Contents

Chapter 1

Philosopherers

(1.5)
$OrderTrichotomy(<, S) := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$
$OrderTransitivity(<, S) := \forall_{x,y,z \in S}\big((x < y \wedge y < z) \implies x < z\big)$
$Order(<, S) := OrderTrichotomy(<, S) \wedge OrderTransitivity(<, S)$

(1.7)
$BoundedAbove(E, S, <) := Order(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(x \leq \beta)$
$BoundedBelow(E, S, <) := Order(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(\beta \leq x)$
$UpperBound(\beta, E, S, <) := Order(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(x \leq \beta)$
$LowerBound(\beta, E, S, <) := Order(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(\beta \leq x)$

(1.8)
$LUB(\alpha, E, S, <) := UpperBound(\alpha, E, S, <) \wedge \forall_{\gamma}\big(\gamma < \alpha \implies \neg UpperBound(\gamma, E, S, <)\big)$
$GLB(\alpha, E, S, <) := LowerBound(\alpha, E, S, <) \wedge \forall_{\beta}\big(\alpha < \beta \implies \neg LowerBound(\beta, E, S, <)\big)$

(1.10)
$LUBProperty(S, <) := \forall_{E}\Big(\big(\emptyset \neq E \subset S \wedge BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$
$GLBProperty(S, <) := \forall_{E}\Big(\big(\emptyset \neq E \subset S \wedge BoundedBelow(E, S, <)\big) \implies \exists_{\alpha \in S}\big(GLB(\alpha, E, S, <)\big)\Big)$

(1.11)
$\boxed{LUBPropertyImpliesGLBProperty}$ $LUBProperty(S, <) \implies GLBProperty(S, <)$

(1) $LUBProperty(S, <) \implies \ldots$ wts: 2

(1.1) $\big(\emptyset \neq B \subset S \wedge BoundedBelow(B, S, <)\big) \implies \ldots$ wts: 1.2

(1.1.1) $Order(<, S) \wedge \exists_{\delta' \in S}\big(LowerBound(\delta', B, S, <)\big)$ from: $BoundedBelow$, 1.1

(1.1.2) $|B| = 1 \implies \ldots$ wts: 1.1.3

(1.1.2.1) $\exists_{u'}(u' \in B)$ ▌ $u := choice(\{u' | u' \in B\})$ ▌ $B = \{u\}$ from: 1.1.2

(1.1.2.2) $GLB(u, B, S, <)$ ▌ $\exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$

(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$

(1.1.4) $|B| \neq 1 \implies \ldots$ wts: 1.1.5

(1.1.4.1) $\forall_{E}\Big(\big(\emptyset \neq E \subset S \wedge BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$ from: $LUBProperty$, 1

(1.1.4.2) $L := \{s \in S | LowerBound(s, B, S, <)\}$

(1.1.4.3) $|B| > 1 \wedge OrderTrichotomy(<, S)$ ▌ $\exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$ from: $Order$, 1.1.1 wts: 1.1.4.7

(1.1.4.4) $b_1 := choice\Big(\{b_1' \in B | \exists_{b_0' \in B}(b_0' < b_1')\}\Big)$ ▌ $\neg LowerBound(b_1, B, S, <)$ from: 1.1.4.2

(1.1.4.5) $b_1 \notin L$ ▌ $L \subset S$

(1.1.4.6) $\delta := choice\big(\{\delta' \in S | LowerBound(\delta', B, S, <)\}\big)$ ▌ $\delta \in L$ ▌ $\emptyset \neq L$ from: 1.1.1

(1.1.4.7) $\emptyset \neq L \subset S$ from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y\in L}\big(LowerBound(y, B, S, <)\big) \blacksquare \forall_{y\in L}\forall_{x\in B}(y \le x)$ from: *LowerBound*, 1.1.4.2 wts: 1.1.4.10

(1.1.4.9) $\forall_{x\in B}\Big(x \in S \land \forall_{y\in L}(y \le x)\Big) \blacksquare \forall_{x\in B}\big(UpperBound(x, L, S, <)\big)$ from: *UpperBound*

(1.1.4.10) $\exists_{x\in S}\big(UpperBound(x, L, S, <)\big) \blacksquare BoundedAbove(L, S, <)$

