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset$ ▌ $\exists_\alpha(\alpha \in A)$ ▌ $\alpha_0 := choice(\{\alpha : \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset$ ▌ $\exists_a(a \in \alpha_0)$ ▌ $a_0 := choice(\{a : a \in \alpha_0\})$ ▌ $a_0 \in \gamma$ ▌ $\gamma \neq \emptyset$

(1.4) $BoundedAbove[A, \mathbb{R}, <_\mathbb{R}]$ ▌ $\exists_\beta(UpperBound[\beta, A, \mathbb{R}, <_\mathbb{R}])$

(1.5) $\beta_0 := choice(\{\beta : UpperBound[\beta, A, \mathbb{R}, <_\mathbb{R}]\})$

(1.6) $UpperBound[\beta_0, A, \mathbb{R}, <_\mathbb{R}]$ ▌ $\forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \beta_0)$ ▌ $\forall_{\alpha\in A}(\alpha \subseteq \beta_0)$ ▌ $\forall_{\alpha\in A}\forall_{a\in\alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma$ ▌ $\forall_{a\in\gamma}(a \in \beta_0)$ ▌ $\gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q}$ ▌ $\gamma \subseteq \beta_0 \subset \mathbb{Q}$ ▌ $\gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q}$ ▌ $CutI[\gamma]$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.10.1) $p \in \gamma$ ▌ $\exists_{\alpha\in A}(p \in \alpha)$ ▌ $\alpha_1 := choice(\{\alpha \in A : p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p$ ▌ $q \in \alpha_1$ ▌ $q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma$ ▌ $\forall_{p\in\gamma}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \gamma)$ ▌ $CutII[\gamma]$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha\in A}(p \in \alpha)$ ▌ $\alpha_2 := choice(\{\alpha \in A : p \in \alpha\})$

(1.12.2) $\alpha_2 \in \mathbb{R}$ ▌ $CutII[\alpha_2]$ ▌ $\exists_{r\in\alpha_2}(p < r)$ ▌ $r_0 := choice(\{r \in \alpha_2 : p < r\})$

(1.12.3) $r_0 \in \alpha_2$ ▌ $r_0 \in \gamma$

(1.12.4) $p < r_0$ ▌ $p < r_0 \wedge r_0 \in \gamma$ ▌ $\exists_{r\in\gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r\in\gamma}(p < r)$ ▌ $\forall_{p\in\gamma}\exists_{r\in\gamma}(p < r)$ ▌ $CutIII[\gamma]$

(1.14) $CutI[\gamma] \wedge CutII[\gamma] \wedge CutIII[\gamma]$ ▌ $\gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha\in A}(\alpha \subseteq \gamma)$ ▌ $\forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \gamma)$

(1.16) $\forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \gamma) \wedge \gamma \in \mathbb{R}$ ▌ $UpperBound[\gamma, A, \mathbb{R}, <_\mathbb{R}]$

(1.17) $\delta <_\mathbb{R} \gamma \implies \ldots$

(1.17.1) $\delta \subset \gamma$ ▌ $\exists_s(s \in \gamma \wedge s \notin \delta)$ ▌ $s_0 := choice(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$

(1.17.2) $s_0 \in \gamma$ ▌ $\exists_{\alpha\in A}(s_0 \in \alpha)$ ▌ $\alpha_3 := choice(\{\alpha \in A : s_0 \in \alpha\})$

(1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta$ ▌ $\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4) $\delta \geq_\mathbb{R} \alpha_3 \implies \ldots$

(1.17.4.1) $\alpha_3 \subseteq \delta$ ▌ $\forall_{s\in\mathbb{Q}}(s \in \alpha_3 \implies s \in \delta)$ ▌ $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4.2) $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$ ▌ $\bot$

(1.17.5) $\delta \geq_\mathbb{R} \alpha_3 \implies \bot$ ▌ $\delta <_\mathbb{R} \alpha_3$ ▌ $\exists_{\alpha\in A}(\delta <_\mathbb{R} \alpha)$ ▌ $\exists_{\alpha\in A}\big(\neg(\alpha \leq_\mathbb{R} \delta)\big)$

(1.17.6) $\neg\forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \delta)$ ▌ $\neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}]$

(1.18) $\delta <_\mathbb{R} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}])$ ▌ $\forall_\delta(\delta <_\mathbb{R} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}])$

(1.19) $UpperBound[\gamma, A, \mathbb{R}, <_\mathbb{R}] \wedge \forall_\delta(\delta <_\mathbb{R} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}])$

(1.20) $LUB[\gamma, A, \mathbb{R}, <_\mathbb{R}]$ ▌ $\exists_{\gamma\in S}(LUB[\gamma, A, \mathbb{R}, <_\mathbb{R}])$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_\mathbb{R}]) \implies \exists_{\gamma\in S}(LUB[\gamma, A, \mathbb{R}, <_\mathbb{R}])$

(3) $\forall_A\Big((\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_\mathbb{R}]) \implies \exists_{\gamma\in S}(LUB[\gamma, A, \mathbb{R}, <_\mathbb{R}])\Big)$ ▌ $LUBProperty[\mathbb{R}, <_\mathbb{R}]$

$+_\mathbb{R}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_\mathbb{R} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

$0_\mathbb{R} := \{x \in \mathbb{Q} : x < 0\}$

$ZeroInR := 0_\mathbb{R} \in \mathbb{R}$

(1) $-1 \in 0_\mathbb{R} \wedge 1 \notin 0_\mathbb{R}$ ▌ $\emptyset \neq 0_\mathbb{R} \subseteq \mathbb{Q}$ ▌ $CutI[0_\mathbb{R}]$

(2) $(x \in 0_\mathbb{R} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_\mathbb{R}$ ▌ $\forall_{x\in 0_\mathbb{R}}\forall_{y\in\mathbb{Q}}(y_0 < x \implies y \in 0_\mathbb{R})$ ▌ $CutII[0_\mathbb{R}]$

(3) $y := x/2$ ▌ $(x \in 0_\mathbb{R}) \implies (x < y < 0) \implies \exists_{y\in 0_\mathbb{R}}(x < y)$ ▌ $\forall_{x\in 0_\mathbb{R}}\exists_{y\in 0_\mathbb{R}}(x < y)$ ▌ $CutIII[0_\mathbb{R}]$

(4) $CutI[0_\mathbb{R}] \wedge CutII[0_\mathbb{R}] \wedge CutIII[0_\mathbb{R}]$ ▌ $0_\mathbb{R} \in \mathbb{R}$

$FieldAdditionClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_\mathbb{R} \beta) \in \mathbb{R}\big)$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_\mathbb{R} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha)\,;\, \exists_b(b \in \beta) \blacksquare a_0 := choice(\{a : a \in \alpha\})\,;\, b_0 := choice(\{b : b \in \beta\}) \blacksquare a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha)\,;\, \exists_y(y_0 \notin \beta) \blacksquare x_0 := choice(\{x : x \notin \alpha\})\,;\, y_0 := choice(\{y : y \notin \beta\})$

(1.5) $\forall_{r \in \alpha}(r < x_0)\,;\, \forall_{s \in \beta}(s < y_0) \blacksquare \forall_{r \in \alpha}\forall_{s \in \beta}(r + s < x_0 + y_0) \blacksquare x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q} \blacksquare CutI[\alpha +_{\mathbb{R}} \beta]$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.7.1) $\exists_{r \in \alpha}\exists_{s \in \beta}(p = r + s) \blacksquare (r_0, s_0) := choice\big((r, s) \in \alpha \times \beta : p = r + s\big)$

(1.7.2) $q < p = r_0 + s_0 \blacksquare (q - s_0) < r_0 \blacksquare (q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta \blacksquare q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta \blacksquare q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} \beta}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta) \blacksquare CutII[\alpha +_{\mathbb{R}} \beta]$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r \in \alpha}\exists_{s \in \beta}(p = r + s) \blacksquare (r_1, s_1) := choice\big(\{(r, s) \in \alpha \times \beta : p = r + s\}\big)$

(1.9.2) $r_1 \in \alpha \blacksquare \exists_{t \in \alpha}(r_1 < t) \blacksquare t_0 := choice(\{t \in \alpha : r_1 < t\})$

(1.9.3) $s_1 \in \beta \blacksquare t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1 \blacksquare \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r) \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} \beta}\exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r) \blacksquare CutIII[\alpha +_{\mathbb{R}} \beta]$

(1.11) $CutI[\alpha +_{\mathbb{R}} \beta] \wedge CutII[\alpha +_{\mathbb{R}} \beta] \wedge CutIII[\alpha +_{\mathbb{R}} \beta] \blacksquare \alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

$FieldAdditionCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s : r \in \alpha \wedge s \in \beta\} = \{s + r : s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

$FieldAdditionAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)\big)$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

$FieldAdditionIdentityOfR := (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \ldots$

(1.1.1) $s < 0 \blacksquare r + s < r + 0 = r \blacksquare r + s < r \blacksquare r + s \in \alpha$

(1.2) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha \blacksquare \forall_{r \in \alpha}\forall_{s \in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}}(p \in \alpha) \blacksquare \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r \in \alpha}(p < r) \blacksquare r_2 := choice(\{r \in \alpha : p < r\})$

(1.4.2) $p < r_2 \blacksquare p - r_2 < r_2 - r_2 = 0 \blacksquare (p - r_2) < 0 \blacksquare (p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha \blacksquare p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare \forall_{p \in \alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

$FieldAdditionInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} : \exists_{r > 0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q} \blacksquare \exists_{s \in \mathbb{Q}}(s \notin \alpha) \blacksquare s_0 := choice(\{s : s \notin \alpha\}) \blacksquare p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha \blacksquare -p_0 - 1 \notin \alpha \blacksquare \exists_{r > 0}(-p_0 - r \notin \alpha) \blacksquare p_0 \in \beta$

(1.4) $\emptyset \neq \alpha \blacksquare \exists_{q \in \alpha} \blacksquare q_0 := choice(\{q \in \mathbb{Q} : q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha \blacksquare -(-q_0) - r = q_0 - r < q_0 \blacksquare -(-q_0) - r < q_0 \blacksquare -(-q_0) - r \in \alpha$

(1.6) $\forall_{r > 0}\big(-(-q_0) - r \in \alpha\big) \blacksquare \neg\exists_{r > 0}\big(-(-q_0) - r \notin \alpha\big) \blacksquare -q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q} \blacksquare CutI[\beta]$

(1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.8.1) $p \in \beta$ ▮ $\exists_{r>0}(-p-r \notin \alpha)$ ▮ $r_0 := choice(\{r > 0 : -p-r \notin \alpha\})$

(1.8.2) $q < p$ ▮ $-p-r < -q-r$

(1.8.3) $-q-r \notin \alpha$ ▮ $q \in \beta$

(1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta$ ▮ $\forall_{p\in\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \beta)$ ▮ $CutII[\beta]$

(1.10) $p \in \beta \implies \ldots$

(1.10.1) $p \in \beta$ ▮ $\exists_{r>0}(-p-r \notin \alpha)$ ▮ $r_1 := choice(\{r > 0 : -p-r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0$ ▮ $r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p$ ▮ $t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -\big(p + (r_1/2)\big) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha$ ▮ $-t_0 - (r_1/2) \notin \alpha$ ▮ $\exists_{r>0}(-t_0 - r \notin \alpha)$ ▮ $t_0 \in \beta$

(1.10.7) $t_0 > p \wedge t_0 \in \beta$ ▮ $\exists_{t\in\beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t\in\beta}(p < t)$ ▮ $\forall_{p\in\beta}\exists_{t\in\beta}(p < t)$ ▮ $CutIII[\beta]$

(1.12) $CutI[\beta] \wedge CutII[\beta] \wedge CutIII[\beta]$ ▮ $\beta \in \mathbb{R}$

(1.13) $(r \in \alpha \wedge s \in \beta) \implies \ldots$

(1.13.1) $s \in \beta$ ▮ $\exists_{t>0}(-s-t \notin \alpha)$ ▮ $t_1 := choice(\{t > 0 : -s-t \notin \alpha\})$ ▮ $-s - t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha$ ▮ $-s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha$ ▮ $r < -s$ ▮ $r + s < 0$ ▮ $r + s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}}$ ▮ $\forall_{(r,s)\in\alpha\times\beta}(r + s \in 0_{\mathbb{R}})$ ▮ $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \ldots$

(1.15.1) $v < 0$ ▮ $w_0 := -v/2$ ▮ $w > 0$

(1.15.2) $\exists_{n\in\mathbb{Z}}\big(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\big)$ ▮ $n_0 := choice\big(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\}\big)$ from: $ARCHIMEDEANPROPERTYOFQ + LUB$???

