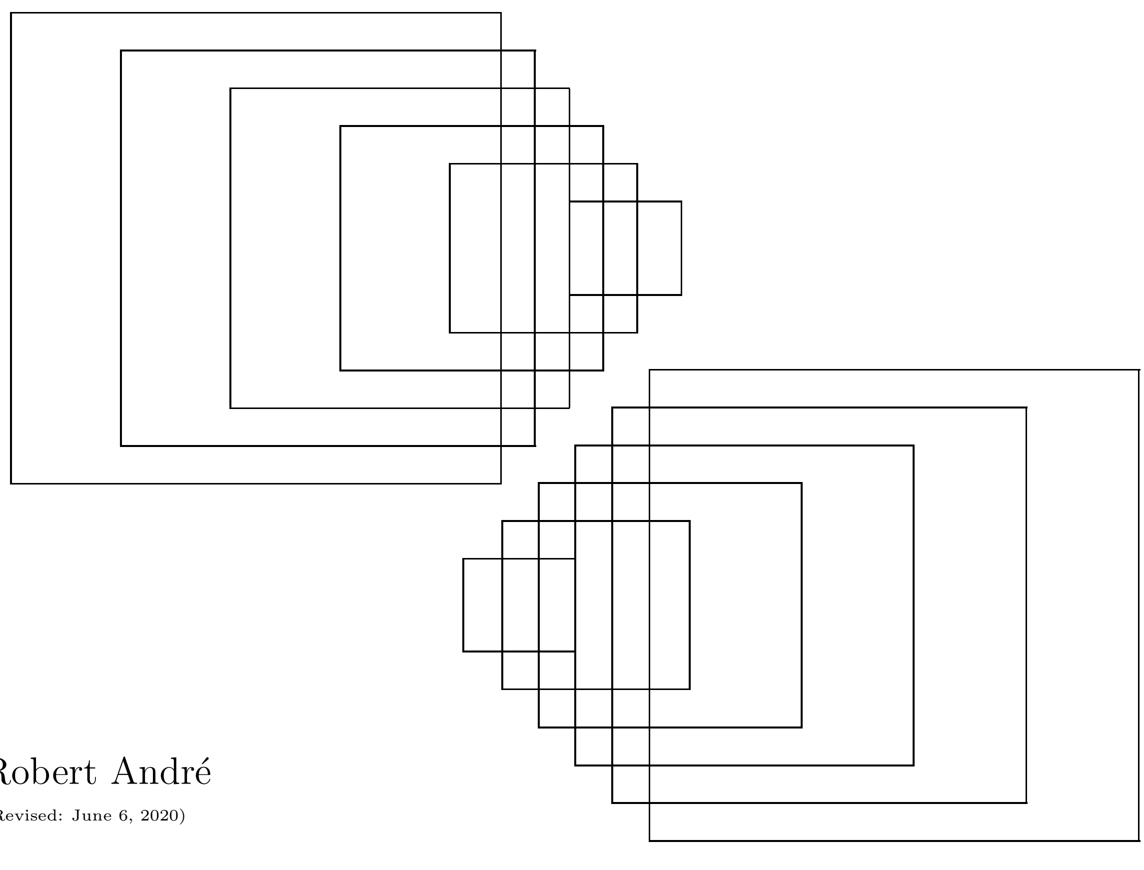
Point-set topology with topics

Basic general topology for graduate studies



Robert André

(Revised: June 6, 2020)

(Revised: June 6, 2020)

ISBN 978-0-9938485-1-3

À

Jinxia, Camille et Isabelle

"Everything has beauty, but not everyone sees it."

Confucius

Preface

This text was prepared to serve as an introduction to the study of general topology. Most students in mathematics are required at some point in their study to have knowledge of some fundamentals of general topology since many of tools and techniques mastered in that field will prove to be useful no matter what choice they make for their area of specialization. "Trust us, it will be good for you" they are told. But, sometimes, strange things happen. Some students develop a fascination for this field sometimes unfortunately described as being "no longer in fashion anymore". Recall that the mathematicians, George Cantor and Felix Hausdorff, were also told that the area of mathematics they spent so much time investigating was, in their time, "not fashionable". Reasons for the continued study of this topic often go beyond the simple perception that it is a practical or useful tool; the intricate beauty of the mathematical structures that are derived in this field become its main attraction. This is, of course, how this writer perceives the subject and served as the main motivation to prepare a textbook that will help the reader enjoy its study. Of course, one would at least hope that an author would write a book about something they are passionate about.

But first, I should at least write a few words about the mathematical content of this textbook. The choice of content as well as the order and pace of the presentation of the concepts found in the text were developed with senior math undergraduate or math graduate students in mind. The targeted reader will have been exposed to some mathematical rigor to a level normally found in an introduction to mathematical analysis texts or as presented in an introduction to linear algebra or abstract algebra texts. The first two sections consist mostly of a review of normed vector spaces and of a presentation of some very basic ideas on metric spaces. These are meant to ease the reader into the main subject matter of general topology. Once we have worked through the most fundamental concepts of topology in chapters one to twenty the reader will be exposed to various more specialized or advanced topics. These are presented in the form of five chapters where it is best to master one chapter before studying the following one. These chapters make excellent reading assignments for graduate students particularly if they are followed by in class presentations so that other students can question and test the understanding of the presenter. A thorough understanding of the first twenty chapters is required.

Each chapter is followed by a list of *Concepts review* type questions. These questions highlight for students the main ideas presented in that section and will help test their understanding of these concepts. The answers to all *Concept review* questions are in the main body of the text. Attempting to answer these questions will help the student discover essential notions which are often overlooked when first exposed to these ideas. Reading a section provides a certain level of understanding, but answering questions, even simple ones, related to its content requires a much deeper understanding. The efforts required in solving correctly various problems or exercise questions takes the reader to where we want to go.

Textbook examples will serve as solution models to most of the exercise questions at the end of each section.

In certain sections, we make use of elementary set theory. A student who feels a bit rusty when facing the occasional references to set theory notions may want to review some of these. For convenience, a summary of the main set theory concepts appear at the end of the text in the form of an appendix to the book. A more extensive coverage of naive set theory is offered in the book "Axioms and Set theory" by this writer. It is highly recommended and will serve as an excellent companion to this book.

As we all know, any textbook, when initially published, will contain some errors, some typographical, others in spelling or in formatting and, what is even more worrisome, some mathematical. Many readers of the text are required to help weed out the most glaring mistakes. If you happen to be a reader who has carefully studied a chapter or two of the book please feel free to communicate to me, by email, any errors you may have spotted, with your name and chapters reviewed. In the preface of further versions of the book, I will gladly acknowledge your help. This will be much appreciated by this writer as well as by feature readers. It is always more pleasurable to study a book which is error-free.

If an instructor wishes to use this book as the main textbook for his or her course and print bounded copies, please first obtain permission.

Robert André
University of Waterloo, Ontario
randre@uwaterloo.ca

ISBN 978-0-9938485-1-3

Contents

Part I

Norms and metrics

1 / Norms on vector spaces.

Summary. *In this section, we review a few basic notions about inner products on vector spaces and how they are used as a mechanism to construct a distance measuring tool called "norm". We then define "norm on an abstract vector space" with no reference to an inner product. We show how to distinguish between norms that are induced by an inner product and those that are not. We then provide a few examples of norms on vectors spaces of continuous functions, $C[a, b]$. We end this section with a formal definition of the compact property in normed vector spaces.*

1.1 Review of inner product spaces.

We begin by reviewing a view basic notions about those vector spaces we refer to as *inner product spaces*. Recall that the set $\mathbb{R}^n = \{(x_1, x_2, x_3, ..., x_n) : x_i \in \mathbb{R}\}$ equipped with two operations, addition and scalar multiplication, is known to be a *vector space*. We can also equip a vector space with a third operation called "dot product" which maps pairs of vectors in $\mathbb{R}^n$ to some real number. The dot-product is a specific real-valued operation on $\mathbb{R}^n$ which belongs to a larger family of vector space operations called *inner products*. We briefly remind ourselves of a few facts about inner products on abstract vector spaces.

Definition 1.1 Let V be a vector space over the reals. An *inner product* is an operation which maps pairs of vectors in V to a real number. We denote an inner product on V by $< \vec{v}, \vec{w} >$. A real-valued function on $V \times V$ is referred to as an *inner product* if and only if it satisfies the following 4 axioms :

IP1 The number $< \vec{v}, \vec{v} >$ is greater than or equal to 0 for all $\vec{v}$ in V. Equality holds if and only if $\vec{v} = 0$. (Hence, if $\vec{v}$ is not 0, $< \vec{v}, \vec{v} >$ is *strictly* larger than 0)

IP2 For all $\vec{v}, \vec{w} \in V$, $< \vec{u}, \vec{v} >=< \vec{v}, \vec{u} >$ (Commutativity)

IP3 For all $\vec{u}, \vec{v}, \vec{w} \in V$, $< \vec{u} + \vec{w}, \vec{v} >=< \vec{u}, \vec{v} > + < \vec{w}, \vec{v} >$

IP4 For all $\vec{u}, \vec{v} \in V$, and $\alpha \in \mathbb{R}, < \alpha\vec{u}, \vec{v} >= \alpha < \vec{u}, \vec{v} >$

A vector space V is called an *inner product space* if it is equipped with some specified inner product.

Definition 1.2 Let V be an inner product space. If $\vec{v}$ is a vector in V we define

$$\|\vec{v}\| = \sqrt{<\vec{v},\vec{v}>}$$

The expression $\|\vec{v}\|$ is called the *norm* (or length) of the vector $\vec{v}$ induced by the inner product $<\vec{u},\vec{v}>$ on V.

Example 1 . It is easily verified that, for $\vec{x} = (x_1, x_2, x_3, ..., x_n)$ and $\vec{y} = (y_1, y_2, y_3, ..., y_n)$ in $\mathbb{R}^n$, its well-known dot-product

$$<\vec{x},\vec{y}> = \vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

satisfies the four axioms above and so is an inner product on the vector space $\mathbb{R}^n$. This is often referred to as the *Euclidean inner product* or the *standard inner product* on $\mathbb{R}^n$. The norm on $\mathbb{R}^n$ induced by the dot product is

$$\begin{aligned} \|\vec{x}\| &= \|(x_1, x_2, x_3, \ldots, x_n)\| \\ &= \sqrt{(x_1, x_2, x_3, \ldots, x_n)\cdot(x_1, x_2, x_3, \ldots, x_n)} \\ &= \sqrt{\sum_{i=1}^{n} x_i^2} \end{aligned}$$

This particular norm is referred to as the *Euclidean norm* on $\mathbb{R}^n$. It is also referred to as the l_2-norm on $\mathbb{R}^n$, in which case, it will be represented as $\|\vec{x}\|_2$. We will use this particular norm to measure distances between vectors in $\mathbb{R}^n$. That is, the distance between $\vec{x} = (x_1, x_2, x_3, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, y_3, ..., y_n)$ is defined to be

$$\|\vec{x}-\vec{y}\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

In the case where $n = 2$ or 3, this represents the usual distance formula between points in 2-space and 3-space, respectively. In the case where $n = 1$, it represents the absolute value of the difference of two numbers.

Example 2 . Consider the vector space, $V = C[a,b]$, the set of all continuous real-valued functions on the closed interval $[a,b]$ equipped with the usual addition and scalar multiplication of functions. We define the following inner product on $C[a,b]$ as:

$$<f,g> = \int_a^b f(x)g(x)\,dx$$

Showing that this operation satisfies the inner product axioms IP1 to IP4 is left as an exercise. In this case, the norm of f, induced by this inner product, is seen to be

$$\|f\| = \sqrt{\int_a^b f(x)^2 \, dx}$$

It is also referred to as the L_2-norm on $C[a,b]$ and we represent it as $\|f\|_2$.

The following theorem called the *Cauchy-Schwarz inequality* offers an important problem solving tool when working with inner product spaces. The norm which appears in the inequality is the one induced by the given inner product. This inequality holds true for any well-defined inner product on a vector space.

Theorem 1.3 *Cauchy-Schwarz inequality.* Let V be a vector space equipped with an inner product and its induced norm. Then, for vectors $\vec{x}$ and $\vec{y}$ in V,

$$| < \vec{x}, \vec{y} > | \leq \|\vec{x}\| \|\vec{y}\|$$

Equality holds true if and only if $\vec{x}$ and $\vec{y}$ are collinear (i.e., $\vec{x} = \alpha\vec{y}$ or $\vec{y} = \alpha\vec{x}$).

Proof: The statement clearly holds true if $\vec{y} = 0$.

Let $\vec{x}, \vec{y}$ be two (not necessarily distinct) vectors in V where $\vec{y} \neq 0$. For any real number t,

$$\begin{aligned} 0 &\leq < \vec{x} - t\vec{y}, \vec{x} - t\vec{y} > \\ &= \|\vec{x}\|^2 - 2t < \vec{x}, \vec{y} > + t^2 \|\vec{y}\|^2 \end{aligned}$$

Choosing

$$t = \frac{< \vec{x}, \vec{y} >}{\|\vec{y}\|^2}$$

in the above equation we obtain

$$0 \leq \|\vec{x}\|^2 - \frac{< \vec{x}, \vec{y} >^2}{\|\vec{y}\|^2}$$

The inequality, $| < \vec{x}, \vec{y} > | \leq \|\vec{x}\| \|\vec{y}\|$, follows.

We now prove the second part of the statement. If $\vec{x}$ and $\vec{y}$ are collinear, say $\vec{x} = \alpha\vec{y}$, then

$$\begin{aligned} |<\vec{x},\vec{y}>| &= |<\alpha\vec{y},\vec{y}>| \\ &= |\alpha| <\vec{y},\vec{y}> \\ &= |\alpha|\|\vec{y}\|\|\vec{y}\| \\ &= \|\alpha\vec{y}\|\|\vec{y}\| \\ &= \|\vec{x}\|\|\vec{y}\| \end{aligned}$$

Conversely, suppose $|<\vec{x},\vec{y}>| = \|\vec{x}\|\|\vec{y}\|$. If $\vec{y}$ is zero then $\vec{0} = t\vec{y}$ and so $\vec{x}$ and $\vec{y}$ are collinear. Suppose $\vec{y} \neq \vec{0}$. Consider $t = \frac{<\vec{x},\vec{y}>^2}{\|\vec{y}\|^2}$.

$$\begin{aligned} <\vec{x}-t\vec{y},\vec{x}-t\vec{y}> &= \|\vec{x}\|^2 - \frac{<\vec{x},\vec{y}>^2}{\|\vec{y}\|^2} \text{ (As described above)} \\ &= 0 \end{aligned}$$

So, by IP1, $<\vec{x}-t\vec{y},\vec{x}-t\vec{y}> = \mathbf{0}$ implies $\vec{x}-t\vec{y} = 0$ so $\vec{x}$ and $\vec{y}$ are collinear. We have shown that equality holds if and only if $\vec{x}$ and $\vec{y}$ are collinear.

We now verify that a norm, $\| \ \|$, which is induced by an inner product $< \ , \ >$ on the vector space V will always satisfy the following three fundamental properties:

1) For all $\vec{x} \in V$, $\|\vec{x}\| \geq 0$, equality holds if and only if $\vec{x} = 0$.
2) For all $\vec{x} \in V$ and scalar $\alpha \in \mathbb{R}$, $\|\alpha\vec{x}\| = |\alpha|\|\vec{x}\|$,
3) For all $\vec{x},\vec{y} \in V$, $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

The first property follows from the fact that the norm is defined as being the square root of a number. The second property follows from the straightforward argument:

$$\begin{aligned} \|\alpha\vec{x}\| &= \sqrt{<\alpha\vec{x},\alpha\vec{x}>} \\ &= \sqrt{\alpha^2}\sqrt{<\vec{x},\vec{x}>} \\ &= |\alpha|\|\vec{x}\| \end{aligned}$$

The third property is referred to as the *triangle inequality*. It's non-trivial proof invokes the Cauchy-Schwarz inequality. We prove it below.

Corollary 1.4 *The triangle inequality for norms induced by inner products.* For any pair of vectors $\vec{x}$ and $\vec{y}$ in an inner product space, $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$. If equality holds then the two vectors $\vec{x}$ and $\vec{y}$ are collinear.

Proof:

$$
\begin{aligned}
\|\vec{x}+\vec{y}\|^2 &= <\vec{x}+\vec{y}, \vec{x}+\vec{y}> \\
&= \|\vec{x}\|^2 + 2 <\vec{x}, \vec{y}> + \|\vec{y}\|^2 \\
&\leq \|\vec{x}\|^2 + 2| <\vec{x}, \vec{y}> | + \|\vec{y}\|^2 \\
&\leq \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|\vec{y}\|^2 \quad \text{(Cauchy-Schwarz)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2
\end{aligned}
$$

Thus $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$, as required.

In the case where we have equality : Suppose we have $\|\vec{x}+\vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$. Then $\|\vec{x}+\vec{y}\|^2 = (\|\vec{x}\| + \|\vec{y}\|)^2$. From the development above, the inequalities must be equalities, so we must have $<\vec{x}, \vec{y}> = \|\vec{x}\|\|\vec{y}\|$. This means $| <\vec{x}, \vec{y}> | = \|\vec{x}\|\|\vec{y}\|$. By the Cauchy-Schwarz theorem, $\vec{x}$ and $\vec{y}$ are collinear.

1.2 Norms on arbitrary vector spaces.

We now show that norms on a vector space can exist independently from inner products.

Definition 1.5 Let V a vector space over the reals. A norm on V is a function, $\| \ \| : V \to \mathbb{R}$, which satisfies the three following norm axioms:

N1 For all $\vec{v} \in V$, $\|\vec{v}\| \geq 0$. The equality $\|\vec{v}\| = 0$ holds true if and only if $\vec{v} = 0$. Hence if $\vec{v}$ is not $\vec{0}$, $\|\vec{v}\| > 0$.

N2 For all $\vec{v} \in V$, $\alpha \in \mathbb{R}$, $\|\alpha\vec{v}\| = |\alpha|\|\vec{v}\|$

N3 For all $\vec{v}, \vec{u} \in V$, $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ (Triangle inequality)

A vector space V is called a *normed vector space* if it is equipped with some specified norm satisfying these three norm axioms. It is denoted by $(V, \| \ \|)$. Note that the vector space V need not be an inner product space, nor need there be any relationship between $\| \ \|$ and some inner product.

This function, $\| \ \| : V \to \mathbb{R}$, will be used to measure the "distance", $\|\vec{x}-\vec{y}\|$, between any two vectors $\vec{x}$ and $\vec{y}$. Note that $\|\vec{x}-\vec{y}\| = \|(-1)(\vec{y}-\vec{x})\| = |-1|\|\vec{y}-\vec{x}\| = \|\vec{y}-\vec{x}\|$. That is, "the distance between $\vec{x}$ and $\vec{y}$ is the same as the distance between $\vec{y}$ and $\vec{x}$".

We provide a few examples of functions known to be norms.

Example 1. The norm $\| \ \|$ induced by the inner product of an inner product space $(V, < \ , \ >)$ has already been shown to satisfy the axioms N1, N2 and N3.

Example 2. The 1-norm on $\mathbb{R}^n$ is defined as

$$\|\vec{x}\|_1 = \|(x_1, x_2, x_3, \ldots, x_n)\|_1 = \sum_{i=1}^{n} |x_i|$$

It can be shown to satisfy the three norm axioms. (Left as an exercise)

Example 3. The ∞-norm on $\mathbb{R}^n$ is defined as

$$\|\vec{x}\|_\infty = \|(x_1, x_2, x_3, \ldots, x_n)\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_n|\}$$

It can be shown to satisfy the three norm axioms. (Proving that N1 and N2 are satisfied is left as an exercise)

We prove that the ∞-norm on $\mathbb{R}^n$ satisfies the triangle inequality N3.

Proof of $\|\vec{x}+\vec{y}\|_\infty \leq \|\vec{x}\|_\infty + \|\vec{y}\|_\infty$: Note that $|x_i| \leq \max_{i=1..n}\{|x_i|\}$ for each i, and $|y_i| \leq \max_{i=1..n}\{|y_i|\}$ for each i Then for each $i = 1$ to n,

$$\begin{aligned} |x_i + y_i| &\leq |x_i| + |y_i| \\ &\leq \max_{i=1..n}\{|x_i|\} + \max_{i=1..n}\{|y_i|\} \\ &= \|\vec{x}\|_\infty + \|\vec{y}\|_\infty \end{aligned}$$

Then $\max\limits_{1..n}\{|x_i - y_i|\} \leq \|\vec{x}\|_\infty + \|\vec{y}\|_\infty$ and so $\|\vec{x}+\vec{y}\|_\infty \leq \|\vec{x}\|_\infty + \|\vec{y}\|_\infty$, as required.

Example 4. Let $p \geq 1$. The p-norm on $\mathbb{R}^n$ is defined as

$$\|\vec{x}\|_p = \|(x_1, x_2, x_3, \ldots, x_n)\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

It can be shown that this function satisfies the three norm axioms. We will not prove this for p in general.

Note that when $p = 2$ the p-norm is simply the Euclidean norm on $\mathbb{R}^n$. Since the Euclidean norm is induced by an inner product it automatically satisfies the three norm axioms. Proving that, for any $p \geq 1$, the p-norm satisfies N1 and N2 is straightforward. But proving that the triangle inequality holds true for all $p \geq 1$ is not easy.

The interested readers can look the proof up in most Real Analysis texts or find it online.

A natural question comes to mind. Are all norms on a vector space V induced by some inner product? The answer is no!

The following theorem tells us how to recognize those norms which are induced by some inner product:

Theorem 1.5.1 Suppose $\| \ \|$ is a norm on a vector space V. There exists an inner product $< , >$ such that $\|\vec{x}\| = \sqrt{< \vec{x}, \vec{x} >}$, for all $\vec{x} \in V$, if and only if, for all $\vec{x}, \vec{y} \in V$, the norm $\| \ \|$ satisfies the parallelogram identity

$$2(\|\vec{x}\|^2 + \|\vec{y}\|^2) = \|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2$$

Proof: The proof involves showing that, if $\| \ \|$ satisfies the parallelogram identity then the identity

$$< \vec{x}, \vec{y} >= \frac{1}{4}\left(\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2\right)$$

is a valid inner product on V which induces $\| \ \|$. The proof is routine and so is not presented here.

Examples of norms on $C[a, b]$.

Example 1. The L_p-norm. Let $V = C[a, b]$ denote the family of all real-valued continuous functions on the closed interval $[a, b]$ equipped with the usual $+$ and scalar multiplication. If $p \geq 1$, then we define the L_p-norm on $C[a, b]$ as follows:

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx\right)^{1/p}$$

In the case where $p = 1$ we have

$$\|f\|_1 = \int_a^b |f(x)| \, dx$$

which is easily shown to satisfy the three norm axioms. In the case where $p = 2$, then this norm is one which is induced by the inner product

$$< f, g >= \int_a^b f(x)g(x) \, dx$$

on $C[a, b]$.

It is straightforward to show that, for all $p \geq 1$, $\|f\|_p$ satisfies N1 and N2. But proving that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, for all $p \geq 1$, is difficult. This inequality is referred to as the *Minkowski inequality.* The interested readers will find a proof in most Real Analysis texts or online.

Example 2. The sup-norm: We define another norm on the vector space, $C[a,b]$, of all continuous real-valued functions on the closed interval $[a,b]$. Recall that the *supremum* of a subset A of an ordered set S, written as "sup A" is the *least upper bound* of A which is contained in S. Note that sup A may or may not belong to A. We define the sup-norm on $C[a,b]$ as

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\}$$

The sup-norm is also referred to as the "infinity-norm" or "uniform norm". Showing that this is a valid norm is left as an exercise.

1.3 Convergence and completeness in a normed vector space.

Convergence of sequences forms and important part of analysis. In what follows, norms on $\mathbb{R}^n$ are always assumed to be the Euclidean norm, unless otherwise stated. An infinite sequence in a normed vector space, V, is a function which maps $\mathbb{N}$ (not necessarily one-to-one) onto a subset, S, of V. This means that a sequence is the indexation of the elements of a countable subset, S, by using the natural numbers. For example, $S = \{\vec{x}_0, \vec{x}_1, \vec{x}_2, ..., \vec{x}_n, ...,\}$. Note that some of these may be repeated (since the function mapping $\mathbb{N}$ into S need not be one-to-one). We generalize the notion of a "convergent sequence and its limit" in $\mathbb{R}$ to a "convergent sequence and its limit" in a normed vector space $(V, \|\ \|)$.

Definition 1.6 Let $(V, +, \alpha, \|\ \|)$ be a vector space equipped with a norm $\|\ \|$. We say a sequence of vectors $\{\vec{x}_n\}$ in V converges to a vector $\vec{a}$ in V with respect to the norm $\|\ \|$ if

For any $\varepsilon > 0$, there exists an integer $N > 0$ such that $\|\vec{x}_n - \vec{a}\| < \varepsilon$ whenever $n > N$.

If the sequence of vectors $\{\vec{x}_n\}$ converges to the vector $\vec{a}$, then we write

$$\lim_{n\to\infty} \vec{x}_n = \vec{a}$$

The following proposition lists the standard properties respected by limits in normed vector spaces.

Proposition 1.7 Let $\{\vec{x}_n\}$ and $\{\vec{y}_n\}$ be two sequences in the normed vector space V which converge to $\vec{a}$ and $\mathbf{b}$, respectively. Let α and β be scalars.

a) $\lim_{n\to\infty} (\alpha\vec{x}_n \pm \beta\vec{y}_n) = \alpha \lim_{n\to\infty} \vec{x}_n \pm \beta \lim_{n\to\infty} \vec{y}_n$.

b) $(\lim_{n\to\infty} \vec{x}_n = \vec{a}) \Leftrightarrow (\lim_{n\to\infty} \|\vec{x}_n - \vec{a}\| = 0)$

c) If $\vec{x} = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$ then $|x_i| \leq \|\vec{x}\|_2$, for $i = 1$ to n.

d) A sequence of vectors $\{\vec{x}_k : k = 1, 2, 3, \ldots, \}$ in $\mathbb{R}^n$ converges to the vector $\vec{a} = (a_1, a_2, a_3, \ldots, a_n)$ (with respect to the Euclidean norm) if and only if the components of the vectors in $\{\vec{x}_k\}$ converge to the corresponding components of $\vec{a}$ (with respect to the absolute value).

Proof: The proof is omitted.

We now define those sequences in a normed vector space called *Cauchy sequences.* As we shall soon see all convergent sequences of $(V, \| \ \|)$ must be Cauchy sequences, but some "subsets" S of V have Cauchy sequences which do not converge in S.

Definition 1.8 A sequence of vectors $\{\vec{x}_n\}$ in a normed vector space V is said to be a *Cauchy sequence* in V, or simply said to be *Cauchy*, if for any $\varepsilon > 0$, there exists an integer $N > 0$ such that $\|\vec{x}_m - \vec{x}_n\| < \varepsilon$ whenever $m, n > N$.

At first glance, the reader may feel that a Cauchy sequence is just another way of referring to a "convergent sequence". In many situations this is indeed the case. But there are sets, S, containing Cauchy sequences which do not converge to a point inside S. For example, we may refer to the sequence $\{1, 1/2, 1/3, \ldots, 1/n, \ldots, \}$ as a sequence in the vector space $\mathbb{R}$, or, as a sequence in the subset $S = \{x : 0 < x \leq 1\}$ where S inherits the absolute value norm from $\mathbb{R}$. The given sequence certainly converges to 0 in $\mathbb{R}$ and so can easily be proven to be Cauchy in $\mathbb{R}$. Using the same norm on S we must conclude that it is also Cauchy in S. However, there is one property of S which clearly distinguishes it from $\mathbb{R}$: It is that $\mathbb{R}$ contains the limit of this Cauchy sequence while S does *not* contain the limit of this same sequence. So we will use Cauchy sequences to help us categorize subsets of a vector space based on whether they contain the limits of all their Cauchy sequences, or not.

This motivates the following definition of *complete subsets* of a normed vector space.

Definition 1.9 A subset S of a normed vector space V is a *complete subset of* V if every Cauchy sequence in S converges to a vector in S.

Definition 1.9.1 A complete normed vector space is referred to as *Banach space*.

Definition 1.9.2 If V is a complete normed vector space whose norm is induced by an inner product then V is referred to as a *Hilbert space*. That is, if $(V, +\alpha, <>, \| \|)$ is complete then V is a Hilbert space.

Even if all Hilbert spaces are Banach spaces not all Banach spaces are Hilbert spaces.

We will not dwell on the notions of convergence and completeness of subsets at this time. We will return to discuss these concepts in the more general contexts of metric spaces and topological spaces.

1.4 The compact property on normed vector spaces.

We will present immediately the formal definition of the *compact property* which applies mostly when we confine ourselves to the universe of normed vector spaces. This is for future reference when, at a point in this book, we will reintroduce the notion of *compactness*, but in a much more general context of topological spaces. This definition will be followed by an important characterization of the compact property which remains valid as long as we confine ourselves to finite dimensional normed vector spaces. The proof is non-trivial and so is omitted here. The complete proof is usually presented in texts which serve as an introduction to Real Analysis courses rather than in a topology text. One of the main reason for an early introduction is to access an important "extreme value theore" which states that "A continuous function on a compact subset, T, of a normed vectors spaces, V, attains its maximum value at a point inside T". The *Heine-Borel theorem* makes it very convenient, in such situations, to refer to *compact subsets* simply as being those that are "closed and bounded". In abstract topological spaces, we will not have access to this tool, since the Heine-Borel theorem applies to finite dimensional normed vector spaces.

Definition 1.10 A subset, T, of a normed vector space, V, is said to be *compact*[1] if and only if every sequence, $\{x_i\}$, in T has a subsequence, $\{x_{f(i)}\}$, which converges to some point, p, which belongs to T.

Theorem 1.11 *The generalized Heine-Borel theorem.* Subsets of a finite dimensional normed vector space are compact if and only if they are both closed and bounded.

Proof: The proof is omitted.

EXERCISES

1. Prove that, if $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^n$, then $|\ \|\vec{x}\| - \|\vec{x} - \vec{y}\|\ | \leq \|\vec{y}\|$.

2. Exercise questions on norms.

   a) Consider the vector space, $C[a, b]$, of all continuous functions on the interval $[a, b]$. Recall that $\|\ \|_1 : C[a, b] \to \mathbb{R}$ is defined as
   $$\|f\|_1 = \int_a^b |f(x)|dx$$
   Show that $\|\ \|_1$ is a valid norm on $C[a, b]$.

   b) Let $\{f_n : n \in \mathbb{N}, n \neq 0\}$ be a sequence of constant functions defined as $f_n(x) = 2 + \frac{1}{n}$ for each n. That is, $f_1(x) = 3$, $f_2(x) = 5/2$, $f_3(x) = 7/3$, and so on. If $\{f_n\}$ is viewed as a subset of $C[-1, 2]$ equipped with the norm $\|\ \|_1$ determine,
   $$\lim_{n\to\infty} \int_{-1}^{2} |f_n(x)|\, dx$$
   Justify all your steps carefully.

3. Recall that $C[a, b]$ is the vector space of all continuous functions on the interval $[a, b]$.

   a) Show that $< f, g >= \int_a^b f(x)g(x)\, dx$ is a valid inner product.

   b) Let $\|\ \|$ be the norm on $C[0, 1]$ which is induced by the inner product given in part a) of this question. Compute $\|e^x\|$.

[1]When we will revisit the property of compactness in our study of topological spaces we will replace the word "compact" with the words "sequentially compact".

4. Recall the definition of "p-norm": $\|\vec{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$, on $\mathbb{R}^n$.

   a) Compute $\|(-1,0,1)\|_3$.

   b) *Hölder's inequality* on $\mathbb{R}^n$ says that "if $p,q \geq 1$ and $1/p + 1/q = 1$, then $\|(x_1y_1, x_2y_2, \ldots, x_ny_n)\|_1 = \sum_{i=1}^n |x_iy_i| \leq \|\vec{x}\|_p\|\vec{y}\|_q$". Note that p and q are not assumed to be integers; Hölder's inequality holds true as long as the condition "$1/p+1/q = 1$" is satisfied. Although it is difficult to see why this inequality is of any interest or is of any value at this point, we will assume it holds true and use it to practice using simpler concepts. Show that the *Cauchy-Schwarz inequality* on $\mathbb{R}^n$, $| < \vec{x}, \vec{y} > | \leq \|\vec{x}\|_2\|\vec{y}\|_2$, (where $< \vec{x}, \vec{y} >$ is the Euclidean inner product) follows from the *Hölder's inequality* on $\mathbb{R}^n$.

   c) *Hölder's inequality* on $C[a,b]$ says that "if $p,q \geq 1$ and $1/p + 1/q = 1$, then $\|fg\|_1 \leq \|f\|_p\|g\|_q$". Show that the *Cauchy-Schwarz inequality* on $C[a,b]$,

   $$| < f, g > | \leq \|f\|_2\|g\|_2$$

   follows from the *Hölder's inequality* on $C[a,b]$.

5. Invoke the Cauchy-Schwarz inequality to show that

$$-\left[\frac{\sqrt{2}-1}{2}\right]^{1/2} \leq \int_0^{\pi/4} \sqrt{\cos x}\sqrt{\sin x}\, dx \leq \left[\frac{\sqrt{2}-1}{2}\right]^{1/2}$$

6. Let f be a function in $C[a,b]$. Show that $\lim_{p\to\infty} \|f\|_p \leq \|f\|_\infty$.

7. A subset S of a normed vector space is bounded if there exists a positive number M such that $\|\vec{x}\| < M$ for all $\vec{x}$ in S. Show that any Cauchy sequence $\{\vec{x}_n\}$ in a normed vector space $(V, \| \ \|)$ when viewed as a set is bounded.

8. Suppose $(V, \| \|)$ is a normed vector space and $\{\vec{x}_n\}$ is a sequence in V which does not converge to $\vec{0}$. If $\delta \in \mathbb{R}$, let $B_\delta(\vec{0}) = \{\vec{x} \in V : \|\vec{x}\| \ < \delta\}$. Show that there exists some $\delta > 0$ and a subsequence $\{x_{n_i}\}_{i=1}^\infty$ of $\{\vec{x}_n\}$ such that $\{\vec{x}_{n_i}\} \cap B_\delta(\vec{0}) = \varnothing$.

9. Convergence of vectors in a normed vector space.

   a) Review the definition of "convergence of a sequence of vectors in a normed vector space". Let $(V, \| \|)$ be an abstract normed vector. Suppose V contains a sequence of vectors, $\{\vec{u}_n : n \in \mathbb{N}\}$, which converges to some vector in V. Show that

   $$\lim_{n\to\infty} \|\vec{u}_n\| = \|\lim_{n\to\infty} \vec{u}_n\|$$

   Hence, whenever the sequence, $\{\vec{u}_n : n \in \mathbb{N}\}$, converges in V then so does the sequence, $\{\|\vec{u}_n\| : n \in \mathbb{N}\}$, of real numbers.

b) Let $\vec{v} \in \mathbb{R}^2$. Construct a sequence $\{\vec{u}_n\}$ in $\mathbb{R}^2$ such that $\lim_{n\to\infty} \|\vec{u}_n\| = \|\vec{v}\|$ where $\lim_{n\to\infty} \vec{u}_n \neq \vec{v}$.

c) Suppose $\{\vec{x}_n\}$ and $\{\vec{u}_n\}$ are two sequences in $\mathbb{R}^n$ which both converge to the same point $\vec{y}$. Show that the sequence $\{\|\vec{x}_n - \vec{u}_n\|\}$ must converge to 0.

10. Let $\{\vec{x}_n : n \in \mathbb{N}\}$ and $\{\vec{y}_n : n \in \mathbb{N}\}$ be sequences in a normed vector space $(V, +, \alpha, \|\ \|)$. Let $\{\alpha_n : n \in \mathbb{N}\}$ be a sequence of real numbers.

   a) Show that, if $\{\vec{x}_n : n \in \mathbb{N}\}$ and $\{\vec{y}_n : n \in \mathbb{N}\}$ converge to $\vec{x}$ and $\vec{y}$, respectively, then the sequence $\{\vec{x}_n + \vec{y}_n : n \in \mathbb{N}\}$ converges to $\vec{x} + \vec{y}$.

   b) Show that, if $\{\vec{x}_n : n \in \mathbb{N}\}$ converges to the vector $\vec{x}$ and $\{\alpha_n : n \in \mathbb{N}\}$ converges to the real number α then the sequence of vectors, $\{\alpha_n\vec{x}_n : n \in \mathbb{N}\}$, converges to $\alpha\vec{x}$.

11. Suppose $(V, \|\ \|)$ is a normed vector space and $\{\vec{x}_n\}$ is a Cauchy sequence in V. Show that if $\{\vec{x}_n\}$ has a convergent subsequence, say $\{\vec{x}_{n_i}\}_{i=1}^{\infty}$, then $\{\vec{x}_n\}$ converges in V.

12. Let $\delta > 0$. Suppose $(V, \|\ \|)$ is a normed vector space and $\{\vec{x}_n\}$ is a sequence in V such that $\|\vec{x}_n\| > \delta$ for all n. Show that if $\{\vec{x}_n\}$ is Cauchy then so is the sequence $\left\{\frac{x_n}{\|x_n\|}\right\}$ on $B = \{\vec{x} \in V : \|\vec{x}\| = 1\}$.

13. Suppose $(V, \|\ \|)$ is a normed vector space and $B = \{\vec{x} \in V : \|\vec{x}\| = 1\}$.

   a) Show that if V is complete with respect to the norm $\|\ \|$ then B is complete with respect to the same norm inherited from V.

   b) Show that if B is known to be complete with respect to the norm $\|\ \ \|$, then V must also be complete with respect to $\|\ \|$.

14. Let M be an n-dimensional subspace of the inner product space $(V, <\ ,\ >, \|\ \|)$. Prove that M is complete with respect to the norm $||\ ||$. Essentially this says that all subspaces of finite dimensional inner product spaces are complete subsets.

15. Suppose $(V, \|\ \|)$ is normed vector space whose norm satisfies the parallelogram identity

$$2(\|\vec{x}\|^2 + \|\vec{y}\|^2) = \|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2$$

   a) Show that $< \vec{x}, \vec{y} >= \frac{1}{4}(\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2)$ (referred to as the *polarizing identity*) satisfies the three inner product axioms IP1, IP2 and IP4.

   b) Show that the polarizing identity in part a) satisfies the inner product property $< \vec{x} + \vec{y}, \mathbf{z} >=< \vec{x}, \mathbf{z} > + < \vec{y}, \mathbf{z} >$ hence the polarizing identity is a valid inner product.

   c) Show that this inner product induces the norm of V.

16. It is known that for any $\vec{x} \in \mathbb{R}^n$, $\|\vec{x}\|_\infty = \lim_{p\to\infty} \|\vec{x}\|_p$. Prove this for the case where $n = 2$.

2 / Metrics on sets.

Summary. *In this section, we define the notion of a "metric" on a set, giving rise to "metric spaces". The metric is defined as a tool for measuring distances between points in the given set. Many metrics can be defined on a set as long as they each satisfy three metric axioms. Metrics are then used to determine whether a given sequence converges to a point or not. The notions of "open ball" and "open sets" are then defined. Functions are then introduced to map points from one metric space to another metric space. We are particularly interested in those functions which are continuous on their respective domains. The topological version of the notion of continuity is presented in terms of "open sets".*

2.1 Measuring distance in arbitrary sets.

We now generalize a few notions of distances between vectors in a normed vector space to distances between points in an arbitrary set. Arbitrary sets, in their most rudimentary form, do not usually appear with some algebraic structure defined on them (such as vector spaces, for example, on which addition and scalar multiplication are defined). When a set is equipped with addition and scalar multiplication at the onset, subtraction $\vec{a} - \vec{b}$ of two points $\vec{a}$ and $\vec{b}$ is easily given meaning. With a previously defined norm, we measured the length, $\|\vec{a} - \vec{b}\|$, of the difference $\vec{a} - \vec{b}$ to obtain the number which represents a notion of distance between $\vec{a}$ and $\vec{b}$. This method is inspired by the way we normally determine the distance between two real numbers, say -7 and 2.5, in the vector space, $\mathbb{R}$, for example. In $\mathbb{R}$, distance between points is expressed by referring to the absolute value. To measure distances between points in arbitrary sets we will proceed differently. Given a set S, rather than defining a "norm" function $\|\ \| : S \to \mathbb{R}$ on S, we will define a function, $\rho : S \times S \to \mathbb{R}$, which maps pairs of points x and y in S to a number which will represent the distance between these two points in a that particular set. This function is what we will call a "metric" on S. There will be certain restrictions on the properties possessed by ρ. We would, of course, not want ρ to give us a distance $x \to y$ which is different from the distance $y \to x$. Also, we definitely do not want ρ to give a "negative" distance between two points. With different metrics, ρ_1 and ρ_2, the set, (S, ρ_1), may be different in nature from the set, (S, ρ_2).

Definition 2.1 Let S be a non-empty set. A metric, ρ on S, is a function, $\rho : S \times S \to \mathbb{R}$, which satisfies three *metric axioms*:

M1 For every $x,\, y \in S,\ \rho(x,y) \geq 0$ and $\rho(x,y) = 0$ if and only if $x = y$.

M2 For every $x,\, y \in S,\ \rho(x,y) = \rho(y,x)$

M3 For every $x,\, y,\, z \in S,\ \rho(x,y) \leq \rho(x,z) + \rho(z,y)$ *(Triangle equality)*

A set S equipped with a metric, ρ, is called a *metric space*. It is expressed as, (S,ρ)

Many metrics can be defined on a given set S. For a given set, S, if ρ_1 and ρ_2 are different metrics then (S,ρ_1) and (S,ρ_2) are considered to be different metric spaces. We provide a few examples.

Example 1. Let $(V, \|\ \|)$ be a normed vector space. Define $\rho : V \times V \to \mathbb{R}$ as $\rho(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$. The function, ρ, is easily seen to satisfy the two first metric axioms, M1 and M2. The triangle inequality for norms guarantees that M3 also holds true. Then ρ, thus defined on V, is a metric induced by a norm and so (V,ρ) is a metric space (It can simultaneously be viewed as a normed vector space depending on the context.) When considering the normed vector space, $(\mathbb{R}^2, \|\ \|_2)$, the metric induced by the norm $\|\ \|_2$ is the distance formula: For $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$,

$$\rho(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

This particular example shows that, given any normed vector space $(V, \|\ \|)$, we can always express it as a metric space, (V,ρ), by defining $\rho(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$. However, one should remember that a metric space (S,ρ) need not be a normed vector space, since S itself need not be a vector space equipped with addition and scalar multiplication, both of which are necessary to define a norm on S.

Example 2. Let $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined as:

$$\rho(\vec{x},\vec{y}) = \rho((x_1,x_2),(y_1,y_2)) = \sup\{|x_1-y_1|, |x_2-y_2|\}$$

It is left to the reader to verify that this function satisfies all three metric axioms, and so is a valid metric on $\mathbb{R}^2$

Example 3. Let S be any non-empty set. The function $\rho : S \times S \to \mathbb{R}$ defined as

$$\rho(x,y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$

can be verified to satisfy all three metric axioms and so is a valid metric on S. Any pair of distinct points are at a distance of one from each other while a point is at a distance zero from itself. This metric is referred to as the *discrete metric*.

Example 4. Consider the set of all integers $\mathbb{Z}$. Let $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ be defined as

$$k(x, y) = \max \{2^n : \; 2^n \text{ divides } x - y \}$$

We define $\rho : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ as follows:

$$\rho(n, m) = \begin{cases} 0 & \text{if} \quad n = m \\ \frac{1}{k(m,n)} & \text{if} \quad m \neq n \end{cases}$$

We verify that ρ is indeed a metric on $\mathbb{Z}$.

- It follows directly from the definition that $\rho(n, m)$ is non-negative and $\rho(n, m) = 0$ if and only if $m = n$. So ρ satisfies property M1.
- We see that 2^n divides $x-y$ if and only if 2^n divides $y-x$. Then $k(x, y) = k(y, x)$. Then, if $m \neq n$, $\rho(m, n) = \frac{1}{k(m,n)} = \frac{1}{k(n,m)} = \rho(n, m)$. So ρ satisfies M2.
- Suppose $\rho(m, n) = 1/2^d$ and $\rho(n, t) = 1/2^e$. Then $2^d|(m - n)$ and $2^e|(n - t)$. We will suppose, without loss of generality, that $e \leq d$. It is easily verified that $2^e|(m - t)$. It follows that $k(m, t) \geq 2^e$, hence

$$\begin{aligned} \rho(m, t) &= \frac{1}{k(m, t)} \\ &\leq \frac{1}{2^e} \\ &= \rho(n, t) \\ &\leq \rho(m, n) + \rho(n, t) \end{aligned}$$

  So $\rho(m, t) \leq \rho(m, n) + \rho(n, t)$. We have shown that ρ satisfies M3.

2.2 Metric subspaces.

We know that certain subsets of a vector space are referred to as "subspaces" provided they satisfy specific conditions. There are no required conditions on a subset, T, of (S, ρ) to be called a "metric subspace" provided we know what its metric will be.

Definition 2.1.1 Suppose (S, ρ) is a metric space and $T \subseteq S$. Then T can inherit the metric ρ of its superset, S, and declare itself to be a metric space $(T, \rho|_T)$, simply by restricting the function $\rho : S \times S \to \mathbb{R}$ to $\rho|_T : T \times T \to \mathbb{R}$. In this case, (T, ρ_T) is referred to as a *metric subspace* of (S, ρ). The metric, ρ_T, is referred to as the *subspace metric.*

For example, suppose $(\mathbb{R}, \|\ \|)$ is a normed vector space. Suppose ρ is the metric on $\mathbb{R}$ induced by its norm, $\|\ \|$, and $T = (-3, 5] \subset \mathbb{R}$. Then (T, ρ_T) is a metric subspace of $\mathbb{R}$.[1]

Of course, metric subspaces of a metric space S are themselves metric spaces.

2.3 Convergence and completeness in a metric space.

We now direct our attention to those subsets of the set S which are sequences. The definition of a *sequence*, $\{x_n\}$, does not involve the notions of "norm" or "metric" and so is precisely as we previously defined it in the section on normed vector spaces. However *convergence* or *divergence* of a sequence depends very much on the tool we use to measure distances in the set. The following definitions of *limits* and convergence in a metric space are in many ways identical to those involving norms.

Definition 2.2 Let (S, ρ) be a metric space. We say that a sequence, $\{x_n\}$, of points in S converges to the point, a, in S with respect to the metric ρ if and only if

> for any $\varepsilon > 0$, there exists an integer $N > 0$ such that $\rho(x_n, a) < \varepsilon$ whenever $n > N$.

If the sequence of points $\{x_n\}$ converges to the point a (with respect to the metric ρ), then we write

$$\lim_{n\to\infty} x_n = a$$

Just as for limits in normed vector spaces, we can say that $\{x_n\}$ converges to the point a with respect to a metric ρ in different ways:

$$\lim_{n\to\infty} x_n = a \quad \Leftrightarrow \quad \lim_{n\to\infty} \rho(x_n, a) = 0$$

Definition 2.2.1 Suppose (S, ρ) is a metric space, $T \subseteq S$ and $\{x_n\}$ is a sequence in S which converges to the point a with respect to ρ. Then we will say that the point a is a *limit point of* T.

Note that, if $\{x_n\} \subseteq T \subseteq (S, \rho)$ and a is the limit of this sequence with respect to ρ, this does not guarantee that a also belongs to T. Consider, for example, the metric

[1] Of course, T is not a vector subspace of the vector space $\mathbb{R}$.

space $(\mathbb{R}, \rho)$ where $\rho(x, y) = |x - y|$ and $T = \{\pi + 1/n : n = 1, 2, 3, \ldots\}$. Then (T, ρ) is also a metric space. We see that the point π is clearly a limit point of T but $\pi \notin T$.

Just as in normed vector spaces, the notion of "Cauchy sequence" can also be defined in terms of metrics.

Definition 2.3 A sequence of points $\{x_n\}$ in a metric space (S, ρ) is said to be a *Cauchy sequence* in S, with respect to the metric ρ (or simply said to be *Cauchy*) if and only if, for any $\varepsilon > 0$, there exists an integer, $N > 0$, such that $\rho(x_m, x_n) < \varepsilon$, whenever $m, n > N$.

Example 1. *Every convergent sequence is a Cauchy sequence.* The following brief argument confirms what we intuitively feel must be true: If $\{x_i\}$ converges to the point, a, with respect to the metric, ρ, then $\{x_i\}$ is a Cauchy sequence. To prove this we let $\varepsilon > 0$. By hypothesis, there exists $N > 0$ such that $n > N \Rightarrow \rho(x_n, a) < \varepsilon/2$. If $m > N$, then, invoking the metric axiom M3,

$$\rho(x_m, x_n) \leq \rho(x_m, a) + \rho(x_n, a) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

So $\{x_i\}$ is Cauchy with respect to the metric ρ.

Definition 2.4 Let (S, ρ) be a metric space. The set S is *complete with respect to the metric* ρ if every Cauchy sequence in S converges to some point in S.

Example 2. Consider the set $S = (0, \infty)$. Let ρ_1 denote the discrete metric on S and ρ_2 be defined as $\rho_2(x, y) = |x - y|$. We compare the two metric spaces (S, ρ_1) and (S, ρ_2).

Claim 1. The metric space, (S, ρ_2), is *not* complete. See that the sequence $\{1/n : n = 1, 2, 3, \ldots, \}$ is a sequence in $(\mathbb{R}, \rho_2)$ which converges to 0, with respect to ρ_2 and so is a Cauchy sequence in $(\mathbb{R}, \rho_2)$. It is then a Cauchy sequence inside S. Since the Cauchy sequence $\{1/n\}$ does not converge in S then S is not a complete subset of S.

Claim 2. The metric space, (S, ρ_1), is a complete metric space. Let $\{x_i\}$ be a Cauchy sequence in S with respect to the discrete metric ρ_1. Let $\varepsilon = 1/2$. Then there exists $N > 0$ such that $m, n > N$ implies $\rho_1(x_m, x_n) < \varepsilon = 1/2$. Then if $m, n > N$, $\rho_1(x_m, x_n) = 0$. This means $x_n = x_{N+1}$, for all $n > N$. Then for any ε, $n > N \Rightarrow \rho_1(x_{N+1}, x_n) = 0 < \varepsilon$ so $\{\vec{x}_i\}$ converges to $x_{N+1} \in S$. We conclude that

(S, ρ_1) is a complete metric space.

Note that a metric space (S, ρ) may be a complete metric space and still have proper subsets which are not complete metric subspaces.

2.4 Open and closed subsets of a metric space.

We now define a few fundamental subsets of metric spaces.

Definition 2.5 Let (S, ρ) be a metric space.

a) If $y \in S$, and $\varepsilon > 0$ we define an *open ball of radius* ε *of center* y as being the set

$$B_\varepsilon(y) = \{x \in S : \rho(y, x) < \varepsilon\}$$

b) Let $U \subseteq S$. We say that U *is an open subset in* S if U is the union of open balls each of which is entirely contained in U. That is, for each $x \in U$, there exists a real number number $\varepsilon_x > 0$ such that

$$U = \cup\{B_{\varepsilon_x}(x) : x \in U\}$$

Instead of saying "U is an open subset of S" we often simply say "U is open in S".

c) Let $F \subset S$. We say that F *is a closed subset of* S if and only if every limit point of F belongs to F. That is,

$$F = \{x \in S : x = \lim_{n\to\infty} x_n \text{ for some sequence } \{x_n\} \subseteq F\}$$

Instead of saying "F is a closed subset of S" we often simply say "F is closed in S".

Example 1. We define the metric ρ on $\mathbb{R}$ as $\rho(x, y) = |x - y|$. If U is the the interval $(-5, 3) = \{x \in \mathbb{R} : -5 < x < 3\}$, then U is an open subset in $\mathbb{R}$ since, for every point $x \in U$, we can find $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq (-5, 3)$. However, $V = (-4, 3]$ is not an open subset in $\mathbb{R}$ since V contains a point 3 such the open ball $(3-\varepsilon, 3+\varepsilon) \not\subseteq V$ no matter how small we choose ε to be.

Example 2. We define the metric ρ on the set $U = (-5, 3]$ as $\rho(x, y) = |x-y|$. Consider the subset $S = (0, 3]$. The subset S is open in the metric space U since, for each $y \in S$ there is ε_y such that, $S = \cup\{B_{\varepsilon_y}(y) : y \in S\}$. Some readers might object, stating that there can be no ε such that $B_\varepsilon(3) \subseteq S$. But we must be careful and see that we are viewing S as a subset of the metric space $U = (-5, 3]$. See that, if we choose $\varepsilon = 1/2$,

$B_\varepsilon(3) = \{x \in U : (3-\varepsilon, 3+\varepsilon)\} = (3-\varepsilon, 3] \subseteq (2,3]$ is indeed entirely contained in S.

Example 3. We define the metric ρ on $\mathbb{R}^2$ as $\rho(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|_2$. The x-axis, $U = \{(x,0) : x \in \mathbb{R}\}$ is closed in $\mathbb{R}^2$ since if $\vec{a} = (a,b)$ is a limit point of U then there must exist a sequence $\{(x_n, 0)\}$ such that $(a,b) = \lim_{n\to\infty}(x_n, 0) = (\lim_{n\to\infty} x_n, \lim_{n\to\infty} 0)$; this implies $b = 0$, hence $(a,b) \in U$. It then follows that U is closed in $\mathbb{R}^2$

Example 4. Let S be an arbitrary non-empty set equipped with the discrete metric ρ. For any $x \in S$, $B_{1/2}(x) = \{x\} \subseteq \{x\}$ hence $\{x\}$ is an open subset of S. Let x be a limit point of $\{x\}$. The only sequence in $\{x\}$ is the constant sequence $\{x, x, x, \ldots, \}$ which converges to $x \in \{x\}$. Hence $\{x\}$ is closed in S.

Remark. We can similarly define the notion of an "open subset S" of a normed vector space $(V, \|\ \|)$ by defining $\rho(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$, in which case U is open in $(V, \|\ \|)$ if and only if U is open in (V, ρ). In $(V, \|\ \|))$, the open ball, $B_\varepsilon(\vec{y})$, centered at $\vec{y}$ is defined as $B_\varepsilon(\vec{y}) = \{\vec{x} \in V : \|\vec{y} - \vec{x}\| = \rho(\vec{y}, \vec{x}) < \varepsilon\}$. Similarly, if $F \subseteq (V, \|\ \|)$, F is said to be closed in V if and only if F contains all its limit points with respect to the norm $\|\ \|$.

Theorem 2.6 Let (S, ρ) be a metric space.

a) Both the empty set, $\varnothing$, and S are open subsets of S.

b) Both the empty set, $\varnothing$, and S are closed subsets of S.

c) Finite intersections of open subsets of S are open subsets of S.

d) Arbitrarily large unions[1] of open subsets of S are open in S.

Proof: The proofs of all four parts are left as an exercise.

Suppose (S, ρ) is a metric space and $T \subseteq S$. Let $\rho_T : T \times T \to \mathbb{R}$ be the restriction of the function ρ to $T \times T$. Then (T, ρ_T) is a metric space. Or we simply say that T is a subspace of (S, ρ). One may wonder what relationship exists between the open sets of the subspace T and the open sets of the space S. The following theorem answers this question.

Theorem 2.7 Suppose (S, ρ) is a metric space and $T \subseteq S$ equipped with the metric ρ_T inherited from the set S. Also suppose $U \subseteq T \subseteq S$. Then U is open in T (with respect to the metric ρ_T) if and only if there exists an open subset U^* in S (with respect to the metric ρ) such that $U = U^* \cap T$.

[1] The words "arbitrarily large unions" include the notion of the union of a countably or uncountably infinite number of sets. For example, $\cup\{A_n : n \in \mathbb{N}\}$

Proof: We are given that (S, ρ) is a metric space and $T \subseteq S$ equipped with the metric ρ_T inherited from the set S. Suppose $U \subseteq T \subseteq S$.

($\Leftarrow$) Suppose U^* is open in S and $U = U^* \cap T$. If $x \in U$, then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$. Now

$$B_\varepsilon(x) \cap T = \{y \in T : \rho(x, y) < \varepsilon)\} = \{y \in T : \rho_T(x, y) < \varepsilon)\} = B'_\varepsilon(x)$$

by definition, an open ball radius ε center x, in T. Since $U = \cup\{B'_\varepsilon(x) : x \in U\}$ then U is open in T.

($\Rightarrow$) Suppose U is an open subset of $T \subseteq S$. If $x \in U$ then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. Let $B'_\varepsilon(x) = \{y \in S : \rho(x, y) < \varepsilon\}$. Then $B'_\varepsilon(x)$ is an open subset of S such that $B'_\varepsilon(x) \cap T = \{y \in T : \rho_T(x, y) < \varepsilon\} = B_\varepsilon(x)$. Then

$$\begin{aligned}
[\cup\{B'_\varepsilon(x) : x \in T\}] \cap T &= \cup\{B'_\varepsilon(x) \cap T : x \in T\} \\
&= \cup\{B_\varepsilon(x) : x \in T\} \\
&= U
\end{aligned}$$

If $U^* = \cup\{B'_\varepsilon(x) : x \in T\}$, then U^* is an open subset of S such that $U = U^* \cap T$.

Theorem 2.8 A subset F is a closed subset of a metric space (S, ρ) if and only if its complement, $S\backslash F$, is an open subset of S.

Proof: Recall that closed subsets of S are those subsets F which contain all their limit points. So the statement says that F is closed in S if and only if $S\backslash F$ contains no limit point of F. The proof is left as an exercise.

2.5 Characterizations of continuous functions on a metric space.

In what follows we study *continuous functions* mapping one metric space, (S_a, ρ_a) to another metric space (S_b, ρ_b).

Definition 2.9 Let (S_a, ρ_a) and (S_b, ρ_b) be two metric spaces.

a) We say that a function $f : S_a \to S_b$ is *continuous at the point* $u \in S_a$ if and only if it satisfies the following condition:

For every $\varepsilon > 0$, there exist $\delta > 0$ such that $\rho_a(u, x) < \delta \Rightarrow \rho_b(f(u), f(x)) < \varepsilon$.

b) If $A \subseteq S_a$, we say that a function $f : A \to S_b$ is *continuous on the set* A if and only if it is continuous at each point in A.

There are various ways of recognizing continuous functions on a subset T of a metric space (S, ρ). The following theorem illustrates the most important ones.

Theorem 2.10 Let (S_a, ρ_a) and (S_b, ρ_b) be two metric spaces and $f : S_a \to S_b$ be a function mapping the S_a into the set S_b. Then the following are equivalent:

1) The function f is continuous on S.

2) Whenever a sequence $\{x_n\}$ in S_a converges to a point $u \in S_a$, then the sequence $\{f(x_n)\}$ in S_b converges to $f(u)$ in S_b.

3) Suppose $f[S_a]$ denotes the range of f on the domain S_a. For any open subset U of $f[S_a] \subseteq S_b$, the subset $f^{\leftarrow}[U] = \{x \in S_a : f(x) \in U\}$ is open in S_a.

Proof: We are given two metric spaces, (S_a, ρ_a) and (S_b, ρ_b) and a function and $f : S_a \to S_b$.

($1 \Rightarrow 2$) Suppose $f : S_a \to S_b$ satisfies the formal definition of continuity on S_a. Let $u \in S_a$ and $\varepsilon > 0$. There exists δ such that $\rho_a(u, y) < \delta \Rightarrow \rho_b(f(u), f(y)) < \varepsilon$. Suppose $\{x_n\}$ is a sequence in S_a such that $\lim_{n\to\infty} x_n = u$. Then the sequence $\{f(x_n)\}$ and the point $f(u)$ are defined in S_b. We claim that $\{f(x_n)\}$ converges to $f(u)$. Since $\{x_n\}$ converges to u, there exist an $N > 0$ such that $n > N$ implies $\rho_a(u, x_n) < \delta$. Then, for $n > N$, $\rho_b(f(u), f(x_n)) < \varepsilon$. It then follows that $\lim_{n\to\infty} f(x_n) = f(u)$, as required.

($2 \Rightarrow 1$) Suppose the function $f : S_a \to S_b$ is such that, for $u \in S_a$ and $\{x_n\} \subseteq S_a$,

$$\lim_{n\to\infty} x_n = u \quad \Rightarrow \quad \lim_{n\to\infty} f(x_n) = f(u)$$

Let $\varepsilon > 0$. Suppose f is not continuous at u. Then there exists $\varepsilon > 0$ such that for any δ there exists $x \in S_a$ such that $\rho_a(u, x) < \delta$ and $\rho_b(f(u), f(x)) > \varepsilon$. Then, for any $\delta_n = \frac{1}{n}$, there exists $x_n \in S_a$ such that $\rho_a(u, x_n) < \delta_n = \frac{1}{n}$ and $\rho_b(f(u), f(x_n)) > \varepsilon$. We can then construct a sequence $\{x_n\}$ in S_a such that $\lim_{n\to\infty}(x_n - u) = 0$ and $\lim_{n\to\infty} f(x_n) \neq f(u)$, contradicting our hypothesis.[1] The source of our contradiction is our supposition that f is not continuous at u. So f must be continuous at u, as required.

[1]Note that the axiom of choice is invoked here.

($1 \Rightarrow 3$) Suppose $f : S_a \to S_b$ satisfies the formal definition of continuity on S_a. Let $u \in S_a$ and $\varepsilon > 0$. Let U be an open subset of $f[S_a] \subseteq S_b$. Let $u \in f^{\leftarrow}[U]$. Then $f(u) \in U$. Since U is open in $f[S_a]$ there exists $\varepsilon > 0$ such that $B_\varepsilon(f(u)) \subseteq U$ where $B_\varepsilon(f(u)) = \{x \in f[S_a] : \rho_b(f(u), x) < \varepsilon\}$. Since f is continuous at u, there exists $\delta > 0$ such that $\rho_a(u, x) < \delta \Rightarrow \rho_b(f(u), f(x)) < \varepsilon$. This means that, if $x \in B_\delta(u)$, then $f(x) \in B_\varepsilon(f(u))$. Then

$$u \in B_\delta(u) \subseteq f^{\leftarrow}[B_\varepsilon(f(u))] \subseteq f^{\leftarrow}[U]$$

So $f^{\leftarrow}[U]$ is open in S_a, as required.

($3 \Rightarrow 1$) Suppose $f : S_a \to S_b$ is a function on S_a such that whenever U is open in $f[S_a]$, $f^{\leftarrow}[U]$ is open in S_a. Let $u \in S_a$ and $\varepsilon > 0$. By hypothesis, $f^{\leftarrow}[B_\varepsilon(f(u))]$ is open in S_a and contains u. Then there exists $\delta > 0$ such that $u \in B_\delta(u) \subseteq f^{\leftarrow}[B_\varepsilon(f(u))]$. This means that if $\rho_a(u, x) < \delta$ then $\rho_b(f(u), f(x)) < \varepsilon$, as required.

Note that the third characterization of a continuous function on a set, "U is open $\Rightarrow f^{\leftarrow}[U]$ is open", is normally not used to determine continuity of f at a *point* in the domain. This characterization "U is open $\Rightarrow f^{\leftarrow}[U]$ is open" is referred to as the *topological definition of continuity on a set* while the second characterization "$\{x_n\} \to a \Rightarrow \{f(x_n)\} \to f(a)$" is referred to as the *sequential definition of continuity at a point.*

EXERCISES

1. Prove parts a) to d) of theorem 2.6.
2. a) Show that the function which appears in Example 2 on page 16 is a valid metric.
   b) Show that the function which appears in Example 3 on page 16 is a valid metric.
3. Define $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $\rho(x, y) = |x - y|$. Prove that, thus defined, ρ is a valid metric on $\mathbb{R}$.
4. Let ρ_1 and ρ_2 be two metrics on the set S.
   a) For a given $k > 0$ we define the function $k\rho : S \times S \to \mathbb{R}$ as $(k\rho)(x, y) = k \times \rho(x, y)$. Show that $k\rho$ is valid metric on S.
   b) If $\rho_1 + \rho_2 : S \times S \to \mathbb{R}$ is defined as $(\rho_1 + \rho_2)(x, y) = \rho_1(x, y) + \rho_2(x, y)$ show that $\rho_1 + \rho_2$ is a valid metric on S.
   c) If $\rho : S \times S \to \mathbb{R}$ is defined as $\rho(x, y) = \min\{1, \rho_1(x, y)\}$ show that ρ is a valid metric on S.

5. Suppose $\{x_n\}$ is a sequence in a metric space (S, ρ). Show that if $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} = b$ where both a and b belong to S then $a = b$.

6. We define two functions $\rho_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and $\rho_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$\begin{aligned} \rho_1((a_1, a_2), (b_1, b_2)) &= |a_1 - b_1| + |a_2 - b_2| \\ \rho_2((a_1, a_2), (b_1, b_2)) &= \max\{|a_1 - b_1|, |a_2 - b_2|\} \end{aligned}$$

   a) Show that ρ_1 thus defined is a valid metric on $\mathbb{R}^2$.
   b) Show that ρ_2 thus defined is a valid metric on $\mathbb{R}^2$.
   c) Show that $\lim_{n\to\infty}\{\vec{x}_n\} = \vec{x} \in \mathbb{R}^2$ with respect to ρ_1 if and only if $\lim_{n\to\infty}\{\vec{x}_n\} = \vec{x} \in \mathbb{R}^2$ with respect to ρ_2.

7. Let a, b be distinct points in the metric space (S, ρ). Find disjoint open sets A and B such that $a \in A$, $b \in B$ and $A \cap B = \varnothing$.

8. Suppose $g : S \to \mathbb{R}$ and $h : S \to \mathbb{R}$ are two continuous functions on the metric space (S, ρ). Show that the set $U = \{x \in S : h(x) < g(x)\}$ is an open subset of M.

9. Prove that a finite subset of a metric space (S, ρ) is always closed in S.

10. Prove theorem 2.8: A subset F is a closed subset of a metric space (S, ρ) if and only if its complement, $S \backslash F$, is an open subset of S.

Part II

Topological spaces: Fundamental concepts

3 / A topology on a set

Summary. *In this section we define the notion of a topological space. We will begin by describing those families of subsets of a set which form a topology, τ, on a set, showing along the way how to recognize "open subsets" and "closed subsets". There can be many topologies on a set. Given a pair of topologies τ_1 and τ_2 on S one can sometimes be seen as being "weaker" or "stronger" than the other. Many examples are provided both in the main body of the text as well as in the given exercises.*

3.1 Introduction.

In our review of metric spaces we saw, in theorem 2.10, that a function, $f : S \to Y$, from a metric space S to another metric space T is said to be *continuous* provided the following condition is satisfied: "The set, $f^{\leftarrow}[U]$, is open in S whenever the set, U, is open in T". We saw that, some sets are described as being "open", others as being "closed", some are both open and closed, while some as neither. Of course, we learned how to distinguish one from the other. For example, we recognized closed sets as being those whose complement is open. We saw that we can also recognize a closed set, F, as being one that "contains the limit point of every convergent sequence in F". So it seems that knowing various characterizations for each type of set is not only useful, but important. How did this notion of an "open set" originate? Recall that we constructed open sets in a set S with the help of a norm or metric previously defined on S. These two distance measuring tools defined on S allowed us to define the notion of an "open ball" in S which, in turn, allowed us to recognize those subsets T of S which are open. We would now like to define "open set" in a much more abstract context, in a way that is independent of any distance measuring tool such as a norm or metric. Our starting point is surprisingly simple. Throughout this book, we will gradually add more structure to what we will call, *topological spaces*. As we move along, the various spaces will not only become more interesting, but more complex and so will require more tools, practice, mathematical skills and techniques to better analyze and understand them. With study, the abstract concepts will conjure mental images that will allow us to better "see" precisely what we are dealing with. This is what makes of mathematics a human endeavour. We begin with a formal definition of what the content of this book is all about.