(1.1.4.11) $\emptyset \ne L \subset S \land BoundedAbove(L, S, <)$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha'\in S}\big(LUB(\alpha', L, S, <)\big) \blacksquare \alpha := choice\Big(\{\alpha' \in S|\big(LUB(\alpha', L, S, <)\big)\}\Big)$ from: 1.1.4.1 wts: 1.1.4.21

(1.1.4.13) $\forall_x\big(x \in B \implies UpperBound(x, L, S, <)\big)$ from: 1.1.4.9 wts: 1.1.4.17

(1.1.4.14) $\forall_x\big(\neg UpperBound(x, L, S, <) \implies x \notin B\big)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$ wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound(\gamma, L, S, <) \blacksquare \gamma \notin B$ from: *LUB*, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B \blacksquare \gamma \in B \implies \gamma \ge \alpha$

(1.1.4.17) $\forall_{\gamma\in B}(\alpha \le \gamma) \blacksquare LowerBound(\alpha, B, S, <)$ from: *LowerBound*

(1.1.4.18) $\alpha < \beta \implies \ldots$ wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y\in L}(y \le \alpha < \beta) \blacksquare \forall_{y\in L}(y \ne \beta)$ from: *LUB*, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta \notin L \blacksquare \neg LowerBound(\beta, B, S, <)$ from: 1.1.4.2

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound(\beta, B, S, <) \blacksquare \forall_{\beta\in S}\big(\alpha < \beta \implies \neg LowerBound(\beta, B, S, <)\big)$

(1.1.4.20) $LowerBound(\alpha, B, S, <) \land \forall_{\beta\in S}\big(\alpha < \beta \implies \neg LowerBound(\beta, B, S, <)\big)$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB(\alpha, B, S, <) \blacksquare \exists_{\epsilon_1\in S}\big(GLB(\epsilon_1, B, S, <)\big)$

(1.1.5) $|B| \ne 1 \implies \exists_{\epsilon_1\in S}\big(GLB(\epsilon_1, B, S, <)\big)$

(1.1.6) $\Big(|B| = 1 \implies \exists_{\epsilon_0\in S}\big(GLB(\epsilon_0, B, S, <)\big)\Big) \land \Big(|B| \ne 1 \implies \exists_{\epsilon_1\in S}\big(GLB(\epsilon_1, B, S, <)\big)\Big)$ from: 1.1.3, 1.1.5

(1.1.7) $(|B| = 1 \lor |B| \ne 1) \implies \exists_{\epsilon\in S}\big(GLB(\epsilon, B, S, <)\big) \blacksquare \exists_{\epsilon\in S}\big(GLB(\epsilon, B, S, <)\big)$

(1.2) $\big(\emptyset \ne B \subset S \land BoundedBelow(B, S, <)\big) \implies \exists_{\epsilon\in S}\big(GLB(\epsilon, B, S, <)\big)$

(1.3) $\forall_B\Big(\big(\emptyset \ne B \subset S \land BoundedBelow(B, S, <)\big) \implies \exists_{\epsilon\in S}\big(GLB(\epsilon, B, S, <)\big)\Big)$

(1.4) $GLBProperty(S, <)$

(2) $LUBProperty(S, <) \implies GLBProperty(S, <)$

(1.12)

$$Field(F, +, *) := \exists_{\mathbb{0},\mathbb{1}\in F}\forall_{x,y,z\in F}\left(\begin{array}{l} x + y \in F \quad\land\quad x * y \in F \quad\land \\ x + y = y + x \quad\land\quad x * y = y * x \quad\land \\ (x + y) + z = x + (y + z) \quad\land\quad (x * y) * z = x * (y * z) \quad\land \\ \mathbb{1} \ne \mathbb{0} \quad\land\quad x * (y + z) = (x * y) + (x * z) \quad\land \\ \mathbb{0} + x = x \quad\land\quad \mathbb{1} * x = x \quad\land \\ \exists_{-x\in F}\big(x + (-x) = \mathbb{0}\big) \land \Big(x \ne \mathbb{0} \implies \exists_{1/x\in F}\big(x * (1/x) = \mathbb{1}\big)\Big) \end{array}\right)$$