(1.15.3) $p_0 := -(n_0 + 2)w_0$ ▮ $-p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha$ ▮ $-p_0 - w_0 \notin \alpha$ ▮ $p_0 \in \beta$

(1.15.4) $n_0 w_0 \in \alpha \wedge p_0 \in \beta$ ▮ $n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta$ ▮ $\forall_{v\in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta)$ ▮ $0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$ ▮ $\alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$ ▮ $\exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

$*_{\mathbb{R}}[\alpha, \beta] :=$ —
$1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$1IsNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ —
$1InR := 1_{\mathbb{R}} \in \mathbb{R}$ —
$FieldMultiplicationClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R}\big)$ —
$FieldMultiplicationCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ —
$FieldMultiplicationAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)\big)$ —
$FieldMultiplicationIdentityOfR := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ —
$FieldMultiplicationInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha\in\mathbb{R}}\big(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}\big)$ —

$FieldDistributativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ —

$FieldWithR := Field[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}]$ —
$OrderedFieldWithR := OrderedField[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$
$QROrderedSubfieldOfR := OrderedSubfield[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —
$QIsomorphicToQR := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ —
$CompletenessOfR := \exists_{\mathbb{R}}(LUBProperty[\mathbb{R}, <_{\mathbb{R}}] \wedge OrderedSubfield[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}])$ —

(1.20)
$ArchimedeanPropertyOfR := \forall_{x,y\in\mathbb{R}}\big(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y)\big)$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \dots$

(1.1) $A := \{nx : n \in \mathbb{N}^+\}$ ▌ $(\emptyset \neq A \subset \mathbb{R}) \wedge \big(a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a)\big)$

(1.2) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \implies \dots$

(1.2.1) $\neg\exists_{n\in\mathbb{N}^+}(nx > y)$ ▌ $\forall_{n\in\mathbb{N}^+}(nx \leq y)$ ▌ $UpperBound[y_0, A, \mathbb{R}, <]$ ▌ $BoundedAbove[A, \mathbb{R}, <]$

(1.2.2) $CompletenessOfR$ ▌ $LUBProperty[\mathbb{R}, <]$

(1.2.3) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (BoundedAbove[A, \mathbb{R}, <])$ ▌ $\exists_{\alpha\in\mathbb{R}}(LUB[\alpha, A, \mathbb{R}, <])$ ...

(1.2.4) ... $\alpha_0 := choice(\{\alpha \in \mathbb{R} : LUB[\alpha, A, \mathbb{R}, <]\})$ ▌ $LUB[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0$ ▌ $\alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge (LUB[\alpha_0, A, \mathbb{R}, <])$ ▌ $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$ ▌ $\exists_{c\in A}(\alpha_0 - x < c)$ ...

(1.2.8) ... $c_0 := choice(\{c \in A : \alpha_0 - x < c\})$ ▌ $(c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge \big(a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a)\big)$ ▌ $\exists_{m\in\mathbb{N}^+}(mx = c_0)$ ...

(1.2.10) ... $m_0 := choice(\{m \in \mathbb{N}^+ : mx = c_0\})$ ▌ $(m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0)$ ▌ $\alpha_0 - x < c_0 = m_0 x$ ▌ $\alpha_0 < m_0 x + x$ ▌ $\alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+$ ▌ $m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge \big(a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a)\big)$ ▌ $(m_0 + 1)x \in A$

(1.2.14) $\big(\alpha_0 < (m_0 + 1)x\big) \wedge \big((m_0 + 1)x \in A\big)$ ▌ $\exists_{c\in A}(\alpha_0 < c)$

(1.2.15) $LUB[\alpha_0, A, \mathbb{R}, <]$ ▌ $UpperBound[\alpha_0, A, \mathbb{R}, <]$ ▌ $\forall_{c\in A}(c \leq \alpha_0)$ ▌ $\neg\exists_{c\in A}(c > \alpha_0)$ ▌ $\neg\exists_{c\in A}(\alpha_0 < c)$

(1.2.16) $\big(\exists_{c\in A}(\alpha_0 < c)\big) \wedge \big(\neg\exists_{c\in A}(\alpha_0 < c)\big)$ ▌ $\bot$

(1.3) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \implies \bot$ ▌ $\exists_{n\in\mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n\in\mathbb{N}^+}(nx > y)$ ▌ $\forall_{x,y\in\mathbb{R}}\big(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y)\big)$

$QDenseInR := \forall_{x,y\in\mathbb{R}}\Big(x < y \implies \exists_{p\in\mathbb{Q}}(x < p < y)\Big)$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \dots$

(1.1) $x < y$ ▌ $(0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $ArchimedeanPropertyOfR \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R})$ ▌ $\exists_{n\in\mathbb{N}^+}\big(n(y - x) > 1\big)$ ...

(1.3) ... $n_0 := choice\big(\{n \in \mathbb{N}^+ : n(y - x) > 1\}\big)$ ▌ $(n_0 \in \mathbb{N}^+) \wedge \big(n_0(y - x) > 1\big)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R})$ ▌ $n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m\in\mathbb{N}^+}\big(m(1) > n_0 x\big)$ ...

(1.6) ... $m_1 := choice\big(\{m \in \mathbb{N}^+ : m(1) > n_0 x\}\big)$ ▌ $(m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m\in\mathbb{N}^+}\big(m(1) > -n_0 x\big)$ ...

(1.8) ... $m_2 := choice\big(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\}\big)$ ▌ $(m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x)$ ▌ $-m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+$ ▌ $|m_1 - (-m_2)| \geq 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge \big(|m_1 - (-m_2)| \geq 2\big)$ ▌ $\exists_{m\in\mathbb{Z}}\big((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\big)$ ...

(1.12) ... $m_0 := choice\big(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\}\big)$ ▌ $(-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$

(1.13) $\big(n_0(y - x) > 1\big) \wedge (m_0 - 1 \leq n_0 x < m_0)$ ▌ $n_0 x < m_0 \leq 1 + n_0 x < n_0 y$ ▌ $n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y)$ ▌ $x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z}$ ▌ $m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y)$ ▌ $\exists_{p\in\mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p\in\mathbb{Q}}(x < p < y)$ ▌ $\forall_{x,y\in\mathbb{R}}\Big(x < y \implies \exists_{p\in\mathbb{Q}}(x < p < y)\Big)$

(1.21)

$RootLemma := (0 < a < b) \implies \Big(b^n - a^n \leq (b - a)nb^{n-1}\Big)$

(1) $(0 < a < b) \implies \dots$

(1.1) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b$ ▌ $b/a > 1$

(1.3) $b/a > 1 \;\blacksquare\; \sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}\left(b^{n-i}a^{i-1}(b/a)^{i-1}\right) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1} \;\blacksquare\; \sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b-a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le (b-a)nb^{n-1} \;\blacksquare\; b^n - a^n \le (b-a)nb^{n-1}$

(2) $(0 < a < b) \implies \left(b^n - a^n \le (b-a)nb^{n-1}\right)$

$\mathit{RootExistenceInR} := \forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}({y_0}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\} \;\blacksquare\; t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$

(1.2) $t_0 := x/(1+x) \;\blacksquare\; \left(t_0 = x/(1+x)\right) \wedge (t_0 \in \mathbb{R})$

(1.3) $0 < x \;\blacksquare\; 0 < x < 1 + x \;\blacksquare\; t_0 = x/(1+x) > 0 \;\blacksquare\; t_0 > 0$

(1.4) $1 = (1+x)/(1+x) > x/(1+x) = t_0 \;\blacksquare\; 1 > t_0$

(1.5) $(t_0 > 0) \wedge (1 > t_0) \;\blacksquare\; 0 < t_0 < 1$

(1.6) $(0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1) \;\blacksquare\; {t_0}^n \le t_0$

(1.7) $0 < x \;\blacksquare\; x > x/(1+x) = t_0 \;\blacksquare\; x > t_0$

(1.8) $({t_0}^n \le t_0) \wedge (x > t_0) \;\blacksquare\; {t_0}^n < x$

(1.9) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge ({t_0}^n < x) \;\blacksquare\; t_0 \in E \;\blacksquare\; \emptyset \neq E$

(1.10) $t_1 := \mathit{choice}(\{t \in \mathbb{R} : t > 1 + x\}) \;\blacksquare\; (t_1 \in \mathbb{R}) \wedge (t_1 > 1 + x)$

(1.11) $x > 0 \;\blacksquare\; t_1 > 1 + x > 1 \;\blacksquare\; t_1 > 1 \;\blacksquare\; {t_1}^n \ge t_1$

(1.12) $({t_1}^n \ge t_1) \wedge (t_1 > 1 + x) \wedge (1 > 0) \;\blacksquare\; {t_1}^n \ge t_1 > 1 + x > x \;\blacksquare\; {t_1}^n > x$

(1.13) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge ({t_1}^n > x) \;\blacksquare\; t_1 \notin E \;\blacksquare\; E \subset \mathbb{R}$

(1.14) $(\emptyset \neq E) \wedge (E \subset \mathbb{R}) \;\blacksquare\; \emptyset \neq E \subset \mathbb{R}$

(1.15) $t \in E \implies \ldots$

(1.15.1) $(t \in E) \wedge \left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \;\blacksquare\; t^n < x$

(1.15.2) $({t_1}^n > x) \wedge (t^n < x) \;\blacksquare\; t^n < x < {t_1}^n \;\blacksquare\; t < t_1$

(1.16) $t \in E \implies t < t_1 \;\blacksquare\; \forall_{t\in E}(t \le t_1) \;\blacksquare\; \mathit{UpperBound}[t_1, E, \mathbb{R}, <] \;\blacksquare\; \mathit{BoundedAbove}[E, \mathbb{R}, <]$

(1.17) $\mathit{CompletenessOfR} \;\blacksquare\; \mathit{LUBProperty}[\mathbb{R}, <]$

(1.18) $(\mathit{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (\mathit{BoundedAbove}[E, \mathbb{R}, <]) \;\blacksquare\; \exists_{y\in\mathbb{R}}(\mathit{LUB}[y, E, \mathbb{R}, <]) \;\ldots$

(1.19) $\ldots\, y_0 := \mathit{choice}(\{y \in \mathbb{R} : \mathit{LUB}[y, E, \mathbb{R}, <]\}) \;\blacksquare\; \mathit{LUB}[y_0, E, \mathbb{R}, <]$

(1.20) $(\mathit{LUB}[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0) \;\blacksquare\; 0 < t_0 \le y_0 \in \mathbb{R} \;\blacksquare\; 0 < y_0 \in \mathbb{R}$

(1.21) ${y_0}^n < x \implies \ldots$

(1.21.1) $k_0 := \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} \;\blacksquare\; k_0 \in \mathbb{R}$

(1.21.2) ${y_0}^n < x \;\blacksquare\; 0 < x - {y_0}^n$

(1.21.3) $(n > 0) \wedge (y_0 > 0) \;\blacksquare\; 0 < n(y_0 + 1)^{n-1}$

(1.21.4) $(0 < x - {y_0}^n) \wedge \left(0 < n(y_0 + 1)^{n-1}\right) \;\blacksquare\; 0 < \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} = k_0 \;\blacksquare\; 0 < k_0$

(1.21.5) $(0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R}) \;\blacksquare\; 0 < \mathit{min}(1, k_0) \in \mathbb{R}$

(1.21.6) $\mathit{QDenseInR} \wedge \left(0, \mathit{min}(1, k_0) \in \mathbb{R}\right) \wedge \left(0 < \mathit{min}(1, k_0)\right) \;\blacksquare\; \exists_{h\in\mathbb{Q}}\left(0 < h < \mathit{min}(1, k_0)\right) \;\ldots$

(1.21.7) $\ldots\, h_0 := \mathit{choice}\left(\{h \in \mathbb{Q} : 0 < h < \mathit{min}(1, k_0)\}\right) \;\blacksquare\; (0 < h_0 < 1) \wedge \left(h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}\right)$

(1.21.8) $(y_0 > 0) \wedge (h_0 > 0) \;\blacksquare\; 0 < y_0 < y_0 + h_0$

(1.21.9) $\mathit{RootLemma} \wedge (0 < y_0 < y_0 + h_0) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}$

(1.21.10) $h_0 < 1 \;\blacksquare\; h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$

(1.21.11) $\left((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}\right) \wedge \left(h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}\right) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}$

(1.21.12) $\left(0 < n(y_0 + 1)^{n-1}\right) \wedge \left(h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}\right) \;\blacksquare\; h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n$

(1.21.13) $\left((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}\right) \wedge \left(h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n\right) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \;\blacksquare\; (y_0 + h_0)^n < x$

(1.21.14) $(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \;\blacksquare\; (y_0 + h_0)^n < x$

(1.21.15) $(0 < y_0\mathbb{R}) \wedge (0 < h_0 < \mathbb{R}) \;\blacksquare\; 0 < y_0 < y_0 + h_0 \in \mathbb{R}$

(1.21.16) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge \left((y_0 + h_0)^n < x\right) \wedge (0 < y_0 + h_0 \in \mathbb{R}) \;\blacksquare\; (y_0 + h_0)^n \in E$

(1.21.17) $\left((y_0+h_0)^n \in E\right) \wedge (y_0 < y_0 + h_0) \;\blacksquare\; \exists_{e\in E}(y_0 < e)$

(1.21.18) $\color{cyan}{LUB}[y_0, E, \mathbb{R}, <] \;\blacksquare\; \color{cyan}{UpperBound}[y_0, E, \mathbb{R}, <] \;\blacksquare\; \forall_{e\in E}(e \le y_0) \;\blacksquare\; \neg\exists_{e\in E}(e > y_0)$

(1.21.19) $\left(\exists_{e\in E}(e > y_0)\right) \wedge \left(\neg\exists_{e\in E}(e > y_0)\right) \;\blacksquare\; \bot$

(1.22) ${y_0}^n < x \implies \bot \;\blacksquare\; {y_0}^n \ge x$

(1.23) ${y_0}^n > x \implies \ldots$

(1.23.1) $k_1 := \frac{{y_0}^n - x}{n{y_0}^{n-1}} \;\blacksquare\; (k_1 \in \mathbb{R}) \wedge (k_1 n {y_0}^{n-1} = {y_0}^n - x)$

(1.23.2) $(0 < x) \wedge (0 < n \in \mathbb{Z}) \;\blacksquare\; {y_0}^n - x < {y_0}^n \le n{y_0}^n \;\blacksquare\; {y_0}^n - x < n{y_0}^n$