Definition 3.1 Let S be a non-empty set.

a) A *topology* on S is a collection, τ, of subsets of S which possesses the following properties:

O1. Both the empty set, $\varnothing$, and S belong to τ.

O2. If $\mathscr{C} \subseteq \tau$, then $\cup\{C \in \mathscr{C}\}$ is a set which also belongs to τ.

O3. If $\mathscr{F}$ is a finite subset of τ, then $\cap\{C \in \mathscr{F}\}$ is also a set which belongs to τ.

We will refer to O1, O2 and O3 as the *open set axioms.*[1]

b) If τ is a topology on a set S (as defined above) then each member, $U \in \tau$, is called an *open subset* of S. We often simply say that "U is *open* in S".

c) Suppose we have defined a topology, τ, on some set S. Then this set, S, when considered together with this topology τ, is called a *topological space* and can be represented as (S, τ). The condition, O2, can also be expressed by the phrase "τ *is closed under arbitrary unions*". The condition, O3, can also be expressed by the phrase "τ *is closed under finite intersections*".

It is important to remember that the elements of a topology τ on S are all subsets of S. To say that S is an open subset of S means that $-$ or at least it is assumed that $-$ a topology τ has been previously defined; that is, that τ satisfies the three properties stated above. There can be many topologies defined on a given set. If τ_1 and τ_2 are two different topologies on a set S, then (S, τ_1) and (S, τ_2) are considered to be two distinct topological spaces even though they contain the same points. When we view (S, τ_1) and (S, τ_2) simply as *sets* they are of course equal; but they are not the same *topological space*. We speak of U as being an open subset of S when it is clearly understood from the context that U is a member of a predefined topology τ on S.

Example 1. Suppose (S, ρ) is a metric space and $B_\varepsilon(y) = \{x \in S : \rho(x, y) < \varepsilon\}$ represents an open ball of radius ε and center y in S. Let

$$\tau_\rho = \{\varnothing\} \cup \{U \subseteq S : U \text{ is the union of open balls in } S\}$$

Then τ_ρ is a topology on S. (Showing this is left as an exercise.) We say that τ_ρ is the *topology induced by the metric* ρ on S.

If a metric, ρ, is defined on S, a set A belongs to τ_ρ if and only if A is open in the metric space (S, ρ). In this sense, every metric space is a topological space.

Note : There are topological spaces whose topology is not derived from a metric ρ. Those topological spaces for which there is some metric which can produce every open set in its topology have a special name.

---

[1] Some readers may notice that, if $\mathscr{C} = \varnothing \subseteq \tau$, then, by O2, $\cup\{C \in \varnothing\} = \varnothing \in \tau$. Also, if the finite subset, $\mathscr{F}$, is empty then, by O3, $\cap\{C \in \varnothing\} = S \in \tau$. So, theoretically, it would be sufficient to axiomatize "open sets" with O2 and O3 where O1 would logically flow from these two. The axiom O1 is normally included for convenience and make it easier to identify a topology.

Definition 3.2 Let (S, τ) be a topological space. We will say that (S, τ) is *metrizable* if there exists a metric, ρ, on S such that $\tau = \tau_\rho$ (where τ_ρ is induced on S by ρ).

For example, the usual metric $\rho(x, y) = |x - y|$ on $\mathbb{R}$ induces a topology, τ_ρ, on $\mathbb{R}$. We normally refer to this topology as being the *usual topology on* $\mathbb{R}$. In many textbooks it is also referred to as the *Euclidean topology* on $\mathbb{R}$. In this case, the open subsets of $\mathbb{R}$ are the sets which are unions of open intervals $B_\varepsilon(y) = (y - \varepsilon, y + \varepsilon)$. For example, the sets $U_1 = \cup\{(n, n+1) : n \in \mathbb{Z}\}$ and $U_2 = (-4, 9)$ can be shown to belong to τ_ρ, but the subset $U_3 = [-7, -3) \notin \tau_\rho$. (It is left to the reader to verify this.)

Example 2. Consider the set $S = \mathbb{R}^2$ equipped with the topology

$$\tau = \{B \subseteq \mathbb{R}^2 : \mathbb{R}^2 \backslash B \text{ is countable}\} \cup \{\varnothing, \mathbb{R}^2\}$$

a) Verify that the family, τ, of subsets of $\mathbb{R}^2$ is indeed a topology on $\mathbb{R}^2$.
b) Verify that the topological space $\mathbb{R}^2$ equipped with the topology, τ, described above is *not* metrizable.

Solution : Given: $\tau = \{B \subseteq \mathbb{R}^2 : \mathbb{R}^2 \backslash B \text{ is countable}\} \cup \{\varnothing, \mathbb{R}^2\}$.

a) Verification that the sets in τ satisfy the three open sets axioms O1, O2, and O3 is left as an exercise.
b) Suppose ρ is a metric on $\mathbb{R}^2$ such that $\tau = \tau_\rho$. We will show that, given $\tau = \tau_\rho$, then $\rho(\vec{x}, \vec{y}) > \rho(\vec{x}, \vec{z}) + \rho(\vec{y}, \vec{z})$, contradicting the fact that ρ is a valid metric on $\mathbb{R}^2$.

   Suppose $\vec{x}$ and $\vec{y}$ are distinct points in $\mathbb{R}^2$ and $\rho(\vec{x}, \vec{y}) = \alpha \neq 0$. Consider the open balls $B_{\alpha/2}(\vec{x})$ and $B_{\alpha/2}(\vec{y})$. Since both balls belong to τ, each ball has a countable complement and so each ball is an uncountable subset. Hence $B_{\alpha/2}(\vec{y})$ cannot be entirely contained in $\mathbb{R}^2 \backslash B_{\alpha/2}(\vec{x})$. That means that $B_{\alpha/2}(\vec{y}) \cap B_{\alpha/2}(\vec{x}) \neq \varnothing$. Let $\vec{z} \in B_{\alpha/2}(\vec{y}) \cap B_{\alpha/2}(\vec{x})$. Then

$$\begin{aligned} \rho(\vec{x}, \vec{z}) + \rho(\vec{y}, \vec{z}) &< \alpha/2 + \alpha/2 \\ &= \alpha \\ &= \rho(\vec{x}, \vec{y}) \end{aligned}$$

   We see that $\rho(\vec{x}, \vec{y}) > \rho(\vec{x}, \vec{z}) + \rho(\vec{y}, \vec{z})$ and so ρ does not satisfy the triangle inequality, contradicting the fact that ρ is a metric on $\mathbb{R}^2$. So there can be no metric ρ on $\mathbb{R}^2$ such such that $\tau = \tau_\rho$. We conclude that the topological space, $(\mathbb{R}^2, \tau)$, is not metrizable.[1]

[1] Later, once we covered the concept of "Hausdorff" this problem will be more easily solved by stating that "this space is not metrizable because it not Hausdorff".

We introduce the following supplementary definition.

Definition 3.2.1 If (S, τ) is a topological space and $x \in U$ where U is an open subset of S, then we say that U is an *open neighbourhood of x*. Furthermore, if U is open and T is a subset of S such that $x \in U \subset T$, then we say that T is a *neighbourhood of x*.

Note that it is required that U be open in S for T to be called a "neighbourhood of x". A point, x, in a topological space, (S, τ), always has at least one open neighbourhood, namely S. If $T = (-2, 4) \cup (4, 7]$ in $\mathbb{R}$, equipped with the usual topology, then we see that T is a neighbourhood of 3 but not a neighbourhood of 4 nor of 7 (since there is no open set U such that $7 \in U \subseteq T$).

Definition 3.3 Suppose τ_1 and τ_2 are two topologies on a given set S. If $\tau_1 \subseteq \tau_2$, then we say that τ_1 is a *weaker topology than* τ_2 on S or that τ_2 is a *stronger topology than* τ_1 on S. We can also say that τ_1 is a *coarser topology than* τ_2 on S, or that τ_2 is a *finer topology than* τ_1 on S. We will say that the two topological spaces (S, τ_1) and (S, τ_2) are *equivalent* if and only if $\tau_1 = \tau_2$.

Example 3. Suppose S is a non-empty set and $\mathscr{P}(S)$ denotes the power set of S (that is, $\mathscr{P}(S)$ denotes the collection of all subsets of S). Let $\tau_d = \mathscr{P}(S)$. Then τ_d is a topology on S. It is left to the reader to verify this. In this case, for every single point $x \in S$, $\{x\}$ is open in S.

Definition 3.3.1 If $\tau_d = \mathscr{P}(S)$, the topology τ_d is referred to as the *discrete topology on S*.

If S is equipped with the discrete topology then, for every $x \in S$, $\{x\}$ is an open neighbourhood of x, so every non-empty subset of S is an open neighbourhood of the elements it contains.

Example 4. For a non-empty set S, if $\tau_i = \{\varnothing, S\}$, then the two-element set, τ_i, is a topology on S.

Definition 3.3.2 If $\tau_i = \{\varnothing, S\}$, the topology τ_i is normally referred to as the *indiscrete topology on S*. In this case, for any $x \in S$, S is its only neighbourhood.

Since all topologies on S must at least contain $\varnothing$ and S, no topology on S can contain fewer than two elements. That is, for any topology, τ, on S which contains more than two sets, $\tau_i \subset \tau$; this means that τ_i is the weakest (coarsest) of all topologies on S. On the other hand, for any topology τ on S, $\tau \subseteq \tau_d = \mathscr{P}(S)$, hence the discrete topology is the strongest (finest) of all topologies on S. For any topology τ on S, we then have $\tau_i \subseteq \tau \subseteq \tau_d$.

It is left for the reader to verify the following important fact.

Theorem 3.3.3 Given a family $\{\tau_k : k \in I\}$ of topologies on a set S, the family $\cap_{k \in I} \tau_k$ is also a topology on S.[1] This fact can also be expressed as: "The set of all topologies is closed under arbitrary intersections."

But $\cup\{\tau_k : k \in I\}$ need not necessarily be a topology on S. For example, verify that $\tau_1 = \{\varnothing, S, \{b\}, \{a, b\}\}$ and $\tau_2 = \{\varnothing, S, \{c\}, \{a, c\}\}$ are two topologies on S but their union, $\tau_1 \cup \tau_2$, is not closed under finite intersections and so cannot be a topology on S.

3.2 Closed subsets of a topological space.

In our brief overview of metric spaces, (S, ρ), we defined a closed subset, F, of S as one which contains all its limit points. We then saw that, in a metric space S, a subset, F, of S is closed if and only if its complement, $S \backslash F$, is open.[2] Metric spaces were equipped with a "metric" so that we could discuss the much needed notions of "convergence of a sequence" and "closed set" in a set S. We now formally define the notions closed subsets in a topological space. We will not discuss the notion of "convergence" in a topological space immediately. We will be doing an in-depth study of this topic in a section later in this book.

Definition 3.4 Let F be a subset of a topological space (S, τ). If the complement, $S \backslash F$, of F is an open subset in S then we say that F is a *closed subset in S*.

The definition of "closed" states that "($S \backslash F$ is open) $\Rightarrow$ (F is closed"). Conversely,

$$\begin{aligned} A \text{ is closed} \quad &\Rightarrow \quad A = S \setminus [S \backslash A] \text{ is closed} \\ &\Rightarrow \quad [S \backslash A] \text{ is open} \end{aligned}$$

---

[1] Caution: However, unions of topologies may not form a topology on a set.

[2] $S \backslash F = \{x \in S : x \notin F\}$

We can then actually write

$$(F \text{ is closed}) \Leftrightarrow (S \backslash F \text{ is open})$$

Suppose (S, τ) is a topological space and $\mathscr{F} = \{A \subset S : A \text{ is closed in } S\}$. Then we can define the topology in terms of $\mathscr{F}$,

$$\tau = \{A : S \backslash A \in \mathscr{F}\} \text{ and } \mathscr{F} = \{A : S \backslash A \in \tau\}$$

Theorem 3.5 Let (S, τ) be a topological space and I be any indexing set. Then,

a) Both $\varnothing$ and S are closed in S.

b) If $\{F_i : i \in I\}$ is a family of closed subsets of S then $\cap\{F_i : i \in I\}$ is a closed subset of S.

c) If $\{F_i : i = 1, 2, 3, \ldots, k\}$ is a finite family of closed subsets of S then $\cup\{F_i : i = 1, 2, 3, \ldots, k\}$ is a closed subset of S.

Proof:

a) Since $\varnothing$ is open, then $S = S \backslash \varnothing$ is closed. Since S is open, $\varnothing = S \backslash S$ is closed.

b) Let $\{F_i : i \in I\}$ be a family of closed subsets of S. Then, for each $i \in I$, $S \backslash F_i$ is open. Since (by De Morgan's law) $S \backslash \cap\{F_i : i \in I\} = \cup\{S \backslash F_i : i \in I\}$ is open (being the union of open sets) then $\cap\{F_i : i \in I\}$ is a closed subset of S.

c) This part is left as an exercise.

Proposition 3.5.1 *Defining a topology on S in terms of "closed sets".* Suppose we are given a set S and family, $\mathscr{F} = \{F : F \subseteq S\}$ of elements from $\mathscr{P}(S)$ which satisfies the following three conditions F1, F2 and F3:

F1. The sets $\varnothing$, S both belong to $\mathscr{F}$.
F2. If $\{F_i : i \in I\}$ is a family of sets in $\mathscr{F}$ then $\cap\{F_i : i \in I\} \in \mathscr{F}$.
F3. If $\{F_i : i = 1, 2, 3, \ldots, k\} \subseteq \mathscr{F}$ then $\cup\{F_i : i = 1, 2, 3, \ldots k\} \in \mathscr{F}$.

Then the family, $\tau = \{S \backslash F : F \in \mathscr{F}\}$, forms a topology on S.

Proof: The proof showing that τ is a topology is left to the reader. It easily follows from an application of De Morgan's law[1].

[1] That is, $S \setminus [\cup_{i \in I} F_i] = \cap_{i \in I}[S \backslash F_i]$ and $S \setminus [\cap_{i \in I} F_i] = \cup_{i \in I}[S \backslash F_i]$

Definition 3.5.2 The conditions F1, F2 and F3 described above are referred to as the *closed set axioms.*

Example 5. Suppose S is a non-empty set and $\mathscr{F} = \{F : F \text{ is a finite subset of } S\} \cup \{\varnothing, S\}$. It is left to the reader to confirm that the set $\mathscr{F}$ satisfies the three closed sets axioms F1, F2 and F3. The set, $\mathscr{F}$, is the set of all closed subsets of a topological space S whose topology is $\tau = \{S \backslash F : F \in \mathscr{F}\}$. The elements of τ are $\varnothing$, S and any subset of S whose complement in S is finite.

Definition 3.5.3 Given a set S, the family of subsets of S, then

$$\tau = \{A : A \subseteq S \text{ and } A \text{ is cofinite, i.e., } S \backslash A \text{ is finite } \} \cup \{S, \varnothing\}$$

is called the *cofinite topology on S* or the *Zariski topology on S.*

3.3 Subspace topology on a subset.

We previously defined the notion of a “metric subspace” as being a subset of a metric space equipped with the subspace metric. In the case of a topological space (S, τ), any subset T can be declared to be a subspace provided the reader understands what topology is defined on T. Suppose H is a non-empty subset of a topological space (S, τ). Then H can inherit its topology from τ, in a natural way, as shown the following theorem.

Theorem 3.6 Let (S, τ) be topological space and $H \subseteq S$.

a) Let

$$\tau_H = \{U \subset H : U = K \cap H, \text{ for some } K \in \tau\}$$

Then τ_H is a topology on H.

b) Let

$$\mathscr{F}_H = \{F \subset H : F = M \cap H \text{ where } M = S \setminus K, K \in \tau\}$$

Then $\mathscr{F}_H$ represents *all* closed subsets of the topological space (H, τ_H)

Proof: The proof is left as an exercise.

Definition 3.7 If (S, τ) is topological space and $H \subseteq S$ and

$$\tau_H = \{U \subset H : U = K \cap H,\ K \in \tau\}$$

then τ_H is called the *subspace topology* or *relative topology* on H induced by S, or inherited from S. In such a case, we will say that (H, τ_H) is a *subspace of* S.

Some subsets of a topological space (S, τ) can be both open and closed. For example, if $(\mathbb{R}, \tau_i)$ is equipped with the indiscrete topology, $\tau_i = \{\varnothing, \mathbb{R}\}$, both $\varnothing$ and $\mathbb{R}$ are the only open subsets of $\mathbb{R}$ and so both $\varnothing$ and $\mathbb{R}$ are the only closed subsets of $\mathbb{R}$. That is, $\varnothing$ and $\mathbb{R}$ are both simultaneously open and closed in $\mathbb{R}$. We consider a less trivial example.

Example 6. Let $H = [3, 5)$. Consider the subset $T = H \cup \{9\}$ of $\mathbb{R}$ where $\mathbb{R}$ is equipped with the usual topology. Suppose the subspace (T, τ_T) is equipped with the subspace topology inherited from τ. Since $H = T \cap [2, 6]$, then H is closed in T with respect to the subspace topology τ_T. Since $H = T \cap (2, 5)$, then H is open in T with respect to the subspace topology τ_T. So the subspace, H, is both open and closed in the subspace, T.

Topologists often use a special adjective to refer to those subsets which are both open and closed. Since its use is fairly common we formally define it below.

Definition 3.7.1 If T is a subset of a topological space (S, τ), and T is both open and closed with respect to τ, we say that T is *clopen* in S.

3.4 Other examples of topological spaces.

We provide a few more examples of topological spaces.

Example 7. Let S be a set and $B \subseteq S$. Let $\tau_B = \{A \in \mathscr{P}(S) : B \subseteq A\} \cup \{\varnothing\}$.

a) Verify that τ_B is indeed a topology.
b) Describe the closed subsets of (S, τ_B).

Solution : We are given that $B \subseteq S$ and $\tau_B = \{A \in \mathscr{P}(S) : B \subseteq A\} \cup \{\varnothing\}$.

a) We verify that τ_B is a topology on S by confirming that it satisfies the open set axioms O1, O2 and O3.

- By definition, $\varnothing \in \tau_B$; also, since $B \subseteq S$ then $S \in \tau_B$.
- Suppose $\mathscr{U}$ is a non-empty subset of τ. Then $B \subseteq U$ for each $U \in \mathscr{U}$. Then $B \subseteq \cup\{U : U \in \mathscr{U}\}$. Hence $\cup\{U : U \in \mathscr{U}\} \in \tau_B$.
- Suppose $\mathscr{F}$ is a finite subset of τ_B. Then $B \subseteq F$ for each $F \in \mathscr{F}$. Then $B \subseteq \cap\{F : F \in \mathscr{F}\}$. Hence $\cap\{F : F \in \mathscr{F}\} \in \tau_B$.

b) We now describe the closed subsets of S. We consider the sets, A, such that $A \cap B = \varnothing$, and those satisfying $A \cap B \neq \varnothing$.

- If $A \subseteq S$ such that $A \cap B = \varnothing$, then $B \subseteq S \setminus A$; this means $S \setminus A$ is open, hence A is a closed subset of S.
- Suppose, on the other hand, that for $A \subseteq S$ and $A \cap B \neq \varnothing$. We claim that S is the only closed subset which contains A. To see this, note that if F is closed then $S \setminus F$ is open and so either contains B or is $\varnothing$. If $A \cap B \neq \varnothing$ and $A \subseteq F$ then $S \setminus F$ cannot contain B so $S \setminus F = \varnothing$, which means that $F = S$.

So the closed subsets of S are the family

$$\mathscr{F} = \{F \subset S : F \cap B = \varnothing\} \cup \{S\}$$

Example 8. Let (S, τ) be a topological space and suppose B is a fixed subset of S. Let

$$\tau_B = \{A \in \mathscr{P}(S) : A = C \cup (D \cap B) \text{ where } C, D \in \tau\}$$

a) Verify that τ_B is another topology on S.

b) Is one of the two topologies, τ_B, τ, stronger than the other? Are these two topologies equivalent topologies?

Solution: We are given that (S, τ) is a topological space and B is a fixed subset of S.

a) We begin by showing that τ_B satisfies the three open set axioms O1, O2 and O3.

- We know $\varnothing, S \in \tau$. So we have $\varnothing = \varnothing \cup (\varnothing \cap B) \in \tau_B$ and $S = S \cup (S \cap B) \in \tau_B$.
- Suppose $\mathscr{U} = \{C_i \cup (D_i \cap B) : i \in I\} \subseteq \tau_B$, where $C_i, D_i \in \tau$ for all $i \in I$. Then $\cup_{i\in I} C_i$, and $\cup_{i\in I} D_i$ both belong to τ. Then

$$\bigcup_{i\in I} [C_i \cup (D_i \cap B)] = [\cup_{i\in I} C_i] \;\cup\; ([\cup_{i\in I} D_i] \cap B) \in \tau_B$$

  Thus τ_B is closed under arbitrary unions.
- Suppose $\mathscr{F} = \{C_i \cup (D_i \cap B) : i = 1, 2, \ldots, n\} \subseteq \tau_B$. Then

$$\begin{aligned}
\bigcap_{i=1,\ldots,n} [C_i \cup (D_i \cap B)] &= \bigcap_{i=1,\ldots,n} [(C_i \cup D_i) \cap (C_i \cup B)] \\
&= [\cap_{i=1,\ldots,n} (C_i \cup D_i)] \;\cap\; [(\cap_{i=1,\ldots,n} C_i) \cup B] \\
&= [\cap_{i=1,\ldots,n} (C_i \cup D_i)] \cap [(\cap_{i=1,\ldots,n} C_i)] \\
&\qquad \cup\; [\cap_{i=1,\ldots,n} (C_i \cup D_i) \cap B] \\
&\in \tau_B
\end{aligned}$$

So τ_B satisfies the three open set axioms O1, O2 and O3.

b) We claim that $\tau \subseteq \tau_B$: See that, if $C \in \tau$, then $C = C \cup (\varnothing \cap B) \in \tau_B$, so $\tau \subseteq \tau_B$. So τ_B is a topology on S which is *finer* (*stronger*) than τ.

We claim that these two topologies are not equivalent: Suppose $B \notin \tau$, $D \in \tau$ and $B \subset D$. Then $\varnothing \cup (D \cap B) = B \notin \tau$. Since $B \in \tau_B$ then τ_B contains elements which are not in τ. So $\tau \subset \tau_B$.

In the above example, we say that the topology τ_B *extends* τ *over* B.

3.5 Free union of topological spaces..

If we are given a family of topological spaces there is way to unite them into one single new topological space without altering their individual topology. This is called taking their "free union". We define this.

Definition 3.8 Let $\{S_i : i \in I\}$ be a family of topological spaces. For each space, S_i, we associate a space, $S_i^* = \{i\} \times S_i$ in such a way that S_i^* and S_i are identical except for the fact that $\{i\} \times S_i$ has label i attached to S_i. This is to guarantee that if $i \neq j$ then S_i^* and S_j^* are entirely different sets and so have empty intersection. This allows us to view the family, $\{S_i^* : i \in I\}$, as being pairwise disjoint, in the sense that no two spaces have elements in common. We define the *free union of the family* $\{S_i : i \in I\}$,[1]denoted as, $\sum_{i \in I} S_i^*$, as being the topological space

$$\sum_{i \in I} S_i^* = \cup\{S_i^* : i \in I\}$$

in which U is open in $\sum_{i \in I} S_i^*$ if and only if $U \cap S_i^*$ is open for each $i \in I$. This topology, thus defined, is referred to as the *disjoint union topology.*

Example 9. For each $n \in \mathbb{N}\backslash\{0\}$, let L_n denote the set,

$$L_n = \left\{(x, y_n) : y_n = g_n(x) = \left(\frac{\sin\left(\frac{\pi}{4n}\right)}{\cos\left(\frac{\pi}{4n}\right)}\right) x,\ x \neq 0\right\}$$

Let $L_0 = \{(x, 0) : x \in \mathbb{R}\}$ and

$$T = \cup\{L_n : n \in \mathbb{N}\backslash\{0\}\} \cup \{L_0\}$$

It is possible to view T as a single subset of $\mathbb{R}^2$ and equip it with the subspace topology, in which case open neighbourhoods of points on L_0 would intersect points on other

[1]Some texts may refer to this set as *direct sum* or *free sum* or *topological direct sum.*

lines. Or, we could view T as a free union of disjoint subspaces of $\mathbb{R}^2$, where each line is equipped with the subspace topology. With the free union topology, an open neighbourhood of a point on a line is restricted to the line itself. Each line would be clopen in T. For example, $\{(x, 0) : 1 < x < 7\}$ would be an open neighbourhood of $(5, 0)$ in T.

3.6 Topics: G-delta and F-sigma sets.

Besides the fundamental *open* and *closed* subsets of a topological space introduced earlier, there are other subsets with special properties that we can present now. These are called G-delta's, F-sigma's. Developing some familiarity with these now will be good practice as well as provide us with the freedom to refer to them in various examples and exercise questions to come.

The G-delta and F-sigma sets in a topological space. We have seen that arbitrary unions of open subsets of a topological space are open. However, the intersection of, at most, finitely many open sets are guaranteed to be open. Similarly, arbitrary intersections of closed sets are closed, but the union of, at most, finitely many closed sets are guaranteed to be closed. Occasionally, the intersection of countably many open sets and the union of countably many closed sets will be sets worthy of interest. Before we continue, we remind the reader that a non-empty set S is said to be "countable" if it is finite or, in the case where it is infinite, the elements of S can be indexed by the natural numbers. That is, $S = \{x_i : i = 0, 1, 2, 3, \ldots\}$. We can also say that S is countable if there exists a function $f : \mathbb{N} \to S$ mapping the natural numbers *onto* S. We now define those special subsets of a topological space we call *G-delta*'s and *F-sigma*'s.

Definition 3.9 The sets in a topological space, (S, τ), which are the intersection of at most countably many open sets are called G_δ-sets (or simply G_δ). Those sets in S which are the union of at most countably many closed sets are called F_σ-sets (or simply F_σ). Neither of these special sets need be open or closed.

Trivially, if F is closed in (S, τ) then F is an F_σ and if U is open then U is a G_δ.

Example 10. If $\mathbb{R}$ is equipped with the usual topology, the set $T = [2, 7]$ is obviously a F_σ. It is also a G_δ since

$$[2, 7] \quad = \quad \cap\{(2 - 1/n, 7 + 1/n) : n = 1, 2, 3, \ldots\}$$

So some sets can be both a G_δ and an F_σ with respect to the same topology τ.

Example 11. On the other hand, suppose (S, τ_i) is equipped with the indiscrete topology, $\tau_i = \{\varnothing, S\}$. If T is a proper non-empty subset of S, we see that $T \nsubseteq \varnothing$ and $T \neq S$; so T is neither a G_δ nor an F_σ with respect to τ_i.

Example 12 . We consider the set of all rationals, $\mathbb{Q}$, as a subset of $\mathbb{R}$ equipped with the usual topology. It is known that $\mathbb{Q}$ is countably infinite and so can be expressed in the form $\mathbb{Q} = \{x_i : i = 1, 2, 3, \ldots\}$. Then $\mathbb{Q} = \cup\{\{x_i\} : i = 1, 2, 3, \ldots\}$ where each $\{x_i\}$ is a closed subset of $\mathbb{R}$. So $\mathbb{Q}$ is an F_σ.[1]

The following theorem exhibits properties respected by each F_σ and G_δ and the families of all G_δ's and F_σ's of a topological space.

Theorem 3.10 Suppose F is an F_σ and G is a G_δ in S.

a) The complement of F in S is a G_δ and the complement of G in S is an F_σ.

b) There exists a sequence, $\{F_i : i = 1, 2, 3, \ldots\}$, of closed subsets of S such that

$$F_i \subseteq F_{i+1} \text{ for all } i = 1, 2, 3, \ldots, \text{ and } F = \cup\{F_i : i = 1, 2, 3, \ldots\}$$

c) There exists a nonincreasing sequence, $\{G_i : i = 1, 2, 3, \ldots\}$, of open subsets of S such that

$$G_{i+1} \subseteq G_i \text{ for all } i = 1, 2, 3, \ldots, \text{ and } G = \cap\{G_i : i = 1, 2, 3, \ldots\}$$

d) Suppose $\mathscr{F}$ denotes the family of all F_σ's in S and $\{F_i : i = 1, 2, 3, \ldots\}$ represents at most countably many elements in $\mathscr{F}$. Then

$$\cup\{F_i : i = 1, 2, 3, \ldots\} \in \mathscr{F} \text{ and } \cap\{F_i : i = 1, 2, 3, \cdots, k\} \in \mathscr{F} \text{ for any } k$$

e) Suppose $\mathscr{G}$ denotes the family of all G_δ's in S and $\{G_i : i = 1, 2, 3, \ldots\}$ represents at most countably many elements in $\mathscr{G}$. Then

$$\cap\{G_i : i = 1, 2, 3, \ldots\} \in \mathscr{G} \text{ and } \cup\{G_i : i = 1, 2, 3, \cdots, k\} \in \mathscr{G} \text{ for any } k$$

Proof: Given: (S, τ) be topological space; F is an F_σ and G is a G_δ in S.

a) Suppose $F = \cup\{K_i : i = 1, 2, 3, \ldots,$ where each K_i is closed in $S\}$

$$\begin{aligned} S\backslash F &= S\backslash(\cup\{K_i\}) \\ &= \cap\{S\backslash K_i\} \quad \text{(By De Morgan's rule)} \\ &= \text{ a } G_\delta\text{-set} \end{aligned}$$

The proof of the second part of a) follows by a similar application of De Morgan's rule.

[1]Once we have the necessary tools we will prove that $\mathbb{Q}$ is not a G_δ

b) The proof is left as an exercise for the reader.

c) The proof is left as an exercise for the reader.

d) For countable unions of F_σ's:

$$\bigcup_{j=1}^{\infty}[\cup\{F_{(i,j)} : i = 1, 2, 3, \ldots, \}] = \cup\{F_{(i,j)} : (i, j) \in \mathbb{N} \times \mathbb{N}\}$$

where $\mathbb{N} \times \mathbb{N}$ is known to be countable.

For finite intersections of F_σ's:

$$\bigcap_{i=1}^{k}[\cup_{j=1}^{\infty}\{F_{(i,j)}\}] = \bigcup_{(j_1,\cdots,j_k)\in\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}}\{F_{(1,j_1)} \cap \cdots \cap F_{(k,j_k)}\}$$

where $\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ is known to be countable.

e) This part is proved similarly to part d).

We summarize two of the results in the above theorem:

"The family, $\mathscr{F}$, of all F_σ's of a topological space is closed under countable unions and closed under *finite* intersections."

"The family, $\mathscr{G}$, of all G_δ's of a topological space is closed under countable intersections and closed under *finite* unions."

3.7 Topics: The family of Borel sets.

Topological spaces each contain a particular family of subsets of $\mathscr{P}(S)$ which plays a role in certain fields of study where topology is applied. Particularly in analysis. Before we formally define it, we begin by defining a special type of subset of $\mathscr{P}(S)$ called a "σ-ring".

A subset, $\mathscr{K}$, of $\mathscr{P}(S)$ is called a *σ-ring* if:

1) For any $A \in \mathscr{K}$, $S \backslash A \in \mathscr{K}$
2) Whenever $\{A_i : i \in \mathbb{N}\} \subseteq \mathscr{K}$ then $\cup\{A_i : i \in \mathbb{N}\} \in \mathscr{K}$.

To summarize, a σ-ring is simply a family of sets which is "closed under complements" and "countable unions". See that $\mathscr{P}(S)$, itself a σ-ring, may contain many σ-rings. Given a topological space, (S, τ), we will consider all those σ-rings in $\mathscr{P}(S)$, which contain τ. To obtain the *unique smallest σ-ring*, $\mathscr{B}$, in $\mathscr{P}(S)$ that contains τ, we then take an intersection,

$$\mathscr{B} = \cap\{\mathscr{K} \subseteq \mathscr{P}(S) : \mathscr{K} \text{ is a } \sigma\text{-ring}, \tau \subseteq \mathscr{K}\}$$

The reader should verify that this "intersection of all σ-rings in $\mathscr{P}(S)$ which contain τ" is itself a σ-ring containing τ; we emphasize, that this intersection is the *unique* and *smallest* such σ-ring. There is a name for this particular set.

Definition 3.11 Given a topological space (S, τ), we call the smallest σ-ring, $\mathscr{B}$, in $\mathscr{P}(S)$ which contains τ, the *family of Borel sets in S*. Each members, $A \in \mathscr{B}$, is referred to as a *Borel set*. That is,

$$\{A = \text{"a } \textit{Borel set} \text{ in } S\text{" }\} \Leftrightarrow \{A \in \mathscr{B}\}$$

Every topological space, S, has its unique family, $\mathscr{B}$, of Borel sets. We identify a Borel set by confirming that it belongs to $\mathscr{B}$. To help us identify Borel sets we list a few properties of $\mathscr{B}$. The reader is left to verify that $\mathscr{B}$ is

- closed under complements,
- closed under countable unions and countable intersections
- contains all G_δ's and all F_σ's.

So the definition of a Borel set in S may make it difficult to recognize such a subset of S. But we at least easily recognize a large subfamily of $\mathscr{B}$. All open subsets (including $\varnothing$ and S itself), all G_δ's and all F_σ's are Borel sets. But there may be others.

Theorem 3.12 Let (S, τ) be a topological space. The family, $\mathscr{B}$, of Borel sets is the *unique smallest subfamily of $\mathscr{P}(S)$*, that

a) contains τ,

b) is closed under complements

c) is closed under countable unions.

Furthermore, $\mathscr{B}$ satisfies the following three properties:

1. $\mathscr{B}$ contains all F_σ's of S,

2. $\mathscr{B}$ is closed under countable intersections,
3. $\mathscr{B}$ contains all G_δ's of S.

Proof: Given: (S, τ) is a topological space.

Suppose $\mathscr{B}$ is the family of Borel sets in $\mathscr{P}(S)$. By definition, $\mathscr{B}$ is the intersection of all σ-rings that

a) contain τ,
b) that are closed under complements and
c) closed under countable unions.

So $\mathscr{B}$ is itself the *unique smallest* σ-ring of subsets of S which satisfies these three properties.

Since $\mathscr{B}$ contains τ, it contains all open subsets of S and since it is closed under complements, it contains all closed subsets of S. Since it is closed under countable unions then it must contain all F_σ's. This establishes property 1.

We now verify that $\mathscr{B}$ is closed under countable intersections: Let $\{A_i : i = 1, 2, 3, \ldots\}$ be a countable family of subsets in $\mathscr{B}$. Then

$$\begin{aligned} \cap\{A_i : i = 1, 2, 3, \ldots\} &= S\backslash S\backslash(\cap\{A_i : i = 1, 2, 3, \ldots\}) \\ &= S\backslash\cup[S\backslash\{A_i : i = 1, 2, 3, \ldots\}] \\ &\in \mathscr{B} \end{aligned}$$

This establishes property 2.

It then follows that, since $\mathscr{B}$ contains all open sets, it follows from property two that it must also contain all G_δ's of S. This establishes property 3.

The above theorem guarantees that every open set, closed set, G_δ and F_σ in a topological space S can be referred to as a Borel set in S. It is sometimes difficult to identify subsets of a topological space (S, τ) which are not Borel sets (with respect to τ). Consider for example, the topological space (S, τ_i) equipped with the indiscrete topology. If A is a non-empty proper subset of S then A is not a Borel set since $\{\varnothing, S\} = \tau_i$ is the smallest σ-ring which contains τ_i and does not contain the element A. On page 58 of this text we provide another example.

Concepts review:

1. Given a set S, what does a topology τ on S represent? How does one verify whether a family of subsets is a topology?
2. Given an open subset, U, of a topological space, (S, τ), what is the relationship between U and τ?
3. What are the three open set axioms of a topological space (S, τ)?
4. Given a metric space, (S, ρ), describe a topology on S which is induced by ρ.
5. Describe the *usual topology* on $\mathbb{R}$.
6. Given a point, x, in a topological space, (S, τ), what is a *neighbourhood* of x?
7. Given two topologies, τ_1 and τ_2, what does it mean to say that τ_1 is weaker than τ_2?
8. Given two topologies, τ_1 and τ_2, what does it mean to say that τ_1 is finer than τ_2?
9. Given a non-empty set, S, describe the discrete topology on S.
10. Given a non-empty set, S, describe the indiscrete topology on S.
11. Suppose F is a closed subset of the topological space (S, τ). What is the relation between F and τ?
12. What are the three closed set axioms, F1, F2, and F3, of a topological space (S, τ).
13. If S is a non-empty set what do we mean by the *cofinite* or *Zariski* topology on S?
14. If T is a subset of the topological space (S, τ) what is the subspace topology on T?
15. Suppose B is a subset of the topological space (S, τ) such that $B \notin \tau$. Describe the topology τ_B which extends τ over B?
16. What does it mean to say that a set is metrizable?
17. What is a G_δ of a topological space? What is an F_σ of a topological space?
18. Describe the family of Borel sets in a topological space (S, τ).
19. Provide a few examples of Borel sets in $\mathbb{R}$ equipped with the usual topology. Is $\mathbb{Q}$ a Borel set? Why?
20. Define the topological space called the "free union" of the spaces, $\{S_i : i \in I\}$.

EXERCISES

1. Prove the statement in theorem 3.5.

2. Consider the open interval $S = (-3, 7)$ in $\mathbb{R}$.

    a) Construct a topology τ on S which contains five elements.

    b) Consider the subset $T = (-2, 4] \subset S$. For the topology τ constructed in part a) what is the subspace topology τ_T on T inherited from S.

    c) Are the open subsets of T necessarily open subsets of S?

3. Consider $\mathbb{R}$ equipped with the usual topology τ (induced by the Euclidian metric). Let $(\mathbb{Q}, \tau_{\mathbb{Q}})$ be the set of all rational numbers equipped with the subspace topology inherited from $\mathbb{R}$. Consider the subset $T = [-\pi, \pi) \cap \mathbb{Q}$. Determine whether T is open in $\mathbb{Q}$, closed in $\mathbb{Q}$, both open and closed in $\mathbb{Q}$, or none of these.

4. Construct a topology other than the *discrete* or *indiscrete topology* on the set $S = \{\triangle, \lozenge, \square\}$.

5. Let $\mathscr{F} = \{A \subseteq \mathbb{R} : A \text{ is countable}\} \cup \{\varnothing, \mathbb{R}\}$. Show that $\mathscr{F}$ satisfies the three conditions F1, F2 and F3 described on page 34. Then use this to construct a topology on $\mathbb{R}$. (This is referred to as being the *cocountable topology.*

6. Suppose τ_A and τ_B are two topologies on a set S. Determine whether $\tau_A \cap \tau_B$ is a topology on S.

7. If $\mathbb{R}$ is equipped with the usual topology and $\mathbb{Z}$ represents the set of all integers determine whether $\mathbb{Z}$ is open or closed (or both or neither) in $\mathbb{R}$.

8. Suppose $\mathbb{R}$ is equipped with the usual topology and $T = [1, 4] \cup (6, 10) \subset \mathbb{R}$ where T is equipped with the subspace topology. Determine whether [1, 4] is open in T, closed in T or both open and closed in T. Determine whether (6, 10) is open in T, closed in T or both open and closed in T.

4 / Set closures, interiors and boundaries.

Summary. *In this section, we introduce the notions of closure and interior of subsets of a topological space. The concept of the boundary of a set is then defined in terms of its interior and closure. Based on their properties, we derive the "closure axioms" and "interior axioms". We then begin viewing closure and interior of sets as being operators on $\mathscr{P}(S)$. From this perspective, we better see how closure and interior operators on $\mathscr{P}(S)$ can be used to topologize a set, providing examples on how this can be done.*

4.1 The closure of a set.

If $T = (2, 7]$ is viewed as a subset of the topological space $\mathbb{R}$ (equipped with the usual topology), we easily see that it is not closed since its complement, $\mathbb{R} \setminus T = (-\infty, 2] \cup (7, \infty)$, is not open in $\mathbb{R}$. And yet we feel that it wouldn't take very much for us to "make it closed": We need only add the element, 2, to T to obtain the closed subset $T^* = [2, 7]$. Adding the *fewest number* of points possible to a set T to obtain a closed set is what we will refer to as obtaining the *closure of* T. The key word here is "fewest number" of points, and no more. In this case, we would say that the "closure of $T = (2, 7]$ is the set $T^* = [2, 7]$, the smallest closed subset of $\mathbb{R}$ which contains all the elements of T. With this example in mind, we will now formally define a concept called "closure of a subset".

Definition 4.1 Let S be a topological space and $T \subseteq S$. We define the *closure of* T *in* S, denoted by, $\mathrm{cl}_S(T)$, as

$$\mathrm{cl}_S(T) = \cap\{F : F \text{ is closed in } S \text{ and } T \subseteq F\}$$

Facts 4.1.1 The reader should first be aware of the following verifiable *facts*. For any subset, T, of the topological space S.

1) *The closure of* T*,* $cl_S(T)$*, is closed in* S. This follows from the fact that arbitrary intersections of closed sets are closed.
2) *The set* $T \subseteq cl_S(T)$. This follows from the definition of closure of T.
3) *The set,* $cl_S(T)$*, is the smallest closed set which contains* T. Suppose A is a closed set containing T. Then $A \in \{F : F \text{ is closed in } S \text{ and } T \subseteq F\}$. Hence $\mathrm{cl}_S(T) \subseteq A$.
4) *If* T *is closed then* $T = cl_S(T)$. This is true since T is the smallest closed set containing T.

Definition 4.1.2 Let A be a non-empty subset of a topological space S.

If x is a point in S such that, for every open neighbourhood U of x, $U \cap A$ contains some point other than x, then we say that x is a *cluster point of* A.

The set of all cluster points of A is called the *derived set of* A.

This definition provides us with another way of describing a closed set. A set A is closed if and only if it contains all its cluster points. For example, if $A = (1,3) \cup (3,5] \cup \{6\}$, the derived set, (that is, the set of all cluster points of A) is $[1,5]$. The element, 6, is not a cluster point of A. The set A is not closed since it doesn't contain the cluster points 1 and 3.

Example 1. Let T be the open interval, $(0,1)$, viewed as a subset of $\mathbb{R}$ equipped with the usual topology. Then $\mathrm{cl}_{\mathbb{R}}(T) = [0,1]$. To prove this we must show that:

1) $[0,1]$ is closed by showing that $\mathbb{R} \backslash [0,1]$ is open.
2) $\{0,1\} \subseteq A$ for any closed set A containing the interval $(0,1)$.

This is left as an exercise.

Example 2. If $\mathbb{Q}$ is the set of all rational numbers then $\mathrm{cl}_{\mathbb{R}}(\mathbb{Q}) = \mathbb{R}$.

Proof: To prove this we will show that, if F is a closed subset of $\mathbb{R}$ such that $\mathbb{Q} \subseteq F$, then $F = \mathbb{R}$.

Suppose F is a closed subset of $\mathbb{R}$ such that $\mathbb{Q} \subseteq F$. Then $\mathbb{R} \backslash F$ is open. Suppose $\mathbb{R} \backslash F \neq \varnothing$. Then $\mathbb{R} \backslash F$ is the union of open intervals each of which must contain a rational number. Since $\mathbb{Q} \cap (\mathbb{R} \backslash F) \neq \varnothing$, this contradicts $\mathbb{Q} \subseteq F$. Then $\mathbb{R} \backslash F = \varnothing$. This means that the only closed set containing $\mathbb{Q}$ is $\mathbb{R}$. So $\mathrm{cl}_{\mathbb{R}}(\mathbb{Q}) = \mathbb{R}$.

Example 3. Suppose S is a topological space induced by the metric ρ (that is, the elements of τ are unions of open balls of the form $B_\varepsilon(x) = \{y : \rho(x,y) < \varepsilon\}$). Suppose F is a non-empty subset of S. We define $\rho(x,F) = \inf\{\rho(x,u) : u \in F\}$. Then $\mathrm{cl}_S(F) = \{x : \rho(x,F) = 0\}$.

Proof: To prove this we must show

1. $F \subseteq \{x : \rho(x,F) = 0\}$,
2. $S \setminus \{x : \rho(x,F) = 0\}$ is open in S,
3. If $F \subseteq A$ where A is a closed subset of S, then $\{x : \rho(x,F) = 0\} \subseteq A$.

Showing these is left as an exercise.

We now list and prove a few of the most fundamental closure properties.

Theorem 4.2 Let A and B be two subsets of a topological space (S, τ). Then,

1) $\text{cl}_S(\varnothing) = \varnothing$.

2) If $A \subseteq B$ then $\text{cl}_S(A) \subseteq \text{cl}_S(B)$

3) $\text{cl}_S(A \cup B) = \text{cl}_S(A) \cup \text{cl}_S(B)$ (Closure "distributes" over finite unions.)

4) $\text{cl}_S(\text{cl}_S(A)) = \text{cl}_S(A)$

Proof:

1) Since $\varnothing$ is closed $\text{cl}_S(\varnothing) \subseteq \varnothing$ so $\text{cl}_S(\varnothing) = \varnothing$.

2) We are given that $A \subseteq B$. If F is closed in S and $B \subseteq F$ then $A \subseteq B \subseteq F$. Then

$$A \subseteq \cap\{F : F \text{ is closed in } S \text{ and } B \subseteq F\} = \text{cl}_S(B)$$

By 1) of *Facts* 4.1.1 above, $\text{cl}_S(B)$ is closed in S and so

$$\text{cl}_S(A) = \cap\{F : F \text{ is closed in } S \text{ and } A \subseteq F\} \subseteq \text{cl}_S(B)$$

We have shown that $\text{cl}_S(A) \subseteq \text{cl}_S(B)$.

3) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ then $\text{cl}_S(A) \subseteq cl_S(A \cup B)$ and $\text{cl}_S(B) \subseteq cl_S(A \cup B)$ (by parts 1) and 2)). So

$$\text{cl}_S(A) \cup cl_S[B) \subseteq \text{cl}_S(A \cup B)$$

Since $A \subset \text{cl}_S(A)$ and $B \subset \text{cl}_S(B)$, $A \cup B \subseteq \text{cl}_S(A) \cup \text{cl}_S(B)$, a closed subset in S. Since $\text{cl}_S(A \cup B)$ is the smallest closed set containing $A \cup B$ then

$$\text{cl}_S(A \cup B) \subseteq \text{cl}_S(A) \cup \text{cl}_S[B)$$

We conclude that $\text{cl}_S(A \cup B) = \text{cl}_S(A) \cup \text{cl}_S(B)$.

4) By part 1) $A \subseteq \text{cl}_S(A) \subseteq \text{cl}_S(\text{cl}_S(A))$. Since $\text{cl}_S(A)$ is closed (see fact 1)), $\text{cl}_S(\text{cl}_S(A)) \subseteq \text{cl}_S(A)$. It then follows that $\text{cl}_S(\text{cl}_S(A)) = \text{cl}_S(A)$.

Example 4. *Closure does not distribute over intersections.* Suppose $A = (2, 5)$ and $B = (5, 7)$ Then $\mathrm{cl}_{\mathbb{R}}(A \cap B) = \mathrm{cl}_{\mathbb{R}}(\varnothing) = \varnothing$. On the other hand, $\mathrm{cl}_{\mathbb{R}}(A) = [2, 5]$ and $\mathrm{cl}_{\mathbb{R}}(B) = [5, 7]$ hence $\mathrm{cl}_{\mathbb{R}}(A) \cap \mathrm{cl}_{\mathbb{R}}(B) = \{5\}$. This shows that $\mathrm{cl}_S(A) \cap \mathrm{cl}_S(B) \neq \mathrm{cl}_S(A \cap B)$ may sometimes occur.

It is however possible to prove that

$$\mathrm{cl}_S(A \cap B) \subseteq \mathrm{cl}_S(A) \cap \mathrm{cl}_S(B)$$

Proving this is left as an exercise.

Remark on closures of arbitrary unions. We have seen in theorem 4.2 that $\mathrm{cl}_S(A \cup B) = \mathrm{cl}_S(A) \cup \mathrm{cl}_S B$, so the "closure distributes over finite unions". This does not hold true for arbitrary unions. Consider the sets of the form $A_i = (\frac{1}{i}, 3]$. Then $\mathrm{cl}_{\mathbb{R}} A_i = [\frac{1}{i}, 3]$. Verify that $\cup\{\mathrm{cl}_{\mathbb{R}}(A_i) : i = 1, 2, 3, \ldots\} = (0, 3]$ (left as an exercise). On the other hand $\cup\{A_i : i = 1, 2, 3, \ldots\} = (0, 3]$ and so $\mathrm{cl}_{\mathbb{R}}[\cup\{A_i : i = 1, 2, 3, \ldots\}] = [0, 3]$ (Exercise). So

$$\cup\{\mathrm{cl}_{\mathbb{R}}(A_i) : i = 1, 2, 3, \ldots\} \subset \mathrm{cl}_{\mathbb{R}}[\cup\{A_i : i = 1, 2, 3, \ldots\}]$$

4.2 Closure viewed as an operator on $\mathscr{P}(S)$.

The closure of a set can be viewed as an action performed on a set. It takes an arbitrary set, A, and associates to it another set, $\mathrm{cl}_S(A)$, obtained by adding sufficiently many points (but no more) so that it is, as a result, closed. It can then be viewed as a function, $K : \mathscr{P}(S) \to \mathscr{P}(S)$, which maps a set to the smallest closed set which contains it. With this in mind, we define the *Kuratowski closure operator*.

Definition 4.2.1 *Kuratowski closure operator.* Suppose S is a non-empty set and $K : \mathscr{P}(S) \to \mathscr{P}(S)$ is a function which satisfies the four conditions:

K1. $K(\varnothing) = \varnothing$ and $A \subseteq K(A)$
K2. If $A \subseteq B$ then $K(A) \subseteq K(B)$
K3. $K(A \cup B) = K(A) \cup K(B)$
K4. $K(K(A)) = K(A)$

A function, $K : \mathscr{P}(S) \to \mathscr{P}(S)$, satisfying these four properties is referred to as a *Kuratowski closure operator* where K1 to K4 are the *Kuratowski closure operator axioms.*

Topologizing a set S by using a closure operator. The reader should notice that, in the above definition, the set S is not described as being a "topological space" since no topology is defined on it. It is just a set. The following theorem shows that, if we are given a Kuratowski operator, $K : \mathscr{P}(S) \to \mathscr{P}(S)$, on $\mathscr{P}(S)$ then we can use K to generate a topology, τ_K, on S such that $K(A) = \text{cl}_S(A)$ for all $A \in \mathscr{P}(S)$ (with respect to the topology τ_K).

Theorem 4.3 Let S be a set and suppose $K : \mathscr{P}(S) \to \mathscr{P}(S)$ satisfies the four *Kuratowski closure operator axioms.* Define $\mathscr{F} \subseteq \mathscr{P}(S)$ as

$$\mathscr{F} = \{A \subseteq S : K(A) = A\}$$

a) Then $\mathscr{F}$, is the set of all closed subsets of some topology, τ_K, on S. That is,

$$\tau_K = \{S \backslash A : \text{ where } A \in \mathscr{F}\}$$

b) Furthermore, in (S, τ_K), $\text{cl}_S(A) = K(A)$, for any $A \subseteq S$.

Proof: Given: The operator $K : \mathscr{P}(S) \to \mathscr{P}(S)$. Also, $\mathscr{F} = \{A \subseteq S : K(A) = A\}$.

a) To prove the statement a) it will suffice to show that $\mathscr{F}$ satisfies the three "closed sets" conditions F1, F2 and F3 on page 34. If so, then we can define τ_K by invoking the statement of Proposition 3.5.1 on page 34.

- Note that $\{\varnothing, S\} \subseteq \mathscr{F}$. To see this note that, by K1, $S \subseteq K(S) \subseteq S \Rightarrow K(S) = S$. Both $\varnothing$ and S belong to $\mathscr{F}$. The set $\mathscr{F}$ satisfies condition F1.
- Suppose $\mathscr{U} = \{F_i : i \in I\}$ is an arbitrarily large family of sets in $\mathscr{F}$. By K1, $\cap\mathscr{U} \subseteq K(\cap\mathscr{U})$. Also, by K2, $K(\cap\mathscr{U}) \subseteq K(F_i) = F_i$ for each $i \in I$. So $K(\cap\mathscr{U}) \subseteq \cap\{F_i : i \in I\} = \cap\mathscr{U}$. Hence $K(\cap\mathscr{U}) \subseteq \cap\mathscr{U}$. So $K(\cap\mathscr{U}) = \cap\mathscr{U}$. We then have $K(\cap\mathscr{U}) \in \mathscr{F}$. The set $\mathscr{F}$ is then closed under arbitrary intersections. The set $\mathscr{F}$ satisfies condition F2.
- We now show that $\mathscr{F}$ is closed under finite unions. We must show that if A and $B \in \mathscr{F}$ then $K(A \cup B) = A \cup B$. Consider $A, B \in \mathscr{F}$. By K2,

$$A \cup B \quad \subseteq \quad K(A \cup B)$$

$$\begin{aligned} A \cup B \subseteq K(A) \cup K(B) \quad &\Rightarrow \quad K(A \cup B) \subseteq K(K(A) \cup K(B)) \quad \text{(By K2.)} \\ &\Rightarrow \quad K(A \cup B) \subseteq K(K(A)) \cup K(K(B)) \quad \text{(By K3.)} \\ &\Rightarrow \quad K(A \cup B) \subseteq A \cup B \quad \text{(By K4.)} \end{aligned}$$

We conclude that $K(A \cup B) = K(A) \cup K(B)$. The set $\mathscr{F}$ satisfies condition F3.

So $\mathscr{F}$ is the set of all closed sets in S. This means the topology τ_K on S is

$$\tau_K = \{S \backslash A : \text{ where } A \subseteq \mathscr{F}\}$$

Hence the set, $\mathscr{F} = \{A \subseteq S : K(A) = A\}$, represents all closed subsets of S (with respect to τ_K).

b) We now prove the second statement, $\text{cl}_S(A) = K(A)$.
Let $A \subseteq (S, \tau_K)$. Then $\text{cl}_S(A) \in \mathscr{F}$. So $K(\text{cl}_S(A)) = \text{cl}_S(A)$. We claim that, from this we can obtain $\text{cl}_S(A) = K(A)$.
Proof of claim: Let $A \subseteq S$.

$$\begin{aligned} K(K(A)) &= K(A) \quad \text{(By K4.)} \\ &\Rightarrow K(A) \in \mathscr{F} \\ &\Rightarrow S \backslash K(A) \in \tau_K \\ &\Rightarrow K(A) \text{ is closed with respect to } \tau_K \\ &\Rightarrow \text{cl}_S(A) \subseteq K(A) \quad \text{(By K2, } A \subseteq K(A).) \end{aligned}$$

$$\begin{aligned} A \subseteq \text{cl}_S(A) &\Rightarrow K(A) \subseteq K(\text{cl}_S(A)) \quad \text{(By K2.)} \\ &\Rightarrow K(A) \subseteq \text{cl}_S(A) \quad \text{(Since } K(\text{cl}_S(A)) = \text{cl}_S(A)) \end{aligned}$$

We conclude that $\text{cl}_S(A) = K(A)$, as claimed.

We have shown that any Kuratowski closure operator $K : \mathscr{P}(S) \to \mathscr{P}(S)$ can be used to construct a topology, τ_K, on S in such a way that, for any $A \subseteq S$, $K(A) = \text{cl}_S(A)$. We illustrate this in the following example.

Example 5. We consider the set $\mathbb{R}^2$, a set with uncountably many elements. We define a function $K : \mathscr{P}(\mathbb{R}^2) \to \mathscr{P}(\mathbb{R}^2)$ as follows:

$$\begin{aligned} K(A) &= A \text{ if } A \text{ is countable} \\ K(A) &= \mathbb{R}^2 \text{ if } A \text{ is uncountable} \\ K(\varnothing) &= \varnothing \end{aligned}$$

Show that K is a Kuratowski operator. Then find the topology on $\mathbb{R}^2$ induced by the operator K.

Solution. Proving that K satisfies the properties K1 to K4 is routine and so is left as exercise. Then, thus defined, K is a Kuratowski closure operator. Then, by theorem 4.3 the set

$$\mathscr{F} = \{A \subseteq \mathbb{R}^2 : K(A) = A\} = \{A \subseteq \mathbb{R}^2 : A \text{ is countable}\} \cup \{\varnothing, \mathbb{R}^2\}$$

represents the set of all closed subsets of the topological space $(\mathbb{R}^2, \tau_K)$. We deduce that

$$\tau_K = \{B \subseteq \mathbb{R}^2 : \mathbb{R}^2\backslash B \text{ is countable}\} \cup \{\varnothing, \mathbb{R}^2\}$$

Definition 4.3.1 We will refer to τ_K in this example as the *cocountable topology* on $\mathbb{R}^2$.

4.3 The interior of a set.

Now that we refer to the smallest closed set in S which contains a set A as the "closure of A", we wish to represent the largest open set in S which is entirely contained in a given set A as its "interior".

Definition 4.4 Let A be a subset of the topological space (S, τ). We say that a point x is an *interior point* of A if there exist an open subset, U, of S such that $x \in U \subseteq A$. We define the *interior of* A, denoted $\text{int}_S(A)$, as follows:

$$\text{int}_S(A) = \{x \in S : x \text{ is an interior point of } A\}$$

If A contains no interior points then we will say that the interior, $\text{int}_S(A)$, of A is empty.

The following theorem shows a relationship between the interior and closure operations. It also proposes a method to determine the interior of a set, A, by considering the closure of its complement, $S\backslash A$.

Theorem 4.5 Let (S, τ) be a topological space and A be a subset of S. Then,

$$S\backslash\text{int}_S(A) = \text{cl}_S(S\backslash A)$$

Proof: Given: (S, τ) is a topological space and A is a subset of S.

Since $\text{int}_S(A) \subseteq A$ then $S\backslash A \subseteq S\backslash\text{int}_S(A)$. But $S\backslash\text{int}_S(A)$ is closed in S, hence

$$\text{cl}_S(S\backslash A) \subseteq S\backslash\text{int}_S(A)$$

Also,

$$\begin{array}{rcl} S\backslash A \subseteq \mathrm{cl}_S(S\backslash A) & \Rightarrow & S\backslash(\mathrm{cl}_S(S\backslash A)) \subseteq A \\ & \Rightarrow & S\backslash(\mathrm{cl}_S(S\backslash A)) \subseteq \mathrm{int}_S(A) \\ & \Rightarrow & S\backslash \mathrm{int}_S(A) \subseteq (\mathrm{cl}_S(S\backslash A)) \end{array}$$

We thus obtain $S\backslash\mathrm{int}_S(A) = \mathrm{cl}_S(S\backslash A)$

Using this theorem, we let the reader verify that the following three statements are equivalent:

a) $\mathrm{int}_S(A) = S\backslash\mathrm{cl}_S(S\backslash A)$,

b) $\mathrm{int}_S(S\backslash A) = S\backslash\mathrm{cl}_S(A)$,

c) $\mathrm{cl}_S(A) = S\backslash(\mathrm{int}_S(S\backslash A))$

Just as for the closure of a set we have four basic similar properties for the interior of sets.

Theorem 4.6 Let (S, τ) be a topological space and A and B be subsets of S.

a) The set $\mathrm{int}_S(A)$ is open in S. Also, $\mathrm{int}_S(A)$ is the largest open subset of S which is entirely contained in A.

b) If $B \subseteq A$ then $\mathrm{int}_S(B) \subseteq \mathrm{int}_S(A)$.

c) The set $\mathrm{int}_S(A \cap B) = \mathrm{int}_S(A) \cap \mathrm{int}_S(B)$. (Int$_S$ "distributes" over finite intersections.)

d) The set $\mathrm{int}_S(\mathrm{int}_S(A)) = \mathrm{int}_S(A)$.

Proof: The proofs of statements a), b) and d) are left as an exercise.

Proof of $\mathrm{int}_S(A \cap B) = \mathrm{int}_S(A) \cap \mathrm{int}_S(B)$:

$$\begin{array}{rcl} S\backslash\mathrm{int}_S(A\cap B) & = & \mathrm{cl}_S(S\backslash(A\cap B)) \\ & = & \mathrm{cl}_S[(S\backslash A)\cup(S\backslash B)] \\ & = & \mathrm{cl}_S(S\backslash A)\cup\ \mathrm{cl}_S(S\backslash B) \\ & = & [S\backslash\mathrm{int}_S(A)]\cup[S\backslash\mathrm{int}_S(B)] \\ & = & S\backslash[\mathrm{int}_S(A)\cap\mathrm{int}_S(B)] \\ & \Rightarrow & \\ \mathrm{int}_S(A\cap B) & = & \mathrm{int}_S(A)\cap\mathrm{int}_S(B) \end{array}$$

Example 6. Given that $\mathbb{R}$ is equipped with the usual topology what is $\text{int}_{\mathbb{R}}(\mathbb{Q})$?

Solution. We consider the subset, $\mathbb{Q}$, of all rationals in $\mathbb{R}$. By theorem 4.6 part a), $\text{int}_{\mathbb{R}}(\mathbb{Q}) \subseteq \mathbb{Q}$. If $\text{int}_{\mathbb{R}}(\mathbb{Q})$ is non-empty, it should be a union of non-empty open intervals. But every open interval contains an irrational; then $\text{int}_{\mathbb{R}}(\mathbb{Q}) = \varnothing$.

Example 7. If $\mathbb{R}$ is equipped with the usual topology, then $\text{int}_{\mathbb{R}}([0,1]) = (0,1)$.

Example 8. The set $\text{int}_{\mathbb{R}}(A \cup B)$ need *not* be equal to $\text{int}_{\mathbb{R}}(A) \cup \text{int}_{\mathbb{R}}(B)$: If $\mathbb{R}$ is equipped with the usual topology, then $\text{int}_{\mathbb{R}}([0,1] \cup [1,2]) = (0,2)$ but $\text{int}_{\mathbb{R}}([0,1]) \cup \text{int}_{\mathbb{R}}([1,2]) = (0,1) \cup (0,2)$.

4.4 The interior viewed as an operator on $\mathscr{P}(S)$.

Just as for closures of sets we can view "int_S" as a function, $I : \mathscr{P}(S) \to \mathscr{P}(S)$.

Definition 4.6.1 Let S be a non-empty set and $I : \mathscr{P}(S) \to \mathscr{P}(S)$ be a function satisfying the properties:

- I1. $I(S) = S$
- I2. $I(A) \subseteq A$, for all $A \subset S$,
- I3. $I(I(A)) = I(A)$, for all $A \subset S$,
- I4. $I(A \cap B) = I(A) \cap I(B)$, for all $A, B \in \mathscr{P}(S)$ (I distributes over finite intersections.)

The function $I : \mathscr{P}(S) \to \mathscr{P}(S)$ satisfying the listed properties is called an *interior operator*. We refer to I1 to I4 as being the *interior operator axioms.*

Topologizing a set S by using an interior operator. Again, just as for the closure operator, the definition of the function, $I : \mathscr{P}(S) \to \mathscr{P}(S)$, doesn't refer to any topology on S. But we will show that the function, I, can be used to define a topology on S by choosing appropriate sets in its range.

Theorem 4.7 Let S be a set and suppose $I : \mathscr{P}(S) \to \mathscr{P}(S)$ satisfies the four *interior operator axioms.* Define $\mathscr{U} \subseteq \mathscr{P}(S)$ as

$$\mathscr{U} = \{A \subseteq S : I(A) = A\}$$

Then,

a) The set $\mathscr{U}$ forms a topology on S.

b) Furthermore, if S is equipped with topology $\mathscr{U}$, $\text{int}_S(A) = I(A)$, for any $A \subseteq S$.

Proof: Let S be a set and $I : \mathscr{P}(S) \to \mathscr{P}(S)$ be a function satisfying the four *interior operator axioms* I1 to I4 listed above. Let $\mathscr{U} = \{A \in \mathscr{P}(S) : I(A) = A\}$.

a) We are required to prove that $\mathscr{U}$ forms a topology on S.

We see that:

- By I1, $I(S) = S$, so $S \in \mathscr{U}$. By I2, $I(\varnothing) \subseteq \varnothing$. Since $\varnothing \subseteq I(\varnothing)$, then $I(\varnothing) = \varnothing$ and so $\varnothing \in \mathscr{U}$
- Suppose A and B belong to $\mathscr{U}$. By property I4, $I(A \cap B) = I(A) \cap I(B) = A \cap B$. So $\mathscr{U}$ is closed under finite intersections.
- To show that $\mathscr{U}$ is closed under arbitrary unions we first verify that $(\ A \subseteq B\ ) \Rightarrow (\ I(A) \subseteq I(B)\ ) \quad (*)$

$$\begin{aligned} A \subseteq B &\Rightarrow A = B \cap A \\ &\Rightarrow I(A) = I(B \cap A) = I(B) \cap I(A) \\ &\Rightarrow I(A) \subseteq I(B) \end{aligned}$$

Let $\{A_i\}_{i \in M}$ be a collection of sets in $\mathscr{U}$. It suffices to show $I(\cup\{A_i\}_{i\in M}) = \cup\{A_i\}_{i\in M}$.

$$\begin{aligned} I(A_i) \subseteq \cup\{I(A_i)\}_{i\in M} &\Rightarrow I(I(A_i)) \subseteq I(\cup\{I(A_i)_{i\in M}) \quad \text{(By *.)} \\ &\Rightarrow I(A_i) \subseteq I(\cup\{I(A_i)\}_{i\in M}) \quad \text{(Since } A_i \in \mathscr{U} \text{ for all } i \in M.) \\ &\Rightarrow \cup\{I(A_i)\}_{i\in M} \subseteq I(\cup\{I(A_i)\}_{i\in M}) \end{aligned}$$

By I2, $I(\cup\{I(A_i)\}_{i\in M}) \subseteq \cup\{I(A_i)\}_{i\in M}$. Then $I(\cup\{I(A_i)\}_{i\in M}) = \cup\{I(A_i)\}_{i\in M}$ so $\cup\{I(A_i)\}_{i\in M} \in \mathscr{U}$.

Then set $\mathscr{U}$ satisfies the three open set axioms O1, O2, and O3. So $\mathscr{U}$ is a topology on S, as required.

We will denote the topology $\mathscr{U}$ on S induced by the operator I, by τ_I.

b) We are now required to show that $I(A) = \text{int}_S(A)$ with respect to τ_I.
Suppose $A \subseteq S$.

By I3, $I(I(A)) = I(A)$, so $I(A) \in \mathscr{U} = \tau_I$; so $I(A)$ is open. Since, $\text{int}_S(A)$ is the largest open subset of S contained in A, and $I(A) \subseteq A$ (by I2),

$$I(A) \subseteq \text{int}_S(A)$$

Also see that, since $\text{int}_S(A) \in \tau_I = \mathscr{U}$ and $\text{int}_S(A) \subseteq A$,

$$\begin{aligned} \text{int}_S(A) &= I(\text{int}_S(A)) \\ &\subseteq I(A) \quad \text{(By * above } A \subseteq B \Rightarrow I(A) \subseteq I(B)) \end{aligned}$$

then $\text{int}_S(A) \subseteq I(A)$.

We conclude that $I(A) = \text{int}_S(A)$.

We provide a few examples.

Example 9. Let $I : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ be a function defined as $I(\mathbb{R}) = \mathbb{R}$ and, if $A \neq \mathbb{R}$, $I(A) = A \backslash \mathbb{Q}$ (the set of all numbers in A that are not rationals).

a) Show that $I : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$, thus defined, is an interior operator on $\mathscr{P}(\mathbb{R})$.

b) Use the interior operator described in part a) to define a topology, τ_I, on $\mathbb{R}$.

c) For the topology, τ_I, on $\mathbb{R}$ shown in part b), describe the open subsets, the closed subsets of $\mathbb{R}$, the closure of sets and the interior of sets.

d) If $\mathscr{F}$ represents the set of all closed subsets with respect to the topology, τ_I, on $\mathbb{R}$, show that $\mathscr{B} = \tau_I \cup \mathscr{F}$ is the smallest σ-ring containing τ_I and so is a family of Borel sets.

Solution.

a) We show I is an interior operator.