*********************** $\big(Field(F, +, *) \land x, y, z \in F\big) \implies \ldots$ ***********************

(1.14)

$\boxed{AdditiveCancellation}$ $(x + y = x + z) \implies y = z$

(1) $y = \mathbb{0} + y = \big(x + (-x)\big) + y = \big((-x) + x\big) + y = (-x) + (x + y) = \ldots$ from: *Field*

(2) $(-x) + (x + z) = \big((-x) + x\big) + z = \big(x + (-x)\big) + z = \mathbb{0} + z = z$ from: *Field*

$\boxed{AdditiveIdentityUniqueness}$ $(x + y = x) \implies y = \mathbb{0}$

(1) $x + y = x = \mathbb{0} + x = x + \mathbb{0}$ from: *Field*

(2) $y = 0$ from: *AdditiveCancellation*

$\boxed{AdditiveInverseUniqueness}$ $(x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$ from: *Field*

(2) $y = -x$ from: *AdditiveCancellation*

$\boxed{DoubleNegative}$ $x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x$ ▮ $\mathbb{0} = (-x) + x$ from: *Field*

(2) $x = -(-x)$ from: *AdditiveInverseUniqueness*

(1.15)

$\boxed{MultiplicativeCancellation}$ $(x \neq 0 \wedge x * y = x * z) \implies y = z$ —

$\boxed{MultiplicativeIdentityUniqueness}$ $(x \neq 0 \wedge x * y = x) \implies y = \mathbb{1}$ —

$\boxed{MultiplicativeInverseUniqueness}$ $(x \neq 0 \wedge x * y = \mathbb{1}) \implies y = 1/x$ —

$\boxed{DoubleReciprocal}$ $(x \neq 0) \implies x = 1/(1/x)$ —

(1.16)

$\boxed{Domination}$ $\mathbb{0} * x = \mathbb{0}$

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x$ ▮ $\mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$ from: *Field*

(2) $\mathbb{0} * x = \mathbb{0}$ from: *AdditiveIdentityUniqueness*

$\boxed{NonDomination}$ $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies \ldots$

(1.1) $(x * y = 0) \implies \ldots$

(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = \big(x * (1/x)\big) * \big(y * (1/y)\big) = (x * y) * \big((1/x) * (1/y)\big) = \mathbb{0} * \big((1/x) * (1/y)\big) = \mathbb{0}$ from: *Field*, *Domination*, 1, 1.1

(1.1.2) $\mathbb{1} = \mathbb{0} \wedge \mathbb{1} \neq \mathbb{0}$ ▮ $\perp$ from: *Field*

(1.2) $(x * y = 0) \implies \perp$ ▮ $x * y \neq 0$

(2) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq 0$

$\boxed{NegationCommutativity}$ $(-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0}$ ▮ $x * y + (-x) * y = \mathbb{0}$ from: *Field*, *Domination* wts: 2

(2) $(-x) * y = -(x * y)$ from: *AdditiveInverseUniqueness*

(3) $x * y + x * (-y) = x * (y + -y) = x * \mathbb{0} = \mathbb{0}$ ▮ $x * y + x * (-y) = \mathbb{0}$ from: *Field*, *Domination* wts: 4

(4) $x * (-y) = -(x * y)$ from: *AdditiveInverseUniqueness*

(5) $(-x) * y = -(x * y) = x * (-y)$ from: 2, 4

$\boxed{NegativeMultiplication}$ $(-x) * (-y) = x * y$

(1) $(-x) * (-y) = -\big(x * (-y)\big) = -\big(-(x * y)\big) = x * y$ from: *NegationCommutativity*, *DoubleNegative*

(1.17)

$$OrderedField(F, +, *, <) := \begin{pmatrix} Field(F, +, *) \quad \wedge \quad Order(<, F) \quad \wedge \\ \forall_{x,y,z \in F}(y < z \implies x + y < x + z) \quad \wedge \\ \forall_{x,y \in F}\big((x > \mathbb{0} \wedge y > \mathbb{0}) \implies x * y > \mathbb{0}\big) \end{pmatrix}$$

************************ $\big(OrderedField(F, +, *, <) \wedge x, y, z \in F\big) \implies \ldots$ ************************

(1.18)