(1.23.3) ${y_0}^n - x < n{y_0}^n \;\blacksquare\; k_1 = \frac{{y_0}^n - x}{n{y_0}^{n-1}} < \frac{n{y_0}^n}{n{y_0}^{n-1}} = y_0 \;\blacksquare\; k_1 < y_0$

(1.23.4) ${y_0}^n > x \;\blacksquare\; 0 < {y_0}^n - x$

(1.23.5) $(n > 0) \wedge (y_0 > 0) \;\blacksquare\; 0 < n{y_0}^{n-1}$

(1.23.6) $(0 < {y_0}^n - x) \wedge 0 < (n{y_0}^{n-1}) \;\blacksquare\; 0 < \frac{{y_0}^n - x}{n{y_0}^{n-1}} = k_1 \;\blacksquare\; 0 < k_1$

(1.23.7) $(k_1 < y_0) \wedge (0 < k_1) \;\blacksquare\; (0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$

(1.23.8) $t \ge y_0 - k_1 \implies \ldots$

(1.23.8.1) $t \ge y_0 - k_1 \;\blacksquare\; t^n \ge (y_0 - k_1)^n \;\blacksquare\; -t^n \le -(y_0 - k_1)^n \;\blacksquare\; {y_0}^n - t^n \le {y_0}^n - (y_0 - k_1)^n$

(1.23.8.2) $\color{cyan}{RootLemma} \wedge (0 < y_0 - k_1 < y_0) \;\blacksquare\; {y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}$

(1.23.8.3) $\left({y_0}^n - t^n \le {y_0}^n - (y_0 - k_1)^n\right) \wedge \left({y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}\right) \;\blacksquare\; {y_0}^n - t^n < k_1 n {y_0}^{n-1}$

(1.23.8.4) $(k_1 n {y_0}^{n-1} = {y_0}^n - x) \wedge ({y_0}^n - t^n < k_1 n {y_0}^{n-1}) \;\blacksquare\; {y_0}^n - t^n < {y_0}^n - x \;\blacksquare\; -t^n < -x \;\blacksquare\; t^n > x$

(1.23.8.5) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge (t^n > x) \;\blacksquare\; t \notin E$

(1.23.9) $t \ge y_0 - k_1 \implies t \notin E \;\blacksquare\; t \in E \implies t < y_0 - k_1 \;\blacksquare\; \forall_{t\in E}(t \le y_0 - k_1) \;\blacksquare\; \color{cyan}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.10) $\left(\color{cyan}{LUB}[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)\right) \;\blacksquare\; \neg\color{cyan}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.11) $(\color{cyan}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg\color{cyan}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \;\blacksquare\; \bot$

(1.24) ${y_0}^n > x \implies \bot \;\blacksquare\; {y_0}^n \le x$

(1.25) $\color{cyan}{Order}[\mathbb{R}, <] \;\blacksquare\; \color{cyan}{OrderTrichotomy}[\mathbb{R}, <]$

(1.26) $(\color{cyan}{OrderTrichotomy}[\mathbb{R}, <]) \wedge ({y_0}^n \ge x) \wedge ({y_0}^n \le x) \;\blacksquare\; {y_0}^n = x$

(1.27) $({y_0}^n = x) \wedge (y_0 \in \mathbb{R}) \;\blacksquare\; \exists_{y\in\mathbb{R}}(y^n = x)$

(1.28) $y_1, y_2 := choice(\{y \in \mathbb{R} : y^n = x\})$

(1.29) $y_1 \ne y_2 \implies \ldots$

(1.29.1) $(\color{cyan}{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_1 \ne y_2) \;\blacksquare\; (y_1 < y_2) \veebar (y_2 < y_1) \;\ldots$

(1.29.2) $\ldots (x = {y_1}^n < {y_2}^n = x) \veebar (x = {y_2}^n < {y_1}^n = x) \;\blacksquare\; (x < x) \veebar (x > x) \;\blacksquare\; \bot \veebar \bot \;\blacksquare\; \bot$

(1.30) $y_1 \ne y_2 \implies \bot \;\blacksquare\; y_1 = y_2 \;\blacksquare\; \forall_{a,b\in\mathbb{R}}\left((a^n = x \wedge b^n = x) \implies a = b\right)$

(1.31) $\left(\exists_{y\in\mathbb{R}}(y^n = x)\right) \wedge \left(\forall_{a,b\in\mathbb{R}}\left((a^n = x \wedge b^n = x) \implies a = b\right)\right) \;\blacksquare\; \exists!_{y\in\mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y\in\mathbb{R}}(y^n = x) \;\blacksquare\; \forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}({y_0}^n = x)$

$\color{red}{RootExistenceInRCorollary} := \forall_{0<a\in\mathbb{R}}\forall_{0<b\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\left((ab)^{1/n} = a^{1/n}b^{1/n}\right)$ —

$$\color{red}{ExtendedRealSystem}[\bar{\mathbb{R}}, +, *, <] := \begin{pmatrix} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \;\wedge\; x - \infty = -\infty \;\wedge\; \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \;\wedge \\ (x > 0) \implies \left(x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty\right) \wedge \\ (x < 0) \implies \left(x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty\right) \end{pmatrix}$$

$\color{red}{\mathbb{C}} := \{\langle a, b\rangle \in \mathbb{R} \times \mathbb{R}\}$
$\color{red}{+_{\mathbb{C}}}[\langle a, b\rangle, \langle c, d\rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d\rangle$
$\color{red}{*_{\mathbb{C}}}[\langle a, b\rangle, \langle c, d\rangle] := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c\rangle$

$\color{red}{FieldC} := \color{cyan}{Field}[\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}}]$ —
$\color{red}{RSubfieldC} := \color{cyan}{Subfield}[\mathbb{R}, \mathbb{C}, +, *]$ —

$\color{red}{i} := \langle 0, 1\rangle \in \mathbb{C}$

$\color{red}{iProperty} := i^2 = -1$ —
$\color{red}{CProperty} := (a, b \in \mathbb{R}) \implies (\langle a, b\rangle = a + bi)$ —

$Conjugate[\overline{a+bi}] := a - bi$

$ConjugateProperties := (w, z \in \mathbb{C}) \implies \ldots$ —

| | |
|---|---|
| (1) | $\overline{z+w} = \overline{z} + \overline{w}$ |
| (2) | $\overline{z*w} = \overline{z} * \overline{w}$ |
| (3) | $Re(z) = (1/2)(z+\overline{z}) \wedge Im(z) = (1/2)(z-\overline{z})$ |
| (4) | $0 \leq z * \overline{z} \in \mathbb{R}$ |

$AbsoluteValueC[|z|] = (z * \overline{z})^{1/2}$
$AbsoluteValueProperties := (z, w \in \mathbb{C}) \implies \ldots$ —

| | |
|---|---|
| (1) | 123123 |

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

# Chapter 2

# Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y} \big((x, y) \in f\big)$

$\big(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y\big) \implies \dots$

| | |
|---|---|
| (1) | $Domain(f) := X; Codomain(f) := Y$ |
| (2) | $Image(f, A) := \{f(a) : a \in A\}; Preimage(f, B) := \{a : f(a) \in B\}$ |
| (3) | $Range(f) := Image\big(Domain(f)\big)$ |

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X} \big(x_1 \neq x_2 \implies f(x_1) \neq f(x_2)\big)$
$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X} \big(y_0 = f(x)\big)$
$Bijective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$
$SurjectiveEquivalent := \big(Range(f) = Codomain(f)\big) \implies Surjective(f)$

$\big(Function(f, X, Y) \wedge Function(g, Y, Z)\big) \implies (f \circ g)(x) := f\big(g(x)\big); Function(f \circ g, X, Z)$

$Propertiesof Functions := \big(Function(f, A, B) \wedge Function(g, B, C) \wedge Function(h, C, D)\big) \implies \dots$

| | |
|---|---|
| (1) | $h \circ (g \circ f) = (h \circ g) \circ f$ |
| (2) | $\big(Injective(f) \wedge Injective(g)\big) \implies Injective(g \circ f)$ |
| (3) | $\big(Surjective(f) \wedge Surjective(g)\big) \implies Surjective(g \circ f)$ |
| (4) | $\big(Bijective(f, A, B)\big) \implies \exists_{f^{-1}} \Bigg(Function(f^{-1}, B, A) \wedge \forall_{a \in A} \Big(f^{-1}\big(f(a)\big) = a\Big) \wedge \forall_{b \in B} \Big(f\Big(f^{-1}(b)\Big) = b\Big)\Bigg)$ |

$|(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}} (b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \dots$

| | |
|---|---|
| (1) | $(a \mid b) \implies a \mid bc$ |
| (2) | $(a \mid b \wedge b \mid c) \implies a \mid c$ |
| (3) | $(a \mid b \wedge b \mid c) \implies a \mid (bx + cy)$ |
| (4) | $(a \mid b \wedge b \mid a) \implies a = \pm b$ |
| (5) | $(a \mid b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$ |
| (6) | $(a \mid b) \iff (m \neq 0 \wedge ma \mid mb)$ |

$DivisionAlgorithm := (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}} (b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \wedge a : b \wedge a : c$
$GCD(a, b, c) := CD(a, b, c) \wedge \forall_d \big((d : b \wedge d : c) \implies d : a\big)$
$GCDEquivalent := 123123$

# Chapter 3

# Linear Algebra

## 3.1 Matrix Operations and Special Matrices

$Matrix[A, m, n] := [a_{i,j}]_{m \times n}$ := m rows, n columns of real numbers
$\mathcal{M}_{m,n} := \{A : Matrix[A, m, n]\}$
$O_{m,n} := (Matrix[O, m, n]) \wedge (a_{i,j} = 0)$
$Square[A, n] := Matrix[A, n, n]$
$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$
$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$
$Diagonal[A, n] := (Square[A, n]) \wedge (i \neq j \implies a_{i,j} = 0)$
$Scalar[A, n, k] := (Diagonal[A, n]) \wedge (a_{i,i} = k)$
$I_n := Scalar[I, n, 1]$

$+(A, B) := \big((Matrix[A, m, n]) \wedge (Matrix[B, m, n])\big) \implies (A + B = [a_{i,j} + b_{i,j}]_{m \times n})$
$*\,(r, A) := \big((r \in \mathbb{R}) \wedge (Matrix[A, m, n])\big) \implies (r * A = [ra_{i,j}]_{m \times n})$
$*\,(A, B) := \big((Matrix[A, m, p]) \wedge (Matrix[B, p, n])\big) \implies \left(A * B = \left[\sum_{k=1}^{p}(a_{i,k} b_{k,j})\right]_{m \times n}\right)$
${}^T[A] := (Matrix[A, m, n]) \implies (A^T = [a_{j,i}]_{n \times m})$

$AddCom := \forall_{A,B \in \mathcal{M}}(A + B = B + A)$

(1) $A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C \in \mathcal{M}}\big((A + B) + C = A + (B + C)\big)$

(1) $(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A \in \mathcal{M}} \exists!_{O \in \mathcal{M}}(A + O = A = O + A)$

(1) $A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$

(2) $A + O_1 = A = A + O_2 \ \blacksquare\ O_1 = O_2$

$AddInv := \forall_{A \in \mathcal{M}} \exists!_{(-A) \in \mathcal{M}}\big(A + (-A) = O = (-A) + A\big)$

(1) $A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$

(2) $A + (-A_1) = O = A + (-A_2) \ \blacksquare\ -A_1 = -A_2 \ \blacksquare\ A_1 = A_2$

$MulAssoc := \forall_{A,B,C \in \mathcal{M}}\big((A * B) * C = A * (B * C)\big)$

(1) $(A * B) * C = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,j})\right] * C = \left[\sum_{k_2=1}^{p_2}\left(\sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,k_2}) c_{k_2,j}\right)\right] = \left[\sum_{k_2=1}^{p_2} \sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,k_2} c_{k_2,j})\right] = \ . . .$

(2) $. . . \left[\sum_{k_1=1}^{p_1} \sum_{k_2=1}^{p_2}(a_{i,k_1} b_{k_1,k_2} c_{k_2,j})\right] = \left[\sum_{k_1=1}^{p_1}\left(a_{i,k_1} \sum_{k_2=1}^{p_2}(b_{k_1,k_2} c_{k_2,j})\right)\right] = \ldots = A * (B * C)$

$MulId := \forall_{A : Square[A,n]}(A * I_n = A = I_n * A)$

(1) $A * I_n = \left[\sum_{k=1}^{n}\left(a_{i,k}\left(\begin{cases}1 & k=j\\ 0 & k\neq j\end{cases}\right)\right)\right] = [a_{i,j}] = A$

(2) $TODO = A$

$ScalAssoc := \forall_{r,s\in\mathbb{R}}\forall_{A\in\mathcal{M}}\big(r(sA) = (rs)A = s(rA)\big)$

(1) $r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$

(2) $(rs)A = [rsa_{i,j}]$

(3) $s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$

$TransCancel := \forall_{A\in\mathcal{M}}\big(A = (A^T)^T\big)$

(1) $A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$

$ScalMulCom := \forall_{r\in\mathbb{R}}\forall_{A,B\in\mathcal{M}}\big((rA) * B = r(A * B) = A * (rB)\big)$

(1) $(rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[\sum_{k=1}^{p}(ra_{i,k}b_{k,j})\right] = r(A * B)$

(2) $A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[\sum_{k=1}^{p}(a_{i,k}rb_{k,j})\right] = \left[\sum_{k=1}^{p}(ra_{i,k}b_{k,j})\right] = r(A * B)$

$ScalDistLeft := \forall_{r,s\in\mathbb{R}}\forall_{A\in\mathcal{M}}\big((r+s)A = rA + sA\big)$