1. By definition, $I(\mathbb{R}) = \mathbb{R}$ so I1 is satisfied.
2. Also, if $A \neq \mathbb{R}$, $I(A) = A \backslash \mathbb{Q} \subseteq A$. So I2 is satisfied.
3. If $A \neq \mathbb{R}$

$$\begin{aligned} I(I(A)) &= I(A) \backslash \mathbb{Q} \\ &= (A \backslash \mathbb{Q}) \backslash \mathbb{Q} \\ &= A \backslash \mathbb{Q} \\ &= I(A) \end{aligned}$$

So $I(I(A)) = A$. So I3 is satisfied.

4. If neither A nor B is $\mathbb{R}$,

$$\begin{aligned} I(A \cap B) &= (A \cap B) \backslash \mathbb{Q} \\ &= (A \backslash \mathbb{Q}) \cap (B \backslash \mathbb{Q}) \\ &= I(A) \cap I(B) \end{aligned}$$

So $I(A \cap B) = I(A) \cap I(B)$. Then I4 is satisfied.

This means that $I : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ is an interior operator.

b) We now use the interior operator described in part a) to topologize $\mathbb{R}$.

Since $I : \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ has been shown to be an interior operator then

$$\begin{aligned}\tau_I &= \{A : I(A) = A\} \\ &= \{A : A\backslash\mathbb{Q} = A\} \cup \{\mathbb{R}\} \\ &= \{A : A \text{ does not contain any points of } \mathbb{Q}\} \cup \{\mathbb{R}\}\end{aligned}$$

is a topology on $\mathbb{R}$.

c) For the topology, τ_I, on $\mathbb{R}$ we now describe the open subsets, the closed subsets of $\mathbb{R}$, the closure of sets and the interior of sets .

Open sets in $\mathbb{R}$. Open sets in $\mathbb{R}$, are $\mathbb{R}$ itself and all sets which do not contain any rationals, including $\varnothing$.

For example, if $\mathbb{J}$ is the set of irrationals and $r \in \mathbb{J}$, then, since $\{r\}$ contains no rationals, $\{r\}$ is an open singleton set. Also, if $q \in \mathbb{Q}$, $\mathbb{R}$ is the only open set containing q. So $\{q\}$ is *not* an open singleton set.

Closed sets in $\mathbb{R}$. Suppose B is not $\mathbb{R}$. We claim that B is closed in $\mathbb{R}$ with respect to τ_I if and only if $\mathbb{Q} \subseteq B$:

$$\begin{aligned}\mathbb{Q} \subseteq B &\Leftrightarrow \mathbb{R}\backslash B = (\mathbb{R}\backslash B)\backslash\mathbb{Q} \\ &\Leftrightarrow \mathbb{R}\backslash B = I(\mathbb{R}\backslash B) \\ &\Leftrightarrow \mathbb{R}\backslash B \text{ is open } {\scriptstyle(\text{with respect to } \tau_I)} \\ &\Leftrightarrow B \text{ is closed } {\scriptstyle(\text{with respect to } \tau_I)}\end{aligned}$$

For example, if $r \in \mathbb{J}$, since $\mathbb{Q} \not\subseteq \{r\}$, the singleton set, $\{r\}$, is *not* a closed set.

Closure of a set . Then taking the closure of a subset A of $\mathbb{R}$ comes down to adding all of $\mathbb{Q}$ to A. That is, if $A \neq \mathbb{R}$, $\text{cl}_{\mathbb{R}} A = A \cup \mathbb{Q}$. For example, if $q \in \mathbb{Q}$, since $\text{cl}_{\mathbb{R}}\{q\} = \mathbb{Q}$, $\{q\}$ is *not* closed.

Interior of a set in $\mathbb{R}$. $\text{Int}_{\mathbb{R}} A = A \cap \mathbb{J}$. Finding the interior of A comes down to removing any trace of $\mathbb{Q}$ in A.

d) *The set, $\mathscr{B} = \tau_I \cup \mathscr{F}$, is the smallest σ-ring containing τ_I and so is a family of Borel sets of* $\mathbb{R}$.

Given: $\mathscr{F}$ is the set of all closed subsets in $\mathbb{R}$ with respect to τ_I. We claim that $\mathscr{B}$ is a σ-ring.

Closure of $\mathscr{B}$ under countable unions.

If $U \in \tau_I$ and $V \in \mathscr{F}$ then $U \cup V \in \mathscr{F}$ (since an open subset union a closed subset containing $\mathbb{Q}$ contains $\mathbb{Q}$. So $U \cup V$ is a closed subset.)

The set $\mathscr{F}$ is closed under countable unions (since closed subsets are those subsets of $\mathbb{R}$ which contain all of $\mathbb{Q}$, arbitrary unions of elements of $\mathscr{F}$ are closed with respect to τ_I). This actually show that $\mathscr{F}$ contains all its F_σ's. Since all F_σ's are closed then all G_δ's are open and so belong to τ_I.

Trivially, τ_I is closed under arbitrary unions and so is closed under countable unions.

Closure of $\mathscr{B}$ under "complements".

Let $U \in \mathscr{B}$. If $U \in \tau_I$ then $\mathbb{R}\backslash U \in \mathscr{F}$. If $U \in \mathscr{F}$ then $\mathbb{R}\backslash U \in \tau_I$.

So $\mathscr{B}$ is a σ-ring, as claimed. Since it must contains τ_I and all complements it is the smallest σ-ring containing τ_I. By definition, it is a family of Borel sets of $\mathbb{R}$ with respect to τ_I.

Remark. Note that, if T is the closed interval $[1, 3]$ in $\mathbb{R}$ then $T \notin \tau_I \cup \mathscr{F}$ (since $[1, 3]$ contains some elements of $\mathbb{Q}$, but does not contain all of $\mathbb{Q}$) and so is not an element of the unique family, $\mathscr{B}$, of all Borel sets, with respect to τ_I. We can then refer to it as a "*non-Borel set*".

4.5 Boundary of a set.

We have seen that, for any subset, A, of a topological space (S, τ),

$$\text{int}_S(A) \subseteq A \subseteq \text{cl}_S(A)$$

Often, $A\backslash\text{int}_S(A) \neq \varnothing$ and $\text{cl}_S(A)\backslash A \neq \varnothing$. We will now briefly discuss the sets whose points belong to $\text{cl}_S(A)\backslash\text{int}_S(A)$.

Definition 4.8 Let A be a subset of a topological space (S, τ). We define the *boundary of A*, denoted as $\text{bd}_S(A)$, as

$$\text{bd}_S(A) = \text{cl}_S(A)\backslash\text{int}_S(A)$$

The expressions, $\text{Fr}_S(A)$, $\partial_S(A)$, $\text{Bd}_S(A)$ are also often used to represent the boundary of A. It is easily verified that

$$\text{bd}_S(A) = \text{cl}_S(A) \cap \text{cl}_S(S\backslash\text{int}_S A)$$

Since the finite intersection of closed sets is closed, we see that the boundary, $\text{bd}_S(A)$ of a set A, is always closed. Furthermore, for any set A in S, both A and $S\backslash A$ share the same boundary (like two adjacent neighbours who share the same fence). It is always the case that $\mathrm{int}_S(A) \cap \mathrm{bd}_S(A) = \varnothing$ and that $\mathrm{int}_S(A)$, $\mathrm{bd}_S(A)$ and $\mathrm{int}_S(\mathrm{cl}_S(S \setminus A))$ are pairwise disjoint sets whose union is S. The reader is left to verify this.

Example 10. If $\mathbb{Q}$ is viewed as a subset of $\mathbb{R}$ equipped with the usual topology then

$$\begin{aligned} \mathrm{bd}_{\mathbb{R}}(\mathbb{Q}) &= \mathrm{cl}_{\mathbb{R}}(\mathbb{Q}) \setminus \mathrm{int}_{\mathbb{R}}(\mathbb{Q}) \\ &= \mathbb{R} \setminus \varnothing \\ &= \mathbb{R} \end{aligned}$$

Example 11. If $B = [0, 1]$ is a closed interval viewed as a subset of $\mathbb{R}$ equipped with the usual topology then $\mathrm{bd}_{\mathbb{R}}(B) = \{0, 1\}$. It is left to the reader to verify this.

Example 12. Let $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be a subset of $\mathbb{R}^2$ equipped with the topology induced by the Euclidean metric. Then

$$\mathrm{bd}_{\mathbb{R}^2}(B) = \mathrm{cl}_{\mathbb{R}^2}(B) \setminus \mathrm{int}_{\mathbb{R}^2}(B) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

The reader is left to verify the details.

Example 13. Suppose the set of natural numbers $\mathbb{N}$ is equipped with the cofinite topology.[1] Let E denote the set of all even natural numbers. Then

$$\begin{aligned} \mathrm{bd}_{\mathbb{N}}(E) &= \mathrm{cl}_{\mathbb{N}}(E) \setminus \mathrm{int}_{\mathbb{N}}(E) \\ &= \mathbb{N} \setminus \varnothing \\ &= \mathbb{N} \end{aligned}$$

Example 14. Suppose $B = (0, 1) \cup (1, 2]$ viewed as a subset of $\mathbb{R}$ equipped with the usual topology. We compute the boundary to be,

$$\begin{aligned} \mathrm{bd}_{\mathbb{R}}(B) &= \mathrm{cl}_{\mathbb{R}}(B) \setminus \mathrm{int}_{\mathbb{R}}(B) \\ &= [0, 2] \setminus (0, 1) \cup (1, 2) \\ &= \{0, 2\} \end{aligned}$$

It is interesting to note that

$$\begin{aligned} \mathrm{int}_{\mathbb{R}}(\mathrm{cl}_{\mathbb{R}}(B)) &= \mathrm{int}_{\mathbb{R}}([0, 2]) \\ &= (0, 2) \neq B \end{aligned}$$

---

[1] The open subsets are those whose complement is finite.

4.6 Dense subsets of a topological space.

Some topological spaces (S, τ) have particular subsets, A, which occupy so much space in S that it's complement, $X \backslash A$, cannot entirely contain any non-empty open subset. That is, $X \backslash A$ contains, at most, boundary points of A. We refer to such subsets as being "dense" in X. We define this formally.

Definition 4.9 Suppose A and B are subsets in a topological space (S, τ). If $A \subseteq B$ and $B \subseteq \text{cl}_S(A)$, then we will say that A *is a dense subset of* B. In the case where $B = S$, then A is dense in S if and only if $\text{cl}_S A = S$. If $A \subseteq S$ is such that $\text{int}_S \text{cl}_S A = \varnothing$ then we say that A is *nowhere dense in* S.

Example 15. Suppose

$$A = \{(x, y) : x^2 + y^2 < 1\} \text{ and } B = A \cup \{(0, 1), (1, 0), (0, -1), (-1, 0)\}$$

Since

$$B \subset \text{ cl}_{\mathbb{R}^2} A = \{(x, y) : x^2 + y^2 \leq 1\}$$

then A is dense in B.

Example 16. *A nowhere dense subset.* Let $T = [-2, 5] \cup \{6\}$ be a subspace of $(\mathbb{R}, \tau)$ equipped with the usual topology, τ. Let $A = \{6\}$ be a subset of T. Then $\text{cl}_T A = \{6\}$ and $\text{int}_T \text{cl}_T A = \varnothing$. So A is nowhere dense in the subspace T. Another example: The set of all integers, $\mathbb{Z}$, is nowhere dense in $\mathbb{R}$, since $\text{int}_{\mathbb{R}} \text{cl}_{\mathbb{R}} \mathbb{Z} = \varnothing$.

Example 17. Verify that the set $C = \{(x, y) : x^2 + y^2 = 1\}$ is nowhere dense in $\mathbb{R}^2$.

The property in the following definition is one which refers to an upper bound for the cardinality of a dense subset of a topological space. Hence, in a way it expresses a restriction on its size.

Definition 4.10 We will say that a topological space (S, τ) is *separable* if and only if S contains a countable dense subset.

For example, the topological space, $(\mathbb{R}, \tau)$, equipped with the usual topology is *a separable space* since, $\mathrm{cl}_{\mathbb{R}}\mathbb{Q} = \mathbb{R}$, and the subset of all rational numbers is a countable subset of $\mathbb{R}$. Notice how the border of $\mathbb{Q}$ is much larger than $\mathbb{Q}$ itself.

4.7 Topic: Regular open sets and regular closed sets.

Suppose A is a non-empty open subset of a topological space S. We know that $A \subseteq \mathrm{cl}_S A$ is always true. Also, it is always true that, if A is open, $A = \mathrm{int}_S A \subseteq \mathrm{int}_S \mathrm{cl}_S A$ (by part b) of theorem 4.6). So for all open subsets, A, of a space, it is always true that

$$A \subseteq \mathrm{int}_S \mathrm{cl}_S A$$

One may wonder whether, in the case where A is open, we must have equality between A and $\mathrm{int}_{\mathbb{R}} \mathrm{cl}_{\mathbb{R}} A$. Let's consider the following simple example.

Suppose $A = (1, 3) \cup (3, 7)$ of $\mathbb{R}$. Then by applying the above reasoning we have,

$$A = (1, 3) \cup (3, 7) \subseteq \mathrm{int}_{\mathbb{R}} \mathrm{cl}_{\mathbb{R}} A = \mathrm{int}_{\mathbb{R}}[1, 7] = (1, 7)$$

So in the case where $A = (1, 7)$ we do have equality; but if $A = (1, 3) \cup (3, 7)$ we have an open subset, A, such that $A \neq \mathrm{int}_{\mathbb{R}} \mathrm{cl}_{\mathbb{R}} A$. Then the best we can then do is affirm that A is always a subset of $\mathrm{int}_S \mathrm{cl}_S A$, and sometimes is a proper subset. We have a special name for those open subsets where equality holds.

Definition 4.11 An open subset, A, of a topological space (S, τ) is called a *regular open subset of* S if and only if $A = \mathrm{int}_S \mathrm{cl}_S A$. In this book, we denote the set of all regular open subsets of S by $\mathscr{R}o(S)$

If $A \subset \mathrm{int}_S \mathrm{cl}_S A$ then $S \setminus (\mathrm{int}_S \mathrm{cl}_S A) \subset \mathrm{cl}_S \, \mathrm{int}_S (S \setminus A)$. Then if A is regular open $S \setminus (\mathrm{int}_S \mathrm{cl}_S A) = S \setminus A$ and $\mathrm{cl}_S \, \mathrm{int}_S (S \setminus A)$ are equal closed subsets. We can then say that if F is the complement of a regular open subset of S,

$$F = \mathrm{cl}_S \, \mathrm{int}_S F$$

We also have a name for the complements of regular open subsets.

Definition 4.12 A closed subset, F, of a topological space (S, τ) is called a *regular closed subset of* S if and only if $F = \mathrm{cl}_S \, \mathrm{int}_S F$. Hence the regular closed subsets of S are precisely the complements of the regular open subsets. We denote the set of all regular closed subsets of S by $\mathscr{R}(S)$

Example 18. Let S be a topological space and $\mathscr{R}o(S)$ be the set of all regular open sets in S.

a) Verify that $\mathscr{R}o(S)$ is closed under finite intersections but is not closed under finite unions.

b) We know that, if $A = \text{int}_S\text{cl}_S A$, then A belongs to $\mathscr{R}o(S)$. Suppose A is some non-empty subset of S which does not belong to $\mathscr{R}o(S)$. Verify that $\text{int}_S\text{cl}_S A$ belongs to $\mathscr{R}o(S)$.

c) Verify that $\mathscr{R}o(S)$ has as minimal element, $\varnothing$. Then show that every subset $\mathscr{U}$ of $\mathscr{R}o(S)$ of has a *smallest maximal element* with respect the inclusion partial ordering, $\subseteq$. That is, show that $\mathscr{R}o(S)$ contains a smallest regular open set, W, which contains every element of $\mathscr{U}$.

d) If $\mathscr{R}(S)$ be the set of all regular closed sets in S, show that $\mathscr{R}(S)$ is closed under finite unions.

Solution : Given: $\mathscr{R}o(S)$ is the set of all regular open sets in S.

a) Suppose $A = \text{int}_S\text{cl}_S A$ and $B = \text{int}_S\text{cl}_S B$. Then

$$\begin{aligned} \text{int}_S\text{cl}_S(A \cap B) &\subseteq \text{int}_S(\text{cl}_S A \cap \text{cl}_S B) \quad \text{By theorem 4.6} \\ &= \text{int}_S\text{cl}_S A \cap \text{int}_S\text{cl}_S B \\ &= A \cap B \end{aligned}$$

Since $A \cap B \subseteq \text{int}_S\text{cl}_S(A \cap B)$ then $A \cap B$ is regular open.

On the other hand, $(1, 5)$ and $(5, 9)$ are both regular open but $(0, 5) \cup (5, 9)$ is not.

b) We begin with supposing that F is closed in S. Then $\text{int}_S\text{cl}_S(\text{int}_S F) = \text{int}_S\text{cl}_S F = \text{int}_S F$ hence $\text{int}_S F$ is regular open. Let A be any non-empty subset of S. We know that $\text{cl}_S A$ is closed in S. Replacing F with $\text{cl}_S A$ in the above argument allows us to conclude that $\text{int}_S\text{cl}_S A$ is regular open, as required.

c) Since $\text{int}_S\text{cl}_S\varnothing = \text{int}_S\varnothing = \varnothing$ then $\varnothing$ is the smallest regular open set with respect to $\subseteq$. Suppose $\mathscr{U}$ is any non-empty subset of $\mathscr{R}o(S)$. We are required to show that $\mathscr{R}o(S)$ contains a smallest regular open set, W, which contains every element of $\mathscr{U}$.

Let $V = \cup\{B : B \in \mathscr{U}\}$ then $V \subseteq \text{int}_S\text{cl}_S V$. So $\text{int}_S\text{cl}_S V$ is a regular open set, (not necessarily equal to V), containing every element of $\mathscr{U}$. We claim that $\text{int}_S\text{cl}_S V$ is the smallest such regular open set. Suppose there is a regular open set, M, containing every element of $\mathscr{U}$. Then $V \subseteq M$. Then $\text{int}_S\text{cl}_S V \subseteq \text{int}_S\text{cl}_S M = M$. This establishes the claim.

d) This part is left an exercise.

Concepts review:

1. Given a topological space (S, τ) and $T \subseteq S$, define the closure, $\mathrm{cl}_S(T)$, in S, with respect to the topology τ.
2. Does cl_S "distribute" over finite unions? How about finite intersections?
3. List the four Kuratowski closure operator axioms, K1 to K4.
4. If $K : \mathscr{P}(S) \to \mathscr{P}(S)$ satisfies the four Kuratowski closure operator axioms describe a topology on S which can be constructed from K.
5. Define the cocountable topology on $\mathbb{R}^2$. Describe the Kuratowski operator used to develop this topology.
6. Given a topological space (S, τ) and $A \subseteq S$, define "interior point" of A with respect to τ. Define the interior, $\mathrm{int}_S(A)$, of A with respect to τ.
7. Does int_S distribute over finite unions? How about finite intersections?
8. State the four interior operator axioms I1 to I4 for $I : \mathscr{P}(S) \to \mathscr{P}(S)$.
9. Given a set S, describe a topology that can be constructed on S by using the operator $I : \mathscr{P}(S) \to \mathscr{P}(S)$.
10. Describe the topology induced on $\mathbb{R}$ by the operator $I(A) = A \backslash \mathbb{Q}$.
11. Define the boundary of a set with respect to a topology τ on S.
12. Define what we mean when we say that "A is dense in B".
13. Define what we mean when we say that "A is nowhere dense in B".
14. Define a regular open subset and a regular closed subset of a space.
15. Show that for any subset U of S, $\mathrm{int}_S\mathrm{cl}_S U$ is regular open in S.

EXERCISES

1. Let $T = (0, 1)$, viewed as a subset of $\mathbb{R}$ equipped with the usual topology. Show that $\mathrm{cl}_{\mathbb{R}}(T) = [0, 1]$. This is Example 1 on page 47.

2. Suppose S is a topological space induced by the metric ρ (that is, the elements of τ are unions of open balls of the form $B_\varepsilon(x) = \{y : \rho(x, y) < \varepsilon\}$). Suppose F is a non-empty subset of S. We define $\rho(x, F) = \inf\{\rho(x, u) : u \in F\}$. Show that $\mathrm{cl}_S(F) = \{x : \rho(x, F) = 0\}$. This is Example 3 on page 47.

3. Consider $\mathbb{R}$ equipped with the usual topology τ (induced by the Euclidian metric). Let $(\mathbb{Q}, \tau_{\mathbb{Q}})$ be the set of all rational numbers equipped with the subspace topology inherited from $\mathbb{R}$. Consider the subset $T = [-\pi, \pi) \cap \mathbb{Q}$. Determine whether T is open in $\mathbb{Q}$, closed in $\mathbb{Q}$, both open and closed in $\mathbb{Q}$, or none of these.

4. In Example 4 on page 49 it is shown that $\mathrm{cl}_S(A) \cap \mathrm{cl}_S(B) \neq \mathrm{cl}_S(A \cap B)$ sometimes occurs. Show that $\mathrm{cl}_S(A \cap B) \subseteq \mathrm{cl}_S(A) \cap \mathrm{cl}_S(B)$ is always true.

5. Let (S, τ) be a topological space and A and B be subsets of S. Show that:

   a) $\mathrm{int}_S(A)$ is open in S. Also, $\mathrm{int}_S(A)$ is the largest open subset of S which is entirely contained in A.
   b) If $B \subseteq A$ then $\mathrm{int}_S(B) \subseteq \mathrm{int}_S(A)$.
   c) $\mathrm{int}_S(A \cap B) = \mathrm{int}_S(A) \cap \mathrm{int}_S(B)$.
   d) $\mathrm{int}_S(\mathrm{int}_S(A)) = \mathrm{int}_S(A)$.

   (This is theorem 4.6. Part c) is already proven.)

6. Let S be a set and $I : \mathscr{P}(S) \to \mathscr{P}(S)$ be an interior operator satisfying the four conditions I1 to I4 listed on page 54. Show that, if $\tau = \{A \in \mathscr{P}(S) : I(A) = A\}$, (S, τ) is a topological space such that $I(A) = \mathrm{int}_S(A)$ for all $A \in \mathscr{P}(S)$.

7. If $B = [0, 1]$ is a closed interval viewed as a subset of $\mathbb{R}$ equipped with the usual topology show that $\mathrm{bd}_{\mathbb{R}}(B) = \{0, 1\}$.

8. Let $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, x > 0\} \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, x \leq 0\}$ be a subset of $\mathbb{R}^2$ equipped with the topology induced by the Euclidean metric. Show that

$$\mathrm{bd}_{\mathbb{R}^2}(B) = \mathrm{cl}_{\mathbb{R}^2}(B) \backslash \mathrm{int}_{\mathbb{R}^2}(B) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

9. Show that A is both open and closed in the topological space (S, τ) if and only if $\mathrm{bd}_S(A)$ is empty.

10. Let A and B be two subsets of the topological space (S, τ). Show that if $\mathrm{bd}_S(A) \cap \mathrm{bd}_S(B) = \varnothing$ then $\mathrm{int}_S(A \cup B) = \mathrm{int}_S(A) \cup \mathrm{int}_S(B)$.

11. Let $B = [0, 7) \cup \{9\}$ where B is equipped with the subspace topology inherited from $\mathbb{R}$ itself equipped with the usual topology. If $A = \{0\} \cup \{3\} \cup (5, 7)$, find i) $\mathrm{cl}_B(A)$, ii) $\mathrm{bd}_B(A)$, iii) $\mathrm{int}_B(A)$.

---

5 / Bases of topological spaces.

Summary. *In this section we define a neighbourhood system of $x \in S$ with respect to a given topology on S. Given a topology, τ, on S, we define a "base for the topology τ". We deduce a set-theoretic property called the "base property" possessed by any base. Those subsets of $\mathscr{P}(S)$ which satisfy the described property will be shown to be a base for some topology on S. We introduce the notion of a "subbase for the topology τ" by describing its properties. We then show how to topologize a set both from a collection of sets which possesses the 'base property" and also from an arbitrary collection of subsets.*

5.1 Neighbourhoods of points.

It is not always easy to confirm that a given subset, $\mathscr{S}$, of $\mathscr{P}(S)$ is a well-defined topology on a set S. Ultimately, we would prefer a topology on S to be described in a way that provides some intuitive idea about what its open subsets are like. One way of representing a topology, τ on S, is to form subsets of smaller sets in τ whose elements are described in reference to points of S. With this in mind, we introduce the notion of a *neighbourhood system* and *neighbourhood base.*

Definition 5.1 Let (S, τ) be a topological space and $x \in A \subseteq S$. We will say that A is a *neighbourhood of x with respect to τ* if $x \in \text{int}_S(A)$. For a given $x \in S$, the set

$$\mathscr{U}_x = \{A \in \mathscr{P}(S) : A \text{ is a neighbourhood of } x\}$$

is called a *neighbourhood system of x with respect to τ*. A subset, $\mathscr{B}_x$, of a neighbourhood system, $\mathscr{U}_x$, such that, for any open set, A, containing x there exists $U_x \in \mathscr{B}_x$ such that $x \in U_x \subseteq A$, is called a *neighbourhood base at x.*

Note that, since its definition refers to the "interior" of sets, a *neighbourhood system* is always expressed with respect to some topology, τ, defined on the whole set S. Also, see that we define two concepts in the above statement. We see that a neighbourhood base, $\mathscr{B}_x$, at x is a subset of a neighbourhood system at x. The set, $\mathscr{B}_x$, must be such that, any open neighbourhood of x, contains an element of $\mathscr{B}_x$. While there can be only one neighbourhood system at x we will see that there can be more than one neighbourhood base at this point.

For example, $B = [-1, 5) \cup [6, \infty)$ is a neighbourhood of 0 with respect to the usual topology of $\mathbb{R}$. The set

$$\mathscr{U}_3 = \{U \in \mathscr{P}(\mathbb{R}) : 3 \in \text{int}_{\mathbb{R}}(U)\}$$

is a neighbourhood system of 3. If A belongs to $\mathscr{U}_3$, the element "3" must belong to the interior of A in the sense that 3 cannot be sitting on A's boundary. For example, $[0,4] \in \mathscr{U}_3$, but $[1,3] \notin \mathscr{U}_3$. Notice that, for a given $x \in S$, its neighbourhood system, $\mathscr{U}_x$, with respect to τ is, by definition, unique.

Based on the definition, we can make the following comments about neighbourhoods of a point in a topological space (S, τ).

- The empty set, $\varnothing$, is not the neighbourhood of any point, so $\varnothing \notin \mathscr{U}_x$ for any $x \in S$.
- In general, a neighbourhood, U, of a point x is not necessarily open, but it must contain x in its interior, $\text{int}_S(U)$.
- If $x \in S$, then $\mathscr{U}_x$ is not empty since S is a neighbourhood of x. If τ is the indiscrete topology, $\{\varnothing, S\}$, S is the only neighbourhood of every point $x \in S$.

The "neighbourhood of a point" concept allows us to come up with another characterization of open sets provided we have predefined "interior of a set". We propose:

$$\text{“} [\ A \text{ is open in } S\ ] \Leftrightarrow [\ x \in A \Rightarrow \exists \text{ neighbourhood, } U_x, \text{ where } x \in U_x \subseteq A\ ]\text{”}$$

This will work since this would imply that

$$A = \cup_{x \in A} \{\text{int}_S(U_x) : U_x \in \mathscr{U}_x \text{ and } U_x \subseteq A\}$$

Or, we could simply say that "A is open in S provided A contains an x-neighbourhood, U_x, for each point x in A". This definition of "open set" (in terms of neighbourhoods) is equivalent to its formal definition of "open set".

5.2 A base for a topology.

In our study of normed vector spaces, S, (as well as of metric spaces) we have seen that, introducing the notion of "open ball, $B_\varepsilon(x)$, center x with radius ϵ", led to a convenient way of constructing a subfamily, $\mathscr{B}_x = \{B_\varepsilon(x) : \varepsilon > 0\}$, of open neighbourhoods of a point x. Every open subset of S can then simply be described as being the union of open neighbourhoods of the form $B_\varepsilon(x)$. The collection of open sets, $\mathscr{B} = \cup\{\mathscr{B}_x : x \in S\}$, is sufficient to generate all open subsets of S. We will now seek a subset, $\mathscr{B}$, of τ which will be sufficient to generate, by itself, every element of τ.

Definition 5.2 Let (S, τ) be a topological space. Suppose $\mathscr{B}$ is a subset of τ satisfying the property:

"For any $A \in \tau$, A is the union of elements of a subset, $\mathscr{C}$, of $\mathscr{B}$."

We call the subset, $\mathscr{B}$, a *base for open sets* or a *base for the topology* τ (or a *basis for the topology* τ).[1] The elements of a base are referred to as *basic open sets*.[2]

Suppose $\mathscr{F}$ is a family of closed subsets of S satisfying the property:

> "For any closed subset B in S, B is the intersection of elements of a subset, $\mathscr{C}$, of $\mathscr{F}$."

We call the subset, $\mathscr{F}$, a *base for closed sets of* S.

A base is generally not unique. Given the topological space $(\mathbb{R}, \tau)$, where τ is the usual topology, both τ and $\mathscr{B} = \{(a, b) : a < b\}$ are bases for $\mathbb{R}$.

It is easily verified that if $\mathscr{F}$ is a base for closed subsets of S then the family, $\mathscr{B}$, of all complements of the elements of $\mathscr{F}$ will form a base for open sets.

How does one go about constructing a useful base for a topology? A good way to start is to establish some properties possessed by a useful open base. The following theorem characterizes a base for a topology on a set S.

Theorem 5.3 Let (S, τ) be a topological space and $\mathscr{B} \subset \tau$. Then the following are equivalent:

1. The family $\mathscr{B}$ is a base for τ.
2. For any $x \in S$, $\mathscr{B}$ contains a neighbourhood base, $\mathscr{B}_x$, of open sets.
3. Whenever $x \in U \in \tau$, there exists $B \in \mathscr{B}$ such that $x \in B \subseteq U$.

Proof: We are given that (S, τ) is a topological space and $\mathscr{B} \subset \tau$.

$(1 \Rightarrow 2)$ Suppose the set $\mathscr{B}$ is a base for a topology τ on S and $x \in S$. Let

$$\mathscr{B}_x = \{B \in \mathscr{B} : x \in B\}$$

Suppose A is an open subset of S which contains x. It suffices to show that $\mathscr{B}_x$ contains an open neighbourhood U of x such that $x \in U \subseteq A$. By hypothesis, $A = \cup\{B : B \in \mathscr{B}\}$. Then, there exists some $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq A$. By definition, $B_x \in \mathscr{B}_x$. So $\mathscr{B}_x$ is an open neighbourhood base of x contained in $\mathscr{B}$.

$(2 \Rightarrow 3)$ This follows immediately from the definition of "neighbourhood base".

$(3 \Rightarrow 1)$ Suppose $A \in \tau$. If A is empty then A is the union of all open sets in $\varnothing \subset \mathscr{B}$. Suppose $x \in A$. By hypothesis, there exists $B \in \mathscr{B}$ such that $x \in B \subseteq A$. Then A is the union of sets from $\mathscr{B}$. So $\mathscr{B}$ is a base for open subsets of S.

[1] Both words "base" and "basis" are commonly used; thus the reader can assume these have the same meaning.

[2] Note that, if $A = \varnothing \in \tau$, then A is the union of all elements from $\mathscr{C} = \varnothing = \{\ \} \subseteq \mathscr{B}$. So $\mathscr{B}$ also generates the empty set.

5.3 The "base property".

We have seen how an arbitrary set can be topologized by two different techniques. One by using a *Kuratowski closure operator* the other by using an *interior operator*. These operators must satisfy certain axioms. If they do, then a particular topology on their domain can be defined. In this section, we will propose two other methods for topologizing a set. We will first discuss another characterization of a base for a topology on a set S.

Theorem 5.4 Let S be a non-empty set and $\mathscr{B} \subseteq \mathscr{P}(S)$. The set $\mathscr{B}$ is a base for a topology τ on S if and only if $S = \cup\{B : B \in \mathscr{B}\}$ and, if $x \in A \cap B$ for some $A, B \in \mathscr{B}$, then there exists $C \in \mathscr{B}$ such that $x \in C \subseteq A \cap B$.

Proof: We are given that S be a non-empty set and $\mathscr{B} \subseteq \mathscr{P}(S)$.

($\Rightarrow$) Suppose the set $\mathscr{B}$ is a base for a topology τ on S.
Since $S \in \tau$ then $S = \cup\{B : B \in \mathscr{C}\}$ where $\mathscr{C} \subseteq \mathscr{B}$. Since $\cup\{B : B \in \mathscr{B}\} \subseteq S$ then $S = \cup\{B : B \in \mathscr{B}\}$.
Suppose $x \in A \cap B$ for some $A, B \in \mathscr{B}$. Then $A, B \in \tau$ and so $A \cap B \in \tau$. Then there exists $C \in \mathscr{B}$ such that $x \in C \subseteq A \cap B$, as required.

($\Leftarrow$) We are given that $\mathscr{B} \subseteq \mathscr{P}(S)$ and satisfies the two property: $S = \cup\{B : B \in \mathscr{B}\}$ and if $x \in A \cap B$ for some $A, B \in \mathscr{B}$ then there exists $C \in \mathscr{B}$ such that $x \in C \subseteq A \cap B$.

Let $\mathscr{T} = \{A \subseteq S : A = \cup\{C : C \in \mathscr{C}\}$ for some subset $\mathscr{C}$ of $\mathscr{B}\}$.

We are required to show that $\mathscr{T}$ is a topology on S and that $\mathscr{B}$ is base for $\mathscr{T}$.
If we show that $\mathscr{T}$ is a topology on S then, by definition, $\mathscr{B}$ is base for $\mathscr{T}$.

O1 The set S belongs to $\mathscr{T}$: Since $S = \cup\{B : B \in \mathscr{B}\}$ then $S \in \mathscr{T}$. The set $\varnothing$ belongs to $\mathscr{T}$: The union of all elements in $\varnothing \subset \mathscr{B}$ is $\varnothing$. So $\varnothing \in \mathscr{T}$.

O2 *Claim*: The set $\mathscr{T}$ is closed under unions. Let $\{A_\alpha : \alpha \in \Gamma\} \subseteq \mathscr{T}$. For $\alpha \in \Gamma$, $A_\alpha = \cup\{B : B \in \mathscr{B}_\alpha \subseteq \mathscr{B}\}$. Then $\cup_{\alpha\in\Gamma}\{A_\alpha\} = \cup_{\alpha\in\Gamma}\{\cup_{B\in\mathscr{B}_\alpha}\{B\}\}$, a union of elements in $\mathscr{B}$. So $\cup_{\alpha\in\Gamma}\{A_\alpha\} \in \mathscr{T}$.

O3 *Claim*: The set $\mathscr{T}$ is closed under finite intersections. Let $A, C \in \mathscr{T}$. It suffices to show that $A \cap C \in \mathscr{T}$. Let $x \in A \cap C$. See that $A = \cup\{B : B \in \mathscr{B}_A \subseteq \mathscr{B}\}$ and $C = \cup\{B : B \in \mathscr{B}_C \subseteq \mathscr{B}\}$. There exist $B_A \in \mathscr{B}_A$ and $B_C \in \mathscr{B}_C$ such that $x \in B_A \cap B_C$. By hypothesis, there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq B_A \cap B_C \subseteq A \cap C$. Then for every $x \in A \cap C$ there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq A \cap B$. Then $A \cap B = \cup_{x\in A\cap C}\{B_x\}$; so $A \cap C \in \mathscr{T}$, as required.

So $\mathscr{T}$ is a topology on S. By definition of $\mathscr{T}$, every element of $\mathscr{T}$ is a union of elements of $\mathscr{B}$ so $\mathscr{B}$ is base of $\mathscr{T}$.

In this text, we will call the special property which is satisfied by $\mathscr{B} \subseteq \mathscr{P}(S)$ in the theorem statement, the "base property". We formally define this below.

Definition 5.4.1 Let S be a non-empty set and let $\mathscr{B}$ be a subset of $\mathscr{P}(S)$.

> *Base property* : If $S = \cup\{B : B \in \mathscr{B}\}$ and, if $x \in A \cap B$ for some $A, B \in \mathscr{B}$, then there exists $C \in \mathscr{B}$ such that $x \in C \subseteq A \cap B$.

The theorem 5.4 guarantees that, given any non-empty set S, if a subset, $\mathscr{B}$, of $\mathscr{P}(S)$ satisfies the *base property* then $\mathscr{B}$ forms a base for some topology, τ, on S. The family, τ, is made precisely of the arbitrary unions of the elements of subsets of $\mathscr{B}$. In this case, we say that $\mathscr{B}$ *generates the topology* τ. The above statement is a powerful tool for topologizing sets. The following example illustrates how the result is used to topologize a subset of $\mathbb{R}^2$.

Example 1. The *Moore plane*. Also called, *Niemytzki's topology*.
Let $S = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. The set $B_\varepsilon(a, b)$ represents the usual open ball of center, (a, b), and radius ε. Let

$$\mathscr{A} = \{B_\varepsilon(x_0, y_0) : x_0 \in \mathbb{R}, y_0 > 0, \text{ and } \varepsilon \leq y_0\}$$

That is, $\mathscr{A}$ is the set of all open balls which are entirely contained in S (and so do not meet the x-axis). For each $x \in \mathbb{R}$ and $\varepsilon > 0$, let

$$D_\varepsilon(x, 0) = \{(x, 0)\} \cup B_\varepsilon(x, \varepsilon)$$

That is, $D_\varepsilon(x, 0)$ is an open ball, $B_\varepsilon(x, \varepsilon)$, of radius ε tangent to the horizontal axis at $(x, 0)$ with the point $(x, 0)$ attached to it. Let

$$\mathscr{D} = \{D_\varepsilon(x, 0) : x \in \mathbb{R}, \varepsilon > 0\}$$

Let $\mathscr{B} = \mathscr{A} \cup \mathscr{D}$. Show that $\mathscr{B}$ is the base for some topology on S.

Solution : If we show that $\mathscr{B}$ satisfies the "base property" then $\mathscr{B}$ is a base which generates a topology, τ, on the half-plane S. It is first easily seen that $S = \cup\{B : B \in \mathscr{B}\}$.

We consider case 1: Suppose $A, B \in \mathscr{B}$ and $(x, y) \in A \cap B$ where $y > 0$. Then the situation is analogous to what occurs in $\mathbb{R}^2$ with the usual topology, and so $(x, y) \in B_\varepsilon(x, y) \subseteq A \cap B$, for some $\varepsilon > 0$.

We consider case 2: Suppose $A, B \in \mathscr{B}$ and $(x, 0) \in A \cap B$. Then $A = \{(x, 0)\} \cup B_{\varepsilon_1}(x, \varepsilon_1)$ and $B = \{(x, 0)\} \cup B_{\varepsilon_2}(x, \varepsilon_2)$. Let $\varepsilon = \frac{\min\{\varepsilon_1, \varepsilon_2\}}{2}$ and $C = \{(x, 0)\} \cup B_{\varepsilon}(x, \varepsilon)$. Then $(x, 0) \in C \subseteq A \cap B$.

By invoking theorem 5.4, we conclude that $\mathscr{B}$ is a base for a topology, τ_M, on S. This well-known topological space, (S, τ_M), is referred to as the *Moore plane*. Some authors also refer to this as *Niemytzki's tangent disc topology* .

Example 2. *The radial plane.* Consider the set $\mathbb{R}^2$. We will equip $\mathbb{R}^2$ with what is called the *radial plane topology.* If $(x, y) \in \mathbb{R}^2$, let $\mathscr{B}_{(x,y)}$ represent the set of all $B \subseteq \mathbb{R}^2$, where B is defined as follows: The point, $(x, y) \in B$, and B is the union of a set of open line segments, precisely one in each direction, each one originating at (x, y). The line segments need not be of the same length.

Show that $\mathscr{B} = \cup\{\mathscr{B}_{(x,y)} : (x, y) \in \mathbb{R}^2\}$ forms a valid base for a topology, τ^*, on $\mathbb{R}^2$. Then show that the topology , τ^*, generated by $\mathscr{B}$ is strictly stronger than the usual topology, τ.

Solution : Clearly, $\mathbb{R}^2 = \cup\{B : B \in \mathscr{B}\}$. Suppose $(x, y) \in U \cap V$ where $U, V \in \mathscr{B}_{(x,y)}$. Let K_U be a line segment in a particular direction originating at (x, y) such that $K_U \subset U$ and K_V be a line segment originating at (x, y) pointing in the same direction as K_U but $K_V \subset V$. If $K_{U\cap V} = K_U \cap K_V$ is an open line segment originating at (x, y), then $K_{U\cap V}$ is contained in $U \cap V$. This applies to open lines in all directions originating at (x, y). So there exists $M \in \mathscr{B}_{(x,y)}$ such that $M \subseteq U \cap V$. So $\mathscr{B}$ satisfies the open base property and so generates a topology, say τ^*.

We claim that the usual topology, τ, is contained in τ^*: Suppose $(x, y) \in U$ an open base element for τ. Then $U = B_\varepsilon(x, y)$. Let $(a, b) \in B_\varepsilon(x, y)$. Then there exists $\delta > 0$ such that $B_\delta(a, b) \subseteq B_\varepsilon(x, y)$. Let V be a set such that all open line segments originating at (a, b) are of length less than $\delta/2$. Then $V \in \mathscr{B}$ and $V \subseteq B_\delta(a, b) \subseteq B_\varepsilon(x, y)$. So $U \in \tau^*$. Then $\tau \subseteq \tau^*$.

We claim that τ^* contains an element which does not belong to τ: Consider the sets $U = B_1(0, 1)$ and $V = B_1(0, -1)$.[1] Then $U \cup V$ is open in the usual topology but the set,

$$S = U \cup \{(x, 0) : x \in \mathbb{R}\} \cup V$$

5.4 The subbase of a topology.

We have seen how, when given an arbitrary set S, a subfamily $\mathscr{B} \subseteq \mathscr{P}(S)$ which satisfies the "base property" can be used to topologize S. As we shall soon see, it is

[1] The set $B_1(0, 1)$ refers to the open ball center $(0, 1)$ and radius 1 while $B_1(0, -1)$ refers to the open ball center $(0, -1)$ with radius 1.

also possible to topologize S from an arbitrary sufamily family, $\mathscr{S} \subseteq \mathscr{P}(S)$, which needs not satisfy any particular property.

Definition 5.5 Let (S, τ) be a topological space. A *subbase for the topology* τ is a non-empty subfamily, $\mathscr{S}$, of τ such that the set, $\mathscr{B}$, defined as

$$\mathscr{B} = \{B : B = \cap\{U : U \in \mathscr{F}\} \text{ where } \mathscr{F} \text{ is a finite subset of } \mathscr{S}\}$$

forms a base for τ.

In the following two examples we are given a particular topology on a set. We identify a subbase for the given topology. To verify that the family of sets we have is indeed a subbase we consider finite intersections of its element to see if we obtain a base for the topology.

Example 3. Consider the sets $X_a = \{x \in \mathbb{R} : a < x\}$ and $Y_b = \{x \in \mathbb{R} : x < b\}$ and the family $\mathscr{S} = \{X_a : a \in \mathbb{R}\} \cup \{Y_b : b \in \mathbb{R}\}$. We see that $\mathscr{S}$ forms a subbase for the usual topology τ on $\mathbb{R}$ since the set of all finite intersections of elements of $\mathscr{S}$ form the set, $\mathscr{B} = \{(a, b) : a < b\}$, known to be a base of τ.

Example 4. Consider $(\mathbb{N}, \tau_d)$ where τ_d is the *discrete topology* (see definition on page 32). Then $\tau_d = \mathscr{P}(\mathbb{N})$. The set of all singleton sets, $\mathscr{B} = \{\{n\} : n \in \mathbb{N}\}$, forms a base for τ_d since every subset of $\mathbb{N}$ is the union of elements from some subset $\mathscr{C} \subseteq \mathscr{B}$. If $N_a = \{n \in \mathbb{N} : n \leq a\}$ and $M_b = \{n \in \mathbb{N} : n \geq b\}$ and

$$\mathscr{S} = \{N_a : a \in \mathbb{N}\} \cup \{M_b : b \in \mathbb{N}\}$$

then $\mathscr{B}$ is a subset of the family of finite intersections of elements of $\mathscr{S}$. So $\mathscr{S}$ is a subbase for the topology τ_d on $\mathbb{N}$.

Example 5. Consider the sets $X_a = \{x \in \mathbb{R} : a < x\}$ and $Y_b = \{x \in \mathbb{R} : x \leq b\}$ and the family $\mathscr{S} = \{X_a : a \in \mathbb{R}\} \cup \{Y_b : b \in \mathbb{R}\}$. We see that the intersections of finite subsets of $\mathscr{S}$ are either $\varnothing$ or of the form $(a, b]$. So

$$\mathscr{B} = \{(a, b] : a < b\} \cup \{\varnothing\}$$

forms a base for a topology, $\tau_{\mathscr{S}}$, on $\mathbb{R}$ generated by the subbase $\mathscr{S}$. This topology is referred to the *upper limit topology* or the *Sorgenfrey line* .

5.5 A subbase as a generator of a topology.

We now show how, from *any* non-empty subset, $\mathscr{S}$, of $\mathscr{P}(S)$ we can construct, from $\mathscr{S}$, a topology, $\tau_{\mathscr{S}}$, on S which has $\mathscr{S}$ as subbase.

Suppose S is a set and $\mathscr{S}$ is a non-empty subset of $\mathscr{P}(S)$. Let,

$$\mathscr{J} = \{\tau_j : j \in I, \mathscr{S} \subseteq \tau_j\}$$

denote *all* topologies on S which contain $\mathscr{S}$. The set $\mathscr{J}$ is non-empty since it at least contains the discrete topology, $\tau_d = \mathscr{P}(S)$, which, by definition, contains $\mathscr{S}$. Since the family of all topologies on a set is closed under intersections (but not necessarily under unions) then the family

$$\tau_{\mathscr{S}} = \cap\{\tau_j : j \in I,\ \mathscr{S} \subseteq \tau_j\}$$

is also a topology on S which contains all elements of $\mathscr{S}$. In fact, it is easily verified that $\tau_{\mathscr{S}}$ *is the smallest possible topology on* S *which contains* $\mathscr{S}$. We are now left with the task of showing that $\mathscr{S}$ is indeed a subbase for the constructed topology, $\tau_{\mathscr{S}}$, on S.

Theorem 5.6 Let S be a non-empty set and $\mathscr{S} \subseteq \mathscr{P}(S)$. Suppose

$$\tau_{\mathscr{S}} = \cap\{\tau_j : j \in I, \tau_j \text{ is a topology on } S,\ \mathscr{S} \subseteq \tau_j\}$$

where I is an indexing set. Then $\mathscr{S}$ is a subbase for the unique smallest topology, $\tau_{\mathscr{S}}$, on S which contains $\mathscr{S}$.

Proof: Given: $\tau_{\mathscr{S}} = \cap\{\tau_j : j \in I, \tau_j \text{ is a topology on } S,\ \mathscr{S} \subseteq \tau_j\}$.

We have already seen that $\tau_{\mathscr{S}}$ is the smallest topology on S with contains $\mathscr{S}$. If $\mathscr{S} \subseteq \mathscr{P}(S)$ let

$$\mathscr{B} = \{B : B = \cap_{M \in \mathscr{F}} M \text{ where } \mathscr{F} \text{ is a finite subset of } \mathscr{S}\}$$

To prove that $\mathscr{S}$ is a subbase of $\tau_{\mathscr{S}}$, it suffices to show that $\mathscr{B}$ is a basis for a topology τ on S.

To prove that $\mathscr{B}$ is a basis for a topology τ on S it suffices to show that $\mathscr{B}$ satisfies the "base property": That is, $\cup\{B : B \in \mathscr{B}\} = S$ and that, if $A, D \in \mathscr{B}$ and $x \in A \cap D$, there exists $E \in \mathscr{B}$ such $x \in E \subseteq A \cap D$.

We first show that $\cup\{B : B \in \mathscr{B}\} = S$: It suffices to show that $S \in \mathscr{B}$. To do this we must show that S is the intersection of all elements of a finite subset of $\mathscr{S}$. Note that, the empty set, $\varnothing$, is a finite subset of $\mathscr{S}$. Then $\cap_{M \in \varnothing} M = \{x \in S : x \in M \text{ for every } M \in \varnothing\} = S$. So $S \in \mathscr{B}$.

Claim: If $A, D \in \mathscr{B}$ and $x \in A \cap D$, there exists $E \in \mathscr{B}$ such that $x \in E \subseteq A \cap D$. Let $A, D \in \mathscr{B}$ and $\mathscr{A}$ and $\mathscr{D}$ be finite subfamilies of $\mathscr{S}$ such that $A = \cap_{M \in \mathscr{A}} M$ and $D = \cap_{M \in \mathscr{D}} M$. Then $\mathscr{A} \cup \mathscr{D}$ is a finite subfamily of $\mathscr{S}$. Suppose $x \in A \cap D$. Then

$$x = (\cap_{M \in \mathscr{A}} M) \cap (\cap_{M \in \mathscr{D}} M) = \cap_{M \in \mathscr{A} \cup \mathscr{D}} M = A \cap D$$

If $E = \cap_{M \in \mathscr{A} \cup \mathscr{D}} M$, $x \in E \subseteq A \cap D$.

Then $\mathscr{B}$ is a base for "some" topology τ on S and so $\mathscr{S}$ is a subbase for the topology τ.
Since $\tau_{\mathscr{S}}$ is the smallest topology which contains $\mathscr{S}$ then $\tau_{\mathscr{S}} \subseteq \tau$. On the other hand, if $U \in \tau$ then U is the union of elements of $\mathscr{B} \subseteq \tau_{\mathscr{S}}$. So $U \in \tau_{\mathscr{S}}$ and $\tau \subseteq \tau_{\mathscr{S}}$. Then $\tau_{\mathscr{S}} = \tau$ and so $\mathscr{S}$ is a subbase of $\tau_{\mathscr{S}}$.

Example 6. Consider the set $\mathscr{S} = \{[a, b] : a < b\} \subset \mathscr{P}(\mathbb{R})$.

We see that $\mathscr{S}$ does not satisfy the "base property" (since there does not exist $[x, y]$ in $\mathscr{S}$ such that $[x, y] \subseteq [a, b] \cap [b, c]$) and so cannot form a base for a topology on $\mathbb{R}$.

However, the theorem guarantees that $\mathscr{S}$ is a subbase for some topology on $\mathbb{R}$. Describe this topology.

Solution : We see that non-empty finite intersections of elements of $\mathscr{S}$ are of the form $[c, d]$ where $c \leq d$. In particular, $[u, x] \cap [x, v] = \{x\}$ is an open base element for all $x \in \mathbb{R}$.

So the subbase $\mathscr{S}$ will generate an open base, $\mathscr{B} = \{\{x\} : x \in \mathbb{R}\}$, for the discrete topology, τ_d. We see that the subbase, $\mathscr{S}$, holds more sets then is really required to generate τ_d. We can reduce its size to

$$\mathscr{S}^* = \{[x - 1, x] : x \in \mathbb{R}\} \cup \{[x, x + 1] : x \in \mathbb{R}\}$$

Since, for each $x \in \mathbb{R}$, $[x - 1, x] \cap [x, x + 1] = \{x\}$, $\mathscr{S}^*$ still generates τ_d.

Example 7. Consider two topological spaces (S, τ_S) and (T, τ_T). Using the sets S and T we can construct a new set, the Cartesian product $S \times T = \{(x, y) : x \in S, y \in T\}$. We can topologize the set $S \times T$ by defining a suitable subbase.

We will proceed as follows. Define the two projection functions $\pi_S : S \times T \to S$ and $\pi_T : S \times T \to T$ as $\pi_S(x, y) = x$ and $\pi_T(x, y) = y$, respectively. We will define as subbase for $S \times T$

$$\mathscr{S} = \{\pi_S^{-1}(U) : U \in \tau_S\} \cup \{\pi_T^{-1}(V) : V \in \tau_T\}$$

where $\pi_S^{-1}(U) = U \times T$ and $\pi_T^{-1}(V) = S \times V$. By referring to the principle $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ (verify this!), the basis $\mathscr{B}$ induced by this subbase is of the form

$$\mathscr{B} = \{U \times V : U \in \tau_S, V \in \tau_T\}$$

The topology on $S \times T$ whose base is $\mathscr{B}$ is called the *product topology on $S \times T$*.

Definition 5.6.1 Two bases $\mathscr{B}$ and $\mathscr{B}^*$ are said to be *equivalent bases* for a particular set S if they generate the same topology.

Note that the usual basis for $\mathbb{R}$ and the basis for the *upper limit topology* described in the example above are not equivalent bases since $(a, b]$ does not belong to the usual topology on $\mathbb{R}$. However, since $\cup\{(a, b - 1/n] : n = 1, 2, 3, \ldots\} = (a, b)$, then every base element for the usual topology belongs to $\tau_{\mathscr{S}}$ and so the usual topology is weaker than $\tau_{\mathscr{S}}$.

5.7 Topic: On countable bases.

We will later see that the cardinality of the two types of bases of a topology space presented above is reflected on some of its properties. Those topological spaces that have *countable neighbourhood bases at every point* x and, those that have a *countable open base* are given particular names. We define these below.

Definition 5.7 Let (S, τ) be a topological space. The topological space, S, is said to be *first countable* if and only if every point, $x \in S$, has a countable neighbourhood base. The topological space, S, is said to be *second countable* if and only if it has a countable base for open sets.

Countability properties. Recall (from definition 4.10) that a space is *separable* if it has a countably dense subspace. The three properties, *separable*, *first countable* and *second countable* are often referred to together in many theorems. We will refer to them as the "countability properties". The size of the smallest bases of a topological space (either the base itself or its neighbourhood bases) is closely related to many topological properties of a space.

Theorem 5.8 A second countable topological space is a first countable space.

Proof: Suppose (S, τ) is a second countable space. Then S has a countable base, $\mathscr{B}$. Then, for each $x \in S$, $\mathscr{B}_x = \{B \in \mathscr{B} : x \in B\}$ is a neighbourhood base at x. Since $\mathscr{B}$ is countable and $\mathscr{B}_x \subseteq \mathscr{B}$, $\mathscr{B}_x$ cannot be uncountable. So (S, τ) is first countable.

However there are some uncountably large topological spaces which are first countable but not second countable. The following example illustrates such a property.

Example 8. Suppose τ_S represents the *upper limit topology* on $\mathbb{R}$ (the Sorgenfrey line) (see example 4 on page 72). Recall that $\mathscr{B} = \{(x, y] : x, y \in \mathbb{R}\}$ is an open base for $\mathbb{R}$, when ever it is equipped with this topology.

We claim that the space, $(\mathbb{R}, \tau_S)$ *is first countable*: For each $x \in \mathbb{R}$, $\mathscr{B}_x = \{(x - \frac{1}{n}, x] : n \in \mathbb{N}\}$ is a countable neighbourhood base at x. So $(\mathbb{R}, \tau_S)$ is first countable, as claimed.

We claim that the space, $(\mathbb{R}, \tau_S)$ *is not second countable*: Let the function $\phi : \mathbb{R} \to \mathscr{B} = \{(x, y] : x, y \in \mathbb{R}\}$ (the base of open sets of τ_S) be defined as

$$\phi(x) = (x - 1, x]$$

So $\phi[\mathbb{R}] \subseteq \mathscr{B}$. If $x \neq y$, then

$$\phi(x) = (x - 1, x] \neq (y - 1, y] = \phi(y)$$

Since $\mathbb{R}$ is uncountable and ϕ is one-to-one on its domain, $\mathbb{R}$, then $\phi[\mathbb{R}]$ is an uncountable subset of $\mathscr{B}$. Since, for example, $(7, 8] = (7, 7.1] \cup (7.1, 8] \in \mathscr{B}$ every element of the range is required in the base.

So $(\mathbb{R}, \tau_S)$ is not second countable, as claimed.

Theorem 5.9 Any metrizable space is first countable.

Proof: Suppose (S, τ) is a metrizable space whose open sets are generated by the metric ρ. Then the set

$$\mathscr{B}_x = \{B_{1/n}(x) : n \in \mathbb{N}, n > 0\}$$

(an open ball center x with radius $1/n$) forms a countable neighbourhood base at x. Hence S is a first countable space.

Example 9. Since $\mathbb{R}$ (equipped with usual topology) is metrizable then it is first countable.

Example 10. Verify that space, $\mathbb{R}$, when equipped with the usual topology, is second countable.

Solution : Suppose U is an open subset of $\mathbb{R}$ and $x \in U$. Then there exists an open interval, $B_\varepsilon(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$, such that $x \in B_\varepsilon(x) \subseteq U$.

We consider two cases.

- If $x \in \mathbb{Q}$, then there exists integer m such that $x \in B(x, \frac{1}{m}) \subseteq B(x, \varepsilon) \subseteq U$.
- If x is an irrational, we know that there is a sequence of rationals which converges to x, so we can find a rational, $y \in B(x, \frac{\varepsilon}{4})$, such that $x \in B(y, \frac{\varepsilon}{2}) \subseteq B(x, \varepsilon)$.

We have just shown that $\mathscr{B} = \{B(y, m) : y, m \in \mathbb{Q}\}$ forms a base for the open sets in $\mathbb{R}$. Since $\mathbb{Q}$ is countable, then so is $\mathbb{Q} \times \mathbb{Q}^{\dagger 2}$ so

$$|\mathscr{B}| = |\{B(y, m) : (y, m) \in \mathbb{Q} \times \mathbb{Q}\}| = |\mathbb{Q} \times \mathbb{Q}| = \aleph_0$$

We have shown that, when equipped with the usual topology, $\mathbb{R}$ has a countable base for open sets.

Example 11. Recall that the radial plane, $(\mathbb{R}^2, \tau_r)$, is equipped with what is called the *radial plane topology*, τ_r, (see the example on page 71). The open sets in the radial plane are defined as follows: the subset U is an open neighbourhood of q if and only if $q \in U$, and U is the union of a set of open line segments, precisely one in each direction, each one originating at q. It is shown on page 71, that this is a valid topology and that it is strictly stronger than the usual topology on $\mathbb{R}^2$. Verify that *the radial plane is not first countable.*

Solution : Let $p \in \mathbb{R}^2$ and suppose p has a countable open neighbourhood base, $\mathscr{B}_p = \{B_i : i \in \mathbb{N}\}$. To establish the claim we will show that that p has some radially open neighbourhoods which are not the union of elements in $\mathscr{B}_p$.

For each $i \in \mathbb{N} \backslash \{0\}$, there is a ray of length r_i originating at p, which remains inside B_i no matter which direction it points to. We then inductively construct a sequence of rays, of lengths, $R = \{r_i/2^i : i \in \mathbb{N} \backslash \{0\}\}$, each of which originates at p. There are rays emanating from p which are not accounted for in R. We will choose infinitely long rays for those that are missing to construct a complete radially open neighbourhood, V, of p. No open neighbourhood B_i can be contained in this V (since the rays become arbitrarily small, shrinking down to p). So $\mathscr{B}_p$ cannot be a open neighbourhood base for p. So p must have an uncountable open neighbourhood base.

2 † See appendix on cardinalities.

5.8 Topic: Relating countable bases with a countable dense subset of a space.

Recall that in definition 4.10, we defined a *separable* topological space as being a space which has a countable dense subset. As example, saw that, since $\mathbb{Q}$ is a countable dense subset of the reals, $\mathbb{R}$ (with the usual topology) is separable. In the following theorem, we see that the separable property is satisfied in all second countable space.

Theorem 5.10 *"Second countable" implies "separable".* Suppose $\mathscr{B}$ is the smallest base of the space (S, τ_S).

a) Then there exists in S a dense subset, D such that $|D| \leq |\mathscr{B}|$.

b) Any second countable topological space is a separable space.

Proof: Let $\mathscr{B} = \{B_i : i \in I\}$ be the smallest base of S.

a) For each $i \in I$, choose $x_i \in B_i$.(AC)[1] Let $D = \{x_i : i \in I\}$. Then $|D| \leq |\mathscr{B}|$.
We claim that D is dense in S:
Let U be a non-empty open subset of S. Then, for $x \in U$ there exists $B_i \in \mathscr{B}$ such that $x \in B_i \subseteq U$. Then $x_i \in B_i \subseteq U$. Then $U \cap D \neq \varnothing$. So every open set in S intersects D. This means that D is dense in S, as claimed.
Since $|D| \leq |\mathscr{B}|$ we are done, for if $|\mathscr{B}| = \aleph_0$, then D is a countable dense subset of S. So S is separable.

b) It follows immediately from part a) that "second countable property" implies the "separable property".

Theorem 5.11 Separable metric spaces are second countable.

Proof: Let (S, ρ) be a separable metric space with metric ρ.

Since S is separable, then S has a countable dense subset, $D = \{x_i : i \in \mathbb{N}\backslash\{0\}\}$.

For each i and n in $\mathbb{N}\backslash\{0\}$ let $B_{(i,n)} = B_{1/n}(x_i)$, an open ball center x_i and radius $1/n$. Consider

$$\mathscr{B} = \{B_{(i,n)} : i, n \in \mathbb{N}\backslash\{0\} :\}$$

For $x \in S$, let U_x be an open neighbourhood of x. Then there exists, $j \in \mathbb{N}\backslash\{0\}$ such that $B_{1/j}(x) \subseteq U_x$. Since D is dense in S, $B_{1/j}(x) \cap D$ is non-empty. Say, $x_k \in B_{\frac{1}{2j}}(x) \cap D$. Then $x \in B_{\frac{1}{2j}}(x_k) \subseteq B_{\frac{1}{2j}}(x) \subseteq U_x$. Then $\mathscr{B}$ is a countable base for open sets in S. So the separable metric space, S, is second countable.

[1] Existence theorems often (but not always) suggest an application of the Axiom of choice in the proof. Keep an eye open for it.

Example 12. From the above statements, we have another proof that the set of all reals equipped with the usual topology is both separable and second countable.

5.9 Topic : Hereditary topological properties.

Some properties on spaces are carried over to their subspaces, while others are not. Those properties that do are called hereditary properties.

Definition 5.12 A topological property, say P, of a space (S, τ_S) is said to be a *hereditary topological property* provided every subspace, (T, τ_T), of S also has P.

Example 13. Metrizability is hereditary. Suppose (S, τ) is metrizable. Then there exists a metric ρ such that (S, τ) and (S, ρ) have the same open sets. Suppose $T \subseteq S$ has the subspace topology and $\rho_t : T \times T \to \mathbb{R}$ is the subspace metric on T. Then (T, τ_t) and (T, ρ_t) have the same open sets and so T is metrizable.

"First countable" is another example of a hereditary topological property. However, "separable" is not a hereditary topological property. Witness the separable space, $\mathbb{R}$ with the usual topology; its subspace of all irrationals, $\mathbb{J}$, has no countable dense subset. However, the reader may want to verify that, if S is separable and V is an open subspace of S, then V is separable.

Theorem 5.13 Suppose (S, τ_S) is a second countable topological space. Then any non-empty subspace of S is also second countable. So "second countable" is a hereditary property.

Proof: Suppose (S, τ_S) has a countable base $\mathscr{B} = \{B_i : i \in \mathbb{N}\}$. Suppose (T, τ) is a non-empty subspace of S. Let U be an open subset of T. Then there exists an open subset U^* of S such that $U = U^* \cap T$. Then there exists $N \subseteq \mathbb{N}$ such that $U^* = \cup\{B_i : i \in N\}$. Then $U = \cup\{B_i \cap T : i \in N\}$. So $\mathscr{B}_T = \{B_i \cap T : i \in \mathbb{N}\}$ is a countable basis of T. Hence T inherits the second countable property from its superset S.

5.10 Topic : The ordinal space.

The set of ordinals plays an important role in general topology. This linearly and well-ordered set often serves as an example or counterexample to certain conjectures that involve large cardinalities. By "well-ordered" we mean that every non-empty subset of ordinals contains its least element. We will topologize the ordinals by defining an appropriate subbase.

Definition 5.14 Let ω_α be an ordinal and $S = [0, \omega_\alpha] = \{\text{ordinals } \alpha : \alpha \leq \omega_\alpha\}$.
Suppose β and μ are both ordinals which belong to S. Let $S_\mu = (\mu, \omega_\alpha] = \{\alpha \in S : \alpha > \mu\}$ and $S_\beta = [0, \beta) = \{\alpha \in S : \alpha < \beta\}$. The standard subbase of S is defined as,

$$\mathscr{S} = \{S_\mu : \mu \in S\} \cup \{S_\beta : \beta \in S\}$$

This subbase will generate a base, $\mathscr{B}$, which in turn will generate the topology, τ_ω, of S. When the space S is equipped with the topology τ_ω, (S, τ_ω), is referred to as an *ordinal space*. The topology that is generated by this subbase is called the *interval topology on the set of ordinals.*

A few facts about an ordinal space. When we say "ordinal space" we mean a set of ordinals with the topology generated by the subbase $\mathscr{S}$. But the best way to memorize the topology of the ordinal space is to remember what the elements of its base for open sets look like. Remember that there are two types of ordinals. Every ordinal number, α, without exception, has an immediate successor, $\alpha+1$, by definition. Some ordinals, γ, have an immediate predecessor, say β, provided

$$\gamma = \beta + 1$$

In this case, $\sup[0, \gamma) = \sup[0, \beta + 1) = \beta$.

While some ordinals, μ don't have an immediate predecessor. In this case,

$$\sup\{\delta : \delta < \mu\} = \sup[0, \mu) = \mu$$

So when we consider the intersection of two elements, $(\mu, \omega_\alpha]$ and $[0, \beta)$, of the subbase, $\mathscr{S}$, we get (μ, β). At this point, we see that there are two possibilities.

- Case 1 : Suppose β has an immediate predecessor, say γ (because $\beta = \gamma + 1$). In this case, we can express (μ, β) as the half-open interval, $(\mu, \gamma]$.
- Case 2 : Suppose β doesn't have an immediate predecessor. Then $\beta = \{\delta : \delta < \beta\}$. In this case, (μ, β) can be expressed as the half-open interval, $(\mu, \beta]$.

In both cases we have an open base element which is a half-open interval. So we conclude that a base, $\mathscr{B}$, for open sets in the ordinal space, $S = [0, \omega_\alpha]$, is the set

$$\mathscr{B} = \{(\alpha, \beta] : \alpha, \beta \in S,\ \alpha < \beta\}$$

This is easier to remember. The interval topology is generated by the open base $\mathscr{B}$.[1]

5.11 Topic : The topology generated by regular open sets.

Suppose we are given a topological space (S, τ). Recall that an open subset, U, of S is called a *regular open* subset if it satisfies the property,

$$U = \text{int}_S\text{cl}_S U$$

See definition 4.12. In the expression, $\text{int}_S\text{cl}_S U$, the interior and closure are with respect to the topology τ. The symbol

$$\mathscr{R}o(S) = \{U \in \tau : U \text{ is regular open}\}$$

represents the set of all regular open subsets of S. So $\mathscr{R}o(S) \subseteq \tau$. But since $\mathscr{R}o(S)$ is not closed under unions then it is not by itself a topology on S.

Clearly, $\varnothing$ and S belong to $\mathscr{R}o(S)$. On page 62, we showed that $\mathscr{R}o(S)$ is closed under finite intersections. Then, if $x \in S$ and $x \in A \cap B$ where $\{A, B\} \subseteq \mathscr{R}o(S)$, since $A \cap B \in \mathscr{R}o(S)$, then $\mathscr{R}o(S)$ satisfies the "base property". Then, by theorem 5.4, $\mathscr{R}o(S)$ is an open base for some topology, say τ_s, on S. The elements of $\mathscr{R}o(S)$ are all open with respect to τ but not all elements in τ_s are necessarily regular open in S. Furthermore there may be open sets in τ which are not unions of elements in $\mathscr{R}o(S)$ and so are not in τ_s. So we have $\mathscr{R}o(S) \subseteq \tau_s \subseteq \tau$.