$\boxed{NegationOnOrder}$ $x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \ldots$

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x$ ▮ $\mathbb{0} > -x$ ▮ $-x < \mathbb{0}$ from: *OrderedField*

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \ldots$

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x$ ▮ $\mathbb{0} < x$ ▮ $x > \mathbb{0}$ from: *OrderedField*

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \land -x < \mathbb{0} \implies x > \mathbb{0}$ ▮ $x > \mathbb{0} \iff -x < \mathbb{0}$ from: 2, 4

PositiveFactorPreservesOrder $(x > \mathbb{0} \land y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \land y < z) \implies \ldots$

(1.1) $(-y) + z > (-y) + y = \mathbb{0}$ ▮ $z + (-y) = \mathbb{0}$ from: *OrderedField*

(1.2) $x * (z + (-y)) > \mathbb{0}$ ▮ $x * z + x * (-y) > \mathbb{0}$ from: *OrderedField*

(1.3) $x * z = \mathbb{0} + x * z = \big(x * y + -(x * y)\big) + x * z = \big(x * y + x * (-y)\big) + x * z = \ldots$ from: *Field*, *NegationCommutativity*

(1.4) $x * y + \big(x * z + x * (-y)\big) > x * y + \mathbb{0} = x * y$ from: *Field*, 1.2

(1.5) $x * z > x * y$ from: 1.3, 1.4

(2) $(x > \mathbb{0} \land y < z) \implies x * z > x * y$

NegativeFactorFlipsOrder $(x < \mathbb{0} \land y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \land y < z) \implies \ldots$

(1.1) $-x > \mathbb{0}$ from: *NegationOnOrder*

(1.2) $(-x) * y < (-x) * z$ ▮ $\mathbb{0} = x * y + (-x) * y < x * y + (-x) * z$ ▮ $\mathbb{0} < x * y + (-x) * z$ from: *PositiveFactorPreservesOrder*

(1.3) $\mathbb{0} < (-x) * (-y + z)$ ▮ $\mathbb{0} > x * (-y + z)$ ▮ $\mathbb{0} > -(x * y) + x * z$ from: *NegationOnOrder*

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \land y < z) \implies x * y > x * z$

SquareIsPositive $(x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ from: *OrderedField*

(2) $(x < \mathbb{0}) \implies \ldots$

(2.1) $-x > \mathbb{0}$ ▮ $x * x = (-x) * (-x) > \mathbb{0}$ ▮ $x * x > \mathbb{0}$ from: *NegationOnOrder*, *OrderedField*, *NegativeMultiplication*

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \lor x < \mathbb{0}) \implies x * x > \mathbb{0}$ ▮ $x \neq \mathbb{0} \implies x * x > \mathbb{0}$ from: *OrderTrichotomy*, 1, 3

OneIsPositive $\mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0}$ ▮ $\mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ from: *Field*, *SquareIsPositive*

ReciprocationOnOrder $(0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \ldots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0}$ ▮ $x * (1/x) > 0$ from: *Field*, *OneIsPositive*

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \land x * (1/x) > 0 \implies \bot$ ▮ $1/x > \mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0}$ ▮ $y * (1/y) > 0$ from: *Field*, *OneIsPositive*

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \land y * (1/y) > 0 \implies \bot$ ▮ $1/y > \mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

(1.5) $(1/x) * (1/y) > \mathbb{0}$ from: *OrderedField*

(1.6) $\mathbb{0} < 1/y = \big((1/x) * (1/y)\big) * x < \big((1/x) * (1/y)\big) * y = 1/x$ from: *OrderedField*, 1, 1.4, 1.5

(1.19)
$\boxed{OrderedFieldQ}$ $OrderedField(\mathbb{Q}, +, *, <)$ —

$Subfield(K, F, +, *) := Field(F, +, *) \wedge K \subset F \wedge Field(K, +, *)$
$OrderedSubfield(K, F, +, *, <) := OrderedField(F, +, *, <) \wedge K \subset F \wedge OrderedField(K, +, *, <)$
$CutI(\alpha) := \emptyset \neq \alpha \subset \mathbb{Q}$
$CutII(\alpha) := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
$CutIII(\alpha) := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} | CutI(\alpha) \wedge CutII(\alpha) \wedge CutIII(\alpha)\}$