(1) TODO

$ScalDistRight := \forall_{r\in\mathbb{R}}\forall_{A,B\in\mathcal{M}}\big(r(A+B) = rA + rB\big)$

(1) TODO

$MulDistRight := \forall_{A,B,C\in\mathcal{M}}\big((A+B) * C = A * C + B * C\big)$

(1) $(A+B) * C = [a_{i,j} + b_{i,j}] * C = \left[\sum_{k=1}^{p}\Big((a_{i,k} + b_{i,k})c_{k,j}\Big)\right] = \ \ldots$

(2) $\ldots \left[\sum_{k=1}^{p}(a_{i,k}c_{k,j} + b_{i,k}c_{k,j})\right] = \left[\sum_{k=1}^{p}(a_{i,k}c_{k,j})\right] + \left[\sum_{k=1}^{p}(b_{i,k}c_{k,j})\right] = A * C + B * C$

$MulDistLeft := \forall_{A,B,C\in\mathcal{M}}\big(C * (A+B) = C * A + C * B\big)$

(1) TODO

$TransAddDist := \forall_{A,B\in\mathcal{M}}\big((A+B)^T = A^T + B^T\big)$

(1) TODO

$TransMulDist := \forall_{A,B\in\mathcal{M}}\big((A * B)^T = B^T * A^T\big)$

(1) $(A * B)^T = \left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]^T = \left[\sum_{k=1}^{p}(a_{j,k}b_{k,i})\right] = \left[\sum_{k=1}^{p}(b_{k,i}a_{j,k})\right] = \left[\sum_{k=1}^{p}(b^T_{i,k}a^T_{k,j})\right] = B^T * A^T$

$Sym[A] := A = A^T$
$SkewSym[A] := A = -A^T$
$Invertible[A] := (Square[A,n]) \wedge \left(\exists_{A^{-1}\in\mathcal{M}}(A * A^{-1} = I_n = A^{-1} * A)\right)$

$SymGen := \forall_{A\in\mathcal{M}}(Sym[A + A^T])$

(1) $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$SkewSymGen := \forall_{A\in\mathcal{M}}(SkewSym[A - A^T])$

(1) $-(A - A^T)^T = -\big(A^T - (A^T)^T\big) = -(A^T - A) = (A - A^T)$

$SymDecomp := \forall_{A \in \mathcal{M}} \exists!_{B:Sym[B]} \exists!_{C:SkewSym[C]} (A = B + C)$

(1) $B := (1/2) * (A + A^T)\ ;\ C := (1/2) * (A - A^T)$

(2) $SymGen[B] \wedge SkewSymGen[C]$

(3) $A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$

(4) $(1/2) * (A_1 + {A_1}^T) = (1/2) * (A_2 + {A_2}^T)\ \blacksquare\ A_1 = A_2$

(5) $(1/2) * (A_3 - {A_3}^T) = (1/2) * (A_4 - {A_4}^T)\ \blacksquare\ A_3 = A_4$

$InvId := \forall_{A:Invertible[A]} \left( \exists!_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A) \right)$

(1) ${A^{-1}}_1 = {A^{-1}}_1 * I_n = {A^{-1}}_1 * (A * {A^{-1}}_2) = ({A^{-1}}_1 * A) * {A^{-1}}_2 = I_n * {A^{-1}}_2 = {A^{-1}}_2$

$InvCancel := \forall_{A:Invertible[A]} \left( (A^{-1})^{-1} = A \right)$

(1) $(A * A^{-1})^{-1} = {I_n}^{-1} = I_n$

(2) $(A^{-1})^{-1} * A^{-1} = I_n\ \blacksquare\ A^{-1})^{-1} = I_n * A = A$

$InvDist := \forall_{A:Invertible[A]} \forall_{B:Invertible[B]} \left( (A * B)^{-1} = B^{-1} * A^{-1} \right)$

(1) $(A * B) * (A * B)^{-1} = I\ \blacksquare\ B * (A * B)^{-1} = A^{-1}\ \blacksquare\ (A * B)^{-1} = B^{-1} * A^{-1}$

$InvTrans := \forall_{A:Invertible[A]} \left( (A^T)^{-1} = (A^{-1})^T \right)\ \blacksquare \Leftarrow$

(1) $A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I\ \blacksquare\ (A^{-1})^T = (A^T)^{-1}$

## 3.2 Elementary Matrices on Invertibility and Systems of Linear Equations

$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$
$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$
$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X (Sol[X, A, B])$
$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$
$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$

$HomoSysProps := (Sys[A, O]) \implies \ldots$

(1) $u_0 := O\ ;\ u_1 := choice(\{X \in \mathcal{M} | X \neq O\})\ ;\ k := choice(\mathbb{R})$

(2) $TrivSol[u_0, A]$

(3) $(NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$

(4) $(TrivSol[\overrightarrow{X}, A]) \implies \left( TrivSol[LC(\overrightarrow{X}), A] \right)$

$ElemMat[E] := (E = Swap[I_n, i, j]) \vee \left( Scale_*(I_n, i, c) \right) \vee \left( Combine_*(I_n, i, c, j) \right)$
$ElemMatProd[E^*] := \exists_{\langle E \rangle} \left( \forall_{E_i \in E^*} (ElemMat[E_i]) \wedge \left( E^* = \Pi_{E_i \in E^*} (E_i) \right) \right)$
$RowEquiv[A, B] := \exists_{E^*} \left( (ElemMatProd[E^*]) \wedge (B = E^* * A) \right)$

$ElemMatInv := \forall_{E \in \mathcal{M}} \left( (ElemMat[E]) \implies (Invertible[E]) \right)$

(1) $E - RowSwap[E] \implies TODO\ ;\ E - RowScale_*(E) \implies TODO\ ;\ E - RowCombine_*(E) \implies TODO$

$ElemMatProdInv := \forall_{E^*} \left( (ElemMatProd[E^*]) \implies (Invertible[E^*]) \right)$

(1) TODO

$RowEquivSys := \forall_{A,B,C,D,X \in \mathcal{M}} \left( \left( (Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]]) \right) \implies (Sol[X, A, B] \iff Sol[X, C, D]) \right)$

(1) $\exists_{E^*:ElemMatProd[E^*]} ([CD] = E^* * [AB])$

(2) $(E^* * A = C) \land (E^* * B = D)$

(3) $Sol[Y, A, B] \implies \ldots$

(3.1) $A * Y = B$

(3.2) $C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D \blacksquare Sol[Y, C, D]$

(4) $Sol[Y, A, B] \implies Sol[Y, C, D]$

(5) $\left(A = (E^*)^{-1} * C\right) \land \left(B = (E^*)^{-1} * D\right)$

(6) $Sol[Z, C, D] \implies \ldots$

(6.1) $C * Z = D$

(6.2) $A * Z = \left((E^*)^{-1} * C\right) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$

(7) $Sol[Z, C, D] \implies Sol[Z, A, B]$

(8) $Sol[X, A, B] \iff Sol[X, C, D]$

$$RowEquivHomoSysSol := \forall_{A,C,X \in \mathcal{M}}\Big((RowEquiv[A, C]) \implies \big((Sol[X, A, O]) \iff (Sol[X, C, O])\big)\Big)$$

(1) Set $B = D = O$

$$RREF[A] := (A \in \mathcal{M}) \land \left(\begin{array}{ll} \text{All zero rows are at the bottom of the matrix.} & \land \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} & \land \\ \text{The leading entry in any nonzero row is 1.} & \land \\ \text{All entries in the column above and below a leading 1 are zero.} & \land \end{array}\right)$$

$$GaussJordanElim := \forall_{A \in \mathcal{M}} \exists!_{B \in \mathcal{M}}\big((RREF[B]) \land (RowEquiv[A, B])\big)$$

(1) Hit $A$ with $ElemMat$'s until it becomes $B$

(2) $(B = E^* * A) \land (RREF[B])$

$$HasZero[A] := \big(Matrix(A, m, n)\big) \land \big(\exists_{i \le m}(A_{i,:} = O)\big)$$

$$HasZeroNonInvertible := \forall_{A \in \mathcal{M}}\big((HasZero[A]) \implies (\neg Invertible[A])\big)$$

(1) $i := choice(\{i \le m | A_{i,:} = O\})$

(2) $(B \in \mathcal{M}) \implies \ldots$

(2.1) $(A * B)_{i,:} = O \ne I_{ni,:} \blacksquare A * B \ne I_n$

(3) $(B \in \mathcal{M}) \implies (A * B \ne I_n) \blacksquare \forall_{B \in \mathcal{M}}(A * B \ne I_n) \blacksquare \neg Invertible[A]$

$$InvIffRowEquivI := \forall_{A \in \mathcal{M}}\big((Invertible[A]) \iff (RowEquiv[A, I_n])\big)$$

(1) $(Invertible[A]) \implies \ldots$

(1.1) $(RREF[B]) \land (RowEquiv[A, B])$

(1.2) $B = E^* * A$

(1.3) $(Invertible[E^*]) \land (Invertible[A]) \blacksquare Invertible[B]$

(1.4) $Invertible[B] \blacksquare \neg HasZero[B]$

(1.5) $(RREF[B]) \land (\neg HasZero[B]) \blacksquare B = I_n$

(1.6) $RowEquiv[A, I_n]$

(2) $(Invertible[A]) \implies (RowEquiv[A, I_n])$

(3) $(RowEquiv[A, I_n]) \implies \ldots$

(3.1) $I_n = E^* * A \blacksquare (E^*)^{-1} = A$

(3.2) $A^{-1} = E^*_{DescSort} \blacksquare Invertible[A]$

(4) $(RowEquiv[A, I_n]) \implies (Invertible[A])$

(5) $(Invertible[A]) \iff (RowEquiv[A, I_n])$

$$RowEquivIIffTrivSol := \forall_{A \in \mathcal{M}}\Big((RowEquiv[A, I_n]) \iff \Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big)\Big)$$

(1) $(RowEquiv[A, I_n]) \implies \ldots$

(1.1) $RowEquiv[A, I_n] \blacksquare Invertible[A]$

(1.2) $(Sol[X, A, O]) \implies \dots$

(1.2.1) $A * X = O \blacksquare X = A^{-1} * O = O \blacksquare X = O$

(1.3) $(Sol[X, A, O]) \implies (X = O)$

(1.4) $(X = O) \implies (Sol[X, A, O])$

(1.5) $(X = O) \iff (Sol[X, A, O]) \blacksquare \forall_X\big((X = O) \iff (Sol[X, A, O])\big)$

(2) $(RowEquiv[A, I_n]) \implies \Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big)$

(3) $\Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big) \implies \dots$

(3.1) $(RREF[B]) \wedge (RowEquiv[A, B])$

(3.2) $Sol[X, B, O]$

(3.3) $(B \neq I_n) \implies \dots$

(3.3.1) $\Big(\exists_{Y \neq X}(Sol[Y, B, O])\Big)$

(3.3.2) $Sol[Y, A, O] \blacksquare Y = X$

(3.3.3) $(Y \neq X) \wedge (Y = X) \blacksquare \perp$

(3.4) $(B \neq I_n) \implies \perp \blacksquare B = I_n$

(3.5) $(RowEquiv[A, B]) \wedge (B = I_n) \blacksquare RowEquiv[A, I_n]$

(4) $\Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big) \implies (RowEquiv[A, I_n])$

(5) $(RowEquiv[A, I_n]) \iff \Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big)$

$$InvIffUniqSol := \forall_{A \in \mathcal{M}}\Big((Invertible[A]) \iff \big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\big)\Big)$$

(1) $(Invertible[A] \wedge B \in \mathcal{M}) \implies \dots$

(1.1) $(Invertible[A]) \wedge (Sys[A, B])$

(1.2) $(X = A^{-1} * B) \iff (Sol[X, A, B]) \blacksquare \exists!_{X \in \mathcal{M}}(Sol[X, A, B])$

(2) $\big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\big) \implies \dots$

(2.1) $X_i := choice(\{X_i | Sol[X_i, A, I_{n:,i}]\})$

(2.2) $A * [X_1 \dots X_n] = [(A * X_1) \dots (A * X_n)] = [I_{n:,1} \dots I_{n:,n}] = I_n$

(2.3) $A^{-1} = [X_1 \dots X_n]$

(3) $\big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\big) \implies (Invertible[A])$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}} \left( \begin{array}{ll} (Invertible[A]) & \iff \\ (RowEquiv[A, I_n]) & \iff \\ \Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big) & \iff \\ \big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\big) & \end{array} \right)$$

## 3.3 Vector Spaces

$$VectorSpace[V, +, *] := \exists_{O \in V} \forall_{\alpha, \beta \in \mathbb{R}} \forall_{u, v, w \in V} \left( \begin{array}{l} (u + v \in V) \ \wedge \ (u + v = v + u) \ \wedge \ \big((u + v) + w = u + (v + w)\big) \ \wedge \\ (u + O = u) \quad \wedge \quad \Big(\exists_{-u \in V}\big(u + (-u) = O\big)\Big) \quad \wedge \\ (\alpha * u \in V) \ \wedge \ \big(\alpha * (\beta * u) = (\alpha\beta) * u\big) \ \wedge \ (1 * u = u) \ \wedge \\ \big(\alpha * (u + v) = (\alpha * u) + (\alpha * v)\big) \wedge \big((\alpha + \beta) * u = (\alpha * u) + (\beta * u)\big) \end{array} \right)$$

$$ZeroVectorUniq := \forall_{O', v \in V}\big((v + O' = v) \implies (O' = O)\big)$$

(1) $O' = O' + O = O + O' = O \blacksquare O' = O$

$$AddInvUnique := \forall_{-v', v \in V}\big((v + -v' = O) \implies (-v' = -v)\big)$$