Note that the topology generated by $\mathscr{R}o(S)$ is weaker than τ, but under certain conditions, may be equivalent to it.

Definition 5.15 Let (S, τ) be a topological space and τ_s denote the topology whose base is $\mathscr{R}o(S)$. Even if (S, τ_s) is the same set as (S, τ) it has a weaker topology.

If τ_s is a proper subset of τ, the topological space, (S, τ_s), is called the *semiregularization of S with respect to τ.*

If $\tau = \tau_s$, then we will call (S, τ) a *semiregular* topological space.[1] That is, a semiregular topological space is a space such that for all $U \in \tau$, U is the union of regular open sets.[2]

[1] A more detailed study of the ordinals in the context of set theory is found in Axioms and set theory, by Robert André (can be found online)

[1] We will specify later in this text that semiregular spaces are assumed to be Hausdorff.

[2] We will show in chapter 9 that regular spaces are semiregular, but there exists semiregular spaces which are not regular.

The set of real numbers, $\mathbb{R}$, equipped with the usual topology is easily seen to be semiregular.

5.12 Topic : The topology generated by clopen sets.

We now consider the set, $\mathscr{B}(S) = \{U \in \mathscr{P}(S) : U \text{ is clopen}\}$, of all clopen sets in the topological space, (S, τ). The set $\mathscr{B}(S)$ is never empty since $\varnothing$ and S are amongst its elements. Furthermore, if U and V belong to $\mathscr{B}(S)$ and $x \in U \cap V$, given that $U \cap V$ also belongs to $\mathscr{B}(S)$ then $\mathscr{B}(S)$ satisfies the "base property".

This means that $\mathscr{B}(S)$ forms a base for some topology, τ_b, on S. Since $\mathscr{B}(S) \subseteq \tau$, then the topology, τ_b, is weaker than τ, but may, under certain conditions, be equivalent to it.

Definition 5.16 Let (S, τ) be a topological space and τ_b denote the topology whose base is $\mathscr{B}(S)$. If $\tau = \tau_b$, then we say that (S, τ) is a *zero-dimensional* topological space.

Example 13. Consider the subspace, $(\mathbb{Q}, \tau)$, of rational numbers with the subspace topology of $\mathbb{R}$ itself equipped with the usual topology. Verify that $\mathbb{Q}$ is a zero-dimensional topological space.

Solution : Consider $\mathscr{U} = \{(a, b) : a < b,\ a \text{ and } b \text{ irrationals}\} \subseteq \mathbb{R}$. This forms a base for open sets in $\mathbb{R}$. Then $\mathscr{U}_Q = \{U \cap \mathbb{Q} : U \in \mathscr{U}\}$ forms a base for open sets for $\mathbb{Q}$. Each element of $\mathscr{U}_Q$ is clopen in $\mathbb{Q}$ so $\mathbb{Q}$ is zero-dimensional.

Concepts review:

1. Define a neighbourhood system of x with respect to τ.
2. Define a neighbourhood base of x with respect to the topology τ.
3. Is $\varnothing$ a neighbourhood of a point x? Is a neighbourhood of x necessarily open?
4. Define a base for a topology τ.
5. Find a base for the usual topology τ on $\mathbb{R}$.
6. Give a characterization of a base of τ in terms of "neighbourhoods".

7. What does it mean to say that a subset $\mathscr{P}(S)$ satisfies the "base property"?

8. Describe the Moore plane and the base for its topology.

9. Given an arbitrary subset $\mathscr{S}$ of $\mathscr{P}(S)$ explain how it can be used to construct a topology on S.

10. Define a *subbase for a topology.*

11. Describe a subbase for the usual topology on $\mathbb{R}$.

12. Describe a topology generated by a subbase $\mathscr{S}$ in terms of other topologies on S.

13. Given two topological spaces S and T and a Cartesian product $S \times T$. Find a useful subbase involving projection maps that can be used to generate a topology on $S \times T$.

14. Describe the *upper limit topology* (or Sorgenfrey topology) on $\mathbb{R}$ in terms of its base and subbase.

15. Is $\mathbb{R}$ with the upper limit topology first countable? Is it second countable.

16. What does it mean to say that two subsets of $\mathscr{P}(S)$ are equivalent topologies for the set S.

17. What can we say about the size of the neighbourhood bases at the points of $\mathbb{R}$ with respect to the usual topology?

18. What can we say about the size of the base of $\mathbb{R}$ with respect to the usual topology?

19. Define *first countable topological space.*

20. Define *second countable topological space.*

21. Is $\mathbb{R}$ equipped with the usual topology first countable? What about second countable?

22. Describe a topological space which is first countable but not second countable.

23. State a relationship between "second countable" and "separable".

24. Describe the topology on an ordinal space.

25. Metrizable space are necessarily first countable. Describe a neighbourhood base.

26. Describe how the set of regular open subsets can be used to generate a topology on a space.

27. What does it mean to say that a property is hereditary?

28. Define a semiregular space.

29. Given a topological space, (S, τ), what is the semiregularization of (S, τ)?

30. Define a zero-dimensional space. As an example provide a subspace of $\mathbb{R}$ which is zero-dimensional.

EXERCISES

1. Show that, if F is a closed subset of the topological space (S, τ) and $x \notin F$, then x has a neighbourhood which does not intersect F.

2. Let $\vec{x} = (a, b) \in \mathbb{R}^2$ equipped with the usual (Euclidean) topology. For each $q \in \mathbb{R}$, let $B_{\vec{x}}(q) = \{(x, y) : \max\{|a - x|, |b - y|\} < q\}$. Show that $\mathscr{B}_{\vec{x}} = \{B_{\vec{x}}(q) : q \in \mathbb{R}, q \geq 0\}$ forms a neighbourhood base at $\vec{x}$.

3. Suppose τ_1 and τ_2 are two topologies on the set S with respective bases $\mathscr{B}_1$ and $\mathscr{B}_2$. Show that $\tau_1 \subseteq \tau_2$ if and only if whenever $x \in B_1$ there exists $B_2 \in \mathscr{B}_2$ such that $x \in B_2 \subseteq B_1$.

4. Let A and B be two infinite sets each equipped with the cofinite topology. Describe a basis of $A \times B$ equipped with the product topology.

5. Let (S, τ) be a topological space with base $\mathscr{B}$. If $A \subseteq S$, show that $\mathscr{B}_A = \{B \cap A : B \in \mathscr{B}\}$ is a base for the open sets of A.

6. Let (S, τ_S) and (T, τ_T) be two first countable topological spaces. Show that $S \times T$ equipped with the product topology is first countable.

7. Prove that, if (A, τ_1) is a subspace of (B, τ_2) and (B, τ_2) is a subspace of (C, τ_3) then (A, τ_1) is a subspace of (C, τ_3).

---

6 / Continuity on topological spaces.

Summary. *In this section we formally define the notion of a continuous function mapping one topological space into another. We discuss various characterizations of these. An important class of continuous functions are those called "homeomorphisms". We will see why these are fundamental in the study of topological spaces. Finally, we examine how a function can be used to topologize sets.*

6.1 Basic notions and notation associated to functions mapping sets to sets.

We begin by establishing the notation and terminology we will use in our discussion of functions. Suppose $f : S \to T$ is a well-defined function mapping a topological space, S, into another topological space, T. For $f : S \to T$. "f is one-to-one and onto" and "f is a bijection between S and T" are simply different ways of conveying the same idea.

The function, $f : S \to T$, induces another function, $f : \mathscr{P}(S) \to \mathscr{P}(T)$, where

$$f[A] = \{y \in T : y = f(x) \text{ for some } x \in A \subseteq S\}$$

Essentially, $f[A]$ is the image of the set A under the function $f : S \to T$. The function $f : S \to T$ also induces the function $f^{\leftarrow} : \mathscr{P}(T) \to \mathscr{P}(S)$ where

$$f^{\leftarrow}[B] = \{x \in S : \text{ where } f(x) \in B\}$$

We can also say that, if $f^{\leftarrow}[B] = D$, then D is the "inverse image" or "pre-image" of B under f, or that the function $f^{\leftarrow}$ "pulls back" the set B onto the set D inside the domain S of f. In the case where $f^{\leftarrow}[\{y\}] = \{x\}$, if there is no risk of confusion, we will simply write $f^{\leftarrow}(y) = x$.[1] In the case where $f : S \to T$ is one-to-one and onto T then $f^{\leftarrow}$ can itself be seen as a well-defined function, $f^{\leftarrow} : T \to S$, and so we can write "$f^{\leftarrow}(x) = y$ if and only if $f(y) = x$" (without using the square brackets).

In this book, if $f : S \to \mathbb{R}$ is a function and $0 \notin f[S]$, f^{-1} will be interpreted as follows:

$$f^{-1}(x) = \frac{1}{f(x)}$$

In the following theorem statement, we review some basic principles on how functions act on sets and on their set-operations. In this process, we are reminded of the following three principles:

1) A function "respects arbitrary unions of sets".

[1] Note that f need not be one-to-one on all of the domain in order for us to speak of $f^{\leftarrow}$ in this way.

2) If a function is *one-to-one* it will "respect arbitrary intersections". Otherwise, a function "does *not* always respect intersections of sets".
3) An inverse function, $f^{\leftarrow}$, "always respects unions, intersections and complements of sets".

Theorem 6.1 Let $f : A \to B$ be a *function* mapping the set A to the set B. Let $\mathscr{A}$ be *a set* of subsets of A and $\mathscr{B}$ be *a set* of subsets of B. Let $D \subseteq A$ and $E \subseteq B$. Then:

a) $f\left[\bigcup_{S\in\mathscr{A}} S\right] = \bigcup_{S\in\mathscr{A}} f[S]$

b) $f\left[\bigcap_{S\in\mathscr{A}} S\right] \subseteq \bigcap_{S\in\mathscr{A}} f[S]$ where equality holds true only if f is one-to-one.

c) $f[A\backslash D] \subseteq B\backslash f[D]$. Equality holds true only if f is one-to-one and onto B.

d) $f^{\leftarrow}\left[\bigcup_{S\in\mathscr{B}} S\right] = \bigcup_{S\in\mathscr{B}} f^{\leftarrow}[S]$

e) $f^{\leftarrow}\left[\bigcap_{S\in\mathscr{B}} S\right] = \bigcap_{S\in\mathscr{B}} f^{\leftarrow}[S]$

f) $f^{\leftarrow}[B\backslash E] = A\backslash f^{\leftarrow}[E]$

Proof:

a)
$$\begin{aligned} x \in f\left[\bigcup_{S\in\mathscr{A}} S\right] &\Leftrightarrow x = f(y) \text{ for some } y \in \bigcup_{S\in\mathscr{A}} S \\ &\Leftrightarrow x = f(y) \text{ for some } y \text{ in some } S \in \mathscr{A} \\ &\Leftrightarrow x = f(y) \in f[S] \text{ for some } S \in \mathscr{A} \\ &\Leftrightarrow x \in \bigcup_{S\in\mathscr{A}} f[S] \end{aligned}$$

b) It will be helpful to first prove this statement for the intersection of only two sets U and V. The use of a Venn diagram will also help visualize what is happening.

So we first prove the statement: $f[U \cap V] \subseteq f[U] \cap f[V]$ with equality only if f is one-to-one on $U \cup V$.

Case 1: We consider the case where $U \cap V = \varnothing$.
Then $f[U \cap V] = \varnothing \subseteq f[U] \cap f[V]$. So the statement holds true.

Case 2: We now consider the case where $U \cap V \neq \varnothing$.

$$\begin{aligned} x \in f[U \cap V] &\Leftrightarrow x = f(y) \text{ for some } y \in U \cap V \\ &\Leftrightarrow x = f(y) \text{ for some } y \text{ contained in both } U \text{ and } V \\ &\Rightarrow x = f(y) \in f[U] \text{ and } f[V] \\ &\Leftrightarrow x \in f[U] \cap f[V] \end{aligned}$$

We now show that if f is one-to-one on $U \cup V$, then $f[U] \cap f[V] \subseteq f[U \cap V]$ and so equality holds true.

– Suppose $x = f(y) \in f[U] \cap f[V]$. Then there exists $u \in U$ and $v \in V$ such that $f(u) = f(v) = f(y)$. Since f is one-to-one, $u = v = y$. This implies $y \in U \cap V$. Hence, $f[U \cap V] = f[U] \cap f[V]$.

The proof of the general statement is left as an exercise.

c) Proof is left as an exercise.

d)
$$\begin{aligned} x \in f^{\leftarrow}\left[\bigcup_{S\in\mathscr{B}} S\right] &\Leftrightarrow x = f(y) \text{ for some } y \in \bigcup_{S\in\mathscr{B}} S \quad \text{(By definition of } f^{\leftarrow}.) \\ &\Leftrightarrow x = f(y) \text{ for some } y \text{ in some } S \in \mathscr{B} \\ &\Leftrightarrow x \in f^{\leftarrow}[\{y\}] \subseteq f^{\leftarrow}[S] \text{ for some } S \in \mathscr{B} \\ &\Leftrightarrow x \in \bigcup_{S\in\mathscr{B}} f^{\leftarrow}[S] \end{aligned}$$

Thus, $f^{\leftarrow}\left(\bigcup_{S\in\mathscr{B}} S\right) = \bigcup_{S\in\mathscr{B}} f^{\leftarrow}(S)$.

e) Proof is left as an exercise.

f) Proof is left as an exercise.

6.2 Continuous functions on topological spaces.

Given two topological spaces, (S, τ_S) and (T, τ_T), we will discuss various types of functions, $f : S \to T$, which map S into T. The reader is already familiar with those functions called "continuous functions" mapping $\mathbb{R}$ to $\mathbb{R}$. We will generalize this notion of continuity to topological spaces.

Our formal definition of a continuous function mapping a topological space into another (in terms involving their topology) is presented below. Those readers who are familiar with the "epsilon-delta" definition of a continuous function (normally presented in any *Introduction to analysis* course) will notice the analytical approach cannot be used in topology since topological spaces, in their most rudimentary form are not equipped with distance functions such as absolute values, norms or metrics.

Definition 6.2 Let $f : S \to T$ be a function mapping (S, τ_S) into (T, τ_T). We say that $f : S \to T$ is *continuous on* S if, for any open subset $U \in T$, $f^{\leftarrow}[U]$ is open in S. If $x \in S$, we will say that f is *continuous at* x if, for any open neighbourhood, U, of $f(x)$ (inside T) there exists and open neighbourhood V of x such that $f(x) \in f[V] \subseteq U$.

Topologists may sometimes speak of the continuity of a function $f : S \to T$ by using the phrase "since f is continuous, f pulls back open sets to open sets". Or one might say "$f : S \to T$ is continuous on S if and only if $f^{\leftarrow}$ maps elements in τ_T into τ_S".

The above continuity of a function is defined in two different contexts. The first describes continuity of f on a set while the second describes continuity of f at a point x. Clearly, we cannot apply the first definition to determine continuity of f at a point. But a set A is simply a collection of points. If a function f can be shown to be continuous at every point x in a set A we would hope that, by referring to both the first and the second definition, it is continuous on the given set. The next theorem confirms that this is the case.

Theorem 6.3 Let (S, τ_S) and (T, τ_T) be two topological spaces and $f : S \to T$ be a function. Then f is continuous on S if and only if f is continuous at every point of S.

Proof:

($\Rightarrow$) Suppose f is continuous on S. Let $x \in S$ and suppose $y = f(x) \in f[S]$. We are required to prove that f is continuous at x. Suppose U is an open neighbourhood of y. By definition of continuity on a set, $f^{\leftarrow}[U]$ an open neighbourhood of $x \in f^{\leftarrow}[\{y\}] \subseteq f^{\leftarrow}[U]$ in S. By definition of neighbourhood, there exists an open $V \subseteq f^{\leftarrow}[U]$ such that $x \in V$. Then $y = f(x) \in f[V] \subseteq U$. So we have found the required open neighbourhood V of x. So f is continuous at x.

($\Leftarrow$) Suppose that f is continuous at every point of S. Let U be a non-empty open subset of $f[S] \subseteq T$. We are required to show that $f^{\leftarrow}[U]$ is open in S. Let $x \in f^{\leftarrow}[U]$. Then $f(x) \in U$. Since f is continuous at x then there exists an open neighbourhood V of x such that $f[V] \subseteq U$. Now $x \in V \subseteq f^{\leftarrow}[f[V]] \subseteq f^{\leftarrow}[U]$. Then $f^{\leftarrow}[U]$ is an open subset of S. So f is continuous on S.

There are other ways of recognizing those functions which are continuous on a set. For example, if $f : S \to T$ satisfies the property,

"$f^{\leftarrow}[F]$ is closed in S whenever F is closed in the codomain T"

then, when U is open in T, $S \setminus f^{\leftarrow}[U] = f^{\leftarrow}[T \setminus U]$ is closed in S. Then $f^{\leftarrow}[U]$ is open in S and so f is continuous on S. The reader is left to verify that the converse also holds true.

Other useful characterizations of continuity on a topological space are given below.

Theorem 6.4 Let (S, τ_S) and (T, τ_T) be two topological spaces and $f : S \to T$ be a function.

a) The function f is continuous on S if and only if f pulls back subbase elements of the topological space T to open sets in S.[1]

b) The function f is continuous on S if and only if f pulls back open base elements of the topological space T to open sets in S.[2]

c) The function f is continuous on S if and only if, for any subset U of S, $f[\mathrm{cl}_S U] \subseteq \mathrm{cl}_T f[U]$.

Proof: The proofs of parts a) and b) are left as an exercise.

c) ($\Rightarrow$) Suppose f is continuous on S. To show that $f[\mathrm{cl}_S U] \subseteq \mathrm{cl}_T f[U]$ it will suffice to show that $x \notin \mathrm{cl}_T f[U]$ implies $x \notin f[\mathrm{cl}_S U]$.

Suppose $x \notin \mathrm{cl}_T f[U]$. Then there musts exist an open V in T such that $x \in V \subseteq T \backslash \mathrm{cl}_T f[U]$. Then $f^{\leftarrow}(x) \subseteq f^{\leftarrow}[V] \subseteq S \backslash U$. By continuity of f, $f^{\leftarrow}[V]$ is open in S and so $\mathrm{cl}_S U \cap f^{\leftarrow}[V] = \varnothing$. Then $f[f^{\leftarrow}(x)] = x \notin f[\mathrm{cl}_S U]$. So $f[\mathrm{cl}_S U] \subseteq \mathrm{cl}_T f[U]$.

The proof of ($\Leftarrow$) is left as an exercise.

Suppose A is a subspace of the topological space, (S, τ), equipped with the subspace topology, τ_A. Suppose $f : S \to T$ is known to be continuous on its domain. Then, when f is restricted to $f|_A$ on A, $f|_A$ preserves the continuity property on A. This is confirmed by the following theorem statement.

Theorem 6.5 Let (S, τ_S), (T, τ_T) and (Z, τ_Z) be topological spaces.

a) If $f : S \to T$ and $g : T \to Z$ are both continuous on their domains then $g \circ f : S \to Z$ is continuous on S. (That is, the composition of continuous functions is continuous.)

b) Suppose $f : S \to T$ is a continuous function on S and $A \subseteq S$. Let $f|_A : A \to T$ denote the restriction of f to the subset A. Then $f|_A$ is continuous on A.

[1]That is, $f^{\leftarrow}[B]$ is open in S whenever B is a subbase element of T.
[2]That is, $f^{\leftarrow}[B]$ is open in S whenever B is a base element of T.

Proof : a) The proof of part a) is left as an exercise.

b) Let U be an open subset of $f[A]$ with respect to the subspace topology $\tau_{f[A]}$ in T. Then there exists $U^* \in \tau_T$ such that $U = U^* \cap f[A]$. Now

$$\begin{aligned} x \in f|_A^{\leftarrow}[U] &\Rightarrow \{x \in A : f(x) \in U\} \\ &\Rightarrow x \in \{x \in S : f(x) \in U^*\} \cap A \\ &\Rightarrow x \in f^{\leftarrow}[U^*] \cap A \end{aligned}$$

Since f is continuous on S, $f^{\leftarrow}[U^*] \cap A$ is an open subset of A with respect to the subspace topology. So $f|_A^{\leftarrow}[U]$ is open in A.

We have seen that continuous functions are those functions which "pull back" open sets to open sets, or, equivalently, "pull back" closed sets to closed sets. We will encounter at least two other similar types of functions, which are not necessarily continuous.

Definition 6.6 Let $f : S \to T$ be a function mapping (S, τ_S) into (T, τ_T). We say that $f : S \to T$ is *an open function on* S, if for any open subset U of S, $f[U]$ is open in T. We say that $f : S \to T$ is *a closed function on* S, if for any closed subset F of S, $f[F]$ is closed in T.

The reader should be alerted to the fact that an *open function need not be a continuous function*; similarly, a closed function need not be continuous. Also, we caution the reader by pointing out that, even if $f : S \to T$ is an open function, it does *not* necessarily follow that $f^{\leftarrow} : T \to S$ is a continuous function on T. (See one of the examples that will follow.) However, for one-to-one functions f, "f is closed if and only if f is open" is true, as we shall now prove.

Theorem 6.6.1 Suppose $f : S \to T$ is *one-to-one and onto* T. Then f is an open function if and only if f is a closed function.

Proof : ($\Rightarrow$) Suppose $f : S \to T$ is a one-to-one and onto open function. We are required to show that f is also a closed function. Let F be a closed subset of S. Then $U = S \backslash F$ is open in S and so $f[U]$ is open in T. Then

$$\begin{aligned} f[U] &= f[S \backslash F] \\ &= f[S] \backslash f[F] \quad \text{(Since } f \text{ is one-to-one.)} \\ &= T \backslash f[F] \end{aligned}$$

Since $T \setminus f[F]$ is open in T then $f[F]$ is closed in T. We conclude that f is a closed function, as required.
Proof of the converse ($\Leftarrow$) is left to the reader.

Example 1. Suppose $(\mathbb{R}, \tau)$ and $(\mathbb{R}, \tau_S)$ denote the real line with the usual topology, τ, and with the the *upper limit topology* (Sorgenfrey line), τ_S, respectively.

Let $i : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_S)$ denote the identity map, $i(x) = x$. Verify that this identity map is open on $(\mathbb{R}, \tau)$ but is not continuous on its domain.

Solution : Since $i[(a, b)] = (a, b)$, i maps the open base element (a, b) of τ to the open set $(a, b) \in \tau_S$ (as we have seen earlier, $\tau \subset \tau_s$) then i maps open sets to open sets hence, i is an open function. But $(a, b\,] \notin \tau$. So $i^{\leftarrow}[(a, b\,]] = (a, b\,] \notin \tau$. So the open identity map i is *not* continuous on $\mathbb{R}$ with respect to τ.

This example illustrates that, if the codomain has more open sets then the domain, then a function will not pass the test of continuity.

Example 2. Let $(\mathbb{R}^2, \tau)$ be equipped with the usual topology. Suppose the open ball center (0, 0) of radius 1,

$$B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

is equipped with the subspace topology, τ_B. We define the function $f : B \to \mathbb{R}^2$ as follows:

$$f(x, y) = (x, y)$$

If U is an open subset of B then $f[U] = U \cap B$ an open subset of $\mathbb{R}^2$. So f is an open map. But f is not a closed map. To see this note that B is a closed subset of itself with respect to τ_B. On the other hand, $f[B] = B$ is not closed in the codomain $\mathbb{R}^2$.

As we shall now see, if f is one-to-one and onto, we have a different conclusion.

Theorem 6.7 Let (S, τ_S) and (T, τ_T) be topological spaces. For a one-to-one and onto continuous function $f : S \to T$

$$[\ f^{\leftarrow} : T \to S \text{ is continuous }] \Leftrightarrow [\ f : S \to T \text{ is open }]$$

Proof : The proof is left as an exercise.

6.3 Homeomorphic topological spaces.

For a continuous function, $f : S \to T$, its inverse function, $f^{\leftarrow} : T \to S$, may, *or may not*, be continuous, even if f is one-to-one. Those one-to-one continuous functions $f : S \to T$ where $f^{\leftarrow}$ is continuous on T have a special name.

Definition 6.8 Let (S, τ_S) and (T, τ_T) be topological spaces and $f : S \to T$ be a function. If f simultaneously satisfies all three of the following conditions,

1. f is one-to-one on S and onto T
2. f is continuous on S
3. $f^{\leftarrow}$ is continuous on T

then the function f is a *homeomorphism from S onto T*. If $f : S \to T$ is a homeomorphism then S and T are said to be *homeomorphic topological spaces*.

Example 3. Let the open interval, $S = (-\pi/2, \pi/2)$, be equipped with the usual subspace topology. The one-to-one and onto function, $\tan : S \to \mathbb{R}$, is well-known to be continuous on its domain, S. Similarly its inverse (arctan) , $\tan^{\leftarrow} : \mathbb{R} \to S$, is continuous on its domain, $\mathbb{R}$. By definition, tan is a homeomorphism. We can then say that $S = (-\pi/2, \pi/2)$ and $\mathbb{R}$ are homeomorphic topological spaces. In fact, as we shall see, any non-empty open interval of $\mathbb{R}$ is homeomorphic to $\mathbb{R}$.

Theorem 6.9 Let (S, τ_S) and (T, τ_T) be topological spaces. Suppose $f : S \to T$ is *one-to-one and onto S*. The following are equivalent:

1. The function f is a homeomorphism.
2. The function f is both continuous and open.
3. The function f is both continuous and closed.

Proof: The proof follows from the statement in theorem 6.7. It is left as an exercise.

Definition 6.9.1 Let (S, τ_S) and (T, τ_T) be topological space and $f : S \to Y$ be a function mapping S onto a subspace $Y = f[S]$ of T. If f is a homeomorphism then we say that "*f embeds S into T*" or, simply that "*f is an embedding*". We also say that "*T contains a homeomorphic copy of S*".

We have seen that, if $f : S \to T$ is a homeomorphism between the two topological spaces (S, τ_S) and (T, τ_T), the one-to-one function, f, maps each open base element in τ_S to a unique set in τ_T, and vice-versa. Since f is one-to-one it respects both arbitrary unions and arbitrary intersections; then every element of τ_S will be paired, under f, to exactly one element in τ_T. This suggests that properties in S which involve open sets will be mirrored inside T. We will refer to such properties as "topological properties". We formally define this notion.

Definition 6.10 A property P, defined in terms of open sets, which, if satisfied in (S, τ_S), is satisfied in every topological space which is homeomorphic to S is called a *topological property* or *topological invariant*.

6.4 Continuity and countability properties.

We have previously defined the three countability properties, "separable", "second countable" and "first countable". We would like to verify whether or not a continuous function will always carry over each of these properties from its domain into its codomain.

Recall that a topological space, (S, τ), is *separable* if and only if S contains a countable dense subset. We expect that, since a subset D is dense in S if and only if $S \backslash D$ does not contain a non-empty open subset, "separable" is a topological property. This fact is confirmed by the following result which shows that separability is carried over by continuous functions. Hence, if S is separable, every topological space which is homeomorphic to S is also separable.

Theorem 6.11 Suppose (S, τ_S) is a separable topological space. Then any continuous image of S is also separable.

Proof: Suppose (S, τ_S) is separable and $f : S \to T$ is a continuous function mapping S onto the topological space (T, τ_T). Then S contains a countable dense subset, say D. Since the cardinality of the image of a function is less than or equal to the cardinality of its domain, then $f[D]$ is a countable subset of T. Suppose U is a non-empty open subset of T. Continuity of f guarantees that $f^{\leftarrow}[U]$ is open and so there must exist $x \in f^{\leftarrow}[U] \cap D$. Then

$$\begin{array}{rcl} f(x) & \in & f[f^{\leftarrow}[U] \cap D] \\ & \subseteq & f[f^{\leftarrow}[U]] \cap f[D] \\ & = & U \cap f[D] \end{array}$$

Hence, every open subset U of T intersects $f[D]$ in a non-empty set. We conclude that $f[D]$ is dense in T. So T is separable.

A continuous function is by itself not quite strong enough to carry over the second countable property or the first countable property from its domain to its codomain. A homeomorphism will certainly do the trick. But we don't need the full power of a homeomorphism to do so. The following theorem shows that a continuous *open* function will suffice.

Theorem 6.12 Let $f : S \to T$ be a continuous *open* function mapping S onto a space T. If the space S is second countable then T is second countable. If the space S is first countable then T is first countable.

Proof: Let (S, τ_S) and (T, τ_T) be topological spaces and $f : S \to T$ be a *continuous open function* mapping S onto T.

If S is second countable then S has a countable basis $\mathscr{B}$. Consider the family, $\mathscr{B}_T = \{f[B] : B \in \mathscr{B}\}$. By hypothesis, $\mathscr{B}_T$ is a set of open subsets of T. We claim that the countable set, $\mathscr{B}_T$, is base for open sets in T.

Proof of claim : Let U be a non-empty open subset of T and let $y \in U$. Let $x \in f^{\leftarrow}(y) \in f^{\leftarrow}[U]$. Since f is continuous, $f^{\leftarrow}[U]$ is open in S, so there exist an open $V \in \mathscr{B}$ such that $x \in V \subseteq f^{\leftarrow}[U]$. Since f is declared to be open, $f(x) = y \in f[V] \in \mathscr{B}_T$, an open subset of U. We can then conclude that U is the union of elements from $\mathscr{B}_T$. So $\mathscr{B}_T$ forms a base for T, so T is second countable.

The proof that the "first countable" property is carried over by continuous open functions is left as an exercise.

6.5 The weak topology induced by a family of functions.

Suppose we are given a function, $f : S \to (T, \tau_T)$, mapping the set S onto the topological space, T, where T is equipped with some topology, τ_T, and S is not yet topologized. With this premise alone, we would normally not discuss the continuity of f since "continuity" is defined in terms of a topology on both the function's domain and codomain. But, theoretically, we could first hypothesize"continuity on f" and then force on S enough open sets on it so that this family of open sets would support this.

But how hard is this to do? In fact, it is quite easy. We need only equip the domain, S of f, with the discrete topology, τ_d. Then, no matter how f is defined, since every subset of S is open, then, f is, by definition, continuous on S. But this is not entirely satisfactory since the discrete topology on S doesn't depend on the function, f, at all. But we can tighten up our argument a bit.

Let $f : S \to (T, \tau_T)$ be any given function mapping the set S into the topological space T. We wish to topologize S so that the function f is guaranteed to be continuous on S. Furthermore, we want the smallest such topology. To do this we will let

$$\mathscr{S}_f = \{f^{\leftarrow}[U] : U \in \tau_T\}$$

be a subbase of S. By taking all finite intersections of elements from $\mathscr{S}_f$, we will generate, from $\mathscr{S}_f$, the smallest open base, $\mathscr{B}_f$, of S. By taking, all unions of elements from $\mathscr{B}_f$ we will generate the smallest topology, τ_f, with subbase $\mathscr{S}_f$. The family τ_f is the weakest topology possible that will guarantee continuity for f. This is why we will refer to τ_f as being the *weak topology on S induced by f*. This is because, eliminating just one element from this topology, would have as effect of producing, for some open subset U of T, a set $f^{\leftarrow}[U]$ which is not open in S. Also the topology we obtain is directly related to the function f. If we consider a different function we will obtain a different topology.

If we want the two functions $f : S \to T$ and $g : S \to T$ to be continuous on S, we would require more open sets on S. That is, the required subbase would have to be,

$$\mathscr{S}_{\{f,g\}} = \{f^{\leftarrow}[U] : U \in \tau_T\} \cup \{g^{\leftarrow}[U] : U \in \tau_T\}$$

In this case the topology generated by the subbase $\mathscr{S}_{\{f,g\}}$ would be called the *weak topology induced by $\{f,\ g\}$*.

We can generalize this even more. Rather than restrict ourselves to only two functions we will formally define a *weak topology on S induced by a family of functions* $\{f_\alpha : S \to T_\alpha\}_{\alpha \in I}$.

Definition 6.13 Let S be a non-empty set and $\{(T_\alpha, \tau_\alpha) : \alpha \in \Gamma\}$ denote a family of topological spaces. For each $\alpha \in \Gamma$, suppose $f_\alpha : S \to T_\alpha$ is a function mapping S onto T_α. Let

$$\mathscr{S} = \{f_\alpha^{\leftarrow}[U_\alpha] : U_\alpha \in \tau_\alpha\}_{\alpha \in \Gamma}$$

We let $\mathscr{S}$ be the subbase which will generate a topology, $\tau_{\mathscr{S}}$, on S. The topology $\tau_{\mathscr{S}}$ is called the *weak topology induced by the family of functions* $\{f_\alpha : \alpha \in \Gamma\}$ on S.

The reader is encouraged to keep this definition in mind. We will refer to the weak topology induced by a set of functions when we will discuss the question of a suitable topology on the Cartesian product of a family of topological spaces.

6.6 Topic: Continuous functions on a dense subset.

We examine particular properties of a continuous function on a dense subset of (S, τ).

Note that, if A is dense in B and $A \subseteq D \subseteq B$, then D must be dense in B. The reader is left to verify this. It is also easy to verify that, if D is dense in S, then $S \backslash D$ cannot contain any non-empty open subsets of S.

Before we state the next theorem we provide the following definition. If $f : S \to T$ and $g : S \to T$ are two functions such that $f(x) = g(x)$ for all $x \in A \subseteq S$ then we will say that "f and g agree on A". We present the following theorem concerning continuous functions which agree on a dense subset of topological space.

Theorem 6.14 Suppose (S, τ) is a topological space and (T, τ_ρ) is a metrizable topological space induced by the metric ρ. Suppose $f : S \to T$ and $g : S \to T$ are two continuous functions which agree on some dense subset D of (S, τ). Then f and g must agree on all of S.[1]

Proof: Let $U = \{x \in S : f(x) = g(x)\}$. By hypothesis, $D \subseteq U$. Since D is dense in S, then U is dense in S. We are required to show that $U = S$. Suppose $a \in S \backslash U$; then $f(a) \neq g(a)$. Then there exists in T, *disjoint* basic open subsets, $B_\varepsilon(f(a))$ and $B_\varepsilon(g(a))$, of S of radius ε, with center $f(a)$ and $g(a)$, respectively. Since both f and g are continuous on S, both $f^{\leftarrow}[B_\varepsilon(f(a))]$ and $g^{\leftarrow}[B_\varepsilon(g(a))]$ are open in S each contain at least the point a. So, the open subset

$$f^{\leftarrow}[B_\varepsilon(f(a))] \cap g^{\leftarrow}[B_\varepsilon(g(a))] \neq \varnothing$$

[1] Once we have introduced the concept of "Hausdorff" this statement generalizes from "metrizable topological spaces" to "Hausdorff topological spaces".

in S. Now for any

$$x \in f^{\leftarrow}[B_\varepsilon(f(a))] \cap g^{\leftarrow}[B_\varepsilon(g(a))]$$

$f(x) \in B_\varepsilon(f(a))$ and $g(x) \in B_\varepsilon(g(a))$ so $f(x) \neq g(x)$ (since $B_\varepsilon(f(a))$ and $B_\varepsilon(g(a))$ are disjoint). So $S\backslash U$ contains a non-empty open neighbourhood, $f^{\leftarrow}[B_\varepsilon(f(a))] \cap g^{\leftarrow}[B_\varepsilon(g(a))]$. Since U is dense in S, this cannot happen. Then there can be no point a in $S\backslash U$. So $U = S$, as required.

6.7 Topic: Continuous real-valued functions.

We now take a quick glance at those continuous functions with range in $\mathbb{R}$. If S is a topological space, we denote the set of all continuous functions, $f : S \to \mathbb{R}$, mapping S into $\mathbb{R}$ by $C(S)$. The set $C(S)$ is normally considered equipped with the algebraic operations $+$, $\cdot$ and scalar multiplication defined as

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (f\cdot g)(x) &= f(x)g(x)\\ (\alpha f)(x) &= \alpha f(x)\end{aligned}$$

We will assume that the reader is able to write proofs showing that $C(S)$ is closed under sums, multiplication and scalar multiplication and that $|f|$ defined as

$$|f|(x) = |f(x)|$$

also belongs to $C(S)$ whenever f is in $C(S)$. The elements of $C(S)$ can be partially ordered with "$\leq$" where $f \leq g$ if and only if $f(x) \leq g(x)$ on S. We also define the operations $\vee$ and $\wedge$ on $C(S)$ as,

$$\begin{aligned}f \vee g &= \max\{f(x), g(x)\}\ x \in S\\ f \wedge g &= \min\{f(x), g(x)\}\ x \in S\end{aligned}$$

Verification of the following formulas is left to the reader.

$$\begin{aligned}f \vee g &= \frac{f+g+|f-g|}{2}\\ f \wedge g &= \frac{f+g-|f-g|}{2}\end{aligned}$$

From which we can conclude that $f \vee g$ and $f \wedge g$ are continuous when f and g are.

Concepts review:

1. Given the function $f : S \to T$ and $A \subseteq S$, define $f[A]$.
2. Given the function $f : S \to T$ and $B \subseteq S$, define $f^{\leftarrow}[B]$.
3. State the formal topological definition of "f is continuous function on the set A".
4. State the formal topological definition of "f is continuous function at the point x".
5. Give a formal theorem statement which link the above two definitions of continuity.
6. A continuous function $f : S \to T$ pulls back open subbase elements in $\mathscr{P}(T)$ to what kind of set in S?
7. A continuous function $f : S \to T$ pulls back open base elements in $\mathscr{P}(T)$ to what kind of set in S?
8. If $f : S \Rightarrow T$ is continuous on S and $A \subseteq S$, show that $f|_A$ is continuous on A.
9. Is it correct to say "$f : S \to T$ is continuous if and only if f pulls back closed sets to closed sets"?
10. What does it mean to say $f : S \to T$ is an open functions?
11. What does it mean to say $f : S \to T$ is a closed functions?
12. Is it okay to say that "open functions are always closed"? What about "continuous"?
13. What does it mean to say that $f : S \to T$ is a homeomorphism?
14. What does it mean to say that S and T are homeomorphic spaces?
15. Give two characterizations of homeomorphic functions.
16. What is a topological property?
17. Is "first countable" a topological property? What about "second countable"?
18. Do continuous functions necessarily carry over the second countable property to its codomain? What about the first countable property?
19. What does it mean to say that A is a dense subset of B?
20. What can we say about two continuous functions which agree on a dense subset of a metrizable topological space?

21. What does it mean to say that a topological space is separable?
22. What can we say about continuous images of separable spaces?
23. What can we say about second countable spaces in reference to the "separable property"?
24. Provide examples of hereditary and non-hereditary topological properties.
25. Define the weak topology induced by a family of functions $\{f_\alpha : S \to T_\alpha\}_{\alpha \in I}$.

EXERCISES

1. Suppose $S = \{a, b\}$ is a set with topology $\tau_S = \{\varnothing, \{a\}, S\}$. Let $T = \{a, b\}$ where T is equipped with the discrete topology τ_d. Let $i : S \to T$ denote the identity map. What can we say about the continuity or non-continuity of the functions $i : S \to T$ and $i^{\leftarrow} : T \to S$?

2. Let $f : S \to T$ be a continuous function mapping (S, τ_S) onto (T, τ_T). If U is a G_δ in T, is $f^{\leftarrow}[U]$ necessarily a G_δ in S? Show that f pulls back F_σ's in T to F_σ's in S.

3. Let $f : (S, \tau_S) \to (T, \tau_T)$ be a continuous function mapping S onto T. Show that f is continuous on S if and only if $f[\mathrm{cl}_S[U]] \subseteq \mathrm{cl}_T f[U]$ for any $U \in \mathscr{P}(S)$.

4. Recall that *infinite countable sets* are those sets which can be mapped one-to-one and onto the natural numbers $\mathbb{N}$. Suppose X and Y are both countable dense subsets of $(\mathbb{R}, \tau)$ where τ is the usual topology. Show that X and Y must be homeomorphic subspaces of $\mathbb{R}$.

5. Consider the topological spaces $S = (\mathbb{R}^2, \tau_1)$ and $T = (\mathbb{R}, \tau)$ where τ_1 and τ represent the usual topology. (The open base of τ_1 are the open balls, $B_\varepsilon(x, y)$, with center (x, y) and radius ε). Consider the function $f : S \to T$ defined as $f(x, y) = x$.

   a) Show that f is an open function.
   b) Show that f is not a closed function. (Hint: See that $F = \{(x, y) : xy = 1\}$ is a closed subset of S. Consider the image of F under f.)

6) Prove: The function $f : S \to T$ is a closed function if and only if whenever F is a closed subset of S then $\{t \in T : f^{\leftarrow}[\{t\}] \cap F \text{ is non-empty}\}$ is a closed subset of T.

7) Suppose $f : S \to T$ is a closed function. Show that whenever V is an open subset of S and $f^{\leftarrow}[\{x\}]$ is a subset of V then $x \in \mathrm{int}_T(f[\mathrm{cl}_S V])$.

8) Suppose $f : S \to T$ is a closed function and F is closed subset of S. Show that the restriction, $f|_F : F \to f[F]$, is also a closed function.

9) Suppose U and V are subsets of S such that $U \cup V = S$ and $x \in U \cap V$. Suppose $f : S \to T$ is a function such that both $f|_U$ and $f|_V$ are continuous on U and V, respectively. Show that f is continuous at x.

10) Suppose that $f : S \to T$ is a one-to-one and onto function. Show that f is a homeomorphism if and only if, for any $U \in \mathscr{P}(S)$, $f[\mathrm{cl}_S U] = \mathrm{cl}_T f[U]$.

7 / Product spaces.

Summary. *In this section we will review some fundamental facts about those sets that are "Cartesian products of sets". We will then discuss how to topologize a Cartesian product of topological spaces and then study some of their most fundamental properties. Finally, we will look at some applications where product spaces play an important role. In particular, we prove that the closed interval* $[0, 1]$ *maps continuously onto the product space,* $[0, 1] \times [0, 1] \times [0, 1]$*, a cube in a three dimensional space.*

7.1 Fundamentals of Cartesian products.

The reader may find this chapter rather lengthy to read, at least compared to most others in this book. This is because, in topology, Cartesian products are an important source of examples of various topological properties as well as a tool to construct new spaces from old ones. For at least this reason, it is crucial to understand them well.

Most students at the high school level are exposed, in some form or other, to the notion of a Cartesian product with two or three factors. For example,

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

The reader will find in Appendix C, for review purposes, the most elementary rules of set operations involving finite Cartesian products.[1] When discussing *infinite* Cartesian products of sets we must proceed cautiously and decide which topology we will adopt to best suit our purposes. We thought it would be best if we begin by presenting a formal definition of products of sets. Those readers already well familiar with these concepts can skim through this section, or go directly to section 7.2.

Definition 7.1 Let $\{S_\alpha : \alpha \in I\}$ be an indexed family of sets. The Cartesian product of these sets, denoted by $\prod_\alpha S_\alpha$, or in more detail as,

$$\textstyle\prod_{\alpha \in I} S_\alpha = \{\ f \mid f : I \to \displaystyle\bigcup_{\alpha \in I} S_\alpha\}$$

is the set of all functions f mapping the index set, I, into the union, $\cup_{\alpha \in I} S_\alpha$, such that, for $\beta \in I$, $f(\beta) \in S_\beta$.[2] So, if u is an element of $\prod_\alpha S_\alpha$, then we can express it in the form

$$u = \{f(\alpha) : \alpha \in I\} = \{m_\alpha : \alpha \in I\}$$

or write it more simply as $\{m_\alpha\}$, where $m_\alpha = f(\alpha)$. If $\beta \in I$, the set, S_β, is called the β^{th} *factor* of the set, $\prod_\alpha S_\alpha$, and m_β is called the β^{th} *coordinate* of the element, $\{m_\alpha\}_{\alpha \in I}$.

[1] These are summarized excerpts from the set theory text *Axioms and set theory*, R. André.

[2] We will assume that a verification involving a combination of the *Axiom of union* and the *Axiom of power set* guarantees that $\prod_\alpha S_\alpha$ is indeed a "set".

For example, if $I = \{1, 2, 3\}$

$$\prod_{\alpha \in I} S_\alpha = \{(a, b, c) : a \in S_1, b \in S_2, c \in S_3\}$$

The size of $\prod_{\alpha \in I} S_\alpha$ depends on the size of the respective sets, S_α, and of the index set.

If we are given an indexed family of sets, say $\{S_\alpha : \alpha \in I\}$, and S_α is the same set S for all α then we often use the following notation:

$$S^I = \prod_{\alpha \in I} S$$

The expression, S^I, is normally interpreted as being "the set of all functions mapping I into S". For example, if $I = \mathbb{N}$ and $S_\alpha = \mathbb{R}$ for all $\alpha \in I$, then $\mathbb{R}^\mathbb{N} = \prod_{\alpha \in \mathbb{N}} \mathbb{R}$; it thus represents all countably infinite sequences of real numbers, $\{a_0, a_1, a_2, a_3, \ldots\}$, or equivalently, the set of all functions mapping $\mathbb{N}$ into $\mathbb{R}$. Another example is, $\mathbb{R}^\mathbb{R}$, which represents the set of all functions mapping $\mathbb{R}$ into $\mathbb{R}$. Since $\mathbb{R}$ is not normally viewed as an index set, the set $\mathbb{R}^\mathbb{R}$ is not normally expressed as a Cartesian product.

A few words on the definition of Cartesian product. In our definition of Cartesian product we refer to an indexed family, $\{S_\alpha : \alpha \in I\}$, of sets. We didn't state explicitly that each one of these is non-empty. Should we include this requirement in the definition? What happens if, say $S_\beta = \varnothing$, for some $\beta \in I$? Since there is nothing in S_β then there cannot exist a function which will map β to some element in S_β and so $\prod_\alpha S_\alpha$ must be empty. This is not catastrophic. So we can leave it as is. On the other hand, if no S_α is empty, are we guaranteed that there exists at least one function, $f : I \to \cup_{\alpha \in I} S_\alpha$, such that f assigns β in I to a particular element in S_β? If so, is there a way to decide which element should be selected by f? The assumption that at least one such function exists invokes the statement in the Axiom of choice:

> "Given any set $\mathscr{A}$ of non-empty sets, there is a rule f which associates to each set A in $\mathscr{A}$ some element $a \in A$".

So the Axiom of choice grants us permission to assume that at least one function f will select a point $f(\beta) = m_\beta$ in S_β for us. However, other than this guarantee that at least one f exists, we have no way of ever determining what that function f is. We are assuming the existence of a mathematical entity we will never ever see. Invoking the Axiom of choice is not ideal, but it is the best we can do. For most people, the fact that the Cartesian product of non-empty sets is non-empty is obvious and so they don't lose any sleep over it. Throughout this section, we will, as a rule, assume the Axiom of choice holds true, and not point out it's application at each place it is involved, unless it is of particular interest to do so.

In this book, when we say the "Cartesian product of topological spaces, $S = \prod_{i \in I} S_i$". it will always mean "the *non-empty* Cartesian product, S, of spaces."

A few set-theoretic properties of Cartesian products.

We have yet to topologize Cartesian products. But before, we present a few of the Cartesian product's most fundamental properties.

Definition 7.1.1 Let $S = \prod_{i \in I} S_i$ be a Cartesian product of sets. The family of functions $\{\pi_i : i \in I\}$ where $\pi_i : \prod_{i \in I} S_i \to S_i$, is defined as

$$\pi_i(\{x_i\}) = x_i$$

We will refer to π_i as being the i^{th} *projection* which maps $\prod_{i \in I} S_i$ *onto* S_i.

Theorem 7.1.2 Let $S = \prod_{i \in I} S_i$ and $T = \prod_{i \in I} T_i$ be Cartesian products of sets.

a) If $\prod_{i \in I} S_i \subseteq \prod_{i \in I} T_i$ then $S_i \subseteq T_i$ for each $i \in I$.
b) For U_i, V_i subsets of S_i,
   i) $\prod_{i \in I} U_i \cap \prod_{i \in I} V_i = \prod_{i \in I} (U_i \cap V_i)$
   ii) $\prod_{i \in I} U_i \cup \prod_{i \in I} V_i = \prod_{i \in I} (U_i \cup V_i)$

Proof : a) Let $\beta \in I$. Since the β^{th} projection map, π_j, is onto the $(S_j)^{\text{th}}$ factor,

$$S_j = \pi_j \left(\prod_{i \in I} S_i\right) \subseteq \pi_j \left(\prod_{i \in I} T_i\right) = T_j$$

The proofs of part b) are left for the reader.

7.2 Topologizing the Cartesian product of topological spaces.

Taking Cartesian products of large numbers of topological spaces is a powerful way to construct new topological spaces from old ones. By old ones, I mean topological spaces that have already been well studied or on which we already have a deep understanding. Of course, this implies that, at some point, we have agreed on a "natural" topology for such products. There are many possibilities for us to choose from, some of which will eventually be more useful than others. The topology which is the most useful in many cases is the one we will refer to as being "standard". For some special topological spaces we may opt for a non-standard topology.

The third example on page 74, illustrates how one might proceed to topologize the Cartesian product of two spaces (S, τ_S) and (T, τ_T). In that example, the topology, τ, on $S \times T$ was generated by choosing, as subbase for τ, the family of sets

$$\mathscr{S} = \{\pi_S^{\leftarrow}[U] : U \in \tau_S\} \cup \{\pi_T^{\leftarrow}[V] : V \in \tau_T\}$$

where $\pi_S : S \times T \to S$ and $\pi_T : S \times T \to T$ are projection maps. So the set of all finite intersections of the elements in $\mathscr{S}$ forms a base, $\mathscr{B}$, for this topology. Thus, the base elements of τ are of the form

$$\begin{aligned} \pi_S^{\leftarrow}[U] \cap \pi_T^{\leftarrow}[V] &= (U \times T) \cap (S \times V) \\ &= (U \cap S) \times (V \cap T) \\ &= U \times V \end{aligned}$$

where $U \in \tau_S$ and $V \in \tau_T$. That is, $\mathscr{B} = \{U \times V : U \in \tau_S, V \in \tau_T\}$.

Note that, in that example, the topology constructed for $S \times T$ can also be viewed as being the *weak topology induced by the family of functions* $\{\pi_S, \pi_T\}$. The "weak topology" approach provides $S \times T$ with the same subbase $\mathscr{S}$. In this way, one could also add that this topology guarantees continuity to each projection map, π_α, on $S \times T$.

Our inspiration for topologizing arbitrary Cartesian products of topological spaces will come from the technique we applied in that example.

Definition 7.2 Let $\{(S_\alpha, \tau_\alpha) : \alpha \in I\}$ be an indexed family of non-empty topological spaces and

$$S = \prod_{\alpha \in I} S_\alpha$$

be the Cartesian product of these spaces. Let $\{\pi_\alpha : \alpha \in I\}$ be the family of the associated projection maps

$$\pi_\beta : \prod_{\alpha \in I} S_\alpha \to S_\beta$$

We define the *product topology* as being the weak topology on $\prod_{\alpha \in I} S_\alpha$ induced by the family of functions $\{\pi_\alpha : \alpha \in I\}$. The Cartesian product of topological spaces, when equipped with this weak topology, is referred to as a *product space*. The family of all sets of the form,

$$\{\pi_\alpha^{\leftarrow}[U_\alpha] : U_\alpha \in \tau_\alpha\}$$

is the subbase, $\mathscr{S}$, for the product space. While each element of the base of open sets, $\mathscr{B}$, is of the form

$$\cap_{\alpha \in F} \{\pi_\alpha^{\leftarrow}[U_\alpha] : U_\alpha \in \tau_\alpha\}$$

where F is a finite subset of I.

Theorem 7.2.1 Given a product space, $S = \prod_{\alpha \in I} S_\alpha$, then the projection map, $\pi_\beta : S \to S_\beta$ is an open map.

Proof: Given: A product space, $S = \prod_{\alpha \in I} S_\alpha$, and a projection map, $\pi_\beta : S \to S_\beta$.

Since functions respect arbitrary unions, it suffices to show that the projection map sends basic open sets to open sets. Let $\cap\{\pi_{\alpha_i}^{\leftarrow}[U_{\alpha_i}] : i = 1, 2, \ldots, k\}$ be a basic open set in S. Then $\pi_\beta[\cap\{\pi_{\alpha_i}^{\leftarrow}[U_{\alpha_i}] : i = 1, 2, \ldots, k\}] = U_{\alpha_j}$ if $\beta = \alpha_j$ and is equal to S_β if $\beta \neq \alpha_i$ for $i = 1, 2, \ldots, k$. So π_β is an open map, as required.

A few remarks. Since $\pi_\alpha^{\leftarrow}[U_\alpha] \cap \pi_\alpha^{\leftarrow}[V_\alpha] = \pi_\alpha^{\leftarrow}[U_\alpha \cap V_\alpha]$, we can assume that all the α's in the finite set, F, are distinct. This assumption does not alter the definition of *product topology.* Also, $\pi_\beta^{\leftarrow}[U_\beta]$ is a subset of $\prod_{\alpha \in I} S_\alpha$ where, if $\alpha \neq \beta$, the α^{th} factor is, S_α, itself, and only the β^{th} factor is U_β. Hence, for any open base element, every factor is S_α itself except for finitely many factors, U_α, as proper subsets of S_α.

It is also worth noting that the product topology is the absolute smallest topology on $S = \prod_{\alpha \in I} S_\alpha$ which guarantees that each and every projection map in

$$\{\pi_\alpha : \textstyle\prod_{\alpha \in I} S_\alpha \to S_\alpha\}$$

is continuous on its domain, S.

Finally, note that the product topology depends on the topology of each of the factors. It does not depend on some topology defined on the index set, I.

Box topology on a Cartesian product.

The product topology is, of course, not the only topology we can define on a product of topological spaces. Some readers may have noticed that the set

$$\mathscr{B}^* = \cap_{\alpha \in I}\{\pi_\alpha^{\leftarrow}[U_\alpha] : U_\alpha \in \tau_\alpha\}$$

satisfies the "base property". Hence, this set will be a base for another topology, τ^* which, in the literature, is referred to as the *box topology.* Notice that every open base element, $B \in \mathscr{B}$, for the product topology is an open base element in $\mathscr{B}^*$. Hence the "box topology" is stronger (finer) than the "product topology". In fact, if every factor of $\prod_{\alpha \in I} U_\alpha$ is a proper open subset of S_α and $\{x_\alpha\} \in \prod_{\alpha \in I} U_\alpha \in \tau^*$, then there does not exist a $V \in \mathscr{B}$ such that $\{x_\alpha\} \in V \subseteq \prod_{\alpha \in I} U_\alpha$. Can you see why? So, $\mathscr{B}^* \nsubseteq \mathscr{B}$.

Note that for the Cartesian product of finitely many spaces, the product topology and the box topology are equivalent topologies.

Example 1. The product space $\mathbb{R}^3$ is metrizable. This can be seen since the basic open sets in $(\mathbb{R}^3, \tau)$ equipped with the product topology are 3-dimensional open rectangular boxes. While the basic open sets of $(\mathbb{R}^3, \rho)$, where $\rho(\vec{x}, \vec{y})$ is the distance between the two given points, are open balls. Since the rectangular boxes can be filled with open balls and vice versa the two spaces are equivalent topological spaces. So $(\mathbb{R}^3, \tau)$ is metrizable.

Example 2. The product space $[0,1]^3$ is metrizable. Since $[0,1]^3$ is a subspace of the metrizable, $\mathbb{R}^3$, then by the example on page 79, $[0,1]^3$ is metrizable.

Theorem 7.3 Let $\{(S_\alpha, \tau_\alpha) : \alpha \in I\}$ be an indexed family of non-empty topological spaces and $\prod_{\alpha \in I} S_\alpha$ be the Cartesian product space equipped with the product topology, τ. For each $\alpha \in I$, $\mathscr{B}_\alpha$ represents a base for open sets of S_α. Then the set

$$\mathscr{B}^* = \{ \cap_{\alpha \in F} \{\pi_\alpha^{\leftarrow}[B_\alpha] : B_\alpha \in \mathscr{B}_\alpha\} \}_{\alpha \in I}$$

where F is finite, forms a base for τ.

Proof: Let $V = \cap_{\alpha \in F}\{\pi_\alpha^{\leftarrow}[U_\alpha] : U_\alpha \in \tau\}$ (with F finite) be an open base element of the product topology, τ. Suppose $\{x_\alpha\} \in V$. Then, for each $\alpha \in F$, there exists an open base element, $B_\alpha \in \mathscr{B}_\alpha$, such that $x_\alpha \in B_\alpha \subseteq U_\alpha$. Then

$$\{x_\alpha\} \in \cap_{\alpha \in F}\{\pi_\alpha^{\leftarrow}[B_\alpha] : B_\alpha \in \mathscr{B}_\alpha\} \subseteq V$$

Then every open base element of τ is the union of elements from $\mathscr{B}^*$. So $\mathscr{B}^*$ forms a base for the product topology.

7.3 On products of spaces with countability properties.

In this text, when we say “product space”, we will always mean the Cartesian product equipped with the product topology. The product topology is to be considered as the default topology unless stated otherwise.

In the following two theorems we investigate when some product space inherits topological properties possessed by each of its factors. Recall that a topological space, (S, τ_S), is *second countable* if it has a countable base for open sets. It is *first countable* if each point, $x \in S$, has a countable open neighbourhood base at x.

Theorem 7.4 Let $\{(S_\alpha, \tau_\alpha) : \alpha \in \mathbb{N}\}$ be an indexed *countable* family of non-empty topological spaces and $S = \prod_{\alpha\in\mathbb{N}} S_\alpha$ be the corresponding product space.

a) Then the product space, S, is second countable if and only if each S_α is second countable.

b) Then the product space, S, is first countable if and only if each S_α is first countable.

Proof : We are given that $S = \prod_{\alpha\in\mathbb{N}} S_\alpha$ is a product space of countably many factors.

We will prove part a). The proof of part b) is similar and so is left as an exercise.

($\Rightarrow$) Suppose S is second countable. Recall from theorem 7.2.1 on page 105, the projection map π_α is continuous and open for each $\alpha \in \mathbb{N}$. Hence by theorem 6.12, each $S_\alpha = \pi_\alpha[S]$ is second countable. We are done with this direction.

($\Leftarrow$) We are given that, for each $\alpha \in \mathbb{N}$, (S_α, τ_α) is a second countable topological space, and $\prod_{\alpha\in\mathbb{N}} S_\alpha$ is equipped with the product topology, τ. By hypothesis, for each $\alpha \in \mathbb{N}$, S_α has a countable base, $\mathscr{B}_\alpha$, of open sets. Then, by theorem 7.3, the elements of τ of the form

$$\cap_{\alpha\in F}\{\pi_\alpha^{\leftarrow}[B_\alpha] : B_\alpha \in \mathscr{B}_\alpha\}$$

when gathered together form a base, $\mathscr{B}$, for open sets in τ. Since $\mathscr{B}_\alpha$ is countable then so is the family

$$\{\pi_\alpha^{\leftarrow}[B_\alpha] : B_\alpha \in \mathscr{B}_\alpha\} \subseteq \mathscr{B}$$

Let $\mathscr{F} = \{F : F \subset \mathbb{N}, F \text{ is finite}\}$. Note that the set, $\mathscr{F}$, has countably many elements.[1]

Since the cardinality of $\mathscr{F} \times \mathscr{B}_\alpha \times \mathbb{N}$ is $\aleph_0$, then the base

$$\mathscr{B} = \{\ \cap_{\alpha\in F}\{\pi_\alpha^{\leftarrow}[B_\alpha] : B_\alpha \in \mathscr{B}_\alpha\}\ \}_{F \text{ finite},\ \alpha\in\mathbb{N}}$$

is countable. This means $\prod_{\alpha\in\mathbb{N}} S_\alpha$ is second countable.

Theorem 7.5 Let $\{(S_\alpha, \tau_\alpha) : \alpha \in I\}$ be a family of sets indexed by the set I, where $|I| \leq c = 2^{\aleph_0}$. Also suppose that S_α is a non-empty non-singleton set, for all $\alpha \in I$. Let $S = \prod_{\alpha\in I} S_\alpha$, be the corresponding product space. Then S is separable if and only if each S_α is separable.

[1]For each $n \in \mathbb{N}$, let $\mathscr{U}_n = \{A \in \mathscr{P}(\mathbb{N}) : \max A \leq n\}$. If F is a finite subset of $\mathbb{N}$, $F \in \mathscr{U}_m$ for some m. Then $\cup_{n\in\mathbb{N}}\{\mathscr{U}_n\}$ contains all finite subsets of $\mathbb{N}$. Each $\mathscr{U}_n$ contains countably ($\leq 2^n$) many sets, and since the countable union of countable sets is countable (See 19.3 of Appendix B.), $\cup_{n\in\mathbb{N}}\{\mathscr{U}_n\}$ is countable.

Proof: We are given that $S = \prod_{\alpha \in I} S_\alpha$ is a product space of non-empty non-singleton sets where $|\, I\,| \leq 2^{\aleph_0}$.

($\Rightarrow$) Suppose S is separable. Recall from theorem 7.2.1 on page 105, the projection map π_α is continuous for each $\alpha \in \mathbb{N}$. Hence by theorem 6.11, each $S_\alpha = \pi_\alpha[S]$ is separable. We are done with this direction.

($\Leftarrow$) Suppose S_α is separable. By hypothesis, the index set, I, has cardinality less than or equal to $|\,[0,1]\,|$. We can view an element of S, as a sequence $\{x_{f(\alpha)} : \alpha \in I\}$, indexed by $f[I]$, a subset of $[0,1]$.

For each α, $S_{f(\alpha)}$ contains a dense subset,

$$D_{f(\alpha)} = \{d_{f(\alpha)_n} : n = 1, 2, 3, \ldots, \}$$

For each $k = 1, 2, 3, \ldots$, let $F_k = \{1, 2, 3, \ldots, k\}$.

For $j \in F_k$, let $J_j = [a_j, b_j]$ where $a_j, b_j \in \mathbb{Q} \cap [0,1]$ and $a_{j+1} > b_j$. So we have a finite sequence of k non-intersecting closed intervals with rational endpoints,

$$\{J_j\}_k = \{J_j : j \in F_k\}$$

along with a chosen corresponding finite sequence of k positive integers

$$\{n_j\}_k = \{n_j : j \in F_k\}$$

Let $\mathscr{D} = \{\, [\,\{J_j\}_k, \{n_j\}_k\,] : k = 1, 2, 3, \ldots, \}$.

Note that, for each $\alpha \in I$ and each k, $f(\alpha)$ belongs to, at most one, of the k, J_j-intervals.

We define a function h which maps each pair, $[\{J_j\}_k, \{n_j\}_k]$ in $\mathscr{D}$, to a point $\{x_{f(\alpha)} : \alpha \in I\}$ in S:

$$h(\ [\{J_j\}_k, \{n_j\}_k]\ ) = \{y_{f(\alpha)} : \alpha \in I\} \text{ where } \begin{cases} y_{f(\alpha)} = d_{f(\alpha)_{n_j}} & \text{if } f(\alpha) \in J_j \\ y_{f(\alpha)} = d_{f(\alpha)_1} & \text{otherwise} \end{cases}$$

The domain, $\mathscr{D}$, of h is countable hence the range, $h[\mathscr{D}] \subseteq S$, is countable.

We claim that $h[\mathscr{D}]$ is dense in S. Let $B = \pi^{\leftarrow}_{\alpha_1}[U_{\alpha_1}] \cap \cdots \cap \pi^{\leftarrow}_{\alpha_k}[U_{\alpha_k}]$ be a basic open neighbourhood of a point in S. Then, for each i, we can find $f(\alpha) \in J_i$ and $d_{f(\alpha)_{n_j}} \in U_{\alpha_i}$. Then B contains a point in the range of $h[\mathscr{D}]$. So the countable range of h is dense in S.

One should be careful about transferring a topological property possessed by all factors of a product to the product itself. Note that the statements of the above two theorems were proven for particular conditions on the number of factors.

7.4 Statements involving product spaces and continuous functions.

We presently know of one family of functions whose members are guaranteed to be continuous on a product space. It is the family of all its projection functions. Determining whether other functions whose domain is a product space are continuous, or not, can sometimes be tricky The following lemma and theorem provide us with useful tools.

Lemma 7.6 Let $Y = \prod_{\alpha \in I} S_\alpha$ be a product space and (Z, τ) be a space. Suppose $g : Z \to Y$ is a function mapping the space Z to Y. Then g is continuous on Z if and only if $\pi_\alpha \circ g : Z \to S_\alpha$ is continuous for all α.

Proof: Recall that, by definition of "product space", each π_α is guaranteed to be continuous on Y.

($\Rightarrow$) If $g : Z \to Y$ is continuous then so is $(\pi_\alpha \circ g) : Z \to S_\alpha$ (since the composition of continuous functions is continuous).
($\Leftarrow$) We fix $\beta \in I$. We are given that $(\pi_\beta \circ g) : Z \to S_\beta$ is continuous. We are required to show that g is continuous. If V_β is open in S_β, then

$$g^{\leftarrow}\left[\pi_\beta^{\leftarrow}[V_\beta]\right] = (\pi_\beta \circ g)^{\leftarrow}[V_\beta]$$

is open in $Y = \prod_{\alpha \in I} S_\alpha$. So g pulls back an open subbase element, $\pi_\beta^{\leftarrow}[V_\beta]$, to an open set. By theorem 6.4, g is continuous on Z, as required.

In the following statement we refer to a function, $g : \prod_{\alpha \in I} S_\alpha \to \prod_{\alpha \in I} T_\alpha$, which is defined in terms of a family. $\{f_\alpha\}$, of other functions, $f_\alpha : S_\alpha \to T_\alpha$. The statement is sometimes summarized as "componentwise continuity implies continuity on the product".

Theorem 7.7 Let $\{S_\alpha\}_{\alpha \in I}$ and $\{T_\alpha\}_{\alpha \in I}$ be two sets of topological spaces and let $\{f_\alpha\}_{\alpha \in I}$ be a family of functions, $f_\alpha : S_\alpha \to T_\alpha$. Then the function, $g : \prod_{\alpha \in I} S_\alpha \to \prod_{\alpha \in I} T_\alpha$, defined as $g(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}_{\alpha \in I}$ is continuous if and only if f_α is continuous for each $\alpha \in I$.

Proof: Suppose $S = \prod_{\alpha\in I} S_\alpha$ and $Y = \prod_{\alpha\in I} T_\alpha$. Let $\pi_{\beta_S} : S \to S_\beta$ and $\pi_{\beta_Y} : Y \to T_\beta$ be β^{th} projection maps.

($\Leftarrow$) Suppose each function f_α is continuous on S_α. To show continuity of g it will suffice to show that g pulls back subbase elements of $\prod_{\alpha\in I} T_\alpha$ to open sets in $\prod_{\alpha\in I} S_\alpha$. Let U_β be an open subset of T_β. Then $\pi^{\leftarrow}_{\beta_Y}[U_\beta]$ is a subbase element for open sets in $Y = \prod_{\alpha\in I} T_\alpha$.

It suffices to show that $g^{\leftarrow}[\pi^{\leftarrow}_{\beta_Y}[U_\beta]]$ is open in S.
See that,

$$\begin{aligned}
\pi^{\leftarrow}_{\beta_S}[f^{\leftarrow}_\beta[U_\beta]] &= \pi^{\leftarrow}_{\beta_S}[\{x \in S_\beta : f_\beta(x) \in U_\beta\}] \\
&= \{\{x_\alpha\} \in S : f_\beta(x_\beta) \in U_\beta\} \\
&= \{\{x_\alpha\} \in S : \{f_\alpha(x_\alpha)\} \in \pi^{\leftarrow}_{\beta_Y}[U_\beta]\} \\
&= \{\{x_\alpha\} \in S : g(\{x_\alpha\}) \in \pi^{\leftarrow}_{\beta_Y}[U_\beta]\} \\
&= g^{\leftarrow}\left[\{\{x_\alpha\} \in S : \{x_\alpha\} \in \pi^{\leftarrow}_{\beta_Y}[U_\beta]\}\right] \\
&= g^{\leftarrow}[\pi^{\leftarrow}_{\beta_Y}[U_\beta]]
\end{aligned}$$

so,

$$g^{\leftarrow}[\pi^{\leftarrow}_{\beta_Y}[U_\beta]] = \pi^{\leftarrow}_{\beta_S}[f^{\leftarrow}_\beta[U_\beta]]$$

Since both π_{β_S} and f_β are continuous then the right-hand side is open, so $g^{\leftarrow}[\pi^{\leftarrow}_{\beta_Y}[U_\beta]]$ is open. So g pulls back open subbase elements to open sets, which implies g is continuous.