$\boxed{CutCorollaryI}$ $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \ldots$
(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ from: $CutII$, 1
(1.2) $q < p \implies q \in \alpha$ ▮ $q \notin \alpha \implies q \geq p$ from: 1
(1.3) $(q \notin \alpha) \implies \ldots$
(1.3.1) $q \geq p$ from: 1.2
(1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp$ ▮ $q \neq p$ from: 1, 1.3
(1.3.3) $q \geq p \wedge q \neq p$ ▮ $p < q$
(1.4) $q \notin \alpha \implies p < q$ ▮ $p < q$ from: 1
(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

$\boxed{CutCorollaryII}$ $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$
(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ from: $CutII$, 1
(1.2) $s \in \alpha \implies \big(r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)\big)$ ▮ $s \in \alpha \implies r \in \alpha$ from: 1, 1.1
(1.3) $r \notin \alpha \implies s \notin \alpha$ ▮ $s \notin \alpha$ from: 1, 1.2
(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$\boxed{OrderTrichotomyR}$ $OrderTrichotomy(\mathbb{R}, <_{\mathbb{R}})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$
(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$
(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$ from: $<_{\mathbb{R}}$, 1.1
(1.1.2) $\exists_{p'} (p' \in \alpha \wedge p' \notin \beta)$ ▮ $p := choice(\{p' | p' \in \alpha \wedge p' \notin \beta\})$
(1.1.3) $q \in \beta \implies \ldots$
(1.1.3.1) $p, q \in \mathbb{Q}$
(1.1.3.2) $q < p$ from: $CutCorollaryI$
(1.1.3.3) $q \in \alpha$ from: $CutII$
(1.1.4) $q \in \beta \implies q \in \alpha$
(1.1.5) $\forall_{q \in \beta} (q \in \alpha)$ ▮ $\beta \subseteq \alpha$
(1.1.6) $\beta \subset \alpha$ ▮ $\beta <_{\mathbb{R}} \alpha$
(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$
(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$ ▮ $(\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$
(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$
(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$
(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$
(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$
(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha,\beta\in\mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $OrderTrichotomy(\mathbb{R}, <_{\mathbb{R}})$

$\boxed{OrderTransitivityR}$ $OrderTransitivity(\mathbb{R}, <_{\mathbb{R}})$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a\in\alpha}(a \in \beta) \wedge \forall_{b\in\beta}(b \in \gamma)$

(1.1.3) $\forall_{a\in\alpha}(\alpha \in \gamma) \blacksquare \alpha \subset \gamma \blacksquare \alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(3) $\forall_{\alpha,\beta,\gamma\in\mathbb{R}}\big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(4) $OrderTransitivity(\mathbb{R}, <_{\mathbb{R}})$

$\boxed{OrderR}$ $Order(<_{\mathbb{R}}, \mathbb{R})$ from: $OrderTrichotomyR, OrderTransitivityR$ wts:

$\boxed{LUBPropertyR}$ $LUBProperty(\mathbb{R}, <_{\mathbb{R}})$

(1) $\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} | \exists_{\alpha\in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset \blacksquare \exists_{\alpha}(\alpha \in A) \blacksquare \alpha_0 := choice(\{\alpha | \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset \blacksquare \exists_{a}(a \in \alpha_0) \blacksquare a_0 := choice(\{a | a \in \alpha_0\}) \blacksquare a_0 \in \gamma \blacksquare \gamma \neq \emptyset$

(1.4) $BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}}) \blacksquare \exists_{\beta}\big(UpperBound(\beta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.5) $\beta_0 := choice(\{\beta | UpperBound(\beta, A, \mathbb{R}, <_{\mathbb{R}})\})$