(1) $-v' = -v' + O = -v' + (v + -v) = (-v' + v) + -v = (v + -v') + -v = O + -v = -v \blacksquare -v' = -v$

$$AddInvGen := \forall_{v \in V}\big((-1) * v = -v\big)$$

(1) $v + (-1) * v = (1-1) * v = 0 * v = O \;\blacksquare\; (-1) * v = -v$

$ZeroVectorGenLeft := \forall_{v \in V}(0 * v = O)$

(1) $0 * v = (0+0) * v = (0 * v) + (0 * v) \;\blacksquare\; O = 0 * v$

$ZeroVectorGenRight := \forall_{r \in \mathbb{R}}(r * O = O)$

(1) $r * O = r * (O + O) = (r * O) + (r * O) \;\blacksquare\; O = r * O$

$ZeroVectorEquiv := \forall_{r \in \mathbb{R}} \forall_{v \in V}\Big((r * v = O) \iff \big((v = O) \vee (r = 0)\big)\Big)$

(1) $(ZeroVectorGenLeft) \wedge (ZeroVectorGenRight) \;\blacksquare\; \big((v = O) \vee (r = 0)\big) \implies (r * v = O))$

(2) $(r * v = O) \implies \ldots$

(2.1) $(r \neq 0) \implies \ldots$

(2.1.1) $r \neq 0 \;\blacksquare\; r^{-1} \in \mathbb{R}$

(2.1.2) $ZeroVectorGenRight \;\blacksquare\; O = r^{-1} * O = r^{-1} * (r * v) = (r^{-1} r) * v = 1 * v = v \;\blacksquare\; O = v$

(2.2) $(r \neq 0) \implies (v = O) \;\blacksquare\; (r = 0) \vee (r \neq 0) \;\blacksquare\; (r = 0) \vee (v = O)$

(3) $(r * v = O) \implies \big((r = 0) \vee (v = O)\big)$

(4) $(r * v = O) \iff \big((r = 0) \vee (v = O)\big)$

## 3.4 Subspaces and Special Subspaces

$Subspace[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (\emptyset \neq S \subseteq V) \wedge (VectorSpace[S, +, *])$

$$SubspaceEquiv := \forall_{V,S}\left(\begin{array}{l} (VectorSpace[V, +, *]) \qquad\qquad\qquad \implies \\ \Big((Subspace[S, V, +, *]) \iff \Big((\emptyset \neq S \subseteq V) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)\Big)\Big) \end{array}\right)$$

(1) $(Subspace[S, V, +, *]) \implies \ldots$

(1.1) $Subspace[S, V, +, *] \;\blacksquare\; \emptyset \neq S \subseteq V$

(1.2) $VectorSpace[S, +, *] \;\blacksquare\; \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)$

(1.3) $(\emptyset \neq S \subseteq V) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)$

(2) $(Subspace[S, V, +, *]) \implies \Big((\emptyset \neq S \subseteq V) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)\Big)$

(3) $\Big((\emptyset \neq S \subseteq V) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)\Big) \implies \ldots$

(3.1) $\big((\alpha, \beta \in \mathbb{R}) \wedge (\emptyset \neq S) \wedge (u, v, w \in S)\big) \implies \ldots$

(3.1.1) $u, v \in V \;\blacksquare\; u + v = v + u$

(3.1.2) $u, v, w \in V \;\blacksquare\; (u + v) + w = u + (v + w)$

(3.1.3) $(ZeroVectorGenLeft) \wedge (u \in S) \;\blacksquare\; 0 * u = O \in S$

(3.1.4) $u \in V \;\blacksquare\; u + O = u$

(3.1.5) $(AddInvGen) \wedge (u \in S) \;\blacksquare\; (-1) * u = -u \in S$

(3.1.6) $u \in V \;\blacksquare\; \alpha * (\beta * u) = (\alpha\beta) * u$

(3.1.7) $u \in V \;\blacksquare\; 1 * u = u$

(3.1.8) $u, v \in V \;\blacksquare\; \alpha * (u + v) = (\alpha * u) + (\alpha * v)$

(3.1.9) $u \in V \;\blacksquare\; (\alpha + \beta) * u = (\alpha * u) + (\beta * u)$

(4) $\Big(\big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)\Big) \implies (Subspace[S, V, +, *])$

(5) $(Subspace[S, V, +, *]) \iff \Big(\big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)\Big)$

$$SumSubContains := \forall_{A,B,V}\left(\begin{array}{l} \big((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])\big) \implies \\ \big((Subspace[A + B, V, +, *]) \wedge (A, B \subseteq A + B)\big) \end{array}\right)$$

(1) $(Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \blacksquare (O \in A) \wedge (O \in B)$

(2) $(SetSum[A+B,A,B,V,+,*]) \wedge (O \in A) \wedge (O \in B) \blacksquare O \in A+B \blacksquare \emptyset \neq A+B$

(3) $(Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \blacksquare A+B \subseteq V \blacksquare \emptyset \neq A+B \subseteq V$

(4) $(u,v \in A+B) \implies \dots$

(4.1) $\left(\exists_{a_1 \in A} \exists_{b_1 \in B}(u = a_1 + b_1)\right) \wedge \left(\exists_{a_2 \in A} \exists_{b_2 \in B}(v = a_2 + b_2)\right)$

(4.2) $u + v = (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$

(4.3) $(a_1 + a_2 \in A) \wedge (b_1 + b_2 \in B) \blacksquare u + v \in A + B$

(5) $(u,v \in A+B) \implies (u+v \in A+B) \blacksquare \forall_{u,v \in A+B}(u+v \in A+B)$

(6) $\left((r \in \mathbb{R}) \wedge (v \in A+B)\right) \implies \dots$

(6.1) $\exists_{a \in A} \exists_{b \in B}(v = a + b)$

(6.2) $r * v = r * (a+b) = r * a + r * b$

(6.3) $(r * a \in A) \wedge (r * b) \in B \blacksquare r * v \in A+B$

(7) $\left((r \in \mathbb{R}) \wedge (v \in A+B)\right) \implies (r * v \in A+B) \blacksquare \forall_{r \in \mathbb{R}} \forall_{v \in A+B}(r * v \in A+B)$

(8) $(\emptyset \neq A+B \subseteq V) \wedge \left(\forall_{u,v \in A+B}(u+v \in A+B)\right) \wedge \left(\forall_{r \in \mathbb{R}} \forall_{v \in A+B}(r * v \in A+B)\right) \blacksquare Subspace[A+B,V,+,*]$

(9) $\left(\forall_{a \in A}(a + O) = a\right) \wedge (O \in B) \blacksquare A \subseteq A+B$

(10) $\left(\forall_{b \in B}(b + O) = b\right) \wedge (O \in A) \blacksquare B \subseteq A+B$

(11) $(A, B \subseteq A+B) \wedge (Subspace[A+B,V,+,*])$

$$SumSubMinContains := \forall_{A,B,V}\left(\begin{array}{l} \left((Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \wedge (SetSum[A+B,A,B,V,+,*])\right) \implies \\ \left(\forall_C\left((Subspace[C,V,+,*]) \wedge (A,B \subseteq C)\right) \implies (A+B \subseteq C)\right) \end{array}\right)$$

(1) $SumSub \blacksquare (A, B \subseteq A+B) \wedge (Subspace[A+B,V,+,*])$

(2) $\left((Subspace[C,V,+,*]) \wedge (A,B \subseteq C)\right) \implies \dots$

(2.1) $(s \in A+B) \implies \dots$

(2.1.1) $\exists_{a \in A} \exists_{b \in B}(s = a + b)$

(2.1.2) $(A, B \subseteq C) \blacksquare a, b \in C$

(2.1.3) $Subspace[C,V,+,*] \blacksquare s = a + b \in C$

(2.2) $(s \in A+B) \implies (s \in C) \blacksquare A+B \subseteq C$

(3) $\left((Subspace[C,V,+,*]) \wedge (A,B \subseteq C)\right) \implies (A+B \subseteq C)$

$NullSpace[N,A,m,n] := (Matrix[A,m,n]) \wedge (N = \{x \in \mathbb{R}^n | A * x = O\})$
$RowSpace[R,A,m,n] := (Matrix[A,m,n]) \wedge (R = \{x^T * A \in \mathbb{R}^n | x \in \mathbb{R}^m\})$
$ColSpace[C,A,m,n] := (Matrix[A,m,n]) \wedge (C = \{A * x \in \mathbb{R}^m | x \in \mathbb{R}^n\})$

$NullSubspace := (NullSpace[N,A,m,n]) \implies (Subspace[N,\mathbb{R}^n,+,*])$

(1) TODO

$RowSubspace := (RowSpace[R,A,m,n]) \implies (Subspace[R,\mathbb{R}^n,+,*])$

(1) TODO

$ColSubspace := (ColSpace[C,A,m,n]) \implies (Subspace[C,\mathbb{R}^m,+,*])$

(1) TODO

## 3.5 Linear Combination, Linear Span, Linear Independence

$LinComb[c,U,K,V,+,*] := (VectorSpace[V,+,*]) \wedge (n \in \mathbb{N}) \wedge (U \in V^n) \wedge (K \in \mathbb{R}^n) \wedge \left(c = \sum_{i=1}^n (k_i * u_i)\right)$

$$LinSpan[S',S,V,+,*] := \left(\begin{array}{l} (VectorSpace[V,+,*]) \wedge (S \in V^n) \wedge \left((S = \emptyset) \implies (S' = \{O\})\right) \wedge \\ \left((S \neq \emptyset) \implies \left(S' = \{c \in V | \exists_{K \in \mathbb{R}^n}(LinComb[c,S,K,V,+,*])\}\right)\right) \end{array}\right)$$

$LinSpanSubContains := \forall_{S',S,V,+,*}\Big((LinSpan[S',S,V,+,*]) \implies \big((Subspace[S',V,+,*]) \land (S \subseteq S')\big)\Big)$

(1) $(S = \emptyset) \implies (S' = \{O\}) \implies (\emptyset \neq S')$

(2) $(S \neq \emptyset) \implies (LinComb[O,S,\{0\}^n,V,+,*]) \implies (O \in S') \implies (\emptyset \neq S')$

(3) $\big((S = \emptyset) \lor (S \neq \emptyset)\big) \implies (\emptyset \neq S') \blacksquare\ \emptyset \neq S'$

(4) $LinSpan[S',S,V,+,*] \blacksquare\ S' \subseteq V \blacksquare\ \emptyset \neq S' \subseteq V$

(5) $(a,b \in S') \implies \dots$

(5.1) $\big(\exists_{K\in\mathbb{R}^n}(LinComb[a,S,K,V,+,*])\big) \land \big(\exists_{L\in\mathbb{R}^n}(LinComb[b,S,L,V,+,*])\big)$

(5.2) $a + b = \sum_{i=1}^{n}(k_i * s_i) + \sum_{i=1}^{n}(l_i * s_i) = \sum_{i=1}^{n}\big((k_i + l_i) * s_i\big) \blacksquare\ a + b = \sum_{i=1}^{n}\big((k_i + l_i) * s_i\big)$

(5.3) $\langle k_i + l_i | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{N}^n$

(5.4) $\Big(a + b = \sum_{i=1}^{n}\big((k_i + l_i) * s_i\big)\Big) \land (\langle k_i + l_i | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{N}^n) \blacksquare\ \exists_{M\in\mathbb{N}^n}\big(a + b = \sum_{i=1}^{n}(m_i * s_i)\big)$

(5.5) $\exists_{M\in\mathbb{N}^n}(LinComb[a+b,S,M,V,+,*]) \blacksquare\ a + b \in S'$

(6) $(a,b \in S') \implies (a + b \in S') \blacksquare\ \forall_{a,b\in S'}(a + b \in S')$

(7) $\big((r \in \mathbb{R}) \land (u \in S')\big) \implies \dots$

(7.1) $\exists_{K\in\mathbb{R}^n}(LinComb[u,S,K,V,+,*]) \blacksquare\ \exists_{K\in\mathbb{R}^n}\big(u = \sum_{i=1}^{n}(k_i * s_i)\big)$

(7.2) $r * u = r * \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{n}\big(r * (k_i * s_i)\big) = \sum_{i=1}^{n}(rk_i) * s_i) \blacksquare\ r * u = \sum_{i=1}^{n}(rk_i) * s_i)$

(7.3) $\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n$

(7.4) $\big(\sum_{i=1}^{n}(rk_i) * s_i)\big) \land (\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n)$

(7.5) $\exists_{M\in\mathbb{R}^n}\big(r * u = \sum_{i=1}^{n}(m_i * s_i)\big) \blacksquare\ \exists_{M\in\mathbb{R}^n}(LinComb[r * u,S,M,V,+,*]) \blacksquare\ r * u \in S'$

(8) $\big((r \in \mathbb{R}) \land (u \in S')\big) \implies (r * u \in S') \blacksquare\ \forall_{r\in\mathbb{R}}\forall_{u\in S'}(r * u \in S')$

(9) $(SubspaceEquiv) \land (\emptyset \neq S' \subseteq V) \land \big(\forall_{a,b\in S'}(a + b \in S')\big) \land \big(\forall_{r\in\mathbb{R}}\forall_{u\in S'}(r * u \in S')\big) \blacksquare\ Subspace[S',V,+,*]$

(10) $(s_j \in S) \implies \dots$

(10.1) $K := \langle\left(\begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}\right)|\mathbb{N}_{1,n}\rangle \in \mathbb{R}^n \blacksquare\ \sum_{i=1}^{n}(k_i * s_i) = s_j$

(10.2) $\dots \blacksquare\ \exists_{K\in\mathbb{R}^n}(LinComb[s_j,S,K,V,+,*]) \blacksquare\ s_j \in S'$