($\Rightarrow$) Suppose $g : \prod_{\alpha\in I} S_\alpha \to \prod_{\alpha\in I} T_\alpha$ is continuous. Then for each β, $(\pi_\beta \circ g)(\{x_\alpha\}) = \pi_\beta(\{f_\alpha(x_\alpha)\}) = f_\beta(x_\beta) \in T_\beta$. Since both π_β and g are continuous on their domain then so is f_β.

It will be important to confirm that, *if two product spaces, S and T, have corresponding factors which are homeomorphic then S and T are homeomorphic*. In the proof of the following theorem we explicitly identify the homeomorphism.

Theorem 7.8 Let $S = \prod_{\alpha\in I} S_\alpha$ and $T = \prod_{\alpha\in I} T_\alpha$ be two product spaces. Suppose that, for each $\alpha \in I$, S_α and T_α are homeomorphic. Then the spaces S and T are homeomorphic.

Proof: For each $\alpha \in I$, let $f_\alpha : S_\alpha \to T_\alpha$ be a homeomorphism.

Let $g : \prod_{\alpha\in I} S_\alpha \to \prod_{\alpha\in I} T_\alpha$ be defined as $g(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\} \in \prod_{\alpha\in I} T_\alpha$. Since each f_α is continuous and one-to-one then so is g (by the theorem above).

We claim that g is open.

By hypothesis, each f_α is an open map. Let $\pi^{\leftarrow}_{\alpha_1}[U_{\alpha_1}] \cap \pi^{\leftarrow}_{\alpha_2}[U_{\alpha_2}] \cdots \cap \pi^{\leftarrow}_{\alpha_k}[U_{\alpha_k}]$ be an open base element in $\prod_{\alpha\in I} S_\alpha$. Since g is one-to-one, see that

$$\begin{aligned} g\left[\pi^{\leftarrow}_{\alpha_1}[U_{\alpha_1}] \cap \pi^{\leftarrow}_{\alpha_2}[U_{\alpha_2}] \cdots \cap \pi^{\leftarrow}_{\alpha_k}[U_{\alpha_k}]\right] &= g\left[\pi^{\leftarrow}_{\alpha_1}[U_{\alpha_1}]\right] \cap g\left[[\pi^{\leftarrow}_{\alpha_2}[U_{\alpha_2}]\right] \cdots \cap g\left[[\pi^{\leftarrow}_{\alpha_k}[U_{\alpha_k}]\right] \\ &= f_{\alpha_1}\left[\pi^{\leftarrow}_{\alpha_1}[U_{\alpha_1}]\right] \times \cdots \times f_{\alpha_k}\left[\pi^{\leftarrow}_{\alpha_k}[U_{\alpha_k}]\right] \\ &\quad \text{with all other factors equal to } T_\gamma. \\ &\quad \text{Since each } f_{\alpha_i} \text{ is open RHS is open in } \textstyle\prod_{\alpha\in I} T_\alpha. \end{aligned}$$

Since the right-hand side is open, then g is open. So g is a homeomorphism.

We have shown that if, for each $\alpha \in I$, S_α and T_α are homeomorphic, then two product spaces, $\prod_{\alpha\in I} S_\alpha$ and $\prod_{\alpha\in I} T_\alpha$, must also be homeomorphic.

Theorem 7.9 Let $S = \prod_{\alpha\in I} S_\alpha$ be a product space. Then, for each $\alpha \in I$, S contains a subspace which is a homeomorphic copy of S_α.

Proof: Let $\beta \in I$. For $\alpha \neq \beta$, we choose and fix $k_\alpha \in S_\alpha$. Let

$$T = \left\{\{x_\alpha\} \in \textstyle\prod_{\alpha\in I} S_\alpha : \alpha \neq \beta \Rightarrow S_\alpha = \{k_\alpha\}\right\}$$

So every factor of T is a singleton set except for S_β. We define $g : S_\beta \to T$ as $g(x) = \{x_\alpha\}$, where $x_\alpha = x$ if $x \in S_\beta$. Then g maps S_β one-to-one and onto $T \subseteq Y$. Then the function, $(\pi_\alpha \circ g) : S_\beta \to T$, maps x to x on S_β if $\alpha = \beta$, and maps x to the constant k_α for $\alpha \neq \beta$. So $\pi_\alpha \circ g$ is continuous on S_β for all α. Invoking theorem 7.6, we conclude that g is continuous on S_β. So g embeds S_β in the proper subset T of $\prod_{\alpha\in I} S_\alpha$ and so embeds S_β in $\prod_{\alpha\in I} S_\alpha$.

The above theorem simply states that every factor of a product space is embedded in the product space itself. We suspected as much, but the proof exhibits explicitly the required homeomorphism.

7.5 The closure of a subset of a product space.

The following theorem confirms that the "cl_S" symbol distributes over to its factors and maintains equality.

Theorem 7.10 Let $S = \prod_{\alpha\in I} S_\alpha$ be a product space and, for each α, let U_α be a non-empty subset of S_α. Then,

$$\text{cl}_S\left(\prod_{\alpha\in I} U_\alpha\right) = \prod_{\alpha\in I} (\text{cl}_{S_\alpha} U_\alpha)$$

Proof: Note that $\prod_{\alpha\in I} (U_\alpha) \subseteq \prod_{\alpha\in I} (\text{cl}_{S_\alpha} U_\alpha)$.

We claim : $\prod_{\alpha\in I} (\text{cl}_{S_\alpha} U_\alpha) \subseteq \text{cl}_S\left(\prod_{\alpha\in I} U_\alpha\right)$.

Let $\{x_\alpha\} \in \prod_{\alpha\in I} (\text{cl}_{S_\alpha} U_\alpha)$. It suffices to show that any open set containing $\{x_\alpha\}$ will intersect $\prod_{\alpha\in I} (U_\alpha)$.

Let $B_{\{x_\alpha\}} = \cap_{\alpha\in F}\{\pi_\alpha^{\leftarrow}[V_\alpha]\}$ be an open base neighbourhood of $\{x_\alpha\}$. Since $x_\alpha \in \text{cl}_{S_\alpha} U_\alpha$ for $\alpha \in F$, then, for each $\alpha \in F$, $V_\alpha \cap U_\alpha$ is non-empty. This implies that $B_{\{x_\alpha\}}$ intersects $\prod_{\alpha\in I} U_\alpha$ in a non-empty set. We can then conclude that $\{x_\alpha\} \in \text{cl}_S\left(\prod_{\alpha\in I} U_\alpha\right)$.

Hence, $\prod_{\alpha\in I} (\text{cl}_{S_\alpha} U_\alpha) \subseteq \text{cl}_S\left(\prod_{\alpha\in I} U_\alpha\right)$, as claimed.

We claim : $\text{cl}_S\left(\prod_{\alpha\in I} U_\alpha\right) \subseteq \prod_{\alpha\in I} (\text{cl}_{S_\alpha} U_\alpha)$.

Let $\{x_\alpha\} \in \text{cl}_S\left(\prod_{\alpha\in I} U_\alpha\right)$. To show that $\{x_\alpha\} \in \prod_{\alpha\in I} (\text{cl}_{S_\alpha} U_\alpha)$, it will suffice to that $x_\alpha \in \text{cl}_{S_\alpha} U_\alpha$, for each α.

Let $\beta \in I$, and V_β be an open base neighbourhood of x_β in $\text{cl}_{S_\beta} U_\beta$. Then $\pi_\beta^{\leftarrow}(x_\beta) \subseteq \pi_\beta^{\leftarrow}[V_\beta]$ is a non-empty open subset of $\text{cl}_S\left(\prod_{\alpha\in I} U_\alpha\right)$ and so its intersection with $\prod_{\alpha\in I} U_\alpha$ must contain at least one point, say $\{y_\alpha\}$. Since $y_\beta \in V_\beta \cap U_\beta$, then it must be that $x_\beta \in \text{cl}_{S_\alpha} U_\alpha$. So $\{x_\alpha\} \in \prod_{\alpha\in I} (\text{cl}_{S_\alpha} U_\alpha)$.

Hence, $\text{cl}_S\left(\prod_{\alpha\in I} U_\alpha\right) \subseteq \prod_{\alpha\in I} (\text{cl}_{S_\alpha} U_\alpha)$, as claimed.

We conclude that $\text{cl}_S\left(\prod_{\alpha\in I} U_\alpha\right) = \prod_{\alpha\in I} (\text{cl}_{S_\alpha} U_\alpha)$.

7.6 Topic: Products of products

We address the question: Can we transform a product space into a product of product spaces? We begin by illustrating what we mean by this question.

Let $I = \{1, 2, \ldots, 12\}$, $A = \{1, 2, 3, 4\}$, $B = \{5, 6\}$, and $C = \{7, 8, \ldots, 12\}$. The set I is the disjoint union the of sets A, B and C. Let $\{(S_\alpha, \tau_\alpha) : \alpha \in I\}$ be a set of twelve topological spaces. Consider the two product spaces S and Y defined as follows:

$$S = \prod_{\alpha\in I} S_\alpha \quad \text{and} \quad Y = \prod_{\alpha\in A} S_\alpha \times \prod_{\alpha\in B} S_\alpha \times \prod_{\alpha\in C} S_\alpha$$

We wonder, "How do the two product spaces compare?". We see that S is a product space with twelve factors, while Y is a product space of only three factors. This observation is sufficient to conclude that $S \neq Y$. On the other hand, we see that each of the three factors of Y are themselves product spaces, all three with factors, S_α, identical to the ones found in S. While we have excluded, the possibility that S equals Y, it would be reasonable to suspect that these two spaces are homeomorphic copies of each other. We will present this conjecture as a proposition (in just a slightly more general form) followed by its proof.

Proposition 7.11 Let $\{(S_\alpha, \tau_\alpha) : \alpha \in I\}$ be a set of non-empty topological spaces. Suppose I is the disjoint union of three non-empty subsets, A, B and C where the elements in $A \cup B \cup C$ respect the order in which they appear in I. Then

$$S = \prod_{\alpha\in I} S_\alpha \quad \text{and} \quad Y = \prod_{\alpha\in A} S_\alpha \times \prod_{\alpha\in B} S_\alpha \times \prod_{\alpha\in C} S_\alpha$$

are homeomorphic product spaces.

Proof: We define three functions, $\theta_A : S \to \prod_{\alpha\in A} S_\alpha$, $\theta_B : S \to \prod_{\alpha\in B} S_\alpha$ and $\theta_C : S \to \prod_{\alpha\in C} S_\alpha$ as follows:

$$\begin{aligned} \theta_A(\{x_\alpha\}_{\alpha\in I}) &= \{x_\alpha\}_{\alpha\in A} \\ \theta_B(\{x_\alpha\}_{\alpha\in I}) &= \{x_\alpha\}_{\alpha\in B} \\ \theta_C(\{x_\alpha\}_{\alpha\in I}) &= \{x_\alpha\}_{\alpha\in C} \end{aligned}$$

All three functions are easily seen to be onto $\prod_{\alpha\in A} S_\alpha$, $\prod_{\alpha\in B} S_\alpha$ and $\prod_{\alpha\in C} S_\alpha$, respectively. Note that

$$\begin{aligned} \text{For } \beta \in A,\ (\pi_\beta \circ \theta_A)(\{x_\alpha\}_{\alpha\in I}) &= x_\beta = \pi_\beta(\{x_\alpha\}_{\alpha\in I}) \\ \text{For } \beta \in B,\ (\pi_\beta \circ \theta_B)(\{x_\alpha\}_{\alpha\in I}) &= x_\beta = \pi_\beta(\{x_\alpha\}_{\alpha\in I}) \\ \text{For } \beta \in C,\ (\pi_\beta \circ \theta_C)(\{x_\alpha\}_{\alpha\in I}) &= x_\beta = \pi_\beta(\{x_\alpha\}_{\alpha\in I}) \end{aligned}$$

Since, for each $\beta \in I$, π_β is continuous then, by theorem 7.6, the functions θ_A, θ_B, and θ_C are all continuous on S.

We now consider the function

$$\theta : \prod_{\alpha\in I} S_\alpha \to \prod_{\alpha\in A} S_\alpha \times \prod_{\alpha\in B} S_\alpha \times \prod_{\alpha\in C} S_\alpha$$

defined as

$$\theta(\{x_\alpha\}_{\alpha\in I}) = \{\, \theta_A(\{x_\alpha\}_{\alpha\in I}),\ \theta_B(\{x_\alpha\}_{\alpha\in I}),\ \theta_C(\{x_\alpha\}_{\alpha\in I}) \,\}$$

Since θ_A, θ_B and θ_C are all three continuous then, by theorem 7.6.1 on page 109, $\theta : S \to Y$ is continuous on its domain S. Also, since I is the disjoint union of A,

B and C, then θ maps S one-to-one and onto Y. (Verify this!) To show that θ is a homeomorphism it now suffices to show that it is open.

Let $V = \pi_{\alpha_1}^{\leftarrow}[U_{\alpha_1}] \cap \pi_{\alpha_2}^{\leftarrow}[U_{\alpha_2}] \cap \cdots \cap \pi_{\alpha_k}^{\leftarrow}[U_{\alpha_k}]$ be an open base element in S. Then

$$\begin{aligned}
\theta[V] &= \{\,\{\theta_A[V],\ \theta_B[V],\ \theta_C[V]\,\} \\
&= \{\theta_\Phi[V]\}_{\Phi\in\{A,B,C\}} \\
&= \left\{\theta_\Phi\left[\,\pi_{\alpha_1}^{\leftarrow}[U_{\alpha_1}] \cap \pi_{\alpha_2}^{\leftarrow}[U_{\alpha_2}] \cap \cdots \cap \pi_{\alpha_k}^{\leftarrow}[U_{\alpha_k}]\,\right]\right\}_{\Phi\in\{A,B,C\}} \\
&= \left\{\theta_\Phi\left[\,\pi_{\alpha_1}^{\leftarrow}[U_{\alpha_1}]\right] \cap \theta_\Phi\left[\pi_{\alpha_2}^{\leftarrow}[U_{\alpha_2}]\right] \cap \cdots \cap \theta_\Phi\left[\pi_{\alpha_k}^{\leftarrow}[U_{\alpha_k}]\,\right]\right\}_{\Phi\in\{A,B,C\}}
\end{aligned}$$

See that

$$\begin{aligned}
\theta_\Phi[\pi_{\alpha_i}^{\leftarrow}[U_{\alpha_i}]] &= U_{\alpha_i} \text{ if } \alpha_i \in \Phi \\
&= \textstyle\prod_{\alpha\in\Phi} S_\alpha \text{ if } \alpha_i \notin \Phi
\end{aligned}$$

So $\theta[V]$ is open in Y.

The given homeomorphism, θ, confirms that the product spaces

$$\textstyle\prod_{\alpha\in A\cup B\cup C} S_\alpha \quad \text{and} \quad \prod_{\alpha\in A} S_\alpha \times \prod_{\alpha\in B} S_\alpha \times \prod_{\alpha\in C} S_\alpha$$

are homeomorphic.

The statement in the proposition above generalizes to product spaces with an infinite number of factors. The proof flows similarly. We can also summarize the statement by saying,

Associativity holds true for product spaces.

For the next example, we wonder how a product space compares with the product of finitely many copies of itself.

Example 3. Let $S = \prod_{\alpha\in I} S_\alpha$ be a product space with a countably infinite index set I. Show that S and $S \times S \times S$ are homeomorphic product spaces.

Solution : Since I is countably infinite then $3I$ is countably infinite and so there exists a one-to-one and onto function $q : I \to 3I$.

We claim that $\prod_{\alpha\in I} S_\alpha$ and $\prod_{\alpha\in 3I} S_\alpha$ are homeomorphic. Define $f : \prod_{\alpha\in I} S_\alpha \to \prod_{\alpha\in 3I} S_\alpha$ as $f(\{x_\alpha\}) = \{x_{q(\alpha)}\}_{\alpha\in I}$. Since $q : I \to 3I$ is one-to-one then so is f. Since

$$(\pi_{q(\alpha)} \circ f)(\{x_\alpha\}) = \pi_{q(\alpha)}(\{x_{q(\alpha)}\}) = x_{q(\alpha)}$$

is a projection map for each $q(\alpha)$, then $\pi_{q(\alpha)}\circ f$ it is continuous for each $q(\alpha)$. Then, by theorem 7.6, f is continuous on $\prod_{\alpha\in I} S_\alpha$. Now $f^{\leftarrow}(\{x_{q(\alpha)}\}) = \{x_\alpha\}$ is easily seen to be one-to-one and continuous. Then $f : \prod_{\alpha\in I} S_\alpha \to \prod_{\alpha\in 3I} S_\alpha$ is a homeomorphism and $\prod_{\alpha\in I} S_\alpha$ and $\prod_{\alpha\in 3I} S_\alpha$ are homeomorphic product spaces, as claimed. Express $3I$ as $3I = I_A \cup I_B \cup I_C$ (the disjoint union of 3 infinite subsets of $3I$).

In the proof of theorem 7.11, we constructed a homeomorphism $\theta : \prod_{\alpha\in I_A\cup I_B\cup I_C} S_\alpha \to \prod_{\alpha\in I_A} S_\alpha \times \prod_{\alpha\in I_B} S_\alpha \times \prod_{\alpha\in I_C} S_\alpha$ defined as

$$\theta(\{x_\alpha\}_{\alpha\in 3I}) = \{\, \theta_A(\{x_\alpha\}_{\alpha\in 3I}),\ \theta_B(\{x_\alpha\}_{\alpha\in 3I}),\ \theta_C(\{x_\alpha\}_{\alpha\in 3I})\,\}$$

Then we have that $\prod_{\alpha\in 3I} S_\alpha$, $\prod_{\alpha\in I} S_\alpha$ and $\prod_{\alpha\in I} S_\alpha \times \prod_{\alpha\in I} S_\alpha \times \prod_{\alpha\in I} S_\alpha$ are three homeomorphic, as required.

The following statement generalizes some of the ideas expressed in above example.

Theorem 7.12 Let $S = \prod_{\alpha\in I} S_\alpha$ and let $q : I \to I$ be a one-to-one map onto I where $q = q(\alpha)$. Then the product spaces $T = \prod_{q\in J} S_q = \prod_{\alpha\in I} S_{q(\alpha)}$ and S are homeomorphic.

Proof: Let $S = \prod_{\alpha\in I} S_\alpha$ and let $q : I \to I$ be a one-to-one map onto I. We are required to show that $T = \prod_{\beta\in J} S_\beta = \prod_{\alpha\in I} S_{q(\alpha)}$ and S are homeomorphic.

Let $q^* : S \to T$ be defined as $q^*(\{x_\alpha\}) = \{x_{q(\alpha)}\}$. Since q is one-to-one and onto I, then both q^* and $q^{*\leftarrow}$ are one-to-one and onto T and S, respectively. Then $\pi_{q(\alpha)}\circ q^* : S \to S_{q(\alpha)}$. Verify that $\pi_{q(\alpha)}\circ q^*$ pulls pack and open subset of $S_{q(\alpha)}$ to an open subset of S_α. So $\pi_{q(\alpha)}\circ q^*$ is continuous for each α. By theorem 7.6, $q^* : S \to T$ is continuous. Similarly $q^{*\leftarrow} : T \to S$ is continuous. So q^* maps S homeomorphically onto T.

The above theorem confirms that "altering the order of the factors of a product space produces another product space which is homeomorphic to the original one". We can summarize the statement by saying that . . . '

product spaces are commutative.

7.7 Topic: A product of ordinal spaces: Tychonoff plank

We end the section on product spaces by briefly discussing the product of two ordinal spaces. In the following example, ω_1 represents the first uncountable ordinal, while, ω_0 represents the first countable infinite ordinal. Let W represent the ordinal space, $[0, \omega_1]$, and T represent the ordinal space, $[0, \omega_0]$. Recall that the elements of the open base for an ordinal space are of the form $(\alpha, \beta]$.

Let $S = W \times T = [0, \omega_1] \times [0, \omega_0]$ be the product space of the two given ordinal spaces. Then the elements of S can be viewed as ordered pairs $(\alpha, \beta) \in W \times T$. Since both sets are linearly ordered, it doesn't hurt to represent the product space, S, as a Cartesian plane of numbers where W represents the horizontal axis and T represents the vertical axis. We would then have $(0, 0)$ in the lower left corner and (ω_1, ω_0) in the top right corner. The topological space $S = [0, \omega_1] \times [0, \omega_0]$ equipped with this topology is commonly referred to by topologists as the

"Tychonoff plank"

The subspace S^* of S defined as $S^* = S \backslash \{(\omega_1, \omega_0)\}$ simply obtained by deleting the top right corner from the Tychonoff plank is appropriately referred to as the

"Deleted Tychonoff plank"

As an open neighbourhood base of the point, $(\beta, \mu) \in S$, we can use elements of the form

$$\mathscr{B}_{(\beta,\mu)} = \{(\alpha, \beta] \times (\gamma, \mu] : \alpha < \beta \text{ and } \gamma < \mu\}$$

In the following few sections we will illustrate how product spaces play an important role, sometimes in unexpected ways, in various applications. We briefly explain and describe the content of the theorem statements. Some of the proofs tend to be a bit involved and, at least initially, might appear a bit intimidating (even to the experienced reader), so, when this is the case, the proofs are given in Appendix A at the end of this text.

7.8 Topic: The embedding theorem.

Firstly, we will show how a family, $\mathscr{F}$, of continuous functions on a topological space, (S, τ), can be used to embed S in a product space whose factors are the range of the functions in $\mathscr{F}$. This method is exhibited in a theorem titled "The embedding theorem". The *embedding theorem* applies only to topological spaces, S, in which singleton sets are closed. In order to understand its statement we must introduce two very important notions in topology. Firstly, we will define what we mean when we say that *a set of functions separates points and closed sets* of S. Secondly, will define an *evaluation map with respect to a family of functions.*

Definition 7.13 Let (S, τ_S) be a topological space and $\{(X_\alpha, \tau_\alpha) : \alpha \in \Gamma\}$ be an indexed family of non-empty topological spaces. Let $\mathscr{F} = \{f_\alpha : \alpha \in \Gamma\}$ be a set of *continuous functions*, $f_\alpha : S \to X_\alpha$, each one mapping S onto its range $f_\alpha[S] \subseteq X_\alpha$.

a) We say that $\mathscr{F}$ *separates points and closed sets* if, whenever F is a closed subset of S and $x \notin F$, then there exists at least one function $f_\beta \in \mathscr{F}$ such that $f_\beta(x) \notin \mathrm{cl}_{X_\beta} f_\beta[F]$.

b) We define a function, $e : S \to \prod_{\alpha\in\Gamma} X_\alpha$, as follows:

$$e(x) = \{f_\alpha(x)\} \in \prod_{\alpha\in\Gamma} f_\alpha[S] \subseteq \prod_{\alpha\in\Gamma} X_\alpha$$

We refer to the function $e : S \to \prod_{\alpha\in\Gamma} X_\alpha$ as the *evaluation map of S into $\prod_{\alpha\in\Gamma} X_\alpha$ with respect to $\mathscr{F}$.*

Theorem 7.14 *The embedding theorem I.* Let (S, τ_S) be a topological space in which every singleton set in S is a closed subset of S. Given an indexed family of non-empty topological spaces, $\{(X_\alpha, \tau_\alpha) : \alpha \in \Gamma\}$, let $\mathscr{F} = \{f_\alpha : \alpha \in \Gamma\}$ be a set of *continuous functions* where each f_α maps S onto its range, $f_\alpha[S]$, inside X_α.

a) If $\mathscr{F}$ *separates points and closed sets* of its domain, S, then the evaluation map,

$$e(x) = \{f_\alpha(x)\} \in \prod_{\alpha\in\Gamma} f_\alpha[S] \subseteq \prod_{\alpha\in\Gamma} X_\alpha$$

with respect to $\mathscr{F}$, is both continuous and one-to-one on S.

b) Furthermore, the function, $e : S \to \prod_{\alpha\in\Gamma} X_\alpha$, maps S homeomorphically onto $e[S] \subseteq \prod_{\alpha\in\Gamma} f_\alpha[S]$ in $\prod_{\alpha\in\Gamma} X_\alpha$. Hence this evaluation map embeds a homeomorphic copy of S into $\prod_{\alpha\in\Gamma} X_\alpha$.

Proof : We are given that (S, τ_S) is a topological space in which all singleton sets, $\{x\}$, are closed in S and a family of topological spaces, $\{X_\alpha : \alpha \in \Gamma\}$. For the set, $\mathscr{F} = \{f_\alpha : S \to X_\alpha\}_{\alpha\in\Gamma}$, of continuous functions on S, we define $e : S \to \prod_{\alpha\in\Gamma} f_\alpha[S] \subseteq \prod_{\alpha\in\Gamma} X_\alpha$ as an evaluation map with respect to $\mathscr{F}$.

a) Note that, for each $\alpha \in \Gamma$ and $x \in S$, $(\pi_\alpha \circ e)(x)) = \pi_\alpha(\{f_\alpha(x)\}) = f_\alpha(x)$. Since, for each $\alpha \in \Gamma$, f_α is continuous then so is $\pi_\alpha \circ e : S \to f_\alpha[S]$. By lemma 7.6, $e : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ is continuous.

We now show that $e : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ is one-to-one on S. Suppose a and b are distinct points in S. Then, since the single set $\{b\}$ is closed and $\mathscr{F}$ separates points and closed sets, there exists $\beta \in \Gamma$ such that $f_\beta(a) \notin \mathrm{cl}_{X_\beta} f_\beta[\{b\}]$. Then the β^{th} component of $e(a) = \{f_\alpha(a)\}$ and $e(b) = \{f_\alpha(b)\}$ are distinct and so $e(a) \neq e(b)$. We conclude that the evaluation map $e : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ is one-to-one on S.

b) To prove that the evaluation map $e : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ embeds S into $\prod_{\alpha\in\Gamma} X_\alpha$, it will suffice to show that it is an open function and then invoke theorem 6.9. Let

U be a non-empty open subset of S with the point $u \in U$. Then $F = S \setminus U$ is closed in S. Since $\mathscr{F}$ separates points and closed sets, there exists $\beta \in \Gamma$ such that $f_\beta(u) \notin \mathrm{cl}_{X_\beta} f_\beta[F]$. That means, $f_\beta(u) \in X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]$.

We now show that $e[U]$ is open in $Y = \prod_{\alpha\in\Gamma} X_\alpha$.
Note that

$$\begin{array}{rcl} (\pi_\beta \circ e)(u) & = & \pi_\beta(e(u)) \\ & = & \pi_\beta(\{f_\alpha(u)\}) \\ & = & f_\beta(u) \\ & \in & X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right] \end{array}$$

Since $e(u) \in \pi_\beta^{\leftarrow}\left[X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right]$ and since π_β is continuous then $\pi_\beta^{\leftarrow}\left[X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right]$ is an open neighbourhood of $e(u)$ in Y. It now suffices to show that

$$\pi_\beta^{\leftarrow}\left[X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right] \subseteq e[U]$$

Suppose $e(a) \in \pi_\beta^{\leftarrow}\left[X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right]$.

$$\begin{array}{rcl} e(a) \in \pi_\beta^{\leftarrow}\left[X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right] & \Rightarrow & (\pi_\beta \circ e)(a) \in \pi_\beta\left[\pi_\beta^{\leftarrow}\left[X_\beta \setminus \mathrm{cl}_{X_\beta} f_\beta[F]\right]\right] \\ & \Rightarrow & f_\beta(a) \in \left[X_\beta \setminus \mathrm{cl}_{X_\beta} f_\beta[F]\right] \\ & \Rightarrow & f_\beta(a) \notin \mathrm{cl}_{X_\beta} f_\beta[F] \\ & \Rightarrow & a \in S \setminus F = S \setminus (S \setminus U) = U \\ & \Rightarrow & e(a) \in e[U] \end{array}$$

So $\pi_\beta^{\leftarrow}\left[X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right]$ is an open neighbourhood of $e(u)$ which is entirely contained in $e[U]$. We conclude $e[U]$ is open and so $e : S \to \prod_{\alpha\in\Gamma} S_\alpha$ is a homeomorphism.

7.9 Topic: What is the Cantor set?

The Cantor set is a notion which is part of general mathematical culture. There are various ways it can be presented. We will formally provide a definition of the Cantor set and then study it from a topological point of view. Before we define it, it will be helpful if we first work through, together, the following example.

Example 4. *A prologue to the Cantor set.* Recall that $\mathbb{Z}^+ = \{1, 2, 3, \ldots, \}$. For each $n \in \mathbb{Z}^+$, let $X_n = D = \{0, 1, 2, \ldots, 8, 9\}$. So $\prod_{n\in\mathbb{Z}^+} X_n = \prod_{n\in\mathbb{Z}^+} D = D^{\mathbb{Z}^+}$ represents the set of all countably infinite ordered strings,

$$\{m_n : m_n = 0, \ldots, 9\}_{n\in\mathbb{Z}^+}$$

of digits from 0 to 9.

Suppose we define the function, $\varphi : \prod_{n\in\mathbb{Z}^+} D \to [0,1]$ (the closed interval in $\mathbb{R}$ from 0 to 1) as follows:

$$\varphi(\{m_n\}) = \sum_{n=1}^{\infty} \frac{m_n}{10^n}$$

Noting that every number x (represented in its infinite decimal expansion $0.m_1m_2m_3\cdots$) in the closed interval, $[0,1]$, can be expressed in the form,

$$x = \sum_{n=1}^{\infty} \frac{m_n}{10^n} = 0.m_1m_2m_3\cdots$$

See that $0 = \sum_{n=1}^{\infty} \frac{0}{10^n}$ and $1.000\ldots = \sum_{n=1}^{\infty} \frac{9}{10^n} = 0.9999\ldots$) and so φ maps $\prod_{n\in\mathbb{Z}^+} D$ *onto* [0,1] but is not necessarily one-to-one (since an endless string of 9's and an endless string of 0's may be mapped to the same element in [0, 1]). But the entire set [0,1] is, indeed, the image of $\prod_{n\in\mathbb{Z}^+} D$ under φ.

However, if $Y = \{0,1,2,5,7,9\}$, then the function, $\varphi : \prod_{n\in\mathbb{Z}^+} Y \to [0,1]$, defined similarly, would produce a range which is a *proper* subset of $[0,1]$ containing multiple gaps in it (since the digits 3, 6, and 8 are lacking in the ordered strings of the domain).

On the other hand, if $Z = \{0,1,2\}$ and every number x in $[0,1]$ is expressed in its triadic expansion form then the function

$$\varphi(\{m_n\}) = \sum_{n=1}^{\infty} \frac{m_n}{3^n} = 0.m_1m_2m_3\cdots$$

would similarly map $\prod_{n\in\mathbb{Z}^+} Z$ onto [0,1] where, for example,

$$\begin{aligned} 0.0000\ldots &= \sum_{n=1}^{\infty} \frac{0}{3^n} \\ 0.2222\ldots &= \sum_{n=1}^{\infty} \frac{2}{3^n} = 1 \\ 0.1111\ldots &= \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2} \end{aligned}$$

Example 5. *The Cantor set: A definition.* Suppose our index set is again $\mathbb{Z}^+ = \mathbb{N}\backslash\{0\}$ and, for each $n \in \mathbb{Z}^+$, $X = \{0,2\}$. Then $\prod_{n\in\mathbb{Z}^+}\{0,2\} = \{0,2\}^{\mathbb{Z}^+}$. Let the function

$$\varphi : \prod_{n\in\mathbb{Z}^+} \{0,2\} \to [0,1]$$

be defined as in example 1. That is,

$$\varphi(\{m_n\}) = \sum_{n=1}^{\infty} \frac{m_n}{3^n}$$

Then φ maps $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ into [0,1], where $\varphi(\{m_n\}) = 0.m_1m_2m_3\ldots$ is expressed in its triadic expansion form. Since the digit 1 is lacking in the ordered strings, the range, $\varphi\,[\,\prod_{n\in\mathbb{Z}^+}\{0,2\}\,]$, would be a *proper* subset of the interval, [0, 1], with multiple gaps in it. The image,

$$\varphi\,[\,\textstyle\prod_{n\in\mathbb{Z}^+}\{0,2\}\,] \subset [0,1]$$

is referred to as the *Cantor set.*[1]

More on the Cantor set.

What does the Cantor set look like? Defining the Cantor set as being the range, $\varphi\,[\,\prod_{n\in\mathbb{Z}^+}\{0,2\}\,]$, is not very useful when trying to visualize what kind of subset of [0, 1] it represents. Graphically, the Cantor set is constructed in stages by successively defining a nested sequence, $C_0, C_1, C_2, \ldots$, of subsets of [0,1] and then defining the Cantor set as being the intersection, $C = \cap_{n\in\mathbb{N}}C_n$, of all of these (after convincing ourselves that C would not be empty). The following describes the procedure.

$C_0 = [0,1]$
$C_1 = C_0\backslash(1,3)$
$C_2 = C_1\backslash(\frac{1}{3^2}\frac{2}{3^2})\cup(\frac{7}{3^2}\frac{8}{3^2})$
$C_3 = C_2\backslash$...open middle thirds in C_2

$\vdots$

The construction of C_n is normally described by saying " . . . to obtain C_n, subtract open middle thirds from C_{n-1}" After inductively obtaining an infinite sequence of nested sets, $\{C_n\}_{n\in\mathbb{N}}$, in this way we define the Cantor set as being the infinite intersection

$$C = \cap_{n=0}^{\infty} C_n$$

We visualize the Cantor set geometrically (up to the construction of C_5) as follows.

[1] Note that, since the digit, one, is not available for the strings, only one string $\{0,0,2,0,0,\ldots,\}$ is mapped to 0.0020000; the string $\{0,0,0,1,1,1,1\}$ does not belong to the domain and so cannot be mapped to $0.000111111\ldots = 0.00200000\ldots$.

Some may still prefer to describe the Cantor set as being the set of all triadic expansions of the numbers in [0,1] that can be expressed using only the digits 0 and 2.[1]

Recall that, in example 2 of page 119, the Cantor set was defined as being a proper subset, C, of $[0,1]$ which is the image of the one-to-one function, $\varphi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to [0,1]$ where

$$\varphi(\{m_n\}) = \sum_{n=1}^{\infty} \frac{m_n}{3^n}$$

The Cantor set was viewed simply as a set. The topology of sets involved was not discussed in our example and so we couldn't speak of the "continuity" of φ on the product $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. Now that we have decided on a topology on product spaces we can discuss the continuity of φ, or lack thereof, on its domain. We will define topologies of all sets involved in the most natural way. We will equip the set $\{0,2\}$ with the discrete topology, the set $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ with the product topology, and finally, the Cantor set C, with the subspace topology inherited from $\mathbb{R}$. We will now show that, with these topologies, φ maps $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ homeomorphically onto C.

Theorem 7.15 The one-to-one function, $\varphi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to C$, defined as,

$$\varphi(\{m_n\}) = \sum_{n=1}^{\infty} \frac{m_n}{3^n}$$

maps the product space $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ homeomorphically onto the Cantor set C.

Proof : The proof is found in Appendix A.

We see that investigating the Cantor set from a topological point of view provides us with a different perspective on the product space $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. Since we have shown it is a homeomorphic copy of C then, topologically speaking, the product space,

[1]Remembering that $0.1000000\cdots = 0.022222\cdots$

$\prod_{n\in\mathbb{Z}^+}\{0,2\}$, "is" the Cantor set. The topological point of view certainly provides much more insight on the nature of C as well as those sets that are linked to it via continuous functions.

Theorem 7.16 There is a continuous function, $\delta : C \to [0,1]$, which maps the Cantor set, C, onto the closed interval $[0,1]$.

Proof: The proof is found in Appendix A.

About the cardinality of C.

The theorem, immediately above, allows us to conclusively arrive at a surprising conclusion about the cardinality of the Cantor set. Our geometric description of the construction of the Cantor set, C, showed that C was obtained by successively removing open middle-third interval from a previous set, leaving behind, at least, the endpoints of countably many closed intervals. The endpoints, all of them of the form, $\frac{m_n}{3_n}$, are rationals and are never removed and so must belong to C. It is impossible to logically deduce from this description of the construction of the Cantor set that C contains anything else but these endpoints of the form $\frac{m_n}{3_n}$ each one left behind in the construction process. So, if we were only to believe our eyes we might conclude that C is countably infinite. But the previous theorem shows that this cannot be so. It states that C can be mapped continuously onto the uncountable set $[0,1]$. The range of a function can never have more points then the number of points in its domain. So C must be uncountable. Then $|C| = 2^{\aleph_0} = c$. We leave the reader with the more challenging question: If x is a point in C which is not an endpoint of a middle third, what does it look like? How can it be that a "non-endpoint" is left behind?

Is the Cantor topological space discrete?

By this question, we are wondering whether every single point of C is both open and closed. We claim that the subspace C cannot be discrete. We know that $\mathbb{R}$ is second countable and in theorem 5.13, we showed that "second countable" was a hereditary property. So C is second countable. If every point of the uncountable set, C, was open then C could not contain a countable dense subset and so would not be separable. Since "second countable" implies "separable" we would have a contradiction. So C must contain some points which are not open. That means that C is not a discrete subspace of $\mathbb{R}$. However the Cantor set does have a base of clopen sets. There are many websites on the internet which discuss various properties of C. Some are at a fairly elementary level while others involve mathematics which are more advanced and aimed at more specialized readers.

7.10 Topic: A curve which contains every point in a cube. (Peano's curve)[1]

As a final example of an application of product spaces we show the somewhat surprising result which states that a cube

$$[0,1]^3 = \{(x,y,z) : x,y,z \in [0,1]\}$$

equipped with the product topology, in $\mathbb{R}^3$ is the continuous image of the closed interval [0, 1] with the usual topology inherited from $\mathbb{R}$. We present this in the form of a solved example question.

Example 6. Find a function which maps the closed interval [0,1] continuously onto the cube $[0,1]^3$.

Solution : The complete solution is found in Appendix A.

Mathematicians specializing in various fields of study refer to the continuous image of the closed interval $[0,1]$ as a *curve*. In this sense, saying that "$[0,1]$ can be mapped continuously onto the cube $[0,1]^3$" is another way of saying "there is a curve that can fill the cube $[0,1]^3$" or "there is a curve that goes through every point of the cube $[0,1]^3$". The reader who follows through the proof carefully will notice that it can be generalized to the statement "For any integer n there is a curve which goes through each point of the cube $[0,1]^n$". The curve is referred to as *Peano's space filling curve.* A quick internet search will lead to many illustrations of curves which fill $[0,1]^2$ or $[0,1]^3$. The figure below (which can be graphed by most sophisticated math software freely available on the market) illustrates part of a curve gradually filling up a cube.

[1]Pronounced: pay-an-o.

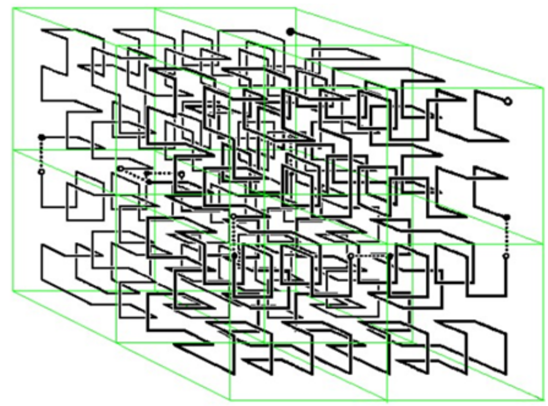


Figure 1: Part of a space filling curve in $[0,1]^3$.

Concepts review:

1. Give a general definition of a Cartesian product of sets.
2. Provide two definitions of the Cantor set.
3. Define the *product topology* on a Cartesian product.
4. Define the *box topology* on a Cartesian product.
5. Define what we mean by *product space*.
6. How does the closure of subsets of the factors of a product space compare with the closure of the associated subset in the product space itself.
7. What do we mean by "partitioning" a product space and how does it change its topology.
8. What effect does changing the order of the factors have on the topology of a product space.
9. Describe the *Tychonoff plank* and its topology.
10. What does it mean to say that a set of functions $\mathscr{F}$ separates points and closed sets of a set S.
11. Define evaluation map with respect to a set of functions $\mathscr{F}$.
12. State the *Embedding theorem*.

13. The embedding theorem describes a homeomorphism between a topological space and a product space. What are these spaces? What is the homeomorphism?
14. The Cantor set equipped with the usual topology is shown to be homeomorphic to a product space. Which one? Describe the homeomorphism.
15. Describe a continuous function which maps the Cantor set, C, continuously onto the closed interval $[0, 1]$.
16. Is the Cantor set a countable subset of $\mathbb{R}$?
17. Is the Cantor set, equipped with the subspace topology, discrete? Is it second countable?
18. Does there exist a continuous function which maps $[0, 1]$ onto the cube, $[0, 1]^3$?

EXERCISES

1. Show that $\prod_{\alpha\in\Gamma} S_\alpha$ is dense in $\prod_{\alpha\in\Gamma} T_\alpha$ if and only if, for each $\alpha \in \Gamma$, S_α is dense in T_α.
2. Suppose that, for each $\alpha \in \Gamma$, $S_\alpha \subseteq T_\alpha$. Then $\prod_{\alpha\in\Gamma} S_\alpha$ is a subset of the product space, $\prod_{\alpha\in\Gamma} T_\alpha$. Show that the product topology on $\prod_{\alpha\in\Gamma} S_\alpha$ is the same as the subspace topology $\prod_{\alpha\in\Gamma} S_\alpha$ inherits from $\prod_{\alpha\in\Gamma} T_\alpha$.
3. Given the product space $\prod_{\alpha\in\Gamma} S_\alpha$, show that the projection map $\pi_\alpha : \prod_{\alpha\in\Gamma} S_\alpha \to S_\alpha$ is an open function. Is it a closed function?
4. If in the product space, $S = \prod_{\alpha\in\Gamma} S_\alpha$, each S_α is discrete, describe the open subsets of S.
5. If in the product space, $S = \prod_{\alpha\in\Gamma} S_\alpha$, each S_α is indiscrete, describe the open subsets of S.
6. Let $X = \prod_{\alpha\in\Gamma} S_\alpha$ and $Y = \prod_{\gamma\in\Phi} T_\gamma$ be two product spaces where Γ and Φ have the same cardinality confirmed by the one-to-one function $q : \Gamma \to \Phi$. For each α, S_α and $T_{q(\alpha)}$ are homeomorphic topological spaces. Show that X and Y are homeomorphic.
7. Suppose we are given a topological space (S, τ_S) and a family of topological spaces, $\{(T_\alpha, \tau_\alpha) : \alpha \in \Gamma\}$. Suppose $\mathscr{F} = \{f_\alpha : \alpha \in \Gamma\}$ is a family of functions, $f_\alpha : S \to T_\alpha$, where each f_α maps its domain S into T_α. Let

$$\mathscr{B} = \{f_\alpha^{\leftarrow}[U] : (\alpha, U) \in \Gamma \times \tau_\alpha\}$$

   Show that $\mathscr{B}$ is a base for open sets of S if and only if $\mathscr{F}$ separates points and closed sets in S.

8. Suppose $U \subseteq S$ and $V \subseteq T$. Show that $\mathrm{int}_{S\times T}(U \times V) = \mathrm{int}_S U \times \mathrm{int}_T V$.
9. Show that, if the product space, $\prod_{\alpha\in\Gamma} S_\alpha$, is first countable, then S_α is first countable for each $\alpha \in \Gamma$.

8 / The quotient topology.

Summary. *In this section we will present a method to topologize the range, T, of a function, f, whose domain, S, is a topological space. The topology on the range is referred to as the "quotient topology induced by f". When the function f is used for this purpose, it is referred to as the quotient map.*

8.1 The strong topology induced by a function.

We have previously discussed a case where we are given a function, f, with, as domain, a set S (of undeclared topology) and codomain, a topological space (T, τ_T). We topologized S in such a way that guarantees the continuity of the function $f : S \to T$ on S. To do this we must be sure that S is provided with enough open sets so that $f : S \to T$ is continuous. The easiest way to do this is to assign to S the discrete topology, since, in this case, it allows any function to be continuous on S. But we preferred a topology which is custom-made for f and τ_T. We then opted for the *weak topology induced by* f and τ_T. Namely,

$$\tau_S = \{U \subseteq S : U = f^{\leftarrow}[V], \text{ for some open } V \text{ in } T\ \}$$

In this section, we will work the other way around. We wish to assign to T a topology, τ_T that will guarantee that f is continuous on S. Again, we could take the easy way out by assigning to T the indiscrete topology, $\{\varnothing, T\}$. It only has one non-empty open set, T, pulled back by f to the open set, $S \in \tau_S$. We again obtain continuity, but one which is independent of the function, f. We opt for choosing a topology on T which is custom made for the given function, f. That is, we will choose the *strongest topology on* T that will guarantee continuity of f on the given topological space (S, τ_S). With this mind, we present the following formal definition.

Definition 8.1 Let $f : S \to T$ be a function mapping the topological space (S, τ_S) *onto* the set T. We assign to the set T the following topology

$$\tau_f = \{U \subseteq T : f^{\leftarrow}[U] \text{ is open in } S\ \}$$

The set, τ_f is referred to as the *quotient topology induced by* f and τ_S. When T is equipped with the quotient topology, then (T, τ_f) is referred to as the *quotient space of* S and $f : S \to T$ is referred to as its associated *quotient map.*[1]

[1]Some authors use the terms "identification topology" instead of *quotient topology* and "identification map" instead of *quotient map*.

Since $f^{\leftarrow}$ respects both infinite unions and intersections of sets then τ_f is a well-defined topology on T. Also since τ_f contains precisely those sets which are pulled back to some set in τ_S, and no other sets, then τ_f is indeed the largest topology that guarantees continuity for f on S. The quotient topology, τ_f, induced by f is then unique.

It is important for the reader to notice that, in the above definition, we declare the function $f : S \to T$ to be *onto* T. This fact may be relevant in some of the proofs that follow. If $f : S \to T$ was not declared to be *onto* T, we would at least have to modify our definition of quotient topology to $\tau_f = \{U \subseteq f[S] : f^{\leftarrow}[U] \in \tau_S\}$.

Suppose we are given two topological spaces (S, τ_S) and (T, τ_T) and a continuous function, f, mapping S onto T. If we are given no more information about τ_T, it may or may not be the quotient topology induced by f. The following theorem shows that there may be various ways of recognizing a quotient topology. By definition, quotient maps must at least be continuous.

Theorem 8.2 Suppose (S, τ_S) and (T, τ_T) are topological spaces and $f : S \to T$ is a continuous function onto T.

a) If $f : S \to T$ is an open map, then τ_T is the quotient topology on T induced by f. That is, $\tau_T = \tau_f$.

b) If $f : S \to T$ is a closed map, then τ_T is the quotient topology on T induced by f. That is, $\tau_T = \tau_f$.

c) If there is a continuous function $g : T \to S$ such that $(f \circ g)(x) = x$ on T, then $\tau_T = \tau_f$.

Proof: We are given that $f : S \to T$ is a continuous function. For all three parts, to show that τ_T is the quotient topology it suffices to show that $\tau_T = \tau_f$. That is, τ_T is the strongest topology that will guarantee continuity to this function, f.

a) Since $f : S \to T$ is continuous, and τ_f is the largest topology on T for which f is continuous, then $\tau_T \subseteq \tau_f$. Let $U \in \tau_f$. By definition of τ_f, $f^{\leftarrow}[U]$ is open in S. Since f is both open and onto, then $f[f^{\leftarrow}[U]] = U \in \tau_T$. So $\tau_f \subseteq \tau_T$. Hence $\tau_T = \tau_f$.

b) The proof of part b) is similar to a) and so is left to the reader.

c) We are given that $g : T \to S$ is continuous such that $(f \circ g)(x) = x$. Suppose U is a subset of T such that $f^{\leftarrow}[U]$ is open in S. To show that $U \in \tau_f$ it suffices to show that U is open in T. We have

$$\begin{aligned} U &= \{x \in T : (f \circ g)(x) = x \in U\} \\ &= (f \circ g)^{\leftarrow}[U] \\ &= g^{\leftarrow}[f^{\leftarrow}[U]] \quad \text{Open in } T \text{ since } g \text{ and } f \text{ are continuous.} \end{aligned}$$

Hence $U \in \tau_f$.

8.2 An equivalence relation induced by a function f.

Any function, f, mapping a set S onto a set T can be used to partition the domain into subsets we call "fibres in S induced by the function f". We will first formally define this set theoretic notion.

Definition 8.3 Let $f : S \to T$ be a function mapping a set S *onto* a set T. If $w \in T$, we will refer to $f^{\leftarrow}[\{w\}]$ as the *fibre of* w *under the map* f. Fibres in the domain are the preimages of singleton sets.[1]

Let $f : S \to T$ be a function which maps the topological space (S, τ_S) *onto* the quotient space, (T, τ_f). Using this function we will construct a new set by defining a relation R_f on S as follows:

$$[\, u \text{ is related to } v\,] \Leftrightarrow [\, f(u) = f(v) \text{ in } T\,]$$

The phrase, "u is related to v" in S, can be more succinctly expressed as,

$$uR_fv \qquad \text{or} \qquad (u, v) \in R_f$$

This essentially means that u and v are related if and only if they both belong to the same fibre under the map $f : S \to T$.[2] The relation, R_f, on S is easily seen to be reflexive, symmetric and transitive, and so R_f is an *equivalence relation* on S. We will denote an equivalence class of x under R_f by

$$S_x = \{y \in S : xR_fy\}$$

[1] The word "fibre" is also written as "fiber". It is usually interpreted in this way in the field of set theory. But it can have an entirely different meaning in other mathematical fields. The meaning is usually determined by the context.

[2] In this context a fibre is a set theoretic concept and is independent of the topologies involved.

and the set of all such equivalence classes in S by

$$S/R_f = \{ S_x : x \in S \}$$

Each element, S_x, is a fibre under, f, and the set, S/R_f, can be viewed as the set of all fibres in S under f. In set theory, S/R_f, is normally called the *quotient set of S induced by R_f*. Just like the set of all fibres of a function, $f : S \to T$, partitions the domain, we see that the set of all equivalence classes partitions the set S. By this we mean that the elements of S/R_f are pairwise disjoint subsets which cover all of S. One should remember that, when S_x is in S, it is a subset of S, but, whenever S_x is in S/R_f, it is viewed as one of its elements.

There is a "natural" function, $\theta : S \to S/R_f$, defined as

$$\theta(x) = S_x$$

mapping the points of the topological space (S, τ_S) onto the "elements" of the set S/R_f. One might more figuratively say that θ collapses each fibre, $f^{\leftarrow}(x)$, in S down to a unique element, S_x, in S/R_f, or, equivalently, maps fibres in S to singleton sets, $\{S_x\}$, in S/R_f.

As long as no topology is declared on S/R_f our discussion remains within the bounds of set theory. We will topologize S/R_f as being the *quotient space induced by the function, $\theta : S \to S/R_f$*, a function we can now refer to as a *quotient map*. So S/R_f will be equipped with the quotient topology, τ_θ. More specifically,

$$\tau_\theta = \{U : \theta^{\leftarrow}[U] \text{ is open in } S\}$$

The topology on S/R_f will be the strongest topology that guarantees the continuity of $\theta : S \to S/R_f$.

We now have two quotient spaces induced by two functions with domain (S, τ_S):

$$\begin{aligned} f : S &\to (T, \tau_f) \\ \theta : S &\to (S/R_f, \tau_\theta) \end{aligned}$$

An insightful reader may already have a feeling that the two quotient spaces are topologically the same. To confirm this, we have to show that they are linked by some homeomorphism. We will connect S/R_f to T by defining a third function, $\phi_f : S/R_f \to T$ as

$$\phi_f(S_x) = f(x) \qquad \text{(Where } S_x = \theta(x).\text{)}$$

which maps points of S/R_f to points of T (remembering that f is *onto* T and so every element of T can be represented by $f(x)$ for some x in S.) The following diagram illustrates the relationship between S, S/R_f and T.

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\theta"'] & & T \\ & S/R_f \arrow[ur, "\phi_f"'] & \end{tikzcd}$$

This allows us to express the function, f, as a composition of functions $\phi_f \circ \theta = f$. The following definition refers to expressions of the form $f^{\leftarrow}[f[U]]$. One should keep in mind that, even though it occurs that $U = f^{\leftarrow}[f[U]]$, it is always the case that $U \subseteq f^{\leftarrow}[f[U]]$.

Definition 8.4 Let $g : S \to T$ be a function mapping the topological space (S, τ_S) *onto* a topological space (T, τ_T). A subset, $U \subset S$, is said to be *g-saturated* whenever U is the *complete* inverse image of some subset V in T. Equivalently, U is g-saturated if and only if $U = g^{\leftarrow}(g(U))$.

Theorem 8.5 Suppose $f : S \to T$ is continuous on S and U is a non-empty open subset of T. Show that $f^{\leftarrow}[U]$ is θ-saturated.

Proof: It suffices to show that $f^{\leftarrow}[U] = \theta^{\leftarrow}[\theta\,[f^{\leftarrow}[U]]]$. Since $f^{\leftarrow}[U] \subseteq \theta^{\leftarrow}[\theta\,[f^{\leftarrow}[U]]]$ we need only show inclusion in the opposite direction.

$$\begin{aligned} x \in \theta^{\leftarrow}[\theta\,[f^{\leftarrow}[U]]] &\Rightarrow \theta(x) \in \theta\,[f^{\leftarrow}[U]] \\ &\Rightarrow \theta(x) \in \{\theta(y) : y \in f^{\leftarrow}[U]\} \\ &\Rightarrow S_x \in \{S_y : f(y) \in U\} \\ &\Rightarrow f(x) \in U \\ &\Rightarrow x \in f^{\leftarrow}[U] \end{aligned}$$

Then $\theta^{\leftarrow}[\theta\,[f^{\leftarrow}[U]]] \subseteq f^{\leftarrow}[U]$. Then $f^{\leftarrow}[U]$ is θ-saturated. That is,

$$\theta^{\leftarrow}[\theta\,[f^{\leftarrow}[U]]] = f^{\leftarrow}[U]$$

Theorem 8.6 Suppose (S, τ_S) is a topological space. Let (T, τ_f) and $(S/R_f, \tau_\theta)$ be two quotient spaces induced by the quotient maps $f : S \to T$ and $\theta : S \to S/R_f$, where $\theta(x) = S_x$. Then the function, $\phi_f : S/R_f \to T$ defined as $\phi_f(S_x) = f(x)$, maps S/R_f homeomorphically onto T.

Proof: We are given that $f : S \to T$ and $\theta : S \to S/R_f$ and are quotient maps, hence are continuous on S. We are required to show that the function, $\phi_f : S/R_f \to T$, defined as, $\phi_f(S_x) = (\phi_f \circ \theta)(x) = f(x)$, is a homeomorphism.

We claim that ϕ_f is one-to-one and onto T: See that, since f is onto T, ϕ_f is onto T. Also, if $S_x \neq S_y$, then $f(x) \neq f(y)$, which implies $\phi_f(x) \neq \phi_f(y)$; so ϕ_f is one-to-one and onto.

We claim that ϕ_f is continuous on S/R_f: Let U be open in T. Since f is continuous $f^{\leftarrow}[U]$ is open in S.

$$\begin{aligned} \phi_f^{\leftarrow}[U] &= \{S_x : \phi_f(S_x) \in U\} \\ &= \{S_x : \phi_f(\theta(x)) \in U\} \\ &= \{\theta(x) : f(x) \in U\} \\ &= \{\theta(x) : x \in f^{\leftarrow}[U]\} \\ &= \theta[f^{\leftarrow}[U]] \end{aligned}$$

We have shown above that $f^{\leftarrow}[U]$ is θ-saturated hence $\theta^{\leftarrow}[\,\theta\,[\,f^{\leftarrow}[U]\,]\,] = f^{\leftarrow}[U]$. Since $\theta : S \to S/R_f$ is a quotient map and $f^{\leftarrow}[U]$ is open then $\theta\,[\,f^{\leftarrow}[U]\,]$ is open. So $\phi_f^{\leftarrow}[U]$ is open. This establishes the claim that ϕ_f is continuous.

We claim that ϕ_f is open on S/R_f: From the diagram above we know that $f = \phi_f \circ \theta$. So $\theta = \phi_f^{\leftarrow} \circ f$. Since θ is continuous then $(\phi_f^{\leftarrow} \circ f) : S \to S/R_f$ is continuous on S. Let U is an open subset of S/R_f.

$$\begin{aligned} \theta^{\leftarrow}[U] &= (\phi_f^{\leftarrow} \circ f)^{\leftarrow}[U] \in \tau_S \\ &\Rightarrow f^{\leftarrow}[(\phi_f^{\leftarrow})^{\leftarrow}[U]] \in \tau_S \\ &\Rightarrow f^{\leftarrow}[\phi_f[U]] \in \tau_S \end{aligned}$$

Since T has the quotient topology induced by f, $\phi_f[U]$ is open in T. We have shown that ϕ_f is an open map, as claimed.

We have shown that the one-to-one onto function, $\phi_f : S/R_f \to T$, is both continuous and open, hence it maps S/R_f homeomorphically onto T.

We summarize the main ideas behind the above result. If we are given a topological space, (S, τ_S), and a function f mapping S onto a set T we have the two main ingredients necessary to topologize T with the quotient topology, τ_f. We do so in a way that guarantees continuity of the function, $f : S \to T$. The fibres of the function, f, covers its domain. By collapsing, via the map θ, each fibre down to a point we create another set, S/R_f, and topologize it with the quotient topology, τ_θ. The expression "identifying the points of the fibres" is also of common usage. This new topological space, $(S/R_f, \tau_\theta)$ has been proven to be a homeomorphic copy of (T, τ_f).

Example 1. Let $\mathbb{J}$ denote the set of all irrational numbers in the closed interval $[0, 1]$ and let $T = \{1\} \cup \mathbb{J}$, a proper subset of $[0, 1]$. We define a function $f : [0, 1] \to T$ as follows

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

Let the domain, $[0, 1]$, be equipped with the usual topology and the range, $T = \{1\}\cup\mathbb{J}$, be equipped with the quotient topology induced by the function f. Describe the open subsets of the quotient space, T, induced by the function f on $[0, 1]$.

Solution. The open subsets of T are defined as

$$\tau_f = \{U \subseteq T : f^{\leftarrow}[U] \text{ is open in } [0,\, 1]\ \}$$

We have $f^{\leftarrow}[\varnothing] = \varnothing$ and $f^{\leftarrow}[T] = [0, 1]$ so $\{\varnothing, T\} \subseteq \tau_f$.

Consider, for example, the open interval $U = (1/5, \pi/4)$. Then $[U \cap \mathbb{J}] \cup \{1\} \subseteq T$. So

$$\begin{aligned} f^{\leftarrow}\,[\,[U \cap \mathbb{J}] \cup \{1\}\,] &= f^{\leftarrow}[U \cap \mathbb{J}] \cup f^{\leftarrow}[\{1\}] \\ &= [\,U \cap \mathbb{J}\,] \cup [\,[0, 1] \cap \mathbb{Q}\,] \end{aligned}$$

Since $[\,U \cap \mathbb{J}\,] \cup [\,[0, 1] \cap \mathbb{Q}\,]$ is not open in $[0, 1]$ then $[\,U \cap \mathbb{J}\,] \cup \{1\}$ is not open in T.

It seems that the complete, open, *preimage*, $f^{\leftarrow}[V]$, of any non-empty subset, V, of T should contain open intervals and, at least, both 0 and 1. Are there such proper subsets of $[0, 1]$? We test the following subset:

$$f^{\leftarrow}[V] = [0, \pi/7) \cup (\pi/7,\, \pi/5) \cup (\pi/5,\, \pi/4) \cup (\pi/4, 1]$$

is open in $[0, 1]$. We seek a set V in T that f pulls back to the subset, $f^{\leftarrow}[V]$. We test

$$f\,[f^{\leftarrow}[V]] = T \setminus \{\pi/7,\, \pi/5,\, \pi/4\}$$

to find

$$f^{\leftarrow}[T\setminus\{\pi/7,\, \pi/5,\, \pi/4\}] = \underbrace{[0, \pi/7) \cup (\pi/7,\, \pi/5) \cup (\pi/5,\, \pi/4) \cup (\pi/4, 1]}_{\text{An open subset of } [0,1].}$$

So $T\setminus\{\pi/7,\, \pi/5,\, \pi/4\} \in \tau_f$.

Then, if K is any countable subset of irrationals in $[0, 1]$, $T\setminus K$ is open in T. So

$$\tau_f = \{\ \varnothing,\ [0, 1]\ \}\ \cup\ \{T\setminus K : K \subset \mathbb{J} \cap [0, 1], K \text{ countable }\}$$

8.3 Decomposition spaces.

We have seen that the quotient space induced by $\theta : S \to S/R_f$ involves the decomposition of the set S into non-overlapping sets, $\{S_x\}$. We will now slightly generalize this notion of "decomposition of a set". Suppose we are given a topological space (S, τ_S). Rather than partition S into equivalence classes, each of which contains all elements which belong to a particular fibre, S_x, of f, we can also partition the set S in an arbitrary way. By this we mean that we express S as the union of non-intersecting subsets. We will denote this set of subsets by, $\mathscr{D}_S$. Each element x of S belongs to some element, labeled D_x, of $\mathscr{D}_S$. Note that, if $y \in D_x$ then D_x and D_y are simply different labels for the same element of $\mathscr{D}_S$. The set

$$\mathscr{D}_S = \{D_x : x \in S\}$$

is reminiscent of a quotient set whose elements are equivalent classes of some relation R on S. We can similarly define a function $\theta : S \to \mathscr{D}_S$ where $\theta(u) = D_u$, the unique element of $\mathscr{D}_S$ which contains u. Then $\theta^{\leftarrow}(D_u) = \{x \in S : x \in D_u\} = D_u \subseteq S$. The set, $\mathscr{D}_S$, is not yet topologized, but we know of a procedure to topologize it, since it is, after all, the range of a function θ on S. We can equip $\mathscr{D}_S$ with the quotient topology,

$$\tau_\theta = \{U \subseteq \mathscr{D}_S : \theta^{\leftarrow}[U] \text{ is open in } S\}$$

induced by θ. We formally define the concepts we have just presented.

Definition 8.7 Let (S, τ_S) be a topological space. If $\mathscr{D}_S$ is collection of pairwise disjoint subsets of S such that every element of S belongs to some element of $\mathscr{D}_S$, then we say $\mathscr{D}_S$ is a *decomposition of* S. The function, $\theta : S \to \mathscr{D}_S$, defined as, $\theta(u) = D_x$ if and only if $u \in D_x$, mapping every element of S onto $\mathscr{D}_S$ is called the *decomposition map of* S *onto* $\mathscr{D}_S$. The function, θ, is also referred to as the *identification map*. The function, θ, is said to *identify the elements of the subset* $D_x \subseteq S$. If τ_θ is the quotient topology induced on $\mathscr{D}_S$ by θ, then we say that $(\mathscr{D}_S, \tau_\theta)$ is a *decomposition space* or *quotient space*.

Why does all of this sound familiar? Recall that, if $f : S \to T$, then the function, f, decomposes its domain, S, into fibres. In this case, $(\mathscr{D}_S, \tau_\theta)$ is a copy of $(S/R_f, \tau_\theta)$. The following theorem will help recognize the open subsets of the decomposition space, $\mathscr{D}_S$. It states that open subsets $\mathscr{U} = \{D_x : D_x \in \mathscr{U}\}$ of $\mathscr{D}_S$ correspond to open θ-saturated subsets $\cup\{D_x : D_x \in \mathscr{U}\}$ of S.

Theorem 8.8 Suppose $(\mathscr{D}_S, \tau_\theta)$ is a decomposition space of the topological space (S, τ_S) and let $\mathscr{U} = \{D_x : x \in I \subseteq S\}$ be a subset of $\mathscr{D}_S$. Then $\mathscr{U}$ is open in $\mathscr{D}_S$ if and only if

$$\cup\{D_x : x \in I\}$$

is open in S.

Proof: We are given that $\mathscr{U} = \{D_x : x \in I \subseteq S\}$ be a subset of $\mathscr{D}_S$.

($\Rightarrow$) Suppose $\mathscr{U}$ is open in $\mathscr{D}_S$.

Then $\theta^{\leftarrow}[\mathscr{U}] = \{x \in S : \theta(x) = D_x \in \mathscr{U}\} = \{x \in S : x \in I\}$ is open in S. We are required to show that $\cup\{D_x : x \in I\}$ is open in S.

$$\begin{aligned} \theta^{\leftarrow}[\mathscr{U}] &= \theta^{\leftarrow}[\cup\{\{D_x\} : x \in I\}] \\ &= \cup\{\theta^{\leftarrow}[\{D_x\}] : x \in I\} \\ &= \cup\{D_x \subseteq S : x \in I\} \\ &\quad \text{Is open since } \theta \text{ is continuous.} \end{aligned}$$

($\Leftarrow$) Suppose $\cup\{D_x \subseteq S : x \in I\}$ is open in S. We are required to show that $\mathscr{U} = \{D_x : x \in I \subseteq S\}$ is open in $\mathscr{D}_S$. To show that $\mathscr{U}$ is open in $\mathscr{D}_S$ it suffices to show that $\theta^{\leftarrow}[\mathscr{U}]$ is open in S (since $\mathscr{D}_S$ is equipped with the quotient topology induced by θ).

$$\begin{aligned} \cup\{D_x \subseteq S : x \in I\} &= \cup\{\theta^{\leftarrow}[\{D_x\}] : x \in I\} \\ &= \theta^{\leftarrow}[\cup\{ D_x : x \in I\}] \\ &= \theta^{\leftarrow}[\mathscr{U}] \end{aligned}$$

Since $\cup\{D_x : x \in I\} = \theta^{\leftarrow}[\mathscr{U}]$ is open then $\mathscr{U}$ is open.

Example 2. Suppose we are given the subset, $S = [0, 2\pi] \times [0, 2\pi]$ of $\mathbb{R}^2$. We will decompose S as follows: For each point $(x, 0)$ on the line $[0, 2\pi] \times \{0\}$ let

$$D_{(x,0)} = \{(x, 0), (2\pi - x, 2\pi)\}$$

So for each $x \in [0, 2\pi]$, $D_{(x,0)}$ and $D_{(2\pi-x,2\pi)}$ represent the same element of the decomposition, $\mathscr{D}_S$. For example, $D_{(0,0)} = D_{(2\pi,2\pi)} = \{(0, 0), (2\pi, 2\pi)\}$.

For each $(x, y) \notin [0, 2\pi] \times \{0\} \cup [0, 2\pi] \times \{2\pi\}$ let

$$D_{(x,y)} = \{(x, y)\}$$

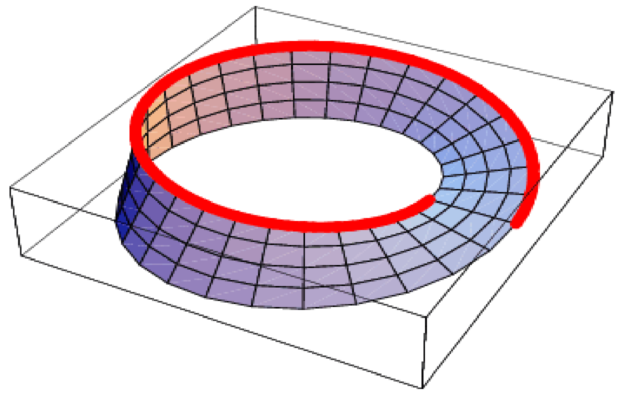


Figure 2: Topological representation of the mobius strip

Each subset of the form $D_{(x,0)}$ is a subset of S which contains two points. All other subsets of the decomposition are singleton sets. The decomposition space, $\mathscr{D}_S$, of the subspace S obtained is referred to as the *mobius strip*. See the figure.