(1.6) $UpperBound(\beta_0, A, \mathbb{R}, <_{\mathbb{R}}) \blacksquare \forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \beta_0) \blacksquare \forall_{\alpha\in A}(\alpha \subseteq \beta_0) \blacksquare \forall_{\alpha\in A}\forall_{a\in\alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \blacksquare \forall_{a\in\gamma}(a \in \beta_0) \blacksquare \gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q} \blacksquare \gamma \subseteq \beta_0 \subset \mathbb{Q} \blacksquare \gamma \subseteq \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q} \blacksquare CutI(\gamma)$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.10.1) $p \in \gamma \blacksquare \exists_{\alpha\in A}(p \in \alpha) \blacksquare \alpha_1 := choice(\{\alpha \in A | p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \blacksquare q \in \alpha_1 \blacksquare q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \blacksquare \forall_{p\in\gamma}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \gamma) \blacksquare CutII(\gamma)$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha\in A}(p \in \alpha) \blacksquare \alpha_2 := choice(\{\alpha \in A | p \in \alpha\})$

(1.12.2) $\exists_{r\in\alpha_2}(p < r) \blacksquare r_0 := choice(\{r \in \alpha_2 | p < r\})$

(1.12.3) $r_0 \in \alpha_2 \blacksquare r_0 \in \gamma$

(1.12.4) $p < r_0 \blacksquare p < r_0 \wedge r_0 \in \gamma \blacksquare \exists_{r\in\gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r\in\gamma}(p < r) \blacksquare \forall_{p\in\gamma}\exists_{r\in\gamma}(p < r) \blacksquare CutIII(\gamma)$

(1.14) $CutI(\gamma) \wedge CutII(\gamma) \wedge CutIII(\gamma) \blacksquare \gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha\in A}(\alpha \subseteq \gamma) \blacksquare \forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma)$

(1.16) $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R} \blacksquare UpperBound(\gamma, A, \mathbb{R}, <_{\mathbb{R}})$

(1.17) $\delta <_{\mathbb{R}} \gamma \implies \ldots$

(1.17.1) $\delta \subset \gamma \blacksquare \exists_{s}(s \in \gamma \wedge s \notin \delta) \blacksquare s_0 := choice(\{s \in \mathbb{Q} | s \in \gamma \wedge s \notin \delta\})$

(1.17.2) $s_0 \in \gamma \blacksquare \exists_{\alpha\in A}(s_0 \in \alpha) \blacksquare \alpha_3 := choice(\{\alpha \in A | s_0 \in \alpha\})$

(1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta \blacksquare \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \ldots$

(1.17.4.1) $\alpha_3 \subseteq \delta \blacksquare \forall_{s\in\mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \blacksquare \neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4.2) $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \blacksquare \perp$

(1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \perp \blacksquare \delta <_{\mathbb{R}} \alpha_3 \blacksquare \exists_{\alpha\in A}(\delta <_{\mathbb{R}} \alpha) \blacksquare \exists_{\alpha\in A}\big(\neg(\alpha \leq_{\mathbb{R}} \delta)\big)$

(1.17.6) $\neg\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \delta) \blacksquare \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}}))$ ▮ $\forall_{\delta}\big(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.19) $UpperBound(\gamma, A, \mathbb{R}, <_{\mathbb{R}}) \wedge \forall_{\delta}\big(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.20) $LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})$ ▮ $\exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(2) $\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(3) $\forall_A\Big(\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)\Big)$ ▮ $LUBProperty(\mathbb{R}, <_{\mathbb{R}})$

$+_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$
$0_{\mathbb{R}} := \{x \in \mathbb{Q} | x < 0\}$

$\boxed{0R}$ $0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}}$ ▮ $\emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q}$ ▮ $CutI(0_{\mathbb{R}})$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}}$ ▮ $\forall_{x \in 0_{\mathbb{R}}}\forall_{y \in \mathbb{Q}}(y < x \implies y \in 0_{\mathbb{R}})$ ▮ $CutII(0_{\mathbb{R}})$

(3) $y := x/2$ ▮ $(x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▮ $\forall_{x \in 0_{\mathbb{R}}}\exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▮ $CutIII(0_{\mathbb{R}})$

(4) $CutI(0_{\mathbb{R}}) \wedge CutII(0_{\mathbb{R}}) \wedge CutIII(0_{\mathbb{R}})$ ▮ $0_{\mathbb{R}} \in \mathbb{R}$