(11) $(s_j \in S) \implies (s_j \in S') \blacksquare\ \forall_{x\in S}(x \in S') \blacksquare\ S \subseteq S'$

(12) $(Subspace[S',V,+,*]) \land (S \subseteq S')$

$LinSpanSubMinContains := \forall_{S',S,V,+,*}\Big((LinSpan[S',S,V,+,*]) \implies \Big(\forall_W\big(((Subspace[W,V,+,*]) \land (S \subseteq W)\big) \implies (S' \subseteq W)\big)\Big)\Big)$

(1) $(s' \in S') \implies \dots$

(1.1) $\exists_{K\in\mathbb{R}^n}(LinComb[s',S,K,V,+,*]) \blacksquare\ s' = \sum_{i=1}^{n}(k_i * s_i)$

(1.2) $(S \subseteq W) \land (Subspace[W,V,+,*]) \blacksquare\ s' = \sum_{i=1}^{n}(k_i * s_i) \in W$

(2) $(s' \in S') \implies (s' \in W) \blacksquare\ S' \subseteq W$

$LinInd[S,V,+,*] := (VectorSpace[V,+,*]) \land (S \in V^n) \land \Big((S \neq \emptyset) \implies \Big(\forall_{K\in\mathbb{R}^n}\big((LinComb[O,S,K,V,+,*]) \implies (K = \{0\}^n)\big)\Big)\Big)$

$ZeroDependent := (O \in S) \implies (\neg LinInd[S,V,+,*])$

(1) $K := \langle\left(\begin{cases} 1 & u_i = O \\ 0 & u_i \neq O \end{cases}\right)|(1 \leq i \leq n) \land (i \in \mathbb{N})\rangle \blacksquare\ K \in \mathbb{R}^n$

(2) $(LinComb[O,S,K,V,+,*]) \land (K \neq \{O\}^n) \blacksquare\ \neg LinInd[S,V,+,*]$

$SingletonNonZeroIndependent := (v \neq O) \implies (LinInd[\langle v\rangle,V,+,*])$

(1) $(r * v = O) \iff \big((r = 0) \lor (v \neq O)\big)$

(2) $v \neq O \blacksquare\ r = 0$

(3) $\forall_{r\in\mathbb{R}}\big((r * v = O) \implies (r = 0)\big)$

$SubIndependent := \forall_{V,A,B}\Big(\big((VectorSpace[V,+,*]) \land (A \subseteq B \in V^m)\big) \implies \big((LinInd[B,V,+,*]) \implies (LinInd[A,V,+,*])\big)\Big)$

(1) $(LinComb[O,A,K,V,+,*]) \implies \dots$

(1.1) $L := \langle\left(\begin{cases} 1 & j \le n \\ 0 & j > n \end{cases}\right)|(1 \le j \le m \land (j \in \mathbb{N}))\rangle \;\blacksquare\; L \in \mathbb{R}^m$

(1.2) $A \subseteq B \;\blacksquare\; \forall_{n \ge j \in \mathbb{N}}(a_j = b_j)$

(1.3) $\forall_{n \ge j \in \mathbb{N}}(a_j = b_j) \;\blacksquare\; \sum_{i=1}^{n}(k_i * a_i)) = \sum_{i=1}^{n}(k_i * a_i)) + O = \sum_{j=1}^{m}(l_j * b_j))$

(1.4) $LinComb[O,A,K,V,+,*] \;\blacksquare\; O = \sum_{i=1}^{n}(k_i * a_i)$

(1.5) $O = \sum_{i=1}^{n}(k_i * a_i) = \sum_{j=1}^{m}(l_j * b_j) \;\blacksquare\; LinComb[O,B,L,V,+,*]$

(1.6) $(LinInd[B,V,+,*]) \land (LinComb[O,B,L,V,+,*]) \;\blacksquare\; L = \{0\}^m$

(1.7) $\big(\forall_{n \ge j \in \mathbb{N}}(a_j = b_j)\big) \land (L = \{0\}^m) \;\blacksquare\; \forall_{n \ge j \in \mathbb{N}}(k_j * a_j = l_j * b * j = l_j * a_j) \;\blacksquare\; K = \{0\}^n$

(2) $(LinComb[O,A,K,V,+,*]) \implies (K = \{0\}^n) \;\blacksquare\; \forall_{K \in \subseteq \mathbb{R}^n}\big((LinComb[O,A,K,V,+,*]) \implies (K = \{0\}^n)\big) \;\blacksquare\; LinInd[A,V,+,*]$

$SuperDependent := \forall_{V,A,B}\Big(\big((VectorSpace[V,+,*]) \land (A \subseteq B \subseteq V)\big) \implies \big((\neg LinInd[A,V,+,*]) \implies (\neg LinInd[B,V,+,*])\big)\Big)$

(1) TODO : A has a non trivial solution $\blacksquare$ use the same non trivial solution in combination with B and L

$LinIndEquiv := \forall_{U,V}\Big((LinInd[U,V,+,*]) \iff \big(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])\big)\Big)$

(1) $\Gamma' = \Gamma \setminus \{j\}$

(2) $(\neg LinInd[U,V,+,*]) \implies \dots$

(2.1) $\Big(\exists_{\Gamma \in \mathbb{R}^{|U|}}\Big(\big(\sum(\gamma_i * u_i) = O\big) \land (\Gamma \ne \{0\}^{|U|})\Big)\Big)$

(2.2) $\exists_{\gamma_k \in \Gamma}(\gamma_k \ne 0)$

(2.3) $\sum(\gamma'_i * u_i) = \sum(\gamma_i * u_i) - \gamma_k * u_k = -\gamma_k * u_j$

(2.4) $u_k = (-1/\gamma_k)\Big(\sum(\gamma'_i * u_i)\Big) = \sum\Big((-\gamma'_i/\gamma_k) * u_i\Big) \;\blacksquare\; \exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])$

(3) $(\neg LinInd[U,V,+,*]) \implies \big(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])\big)$

(4) $\big(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])\big) \implies (LinInd[U,V,+,*])$

(5) $\big(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])\big) \implies \dots$

(5.1) $\exists_{j \in U}\big(j = \sum(\gamma'_i * u_i)\big)$

(5.2) $\Gamma := \Gamma' \cup \{-1\}$

(5.3) $\Big(\sum(\gamma_i * u_i) = \sum(\gamma'_i * u_i) + (-1) * \gamma_j = O\Big) \land (\Gamma \ne \{0\}^n) \;\blacksquare\; \neg LinInd[U,V,+,*]$

(6) $\big(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])\big) \implies (\neg LinInd[U,V,+,*])$

(7) $(LinInd[U,V,+,*]) \implies \big(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])\big)$

(8) $(LinInd[U,V,+,*]) \iff \big(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])\big)$

$LinIndSuperspace := \forall_{U,V}\Big((Subspace[U,V]) \implies \big(\forall_W\big((LinInd[W,U,+,*]) \implies (LinInd[W,V,+,*])\big)\big)\Big)$

(1) $(\neg LinInd[W,V,+,*]) \implies \dots$

(1.1) $\exists_{j \in W}(LinComb[j, W \setminus \{j\}, +, *]) \;\blacksquare\; \neg LinInd[W,U,+,*]$

(1.2) $(LinInd[W,U,+,*]) \land (\neg LinInd[W,U,+,*]) \;\blacksquare\; \bot$

(2) $(\neg LinInd[W,V,+,*]) \implies \bot \;\blacksquare\; LinInd[W,V,+,*]$

$Spans[S,V,+,*] := LinSpan[V,S,V,+,*]$

$FinDim[V,+,*] := \exists_{S \in V^n}(Spans[S,V,+,*])$

$$LinDepLemma := \forall_{S,V}\left(\begin{array}{l}(\neg LinInd[S,V,+,*]) \qquad\qquad\qquad\qquad\qquad\qquad \implies \\ \exists_{j\in\mathbb{N}_{1,n}}\Big((s_j \in LinSpan[P_1,S_{1,j-1},V,+,*]) \wedge (LinSpan[P_2,S,V,+,*] = LinSpan[P_3,S\setminus\{s_j\},V,+,*])\Big)\end{array}\right)$$

(1) $\neg LinInd[S,V,+,*]$ ▮ $\exists_{K\in\mathbb{R}^n}\big((LinComb[O,S,K,V,+,*]) \wedge (K \neq \{0\}^n)\big)$

(2) $\exists_{j\in\mathbb{N}_{1,n}}\Big((k_j \neq 0) \wedge \Big(\forall_{i\in\mathbb{N}_{1,n}}\big((i > j) \implies (k_i = 0)\big)\Big)\Big)$

(3) $s_j = (-1/k_j)\sum_{i=1}^{j-1}(k_i * s_i) = \sum_{i=1}^{j-1}\Big((-k_i/k_j) * s_i\Big)$

(4) $\langle -k_i/k_j | i \in \mathbb{N}_{1,j-1}\rangle \in \mathbb{R}^{j-1}$

(5) $\exists_{M\in\mathbb{R}^{j-1}}(LinComb[s_j, S_{1,j-1}, M, V,+,*])$ ▮ $s_j \in LinSpan[P_1,S_{1,j-1},V,+,*]$

(6) $(v \in P_2) \iff (v \in LinSpan[P_2,S,V,+,*]) \iff \ ...$

(7) $...\ \big(v = \sum_{i=1}^{n}(k_i * s_i)\big) = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^{n}(k_i * s_i)) + k_j * s_j = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^{n}(k_i * s_i)) + k_j * \sum_{i=1}^{j-1}\Big((-k_i/k_j) * s_i\Big) \iff$ $...$

(8) $(v \in LinSpan[P_3, S\setminus\{s_j\},V,+,*]) \iff (v \in P_3)$ ▮ $(v \in P_2) \iff (v \in P_3)$ ▮ $P_2 = P_3$

$$LinIndLengthLeqSpan := \forall_{L,S}\Big(\big((LinInd[L,V,+,*]) \wedge (Spans[S,V,+,*])\big) \implies (|L| \leq |S|)\Big)$$

(1) TODO : form $B = L \cup S$, remove dependent elements in $S$ such that $(Spans[B,V,+,*]) \wedge (|B| = |S|)$ by $LinDepLemma$, $|L| \leq |B| = |S|$

(2) $\forall_{l_i\in L} \cdots$

(2.1) $l_i \in V$ ▮ $LinComb[l_i,S,K,V,+,*]$ ▮ $\neg LinInd[\langle l_i\rangle \cup S,V,+,*]$

(2.2) $LinDepLemma$ ▮ $\exists_{j\in\mathbb{N}_{1,n}}(LinSpan[V,S,V,+,*] = LinSpan[V,S\setminus\{s_j\},V,+,*])$

(2.3) $B := \langle l_i\rangle \cup S\setminus\{s_j\}$ ▮ $|B| = 1 + |S| - 1 = |S|$

(3) $|L| \leq |B| = |S|$ ▮ $|L| \leq |S|$

$$FinSubSpace := \forall_{U,V}\Big(\big((Subspace[U,V,+,*]) \wedge (FinDim[V,+,*])\big) \implies (FinDim[U,+,*)]\Big)$$

(1) TODO : take $Spans[S,V,+,*]$, remove all $s_j \in S$ such that $U \subseteq LinSpan[S\setminus\{s_j\}]$ ▮ $S' = S\setminus\{s_j\}$ $(LinSpan[U,S',V,+,*]) \wedge (|S'| \leq |S|)$ ▮ $FinDim[U,+,*]$

(2) $FinDim[V,+,*]$ ▮ $\exists_{S\in V^n}(Spans[S,V,+,*])$

(3) $\forall_{(u_j\in U)\wedge(\neg LinSpan[U,S,V,+,*])} \cdots$

(3.1) $\neg LinSpan[U,S,V,+,*]$ ▮ $\exists_{u_j\in U}(\neg LinComb[u_j,S_{1,j-1},K_{1,j-1},V,+,*])$

(3.2) $B := S_{1,j-1}$ ▮ $|B| = |S| - 1 < |S|$

(4) $LinSpan[U,B,V,+,*]$ ▮ $\exists_{B\in V^M}(Spans[B,U,+,*])$ ▮ $FinDim[U,+,*$

## 3.6 Bases and Dimensions

$$Basis[S,V,+,*] := (Spans[S,V,+,*]) \wedge (LinInd[S,V,+,*])$$

$$BasisEquiv := \forall_{S,V}\big((Basis[S,V,+,*]) \iff (\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])\big)$$

(1) $(Basis[S,V,+,*]) \implies \ldots$

(1.1) $(v \in V) \implies \ldots$

(1.1.1) $Basis[S,V,+,*]$ ▮ $Spans[V,S,+,*]$ ▮ $\exists_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])$

(1.1.2) $\big((K_1,K_2 \in \mathbb{R}^n) \wedge (LinComb[v,S,K_1,V,+,*]) \wedge (LinComb[v,S,K_2,V,+,*])\big) \implies \ldots$

(1.1.2.1) $\big(v = \sum(k_{1_i} * s_i)\big) \wedge \big(v = \sum(k_{2_i} * s_i)\big)$

(1.1.2.2) $O = v - v = \sum(k_{1_i} * s_i) - \sum(k_{2_i} * s_i) = \sum\big((k_{1_i} - k_{2_i}) * s_i\big)$

(1.1.2.3) $L := \langle k_{1_i} - k_{2_i} | i \in \mathbb{N}_{i=1}^{n}\rangle \in \mathbb{R}^n$

(1.1.2.4) $(LinInd[S,V,+,*]) \wedge (LinComb[O,S,L,V,+,*])$ ▮ $L = \{0\}^n$ ▮ $K_2 = K_1$