The two lines $L_1 = \{(x,0) : 0 \leq x \leq 2\pi\}$ and $L_2 = \{(x,2\pi) : 0 \leq x \leq 2\pi\}$ are collapsed together (after inverting one of the lines) to form the mobius strip.

Without the inversion of one of the lines, the decomposition space becomes a topological representation of a cylindrical shell.

Moore plane decomposition

Example 3. Let (S, τ_S) denote the Moore plane. (Review the description of the Moore plane by looking over the example on page 70.) Let $F = \{(x,0)|x \in \mathbb{R}\}$ and $W = S \backslash F$. We can then express S as the disjoint union of the sets F and W. Let $\mathscr{D} = W \cup \{F\}$, be the decomposition space of S which results from collapsing the subset, F, down to a point and identifying all other points to themselves.

a) Describe the open subsets of the decomposition space $\mathscr{D}$.
b) If $g : \mathscr{D} \to [0,1]$ is a continuous function mapping $\mathscr{D}$ onto $[0,1]$, where g is constant on some neighbourhood of F, construct a continuous function, $f : S \to [0,1]$, mapping S onto $[0,1]$.

Solution : We are given the Moore plane, (S, τ_S), and the decomposition $\mathscr{D} = W \cup \{F\}$. It easily verified that points in S are closed and so S is T_1.

a) If $\theta : S \to \mathscr{D}$ is the corresponding quotient map, then $\theta(a,b) = (a,b)$ if $(a,b) \in W$ and $\theta(u,v) = F$ when $(u,v) \in F$. The quotient topology on $\mathscr{D}$ is, by definition,

$$\tau_\theta = \{U \subseteq \mathscr{D} : \theta^{\leftarrow}[U] \text{ is open in } S\}$$

If τ_W is the subspace topology on the open subspace, W, inherited from S, then $\tau_W \subseteq \tau_\theta$.
We now describe an open neighbourhood base for the element, $F \in \mathscr{D}$.
Let $\mathbb{N}_0 = \mathbb{N}\backslash\{0\}$. Consider the set, $\mathbb{N}_0^{\mathbb{R}}$, of all functions which map $\mathbb{R}$ into $\mathbb{N}_0$.
We now construct a basic open neighbourhood of the element F. For $f \in \mathbb{N}_0^{\mathbb{R}}$, let

$$U_f = \cup \left\{ B_{\frac{1}{f(x)}}\left(x, \frac{1}{f(x)}\right) : (x,0) \in \mathbb{R} \times \{0\} \right\}$$

where $\frac{1}{f(x)}$ is the radius of each open ball of center $\left(x, \frac{1}{f(x)}\right)$ tangent to the x-axis. Since, for each f, $\theta^{\leftarrow}[U_f \cup \{F\}] = U_f \cup F$ is open in S, then

$$\mathscr{B}_F = \{U_f \cup \{F\} : f \in \mathbb{N}_0^{\mathbb{R}}\}$$

is an uncountably large base of open neighbourhoods for F, since for any open neighbourhood V of the x-axis, there exists $f \in \mathbb{N}_0^{\mathbb{R}}$ such that $\{(x,0) : x \in \mathbb{R}\} \subseteq \mathscr{B}_F \subseteq V$.

b) We are given that $g : \mathscr{D} \to [0,1]$ is continuous on $\mathscr{D}$ and is onto $[0,1]$. We are also given that g is constant on some open neighbourhood, say, $\mathscr{B}_f \cup \{F\}$, of F. Say, $g[\mathscr{B}_f \cup \{F\}] = k \in [0,1]$. Define $f : S \to [0,1]$ as $f = g \circ \theta$. Since both θ and g are continuous then $g \circ \theta$ is continuous on S so is f is continuous on S. Then f and g have the same value for all $(x,y) \in W$ and $f[F] = g(F) = \{k\}$.

Concepts review:

1. Define the quotient topology induced by a function.
2. If τ_T is a topology on T for which $f : S \to T$ is continuous, how does τ_T compare with the quotient topology, τ_f, induced by f?
3. If τ_T is a topology on T for which $f : S \to T$ is a continuous open map, how does τ_T compare with the quotient topology, τ_f, induced by f?
4. If τ_T is a topology on T for which $f : S \to T$ is a continuous closed map, how does τ_T compare with the quotient topology, τ_f, induced by f?
5. What is a *fibre* of a point under a map f?
6. Define the set S/R_f of all equivalence classes induced by a function $f : S \to T$ by referring to a function $\theta : S \to S/R_f$.
7. What topology is defined on S/R_f?

8. How is the function $\phi_f : S/R_f \to T$ defined?

9. If $g : S \to T$ is a function and $U \subseteq S$, what does it mean to say that U is g-saturated?

10. If $g : S \to T$ is a function and $U \subseteq S$ is g-saturated, is there a way to describe the subset U in terms of g?

11. Given $f : S \to T$ and $\theta : S \to S/R_f$ describe the homeomorphism which links S/R_f to T.

12. Describe how to construct a decomposition space, $\mathscr{D}_S$, from a topological space (S, τ_S).

EXERCISES

1. Let (S, τ_S) be a topological space. We define a relation, R, on S as follows: $(u, v) \in R$ if and only if $\mathrm{cl}_S\{u\} = \mathrm{cl}_S\{v\}$.

   a) Show that R is an equivalence relation on S.

   b) Let $(S/R, \tau_\theta)$ denote the quotient space induced by the natural map $\theta : S \to S/R$.

      i. If S_x is an element in S/R, is $\{S_x\}$ necessarily a closed subset of S/R?

      ii. If S_x and S_y are distinct elements of S/R, does there exist an open set in τ_θ which contains S_x but not S_y?

2. Describe the quotient space, $\mathbb{R}/R$, if $(u, v) \in R$ if and only if $x - y$ is an integer.

3. Suppose $\mathscr{D}_{\mathbb{R}^2}$ is a decomposition of $\mathbb{R}^2$ (with the usual topology) whose elements are circles with center at the origin. Show that the corresponding decomposition space, $(\mathscr{D}_{\mathbb{R}^2}, \tau_\theta)$, and the set of all non-negative real numbers, $\{x \in \mathbb{R} : x \geq 0\}$, equipped with the usual topology are homeomorphic topological spaces.

Part III

Topological spaces: Separation axioms

9 / Separation with open sets.

Summary. *In this section we will present five classes of topological spaces distinguished from each other by the separation axiom they each satisfy. They differ by the way their open sets can be used to separate their points and their closed sets. For each of these spaces, we present a few of their characterizations and basic properties. These five types of topological spaces are called, T_i-spaces, where $i = 0, 1, 2, 3, 4$.*

9.1 The T_0 separation axiom.

In this chapter we will add more restrictions on our topological spaces. We will then see how these restrictions modify the properties and character of those spaces on which they are imposed. We will require that some of our topological spaces have sufficient amounts of open sets to separate distinct points and closed sets in various ways and to different degrees. There are many ways to do this. An important source of inspiration on how to proceed will come from the well researched and deep understanding of the Euclidean spaces, $(\mathbb{R}^n, \tau)$, when equipped with the usual topology. Our experience with these particular spaces allow us to metaphorically visualize "open balls" that will separate distinct points, or a point from a closed set in more abstract topological spaces. Some of the spaces we will study have less topological structure than $\mathbb{R}^n$ while others will be much more complex and so investigating some of their properties may be more challenging. As we present each of the separation axioms, we will assume that it is obvious to most readers that all of these axioms are topological *invariants* in the sense that their properties are automatically carried over from one space, S, to all others which are homeomorphic to it. With each separation property, we also verify whether it carries over to other spaces via continuous function, from factors to their product spaces, or whether it is inherited by their subspaces.

Definition 9.1 Let (S, τ_S) be a topological space. We say that S is a T_0-*space* if, for every pair of distinct points u and v, there exists at least one open set in τ_S, which contains one of these two points, but not the other.[1]

Most of the topological spaces we have been exposed to are T_0-spaces, so finding an example of a T_0-space is easy, as long as we can justify why a particular space is declared to be T_0. The space, $\mathbb{R}$, with the usual topology, is T_0 since, for any $a, b \in \mathbb{R}$

[1] A T_0-space is also known as a *Kolmogorov space* (named after the mathematician Andrey Kolmogorov).

where $a < b$, $(a-1, b)$ contains a but not b. We will soon see that the Euclidean space, $\mathbb{R}$, can separate points in much more intricate ways. Most of the topological spaces that are of interest, are, at the very least, T_0. Are there any non-T_0-spaces? Let's consider the space which has the fewest open sets possible; the indiscrete space (S, τ_i) with topology, $\tau_i = \{\varnothing, S\}$. If $a, b \in S$, since S is the only non-empty open set, it contains all points and so is unable to separate a point a from a point b in the way prescribed by the T_0-property. So, for this reason, there is simply not much else to say about the indiscrete spaces.

Combining the T_0 separation property with other topological properties.

Even though T_0-spaces by themselves may be of little interest, it is good practice to see what happens when we combine the T_0-property with other topological properties. Consider the following example.

Example 1. Suppose (S, τ_S) is an infinite second countable T_0-space. Show that the cardinality, $|S|$, of S is less than or equal to $2^{\aleph_0}$.

Solution: Since S is an infinite second countable space, then it has an open base, $\mathscr{B}$, such that $|\mathscr{B}| = \aleph_0$. For each $x \in S$, let $\mathscr{B}_x = \{B_x \in \mathscr{B} : x \in B_x\}$. If $y \neq x$ then $\mathscr{B}_x$ cannot equal $\mathscr{B}_y$, for, if it did, then $y \in B_x$ for all $B_x \in \mathscr{B}_x$, and there would be no B_x (B_y) which can separate x from y, a contradiction of the T_0 property. So there is a one-to-one function, $f : S \to \mathscr{P}(\mathscr{B})$, defined as, $f(x) = \mathscr{B}_x$. Since the cardinality of $\mathscr{P}(\mathscr{B})$ is less than or equal to $2^{\aleph_0}$ then $|S| \leq 2^{\aleph_0}$.

9.2 The T_1 separation axiom.

In general, we prefer topological spaces to have points which are closed. However, because of our preference to make the definition of "topological space" as simple as possible, we didn't make this a requirement. Some spaces may have points, x, such that $\{x\}$ is not closed. The next separation axiom called, T_1, requires, as we shall see, that τ_S has the open sets needed to guarantee that singleton sets, $\{x\}$, must be closed subsets.

Definition 9.2 Let (S, τ_S) be a topological space. We say that S is a T_1-*space* if, for every pair of distinct points u and v, there exists an open set, U, which contains u but not v, *and* an open set, V, which contains v but not u.[1]

---

[1] The T_1-spaces are also referred to as Fréchet spaces, named after the mathematician Maurice Fréchet.

Of course, every T_1-space is T_0. Also, $\mathbb{R}^n$, when equipped with the usual topology, is a T_1-space. Are there T_0-spaces which are not T_1? The following example we present, a non-T_1-space, which is T_0.

Example 2. Let S be a set which contains more than a single point. Suppose $u \in S$ and $\mathscr{F} \subseteq \mathscr{P}(S)$ where

1. For all $F \in \mathscr{F}$, $F \neq \varnothing$, and $[\, F \in \mathscr{F}\,] \Leftrightarrow [\, u \in F\,]$.
2. $\varnothing \in \mathscr{F}$.

Verify that the two given conditions on $\mathscr{F}$ describes all closed subsets with respect to a particular topology, τ_S, on S. Then show that S is T_0, but not T_1.

Solution. We claim that $\mathscr{F}$ describes all closed subsets with respect to the topology $\tau_S = \{U : u \notin U\} \cup \{S\}$. Clearly, $\{S, \varnothing\} \subseteq \mathscr{F}$. Finite unions of sets each of which contains the point u will contain the point u and arbitrary intersections of sets each of which contain u will also contain u. Then $\mathscr{F}$ satisfies the closed set axioms. Then the set, $\mathscr{F}$, represents all closed sets for the topology

$$\tau_S = \{U : u \notin U\} \cup \{S\}$$

We see that, if $x \neq u$, then $S\backslash\{u\}$ is an open neighbourhood of x and does not contain u. So S is T_0. But since the only open neighbourhood of u is all of S, then S cannot be T_1. This is what was required.

Characterizations of T_1-spaces.

The most useful characterization of T_1-spaces is given below. This characterization is such that some may use it as a definition of T_1.

Theorem 9.3 Let (S, τ_S) be a topological space. The space S is T_1 if and only if every singleton set, $\{x\}$, of S is a closed subset of S.

Proof : Let (S, τ_S) be a topological space. If S is a singleton set then S is easily verified to be T_1 and its only singleton set is S, which is closed. Suppose S contains more than one point.

($\Rightarrow$) Suppose S is T_1. Let $u \in S$. We claim that $\{u\}$ is closed. For each, $x \neq u$, there exists open, U_x such that, $x \in U_x$ and $u \in S\backslash U_x$. Then $\{u\} = \cap\{S\backslash U_x : x \neq u\}$, the intersection of closed subsets of S. So $\{u\}$ is closed in S.

($\Leftarrow$) Suppose every point of S is closed. Then, then for any $u \in S$, if $y \neq u$, $S\backslash\{u\}$ is an open neighbourhood of y which does not contain u. Hence S is T_1.

Theorem 9.4 Let (S, τ_S) and (T, τ_T) be topological spaces.

a) If S is a T_1-space then so are all its subspaces. So T_1 is a hereditary property.

b) Suppose S is a T_1 space and $f : S \to T$ is a closed function. Then $f[S]$ is a T_1 subspace of T.

c) Suppose $S = \prod_{i \in I} S_i$ is a product space. The space S is a T_1 product space if and only if each of its S_i-factors is a T_1-space.

Proof: Let (S, τ_S) and (T, τ_T) be topological spaces.

a) Suppose S is T_1 and $u \in W \subseteq S$. Since $\{u\}$ is a closed subset of S it is a closed subset of its subspace W (with respect to the subspace topology). So W is T_1 (by theorem 9.3).

b) Suppose S is T_1 and $f : S \to f[S]$ is a closed map. Let $v \in f[S]$. Then there exists $x \in S$ such that $f[\{x\}] = \{v\}$. By hypothesis, $\{x\}$ is closed and f is closed so, $\{v\}$ is closed. Hence $f[S]$ is T_1.

c) ($\Rightarrow$) Suppose $S = \prod_{i \in I} S_i$ is a T_1 product space. Recall that each factor, S_i, is homeomorphic to a subspace, U_i, of S. By part a), U_i is T_1 and so S_i is T_1.

($\Leftarrow$) Suppose each factor, S_i, of $S = \prod_{i \in I} S_i$ is T_1. Let $q = \{a_i\}_{i \in I}$ be a point in S. Then no S_i is empty since, for each $i \in I$, $a_i \in S_i$. Since each S_i is T_1, for each coordinate a_i, the singleton set, $\{a_i\}$, is a closed subset of S_i. For all $j \in I$, $\pi_j : S \to S_j$ is continuous; then $\pi_j^{\leftarrow}(\{a_j\})$ is closed in S. Then $\{q\} = \cap_{j \in I} \left\{ \pi_j^{\leftarrow}(\{a_j\}) \right\}$ is closed in S. So S is T_1.

In the following example, we show that the decomposition space, $\mathscr{D}$, whose elements are the fibres of $f : S \to T$, is T_1 if and only if the fibres of f are closed subsets of S.

Example 3. Let $f : S \to T$ be a function. Suppose τ_f is the quotient topology on S induced by the quotient map f. Let $\mathscr{D}_S = \{D_x : x \in T\}$ be a decomposition space of S, with elements, $D_x = f^{\leftarrow}(x)$. Show that $\mathscr{D}_S$ is T_1 if and only if each of its elements, $D_x = f^{\leftarrow}(x)$, is a closed subset of S.

Solution : Let $f : S \to (T, \tau_T)$ be a function. Let $\mathscr{D}_S = \{D_x : x \in T\}$ be a decomposition space of S induced by f, equipped with the quotient topology τ_f. That is, $f^{\leftarrow}(x) = D_x$ for each $x \in T$. Let $q : S \to \mathscr{D}_S$ be the quotient map $q(x) = D_x$.

($\Leftarrow$) We are given that, for each $x \in T$, $D_x = f^{\leftarrow}(x)$ is a closed subset of S. We are required to show that $\mathscr{D}_S$ is T_1. Let $S_y \in \mathscr{D}$. It suffices to show that $\{S_y\}$ is closed in $\mathscr{D}_S$. See that

$$q^{\leftarrow}[\mathscr{D}_S \backslash \{S_y\}] = S \backslash f^{\leftarrow}[\{y\}]$$

an open subset of S. Since $\mathscr{D}_S$ is equipped with the quotient topology, if $q^{\leftarrow}[\mathscr{D}_S \backslash \{S_y\}]$ is open then $\mathscr{D}_S \backslash \{S_y\}$ is open in $\mathscr{D}_S$. So $\{S_y\}$ is closed in $\mathscr{D}_S$. This implies $\mathscr{D}_S$ is T_1.

($\Rightarrow$) We are given that $\mathscr{D}_S$ is a T_1 quotient space. Then, for each $x \in T$, $\{D_x\}$ is a closed subset of $\mathscr{D}_S$. We are required to show that $D_x = f^{\leftarrow}(x)$ is a closed subset of S. Since $q : S \to \mathscr{D}_S$ is continuous, then $q^{\leftarrow}[\{D_x\}] = f^{\leftarrow}(x)$ is closed in S, as required.

9.3 T_2-spaces or Hausdorff topological spaces

We will now define a T_2-space. The terminology "T_2-space" simply helps remind us that $T_2 \Rightarrow T_1 \Rightarrow T_0$. Although both T_2 and Hausdorff mean the same thing, the word "Hausdorff" is more commonly used.[1]

Definition 9.5 Let (S, τ_S) be a topological space. We say that S is a *T_2-space* or *Hausdorff space* if, for every pair of distinct points u and v, there exists non-intersecting open neighbourhoods, U and V such that $u \in U$ and $v \in V$.

Clearly, any Hausdorff space, (S, τ_S), satisfies the T_1 axiom. Hence,

"If S is Hausdorff then, for each $x \in S$, $\{x\}$ is a closed subset of S".

Example 4. An infinite set, S, with the cofinite topology (also referred to as the Zariski topology, see page 35) is T_1 but not Hausdorff since every pair of open neighbourhoods intersect in infinitely many points. However, it is easily verified to be T_1.

Example 5. Any metrizable topological space, (S, τ_ρ), is Hausdorff since given distinct points a and b, $B_\varepsilon(a) \cap B_\varepsilon(b) = \varnothing$ where $\varepsilon < \frac{\rho(a,b)}{3}$. A standard approach to verifying whether a topological space is metrizable is to first check if it is Hausdorff. If it is not Hausdorff then the question is settled.

Hausdorff characterizations

We present a few of the more common characterizations of the Hausdorff property.

[1] Sadly, the talented mathematician, Felix Hausdorff, did not survive the Nazi regime in Germany.

Theorem 9.6 Let (S, τ_S) be a topological space. The following statements are equivalent.

a) The space, S, is Hausdorff.

b) If u and v are distinct points in S then there exists an open neighbourhood, U, of u such that $v \notin \text{cl}_S U$.

c) If $u \in S$, then $\cap\{\text{cl}_S U : U$ is an open neighbourhood of $u\} = \{u\}$

d) The set $D = \{(u, u) : u \in S\}$ is closed in the product space, $S \times S$.

Proof: Let (S, τ_S) be a topological space.

(a $\Rightarrow$ b) We are given that S is Hausdorff. Suppose u and v are distinct points in S. Then there exists disjoint open neighbourhoods, U and V, that contain u and v, respectively. See that $U \subseteq S \backslash V$, a closed subset of S; hence $\text{cl}_S U \subseteq S \backslash V$. Then $v \notin U \subseteq \text{cl}_S U$.

(b $\Rightarrow$ c) Suppose that, if $u \neq v$ in S, there exists an open neighbourhood, U, of u such that $v \notin \text{cl}_S U$. Then $v \in \cap\{\text{cl}_S U$: U is a neighbourhood of u $\}$ is impossible. So $\cap\{\text{cl}_S U$: U is a neighbourhood of u $\} = \{u\}$.

(c $\Rightarrow$ d) Suppose that $\cap\{\text{cl}_S U$: U is a neighbourhood of u $\} = \{u\}$, for all $u \in S$. Let $D = \{(a, a) \in S \times S : a \in S\}$. We are required to show that D is closed in $S \times S$. Let $(u, v) \in (S \times S) \backslash D$. Then u and v are distinct elements of S. By hypothesis, there exists an open neighbourhood, U, of u such that $v \notin \text{cl}_S U$. Then $U \times (S \backslash \text{cl}_S U)$ is an open neighbourhood of (u, v). If $a \in U$, then $a \notin S \backslash \text{cl}_S U$ so $(a, a) \notin U \times (S \backslash \text{cl}_S U)$. So $U \times (S \backslash \text{cl}_S U)$ does not intersect D. So $(S \times S) \backslash D$ is open. Hence D is closed.

(d $\Rightarrow$ a) Suppose D is closed in $S \times S$. Let u and v be distinct points in S. Then $(u, v) \in (S \times S) \backslash D$. Since D is closed there exists and open neighbourhood, $U \times V$ of (u, v) which does not intersect D. Then U and V are disjoint open neighbourhoods of u and v, respectively. So S is Hausdorff.

In the following theorem, we show that a subspace will always inherit the Hausdorff property from its topological space. Also, the Hausdorff property is carried over from a set of factors, $\{S_i\}_{i \in I}$, to the Cartesian product space, $\prod_{i \in I} S_i$, they generate. The Hausdorff property is also carried over from the domain, S, of a one-to-one closed function, f, to its range, $f[S]$. However, continuous images of Hausdorff spaces need not necessarily be Hausdorff, as the example following the theorem below will show.

Theorem 9.7 Let (S, τ_S) be a Hausdorff topological space and $\{S_i\}_{i\in I}$ be a family of Hausdorff topological spaces.

a) If U is a subspace of S, then U is Hausdorff. That is, "Hausdorff" is a hereditary property.

b) If $f : S \to T$ is a one-to-one closed function onto T, then T is Hausdorff.

c) The product space, $\prod_{i\in I} S_i$, generated by $\{S_i\}_{i\in I}$ is Hausdorff.

Proof: Let (S, τ_S) be a Hausdorff topological space.

a) Suppose u and v are distinct points in the subspace, W, of S. Then there exists disjoint open neighbourhoods, U and V, which respectively contain the points u and v. Then $U \cap W$ and $V \cap W$ are disjoint open neighbourhoods in W, which respectively contain the points u and v.

b) Suppose $f : S \to T$ is a one-to-one closed function mapping S onto T. Recall that a one-to-one closed function is also an open function. (See the corresponding theorem on page 90). Then disjoint open neighbourhoods U and V of u and v are mapped to disjoint open neighbourhoods of $f[U]$ and $f[V]$ of $f(u)$ and $f(v)$.

c) We are given that $\{S_i\}_{i\in I}$ is a family of Hausdorff topological spaces. Let $\{u_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$ be distinct points of $\prod_{i\in I} S_i$. Then, for some $j \in I$, $u_j \neq v_j$. Let U_j and V_j be disjoint open neighbourhoods of u_j and v_j in S_j. Then $\pi_j^{\leftarrow}[U_j]$ and $\pi_j^{\leftarrow}[V_j]$ are disjoint open base elements containing $\{u_i\}_{i\in I}$ and $\{v_i\}_{i\in I}$. So $\prod_{i\in I} S_i$ is Hausdorff.

Example 6. Consider the space $S = \{0, 1\}$ equipped with the discrete topology, $\tau_d = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}$, and $T = \{0, 1\}$ equipped with the topology, $\tau_T = \{\varnothing, \{0\}, \{0, 1\}\}$. Let $f : S \to T$ be a one-to-one onto function defined as $f(x) = x$ on S. Since S is discrete, each point can serve as it own open neighbourhood and so is Hausdorff. Also, every function is continuous on a discrete space so f is continuous on S. Use this example to show that the continuous image of a Hausdorff space need not be Hausdorff.

Solution: In τ_T, there is no open neighbourhood that contains 1 but not 0. So T is not Hausdorff. In fact, T is not even T_1. So continuous images of Hausdorff spaces need not be Hausdorff.

Example 7. Show that, if a continuous function, $f : U \to V$, maps U one-to-one into a Hausdorff space V then its domain, U, must also be Hausdorff.

Solution. Let $f : U \to V$ be a one-to-one continuous function mapping U into a Hausdorff space, V. Then, since Hausdorff is a hereditary property, $f[U]$ is a Hausdorff subspace of V. Also, since f is continuous, $f^{\leftarrow} : f[U] \to U$ is a one-to-one closed function onto U. By the theorem 9.7, part b), the space U must be Hausdorff.

In the following two examples we show that the Hausdorff characterization

$$\text{“}S \text{ is Hausdorff} \Leftrightarrow D = \{(u, u) : u \in S\} \text{ is a closed subset of } S \times S\text{”}$$

can serve as a useful tool for solving certain types of problems.

Example 8. Show that, if f and g are two continuous functions each mapping a topological space S into a Hausdorff space T, then the set, $U = \{u \in S : f(u) = g(u)\}$, is a closed subset of S. That is, if two continuous functions map a space, S, into a Hausdorff space, T, then the set on which they agree is closed in S.

Solution. Let $h : S \to T \times T$ be a function defined as, $h(x) = (f(x), g(x))$, and let $U = \{u \in S : f(u) = g(u)\}$. By theorem 9.6, the statement,

$$\text{“ } D = \{(u, u) : u \in T\} \text{ is a closed subset of } T \times T \text{ ”}$$

characterizes Hausdorff spaces T. Since both f and g are continuous on S then so is h (see theorem on page 109). Hence $h^{\leftarrow}[D]$ is closed in S. Since

$$\begin{aligned} h^{\leftarrow}[D] &= \{u \in S : h(u) = (f(u), g(u)) \in D\} \\ &= \{u \in S : f(u) = g(u)\} \\ &= U \end{aligned}$$

the set, U, is closed in S, as required.

Example 9. Show that, if $f : S \to T$ is continuous, where T is Hausdorff then the graph of f,

$$G = \{(u, f(u)) : u \in S\} \subseteq S \times T$$

is closed in $S \times T$. That is, if the codomain is Hausdorff, the graph of a continuous function is closed.

Solution. Let $t : S \times T \to T \times T$ be defined as

$$t(u, v) = (f(u), v)$$

Then, since f and the identity map are continuous on S, t is continuous on $S \times T$. Hence, since T is Hausdorff, by theorem 9.6, $D = \{(u, u) : u \in T\}$ is a closed subset of $T \times T$. Thus $t^{\leftarrow}[D]$ is closed in $S \times T$. But

$$\begin{aligned} t^{\leftarrow}[D] &= \{(u,v) \in S \times T : t(u,v) \in D\} \\ &= \{(u,v) \in S \times T : (f(u),v) \in D\} \\ &= \{(u,v) \in S \times T : f(u) = v\} \\ &= \{(u,f(u)) \in S \times T : u \in S\} \\ &= G \end{aligned}$$

So the graph G is closed in $S \times T$.

Definition 9.8 Let (S, τ_S) be a topological space. We say that a space is *completely Hausdorff* if and only if for any pair of points a and b in S, there exist open neighbourhoods U and V of a and b, respectively, such that $\mathrm{cl}_S U \cap \mathrm{cl}_S V = \varnothing$. The completely Hausdorff property is also represented as $T_{2\frac{1}{2}}$.

Clearly every completely Hausdorff space is Hausdorff. But,

"Hausdorff $\not\Rightarrow$ completely Hausdorff"

as the following example shows.

Example 10. Let T be the interior of the unit square $[0,1] \times [0,1]$ and $S = T \cup \{(0,0),(1,0)\}$. The points in T have basic neighbourhoods which are the usual Euclidean open balls. The basic open neighbourhoods of the point $(0,0)$ are open rectangles of the form

$$\{(0,1/2) \times (0,1/n) \cup \{(0,0)\} : n > 0\}$$

The basic open neighbourhoods of the point $(1,0)$ are open rectangles of the form

$$\{(1/2,1) \times (0,1/m) \cup \{(1,0)\} : m > 0\}$$

Verify that this topology is Hausdorff but not completely Hausdorff. (Hint: Consider disjoint open neighbourhoods of the two points (0, 0) and (1, 0)).

9.4 T_3-spaces and regular spaces.

The three separation axioms, T_0, T_1 and T_2 illustrate three different ways that open sets can separate two sets of points. The Hausdorff (T_2-space) space imposes the strongest conditions since $T_2 \Rightarrow T_1 \Rightarrow T_0$.

We now want to use open sets to separate a non-empty closed set, F, from a point, x, such that $x \in S \backslash F$. To do this we will define a T_3-space by using the definition of a T_2-space as a model.

Definition 9.9 Let (S, τ_S) be a topological space.

a) We say that S is a T_3-*space* if, for any non-empty closed subset, F and point $v \in S \backslash F$, there exists non-intersecting open subsets, U and V such that $F \subseteq U$ and $v \in V$.

b) If (S, τ_S) is both T_1 and T_3 then we will say that S is a *regular space*.

Let's consider the most trivial of topological spaces, the indiscrete space, (S, τ_i), where $|S| > 1$. There does not exist a "closed F and $x \in S \backslash F$" in S and so S is (vacuously) a T_3-space. But distinct points u and v in S are not contained in disjoint open sets and so, S is not Hausdorff. So we have a situation where $T_3 \not\Rightarrow T_2$. We wanted to rectify this situation. We did do so by defining "regular space = $T_1 + T_3$". If S is both T_1 and T_3 and u, v are distinct points then, since S is T_1, they are both closed, and, since S is T_3, they are, respectively, contained in disjoint open sets and so S is T_2.

Then "$[T_3 + T_1] \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$". Equivalently, "Regular $\Rightarrow$ Hausdorff $\Rightarrow T_1 \Rightarrow T_0$".

Example 11. Let $(\mathbb{R}, \tau)$ we the space of real numbers equipped with the usual topology, τ. If $\mathscr{B} = \{(a, b) \subset \mathbb{R} : a < b\}$ and $\mathscr{S} = \mathscr{B} \cup \{\mathbb{Q}\}$, we define $\tau_{\mathscr{S}}$ as being the topology on $\mathbb{R}$ generated by the subbase $\mathscr{S}$. Show that the topological space, $(\mathbb{R}, \tau_{\mathscr{S}})$, is Hausdorff but not regular.

Solution. Note that $\mathbb{Q}$ belongs to $\tau_{\mathscr{S}}$ but is not an element of the usual topology, τ, and so $\mathscr{S} = \mathscr{B} \cup \{\mathbb{Q}\} \not\subseteq \tau$. Since every element of $\mathscr{B}$ is a base element of τ, then $\tau \subset \tau_{\mathscr{S}}$. So $\tau_{\mathscr{S}}$ is strictly stronger then the usual topology on $\mathbb{R}$. Since $(\mathbb{R}, \tau)$ is Hausdorff then so is $(\mathbb{R}, \tau_{\mathscr{S}})$.

We now show that $(\mathbb{R}, \tau_{\mathscr{S}})$ is not regular. See that $\mathbb{R} \backslash \mathbb{Q} = \mathbb{J}$ is a complement of the open subset, $\mathbb{Q}$, and so is a closed subset of $\mathbb{R}$. Since $\mathbb{R}$ is Hausdorff, $\mathbb{R}$ is T_1. For any open open neighbourhood, U, of 1 there will exist a, b such that $1 \in (a, b) \subseteq U$ or $1 \in (a, b) \cap \mathbb{Q} \subseteq U$. In either case $U \cap \mathbb{J} \neq \varnothing$. So no pair of disjoint open subsets can separate 1 from the closed subset $\mathbb{J}$. So $(\mathbb{R}, \tau_{\mathscr{S}})$ is not regular.

We now present two useful characterizations of "regular".

Theorem 9.10 Let (S, τ_S) be a T_1 topological space. The following statements about S are equivalent.

a) The space S is regular.

b) For every $x \in S$ and open neighbourhood, U of x, there exists an open neighbourhood, V of x, such that $x \in \mathrm{cl}_S V \subseteq U$.

c) For every $x \in S$ and closed subset F disjoint from $\{x\}$, there exists an open neighbourhood, V of x, such that $\mathrm{cl}_S V \cap F = \varnothing$.

Proof: Let (S, τ_S) be a T_1 topological space.

(a $\Rightarrow$ b) We are given that S is regular and that U is an open neighbourhood of $x \in S$. Then $S \backslash U$ and $\{x\}$ are disjoint closed subsets of S. Since S is regular, there exists disjoint open sets, V and W, containing x and $S \backslash U$, respectively. Then $S \backslash W$ is a closed set, entirely contains V and is entirely contained in U. So $\mathrm{cl}_S V \subseteq S \backslash W \subseteq U$.

(b $\Rightarrow$ c) We are given that for every $x \in S$ and open neighbourhood, U of x, there exists an open neighbourhood, V of x, such that $x \in \mathrm{cl}_S V \subseteq U$. Let $x \in S$ and F be a closed subset disjoint from $\{x\}$. By hypothesis, there exists an open neighbourhood, V, of x such that $\mathrm{cl}_S V \subseteq S \backslash F$.

(c $\Rightarrow$ a) We are given that, for every $x \in S$ and closed subset F disjoint from $\{x\}$, there exists an open neighbourhood, V of x, such that $\mathrm{cl}_S V \cap F = \varnothing$. Let $x \in S$ and F be a closed subset disjoint from $\{x\}$. By hypothesis, there exists an open neighbourhood, V, of x such that $\mathrm{cl}_S V \cap F = \varnothing$. Then V and $S \backslash (\mathrm{cl}_S V)$ are disjoint open sets of containing x and F respectively. So S is T_3. Since S is Hausdorff, it is T_1, hence S is regular.

Example 12. Show that any metrizable space is regular.

Solution: Suppose (S, τ_ρ) is a metrizable space whose open sets are generated by a metric, ρ. Metrizable spaces are known to be Hausdorff and so are T_1. Let $y \in S$ and U be an open neighbourhood of y. Then there exists $\varepsilon > 0$ such that the ball, $B_\varepsilon(y)$, (center y and radius ε) is entirely contained in U. Then, if $V = B_{\varepsilon/3}(y)$, $\mathrm{cl}_S V = \{x : \rho(x, y) \leq \varepsilon/3\} \subseteq B_\varepsilon(x) \subseteq U$. By theorem 9.10, S is regular.

Basic properties of regular spaces.

In the following theorems we confirm that "regular" is a hereditary property and that it carries over products, in both directions.

Theorem 9.11 Subspaces of regular spaces are regular.

Proof: Let (S, τ_S) be a regular topological space and (T, τ_T) be a subspace of S.

Any point, x, in T is a point in S and so is closed in S. Then $\{x\}$ is closed in T. We must conclude tha T is T_1.

Let $\{x\}$ and F be disjoint closed subsets of T. Then there exists a closed subset, F^*, of S such that $F = F^* \cap T$. By hypothesis, there exists disjoint open subsets, U and V, of S which contain x and F^*, respectively. Then the disjoint open subsets, $U \cap T$ and $V \cap T$, of T contain x and F, respectively. We can then conclude that T is regular and so "regular" is a hereditary topological property.

Recall that a set, U, is regular open in S, if $U = \text{int}_S\text{cl}_S U$. Also, from definition 5.15, we have that a space (S, τ) is said to be *semiregular* if the set, $\mathscr{R}o(S)$, of all regular open subsets of S forms a base for open set of S. Although it is not explicitly stated in the original definition 5.15, semiregularity is traditionally defined on spaces assumed to be Hausdorff only. We will adopt this tradition here. The definition does not suggest that semiregularity can be viewed as a separation axiom. The following theorem shows that it can be.

Theorem 9.12 A regular space is semiregular.

Proof: Let (S, τ) be a regular space. To prove that S is semiregular it will suffice to show that $\mathscr{R}o(S)$ is a base for the open sets of S. Suppose $u \in S$ and B is an open neighbourhood of u. By theorem 9.10, there exists an open subset, U, of S such that $u \in \text{cl}_S U \subseteq B$. Then $u \in \text{int}_S\text{cl}_S U \subseteq B$. Since $\text{int}_S\text{cl}_S U \in \mathscr{R}o(S)$, then $\mathscr{R}o(S)$ is a base for open sets of S, as required.

Theorem 9.13 A regular space is completely Hausdorff.

Proof: The proof is straightforward and so is left an exercise.

In example 10 we present a topological space which is easily seen to be semiregular but is **not** completely Hausdorff (the rectangular neighbourhoods do not have closures which separate the points (0, 0) and (1, 0). "semiregular" does not imply "completely Hausdorff".

We then have the chain of implications,

$$
\begin{array}{ccccccccc}
 & & \text{completely Hausdorff} & & & & & & \\
 & & \not\Uparrow & & & & & & \\
\text{regular} & \Rightarrow & \text{semiregular} & \Rightarrow & \text{Hausdorff} & \Rightarrow & T_1 & \Rightarrow & T_0 \\
\Downarrow & & & & & & & & \\
\text{completely Hausdorff} & & & & & & & &
\end{array}
$$

Theorem 9.14 Let $\{S_i\}_{i\in I}$ be a family of topological spaces and $S = \prod_{i\in I} S_i$ be a corresponding product space. Then S is regular if and only if each factor, S_i, is regular.

Proof : In theorem 9.4, it is shown that the T_1-property carries over both from products to factors and from factors to products. It will then suffice to show that this holds true for the defining property of T_3-spaces.

($\Rightarrow$) We are given that $\prod_{i\in I} S_i$ is a T_3 space. We have shown that $\prod_{i\in I} S_i$ contains a homeomorphic copy of each S_i. Since "regular" is a hereditary property each S_i is T_3 and so is regular.

($\Leftarrow$) We are given that $S = \prod_{i\in I} S_i$ is a product space and each S_i is T_3.

Let $\{u_i\}_{i\in I} \in S$ and W be any open neighbourhood of $\{u_i\}_{i\in I}$. Then there exists a base element, $\cap\{\pi_i^{\leftarrow}[U_i] : i \in F_{\text{finite}} \subseteq I\}$, of S such that

$$\{u_i\} \in \cap\{\pi_i^{\leftarrow}[U_i] : i \in F_{\text{finite}} \subseteq I\} \subseteq W$$

Since each S_i is regular there exists open S_i-neighbourhoods, $\{V_i\}_{i\in F\subseteq I}$, such that $u_i \in V_i \subseteq \text{cl}_{S_i} V_i \subseteq U_i$ for $i \in F_{\text{finite}}$. Then

$$
\begin{aligned}
\{u_i\}_{i\in I} \in \cap\{\pi_i^{\leftarrow}[V_i] : i \in F\} &\subseteq \text{cl}_S\left[\cap\{\pi_i^{\leftarrow}[V_i] : i \in F\}\right] \\
&= \cap\{\text{cl}_S \pi_i^{\leftarrow}[V_i] : i \in F\} \\
&= \cap\{\pi_i^{\leftarrow}[\text{cl}_{S_i} V_i] : i \in F\} \\
&\subseteq \cap\{\pi_i^{\leftarrow}[U_i] : i \in F\} \\
&\subseteq W
\end{aligned}
$$

Since

$$\{u_i\}_{i\in I} \in \cap\{\pi_i^{\leftarrow}[V_i] : i \in F\} \subseteq \text{cl}_S\left[\cap\{\pi_i^{\leftarrow}[V_i] : i \in F\}\right] \subseteq W$$

then, by the characterization theorem above, S is T_3 and so is regular.

The above theorem shows that most of the topological spaces we are familiar with are regular.

The following example illustrates an interesting property for all regular spaces. It shows that, if T is any infinite subspace of a regular space, then T contains a countably set of closed neighbourhoods, none of which intersects with another of that set.

Example 13. Suppose (S, τ_S) is regular and (T, τ_T) is an infinite subspace of S. Show that there exists a countably infinite set, $\{U_i : i \in \mathbb{N}\}$, of open sets such $\{\mathrm{cl}_S U_i \cap T : i \in \mathbb{N}\}$ forms a pairwise disjoint set of closed sets in T.

Solution : The statement can be proven by a standard application of the principle of mathematical induction, starting with the base step, $U_0 = \varnothing$ followed by an induction hypothesis. The details are left to the reader as an exercise.

Example 14. Show that the *Moore plane* is regular. (For a description of the Moore plane see the example on page 70.)

Solution : Let (S, τ_S) denote the Moore plane. The proof that S is T_1 is straightforward and so is left as an exercise. We now show that S is T_3.

In the Moore plane there are two types of points. Those points, (x, y), whose basic open neighbourhood is of the form $B_\varepsilon(x, y)$ and the points, $(x, 0)$, whose basic open neighbourhood is of the form $B_\varepsilon(x, y) \cup \{(x, 0)\}$ where $\varepsilon = y$. So we consider these two cases separately.

Let U be an open base neighbourhood, $B_\varepsilon(x, y)$, of (x, y), $y \neq 0$. Then $(x, y) \in \mathrm{cl}_S B_{\varepsilon/3}(x, y) \subset U$.

Let V be an open base neighbourhood of $(x, 0)$ of the form, $B_\varepsilon(x, y) \cup \{(x, 0)\}$, where $\varepsilon = y$. Then $(x, 0) \in \mathrm{cl}_S B_{\varepsilon/3}(x, y/3) \cup \{(x, 0)\} \subset V$.

We conclude that the Moore plane is T_3. So S is regular.

Example 15. Recall from definition 5.16 that a space is said to be zero-dimensional if it has a base of clopen sets. Show that a zero-dimensional T_1-space is regular.

Solution : Let F be a closed subset of a zero-dimensional T_1-space S. Then S has a base of clopen subsets. If $x \in S \backslash F$, then there is a subset, B, which is both open and closed such that $x \in B \subseteq S \backslash F$. Then B is an open neighbourhood of x which is disjoint from the open neighbourhood, $S \backslash B$ of F. So S is regular.

9.5 T_4-spaces and normal spaces.

Our final axiom of separation "by open sets" involves, what we will call, T_4-space, from which we define the slightly more restrictive, "normal space" $= T_4 + T_1$.

Definition 9.15 Let (S, τ_S) be a topological space. We say that S is a T_4-*space* if, for any pair of non-empty disjoint closed subsets, F and W, in S, there exists non-intersecting open subsets, U and V containing F and W, respectively.

As our first example of a T_4 space we consider the topological space, $(\mathbb{R}, \tau_u)$, where $\tau_u = \{(x, \infty) : x \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$. Since the closed subsets of $\mathbb{R}$ are of the form $(-\infty, y]$, disjoint pairs of non-empty closed sets do not exist in $\mathbb{R}$ and so $\mathbb{R}$ is vacuously T_4. Consider the subsets, $F = (-\infty, 5]$ and $\{6\}$. Any open subset U which contains F will contain the point 6. So $(\mathbb{R}, \tau_u)$ is not T_3. That is, $T_4 \not\Rightarrow T_3$.

To rectify this situation we will define a new class of spaces called "normal" as, "normal" $= T_4 + T_1$. Let us verify why this will work. If S is $T_1 + T_4$ and F is a non-empty closed subset of S, for $\{u\} \not\subset F$, we would be guaranteed the existence of disjoint open subsets, U an V, which, respectively, contain the closed subsets $\{u\}$ and F, hence S would be both T_1 and T_3; by definition, S would be regular. So we would have the desired chain of implications,

$$\text{“normal} \Rightarrow \text{regular} \Rightarrow \text{Hausdorff} \Rightarrow T_1 \Rightarrow T_0\text{”}$$

We will then proceed with this definition of a *normal topological space*.

Definition 9.16 If (S, τ_S) is both T_1 and T_4 then we will say that S is a *normal space*.[1]

Before we proceed with various examples, we provide a few characterizations of the spaces called "normal".

[1]Unfortunately, readers will find that, in the literature, many authors invert the definitions of "T_4" and "normal" as well as for "T_3" and "regular". So when consulting other texts we caution the reader to verify carefully which version the particular author has a preference for. Some writers may feel strongly about their chosen version. But as long as we are aware of this, most readers will easily adapt to a particular version, as long as the author is consistent with its use throughout the body of the text.

Theorem 9.17 Let (S, τ_S) be a topological space. The following statements about S are equivalent.

a) The space S is normal.

b) For every closed subset, F in S, and open set, U containing F, there exists an open set, V containing F, such that $F \in \text{cl}_S V \subseteq U$.

c) For every pair of disjoint closed sets, F and W in S, there exists and open subset U containing F such that $\text{cl}_S U \cap W = \varnothing$.

d) For every pair of disjoint closed sets, F and W in S, there exists disjoint open subsets U and V containing F and W, respectively, whose closures, $\text{cl}_S U$ and $\text{cl}_S V$, do not intersect.

Proof: The proofs are straightforward and so are left as an exercise for the reader.

Theorem 9.18 Let (S, τ_ρ) be a metrizable topological space whose open sets are generated by the metric ρ. Then S is a normal space.

Proof: Metrizable spaces are Hausdorff, hence T_1.

We now show that S is T_4. Let F and W be disjoint closed subsets of S. If $x \in F$ and $y \in W$ there exists α_x and β_y such that $B_{\alpha_x}(x) \cap W = \varnothing$ and $B_{\beta_y}(y) \cap F = \varnothing$. Then

$$\begin{aligned} U &= \cup\{B_{\alpha_x/3}(x) : x \in F\} \\ V &= \cup\{B_{\beta_y/3}(y) : y \in W\} \end{aligned}$$

are open sets containing F and W, respectively.

We claim that $U \cap V = \varnothing$. Suppose not. Suppose $q \in U \cap V$. Then there exists some $a \in F$ and $b \in W$ such that $q \in B_{\alpha_a/3}(a) \cap B_{\beta_b/3}(b)$. Suppose, WLOG, $\alpha_a \geq \beta_b$. Then

$$\begin{aligned} \rho(a, b) &\leq \rho(a, q) + \rho(b, q) \\ &< \alpha_a/3 + \beta_b/3 \\ &\leq 2\alpha_a/3 \\ &< \alpha_a \end{aligned}$$

But α_a was chosen so that $B_{\alpha_a}(a) \cap W = \varnothing$. Contradiction. So $U \cap V = \varnothing$, as claimed. So S is T_4. We conclude that S is normal, as required.

Since, for every $n \geq 1$, $\mathbb{R}^n$ equipped with the usual topology is metrizable then for every natural number n, $\mathbb{R}^n$ is a normal topological space.

Basic properties of normal spaces.

We will see that normal spaces are sometimes less "well-behaved" than the weaker separation axioms. First, they are not hereditary, which is unlike the regular spaces. But they are closed-hereditary (that is, closed subsets inherit the normal property). Also, unlike regular spaces, the normal property does not generally carry over from factors to their product space even when the product is finite. However, if the product space is normal then so will be each of its factors.

Theorem 9.19 Let (S, τ_S) be a normal topological space.

a) If the function $f : S \to f[S] = T$ is a closed continuous function onto the space (T, τ_T) then T is normal.

b) If T is a closed subspace of S then T is normal.

c) If $S = \prod_{i \in I} S_i$ is a normal product space of topological spaces $\{S_i : i \in I\}$, then each of its factors, S_i, is normal.

Proof: Let (S, τ_S) be a normal topological space.

a) Suppose $f : S \to f[S] = T$ is a closed continuous function. Since S is T_1 and f is closed then T is T_1.

   Let F and W be two disjoint closed subsets of T. Continuity of f guarantees that $f^{\leftarrow}[F]$ and $f^{\leftarrow}[W]$ are disjoint closed subsets of S. Then there exists disjoint open sets U and V containing F and W, respectively. Let $A = T \backslash f[S \backslash U]$ and $B = T \backslash f[S \backslash V]$. Since f is closed A and B are disjoint open neighbourhoods of F and W, respectively. It quickly follows that T is T_4. So T is normal.

b) Let T be a closed subspace of S. Then T is T_1. If F and W are disjoint closed subsets of the closed subspace T, then they are disjoint closed subsets of S. It quickly follows that T is T_4. So T is normal.

c) We are given that $S = \prod_{i \in I} S_i$ is normal. There exists a homeomorphism, $h : S_i \to S$, which embeds each S_i in S (theorem 7.9). Then $S \backslash h[S_i]$ is easily verified to be open in S and so $h[S_i]$ is a closed subspace of S. Since closed subspaces of normal spaces are normal, then $h[S_i]$ is normal and so, S_i is normal.

Ordinal spaces of the form $[0, \omega_\alpha]$ or $[0, \omega_\alpha)$ are normal.

We now investigate whether ordinal spaces such as $[0, \omega_\alpha]$ or $[0, \omega_\alpha)$ satisfy the normal property. The ordinal space is a bit more difficult to grapple with since it is a linearly ordered well-ordered space, so it makes it harder to visualize disjoint closed subsets, let alone disjoint open subsets. The main property that we use, which is proper to the ordinal space, is that it has a topology, τ_ω, which has an open base whose elements are all of the form $(\alpha, \beta]$. Also, since the ordinal space is well-ordered, non-empty sets must have a supremum. Existence of suprema allows us to handle closed subsets.

Example 17. Let ω_α be any ordinal number and let $S = [0, \omega_\alpha]$. Show that the ordinal space (S, τ_ω) is normal, independent of the value ω_α. The proof that shows $[0, \omega_\alpha)$ that is normal is practically the same. (See the reference to "ordinal space" at definition 5.14 on page 80.)

Solution : Verifying that S is T_1 is left to the reader. Let F and W be disjoint non-empty closed subsets of S. If $\mu \in F$ and $\mu \neq 0$ let

$$S_\mu = \{x \in W : x < \mu\} = [0, \mu) \cap W$$

Let $x_\mu = \sup S_\mu$.[1] Since W is closed, then $x_\mu \in W$. Then the open set, $(x_\mu, \mu]$ is an open neighbourhood of $\mu \in F$ which does not intersect W.

This is repeated for each element, μ, of F, to obtain, the open neighbourhood

$$U_F = \cup\{(x_\mu, \mu] : \mu \in F\}$$

of F. The set U_F clearly contains no elements of W.

By applying precisely the same procedure we obtain an open neighbourhood

$$V_W = \cup\{(x_\alpha, \alpha] : \alpha \in W\}$$

of W, which contains no elements of F.

We need now only verify that $U_W \cap V_F = \varnothing$. If so, then S is T_4 and hence is normal. The standard approach is to suppose $U_F \cap V_W \neq \varnothing$. Then there exists some $\mu \in F$ and some $\alpha \in W$, such that $(x_\mu, \mu] \cap (x_\alpha, \alpha]$ is non-empty. Without loss of generality, suppose $\alpha < \mu$. But, this can't be, since this would force $\alpha \in (x_\mu, \mu]$, when we know that $(x_\mu, \mu] \cap W = \varnothing$.

So $U_W \cap V_F = \varnothing$. We conclude that S is T_4 hence is normal. Proceed similarly for $S = [0, \omega_\alpha)$

[1] The existence of the supremum can be justified by the fact that the class of all ordinals is well-ordered. See the related statements from set theory in the appendix.

Example 18. Show that the deleted Tychonoff plank $T = [0, \omega_1) \times [0, \omega_0)$ is not normal.

Solution : Recall that T is topologized as a product space. Consider $U = [0, \omega_1)$ viewed as a closed interval subset of $[0, \omega_1)$ itself and consider $V = \{\omega_0\}$, a singleton set. Then the subset $F = U \times V$ is a closed subset (the top edge) of T with respect to the subspace topology (since its complement is easily seen to be open). Similarly consider the closed subset, $K = \{\omega_1\} \times [0, \omega_0) \subseteq T$. If T is normal then we should be able to separate F from K with a pair of non-intersecting open sets.

Suppose M is an open set in T which contains K.

A basic neighbourhood of u in K which would be entirely contained in M would be of the form

$$W_u = (\alpha_u, \gamma_u] \times (\beta_u, \mu_u]$$

If $W_u = (\alpha_u, \gamma_u] \times (\beta_u, \mu_u] \subseteq M$ for each $u \in K$

$$K \subseteq M = \cup\{W_u : u \in K\}$$

We claim that $M \cap F \neq \varnothing$. Since there can be at most countably many u's in K,

$$\sup\{\alpha_u : u \in K\} = \rho < \omega_1$$

since the sup of a countable set cannot reach the uncountable ordinal ω_1. So $(\rho, \omega_1) \cap M \neq \varnothing$. So no open neighbourhood M of K can miss F, as claimed.

So T is not normal.

Example 19. Show that the product of normal spaces need not be normal.

Solution : We showed in the example above that the open interval ordinal space is normal. Since we showed that the deleted Tychonoff plank (the product of two open interval ordinal spaces) is not normal, we can conclude that the product of normal spaces need not be normal.

Concepts review:

1. What is the formal definition of a T_0-space?
2. Is there a topological space which is not T_0? If so provide an example.
3. Show that an infinite second countable T_0-space cannot have a cardinality larger than $2^{\aleph_0}$.

4. What is the formal definition of a T_1-space, emphasizing in the process how it is different from a T_0-space.

5. Give a characterization of a T_1-space.

6. Does a subspace of a T_1-space necessarily inherit the T_1 property?

7. Does a closed function f carry the T_1 property of its domain, S, to its range $f[S]$.

8. Does the T_1 property of all its factors, S_i, carry over to the product space, $\prod_{i\in I} S_i$?

9. If $\prod_{i\in I} S_i$ possesses the T_1 property does each of the factors necessarily possess the T_1 property?

10. Define the Hausdorff property (T_2 property) on a topological space.

11. Provide three different characterizations of the Hausdorff topological property.

12. Is the Hausdorff property carried over from the domain of a continuous function, $f : S \to T$, to its range $f[S]$? What if the function f was just closed? How about if f was both one-to-one and closed?

13. If all factors of a product space are Hausdorff, is the product space in question necessarily Hausdorff?

14. If $f : S \to T$ is a function and $G = \{(u, f(u) : u \in S\}$ is the graph of f in $S \times T$, give a condition on S or T that will guarantee that the graph, G, is closed in $S \times T$.

15. Define both a T_3-space and a regular space.

16. Does T_3 imply Hausdorff? Does "regular" imply Hausdorff?

17. Give two characterizations of "regular space".

18. Are metrizable spaces necessarily "regular"? Are regular spaces always metrizable?

19. Is "regular" a hereditary property?

20. Does the "regular" property always carry over from a product to its factors? What about, from the factors to the product?

21. Define both a T_4-space and a normal space.

22. Does T_4 imply "regular"? Does "normal" imply "regular"?

23. Give three characterizations of "normal space".

24. Are metrizable spaces necessarily "normal"?

25. What kind of function will always carry a normal property from its domain to its range?
26. Is the normal property hereditary? If not, what kind of subspace will inherit the normal property from its superset?
27. Is the product of normal spaces necessarily normal?
28. Are open ordinal spaces $[0, \omega_\alpha)$ normal spaces?
29. Is the deleted Tychonoff plank normal?

EXERCISES

1. Let (S, τ_S) and (T, τ_T) be two topological spaces and $f : S \to T$ and $g : T \to S$ be two continuous functions satisfying the property: $(g \circ f)(x) = x$ for all x in S.
   a) Show that if T is Hausdorff, then S must also be Hausdorff.
   b) Show that $f[S]$ is closed subset of T.
2. Suppose $S = \{x_0, x_1, \ldots, x_n\}$ is a finite set equipped with a topology, τ_S, which makes of (S, τ_S) a Hausdorff space. Describe all such topologies on S.
3. Let (S, τ_S) and (T, τ_T) be topological spaces where S is Hausdorff and contains a dense subset D. Suppose $f : S \to T$ is a continuous function such that $f|_S : D \to f[D]$ is a homeomorphism. Show that $f[S \backslash D] \subseteq T \backslash f[D]$.
4. Suppose (S, τ_S) is regular and (T, τ_T) is a topological space. Suppose the function $f : S \to T$ is a continuous function mapping S onto T. Show that if f is both an open and closed function as well, then T is Hausdorff.
5. Suppose (S, τ_S) is a topological space and K is a non-empty subset of S. Suppose $\tau_S = \{U \subseteq S : K \subseteq U\} \cup \{\varnothing, S\}$. Verify that this is indeed a valid topology on S. If so, are there any other properties that must be satisfied if we want S to be regular?
6. Suppose (S, τ_S) is a finite regular topological space. Is (S, τ_S) necessarily normal? Why?
7. Suppose (S, τ_S) is a regular topological space and K is an infinite subset of S. Show that there exists an infinite family of open subsets, $\{U_i \subseteq S : i \in \mathbb{N},\ U_i \cap K \neq \varnothing\}$, such that, if $i \neq j$, then $\mathrm{cl}_S U_i \cap \mathrm{cl}_S U_j = \varnothing$.
8. Suppose (S, τ_S) is a normal topological space and that R is an equivalence relation on S. Let (S, τ_θ) be the quotient space induced by the natural map $\theta : S \to S/R$. Show that, if θ is both open and closed, then (S, τ_θ) is a normal topological space.

9. Let $S = [0,1]$ and τ denote the usual topology on S. If $T = \mathbb{Q} \cap [0,1]$, let τ_S be the topology on S which is generated by the subbase, $\mathscr{S} = \tau \cup \mathscr{P}(T)$. Show that (S, τ_S) is a normal topological space.

---

10 / Separation with continuous functions.

Summary. *In this section we will investigate another method for separating closed sets. We can sometimes define a function, $f : S \to [0,1]$, which continuously maps a topological space, S, onto the closed interval $[0,1]$ in such a way that f has different constant values on a pair of disjoint closed subsets. We study those spaces where separation of disjoint closed sets can be performed in this way by such functions.*

10.1 Introduction

Suppose we are given a topological space (S, τ_S) and a continuous function $f : S \to [0,1]$ such that, for a pair of disjoint closed subsets A and B of S, $A \subseteq f^{\leftarrow}[\{0\}]$ and $B \subseteq f^{\leftarrow}[\{1\}]$. Then we will say that f *separates A and B*. The existence of such a separating *continuous* function for the particular sets, A and B, suggests that we can easily generate a pair of disjoint open sets, $U = f^{\leftarrow}[\,[0,1/2)\,]$ and $V = f^{\leftarrow}[\,(1/2,1]\,]$, from which we can immediately conclude that S is a normal topological space. The difficulty lies in proving the existence of such a function no matter which pair of disjoint closed subsets we are presented with. In most cases it is easier to prove normality of a space then proving the existence of a separating continuous function.

Let us consider the converse of the statement we have just discussed. Suppose we are presented with a topological space, S, along with two of its disjoint closed subsets, F and W. Given that S is normal, can we assume the existence of a "separating function", $f : S \to [0,1]$, for F and W? The answer is not obvious. Proving the existence of a separating real-valued continuous function on S which serves a particular purpose is not a trivial task. In fact, it is quite hard. We should also note that, on some topological spaces S, the only functions which are continuous on S are ones which are constant, and so such spaces contain no separating functions. The statement which proves that "For each pair of disjoint closed subsets of S there exists, for this pair, a separating function if and only if S is normal" is titled *Urysohn's lemma*.[1]

In the proof of the lemma, we will invoke the following characterization of the normal property:

> The topological space S is normal if and only if, for every non-empty closed set F and open neighbourhood U of F, there exists another open neighbourhood V of F such that $\text{cl}_S V \subseteq U$.

[1]Named after the Russian mathematician Pavel Urysohn (1898-1924).

10.2 Urysohn's lemma

The proof of Urysohn's lemma will be presented in two parts. We will first discuss a method for constructing a function on a topological space and then prove its continuity. The proof of Urysohn's lemma will follow.

Dyadic rationals.

We will begin by highlighting a few facts about *dyadic rationals*. Dyadic rationals are the rational numbers of the form,

$$\left\{ \frac{m}{2^n} : n \in \mathbb{N}\backslash\{0\},\ m = 1, 2, 3, \ldots, 2^n - 1 \right\}$$

For example, $\{\frac{1}{2^{100}}, \frac{2}{2^{100}}, \frac{3}{2^{100}}, \ldots, \frac{2^{100}-1}{2^{100}}\}$ is a set of dyadic rationals in $[0, 1]$. Let J denote the set of all dyadic rationals in $[0, 1]$.[1] Then J is countably infinite, dense and ordered in $[0, 1]$.[2] That is, $[0, 1]\backslash J$ contains no intervals and, even though J *contains neither 0 nor 1*, $0 = \inf(J)$ and $1 = \sup(J)$. So J can be used as a countably infinite indexing set.[3]

Defining of a function $f : S \to [0, 1]$ on S.

We will make the following two assumptions on (S, τ_S): Firstly, it allows the construction of a nested set of open sets, $\{U_i : i \in J\}$, indexed by the dyadic rationals, J, in $[0, 1]$, where, if $t < s$, $U_t \subseteq \mathrm{cl}_S U_t \subset U_s$. Secondly, both $U_0 = \cap\{U_i : i \in J\}$ and $U_1 = \cap\{S \setminus \mathrm{cl}_S U_i : i \in J\}$ are non-empty.

So we have the strictly increasing chains of inclusions,

$$U_0 \subset \cdots \subset U_{\frac{k-1}{2^n}} \subset \cdots \subset U_{\frac{k}{2^n}} \subset \cdots \subseteq \cup\{U_i\}_{i \in J} \subseteq S\backslash U_1$$

$$U_0 \subset \cdots \subset \mathrm{cl}_S U_{\frac{k-1}{2^n}} \subset \cdots \subset \mathrm{cl}_S U_{\frac{k}{2^n}} \subset \cdots \subseteq \cup\{\mathrm{cl}_S U_i\}_{i \in J} \subset S\backslash U_1$$

We will define a function as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in U_n \text{ for all } n \\ \sup\{i : x \notin U_i\} & \text{otherwise} \end{cases}$$

If $x \in U_0$ then, by definition, $f[U_0] = 0$. Verify that, in fact, we can make the stronger statement, $U_0 = f^{\leftarrow}[\{0\}]$.
If $x \in U_1 = \cap\{S \setminus \mathrm{cl}_S U_i : i \in J\}$, then $x \notin U_i$ for all $i \in J$, so $f(x) = 1$. Hence

[1] To see J: For $n > 0$, let $V_n = \{\frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{2^n-1}{2^n}\}$. Let $J = \cup\{V_n : n = 1, 2, 3, \ldots\}$.

[2] To see "dense": For any (a, b), $\frac{1}{2^n} < b - a$, for some n. Then, $1 < b2^n - a2^n$ implies $a2^n + 1 < b2^n$, hence there must exist an integer, m, such that $a2^n < m < b2^n$.

[3] One might ask: Why not simply use the indexing set $\mathbb{N}$? The significant differences between $\mathbb{N}$ and J are that J is confined to $[0, 1]$ and that every element larger than 0 in $\mathbb{N}$ has an immediate predecessor. While no element in J has an immediate predecessor. The need for such an indexing set is critical in the version of the proof we provide for the Urysohn lemma.

$f[U_1] = 1$. Again verify that, $U_1 = f^{\leftarrow}[\{1\}]$.

Proof of continuity of the above defined function $f : S \to [0, 1]$

Let's verify whether f, thus defined, is continuous on S. Recall (as previously shown) that a function, $f : S \to [0, 1]$, is continuous if f pulls back to open sets, each subbase element. These are of the form,

$$\mathscr{S} = \{\, [0, k)\,\}_{k\in(0,1)} \;\cup\; \{\, (k, 1]\,\}_{k\in(0,1)}$$

Note that,

$$\begin{aligned} f^{\leftarrow}\,[\,[0, k)\,] &= \{x \in S : f(x) < k\} \\ &= \{x \in S : \sup\{i : x \notin U_i\} < k\} \\ &= \{x \in S : x \in U_i \text{ for some } i < k\} \\ &= \cup\{U_i : i < k\} \end{aligned}$$

Also,

$$\begin{aligned} f^{\leftarrow}\,[\,(k, 0]\,] &= \{x \in S : f(x) > k\} \\ &= \{x \in S : \sup\{i : x \notin U_i\} > k\} \\ &= \{x \in S : x \notin U_i \text{ for some } i > k\} \\ &= \{x \in S : x \notin \mathrm{cl}_S U_j \text{ for some } i > j > k\} \\ &= \cup\{S\backslash \mathrm{cl}_S U_i : i > k\} \end{aligned}$$

Since both $f^{\leftarrow}\,[\,[0, k)\,]$ and $f^{\leftarrow}\,[\,(k, 0]\,]$ are open then f is continuous on S.

With these few arguments in mind, we are now set to formally state and prove Urysohn's lemma. The discussion above will significantly shorten the proof for the lemma. The reader is urged to understand it well. You will see that it is well worth the effort. Urysohn's proof is quite impressive. Even though we presented it as a theorem it is generally referred to a lemma.

Theorem 10.1 *Urysohn's lemma.* The topological space (S, τ_S) is *normal* if and only if given a pair of disjoint non-empty closed sets, F and W, in S there exists a continuous function $f : S \to [0, 1]$ such that, $F \subseteq f^{\leftarrow}[\{0\}]$ and $W \subseteq f^{\leftarrow}[\{1\}]$.

Proof : Let (S, τ_S) be a topological space.

($\Leftarrow$) If, for any pair of disjoint closed sets, F and W, there exists a continuous function $f : S \to [0,1]$ such that $F \subseteq f^{\leftarrow}[\{0\}]$ and $W \subseteq f^{\leftarrow}[\{1\}]$, then trivially S is normal.

($\Rightarrow$) Suppose S is normal and F and W are disjoint non-empty closed sets in S. Since $S\backslash W$ is an open neighbourhood of F there exists and open subset $U_{1/2}$ such that $F \subseteq U_{1/2} \subseteq \mathrm{cl}_S U_{1/2} \subseteq S\backslash W$. Normality of S allows us to repeat this step with dyadic rationals, $\{\frac{1}{2^2}, \frac{2}{2^2}, \frac{3}{2^2}\}$ to obtain the chain

$$F \subseteq U_{1/4} \subseteq \mathrm{cl}_S U_{1/4} \subseteq U_{2/2} \subseteq \mathrm{cl}_S U_{2/2} \subseteq U_{3/4} \subseteq \mathrm{cl}_S U_{3/4} \subseteq S\backslash W$$

Once we have, inductively, continued this process ranging over the set of all dyadic rationals in [0, 1], we obtain the chain,

$$F \subset \cdots \subset U_{\frac{k-1}{2^n}} \subset \cdots \subset U_{\frac{k}{2^n}} \subset \cdots \subset S\backslash W$$

Let $U_0 = \cap\{U_i\}_{i\in J}$ and $U_1 = \cap\{S \setminus \mathrm{cl}_S U_i\}_{i\in J}$. We then have $U_i \subset \cup\{\mathrm{cl}_S U_i\} \subseteq S\backslash W$ implies $S\backslash U_1 \subseteq S\backslash W$. We then obtain the chain,

$$F \subset U_0 \subseteq \mathrm{cl}_S U_0 \subset \cdots \subset \mathrm{cl}_S U_{\frac{k-1}{2^n}} \subset \cdots \subset \mathrm{cl}_S U_{\frac{k}{2^n}} \subset \cdots \subset S\backslash U_1 \subset S\backslash W$$

We will define a function $f : S \to [0,1]$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in U_n \text{ for all } n \\ \sup\{i : x \notin U_i\} & \text{otherwise} \end{cases}$$

So as shown in the preface to this theorem, $f : S \to [0,1]$ is a continuous function where $F \subseteq U_0 \subseteq f^{\leftarrow}\{0\}$ and $S\backslash U_1 \subseteq S\backslash W$ implies $W \subseteq U_1 \subseteq f^{\leftarrow}\{1\}$.