$\boxed{Field + R}$ $Field^{+}(\mathbb{R})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha)\,;\,\exists_b(b \in \beta)$ ▮ $a_0 := choice(\{a | a \in \alpha\})\,;\, b_0 := choice(\{b | b \in \beta\})$ ▮ $a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha)\,;\,\exists_y(y \notin \beta)$ ▮ $x_0 := choice(\{x | x \notin \alpha\})\,;\, y_0 := choice(\{y | y \notin \beta\})$

(1.5) $\forall_{r \in \alpha}(r < x_0)\,;\,\forall_{s \in \beta}(s < y_0)$ ▮ $\forall_{r \in \alpha}\forall_{s \in \beta}(r + s < x_0 + y_0)$ ▮ $x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q}$ ▮ $CutI(\alpha +_{\mathbb{R}} \beta)$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.7.1) $\exists_{r \in \alpha}\exists_{s \in \beta}(p = r + s)$ ▮ $(r_0, s_0) := choice((r, s) \in \alpha \times \beta | p = r + s)$

(1.7.2) $q < p = r_0 + s_0$ ▮ $(q - s_0) < r_0$ ▮ $(q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta$ ▮ $q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta$ ▮ $q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta$ ▮ $\forall_{p \in \alpha +_{\mathbb{R}} \beta}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta)$ ▮ $CutII(\alpha +_{\mathbb{R}} \beta)$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r \in \alpha}\exists_{s \in \beta}(p = r + s)$ ▮ $(r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta | p = r + s\})$

(1.9.2) $r_1 \in \alpha$ ▮ $\exists_{t \in \alpha}(r_1 < t)$ ▮ $t_0 := choice(\{t \in \alpha | r_1 < t\})$

(1.9.3) $s_1 \in \beta$ ▮ $t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1$ ▮ $\exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▮ $\forall_{p \in \alpha +_{\mathbb{R}} \beta}\exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▮ $CutIII(\alpha +_{\mathbb{R}} \beta)$

(1.11) $CutI(\alpha +_{\mathbb{R}} \beta) \wedge CutII(\alpha +_{\mathbb{R}} \beta) \wedge CutIII(\alpha +_{\mathbb{R}} \beta)$ ▮ $\alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

(3) $\alpha +_{\mathbb{R}} \beta = \{r + s | r \in \alpha \wedge s \in \beta\} = \{s + r | s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

(4) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(4.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

(4.2) $\{a + (b + c) | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(5) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(6) $\alpha \in \mathbb{R} \implies \ldots$

(6.1) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \ldots$

(6.1.1) $s < 0$ ▮ $r + s < r + 0 = r$ ▮ $r + s < r$ ▮ $r + s \in \alpha$

(6.2) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha$ ▮ $\forall_{r \in \alpha}\forall_{s \in 0_{\mathbb{R}}}(r + s \in \alpha)$

(6.3) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▮ $\forall_{p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}}(p \in \alpha)$ ▮ $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(6.4) $p \in \alpha \implies \ldots$

(6.4.1) $\exists_{r \in \alpha}(p < r)$ ▮ $r_2 := choice(\{r \in \alpha | p < r\})$

(6.4.2) $p < r_2 \;\blacksquare\; p - r_2 < r_2 - r_2 = 0 \;\blacksquare\; (p - r_2) < 0 \;\blacksquare\; (p - r_2) \in 0_{\mathbb{R}}$

(6.4.3) $r_2 \in \alpha \;\blacksquare\; p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(6.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; \forall_{p\in\alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \;\blacksquare\; \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(6.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(6.7) EXISTS ADDITIVE INVERSE

$\boxed{ExistenceOfR}$ $\exists_{\mathbb{R}}\big(LUBProperty(\mathbb{R}, <) \wedge OrderedSubfield(\mathbb{Q}, \mathbb{R}, +, *, <)\big)$

(1) 123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

First Chapter

(1) First

(1.1) Second

(1.2) Third

(2) Fourth

This will be an empty chapter and I will put some text here

$$\sum_{i=0}^{\infty} a_i x^i \tag{2.1}$$

The equation 2.1 shows a sum that is divergent. This formula will later be used in the page **??**.
For further references see Something Linky or go to the next url: http://www.sharelatex.com or open the next file File.txt
It's also possible to link directly any word or any sentence in your document. supwithitSup With It Theorem
If you read this text, you will get no information. Really? Is there no information?

For instance this sentence. supwithit