(1.1.3) $\big((K_1,K_2 \in \mathbb{R}^n) \wedge (LinComb[v,S,K_1,V,+,*]) \wedge (LinComb[v,S,K_2,V,+,*])\big) \implies (K_1 = K_2)$

(1.1.4) $\forall_{K_1,K_2\in\mathbb{R}^n}\big((LinComb[v,S,K_1,V,+,*]) \wedge (LinComb[v,S,K_2,V,+,*]) \implies (K_1 = K_2)\big)$

(1.1.5) $\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])$

(1.2) $(v\in V) \implies \big(\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])\big)$

(2) $(Basis[S,V,+,*]) \implies \big(\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])\big)$

(3) $\big(\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])\big) \implies \dots$

(3.1) $\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*]) \;\blacksquare\; \forall_{v\in V}\exists_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*]) \;\blacksquare\; Spans[S,V,+,*]$

(3.2) $O\in V \;\blacksquare\; \exists!_{K\in\mathbb{R}^n}(LinComb[O,S,K,V,+,*])$

(3.3) $(K\neq\{0\}^n) \implies \big(\neg\exists!_{K\in\mathbb{R}^n}(LinComb[O,S,K,V,+,*])\big) \implies \bot \;\blacksquare\; K=\{0\}^n$

(3.4) $\big(\exists!_{K\in\mathbb{R}^n}(LinComb[O,S,K,V,+,*])\big)\wedge(K=\{0\}^n) \;\blacksquare\; LinInd[S,V,+,*]$

(3.5) $(Spans[S,V,+,*])\wedge(LinInd[S,V,+,*]) \;\blacksquare\; Basis[S,V,+,*]$

(4) $\big(\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])\big) \implies (Basis[S,V,+,*])$

$$SpanReduceBasis := \forall_{S,V}\Big(Spans[S,V,+,*] \implies \exists_B\big((B\subseteq S)\wedge(Basis[B,V,+,*])\big)\Big)$$

(1) TODO : Remove all dependent $s_j\in S \;\blacksquare\; (S'=S\setminus\{s_j\})\wedge(LinSpan[S']=LinSpan[S])$ until $LinInd[S'] \;\blacksquare\; Basis[S']$

$$FinDimBasis := \forall_V\Big((FinDim[V,+,*]) \implies \big(\exists_B(Basis[B,V,+,*])\big)\Big)$$

(1) $FinDim[V,+,*] \;\blacksquare\; \exists_{S\in V^n}(Spans[S,V,+,*])$

(2) $(SpanReduceBasis)\wedge(Spans[S,V,+,*]) \;\blacksquare\; \exists_B(Basis[B,V,+,*])$

$$LinIndExpandBasis := \forall_{L,V}\Big(LinInd[L,V,+,*] \implies \exists_B\big((L\subseteq B)\wedge(Basis[B,V,+,*])\big)\Big)$$

(1) TODO : $FinDimBasis \;\blacksquare\; \exists_A(Basis[A,V,+,*])$, form $B=L\cup A \;\blacksquare\; Span[B]$,
use $SpanReduceBasis$ call it $B'$, $(L\subseteq B')\wedge(Basis[B'])$

$$BasisLinearIndCard := \forall_{S,T,V}\Big(\big((Basis[S,V,+,*])\wedge(LinInd[T,V,+,*])\big) \implies (|T|\le|S|)\Big)$$

(1) $(Basis[S,V,+,*]) \implies \dots$

(1.1) $(|T|>|S|) \implies \dots$

(1.1.1) $(Spans[S,V,+,*])\wedge(T\subseteq V) \;\blacksquare\; t_{1\dots t_j}=\sum(\gamma_i * s * i)\ \dots$

(1.1.2) $\dots t_j=\sum(\gamma_i' * t_i) \;\blacksquare\; \neg LinInd[T,V,+,*]$

(1.2) $(|T|>|S|) \implies (\neg LinInd[T,V,+,*]) \;\blacksquare\; (LinInd[T,V,+,*]) \implies (|T|\le|S|)$

(2) $\big((Basis[S,V,+,*])\wedge(LinInd[T,V,+,*])\big) \implies (|T|\le|S|)$

$$BasisCard := \forall_{S,T,V}\Big(\big((Basis[S,V,+,*])\wedge(Basis[T,V,+,*])\big) \implies (|T|=|S|)\Big)$$

(1) $Basis[S,V,+,*] \;\blacksquare\; LinInd[S,V,+,*]$

(2) $(Basis[T,V,+,*])\wedge(LinInd[S,V,+,*]) \;\blacksquare\; |S|\le|T|$

(3) $Basis[T,V,+,*] \;\blacksquare\; LinInd[T,V,+,*]$

(4) $(Basis[S,V,+,*])\wedge(LinInd[T,V,+,*]) \;\blacksquare\; |T|\le|S|$

(5) $(|S|\le|T|)\wedge(|T|\le|S|) \;\blacksquare\; |T|=|S|$

$$Dim[d,V,+,*] := \big(\exists_B(Basis[B,V,+,*])\big)\wedge\big((V=\{O\}) \implies (d=0)\big)\wedge\big((V\neq\{O\}) \implies (d=|B|)\big)$$

$$LinIndLengthDim := \forall_{U,V}\Big(\big((LinInd[U,V,+,*])\wedge(Dim[|U|,V,+,*])\big) \implies (Basis[U,V,+,*])\Big)$$

(1) $(LinIndExpandBasis)\wedge(LinInd[U,V,+,*]) \;\blacksquare\; \exists_B\big((U\subseteq B)\wedge(Basis[B,V,+,*])\big)$

(2) $(BasisCard)\wedge(Dim[|U|,V,+,*])\wedge(Basis[B,V,+,*]) \;\blacksquare\; |B|=|U| \;\blacksquare\; B=U \;\blacksquare\; Basis[U,V,+,*]$

$$SpanLengthDim := \forall_{U,V}\Big(\big((Spans[U,V,+,*])\wedge(Dim[|U|,V,+,*])\big) \implies (Basis[U,V,+,*])\Big)$$

(1) $(SpanReduceBasis)\wedge(Spans[U,V,+,*]) \;\blacksquare\; \exists_B\big((B\subseteq U)\wedge(Basis[B,V,+,*])\big)$

(2) $(BasisCard)\wedge(Dim[|U|,V,+,*])\wedge(Basis[B,V,+,*]) \;\blacksquare\; |B|=|U| \;\blacksquare\; B=U \;\blacksquare\; Basis[U,V,+,*]$

$$LinDepLengthDim := \forall_{U,V}\Big(\big((U \subseteq V) \land (|U| > Dim[V])\big) \implies (\neg LinInd[U,V,+,*])\Big)$$

(1) Contrapositive of $BasisLinearIndCard$

$$LinDepLengthDim := \forall_{U,V}\Big(\big((U \subseteq V) \land (|U| < Dim[V])\big) \implies (\neg Spans[U,V,+,*])\Big)$$

(1) Suppose $Spans[U,V,+,*]$, $B = SpanReduceBasis[U]$ to form a basis, $(|B| \leq |U| < Dim[V]) \land |B| = Dim[V]$ $\blacksquare\ \bot$

(2) $\neg Spans[U,V,+,*]$

## 3.7 Rank

$Nullity[n, A] := (NullSpace[N, A]) \land (Dim[n, N, +, *])$
$Rank[r, A, m, n] := (Matrix[A, m, n]) \land (RowSpace[R, A, m, n]) \land (Dim[r, R, A, +, *])$

$RowRankEqColRank := \forall_A(TODO)$

(1) TODO

$$RankNullity := \forall_A\big((Matrix[A,m,n]) \implies (Rank[A] + Nullity[A] = n)\big)$$

(1) TODO

$$RankInv := \forall_A\Big((Matrix[A,m,n]) \implies \big((Rank[A] = n) \iff (Inv[A])\big)\Big)$$

(1) TODO

$$RankNonTrivialSol := \Big(\exists_X\big((A * X = O) \land (X \neq O)\big)\Big) \iff (Rank[A] < n)$$

(1) TODO

$$RankUniqueSol := \big(\forall_{B\in\mathcal{M}}\exists!_{X\in\mathcal{M}}(Sol[X,A,B])\big) \iff (Rank[A] = n)$$

(1) TODO

$$SquareTheorems_{()} := \forall_{A\in\mathcal{M}}\left(\begin{array}{ll} (Invertible[A]) & \iff \\ (RowEquiv[A, I_n]) & \iff \\ \Big(\forall_X\big((X = O) \iff (Sol[X,A,O])\big)\Big) & \iff \\ \big(\forall_{B\in\mathcal{M}}\exists!_{X\in\mathcal{M}}(Sol[X,A,B])\big) & \iff \\ (Rank[A] = n) & \iff \\ (Nullity[A] = 0) & \iff \\ \big(\text{The rows form a linearly independent set of vectors (to get full rank)}\big) & \iff \\ \big(\text{The columns form a linearly independent set of vectors (to get full rank)}\big) & \iff \end{array}\right)$$

## 3.8 Linear Transformations

$$LinTrans[L,V,+_v,*_v,W,+_w,*_w] := \left(\begin{array}{c} (Function[f,V,W]) \land (VectorSpace[V,+_v,*_v]) \land (VectorSpace[W,+_w,*_w]) \land \\ \Big(\forall_{\alpha,\beta\in V}\big(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta)\big)\Big) \land \Big(\forall_{r\in\mathbb{R}}\forall_{\alpha\in V}\big(L(r *_v \alpha) = r *_w L(\alpha)\big)\Big) \end{array}\right)$$

$LinOp[L,V,+_v,*_v] := LinTrans[L,V,+_v,*_v,V,+_v,*_v]$
$\mathcal{L}[V,W] := \{L | LinTrans[L,V,+_v,*_v,W,+_w,*_w]\}$

$$ZeroMapsToZero := \forall_{L,V,W}\Big((LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \implies \big(L(O_v) = O_w\big)\Big)$$

(1) $L(O_v) = L(O_v +_v O_v) = L(O_v) +_w L(O_v)$

(2) $O_w = L(O_v) - L(O_v) = L(O_v)$

$$SplitAddInv := \forall_{L,V,W}\bigg((LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \implies \Big(\forall_{\alpha,\beta\in V}\big(L(\alpha -_v \beta) = L(\alpha) -_w L(\beta)\big)\Big)\bigg)$$

(1) $L(\alpha-\beta)=L\big(\alpha+(-\beta)\big)=L(\alpha)+L(-\beta)=L(\alpha)+(-1)*L(\beta)=L(\alpha)-L(\beta)$

$$BasisDomainInduceLinTrans := \forall_{V,W}\left(\begin{array}{l}\big((Basis[A,V,+_v,*_v])\wedge(B\subseteq W)\wedge(n=|B|=|A|)\wedge(VectorSpace[W,+_w,*_w])\big)\implies\\ \left(\exists!_T\left((LinTrans[T,V,+_v,*_v,W,+_w,*_w])\wedge\left(\forall_{i\in\mathbb{N}_{1,n}}\big(T(a_i)=b_i\big)\right)\right)\right)\end{array}\right)$$

(1) $T\big(\sum_{i=1}^n(k_i*a_i)\big) := \sum_{i=1}^n(k_i*b_i)$

(2) $(i\in\mathbb{N}_{1,n})\implies\ldots$

(2.1) $L := \langle\left(\begin{cases}1 & j=i\\ 0 & j\neq i\end{cases}\right)|j\in\mathbb{N}_{1,n}\rangle \;\blacksquare\; L\in\mathbb{R}^n$

(2.2) $T(a_i)=T\big(\sum_{i=1}^n(l_i*a_i)\big)=\sum_{i=1}^n(l_i*b_i)=b_i \;\blacksquare\; T(a_i)=b_i$

(3) $(i\in\mathbb{N}_{1,n})\implies\big(T(a_i)=b_i\big) \;\blacksquare\; \forall_{i\in\mathbb{N}_{1,n}}\big(T(a_i)=b_i\big)$

(4) $(BasisEquiv)\wedge(Basis[A,V,+_v,*_v]) \;\blacksquare\; \forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,A,K,V,+,*]) \;\blacksquare\; T: Span[A]\to Span[B] \;\blacksquare\; Function[T,V,W]$

(5) $(\alpha,\beta\in V)\implies\ldots$

(5.1) $\left(\exists_{K_\alpha}(LinComb[\alpha,A,K_\alpha,V,+_v,*_v])\right)\wedge\left(\exists_{K_\beta}(LinComb[\beta,A,K_\beta,V,+_v,*_v])\right) \;\blacksquare\; \big(\alpha=\sum_{i=1}^n(k_{\alpha i}*a_i)\big)\wedge\left(\beta=\sum_{i=1}^n(k_{\beta_i}*a_i)\right)$

(5.2) $T(\alpha+\beta)=T\left(\sum_{i=1}^n(k_{\alpha i}*a_i)+\sum_{i=1}^n(k_{\beta_i}*a_i)\right)=T\left(\sum_{i=1}^n\left((k_{\alpha i}+k_{\beta_i})*a_i\right)\right))=\sum_{i=1}^n\left((k_{\alpha i}+k_{\beta_i})*b_i\right)=\ \ldots$

(5.3) $\ldots\sum_{i=1}^n(k_{\alpha i}*b_i)+\sum_{i=1}^n(k_{\beta_i}*b_i)=T\big(\sum_{i=1}^n(k_{\alpha i}*a_i)\big)+T\left(\sum_{i=1}^n(k_{\beta_i}*a_i)\right)=T(\alpha)+T(\beta)$