Urysohn's lemma provides us with another way of recognizing normal topological spaces. Given a pair of disjoint closed sets, F and W, in a space S, we will refer to the continuous function, $f : S \to [0,1]$, introduced in Urysohn's proof, as a *Urysohn separating function for F and W*. We also say that this function is one which *separates the disjoint closed sets F and W*.
We define these formally.

Definition 10.2 Suppose A and B are disjoint non-empty subsets of a space S. We will say that A and B are *completely separated in S* if and only if there is a continuous function $f : S \to [0,1]$ such that $A \subseteq f^{\leftarrow}(0)$ and $B \subseteq f^{\leftarrow}(1)$. In such a case, we say the function, f, separates A and B.

10.3 Completely regular spaces.

Note that, if there exists a Urysohn separating function for each pair of disjoint closed subsets in a normal space, then we can find a Urysohn function that can separate a point from a closed subset or separate any pair of points in any T_1 space. But there is nothing in what we have seen that allows us to conclude that a regular or Hausdorff space can generate its own Urysohn separating function like a normal space can. In fact, as we shall soon see, Hausdorff and regular spaces cannot, in general, produce, by themselves, such a function. If we are presented with a regular space in which it is shown that, for any point and disjoint closed set, there exists a Urysohn separating function, then this topological space belongs to a separate class of topological spaces called "completely regular spaces". We formally define this class.

Definition 10.3 If (S, τ_S) is a T_1-space in which, for a given point x and a non-empty closed subset F disjoint from $\{x\}$, there exists a continuous function, $f : S \to [0, 1]$, such that $\{x\} \subseteq f^{\leftarrow}\{0\}$ and $F \subseteq f^{\leftarrow}\{1\}$, then such a space, S, is called a *completely regular space*, or a *Tychonoff space*[1] (or, in some texts, a $T_{3\frac{1}{2}}$-*space*).

We add a few remarks concerning this definition. In this text, our definition emphasizes that only T_1 spaces are completely regular. (Some authors may not require the T_1 condition.) Obviously, since a Urysohn separating function separates disjoint closed sets, and since a regular space is T_1, then

$$\text{normal} \Rightarrow \text{completely regular}$$

Also, if S is completely regular, there exists a Urysohn separating function, f, such that $\{x\} \subseteq f^{\leftarrow}[\{0\}]$ and $F \subseteq f^{\leftarrow}[\{1\}]$. It follows that $\{x\} \subseteq f^{\leftarrow}[\,[0, 1/3)\,]$ and $F \subseteq f^{\leftarrow}[\,(2/3, 1]\,]$ where $[0, 1/3)$ and $(2/3, 1]$ are open in $[0, 1]$. Then, (see implication chart on page 153),

$$\text{completely regular} \Rightarrow \text{regular} \Rightarrow \text{semiregular} \Rightarrow \text{Hausdorff}$$

$$\text{regular} \Rightarrow \text{completely Hausdorff} \Rightarrow \text{Hausdorff}$$

There are regular spaces which are not completely regular. Then,

$$\text{regular} \not\Rightarrow \text{completely regular}$$

[1] Named after the Soviet and Russian mathematician Andreï Tychonoff, (1906-1993).

The standard example used in most topology texts of a regular non-completely regular space is called the *Tychonoff corkscrew space*[1]. This space is very involved and lengthy to describe. So we will not describe it in this text. So, from *Tychonoff corksrew*,

$$\text{regular} \not\Rightarrow \text{completely regular}$$

Also, there are completely regular spaces which are *not* normal. The Moore plane (Niemytzki's plane) is such an example.
At the end of this section, we prove that the "Moore plane is completely regular but not normal". From the Moore plane, we deduce

$$\text{completely regular} \not\Rightarrow \text{normal}$$

This means, a $T_{3\frac{1}{2}}$-space is strictly in between T_3 and T_4 (which partially explains the tongue-in-cheek terminology, " $T_{3\frac{1}{2}}$-space", used by some authors).

The description of the *completely regular property* does not simply refer to the separation of closed sets by open sets. It is defined in terms of the existence of a continuous function. If we want to refer to "completely regular" as a topological property or topological invariant, we should prove that it is first. Fortunately, the proof is fairly straightforward (which does not mean, we can overlook it). Suppose S is completely regular and $h : S \to T$ is a homeomorphism mapping S onto the space T. Suppose the singleton set, $\{x\}$, and the closed subset, F, are disjoint in T. Then there exists a function $f : S \to [0,1]$ such that $h^{\leftarrow}[\{x\}] \subseteq f^{\leftarrow}[\{0\}]$ and $h^{\leftarrow}[F] \subseteq f^{\leftarrow}[\{1\}]$. Let $g = f \circ h^{\leftarrow}$. Then $g[\{x\}] = \{0\}$ and $g[F] = \{1\}$. So "completely regular" is indeed a topological property, as expected.

Theorem 10.4 If (S, τ) is a metrizable space then it is completely regular.

Proof: We are given that (S, τ) is metrizable. Then there is a metric, ρ, such that (S, ρ) and (S, τ) are equivalent topological spaces.

Consider the function, $d_F : S \to \mathbb{R}$ defined as, $d_F(u) = \rho(u, F) = \inf \{\rho(u, y) : y \in F\}$.

Claim: d_F is a continuous function on S.

Proof of claim : To see this, consider, $\rho(x, u) < \delta = \varepsilon/3$. It suffices to show that $\rho(d_F(x), d_F(u)) < \varepsilon$, equivalently, $|d_F(x) - d_F(u)| < \varepsilon$.

Then with the variables, x, u and y, we write the two possible triangle inequalities

[1]Interested readers will easily find a description of the Tychonoff corkscrew online.

with y on one side.

$$\begin{array}{rcl}
\rho(x,y) & \leq & \rho(x,u)+\rho(u,y) \\
\rho(u,y) & \leq & \rho(x,u)+\rho(x,y) \\
& \Rightarrow & \\
\inf\{\rho(x,y):y\in F\} & \leq & \rho(x,u)+\inf\{\rho(u,y):y\in F\} \\
\inf\{\rho(u,y):y\in F\} & \leq & \rho(x,u)+\inf\{\rho(x,y):y\in F\} \\
& \Rightarrow & \\
d_F(x) & \leq & \varepsilon/3+d_F(u) \\
d_F(u) & \leq & \varepsilon/3+d_F(x) \\
& \Rightarrow & \\
-\varepsilon<-\varepsilon/3\leq & d_F(x)-d_F(u) & \leq\varepsilon/3<\varepsilon
\end{array}$$

So $|d_F(x)-d_F(u)|<\varepsilon$. Then d_F is continuous on S, *as claimed.*

Let F be a non-empty closed subset of S and a fixed point, x in $S\backslash F$. Since F is closed then $d_F(x)>0$. Since $d_F(x)>0$ and $d_F[F]=0$ then S is completely regular.

We conclude that every metrizable space is completely regular.

The following theorems confirm that completely regular spaces are hereditary and carry over from factors to their products.

Continuous images of completely regular spaces are not always completely regular. A standard example of this fact is to produce a continuous function on the completely regular *Moore plane* which maps it onto a Hausdorff, but non-regular space.

Theorem 10.5 Let S be a completely regular space and T be a subspace of S. Then T is completely regular.

Proof : We are given that S is a completely regular space and T is a subspace of S.

First note that T is clearly T_1. Let $F\cap T$ be a closed subset of T, where F is closed in S, and let $u\in T\backslash F$. Since S is completely regular there exists a continuous function $f:S\to[0,1]$ such that $x\in f^{\leftarrow}[\{0\}]$ and $F\subseteq f^{\leftarrow}[\{1\}]$. Then the function, $f|_T:T\to[0,1]$ separates x and $F\cap T$ and so T is completely regular.

Theorem 10.6 Let $\{S_i : i \in I\}$ be a family of topological spaces and $S = \prod_{i \in I} S_i$ be a product space. Then S is completely regular if and only if each factor, S_i, is completely regular.

Proof: We are given that $\{S_i : i \in I\}$ is a family of topological spaces and $S = \prod_{i \in I} S_i$ is a product space.

($\Rightarrow$) Suppose the product space, $S = \prod_{i \in I} S_i$, is completely regular. By theorem 7.9, each S_i is homeomorphically embedded in the product space, S. By the preceding theorem, every subspace is completely regular, so each S_i is completely regular.

($\Leftarrow$) We are given that each S_i is completely regular. Let K be a closed subset of $S = \prod_{i \in I} S_i$ and $\{x_i\}$ be a point in $S \backslash K$.

Then $\{x_i\} \in \cap \{\pi_i^{\leftarrow}[U_i] : i \in F\} \subseteq S \backslash K$ for some finite $F \subseteq I$ and open U_i's in S_i, where $x_i \in U_i$ for $i \in F$. Then, for each $i \in F$, there exists a continuous function, $f_i : S_i \to [0, 1]$, such that $x_i \in f_i^{\leftarrow}[\{1\}]$ and $S_i \backslash U_i \subseteq f_i^{\leftarrow}[\{0\}]$. We define $h : S \to [0, 1]$ as

$$h(\{x_i\}) = \inf \{f_i(x_i) : i \in F\}$$

Then $h = \inf \{f_i \circ \pi_i : i \in F\}$ is a continuous function. Note that $h(\{x_i\}) = 1$ and $S \backslash K \subseteq h^{\leftarrow}(0)$. So S is completely regular.

10.4 Topic: On zero-sets in normal spaces.

Recall that, in the proof of Urysohn's lemma (in theorem **??**), we showed that, for disjoint closed subsets F and W of the normal space, S, there is a continuous function, $f : S \to [0, 1]$, such that

$$\begin{array}{rclcl} F \subseteq U_0 & = & \cap \{U_i\}_{i \in J} & = & f^{\leftarrow}(0) \\ W \subseteq U_1 & = & \cap \{S \setminus \mathrm{cl}_S U_i\}_{i \in J} & = & f^{\leftarrow}(1) \end{array}$$

The continuous real-valued function, f, on S was referred to as a *Urysohn separating function for the closed subsets F and W*.

The indexing set, J, we used is a countably infinite set so both U_0 and U_1 are G_δ's (countable intersection of open sets). Note that we did not prove that F and W are equal to the respective G_δ's, U_0 and U_1. Usually, they are not, nor are they required to be in the Urysohn lemma statement.

We will briefly continue our discussion of normal spaces. We will first need the following definition.

Definition 10.7 A subset of a topological space, S, of the form, $f^{\leftarrow}[\{0\}]$, for some continuous real-valued function, f, on S is called a *zero-set*. Such a set is denoted by, $Z(f)$.

A *cozero-set* of S is a set of the form $S\backslash Z(f)$ for some continuous real-valued function f. The cozero-set of f is denoted by $\text{coz}(f)$.

For the definitions of zero-set and cozero-set, the function f need not necessarily be bounded.

We note a few interesting facts about zero-sets.

When we speak of a zero-set in a topological space, S, by definition, we are talking about a specific subset, A, of S which is associated to some function $f : S \to \mathbb{R}$ such that $A = f^{\leftarrow}(0) = Z(f)$. Often, there are quick ways of recognizing a zero-set when we see one. A zero-set is sometimes described as the "fibre" of a continuous real-valued function $f : S \to \mathbb{R}$. For example, if $f(u) = r \in \mathbb{R}$, then $u \in f^{\leftarrow}(r)$; if $g : S \to \mathbb{R}$ is the function defined as $g(x) = f(x) - r$ then

$$\begin{aligned} g(u) &= f(u) - r \\ &= r - r \\ &= 0 \end{aligned}$$

so $f^{\leftarrow}(r)$ is the zero set, $Z(g)$, which contains the element u. So the fibre of a continuous real-valued function, f, on S is a zero-set.

Also see that, for any zero-set, $Z(f)$ of the continuous function, f,

$$Z(f) = \cap\{\ f^{\leftarrow}[\,(-1/n, 1/n)\,]\ \}_{n>0}$$

Since f is continuous,

$$\ldots \textit{any zero-set, } Z(f), \textit{ is a } G_\delta$$

Countable intersections of zero-sets. We know that any zero-set is a countable intersection of open sets. What can we say about countable intersections of zero-sets? To answer this question we prove the general statement,

"*The countable intersection of zero-sets is a zero-set*"

Let $\{Z(f_n) : n \in \mathbb{N}\}$ be a countable family of zero-sets with non-empty intersection. For each f_n, let $h_n = |f_n| \wedge \frac{1}{2^n}$. We see that $h_n(x) \leq \frac{1}{2^n}$ on S. Since $\sum_{n\in\mathbb{N}} \frac{1}{2^n}$ converges, $\sum_{n\in\mathbb{N}} h_n(x)$ converges uniformly to a continuous function $h(x)$ on S. It is easily verified that,

$$Z(h) = \cap\{Z(h_n) : n \in \mathbb{N}\} = \cap\{Z(f_n) : n \in \mathbb{N}\}$$

as required.

In the proof of Urysohn's lemma we showed that $F \subseteq f^{\leftarrow}(0)$ and $W \subseteq f^{\leftarrow}(1)$. This means $F \subseteq Z(f)$ and $W \subseteq Z(f-1)$. We can then confidently reformulate one direction of the Urysohn lemma statement as follows:

> If (S, τ_S) is a normal topological space and F and W are disjoint non-empty closed subsets of S, then F and W are contained in disjoint zero-sets.

The leads to an interesting question: Is the converse of this statement true? We show that it is true, in general. In the following characterization of a normal space we restate Urysohn's lemma in terms of zero-sets rather than "...the existence of a function $f : S \to [0, 1]$ such that ..."

Theorem 10.8 The topological space (S, τ_S) is normal if and only if disjoint non-empty closed sets, F and W, in S are contained in disjoint zero-sets.

Proof: We are given that (S, τ_S) is a topological space.

($\Rightarrow$) This direction follows immediately from Urysohn's lemma.

($\Leftarrow$) We are given that disjoint closed subsets F and W are contained in disjoint zero-sets, $Z(f)$ and $Z(g)$, respectively. Since the two zero-sets are disjoint $0 \notin f[Z(g)]$ and $0 \notin g[Z(f)]$ (since, if $0 \in f[Z(g)]$, there some $u \in Z(g)$ such that $f(u) = 0$ hence $u \in Z(f) \cap Z(g)$).

We are required to show that S is normal. To do this we seek a Urysohn separating function, $h : S \to \mathbb{R}$, for F and W.

Let the function $h : S \to \mathbb{R}$, be defined as

$$h(x) = \frac{|f(x)|}{|f(x)| + |g(x)|}$$

Then h is continuous and real-valued on S with range in [0, 1].

If $x \in F \subseteq Z(f)$ then $h(x) = \frac{0}{0+|g(x)|} = 0$, so $F \subseteq Z(f) \subseteq h^{\leftarrow}(0)$.
If $x \in W \subseteq Z(g)$ then $h(x) = \frac{|f(x)|}{|f(x)|+0} = 1$, so $W \subseteq Z(g) \subseteq h^{\leftarrow}(1)$.

So h separates F and W. By theorem **??** S is normal.

A more succinct − and often used − way of expressing the property described in the above characterization of a normal space is given in the following definition. This is a redefinition of what appears in definition 10.2.

Definition 10.9 If the disjoint non-empty subsets, F and W, of a topological space, S, are contained in disjoint zero-sets, we say that F and W are *completely separated in* S.

This definition allows us to restate Urysohn's lemma as: "The space S is normal if and only if any pair of disjoint closed subsets of S are completely separated".

We mentioned above that zero-sets are G_δ's. Can we turn this phrase around and say that "G_δ's are zero-sets"? The answer is no. The statement "Closed G_δ's are zero-sets" is however true in, at least, one class of topological spaces, namely, normal spaces.

Theorem 10.10 Suppose F is a non-empty closed subset of the normal topological space, (S, τ). Then F is a G_δ if and only if F is a zero-set produced by some continuous function $f : S \to [0, 1]$.

Proof: We are given that (S, τ_S) is normal.

($\Leftarrow$) This direction is obvious since, if $F = Z(f)$ then $F = \cap\{f^{\leftarrow}[\,[0, 1/i)\,] : i \in \mathbb{N}\backslash\{0\}\,\}$, a G_δ.

($\Rightarrow$) Suppose F is a G_δ in the space S. If F is open then it is clopen and so is easily seen to be a zero-set. Suppose the G_δ, F, is not open. Then, by definition, there exists a countably infinite family, $\{V_i : i = 1, 2, 3, \ldots, V_i \text{ is open in } S\,\}$ such that $F = \cap\{V_i\}$. Suppose

$$\begin{aligned} U_1 &= V_1 \\ U_2 &= V_1 \cap V_2 \\ &\vdots \\ U_n &= V_1 \cap V_2 \cap \ldots \cap V_n \\ &\vdots \end{aligned}$$

Then $\{U_i : i = 1, 2, 3, \ldots\} \subset \tau_S$, $\;F \subseteq U_{i+1} \subseteq U_i$, for all $i > 0$, and $F = \cap\{U_i\}$.

Normality of S implies that, for all $i > 0$, there is a continuous function, $f_i : S \to [0, 1]$, such that $F \subseteq f_i^{\leftarrow}\{0\}$ and $S \backslash U_i \subseteq f_i^{\leftarrow}[\{1\}]$.

Consider the function, $f : S \to \mathbb{R}$, defined as,

$$f(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{2^i} \qquad (*)$$

See that, since $\frac{f_i(x)}{2^i} \leq \frac{1}{2^i}$ on S, and $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$, then, by the Weierstrass M-test, the series $(*)$ converges uniformly to the function $f(x)$ on S. Uniform convergence guarantees continuity of f on S.

Also see that, since $F = \cap\{V_i\}$, $F = f^{\leftarrow}[\{0\}]$. So the G_δ, F, is the zero-set, $Z(f)$, produced by the continuous function, $f : S \to [0, 1]$, described in $(*)$.

Theorem 10.11 If (S, τ) is completely regular then,

a) The family $Z[S] = \{Z(f) : f \in C(S)\}$ of all zero-sets forms a base for the closed sets in its topology.

b) The family $\text{coz}[S] = \{S \backslash Z(f) : f \in C(S)\}$ of all cozero-sets forms a base for the open sets in its topology.

Proof: a) Let F be a non-empty proper closed subset in S and $p \in S \backslash F$. Given that S is completely regular, there exists a function $f \in C(S)$ such that $F \subseteq Z(f)$ and $p \in Z(f - 1)$. Repeating this argument for every $p \in S \backslash F$ shows that F is the intersection of zero-sets. So $Z[S]$ forms a base for the closed sets in S.

It is easily seen that a) and b) are equivalent statements.

Example 1. Show that if S is a topological space in which the family of all zero-sets forms a base for closed sets then S must be completely regular.

Solution: Suppose F is a closed subset of S and p be a point not in F. By hypothesis, the closed set F is the intersection of zero-sets. Then, for every $p \in S \backslash F$, there exists a zero-set $Z(f)$ such that $F \subseteq Z(f) \subseteq S \backslash \{p\}$. Let $h : S \to \mathbb{R}$ be defined as $h(x) = \left(\frac{1}{f(p)}\right) f(x)$, Then h is a continuous real-valued function which maps F to $\{0\}$ and p to 1. By definition, S is completely regular.

Example 2. Show that the closed subsets of a metric space S are zero-sets.

Solution: Suppose F is a closed subset of a metric space, (S, ρ). Define a real-valued function $f : S \to \mathbb{R}$ as $f(x) = \inf\{\rho(x, u) : u \in F\}$. Then $f[F] = \{0\}$ and $f(x) \neq 0$ for $x \notin F$. Verifying that f is continuous on S is straightforward. So $F = Z(f)$.

10.5 Topic: Perfectly normal topological spaces.

Theorem 10.10 describes a particular property possessed by normal spaces. We know that in any space a zero-set is a G_δ. We now know that, provided a space is normal, closed G_δ's are zero-sets. We should, however, note something that we cannot conclude from this statement. It doesn't say that, in a normal space, non-empty closed subsets are zero-sets and so are G_δ's. We will exhibit below, a normal space which contains a non-G_δ singleton set. Those normal spaces in which all non-empty closed subsets are G_δ's form a special class of topological spaces which we will define now.

Definition 10.12 A normal topological space, (S, τ_S), is said to be *perfectly normal* if and only if every closed subset of S is a G_δ.

Note that, to say that every closed subset is a G_δ is equivalent to saying that every open set is an F_σ (a countable union of closed sets). To see this suppose F is closed in S and U_i is open in S, and $U = S\backslash F$ and $F_i = S\backslash U_i$.

$$\begin{aligned} G_\delta = F = \cap\{U_i : i \in \mathbb{N}\} \quad &\Leftrightarrow \quad S\backslash F = S\backslash \cap\{U_i : i \in \mathbb{N}\} \\ &\Leftrightarrow \quad U = \cup\{S\backslash U_i : i \in \mathbb{N}\} \\ &\Leftrightarrow \quad U = \cup\{F_i : i \in \mathbb{N}\} = F_\sigma \end{aligned}$$

Example 3. Suppose S is a normal second countable space. Show that S is perfectly normal.

Solution : Let V be an open non-empty subset of S. If $x \in V$, then there exist an open base element, U_x, of x such that $x \in \mathrm{cl}_S U_x \subseteq V$ (since S is normal). Since S is second countable, we can choose $\{U_x : x \in S\}$ to be countable. Then $\{\mathrm{cl}_S U_x : x \in V\}$ is a countable set of closed subsets. Then $V = \cup\{\mathrm{cl}_S U_x : x \in V\}$, an F_σ in S. So every closed set is a G_δ. Then S is perfectly normal.

Theorem 10.13 Suppose (S, τ_S) is a T_1 topological space. Then S is perfectly normal if and only if S is normal and every non-empty closed subset of S is a zero-set.

Proof : We are given that (S, τ_S) is T_1.

($\Rightarrow$) We are given that S is perfectly normal. By definition, only normal sets can be perfectly normal. Suppose F is a non-empty closed subset. By hypothesis, F is a G_δ. By theorem 10.10, since F is a G_δ in a normal space then F is a zero-set, $Z(f)$, for some continuous $f : S \to [0,1]$.

$(\Leftarrow)$ Suppose S is normal and every non-empty closed subset F is a zero-set. Zero-sets have been shown to be G_δ's. So S is perfectly normal.

Example 4. Show that a metric space is perfectly normal.

Solution : By theorem 9.18, a metric space is normal. In an example on page 174 it is shown that closed subsets of a metric space are zero-sets. By the above theorem a metric space is perfectly normal.

Normal but still not perfect

Example 5. *A normal space which is not perfectly normal.* Let ω_1 denote the first uncountable ordinal. Show that the ordinal space, $S = [0, \omega_1]$, shown previously to be normal (on pages 80 and 158) is not perfectly normal.

Solution : Let (S, τ_ω) denote the ordinal space $[0, \omega_1]$ where ω_1 is the first uncountable ordinal. We will show that the closed subset, $\{\omega_1\}$, of S is not a G_δ and hence S is not perfect.

Recall that $(\mu, \omega_1]$ is an open neighbourhood base element of ω_1. Consider the countable family, $\{U_i : i \in \mathbb{N}\}$, of open neighbourhoods of ω_1. Then for each U_i, there exists μ_i such that $(\mu_i, \omega_1] \subseteq U_i$. Let $U = \{\mu_i : i \in \mathbb{N}\}$ and $\gamma = \sup U$. If γ is a limit ordinal then $\gamma = \cup\{\mu_i : i \in \mathbb{N}\}$, a countable union of countable ordinals; hence γ is a countable ordinal. If γ is not a limit ordinal then $\gamma \in U$. So again γ is a countable ordinal. That is $\gamma < \omega_1$. So $(\gamma, \omega_1] \subseteq \cap\{U_i : i \in \mathbb{N}\}$. So $\{\omega_1\}$ cannot be a G_δ. So S is not perfect.

Example 6. *A normal space which is not perfectly normal.* Let $S = \mathbb{R}^2$ and $p = (0,0)$ and suppose S has a topology defined as follows:

$$\tau = \{T \subseteq S : S \backslash T \text{ is finite or } p \notin T\}$$

Show that S is normal but not perfectly normal.

Solution : We claim that S is T_1. The set $\{p\}$ is closed and if $x \neq p$ then $S \backslash \{x\}$ is open so $\{x\}$ is closed. So S is T_1, as claimed.

We claim that S is T_4. Suppose F and K are disjoint closed subsets of S. If $p \notin K \cup F$ then K and F are both open. So F and K are clopen and so separated by open sets. Suppose $p \in F$. Then K is clopen and finite. Now $S \backslash K$ is open since K is finite. So the F and K are contained in the disjoint open sets $S \backslash K$ and K. So S is normal, as claimed.

We claim that S is not perfect. Let $\{U_i : i \in \mathbb{N}\}$ be a countable family of sets in S each containing the point p. Then each U_i is closed in S and $p \in \cap\{U_i : i \in \mathbb{N}\}$. If each U_i is also open $S\backslash U_i$ if finite. So each U_i is uncountable. Then $S\backslash\cap\{U_i : i \in \mathbb{N}\} = \cup\{S\backslash U_i : i \in \mathbb{N}\}$ a countable subset of S. So $\cap\{U_i : i \in \mathbb{N}\}$ is uncountable. So $p \neq \cap\{U_i : i \in \mathbb{N}\}$. So p is not a G_δ. So S is not perfectly normal, as claimed.

By definition perfectly normal bases are normal but our example confirms there can be normal spaces which are not perfect. So have, the implications,

$$\text{normal} \not\Rightarrow \text{perfectly normal}$$

$$\begin{array}{c} \text{perfectly normal} \Rightarrow \text{normal} \Rightarrow \text{completely regular} \Rightarrow \text{regular} \Rightarrow \text{semiregular} \Rightarrow \text{Hausdorff} \\ \Downarrow \\ \text{completely Hausdorff} \Rightarrow \text{Hausdorff} \end{array}$$

Example 7. Suppose S is a normal space. Show that S is perfectly normal if and only if, for any closed subset F of S, there exists a continuous function, $f : S \to T$, mapping S into a perfectly normal space T such that $F = f^{\leftarrow}[K]$ is a zero-set for some closed subset K of T.

Solution : We are given that S is normal and F is closed in S.

($\Leftarrow$) Suppose that there exists a continuous function, $f : S \to T$, mapping S into the perfectly normal space, T, such that $F = f^{\leftarrow}[K]$ for some closed subset K of T. Then K is a zero-set, say $Z(h)$, in T. Then $h[K] = \{0\}$. Then $(h \circ f)^{\leftarrow}(0) = f^{\leftarrow}(h^{\leftarrow}(0)) = F$. So F is a zero-set in S. Since F is an arbitrary closed subset, S is perfectly normal.

($\Rightarrow$) Suppose S is perfectly normal and F is closed subset of S. Then F is a zero-set and so $F = f^{\leftarrow}(\{0\})$, for some $f \in C(S)$. Since $\mathbb{R}$ is a metrizable it is perfectly normal. Let $K = \{0\}$ a closed subset of $\mathbb{R}$. Then $F = f^{\leftarrow}[K]$, as required

10.6 Topic: "Completely regular but not normal" example.

In the following two examples we describe a completely regular topological space which is not normal. This is the Moore plane.

Example 8. Show that the *Moore plane* is completely regular.

Solution : Let (S, τ_S) denote the Moore plane. We have already shown in a previous example (on page 154) that the Moore plane is regular.

We subdivide the Moore plane in two subsets: The closed subset, $F = \{(x, 0) : x \in \mathbb{R}\}$ and its complement, the open subset, $W = S \setminus F$.

Case 1: Suppose $(a, b) \in W$ and K be a non-empty closed subset disjoint from $\{(a, b)\}$ in S.

Without loss of generality, we can enlarge K so that it contains a closed neighbourhood, say M, of the subset F. Then $K \cap W \neq \varnothing$. Since (by theorem 9.19) the subspace W is normal there exists a Urysohn separating function in W, say $g : W \to [0, 1]$, such that $\{(a,b)\} \subseteq g^{\leftarrow}[\{1\}]$ and $K \cap W \subseteq g^{\leftarrow}[\{0\}]$. Since M is a neighbourhood of F and $g[M \cap W] = \{0\}$, then we can extend g to a continuous function, $g_s : S \to [0,1]$, on S where $g_s[K \cup F] = \{0\}$ and $\{(a,b)\} \subseteq g_s^{\leftarrow}[\{1\}]$. Continuity of g_s on F can be verified by applying the definition of continuity at each point of F. Then g_s is a Urysohn separating function for (a,b) and K in S. We are done with case 1.

Case 2: We must also consider the case where $(a,b) \in F$ and K is a non-empty closed subset of W such that $\mathrm{cl}_S K$ is disjoint from $\{(a,b)\}$ in S. Without loss of generality, suppose $\mathrm{cl}_S K$ contains the closed subset, $F \backslash \{(a,b)\}$.[1] Since S is regular, there exists a neighbourhood,

$$B = B_\varepsilon(a,\varepsilon) \cup \{(a,b)\} \subseteq \mathrm{cl}_S B \subseteq S \backslash \mathrm{cl}_S K$$

Since W is dense in S, then $\mathrm{cl}_S B \cap W$ is non-empty in W.

Since W is normal there exists a Urysohn separating function, $g : W \to [0,1]$, such that $\mathrm{cl}_S B \cap W \subseteq g^{\leftarrow}[\{0\}]$ and $K \cap W \subseteq g^{\leftarrow}[\{1\}]$. Given $g[B \cap W] = \{0\}$, then we can extend g to a continuous function, $g_s : S \to [0,1]$, on S where $g_s[\{(a,b)\}] = \{0\}$ and $g_s[\mathrm{cl}_S K] = \{1\}$.

Then $g_s : S \to [0,1]$ is a Urysohn separating function for $\{(a,b)\}$ and K in S, where $K \subseteq \mathrm{cl}_S K \subseteq g_s^{\leftarrow}[\{1\}]$ and $\{(a,b)\} \subseteq \mathrm{cl}_S B \subseteq g_s^{\leftarrow}[\{0\}]$.

We conclude that the Moore plane is indeed completely regular.

But the Moore plane is not normal. As we now show in this solved example.

Example 9. Show that the *Moore plane* is not normal.

Solution : Let (S, τ_S) denote the Moore plane. We claim S is not normal. Suppose we assume S is normal.

Let $F = \{(x,0) : x \in \mathbb{R}\}$ and $W = S \setminus F$ and let $D = (\mathbb{Q} \times \mathbb{Q}) \cap W$, a countable dense subset of S.

For each $(a,0) \in F$ the basic open neighbourhood of $(a,0)$ is of the form

$$B_a = B_\varepsilon(a,\varepsilon) \cup \{(a,0)\} \text{ for some } \varepsilon$$

We claim that each non-empty subset of F is closed in S:

Suppose $T \subset F$ and $(a,0) \in F \backslash T$. Then $(a,0)$ belongs to some B_a. Then

$$F \backslash T \subseteq U_{F \backslash T} = \cup \{B_a : (a,0) \in F \backslash T\}$$

and $T = (S \backslash U_{F \backslash T}) \cap F$, a closed subset of S. So both T and $F \backslash T$ are closed in S, as claimed.

[1] $F \backslash \{(a,b)\}$ is the intersection of the closed set, F, with the complement of the basic open neighbourhood $B_\varepsilon(a,\varepsilon) \cup \{(a,b)\}$ of (a,b).

Since S is assumed to be normal, we can choose from τ_S, a pair of disjoint open neighbourhoods,

$$\begin{aligned} U_T &= \cup\{B_a : (a,0) \in T\} \\ U_{F\setminus T} &= \cup\{B_a : (a,0) \in F\setminus T\} \end{aligned}$$

of T and $F\setminus T$, respectively. For each $T \in \mathscr{P}(F)$, let

$$D_T = \cup\{B_a \cap D : B_a \subseteq U_T\}$$

Let $f : \mathscr{P}(F) \to \mathscr{P}(D)$ be defined as $f(T) = D_T$.

We claim that f maps $\mathscr{P}(F)$ one-to-one into $\mathscr{P}(D)$:

Suppose K and T are distinct elements in $\mathscr{P}(F)$ such that $(b,0) \in K\setminus T$. Then $(b,0) \in K\setminus T \subseteq F\setminus T$. Then $B_b \in U_{F\setminus T}$. Since $U_T \cap U_{F\setminus T} = \varnothing$, $B_b \cap U_T = \varnothing$. So $(B_b \cap D) \cap D_T = \varnothing$. Since $B_b \cap D \subseteq D_K$, then

$$f(K) = D_K \neq D_T = f(T)$$

So f maps $\mathscr{P}(F)$ one-to-one into $\mathscr{P}(D)$. As claimed.

The cardinality of $\mathscr{P}(F)$ is 2^c while the cardinality of $\mathscr{P}(D)$ is $2^{\aleph_0} = c$ Then from the claim we obtain $2^c \leq c$. Contradiction.

So the Moore plane is *not* normal, but *is completely regular.*

10.7 Topic: The embedding theorem revisited.

Recall the statement in theorem 7.14, titled, *The embedding theorem I*,[1] we presented earlier as an application of Cartesian products. After defining what it means for a set, $\mathscr{F}$, of functions to separate points and closed sets of a topological space, S, and defining an evaluation function with respect to $\mathscr{F}$ on S, we proved the *Embedding theorem I* which states that the evaluation map e, with respect to $\mathscr{F}$, embeds a homeomorphic copy, $e[S]$, of S in the product space, $\prod_{\alpha\in\Gamma} X_\alpha$. Each of the factors, X_α, of the product is the codomain of the function, f_α, in $\mathscr{F}$.

Having now introduced the completely regular space, in this new context we can take the Embedding theorem I a step further. But first, there is a preliminary theorem we must take care of.

We present an interesting characterization of a completely regular space. It states that "…, *the completely regular topological spaces, S, are precisely those spaces S whose topology is the weak topology induced by $C^*(S)$.*"

[1]The word "embedding" is sometimes spelled "imbedding".

Theorem 10.14 Suppose (S, τ) is a T_1 topological space and $C^*(S)$, represents the set of all continuous bounded real-valued functions on S. The second statement in part b) is the converse of the first statement which appears in part a).

a) If the topological space, (S, τ), is completely regular then its topology, τ, is the weak topology induced by $C^*(S)$.

b) If (S, τ) is a space whose topology, τ, is the weak topology induced by $C^*(S)$ then S is a completely regular space.

Proof: We are given that (S, τ) is a T_1 topological space. Let

$$\mathscr{S} = \{f^{\leftarrow}[U] : f \in C^*(S),\ U \text{ an open subset of } \mathbb{R}\}$$

a) Suppose S is completely regular. To show that τ is the weak topology induced by $C^*(S)$ we are required to show that $\mathscr{S}$ is a subbase which generates τ (by definition 6.13). So we must show that $\tau_{\mathscr{S}} = \tau$.

   Let $x \in V \in \tau$. We claim that $V \in \tau_{\mathscr{S}}$. Then, since S is completely regular, there exists $f : S \to [0, 1]$ (in $C^*(S)$) such that $x \in f^{\leftarrow}[\{0\}]$ and $S \backslash V \subseteq f^{\leftarrow}[\{1\}]$. Then $f^{\leftarrow}[\,[0, 1/3)\,]$ is an open neighbourhood of x entirely contained in V. Then, by definition, $V \in \tau_{\mathscr{S}}$. This means $\tau \subseteq \tau_{\mathscr{S}}$.

   Suppose $f^{\leftarrow}[U] \in \mathscr{S}$. Since f is continuous on S, $f^{\leftarrow}[U] \in \tau$. So $\mathscr{S} \subseteq \tau$, hence $\tau_{\mathscr{S}} \subseteq \tau$.

   We conclude that $\tau_{\mathscr{S}} = \tau$. So τ is the smallest topology on S such that every function in $C^*(S)$ is continuous on S, as required.

b) Suppose τ is the weak topology on S, induced by the family of functions, $C^*(S)$. Then the set, $\mathscr{S}$, forms a subbase for τ. We are required to show that S is completely regular.

   Let U be a non-empty open set and $x \in U$. To show complete regularity we must construct a continuous function, $h : S \to [0, 1]$, which separates $\{x\}$ from $S \backslash U$. A subbase element in $\mathscr{S}$ is of the form

$$V = \{f^{\leftarrow}[U] :\ \text{some } f \in C^*(S),\ U \text{ is a subbase element of } \mathbb{R}\}$$

   Since τ is generated by the subbase $\mathscr{S}$, there exists finitely many subbase elements, $\{V_i : i \in F = \{1, \ldots, n\}\ \} \subset \mathscr{S}$, such that

$$x \in \cap\{V_i : i \in F = \{1, \ldots, n\}\ \} \subseteq U \qquad (*)$$

   If $V_i = f^{\leftarrow}[U_i]$, then U_i is of the form (a_i, ∞) or $(-\infty, a_i)$. To simplify our expression, note that, if $U_j = (-\infty, a_j)$ then

$$f_j^{\leftarrow}[U_j] = f_j^{\leftarrow}[(-1)(-a_j, \infty)] = (-f)_j^{\leftarrow}[(-a_j, \infty)]$$

so we can assume that all U_i's are of the form $U_i = (a_i, \infty)$ after having made the required adjustments on the functions, f_i. So $(*)$ can be expressed as

$$x \in \cap\{f_i^{\leftarrow}[(a_i, \infty)] : i \in F\ \} \subseteq U \qquad (**)$$

For each i in $F = \{1, \ldots, n\}$, let $t_i : S \to \mathbb{R}$ be defined as: $t_i(y) = f_i(y) - a_i$. See that $t_i(y) > 0$ on $f_i^{\leftarrow}[(a_i, \infty)]$, a subset of U. So $x \in t_i^{\leftarrow}[(0, \infty] \subseteq U$. Also see that $t_i(y) \leq 0$ everywhere else, including on $S \backslash U$.

For each i in $F = \{1, \ldots, n\}$, let

$$t_i^* = t_i \vee 0^1$$

So $t_i^* \geq 0$, and

$$x \in \cap\{t_i^{*\leftarrow}[(0, \infty)] : i \in F\ \} \subseteq U$$

We define $t : S \to \mathbb{R}$, as

$$t(y) = t_1^*(y)t_2^*(y)t_3^*(y) \cdots t_n^*(y)$$

We then have a real-valued function such that $x \in t^{\leftarrow}[(0, \infty)] \subseteq U$ and $S \backslash U \subseteq t^{\leftarrow}(0)$. Suppose $t(x) = \delta$. Then $h = (t/\delta) \wedge 1\,^2$ maps S into $[0, 1]$ and $x \in h^{\leftarrow}(1)$. So h separates x and $S \backslash U$.

This allows us to conclude that S is completely regular.

In the *Embedding theorem II* statement, we will speak of a "cube". Normally, when we casually speak of a cube we think of a three dimensional space, $[a, b]^3$. In topology, we find it convenient to call any Cartesian product, $\prod_{i \in J}[a_i, b_i]$, of closed intervals, $[a_i, b_i]$ in $\mathbb{R}$, a *cube*. Since any closed and bounded interval $[a, b]$ is homeomorphic to the closed unit interval, $[0, 1]$, then, by theorem 7.8, any cube, $\prod_{i \in J}[a_i, b_i]$, is homeomorphic to a product space, $\prod_{i \in J}[0, 1]$, of the unit interval $[0, 1]$.

If one is speaking of a cube, it may sometimes be relevant to state explicitly its dimension.

Theorem 10.15 *The embedding theorem II* . Suppose S is a T_1 topological space. The space S is completely regular if and only if a homeomorphic copy of S is embedded in some cube, $\prod_{i \in J}[a_i, b_i]$.

[1] $(f \vee 0)(y) = \max\{f(y), 0(y)\}$
[2] $(t/\delta) \wedge 1)(y) = \min\{(t/\delta)(y), 1(y)\}$

Proof: We are given that S is a T_1 topological space.

($\Rightarrow$) Suppose S is embedded in a cube, $\prod_{i\in J}[a_i, b_i]$. Since each $[a_i, b_i]$ is a metric space, then, by theorem 10.4, each $[a_i, b_i]$ is completely regular. By theorem 10.6, the cube, $\prod_{i\in J}[a_i, b_i]$, is completely regular. By theorem 10.5, S is completely regular.

($\Leftarrow$) Suppose S is completely regular. We will now apply the *Embedding theorem I*. By theorem 10.14, the completely regular spaces, S, are precisely those spaces, S, whose topology is the weak topology induced by the family, $C^*(S)$; this means that $C^*(S)$ provides the open sets necessary to separate points and closed sets of S. Each function, f_i, in $C^*(S)$ maps S into some closed interval, $[a_i, b_i]$, of $\mathbb{R}$. So, by the *Embedding theorem I*, the evaluation map, $e : S \to \prod_{i\in J}[a_i, b_i]$, with respect to $C^*(S)$, homeomorphically embeds S into the cube, $\prod_{i\in J}[a_i, b_i]$.

Since $\prod_{i\in J}[a_i, b_i]$ and $\prod_{i\in J}[0, 1]$ are homeomorphic spaces (by theorem 7.8) then the compact space, $\prod_{i\in J}[0, 1]$, contains a homeomorphic copy of S.

Notation: If S is a space and $f : S \to [0, 1]$ then $f \in [0, 1]^S = \prod_{x\in S}[0, 1]$.

Concepts review:

1. Define "dyadic rationals" used in the proof of Urysohn's lemma.
2. State Urysohn's lemma.
3. What are we referring to when we speak of a *Urysohn separating function for two closed sets.* Describe such a function.
4. Define a completely regular space (Tychonoff space).
5. How does a completely regular space differ from a $T_{3\frac{1}{2}}$-space?
6. Is a normal space necessarily completely regular? Is a completely regular space necessarily regular? Why?
7. Define a perfectly normal space.
8. What are we referring to when we speak of a zero-set in a topological space?
9. It was proven that, in a certain type of space, G_δ's and zero-sets mean the same thing. What type of space did we refer to?

10. Provide an example of a completely regular space that is not normal.
11. Provide an example of a perfectly normal topological space.
12. Provide an example of a normal space which is not perfectly normal.
13. Describe the *Moore plane* and its topology.
14. Describe an *ordinal space* and its base for open sets.
15. Is the completely regular property hereditary?
16. Suppose $S = \prod_{i \in J} S_i$. If S is completely regular does it follow that each S_i is completely regular?
17. Suppose $S = \prod_{i \in J} S_i$. If each S_i is completely regular does it follow that S is completely regular?
18. The completely regular topology is the weak topology induced by which family of functions?
19. The evaluation map, e, which is used embed S into a cube is with respect to which family of functions?
20. Are continuous images of a completely regular space always completely regular?

EXERCISES

1. Let (S, τ_S) be a normal topological space which contains a subset T which is an F_σ. Verify whether T is a normal subspace.
2. Are there any normal spaces which are not separable? If so, show one.
3. Is "perfectly normal" a hereditary property?
4. Is it true that a topological space, (S, τ_S), is normal if and only if any two non-empty disjoint closed subsets have disjoint closed neighbourhoods? Prove it.
5. Let (S, τ_S) be a normal topological space which contains the non-empty closed subset F. If U is an open neighbourhood of F, show that $F \subset V \subset U$ for some F_σ-set, V.
6. Let (S, τ_S) be a countable completely regular space. Show that S is normal.
7. Let (S, τ_S) be a completely regular topological space. Let V be an open neighbourhood of a point x. Show that there exists a Urysohn separating function, $f : S \to [0, 1]$, for $\{x\}$ and $S \backslash V$ if and only if x is a G_δ.

8. We showed that closed subspaces of normal spaces are normal. A topological space (S, τ_S) is said to be *completely normal* if and only if *every one* of its subspaces is a normal space. Show that S is completely normal if and only if, whenever $\mathrm{cl}_S U \cap V = \varnothing = U \cap \mathrm{cl}_S V$, there exists disjoint open neighbourhoods for U and V, respectively. Note: To prove "$\Rightarrow$" consider $S \backslash (\mathrm{cl}_S U \cap \mathrm{cl}_S V)$.

Part IV

Limit points in topological spaces

11 / Limit points in first countable spaces.

Summary. *In this section we will investigate whether the notion of limit points and there sequences can be applied to topological spaces. We know that, in metric spaces, closed subsets can be characterized by the limit points they possess and continuous functions can be characterized in terms of how they act on sequences and their limits. We will show that topological spaces which are first countable are receptive to notions of sequences and their limits points as we know them.*

11.1 Introduction.

Limit points of sets play an important role in the fields of normed spaces and metric spaces. In particular, limit points are involved in characterizations of closed sets and the sequential definition of continuity. References to limit points have a common link. They are assumed to be the limit of ordered sets called *sequences*, all indexed by countably infinite linearly ordered sets. We will investigate whether the notion of *limit points* can also play a role in our study or topological spaces. Certainly, they are seen to be relevant in those topological spaces which our metrizable. We want to determine how to adapt their definition so that they have similar meanings in more general contexts.

11.2 Sequences and limit points in topological spaces.

We will first start with the formal definition of a sequence of elements in a topological space. It is identical to the definitions introduced in normed vector spaces and metric spaces.

Definition 11.1 Let S be any topological space.

a) A *sequence* in S is a function, $f : \mathbb{N} \to S$, mapping $\mathbb{N}$ into S. It can be denoted as, $\{f(i)\}$, $\{x_0, x_1, x_2, x_3, \ldots, x_n, \ldots\}$ where $x_i = f(i)$ for each $i \in \mathbb{N}$, or, $\{x_i : i = 0, 1, 2, 3, \ldots\}$ or, simply, $\{x_i\}$. If $T_j = \{i \in \mathbb{N} : i \geq j\}$ for some j, we will say that $f[T_j] = \{x_i : i \geq j\}$ is a *tail end of the sequence* $\{f(i)\}$.

b) A sequence, $\{x_i\}$, is said to *converge to a point* p in S if, for every open neighbourhood U of p, there exists, N, such that, when $n > N$, x_n belongs to U. We also say "...if and only if every open neighbourhood of p contains a tail end of the sequence".[1] The expression, $\{x_i\} \to p$, is used to indicate that the sequence converges to p.

c) If $\{x_i\} \to p$, we say that p is a *limit point of the sequence* $\{x_i\}$.

[1] The expressions "$\{x_i\}$ converges to p if the sequence eventually belongs to every neighbourhood, U of p" is also used.

d) If F is a non-empty set, we say that *p is a limit point of the set, F*, if and only if p is the limit point of some sequence, $\{x_i\}$, which is entirely contained in F.

e) We say that k is an *accumulation point* of the sequence, $A = \{f(i)\}$, if every open neighbourhood of k has a non empty intersection with every tail end, $f[T_j] = \{x_i : i \geq j\}$, of A.

The reader is no doubt familiar with the following characterization of "closed set" given in the study of metric spaces:

> "A non-empty subset, F, of the metric space, (M, ρ), is closed in M, if and only if, F contains the limit point of each and every sequence contained in F."

Also, we have another definition which involves sequences, namely the sequential definition of continuity for functions mapping a metric space to a metric space.

> "A function, $f : M_1 \to M_2$, mapping a metric space, M_1, into a metric space, M_2, is continuous at a point p if and only if, whenever p is the limit point of a sequence, $\{x_i\}$, in M_1 $f(p)$ is the limit point of the corresponding sequence, $\{f(x_i)\}$, in M_2."

These two concepts may not make sense, if we simply generalize the statements from "metric spaces" to all topological spaces. We should identify what are the fundamental characteristics of metrizable spaces that give the meaning we expect from these concepts.

We will show that the transition from sequences in metric spaces to sequences in topological spaces flows smoothly provided the topological spaces are first countable.

We begin by providing the following definition.

Definition 11.2 Let S and T be topological spaces.

We say that a function $f : S \to T$ is *sequentially continuous* at p if and only if, whenever p is limit point of a sequence, $\{x_i\}$, in S, then $f(p)$ is the limit point of the corresponding sequence, $\{f(x_i)\}$ in T.

Theorem 11.3 Let (S, τ_S) and (T, τ_T) both be *first countable* topological spaces.

a) A non-empty subset, F of S, is closed in S if and only if every limit point of F belongs to F.

b) A function mapping S into T is continuous on S if and only if it is sequentially continuous at each point, p, in S.

Proof: Let (S, τ_S) and (T, τ_T) both be first countable topological spaces.

a) ($\Rightarrow$) Suppose F is a closed subset of S and let p be a limit point of F.

Then, by definition, there is a sequence, $\{x_i\} \subseteq F$, which converges to p. Suppose $p \in S \backslash F$. Then, if U is any open neighbourhood of p, $U \cap (S \backslash F)$ contains a tail end of the sequence $\{x_i\}$. Since this is impossible, then $p \in F$.

($\Leftarrow$) Suppose F contains all its limit points.

Suppose there exists a point $p \in \mathrm{cl}_S F \backslash F$. By hypothesis, there exists, $\{U_i : i \in \mathbb{N}\}$, a countable open neighbourhood base of p. If, for $n \in \mathbb{N}$, $V_n = U_1 \cap U_2 \cap \cdots \cap U_n$, then $\{V_n : n \in \mathbb{N}\}$ is a countable neighbourhood base of open sets of p, such that $V_{n+1} \subseteq V_n$. Then $\{p\} = \cap\{V_n : n \in \mathbb{N}\}$.

Since p is a boundary point of F, we can choose, for each $i \in \mathbb{N}$, $x_i \in V_i \cap F$ (Choice function). Since every V_i contains a tail end of the sequence $\{x_i\}$, then p is a limit of the sequence in F; so it is a limit point of F. By hypothesis, p, being a limit of F, should belong to F. This contradicts, $p \in \mathrm{cl}_S F \backslash F$. To avoid the contradiction we must have, $\mathrm{cl}_S F \backslash F = \varnothing$, which implies $F = \mathrm{cl}_S F$, and so F is closed.

b) ($\Rightarrow$) Suppose $f : S \to T$ is continuous on S and let $p \in S$. Also, suppose that p is the limit point of a sequence, $\{x_i\}$, in S.

If V is an open neighbourhood of $f(p)$, then there exists an open neighbourhood U of p such that $f[U] \subseteq V$.

Then U contains some tail end, say $\{x_i : i > N\}$, of the sequence $\{x_i\}$. Then $\{f(x_i) : i > N\} \subseteq f[U] \subseteq V$. So V contains a tail end of the sequence $\{f(x_i)\}$. Since V is an arbitrary open neighbourhood of $f(p)$, then $\{f(x_i)\} \to f(p)$. This establishes the proof of ($\Rightarrow$).

($\Leftarrow$) Suppose $f : S \to T$ is such that $\{x_i\} \to p$ in S implies $\{f(x_i)\} \to f(p)$.

We are required to prove that f is continuous at p. By hypothesis, there exists, $\{U_i : i \in \mathbb{N}\}$, a countable open neighbourhood base of p, where $U_{i+1} \subseteq U_i$. Then $\cap\{U_i : i \in \mathbb{N}\} = \{p\}$.

We will proceed by contradiction. That is, suppose f is not continuous at p. Then there exists an open neighbourhood, V, of $f(p)$ such that, for each i, $f(x_i) \backslash V \neq \varnothing$. Then, for each j, there exists $y_j \in U_j$ such that $f(y_j) \notin V$. Then $\{y_j\} \to p$; but the tail end of the sequence, $\{f(y_j)\}$, is excluded from V and so can't converge to $f(p)$, a contradiction. So f is continuous at p.

The above theorem allows us to say that, in a first countable space S, if F is a subset of S, then the closure, $\mathrm{cl}_S F$, of F is the set of all the limit points of F. Also, in first countable spaces, to say that a function, $f : S \to T$, is continuous on S is to say that f is "sequentially continuous" at each point p in S.

11.3 Subsequences of a sequence.

Sequences in a first countable topological space don't always converge to a point. But certain infinite subsets of such a sequence might still converge. These infinite subsets have a formal definition.

Definition 11.4 Let S be a first countable topological space. Let

$$A = \{f(i) : i = 1, 2, 3, \ldots, \} = \{x_i\}$$

be a sequence in S. Suppose $g : \mathbb{N}\backslash\{0\} \to \mathbb{N}\backslash\{0\}$ is a *strictly increasing function*. Then the sequence,

$$B = \{f(g(i)) : i = 1, 2, 3, \ldots, \} = \{x_{g(i)}\}$$

is a called a *subsequence of the sequence* A. By strictly increasing we mean $i > j \Rightarrow g(i) > g(j)$.

Note that the notation, $x_{g(i)}$, means that $x_{g(i)}$ is the element of the sequence $\{x_i\}$ with the index number $g(i)$. The terms in a subsequence *always* respect the order in which they appear in the sequence.

Example 1. If $A = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots, \}$, where $a_n = 1/n$, and $g(n) = 2n$, then $B = \{a_{g(n)}\} = \{a_{2n}\}$ is the subsequence

$$B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2n}, \ldots, \right\}$$

Every element of a subsequence must always be an element of the sequence from which it is derived. The key is that the index function must be strictly increasing. Also, remember that a sequence always has infinitely many terms (but not necessarily have infinitely many distinct elements). In this example, the sequence $\{\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \ldots, \}$ is not a subsequence of the sequence A since the indexing function used is not strictly increasing.

Example 2. Let $A = \{1, 1, 2, 1, 3, 1, 4, 1, \ldots, \}$. Then $B = \{1, 2, 3, 4, 5, \ldots, \}$ and $C = \{1, 1, 1, 1, 1, 1, 1, \ldots, \}$ are both subsequences of A. The subsequence B does not converge and does not have a subsequence which converges while subsequence C converges to 1. So the element, 1, is, by definition an accumulation point of the sequence A. Note that, if we view the sequence A as simply a set then 1 is not a "cluster point" of the set A while even though 1 is an accumulation point of A, since, in this case C is seen as a singleton set, $\{1\}$, not an infinite set. It will be understood in this textbook that "cluster point" refers to a point in relation to a set. While accumulation point is used in relation to sequences (or later, to nets). Some authors use cluster point or accumulation point in interchangeably, which may cause some confusion in some theorem proofs.

Theorem 11.5 Let $A = \{f(i) : i \in \mathbb{N}\} = \{x_i\}$ be a sequence in a *first countable* topological space S. Then p is an accumulation point of A if and only if it is a limit point of some subsequence, $\{f(g(i)) : i \in \mathbb{N}\} = \{x_{g(i)}\}$, of A.

Proof : Let $A = \{f(i) : i \in \mathbb{N}\} = \{x_i\}$ be a sequence in a *first countable* topological space S.

($\Leftarrow$) Suppose A has a subsequence, T, which converges to the point p. Then every open neighbourhood of p contains a tail end of T, and so intersects a tail end of A. So p is an accumulation of A.

($\Rightarrow$) Suppose p is an accumulation point of A.

Since S is first countable, p has a countable open neighbourhood base, $\mathscr{B}_p = \{B_i : i = 1, 2, 3, \ldots\}$ such that $B_{i+1} \subseteq B_i$ for all i. If $T_j = \{i : i \geq j\}$ is a tail end of $\mathbb{N}$, for all i and j, $f[T_j] \cap B_i \neq \varnothing$. Let $g(1)$ be the smallest number in T_1, such that $f(g(1)) \in B_1$. Suppose that, for $g(1) < g(2) < \cdots < g(j)$, $f(g(i)) \in B_i$. Let $g(j+1)$ be the smallest number in $T_{g(j+1)}$ such that $f(g(j+1)) \in B_{j+1}$. Then the subsequence $\{f(g(i)) : i = 1, 2, 3, \ldots, \}$ converges to p.

We will later see that, if S is not first countable, the statement above will fail to be true, in general. So we will have to redefine our notions about "sequence".

Concepts review:

1. If S is a first countable topological space what does it mean to say that $\{x_i\}$ is a sequence in S?
2. If S is a first countable topological space what does it mean to say that p is limit point of $\{x_i\}$, a sequence in S?
3. If S is a first countable topological space what does it mean to say that p is a limit of a subset, U, in S?
4. If S is a first countable topological space and the subset, U of S, contains all its limit points state a topological property possessed by U.
5. If S is a first countable topological space and F is a closed subset of S what can we say about the set of all limit points of F?
6. If S is a first countable topological space define "the closure of F" in terms of limit points.
7. Define a subsequence of a sequence in a first countable topological space.
8. Define an accumulation point of a sequence in a first countable topological space.
9. If p is and accumulation point of a sequence is p a limit point of the sequence? Explain.

EXERCISES

1. Let (S, τ_S) be a first countable topological space. Define the "interior of the subset F" in terms of limit points.

---

12 / Limit points of nets.

Summary. *In this section, we will expand the definition of "limit point" so that it applies to topological spaces, in general. The indexing set of all natural numbers used for sequences will be substituted by a more general indexing set called "directed set". The sets which are ordered by a directed set will be called "nets". We will finally prove that nets serve the purpose we want them to have. That is, they will serve as a tool to better recognize closed subsets and continuous functions in arbitrary topological spaces. They will eventually be used to identify sets which are "compact".*

12.1 Directed sets.

Suppose "$\leq$" is a *partial order relation* on a set, S. That is,

- for all $x \in S$, $x \leq x$, (reflexive property)
- if $x \leq y$ and $y \leq x$ then $x = y$, (anti-symmetric property)
- if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitive property)

The second condition (referred to as the "anti-symmetric property") is not absolutely required in what follows, but it is easier to set up the stage in this way to introduce the notion of a "directed set" whose definition requires some sort of predefined order relation on a given set.

Definition 12.1 Let D be a non-empty infinite set on which is defined an order relation "$\leq$" satisfying the property,

If $x, y \in D$, there exists $z \in D$ such that $x \leq z$ and $y \leq z$ (The directed set property)

A *directed set* is the pair $(D, \leq)$.

We already know of a directed set, namely, $(\mathbb{N}, \leq)$. Simply see that, if $m, n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $m \leq k$ and $n \leq k$. So a directed set is simply a generalization of the linearly ordered countable sets we normally use as indexing sets for sequences. As we can see, the definition of directed set does not require that its order relation, "$\leq$", be linear. It is usually some form of a "partial ordering".

Example 1. Given a set, S, its power set, $(\mathscr{P}(S), \subseteq)$, is partially ordered by the inclusion relation. This order relation can be seen to satisfy the directed set property.

Then $(\mathscr{P}(S), \subseteq)$ is a directed set.

Example 2. Consider the set of all finite partitions, $\mathscr{U}$, of the closed interval $[0, 10]$ of real numbers. For example, $M = \{[0, 4] \cup [4, 7] \cup [7, 10]\} \in \mathscr{U}$. We define a partial order on $\mathscr{U}$ as follows: $M \leq K$ if and only if K is a finer partition then M. That is, each closed interval of M is a union of closed intervals in K. For example,

$$M = \{[0, 4] \cup [4, 7] \cup [7, 10]\} \leq \{[0, 2] \cup [2, 4] \cup [4, 7] \cup [7, 9] \cup [9, 10]\} = K$$

It is easy to verify that "$\leq$" respects the directed set property on $\mathscr{U}$. For example, suppose $M = \{[0, 5] \cup [5, 10]\}$ and $J = \{[0, 2] \cup [2, 10]\}$. If

$$K = \{[0, 2] \cup [2, 5] \cup [5, 7] \cup [7, 10] \cup [9, 10]\}$$

then $M \leq K$ and $J \leq K$. Then $(\mathscr{U}, \leq)$ is a directed set.

We will now show how directed sets can be used to establish a direction in arbitrary subsets of a topological space. The definition of a net, in terms of a function, is similar to the definition of a sequence except that the index set is not necessarily linear and is usually larger than $\mathbb{N}$.

Definition 12.2 Let S be a set.

a) A *net in* S is a function, $f : D \to S$, mapping a directed set, D, into S. It is usually denoted as $\{f(i)\}_{i \in D} = \{x_i\}_{i \in D}$, or, simply, by $\{x_i\}$ whenever it is obvious that the index set is the directed set, D.

b) If $\{f(i)\}_{i \in D}$ is a net in S and $T_k = \{i \in D : i \geq k\}$, we will refer to a subset of the form, $f[T_k] = \{f(i) : i \geq k\}$ as a *tail end* of the net, $\{f(i)\}$.

c) Suppose (S, τ) is a topological space and $A = \{x_i\}$ is a subset ordered by a directed set, D. Then A is a net in S. The net, A, is said to *converge to a point* p in S (with respect to τ and the given ordering) if and only if every open neighbourhood, U of p, contains some tail end of the net, $\{x_i\}$. The expression, $\{x_i\} \to p$, is used to indicate that this net converges to p.

d) If a net, $\{x_i\}$ in a topological space, S, converges to the point p, we say that p is a *limit point of the net* $\{x_i\}$.

e) If F is a non-empty set in a topological space, S, we say that *p is a limit point of the set, F*, if and only if p is the limit point of some net, $\{x_i\}$, which is entirely contained in F.

f) Suppose (S, τ) is a topological space. We will say that y is an *accumulation point of the net,* $A = \{f(i)\}_{i \in D}$ in S, or that A *accumulates at* y, if for *every* tail end $f[T_k] = \{f(i) : i \geq k\}$, of A and *every* open neighbourhood B_y of y,

$$f[T_k] \cap B_y \neq \varnothing$$

Equivalently, each tail end, $f[T_k]$, is cofinally in every neighbourhood B_y of y.

Since $(\mathbb{N}, \leq)$ is a directed set, then a sequence, $\{x_i : i \in \mathbb{N}\}$, is a particular kind of net.

Example 3. Consider the net (sequence),

$$A = \{n + (-1)^n n : n \in \mathbb{N} \backslash \{0\}\}$$

with directed set $\mathbb{N} \backslash \{0\}$ in $\mathbb{R}$ (with the usual topology). This net does not have a limit point, since a small open neighbourhood of any p cannot contain a tail end of A. However, any neighbourhood of the number, 0, intersects every tail end of the net; so 0 is an accumulation point.[2]

It is important to remember that there can be no limit points or convergence of a net on a set without a stated topology on that set, since the notion of convergence is defined with a particular topology in mind. Given a net, $A = \{f(i)\}_{i \in D}$, in a topological space, if we change the topology on the set, then we me may be altering its convergence properties.

However, the directed set, $(D, \leq)$, which indexes the net elements is *not* topologized.

12.2 Subnet of a net

We know that sequences can have "subsequences". Can nets can have "subnets"? They do. We formally define this concept in such a way that it generalizes the concept of a subsequence.

Definition 12.3 Let S be a set and $A = \{f(i) : i \in D\}$ be a net in S.

a) Suppose U is a subset of S such that $U \cap A \neq \varnothing$. We say that U is *cofinal in the net* A if, for every $j \in D$, $U \cap f[T_j] \neq \varnothing$, where $T_j = \{i : i \geq j\}$.

[2] In some texts, authors use the word "cluster of a net" instead of "accumulation point of a net". This may lead to confusion since the word cluster is already defined "cluster point of a set" (see page 47). Note that the net A in this example, when viewed as a set, has no cluster points but does have an accumulation point.

b) Let $\mathscr{D}$ be a directed set (possibly different from D). Suppose $g : \mathscr{D} \to D$ is an *increasing* function on $\mathscr{D}$ so that $g[\mathscr{D}]$ is cofinal in D, hence $(f \circ g)[\mathscr{D}]$ is cofinal in A. (By "increasing" we mean, $i \geq j \Rightarrow g(i) \geq g(j)$.)

Then the net,

$$B = \{f(g(i)) : i \in \mathscr{D}\} = \{x_{g(i)} : i \in \mathscr{D}\}$$

is a called a *subnet of the net* A.

A few remarks on the notion of a subnet. The definition above confirms that given a sequence, $A = \{f(i) : i \in \mathbb{N}\}$, and $g : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, then g is an increasing function and so $B = \{f(g(i)) : i \in \mathbb{N}\}$ satisfies the definition of "subnet". By definition, a sequence is a net. The above definition generalizes the well-known notion of subsequence. Given a sequence A, a careful reading of the definition of a "subnet" B for A, leads us to realize that B may not necessarily be a subsequence of A. In the usual definition of a *subsequence* we require that the function g be "strictly" increasing. The definition of "subnet" requires that g be only increasing. We verify how this may make a difference. If $f(n) = 1/n$, so that the sequence, $A = \{f(n)\} = \{1, 1/2, 1/3, 1/4, \ldots, \}$, and $\{g(n)\} = \{1, 1, 2, 2, 3, 3, \ldots, \}$, then we end up with a subnet $B = \{f(g(n))\} = \{1, 1, 1/2, 1/2, 1/3, 1/3, \ldots\}$, a subset of A, but clearly not a subsequence of A (at least not as we have learned it). So B *is a subnet of* A*, but* B *does not satisfy the definition of a subsequence of* A.

There is another point we should emphasize. The definition states that "$\ldots g$ maps a directed set $\mathscr{D}$ into the directed set D". So the the indexing set of the subnet, $B = \{f(g(i)) : i \in \mathscr{D}\}$, may differ from the indexing set, D, of A. The set, $\mathscr{D}$, may even have a different cardinality. A non-trivial example will follow the next important theorem

Given our experience with subsequences, we expect that, if a net has an accumulation point x, the net will have a subnet which will converge to it. We confirm that this holds true in the next theorem. It is well worth the effort required to read through it carefully.

Theorem 12.4 Let S be a topological space which contains a net, $A = \{f(i) : i \in D\}$. The net A has an accumulation point, u, if and only if A has a subnet converging to u.

Proof: Let S be a topological space which contains a net $A = \{f(i) : i \in D\}$.

($\Leftarrow$) Suppose $A = \{f(i) : i \in D\} = \{x_i\}$ has a subnet $B = \{f(g(i)) : i \in \mathscr{D}\} = \{x_{g(i)}\}$ which converges to u (where $\mathscr{D}$ is a directed set). We must show that u is an accumulation point of the net, A. Let B_u be any open neighbourhood of u. Then B_u contains a tail end,

$$f[g[T_r]] = \{f(g(i)) : i \geq r \in \mathscr{D}\}$$

of B. Let

$$f[T_k] = \{f(i) : i \geq k\}$$

be a some tail end of A. Then there exist q such $g(q) > \max\{k, g(r)\}$. Then $f(g(q)) \in f[T_k] \cap f[g[T_r]] \subseteq B_u$. So every tail end of A intersects every open neighbourhood of u. So, by definition, u is an accumulation point of A.

($\Rightarrow$) Suppose u is an accumulation point of A. We are required to find a subnet of A which converges to u.

For every $j \in D$, let $T_j = \{i \in D : i \geq j\}$. The set, $\{T_j : j \in D\}$, represents all tail ends of D. Let $\mathscr{B}_u$ be an open neighbourhood base of u in S.

By definition of accumulation point of A, if $U_u \in \mathscr{B}_u$, U_u is cofinal in A; that is, $U_u \cap f[T_j]$ is non-empty, for all $j \in D$.

Defining the set $\mathscr{D}$. For $U_u \in \mathscr{B}_u$, let

$$\mathscr{D}_{U_u} = f^{\leftarrow}[U_u] \times \{U_u\} \subseteq D \times \{U_u\}$$

Then let $\mathscr{D} = \cup\{\mathscr{D}_{U_u} : U_u \in \mathscr{B}_u\}$.

Directing the elements of $\mathscr{D}$. We will order the elements of $\mathscr{D}$ with, "$\leq$", as follows:

$$(i,\, V_u) \leq (j,\, U_u) \;\Leftrightarrow\; [\, i \leq j \text{ and } f(j) \in U_u \subseteq V_u\,]$$

thus establishing a direction on the set, $\mathscr{D}$. We now have the direction set, $(\mathscr{D}, \leq)$.

Constructing the subnet B. Define $g : \mathscr{D} \to D$ as, $g(j, U_u) = j$. It is easily seen that g is an increasing function on $\mathscr{D}$ and $g[\mathscr{D}]$, is cofinal in D (as it respects the order established by D). So, $(f \circ g) : \mathscr{D} \to A$ is cofinal in A.

Then

$$B = \{f(g(i, U_u)) : (i, U_u) \in \mathscr{D}\}$$

satisfies the definition of a subnet of A.

Claim that B converges to u. Let $U_u \in \mathscr{B}_u$. We are required to find a tail end of B which is entirely contained in U_u.

We can choose $f(j) \in U_u \cap f[T_j] \neq \varnothing$. Then $(j, U_u) \in f^{\leftarrow}[U_u] \times \{U_u\} \subseteq \mathscr{D}$.