(6) $(\alpha,\beta\in V)\implies\big(L(\alpha+_v\beta)=L(\alpha)+_w L(\beta)\big) \;\blacksquare\; \forall_{\alpha,\beta\in V}\big(L(\alpha+_v\beta)=L(\alpha)+_w L(\beta)\big)$

(7) $\big((r\in\mathbb{R})\wedge(\alpha\in V)\big)\implies\ldots$

(7.1) $\exists_K(LinComb[\alpha,A,K,V,+_v,*_v]) \;\blacksquare\; \alpha=\sum_{i=1}^n(k_i*a_i)$

(7.2) $L(r*_v\alpha)=L\big(r*_v\sum_{i=1}^n(k_i*_v a_i)\big)=L\left(\sum_{i=1}^n\big((rk_i)*_v a_i\big)\right)=\ \ldots$

(7.3) $\ldots\sum_{i=1}^n\big((rk_i)*_w b_i\big)=r*_w\sum_{i=1}^n(k_i*_w b_i)=r*_w L\big(\sum_{i=1}^n(k_i*_v a_i)\big)=r*_w L(\alpha)$

(8) $\big((r\in\mathbb{R})\wedge(\alpha\in V)\big)\implies\big(L(r*_v\alpha)=r*_w L(\alpha)\big) \;\blacksquare\; \forall_{r\in\mathbb{R}}\forall_{\alpha\in V}\big(L(r*_v\alpha)=r*_w L(\alpha)\big)$

(9) $\left(\forall_{i\in\mathbb{N}_{1,n}}\big(T(a_i)=b_i\big)\right)\wedge(Function[T,V,W])\wedge\left(\forall_{\alpha,\beta\in V}\big(L(\alpha+_v\beta)=L(\alpha)+_w L(\beta)\big)\right)\wedge\left(\forall_{r\in\mathbb{R}}\forall_{\alpha\in V}\big(L(r*_v\alpha)=r*_w L(\alpha)\big)\right)\wedge\ \ldots$

(10) $\ldots(VectorSpace[V,+_v,*_v])\wedge(VectorSpace[W,+_w,*_w]) \;\blacksquare\; LinTrans[T,V,+_v,*_v,W,+_w,*_w]$

$$Ker[ker_L,L,V,+_v,*_v,W,+_w,*_w] := (LinTrans[L,V,+_v,*_v,W,+_w,*_w])\wedge\big(ker_L=\{\alpha\in V|L(\alpha)=O_w\}\big)$$

$$KerSub := \forall_{L,V,W}\big((Ker[ker_L,L,V,+_v,*_v,W,+_w,*_w])\implies(Subspace[ker_L,V,+_v,*_v])\big)$$

(1) $ZeroMapsToZero \;\blacksquare\; L(O_v)=O_w \;\blacksquare\; O_v\in ker_L \;\blacksquare\; \emptyset\neq ker_L \;\blacksquare\; \emptyset\neq ker_L\subseteq V$

(2) $(\alpha,\beta\in ker_L)\implies\ldots$

(2.1) $\big(L(\alpha)=O_w\big)\wedge\big(L(\beta)=O_w\big)$

(2.2) $L(\alpha+\beta)=L(\alpha)+L(\beta)=O_w+O_w=O_w \;\blacksquare\; L(\alpha+\beta)\in ker_L$

(3) $(\alpha,\beta\in ker_L)\implies(\alpha+\beta\in ker_L) \;\blacksquare\; \forall_{\alpha,\beta\in ker_L}(\alpha+\beta\in ker_L)$

(4) $\big((r\in\mathbb{R})\wedge(\alpha\in ker_L)\big)\implies\ldots$

(4.1) $L(\alpha)=O_w \;\blacksquare\; L(r*\alpha)=r*L(\alpha)=r*O_w=O_w \;\blacksquare\; r*\alpha\in ker_L$

(5) $\big((r\in\mathbb{R})\wedge(\alpha\in ker_L)\big)\implies(r*\alpha\in ker_L) \;\blacksquare\; \forall_{r\in\mathbb{R}}\forall_{\alpha\in ker_L}(r*\alpha\in ker_L)$

(6) $(SubspaceEquiv)\wedge(\emptyset\neq ker_L\subseteq V)\wedge\left(\forall_{\alpha,\beta\in ker_L}(\alpha+\beta\in ker_L)\right)\wedge\left(\forall_{r\in\mathbb{R}}\forall_{\alpha\in ker_L}(r*\alpha\in ker_L)\right) \;\blacksquare\; Subspace[ker_L,V,+_v,*_v]$

$$KerInjective := \forall_{L,V,W}\Big((Ker[ker_L,L,V,+_v,*_v,W,+_w,*_w])\implies\big((Injective[L,V,W])\iff(ker_L=\{O_v\})\big)\Big)$$

(1) $(Injective[L,V,W])\implies\ldots$

(1.1) $ZeroMapsToZero \;\blacksquare\; L(O_v)=O_w$

(1.2) $O_v\in ker_L \;\blacksquare\; \{O_v\}\subseteq ker_L$

(1.3) $(v\in ker_L)\implies\ldots$

(1.3.1) $L(v) = O_w$

(1.3.2) $(Injective[L, V, W]) \wedge \left(L(O_v) = O_w\right) \;\blacksquare\; O_v = v$

(1.4) $(v \in ker_L) \implies (v = O_v) \;\blacksquare\; ker_L \subseteq \{O_v\}$

(1.5) $(\{O_v\} \subseteq ker_L) \wedge (ker_L \subseteq \{O_v\}) \;\blacksquare\; ker_L = \{O_v\}$

(2) $(Injective[L, V, W]) \implies (ker_L = \{O_v\})$

(3) $(ker_L = \{O_v\}) \implies \dots$

(3.1) $\Big((u, v \in V) \wedge \left(L(u) = L(v)\right)\Big) \implies \dots$

(3.1.1) $O_w = L(u) - L(v) = L(u - v) \;\blacksquare\; u - v \in ker_L$

(3.1.2) $ker_L = \{O_v\} \;\blacksquare\; u - v = O_v \;\blacksquare\; u = v$

(3.2) $\Big((u, v \in V) \wedge \left(L(u) = L(v)\right)\Big) \implies (u = v) \;\blacksquare\; \forall_{u,v \in V}\Big(\left(L(u) = L(v)\right) \implies (u = v)\Big) \;\blacksquare\; Injective[L, V, W]$

(4) $(ker_L = \{O_v\}) \implies (Injective[L, V, W])$

(5) $(Injective[L, V, W]) \iff (ker_L = \{O_v\})$

$Rng[rng_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge \Big(rng_L = \{\beta \in W | \exists_{\alpha \in V}\left(\beta = L(\alpha)\right)\}\Big)$

$RangeSub := \forall_{L,V,W}\big((Ran[rng_L, L, V, +_v, *_v, W, +_w, *_w]) \implies (Subspace[rng_L, W, +_w, *_w])\big)$

(1) $ZeroMapsToZero \;\blacksquare\; O_w = L(O_v) \;\blacksquare\; \exists_{\alpha \in V}\left(O_w = L(\alpha)\right) \;\blacksquare\; O_w \in rng_L \;\blacksquare\; \emptyset \neq rng_L \;\blacksquare\; \emptyset \neq rng_L \subseteq W$

(2) $(\alpha, \beta \in rng_L) \implies \dots$

(2.1) $\Big(\exists_{u \in V}\left(\alpha = L(u)\right)\Big) \wedge \Big(\exists_{v \in V}\left(\beta = L(v)\right)\Big)$

(2.2) $\alpha + \beta = L(u) + L(v) = L(u + v) \;\blacksquare\; \exists_{w \in V}\left(\alpha + \beta = L(w)\right) \;\blacksquare\; \alpha + \beta \in rng_L$

(3) $(\alpha, \beta \in rng_L) \implies (\alpha + \beta \in rng_L) \;\blacksquare\; \forall_{\alpha,\beta \in rng_L}(\alpha + \beta \in rng_L)$

(4) $\big((r \in \mathbb{R}) \wedge (\alpha \in rng_L)\big) \implies \dots$

(4.1) $\exists_{v \in V}\left(\alpha = L(v)\right) \;\blacksquare\; L(r * v) = r * L(v) = r * \alpha \;\blacksquare\; \exists_{w \in V}\left(r * \alpha = L(w)\right) \;\blacksquare\; r * \alpha \in rng_L$

(5) $\big((r \in \mathbb{R}) \wedge (\alpha \in rng_L)\big) \implies (r * \alpha \in rng_L) \;\blacksquare\; \forall_{r \in \mathbb{R}}\forall_{\alpha \in rng_L}(r * \alpha \in rng_L)$

(6) $(SubspaceEquiv) \wedge (\emptyset \neq rng_L \subseteq W) \wedge \Big(\forall_{\alpha,\beta \in rng_L}(\alpha + \beta \in rng_L)\Big) \wedge \Big(\forall_{r \in \mathbb{R}}\forall_{\alpha \in rng_L}(r * \alpha \in rng_L)\Big) \;\blacksquare\; Subspace[rng_L, W, +_w, *_w]$

$RankKer := \forall_{L,V,W}\big((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies (Dim[V] = Dim[ker_L] + Dim[rng_L])\big)$

(1) $\exists_U(Basis[U, ker_L, +_v, *_v]) \;\blacksquare\; Dim[ker_L] = |U|$

(2) $(LinIndSuperspace) \wedge (LinInd[U, ker_L, +_v, *_v]) \;\blacksquare\; LinInd[U, V, +_v, *_v]$

(3) $LinIndExpandBasis \;\blacksquare\; \exists_B\big((U \subseteq B) \wedge (Basis[B, V, +_v, *_v])\big) \;\blacksquare\; Dim[V] = |B|$

(4) $T := B \setminus U \;\blacksquare\; B = U \cup T$

(5) $m := |U|\,;\, n := |T|\,;\, p := |B|$

(6) $L(T) := \langle L(t_i) | i \in \mathbb{N}_{1,n}\rangle \subseteq W^n$

(7) $(w \in W) \implies \dots$

(7.1) $\exists_{v \in V}\left(w = L(v)\right)$

(7.2) $Basis[B, V, +_v, *_v] \;\blacksquare\; \exists_{K \in \mathbb{R}^p}\Big(v = \sum_{i=1}^{p}(k_i * b_i)\Big)$

(7.3) $v = \sum_{i=1}^{p}(k_i * b_i) = \sum_{i=1}^{m}(k_i * u_i) + \sum_{i=1}^{n}(\kappa_i * t_i)$

(7.4) $w = L(v) = L\big(\sum_{i=1}^{m}(k_i * u_i) + \sum_{i=1}^{n}(k_i * t_i)\big) = \sum_{i=1}^{m}\big(k_i * L(u_i)\big) + \sum_{i=1}^{n}\big(k_i * L(t_i)\big) = O_w + \sum_{i=1}^{n}\big(k_i * L(t_i)\big) = \sum_{i=1}^{n}\big(k_i * L(t_i)\big)$

(7.5) $\exists_K\big(LinComb[w, L(T), K, W, +_w, *_w]\big)$

(8) $(w \in W) \implies \Big(\exists_K\big(LinComb[w, L(T), K, W, +_w, *_w]\big)\Big) \;\blacksquare\; \forall_{w \in W}\Big(\exists_K\big(LinComb[w, L(T), K, W, +_w, *_w]\big)\Big)$

(9) $Spans[L(T), W, +_w, *_w]$

(10) $\Big((K \in \mathbb{R}^n) \wedge \big(LinComb[O_w, L(T), K, W, +_w, *_w]\big)\Big) \implies \dots$

(10.1) $O_w = \sum_{i=1}^{n}\big(k_i * L(t_i)\big) = L\big(\sum_{i=1}^{n}(k_i * t_i)\big) \;\blacksquare\; \sum_{i=1}^{n}(k_i * t_i) \in ker_L$

(10.2) $Basis[U, ker_L, +_v, *_v] \;\blacksquare\; \exists_{D \in \mathbb{R}^m}\big(\sum_{i=1}^{n}(k_i * t_i) = \sum_{i=1}^{m}(d_i * u_i)\big)$

(10.3) $(LinInd[T \cup U]) \wedge \left(\sum_{i=1}^{n}(k_i * t_i) = \sum_{i=1}^{m}(d_i * u_i)\right) \blacksquare\ K = D = \{O\} \blacksquare\ K = \{O\}$

(11) $\left((K \in \mathbb{R}^n) \wedge \left(LinComb[O_w, L(T), K, W, +_w, *_w]\right)\right) \implies (K = \{O\})$

(12) $\forall_{K \in \mathbb{R}^n}\left(\left(LinComb[O_w, L(T), K, W, +_w, *_w]\right) \implies (K = \{O\})\right)$

(13) $\forall_{K \in \mathbb{R}^n}\left(\left(LinComb[O_w, L(T), K, W, +_w, *_w]\right) \iff (K = \{O\})\right) \blacksquare\ LinInd[L(T), W, +_w, *_w]$

(14) $Basis[L(T), W, +_w, *_w] \blacksquare\ Dim[V] = |B| = |U| + |L(T)| = Dim[ker_L] + Dim[rng_L]$

$LTSurInj := \forall_{L,V,W}\left(\left((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (Surjective[L, V, W])\right) \implies (Injective[L, V, W])\right)$

(1) TODO

$LTInjSur := \forall_{L,V,W}\left(\left((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (Injective[L, V, W])\right) \implies (Surjective[L, V, W])\right)$

(1) TODO

$LTInjLinInd := \forall_{L,V,W}\left((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies \left((Injective[L, V, W]) \iff (\text{LI In W are LI IN V})\right)\right)$

(1) TODO

TODO coordinatesb