See that, for any $(k, V_u) \geq (j, U_u)$, $k \geq j$ and $f(k) \in V_u \subseteq U_u \cap f[T_k]$. So $(f \circ g)(k, V_u) = f(k) \in U_u$. So the tail end of all elements in B passed the point $(f \circ g)(k, V_u))$ belongs to U_u. So the subnet B converges to u, as claimed.

Then the subnet B converges to the accumulation point, u, as required.

Remark . In the above theorem, the net A may be either a sequence or an uncountable net. The cardinality of A does not play a role in the proof, so the statement holds true for countable and uncountable cases.

Example 4. Recall that we can divide the class of all ordinals into two types of ordinals: Those ordinals, α, which have an immediate predecessor, say β, where $\beta + 1 = \alpha$ and limit ordinals, γ, where $\gamma = \sup\{\alpha : \alpha < \gamma\}$ or, equivalently, γ is an ordinal which does not contain a maximal ordinal. Examples of the smaller countable limit ordinals are:

$$\begin{aligned}
\omega_0 &= \{0, 1, 2, 3, \ldots\} \\
2\omega_0 &= \{0, 1, 2, 3, \ldots, \omega_0, \omega_0 + 1, \omega_0 + 2, \ldots, \omega_0 + n, \ldots, \} = [0, 2\omega_0) \\
3\omega_0 &= \{0, 1, 2, \ldots, \omega_0, \omega_0 + 1, \omega_0 + 2, \ldots, 2\omega_0, 2\omega_0 + 1, 2\omega_0 + 2, \ldots, \} = [0, 3\omega_0) \\
&\vdots \\
n\omega_0 &= \{0, 1, 2, \ldots, \omega_0, \omega_0 + 1, \omega_0 + 2, \ldots, (n-1)\omega_0, (n-1)\omega_0 + 1, \ldots, \} = [0, n\omega_0) \\
&\vdots \\
\omega_0\omega_0 &= [0, \omega_0\omega_0)
\end{aligned}$$

The first few limit ordinals are $\{\omega_0, 2\omega_0, 3\omega_0, \ldots, n\omega_0 \ldots, \omega_0\omega_0\}$, all of which are countably infinite.

We can use $D = [0, \omega_0\omega_0)$ as an index set for a net.
Define $f : [0, \omega_0\omega_0) \to \mathbb{Q} \cap [0, \infty)$ as follows, where $n, m \in \mathbb{N}$, $m \geq 1$:

$$f(i) = \begin{cases} = 0 & \text{if } i = n \\ = n + \frac{1}{m} & \text{if } i = m\omega_0 + n \end{cases}$$

The set. $\mathbb{Q} \cap [0, \infty)$, is equipped with the subspace topology inherited from the usual topology on $\mathbb{R}$. The set, $D = [0, \omega_0\omega_0)$, is not topologized.

Then $A = \{f(i) : i \in [0, \omega_0\omega_0)\}$ is a net with the directed set $[0, \omega_0\omega_0)$.

If we fix $k \geq 1$, we see that the terms of the subnet,

$$f[\{k\omega_0 + n : n \geq 0\}] = \{n + \frac{1}{k} : n = 0, 1, 2, 3, \ldots, \} = \{\ldots, n + \frac{1}{k}, (n+1) + \frac{1}{k}, \ldots, \}$$

increase with no upper bound, so the net cannot have a limit point. However, if we fix n we see that as $m \in \mathbb{N}$ increases, the terms of the subnet,

$$f[\{m\omega_0 + n : m \geq 0\}] = \{n + \frac{1}{m} : m = 1, 2, 3, \ldots, \} = \{\ldots, n + \frac{1}{m}, n + \frac{1}{m+1}, \ldots, \}$$

decrease to n. So each $n \in \mathbb{N}$ is an accumulation point of the the net

$$A = \{f(i) : i \in [0, \omega_0\omega_0)\}$$

Study the following example carefully. It is a good illustration on why, in certain topological spaces, the theory governing sequences and their subsequences is inadequate.

Example 5. Let $S = \{(a, b) : a, b \in [0, 1]\}$. We will equip S with the usual topology inherited from $\mathbb{R}^2$. Let

$$D = \{(c, d) : c, d \in (0, 1), \text{ both } c \text{ and } d \text{ are irrationals}\}$$

lexicographically ordered, untopologized. By "lexicographically ordered" we mean that $(e, k) \leq (c, d)$ if $e < c$, and, when $e = c$, then $k \leq d$.[2] Then, with this ordering, $(D, \leq)$ will serve as a directed set for a net in S. The net,

$$T = \{f(c, d) : (c, d) \in D\}$$

in the space, S, is defined as follows: $f(c, d) = (c, d)$. Then T linear orders the irrationals (c, d) in the space S, where $(0, 0) < (c, d) < (1, 1)$.

a) Is the point, $p = (1, 1)$ in S, is an accumulation point of the net, T, with respect to the usual topology?
b) Is the point, $p = (1, 1)$ in S, a limit point of the net, T, with respect to the usual topology?
c) How should we modify S so that p is a limit point of the net T.

Solution :

a) We claim that $p = (1, 1)$ is an accumulation point of the net, T. By the theorem above it suffices to show that T has a subnet converging to (1, 1). Let $g : T \to T$ be defined as: $g(a, b) = (a, a)$ if $a \geq b$ and $g(a, b) = (b, b)$ if $a < b$. We see that g is an increasing function with respect to the lexicographic ordering of the irrational pairs. So $f \circ g : T \to \{(a, a) : a \text{ irrational }\}$ defines a subnet of T which is easily seen to converge to $(1, 1)$.
b) The point $p = (1, 1)$ in S is not a limit point of the net T since if $(a, b) \in B_\varepsilon(1, 1)$, then $(a + \frac{1-a}{2}, b - 2\varepsilon) > (a, b)$ and does not belong to $B_\varepsilon(1, 1)$. The point p is a limit point of S, if and only if B contains some complete tail end of the net, S.
c) If $S = (0, 1] \times (0, 1)$ then $\{(1, \alpha) : 0 < \alpha < 1\}$ would contain a tail end of T converging to $(1, 1)$.

[2]This means, we order the ordered pairs by using the same principle as the one used to order words in the dictionary.

Example 6. Consider the product space of ordinal spaces, $S = [0, \omega_1) \times [0, \omega_1)$, where ω_1 is the first uncountable ordinal. The set,

$$\mathscr{B} = \{(\alpha, \omega_1] \times (\beta, \omega_1] : \alpha < \omega_1, \beta < \omega_1\}$$

forms a neighbourhood base of open sets for the point $p \in S$. Show that $p = (\omega_1, \omega_1)$ is a limit point of the set S.

Solution : For $U, V \in \mathscr{B}$ we define $U \leq V$ if and only if $V \subseteq U$. Then "$\leq$", thus defined on $\mathscr{B}$, is a partial ordering that satisfies the directed set property, and so is a directed set, $(\mathscr{B}, \leq)$.

Let $f : \mathscr{B} \to S$ be a choice function which maps each $U \in \mathscr{B}$ to one point of our choice, say $f(U) = x_U \in U$. We thus obtain, by definition, a net

$$\{f(U) : U \in \mathscr{B}\} = \{x_U : U \in \mathscr{B}\}$$

an uncountable subset of the product space, $S = [0, \omega_1) \times [0, \omega_1)$, with directed set $\mathscr{B}$.

We claim that, $p = (\omega_1, \omega_1)$ is a limit point of the set, S. It suffices to show that, $\{x_U\} \to p$. Let $K = (\alpha, \omega_1] \times (\beta, \omega_1]$ be an arbitrary open neighbourhood of p in S. There exists γ, μ such that $V = (\gamma, \omega_1] \times (\mu, \omega_1] \subset K$. Then $V \geq K$ and so $x_V \geq x_K$. That is, a tail end, $f[T_K] = \{x_U : U \geq K\}$ is contained in the open neighbourhood, K of p. Hence the tail end, $f[T_K]$, of the net, $\{x_U : U \in \mathscr{B}\}$, is a subset of the open neighbourhood, K, of p. Hence $\{x_U\} \to p$.

Since S contains a net which converges to p then p is a limit point of S, as claimed.

Example 7. We will be referring to the directed set, $(\mathscr{U}, \leq)$, of all finite partitions of the closed interval $[0, 10]$ introduced in example on page 194. Given two partitions P and Q in $\mathscr{U}$, $P \leq Q$ if Q is a finer partition than P. That is, each closed interval in P is a union of closed intervals in Q.

Let $f : [0, 1] \to \mathbb{R}$ be any function on $[0, 1]$. Let $R_L : \mathscr{U} \to \mathbb{R}$ and $R_U : \mathscr{U} \to \mathbb{R}$ be two functions where R_L maps a partition, P, of $[0, 10]$ to the value of the *lower Riemann sum of* f, over the partition P, and R_U maps the partition P to the value of the *upper Riemann sum of* f, over the partition P. That is,

$$\begin{aligned} R_L(P) &= \int_{L_P} f(x)\, dx = x_P \\ R_U(P) &= \int_{U_P} f(x)\, dx = y_P \end{aligned}$$

The set $\mathscr{U}$ is a directed set. The functions $R_L(P) : \mathscr{U} \to \mathbb{R}$ and $R_U(P) : \mathscr{U} \to \mathbb{R}$ are two functions used to define the two nets

$$\begin{aligned} \{R_L(P) : P \in \mathscr{U}\} &= \{x_P : P \in \mathscr{U}\} \\ \{R_U(P) : P \in \mathscr{U}\} &= \{y_P : P \in \mathscr{U}\} \end{aligned}$$

in $\mathbb{R}$. The theory shows that whenever both of the nets converge in $\mathbb{R}$

$$\begin{array}{ccc} \{x_P\} & \to & k \\ \{y_P\} & \to & m \end{array}$$

then their limits k and m are equal and

$$\int_0^{10} f(x)\,dx = k$$

Example 8. Let the space, $\mathbb{R}^2$, be equipped with the radial topology τ_r. Recall that $(\mathbb{R}^2, \tau_r)$ is called the *radial plane.* This topology is described in example 2 on page 71. In example 11 on page 77, we show that the radial plane is not first countable. So convergence in $\mathbb{R}^2$ with respect to τ_r, is best done in terms of nets rather than sequences. Consider the set

$$S = \mathbb{R}^2 \backslash (0,0)$$

an uncountable set equipped with the subspace topology inherited from τ_r. In what follows, (r, θ), represents the radial coordinates of a point in $\mathbb{R}^2$. Show that there is a net which which converges to $p = (0,0)$.

Solution : Let $D = \{(r, \theta) : r > 0,\ \theta > 0\}$ be an uncountably infinite set linearly ordered as follows:

$$\begin{cases} (r_1, \theta_1) & > & (r_2, \theta_2) & \text{if} & r_1 < r_2 \\ (r_1, \theta_1) & = & (r_2, \theta_2) & \text{if} & r_1 = r_2 \text{ and } \theta_2 = \theta_1 + 2n\pi \\ (r_1, \theta_1) & < & (r_2, \theta_2) & \text{if} & r_1 = r_2 \text{ and } \theta_2 + 2n\pi > \theta_1 + 2n\pi \end{cases}$$

Then D is a directed set. Suppose $f : D \to S$ is defined as $f(r, \theta) = (\frac{r}{\theta}, \theta)$. Then

$$A = \{f(r,\theta) : (r,\theta) \in D\} = \{(\tfrac{r}{\theta}, \theta) : (r, \theta) \in D\}$$

describes an uncountable net in S.

We claim that A accumulates at $p = (0,0)$. Suppose B_p is an open neighbourhood of p. Let

$$f[T_q] = \{f(r,\theta) : (r,\theta) \geq (r_q, \theta_q)\} = \{(\tfrac{r}{\theta}, \theta) : (r,\theta) \geq (r_q, \theta_q)\}$$

be a tail end of A. Let $k = (r_k, \theta_k) \in B_p$. For a suitable m, define $\theta_s = \theta_k + 2m\pi$ and $r_s < \frac{r_k}{2\theta_s}$. Then $f(r_s, \theta_s) = (\frac{r_s}{2\theta_s}, \theta_s) \in B_p \cap A$. That is, the tail end $f[T_q]$ intersects B_p. So the net A accumulates at $p = (0,0)$, as claimed.

By theorem 12.4, A has a subnet which converges to p, as required.

12.3 Closed sets and continuity in terms of nets.

We now present the results which describe the closure of a set and the continuity of a function in terms net convergence.

Theorem 12.5 Let (S, τ_S) and (T, τ_T) both be topological spaces.

a) Let B be a subset of S. A point p belongs to $\text{cl}_S B$ if and only if there is a net in B which converges to p.

b) A function, $f : S \to T$, mapping S into T is continuous at p in S if and only if whenever p is limit point of a net, $\{x_i\}$, in S then $f(p)$ is the limit point of the corresponding net, $\{f(x_i)\}$ in T.

Proof: Let (S, τ_S) and (T, τ_T) both be topological spaces.

a) ($\Rightarrow$) Suppose p belongs to $\text{cl}_S B$. Then for each open neighbourhood, U, of p we can choose x_U in $U \cap B$. We proceed as in example 3 to construct a net, $\{x_U : U \text{ is an open neighbourhood of } p\}$, which converges to p.
($\Leftarrow$) Suppose $\{x_i\}$ is a net in B which converges to p.
Then each neighbourhood of p intersects B. Then p cannot belong to the open set $S \backslash \text{cl}_S B$. So $p \in \text{cl}_S B$.

b) Left as an exercise for the reader.

The above result shows that, given a subset, F, of a topological space, S, we can use nets as a tool to determine the closure, $\text{cl}_S F$, of the set F. We choose the sequences or nets which makes the task the easiest or is the most interesting to the writer or the reader.

12.4 Convergence of functions

We will now consider the notion of “convergence to a limit point” for the particular case of a set, S, whose elements are functions. Since convergence to a limit point depends on the topology defined on the set, our first step is to decide which topology on S is the most suitable for our purposes.

Consider, for example, the set,

$$S = \mathbb{R}^{[0,1]}$$

representing the family of all functions, $f : [0,1] \to \mathbb{R}$, mapping $[0,1]$ into $\mathbb{R}$ (where the functions are not necessarily continuous). Our inspiration has, as source, the standard topology on Cartesian products, $\prod_{i\in I} S_i$, which can be described as being the set of all functions mapping the index set, I, into $\cup_{i\in I} S_i$. We can try applying the product topology to $\prod_{i\in[0,1]} \mathbb{R}_i$, where each factor, $\mathbb{R}_i$ is $\mathbb{R}$. An element of the product space $\prod_{i\in[0,1]} \mathbb{R}$ can be viewed as an element of $\mathbb{R}^{[0,1]}$.

We examine what the product topology on $S = \mathbb{R}^{[0,1]} = \prod_{i\in[0,1]} \mathbb{R}$ looks like.

Suppose $f = \{y_i : i \in [0,1]\} = \{f(i) : i \in [0,1]\}$ is an element in S. Then, for $f \in S$ and $j \in [0,1]$, $\pi_j(f) = f(j)$ and, for $k \in \mathbb{R}$, $\pi_j^{\leftarrow}(k) = \{f \in S : f(j) = k\}$.

With the product topology on S, a basic open neighbourhood, B_f, of f is of the form

$$B_f = \cap\{\pi_i^{\leftarrow}[U_i] : i \in F_{\text{finite}} \subseteq [0,1],\ U_i \subseteq \mathbb{R}\}$$

where F is a finite subset of $[0,1]$ and U_i is an open neighbourhood of y_i in $\mathbb{R}$. The product topology declares that $\pi_i[B] = \mathbb{R}$ for all $i \notin F$ and $y_i \in \pi_i[B] = U_i$, for all $i \in F$. If $g = \{x_i : i \in [0,1]\} \in B_f$ (where $g(i) = x_i$) this means that $x_i \in U_i$ for each $i \in F$ and x_i is anything else in $\mathbb{R}$ for $i \notin F$. This means the element g is "close" to f if $g(i)$ and $f(i)$ both belong to U_i for $i \in F_{\text{finite}}$.

The basic open neighbourhood, B_f, of $f = \{y_i\}$ establishes restrictions on convergence of a net only on finitely many index elements at time, not on the whole set $\{y_i : i \in [0,1]\}$ simultaneously.

So a sequence or a net of functions, $\{g_\alpha : \alpha \in J\}$, will converge to a limit point, $f \in S = \mathbb{R}^{[0,1]} = \prod_{i\in[0,1]} \mathbb{R}$ provided $\{g_\alpha(i)\} \to f(i)$ at each point $i \in [0,1]$. We will refer to this type of function convergence as "point-wise convergence".

On the other hand, if we equip $S = \mathbb{R}^{[0,1]} = \prod_{i\in[0,1]} \mathbb{R}$ with the (stronger) box topology an open neighbourhood of $f = \{y_i : i \in [0,1]\} = \{f(i) : i \in [0,1]\}$ is of the form

$$B_f = \cap\{\pi_i^{\leftarrow}[U_i] : i \in [0,1],\ U_i \subseteq \mathbb{R}\}$$

where U_i is an open neighbourhood of y_i in $\mathbb{R}$. In this case, $g = \{x_i : i \in [0,1]\}$ belongs to B_f provided $x_i = g(i) \in U_i$ for each $i \in [0,1]$.

When equipped with the box topology, convergence in $S = \mathbb{R}^{[0,1]} = \prod_{i\in[0,1]} \mathbb{R}$ is referred to as "uniform convergence".

Notice that a topology on $[0,1]$ is not relevant in the choice of topology on S. We formally define these in a more general context, $S = T^A$.

Definition 12.6 Let A be a set and T be a topological space. Let $S = T^A$ represent the set of all functions which map A into the topological space, T.

Suppose S is expressed as $S = \prod_{i \in A} T$ and is equipped with the product topology.
In this case a basic open neighbourhood of $f \in S$, is of the form

$$B_f = \cap\{\pi_i^{\leftarrow}[U_i] : i \in F_{\text{finite}} \subseteq A,\ U_i \subseteq T\}$$

Then convergence of a sequence or net, $\{g_\alpha : \alpha \in J\}$ to f will occur if $\{g_\alpha(i)\}$ converges to $f(i)$ at finitely many values of $i \in A$ at a time. Convergence in $S = T^A$, equipped with the product topology, is called *point-wise convergence*.
Suppose $S = T^A$ is equipped with the box topology.
In this case a basic open neighbourhood of $f \in S$, is of the form

$$B_f = \cap\{\pi_i^{\leftarrow}[U_i] : i \in A,\ U_i \subseteq T\}$$

In this case, g is considered to be near f if $g(i) \in B_f$ for all $i \in A$. Then convergence of a sequence or net, $\{g_\alpha : \alpha \in J\}$ to f will occur if $\{g_\alpha(i)\}$ converges to $f(i)$ for all values of $i \in A$ simultaneously. In this case, we say that this sequence or net *converges to f uniformly*.

Example 9. Let $S = [0,1]^{[0,1]}$ be the topological space of all functions mapping $[0,1]$ into $[0,1]$ equipped with the product topology (that is, the topology in which convergence is pointwise). Let $H = \{f \in S : f[0,1] \subseteq \{0,1\}\}$.

a) Construct an uncountable net, A, in the subset, H of S, which accumulates at the zero function, $z = 0(x)$, but which does not converge to z with respect to the product topology.

b) Let $H^* = \{f \in H : |f^{\leftarrow}(0)| \in \mathbb{N}\backslash\{0\}\}$. Show that no sequence in H^* can converge to z.

c) Let $\mathbb{J}$ denote the set of all irrationals and $V = \{f \in H : \mathbb{J} \cap [0,1] \subseteq f^{\leftarrow}(0)\}$. Construct a sequence in V which accumulates at $z = 0(x)$. Verify whether your sequence converges to z.

Solution : We are given that $S = [0,1]^{[0,1]}$ is equipped with the product topology and $H = \{f \in S : f[0,1] \subseteq \{0,1\}\}$.

a) If $f \in H$, we define the function, card : $H \to \mathbb{N}$, as card(f) = "cardinality of $f^{\leftarrow}(0)$". Let

$$D = \{f \in H :\ \text{card}(f) \in \mathbb{N}\}$$

Since $[0,1]$ is uncountable, the set of all finite subsets of $[0,1]$ is uncountable, so D is uncountable. We will define "$\leq$" on D as follows:

$$\begin{array}{lll} f \leq p & \text{if and only if} & \text{card}(f) \leq \text{card}(p) \\ f = p & \text{if and only if} & \text{card}(p) = \text{card}(f) \end{array}$$

If $s, t \in D$ are such that and $\text{card}(s) = m < k = \text{card}(t)$ then there exists $r \in D$ such that $\text{card}(r) = k + 1$, so $r > s$ and $r > t$. So D is a directed set. A tail end in D is as follows: If $g \in D$, $T_g = \{f : \text{card}(g) \leq \text{card}(f) < \aleph_0\}$.

Let $h : D \to H$ be defined as $h(f) = f$ and $A = \{h(f) : f \in D\}$ be a net in H.

Claim 1: That the net A accumulates at $0(x)$. Consider an arbitrary basic open neighbourhood B_F of $0(x)$. Then, there is $F \subset [0, 1]$, where $|F| = k$,

$$B_F = \prod_{i\in[0,1]} A_i$$

where, for $i \in F$, $A_i = \{0\}$, $A_i = \{0, 1\}$, otherwise.
Let $g \in A$ such that $\text{card}(g) = m$. Then

$$h[T_g] = \{f \in A : m \leq \text{ card}(f) < \aleph_0\}$$

is a tail end of A.

There exists $h \in A$ such that $\text{card}(h) > \max\{m, k\}$, such that $h(i) = 0$ for $i \in F$. Then $h \in B_F$. So $h[T_g] \cap B_F \neq \varnothing$. So $0(x)$ is an accumulation point of the net A, as required for claim 1.

Claim 2: That the net A does *not* converge to $0(x)$. Consider an arbitrary basic open neighbourhood B_F of $0(x)$. Then, there is $F \subset [0, 1]$, where $|F| = k$,

$$B_F = \prod_{i\in[0,1]} A_i$$

where, for $i \in F$, $A_i = \{0\}$, $A_i = \{0, 1\}$, otherwise.

It suffices to show that B_F does not contain a tail end of A. If $g \in B_F$, let $h[T_g] = \{f \in A : f \geq g\}$ be a tail end in A. Then for $i \in F$, $g(i) = 0$. Suppose $F^* \subset [0, 1]$ such that $|F^*| = k + 1$ and $F \backslash F^* \neq \varnothing$. Suppose $h \in A$ such that, for $i \in F^*$, $h(i) = 0$ for $i \in F^*$ and $h(i) = 1$ otherwise. Then, since $\text{card}(h) > \text{card}(g)$, $h > g$, so $h \in h[T_g]$. But since $h(i) = 1$ for some $i \in F \backslash F^*$ then $h \notin B_F$. So B_F cannot contain a tail end of A. So the net, A in H, does not converge to $0(x)$, as required.

b) Let $A = \{f_k : k \in \mathbb{N}\}$ be a sequence in H^*. For each $k \in \mathbb{N}$, let $B_k = f_k^{\leftarrow}(0)$. Since each B_k is a finite subset of $[0, 1]$ then $U = \cup\{B_k : k \in \mathbb{N}\}$ is a countable proper subset of $[0, 1]$. Let F be a finite non-empty subset of $[0, 1]$ such that $F \cap U = \varnothing$. Let B_F be a basic open neighbourhood of $0(x)$ where the i^{th} factor is $\{0\}$ if and only if $i \in F$. Then $f_k(i) = 1$ for all $i \in F$ and $k \in \mathbb{N}$. Such f_k's cannot belong to the basic open neighbourhood, B_F, of $0(x)$. So no tail end of A is contained is contained in B_F. So the sequence A cannot converge to $0(x)$.

c) Given: That $\mathbb{J}$ represents the set of all irrationals and

$$V = \{f \in H : \mathbb{J} \cap [0, 1] \subset f^{\leftarrow}(0)\}$$

Let $N = \{1, 2, 3, \ldots, \}$. For $f \in V$, let card(f) = "cardinality of $f^{\leftarrow}(0) \cap \mathbb{Q}$". We define

$$V_m = \{f \in V : \text{ card}(f) = m\}$$

Since $\mathbb{Q}$ is countable, there are countably many subsets of cardinality m in $\mathbb{Q} \cap [0, 1]$. Then V_m is countable for each m. Then the elements of V_m can be indexed by N as,

$$V_m = \{f(m, i) : i \in N\}$$

where $f(m, i)$ is the i^{th} element in V_m. Furthermore,

$$V = \cup\{V_m : m \in N\} = \{f(m, i) : (m, i) \in N \times N\}$$

and so V is a countable set of functions in $H = \{f \in S : f[0, 1] \subseteq \{0, 1\}\}$.

We will now index the elements of V. Let $D = \{(m, i) : (m, i) \in N \times N\}$.

We define ">" on D as:

$$(m, i) > (r, j) \;\Rightarrow\; \left\{ \begin{array}{c} \text{If } m + i > r + j. \\ \text{\scriptsize In the case where} \\ m + i = r + j, \quad m > r \end{array} \right\}$$

Equality, $(m, i) = (r, j)$, holds if and only if $(m, i) = (r, j)$.

Then every element of D has an immediate predecessor and so D can be viewed as a copy of $\mathbb{N}$. Every element of V is accounted for in $\{f(m, i) : (m, i) \in D\}$ so when ordered in this way, it is essentially a sequence.[1]

We claim that V, when ordered in this way, accumulates at $z = 0(x)$. Consider an arbitrary basic open neighbourhood B_F of $0(x)$. Then, there is $F \subset [0, 1]$, where $|F| = k$, and

$$B_F = \textstyle\prod_{i \in [0,1]} A_i$$

where, for $i \in F$, $A_i = \{0\}$, $A_i = \{0, 1\}$, otherwise.

Consider the tail end, $f[T_{(r,j)}] = \{f(m, i) : f(m, i) \geq f(r, j)\}$, in V. We can choose, d, t such that $d + t > r + j$ and such that $f(d, t)^{\leftarrow}(0)$ contains F. So $f(d, t) \in B_F$. This means the tail end, $f[T_{(r,j)}]$, intersects B_F. So the sequence, V, accumulates at z.

We claim that V does not converge to z. We can choose t such that $1 + t > r + j$ and so $f(1, t)$ belongs to the tail end, $f[T_{(r,j)}]$. Since F is finite, the number t can be chosen such that $f(1, t)^{\leftarrow}(0) \notin F$. So $f(1, t)$ is not in B_F. So V does not converge to z.

[1] For example, $(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), (2, 3), (3, 2), (4, 1), (1, 5), (2, 4), (3, 3), (4, 2), (5, 1) \ldots,$

Concepts review:

1. Define a directed set.
2. Are the natural numbers a directed set?
3. Show that the set of all finite partitions of a closed interval of real numbers is a directed set.
4. Define a net of elements in a set.
5. Define a tail end of a net in a set.
6. What does it mean to say that a net converges to a point p in a topological space?
7. What is a limit point of a net.
8. What does it mean to say that a set is cofinal in a net?
9. What does it mean to say that a set B is a subnet of a net A?
10. If B is a subnet of the sequence A is B necessarily a subsequence of A?
11. Is it true that a point p belongs to the closure of a set B if and only if B contains a net which converges to p?
12. Define an accumulation point of a net.
13. If the sequence A has an accumulation p is there necessarily a subnet which converges to p?
14. If the sequence A has an accumulation p is there necessarily a subsequence which converges to p?
15. Complete the statement in terms of nets: A function $f : S \to T$ is continuous at p if and only if...
16. Complete the statement in terms of subnets: The point u is an accumulation point of the net A if and only if...
17. Describe two topologies used on sets of function T^A and describe the convergence properties with respect to these particular topologies.

EXERCISES

1. Show that the subnet of a subnet of a net A is also a subnet of A.
2. Show that a net, $\{x_i : i \in D\}$, in a product space, $S = \prod_{\alpha\in A} S_\alpha$, converges to p if and only if, for any $\mu \in A$, the net $\{\pi_\mu(x_i) : i \in D\}$ converges to $\pi_\mu(p)$ in S_μ.
3. Let $\mathbb{J}$ denote the set of all irrationals in $\mathbb{R}$ and let $\vec{p} = (\pi, \pi)$. If $S = \mathbb{J}^2 \backslash \{\vec{p}\}$ we will define a partial ordering, "$\leq$", on S as:

$$\vec{a} \leq \vec{b} \text{ whenever } ||\,\vec{b} - \vec{p}\,|| \leq ||\,\vec{a} - \vec{p}\,||$$

   where "$||\ ||$" refers to the distance between points in $\mathbb{J}^2$.

   a) Verify that the pair, $(\,S, \leq)$, forms a directed set in S.
   b) Suppose $f : \mathbb{J}^2 \to \mathbb{R}$ is a real-valued function on $\mathbb{J}^2$. Then, when f is restricted to S, the set, $A = \{\,f(\vec{x}),\ \vec{x} \in S\,\}$, defines a net in $\mathbb{R}$. Verify that the net, A, converges to a limit L if and only if $\lim_{\vec{x}\to\vec{p}} f(\vec{x}) = L$ (in the usual sense).

13 / Limit points of filters.

Summary. *In this section we will introduce a form of convergence by sets rather than the convergence by points of a set whose elements are linearly, or directionally, ordered. These families of sets are called filters. The construction of a filter is modeled closely on the structure of a neighbourhood base at a point. We will formally define a "filter of sets" and explain how to determine whether it has a limit point or not. We will also define an "accumulation point" which belongs to the closure of all sets in a filter. Finally, we will introduce "ultra-filters" and a few of their characterizations; these will help us distinguish them from regular filters.*

13.1 Filters and some of their properties.

Recall that a net is essentially a set whose elements are indexed by a linearly or partially ordered set. The ordering is a tool which, given the properties of the topological space the set lives in, points the elements of the set in a particular direction towards a place where we may find, or not find, a "limit point". Now we wonder if we can bypass the construction of the directed set and simply describe "convergence" (or lack of convergence) in terms of the properties of the neighbourhood base itself. This type of convergence is much more topological in nature since convergence depends strictly on how a family of sets behaves, given the properties of the space it belongs to. Essentially, neighbourhood bases are what inspired the following definitions of "filter" and its "filter base".

Definition 13.1 Let S be a set. Let $\mathscr{F} \subseteq \mathscr{P}(S)$. If $\mathscr{F}$ satisfies the two properties:

1) $\varnothing \notin \mathscr{F}$,

2) For every $U, V \in \mathscr{F}$, there exists $F \in \mathscr{F}$ such that $F \subseteq U \cap V$.

then we will say that $\mathscr{F}$ is a *filter base* of sets.

If $\mathscr{F}$ is a filter base which also satisfies the condition:

3) Whenever $U \in \mathscr{F}$ and there is $V \in \mathscr{P}(S) \backslash \{\varnothing\}$ such that $U \subseteq V$, then $V \in \mathscr{F}$.

Then $\mathscr{F}$ is called a *filter of sets*.

A quick way of describing a filter of sets is to say: "A filter is a subset of $\mathscr{P}(S)\backslash\{\varnothing\}$ which is closed under finite intersections and supersets."[1] Note that the notion of a filter is a set-theoretic concept, not a topological one. Once we want to discuss "convergence" we will be in the context of a filter in some topological space.

A filter is, itself, a filter base, while a filter base, if not a filter, can be completed to become a filter. Given a filter base, $\mathscr{F}$, we define the larger set, $\mathscr{F}^*$, as follows:

$$\mathscr{F}^* = \{U \in \mathscr{P}(S) : F \subseteq U, \text{ for some } F \in \mathscr{F}\}$$

We have simply adjoined to $\mathscr{F}$ all its supersets. It is easily verified that $\mathscr{F}^*$ is a filter. We will say that the filter base, $\mathscr{F}$, *generates the filter* $\mathscr{F}^*$.

Example 1. For a topological space, S, and a point $x \in S$, a neighbourhood base of open sets, $\mathscr{B}_x$, is clearly a filter base of sets in S since it satisfies the filter base property,

> " For any two open neighbourhoods U and V of $\mathscr{B}_x$ there exists $W \in \mathscr{B}_x$ such that $W \subseteq U \cap V$".

In the following proposition we show that a filter which is generated by a filter base is the intersection of all filters which contain the filter base in question.

Proposition 13.2 If $\mathscr{F}^*$ is a filter generated by the filter base, $\mathscr{F}$ then,

$$\mathscr{F}^* = \cap\{\mathscr{H} : \mathscr{H} \text{ is a filter and } \mathscr{F} \subseteq \mathscr{H}\}$$

Proof : The proof is left as an exercise to the reader.

Definition 13.3 Let S be a topological space and $\mathscr{F}$ be a filter base of sets in $\mathscr{P}(S)$.

a) If $\cap\{\text{cl}_S F : F \in \mathscr{F}\} \neq \varnothing$ then we will say that $\mathscr{F}$ is *fixed*.

b) Otherwise, we will say that $\mathscr{F}$ is *free*.[2]

[1] It is worth noting that the set, $\mathscr{P}(S)\backslash\{\varnothing\}$, is not a filter. Why?

[2] This definition is not universal in the literature. Some authors define "$\mathscr{F}$ is fixed if $\cap\{F : F \in \mathscr{F}\} \neq \varnothing$ and free otherwise".

Example 2. Let S be a topological space containing the non-empty subset K. Let $\mathscr{F}^*$ be the set of all subsets, U, of S such that $K \subseteq \text{int}_S U$.

a) Show that $\mathscr{F}^*$ is a filter of sets in S.
b) If $K = \{x\}$, describe a filter base, $\mathscr{F}$, of $\mathscr{F}^*$.
c) Is the filter base $\mathscr{F}$ free or fixed?

Solution : We are given $\mathscr{F}^* = \{U \in \mathscr{P}(S) : K \subseteq \text{int}_S U\}$ for a fixed K.

Let $U, V \in \mathscr{F}^*$ and $M \in \mathscr{P}(S)$ such that $U \subseteq M$.

a)
- Since $K \not\subseteq \text{int}_S \varnothing$, then $\varnothing \notin \mathscr{F}^*$.
- If $K \subseteq \text{int}_S U$ and $K \subseteq \text{int}_S V$ then $K \subseteq \text{int}_S U \cap \text{int}_S V = \text{int}_S (U \cap V)$.
  So $\mathscr{F}^*$ is closed under finite intersections.
  So $\mathscr{F}^*$ is a filter base of sets in $\mathscr{P}(S)$.
- $K \subseteq \text{int}_S U \subseteq \text{int}_S M$ implies $M \in \mathscr{F}^*$.
  So $\mathscr{F}^*$ is closed under supersets.
  So $\mathscr{F}^*$ is filter of sets.

b) Suppose $K = \{x\}$.
Then $\mathscr{F}$ is the set, $\mathscr{B}_x$, of all neighbourhoods of x. The set, $\mathscr{F}$, satisfies the property: "If A and B are two neighbourhoods of x in $\mathscr{F}$ then there exists a neighbourhood C in $\mathscr{F}$ such that $x \in C \subseteq A \cap B$". So the filter base, $\mathscr{F} = \mathscr{B}_x$, is a neighbourhood base of x.

c) Since $K \subseteq \text{int}_S U$ for all $U \in \mathscr{F}$ then $K \subseteq \cap\{\text{cl}_S U : U \in \mathscr{F}\}$ so $\mathscr{F}$ is fixed.

13.2 Limit points and accumulation points of filters.

As we have done for sequences and nets, we define the limit point and accumulation point of a filter base and of a filter. In what follows, filters are family of subsets of some topological space.

Definition 13.4 Let $\mathscr{F}$ be a filter base of sets on the topological space S, and $x \in S$. Let $\mathscr{B}_x$ be a neighbourhood base at x.

a) We will say that the filter base, $\mathscr{F}$, *converges to* x if and only if, for every open neighbourhood, B of x, there is some $U \in \mathscr{F}$ such that $U \subseteq B$. This implies that, if $\mathscr{B}_x$ is an open neighbourhood base of x, $\mathscr{B}_x \subseteq \mathscr{F}^*$ (which is the filter generated by

$\mathscr{F}$). So the filter, $\mathscr{F}^*$, is finer than $\mathscr{B}_x$. We abbreviate the "convergence of $\mathscr{F}$ to x" by the expression, $\mathscr{F} \to x$. In this case, we will say that x *is a limit point of the filter base* $\mathscr{F}$.

b) We will say that x is an *accumulation point* for $\mathscr{F}$ if and only if every neighbourhood of x intersects each $F \in \mathscr{F}$. That is, $x \in \mathrm{cl}_S F$ for all $F \in \mathscr{F}$.

c) The set of all accumulation points of $\mathscr{F}$ is called that the *adherence of* $\mathscr{F}$. The adherence of $\mathscr{F}$ is denoted by $a(\mathscr{F})$. That is, $a(\mathscr{F}) = \cap\{\mathrm{cl}_S F : F \in \mathscr{F}\}$.

It is worth verifying that, if $\mathscr{F}$ is a filter base, then $a(\mathscr{F}) = a(\mathscr{F}^*)$. Also note that, by definition, a limit point of a filter base, $\mathscr{F}$, is an accumulation point of $\mathscr{F}$.

Observe that x need not be an element of any of the subsets in a filter base to be one of its limit points. Witness the following example.

Example 3. Consider the set, $\mathscr{F} = \{(4, 4+\delta] : \delta > 0\}$. See that $\mathscr{F}$ doesn't contain $\varnothing$ and satisfies the filter base property and so qualifies as a filter base. But if we take any open neighbourhood of 4, say $B = (4-\varepsilon, 4+\mu)$ there exists, say $U = (4, 4+\mu/4)$ in $\mathscr{B}$, such that $U \subseteq B$. So 4 is a limit point of the filter base $\mathscr{F}$. However, if we consider the filter, $\mathscr{F}^*$, generated by this filter base, the set, $B = (4-\varepsilon, 4+\mu)$, will belong to $\mathscr{F}^*$.

Furthermore, distinct filter bases can converge to the same point. Witness such a case in the following example.

Example 4. Consider the sets, $\mathscr{F}_1 = \{(4, 4+\delta] : \delta > 0\}$ and $\mathscr{F}_2 = \{[4-\delta, 4) : \delta > 0\}$. See that both $\mathscr{F}_1$ and $\mathscr{F}_2$ are filter bases which converge to the limit point 4.

By definition, any filter base that has an accumulation point is fixed. Filter bases can have multiple accumulation points.

Example 5. Consider the set, $A = \{2\} \cup (3, 4]$ in $\mathbb{R}$, equipped with the usual topology. Let $\mathscr{F} = \{U \in \mathscr{P}(\mathbb{R}) : A \subseteq U\}$. Every point, $x \in A$, belongs to every $F \in \mathscr{F}$. So every point in A is an accumulation point of $\mathscr{F}$. But also, every open neighbourhood of 3 will intersect every set which contains A, so 3 is also an accumulation point. The filter base, $\mathscr{F}$, has no limit point since any open interval of a point in A with radius less than 1 cannot be contained in a set which contains all of A.

However, verify that if, $\mathscr{F}_1 = \{U \in \mathscr{P}(\mathbb{R}) : 2 \in U\}$, is a fixed filter where $\mathscr{F} \subseteq \mathscr{F}_1$.

13.3 Filters derived from nets.

Suppose we are given a net in a topological space S. We will show how a filter can be constructed from this net. Eventually, we will show they have the same limit points.

Theorem 13.5 Let S be a set and D be a directed set. Suppose the function, $f : D \to S$, defines a net, $A = \{f(i) : i \in D\}$, in S. If $u \in D$, let $T_u = \{i \in D : i \geq u\}$, a tail end of D. Then,

$$f[T_u] = \{f(i) : i \geq u \text{ in } D\}$$

represents a tail end of the net, A, starting with $f(u)$. Let

$$\mathscr{F}_f = \{f[T_u] : u \in D\}$$

be the set of all tail ends of the net, A. Then $\mathscr{F}_f$ is a filter base of sets in S.

Proof: We are given $\mathscr{F}_f = \{f[T_u] : u \in D\}$ where $A = \{f(i) : i \in D\}$ is a net in the set S. First note that $\varnothing \notin \mathscr{F}_f$. We claim that $\mathscr{F}_f$ is closed under finite intersection: If $j, k \in D$, then there exist, $h \in D$, such that $j \leq h$ and $k \leq h$. Then $T_h \subseteq T_j \cap T_k$ and so, $f[T_h] \subseteq f[T_j] \cap f[T_k]$. So $\mathscr{F}_f$ is closed under finite intersections, as claimed. So the set, $\mathscr{F}_f$, of all tail ends of the net, A, is a filter base of sets in S.

Definition 13.6 Let S be a set and $A = \{f(i) : i \in D\}$ be a net in S. Let $T_u = \{i \in D : i \geq u\}$. The filter base, $\mathscr{F}_f = \{f[T_u] : u \in D\}$, formed of tail ends, of the net, A, in S, is referred to as the *filter base in S which is determined by the net A*.

13.4 A net derived from a filter.

Having seen how we can construct a filter from a net, we now show how a net can be constructed given any filter.

Definition 13.7 Let S be a set and $\mathscr{F}$ be a filter of sets. We define the relation, $\leq$, on the set,

$$D_{\mathscr{F}} = \{(u, F) \in S \times \mathscr{F} : u \in F\}$$

as follows: $(s, F) \leq (u, H)$ if and only if $u \in H \subseteq F$. Then $(D_{\mathscr{F}}, \leq)$ forms a directed set.

With this directed set in hand, and the function, $f : D_{\mathscr{F}} \to S$, defined as, $f(u, F) = u$, we define the net

$$A_{\mathscr{F}} = \{f(u, F) : (u, F) \in D_{\mathscr{F}}\}$$

in the set, S.

The net $A_{\mathscr{F}}$ is called a *net generated by the filter* $\mathscr{F}$.

13.5 Limit points of corresponding filters and nets.

We will now confirm that the limit point or accumulation point of a net will always be a limit point or accumulation point of the filter corresponding to this net, and vice-versa.

Theorem 13.8 Let S be a topological space. Suppose $\mathscr{F}$ is a filter base of sets in S and $A = \{h(i) : i \in D\}$ is a net.

a) Suppose the net, A, is a net generated by $\mathscr{F}^*$. The point, u, in S is a limit point (accumulation point) of $\mathscr{F}$ if and only if u is a limit point (accumulation point) of A.

b) Suppose the filter base, $\mathscr{F}$, is determined by the net, A. The point, u, in S is a limit point (accumulation point) of $\mathscr{F}$ if and only if u is a limit point (accumulation point) of A.

Proof: We are given that S is a topological space.

a) We are given that $\mathscr{F}$ is a filter base and the net, $A = \{h(u, F) : (u, F) \in D_{\mathscr{F}}\}$, is one which is generated by filter $\mathscr{F}^*$.

($\Rightarrow$) *Case: Limit point.* Suppose u is limit point of the filter base $\mathscr{F}$. We are required to show that u is a limit point of the net A. Then u is a limit point of $\mathscr{F}^*$. Let U be an open neighbourhood of u. To show that A converges to u we must show that U contains a tail end of A.

There exists $F \in \mathscr{F}$ such that $F \subseteq U$. Then U belongs to the filter, $\mathscr{F}^*$. We can choose some point $v \in U$, so that the pair, (v, U) belongs to $D_{\mathscr{F}}$.

We claim that the tail end, $h[T_{(v,U)}]$ in A, is contained in U: Let $(x, V) \in D_{\mathscr{F}}$ such that $(v, U) \leq (x, V)$. Then $T_{(x,V)} \in T_{(v,U)}$ and $x \in V \subset U$. Then $h(x, V) = x \in U$. So U contains the tail end of $h[T_{(v,U)}]$, as claimed. So A converges to u.

($\Rightarrow$) *Case: Accumulation point.* Suppose u is an accumulation point of the filter base $\mathscr{F}$ and $A = \{h(u, F) : (u, F) \in D_{\mathscr{F}}\}$, the net generated by $\mathscr{F}$. We are required to show that u is an accumulation point of A. Let U be an open neighbourhood of u. By hypothesis, $U \cap F \neq \varnothing$ for all $F \in \mathscr{F}$ (equivalently, $u \in \mathrm{cl}_S F$ for each $F \in \mathscr{F}$).

To show that u is an accumulation point of the net, A, we must show that U intersects each tail end of A. Let $h[T_{(v,K)}]$ be a tail end in A and $M \in \mathscr{F}$. Then $V = M \cap K$ is an element of $\mathscr{F}$ which is a subset of K. Then, by hypothesis, there exists

$$t \in U \cap V \subseteq V \subseteq K$$

The expression, $t \in V \subseteq K$, translates to, $(v, K) \leq (t, V)$. Hence

$$t = h(t, V) \in h[T_{(v,K)}] \cap U$$

So U intersects the tail-end $h[T_{(v,K)}]$. We conclude that u is an accumulation point of the net corresponding to $\mathscr{F}$.

($\Leftarrow$) *Case: Limit point.* Suppose u is a limit point of the net A. We are required to show that the filter, $\mathscr{F}$, generated by A converges to u. Let U be an open neighbourhood of u. To show that $\mathscr{F}$ converges to u we must show that U contains an element of $\mathscr{F}$.

By hypothesis, U contains a tail end of A. That is, $h[T_{(s,F)}] \subseteq U$ for some (s, F). Then $h(r, F) \in U$ for all $r \in F$. But $h(r, F) = r$. So $F \subseteq U$. Since F belongs to $\mathscr{F}$ then $\mathscr{F}$ converges to u.

($\Leftarrow$) *Case: Accumulation point.* The proof is left as an exercise for the reader.

b) We are given that the filter base, $\mathscr{F}_h$, is determined by the net $A = \{h(i) : i \in D\}$. Recall that $\mathscr{F}_h = \{h[T_u] : i \geq u\}$ where $T_u = \{i \in D : i \geq u\}$. That is, it is the set of the tail ends of the net, $\{h(i) : i \in D\}$.

($\Rightarrow$) *Case: Limit point.* Suppose x is a limit point of the filter base, $\mathscr{F}_h$. We must show that the net A also converges to x. Suppose U is any open neighbourhood of x. Then there is $a \in D$ such that $h[T_a] \subseteq U$. Then every open neighbourhood contains a tail end of the net, $\{h(i) : i \in D\}$. So the net converges to x.

($\Leftarrow$) Proving the converse of the "Case, limit point", is left as an exercise.

($\Rightarrow$) *Case: Accumulation point.* Suppose y is an accumulation point point of the filter base, $\mathscr{F}_h$. We must show that y is an accumulation point of the net A. Let U be any open neighbourhood of y. Then $U \cap h[T_u] \neq \varnothing$ for all $u \in D$. That is, U contains an element of all tail ends of the net, $\{h(i) : i \in D\}$. Then y is an accumulation point of this net.

($\Leftarrow$) Proving the converse of the "Case, accumulation point" is left as an exercise.

13.6 Other properties of filters.

In the following theorem, we see that "uniqueness of limits of filters (nets)" characterizes the Hausdorff property on a topological space.

Theorem 13.9 Let S be a topological space. The space S is Hausdorff if and only if a convergent filter, $\mathscr{F}^*$, has only one limit point.

Proof : We are given that S is a space.

($\Rightarrow$) Suppose S is Hausdorff. Suppose the points x and p are limits of the filter base of sets, $\mathscr{F}$, in S. Then neighbourhood bases, $\mathscr{B}_x$ and $\mathscr{B}_p$, are subsets of the filter, $\mathscr{F}^*$, generated by $\mathscr{F}$. If $x \neq p$, then, by hypothesis, we can choose disjoint open neighbourhoods U_x and V_p which separate p from x. Since $\mathscr{F}^*$ converges to p and x there exists, W and M, in $\mathscr{F}^*$ which are contained in U_x and V_p, respectively. But then $W \cap M = \varnothing$, contradicting the fact that $\mathscr{F}^*$ is a filter. So $x = p$.

($\Leftarrow$) We are given that limits of any filter, $\mathscr{F}^*$, of sets in S are unique. Suppose S is not Hausdorff. Then there is a pair of points, x and p, with open neighbourhood bases, $\mathscr{B}_x$ and $\mathscr{B}_p$ such that, $U \cap V \neq \varnothing$ for all U and V in $\mathscr{B}_x$ and $\mathscr{B}_p$. It easily follows that $\mathscr{F} = \{U \cap V : U \in \mathscr{B}_x \text{ and } V \in \mathscr{B}_p\}$ is a filter base of sets in S.

We claim that $\mathscr{F}^*$ converges to both x and p. Let W_x and D_p be elements of $\mathscr{B}_x$ and $\mathscr{B}_p$, respectively. It suffices to show that there are elements, F_1 and F_2 in $\mathscr{F}^*$ such that $F_1 \subseteq W_x$ and $F_2 \subseteq D_p$, respectively. This would contradict convergence to a single element.

There exists $U_x \in \mathscr{B}_x$ and $V_p \in \mathscr{B}_p$ such that

$$x \in F_1 = U_x \cap V_p \subseteq W_x$$

and $T_p \in \mathscr{B}_p$ and $C_x \in \mathscr{B}_x$ such that,

$$p \in F_2 = T_p \cap C_x \subseteq D_p$$

By definition of $\mathscr{F}^*$ both F_1 and F_2 belong to $\mathscr{F}^*$. By definition of convergence, $\mathscr{F}^*$ converges to both x and p. This contradicts our hypothesis. So S must be Hausdorff.

The statement in the above theorem translates to nets. That is, limit points of nets in a topological space, S, are unique if and only if S is Hausdorff.

In the following theorem we show that, for any accumulation point, p, of a filter, $\mathscr{F}$, it is possible to increase the size of $\mathscr{F}$ to a larger filter which will converge to p. So every accumulation point of a filter is the limit point of a larger filter which contains it.

Theorem 13.10 Suppose S is a topological space and p is an accumulation point for a filter base, $\mathscr{F}$, of sets in S. Then p is the limit point of some filter, $\mathscr{H}^*$, which contains $\mathscr{F}^*$.

Proof: We are given that S is a space and that p is an accumulation point of $\mathscr{F}$.

This means every open neighbourhood of p intersects every element of $\mathscr{F}$. We can then consider the family

$$\mathscr{H} = \{U \cap F : U \in \mathscr{B}_p, F \in \mathscr{F}\}$$

itself a filter base. Let $\mathscr{H}^*$ be the filter generated by $\mathscr{H}$.

We claim that $\mathscr{H}$ converges to p. If V is an element of $\mathscr{B}_p$, for $F \in \mathscr{F}$, $V \cap F$ is non-empty so is an element of $\mathscr{H}$. Since $V \cap F$ is a subset of V, by definition of convergence, $\mathscr{H}$ converges to p, as claimed.

Suppose $F \in \mathscr{F}$. Since $U \cap F \subseteq F$ then $F \in \mathscr{H}^*$ so $\mathscr{F} \subseteq \mathscr{H}^*$. In fact, $\mathscr{F}^* \subseteq \mathscr{H}^*$.

We conclude that, if p is accumulation point of $\mathscr{F}$, then it is the limit point of $\mathscr{H}^*$, a filter which contains $\mathscr{F}^*$.

Theorem 13.11 Suppose S is a topological space. Suppose $\mathscr{F}_1$ and $\mathscr{F}_2$ are two fixed filter bases in S. If $\mathscr{F}_1$ and $\mathscr{F}_2$ have the same adherence set then every element of the filter, $\mathscr{F}_1$, intersects every element of the filter, $\mathscr{F}_2$.

Proof: We are given that $\mathscr{F}_1$ and $\mathscr{F}_2$ are two fixed filter bases in S.

Suppose $a(\mathscr{F}_1) = a(\mathscr{F}_2)$. Then, if $p \in a(\mathscr{F}_1)$, p is an accumulation point of both $\mathscr{F}_1$ and of $\mathscr{F}_2$. There exists a filter, $\mathscr{H}^*$, which contains both $\mathscr{F}_1$ and of $\mathscr{F}_2$ and which converges to p. There are some straightforward details involving theorem 13.10 to work out for proving this; these are left as an exercise. Since $\mathscr{H}^*$ is closed under finite intersections, every element of $\mathscr{F}_1$ intersects every element of $\mathscr{F}_2$.

13.7 Filter bases describe closures of sets and continuity.

Not surprisingly, given our experience with sequences and nets, the closure of a set can be described in terms of the limit points of its filter bases.

Theorem 13.12 Let F be a non-empty subset of a topological space, S. Then a point u belongs to the closure, $\text{cl}_S F$, of F if and only if u is the limit point of some filter base in F.

Proof: We are given that S is a space and F is a non-empty subset of S.

($\Rightarrow$) Suppose $u \in \text{cl}_S F$. Let $\mathscr{B}_u$ be a neighbourhood base of open sets for u. Then

$$\mathscr{F} = \{U \cap F : U \in \mathscr{B}_u\}$$

forms a filter base of sets in F which converges to u.

($\Leftarrow$) Suppose $\mathscr{F}$ is a filter base in F converging to some $u \in S$. Let U be any open neighbourhood of u. Then there exists some $V \in \mathscr{F}$ such that $V \subseteq U$. Since $V \subseteq F$, U intersects F and so $u \in \text{cl}_S F$.

Just as for nets, continuity of a function $f : S \to T$ can be expressed in terms of filters. But first we must determine how a filter in the range of a function corresponds appropriately to a filter in its domain.

The image, $f[\mathscr{F}^*]$, of a filter, $\mathscr{F}^*$. Given a function $f : S \to T$ and given that $\mathscr{F}^*$ is a filter of sets in S we define,

$$f[\mathscr{F}^*] = \{f[F] : F \in \mathscr{F}^*\}$$

We claim $f[\mathscr{F}^*]$ is a filter base in the range of the function, f. Let $f[U]$ and $f[V]$ be two elements of $f[\mathscr{F}^*]$ where U and V are elements of $\mathscr{F}^*$. Then there exists non-empty $W \in \mathscr{F}^*$ such that $W \subseteq U \cap V$. Then $f[W] \subseteq f[U \cap V] \subseteq f[U] \cap f[V] \neq \varnothing$. So,

$$f[\mathscr{F}^*] \text{ is a filter base which generates the filter } f[\mathscr{F}^*]^*$$

Theorem 13.13 Let S and T be two topological spaces. Let $f : S \to T$ be a function and $u \in S$. Let $\mathscr{F}^*$ be a filter. The function f is continuous at u if and only if whenever u is a limit point of the filter, $\mathscr{F}^*$, then $f(u)$ is a limit point of the filter, $f[\mathscr{F}^*]^*$.

Proof : We are given that S and T are two topological spaces and $f : S \to T$ is a function.

($\Rightarrow$) Suppose f is continuous at $u \in S$ and $\mathscr{F}^*$ converges to u. We are required to show that $f[\mathscr{F}^*]^*$ converges to $f(u)$. Suppose V is an open neighbourhood of $f(u)$. It suffices to show that an element of $f[\mathscr{F}^*]^*$ is contained in V. Since f is continuous at u, then there exists an open neighbourhood U of u such that $f[U] \subseteq V$. Since $\mathscr{F}^*$ is a filter then U belongs to the filter $\mathscr{F}^*$. Since $f[U] \in f[\mathscr{F}^*]$ and is contained in V, then $V \in f[\mathscr{F}^*]^*$. So $f[\mathscr{F}^*]^*$ converges to $f(u)$.

($\Leftarrow$) Suppose that whenever $\mathscr{F}^*$ converges to u then $f[\mathscr{F}^*]^*$ converges to $f(u)$. Let $\mathscr{F}^*$ be the neighbourhood filter of u. Since $f[\mathscr{F}^*]^*$ converges to $f(u)$, each neighbourhood V of $f(u)$ contains some element of $f[\mathscr{F}^*]^*$. Then for some neighbourhood U of u, $f[U] \subseteq V$. This shows that f is continuous at u.

We now provide the following familiar example, but solved by referring to filters rather than nets.

Example 6. Let $S = [0,1]^{[0,1]}$, the set of all functions mapping $[0,1]$ into $[0,1]$, be equipped with the product topology. Let $H \subseteq \{0,1\}^{[0,1]} \subseteq S$ be a subspace of S defined as follows:

$$H = \{h \in S : h(x) = 0 \text{ for finitely many values, otherwise } h(x) = 1.\}$$

a) Construct a filter in H, which accumulates at the zero element, z.

b) Describe a filter on H which contains the filter in part a) and which converges to the zero element, z

Solution : Given $S = [0,1]^{[0,1]}$, the set of all functions mapping $[0,1]$ into $[0,1]$, equipped with the product topology and

$$H = \{h \in S : h^{\leftarrow}(0) \text{ is finite, otherwise } h(x) = 1\}$$

a subspace of $\{0,1\}^{[0,1]}$. Let z represent the zero function, $0(x)$.

a) Let $T_n = \{f \in H : n \leq |\, f^{\leftarrow}(0)\,| < \aleph_0\}$. Let

$$\mathscr{F} = \{T_n : n = 1, 2, 3, \ldots, \}$$

where $T_n \subseteq H$ for each n. If $f \in T_m$, $f(i) = 0$ for m or more values of i in $[0, 1]$. So $T_m \subseteq T_{m-1}$. By finite induction, if $n > m$ then $T_n \subseteq T_m$. Hence, if $m > n$ then $T_m \cap T_n = T_m \neq \varnothing$. Since T_n is never empty, $\mathscr{F}$ is then a filter base of sets in S.

Let B_F be a basic open neighbourhood of z in S where $|F| = q$, for some $q \in \mathbb{N}\backslash\{0\}$ and $f(x) = 0$ for the q values of x in $F \subset [0, 1]$.

To show z is an accumulation point of $\mathscr{F}$, we must show that, given $T_k \in \mathscr{F}$, $B_F \cap T_k \neq \varnothing$. If $k > q$, then, for some $f \in T_k$, $f \in B_F \cap T_k \neq \varnothing$. If $k < q$, since $T_q \subseteq T_k$ then, again, for some $f \in T_k$, $f \in B_F \cap T_k \neq \varnothing$. So $B_F \cap T_k \neq \varnothing$. Then z is an accumulation point of the filter base, $\mathscr{F}$.

b) In part a) we showed that, if $T_n = \{f \in S : n \leq |\, f^{\leftarrow}(0)\,| < \aleph_0\}$,

$$\mathscr{F} = \{T_n : n = 1, 2, 3, \ldots, \}$$

is a filter in H which accumulates at z with respect to the product topology.

Let $\mathscr{B}_z$ denote a neighbourhood base of open sets at z.

We can then consider the family,

$$\mathscr{H} = \{B \cap T_n : B \in \mathscr{B}_z, n = 1, 2, 3, \ldots, \}$$

of subsets of S. If $f \in B \cap T_n$, then $n \leq |f^{\leftarrow}(0)| \leq k$ for some number k. So every element of $\mathscr{H}$ is a subset of H.

It is easily verified that $\mathscr{H}$ is itself a filter base. Since $B \cap T_n \subseteq T_n$, then $T_n \in \mathscr{H}^*$, the filter generated by $\mathscr{H}$. So $\mathscr{F} \subseteq \mathscr{H}^*$.

We claim that the filter, $\mathscr{H}^*$, converges to z. Suppose B_F is an element of $\mathscr{B}_z$. If $T_n \in \mathscr{F}$ then, since z is an accumulation point of $\mathscr{F}$, $B_F \cap T_n$ is non-empty and so is an element of $\mathscr{H}$ which is a subset of B_F. So, by definition of convergence, z is a limit point of $\mathscr{H}$. So $\mathscr{F}$ is contained in a filter $\mathscr{H}$ which converges to z.

13.8 Partial ordering the family, $\mathcal{F}$, of all filters in $\mathscr{P}(S)$.

We will now discuss some properties possessed by families of filters of sets. The reader can assume that all filters discussed here are subsets of $\mathscr{P}(S)$ for some predefined topological space. Let

$$\mathcal{F} = \{\mathscr{F} \subseteq \mathscr{P}(S) : \mathscr{F} \text{ is a filter of sets of } S\}$$

denote the family of all filters on a space S. We will define a partial ordering of the elements in $\mathcal{F}$ by inclusion, "$\subseteq$".

Given two filters, $\mathscr{F}$ and $\mathscr{H}$, in $\mathcal{F}$, if $\mathscr{F} \subseteq \mathscr{H}$, we will say that,

"$\mathscr{H}$ is *finer* than $\mathscr{F}$" or "$\mathscr{F}$ is *coarser* than $\mathscr{H}$"

For example, the sets $\mathscr{F}^* = \{(2-1/n, 2+1/n) : n \in \mathbb{N}\backslash\{0\}\}^*$ and $\mathscr{H}^* = \{(2-\varepsilon, 2+\varepsilon) : \varepsilon > 0\}^*$ are easily verified to be two filters both converging to 2. We see that since $\mathscr{F}^* \subseteq \mathscr{H}^*$, then $\mathscr{H}^*$ is finer than $\mathscr{F}^*$.[1]

13.9 Ultrafilters.

We have seen that a filter base can contain an other smaller filter base and often be itself contained by some other larger filter of sets. Filter bases with large adherence sets can be built up so that they eventually will converge to at most one of those points. But, as we shall soon see, at some point, a filter of sets will have attained its maximum size. We often call such filters, "maximal filters". However, the term which is more commonly used is "ultrafilter". We will formally define an "ultrafilter" and prove some of its characterizations. These characterizations will make it easier to recognize an ultrafilter when we see one. We will then discuss some of their properties. Ultrafilters play an important role in certain branches of general topology.

Definition 13.14 Let S be a topological space and $(\mathcal{F}, \subseteq)$ denote the set of all filters in $\mathscr{P}(S)$, partially ordered by inclusion. Suppose $\mathscr{F}$ be an element of $\mathcal{F}$. We will say that the filter $\mathscr{F}$ is an *ultrafilter* if and only if, whenever $\mathscr{H} \in \mathcal{F}$ is such that $\mathscr{F} \subseteq \mathscr{H}$, then $\mathscr{F} = \mathscr{H}$. That is, $\mathscr{F}$ cannot be properly contained in some other filter.

13.10 Ultranets: A close relative of ultrafilters.

We will now formally define what is also known as maximal nets.

Definition 13.15 Let S be a topological space and $A = \{f(i)\}_{i \in D}$ be a net in S. We will say that A is an *ultranet* if, for any subset, B, in S, a tail end of A either belongs to B and, if not, belongs to its complement, $S \backslash B$.

[1] Of course, "$\subseteq$" is a partial ordering of $\mathcal{F}$ since it is not always the case that one filter is a subset of another.

A net which is constant on its tail end, is an example of an ultranet. It is not immediately obvious from this definition whether other types of ultranets actually exists. We will be able to address this question at the end of this section.

Theorem 13.16 Let $A = \{f(i) : i \in D\}$ be an ultranet in a space S. If E is a subnet of A then E is a also an ultranet.

Proof: Let S be a topological space which contains an ultranet, $A = \{f(i) : i \in D\}$.

Suppose E is a subnet of A. Let B be a non-empty subset of S. Then B or $S\setminus B$ contains a tail end of A. Suppose, without loss of generality, that B contains a tail end of A. Then B must contain the tail end of any subnet of A. So E is an ultranet.

The most commonly used characterizations of ultrafilters of sets in $\mathscr{P}(S)$ are stated and proved below. These are extremely useful. It is a good exercise to try to prove them before reading their proofs.

Theorem 13.17 Let S a topological space and $\mathscr{F}$ be a filter of sets in $\mathscr{P}(S)$. The following statements are equivalent.

a) The set $\mathscr{F}$ is an ultrafilter in $\mathscr{P}(S)$.

b) If $U \in \mathscr{P}(S)$ is such that it intersects every element of $\mathscr{F}$, then $U \in \mathscr{F}$.

c) For any $U \in \mathscr{P}(S)$, if $U \notin \mathscr{F}$ then $S\setminus U \in \mathscr{F}$

Proof: We are given that S is a topological space.

($a \Rightarrow b$) We are given that $\mathscr{F}$ is an ultrafilter. Suppose $U \in \mathscr{P}(S)$ is such that $U \cap F \neq \varnothing$ for all $F \in \mathscr{F}$. We are required to show that $U \in \mathscr{F}$. Let

$$\mathscr{H} = \{U \cap F : F \in \mathscr{F}\}$$

The set, $\mathscr{H}$, is easily seen to be a filter base of sets in S. Then the filter, $\mathscr{H}^*$, which is generated by $\mathscr{H}$, contains both U and the set F . Then, $\mathscr{F} \subseteq \mathscr{H}^*$. But, since $\mathscr{F}$ is a maximal filter, then $\mathscr{F} = \mathscr{H}^*$. Then $U \in \mathscr{F}$, as required.

($b \Rightarrow c$) Suppose that, whenever a set U meets every element of $\mathscr{F}$, then $U \in \mathscr{F}$. Suppose $U \notin \mathscr{F}$. It suffices to show that $S\backslash U \in \mathscr{F}$. Then $U \cap F = \varnothing$ for some $F \in \mathscr{F}$. Then $F \subseteq S\backslash U$. If $F_1 \in \mathscr{F}$ and $F_1 \neq F$, then $F_1 \cap F \subseteq S\backslash F$. Since $F_1 \cap F$ is non-empty and is a subset of $S\backslash U$, then $S\backslash U$ intersects every element of $\mathscr{F}$. By hypothesis, $S\backslash U \in \mathscr{F}$.

($c \Rightarrow a$) We are given that, if $U \notin \mathscr{F}$ then $S\backslash U \in \mathscr{F}$. Let $\mathscr{H}^*$ be any filter such that $\mathscr{F} \subseteq \mathscr{H}^*$. If $V \in \mathscr{H}^*$ such that $V \notin \mathscr{F}$ then $S\backslash V \in \mathscr{F}$. This means that $\mathscr{H}^*$ contains both V and its complement. This would imply that $\mathscr{H}^*$ is not a filter. Then there can be no such V. So $\mathscr{H}^* \subseteq \mathscr{F}$. This means that $\mathscr{F}$ is an ultrafilter.

Example 7. Let u be an element of the space S. Let $\mathscr{F} = \{U \in \mathscr{P}(S) : u \in U\}$. We claim that $\mathscr{F}$ is an ultrafilter. To see this, let $U \in \mathscr{P}(S)$. If $u \in U$ then $U \in \mathscr{F}$; if $u \notin U$, then $u \in S\backslash U \in \mathscr{F}$. So, by the above theorem, $\mathscr{F}$ is an ultrafilter.

Does every filter $\mathscr{H}$ have a maximal filter which contains it? We have seen that $(\mathcal{F}, \subseteq)$ is partially ordered by inclusion. We are then wondering whether a maximal element exists in a partially ordered chain of elements in $(\mathcal{F}, \subseteq)$. For many readers this may ring a bell because it is an “existence statement”. It suggests that, to prove it we will have to invoke Zorn's lemma[1]. We remind ourselves what Zorn's lemma formally states:

> “If every chain in a partially ordered set, S, has a maximal element with respect to its partial order, $\leq$, then S has a maximal element.”

Theorem 13.18 Let S be a topological space and $(\mathcal{F}, \subseteq)$, denote the set of all filters in $\mathscr{P}(S)$, partially ordered by inclusion. If $\mathscr{F} \in \mathcal{F}$, then there exists an ultrafilter, $\mathscr{U}$, in $\mathcal{F}$ which contains $\mathscr{F}$.

Proof: We are given that $\mathscr{F} \in \mathcal{F}$. We wish to prove the existence of an ultrafilter which contains $\mathscr{F}$. We set up the problem so that Zorn's lemma applies.

For a given $\mathscr{F} \in \mathcal{F}$, consider the set $\mathbb{S}_{\mathscr{F}} = \{\mathscr{H} \in \mathcal{F} : \mathscr{F} \subseteq \mathscr{H}\}$ of all filters which contain $\mathscr{F}$. We are required to show that some filter in $\mathbb{S}_{\mathscr{F}}$ is an ultrafilter (i.e., $\mathbb{S}_{\mathscr{F}}$ has a maximal element). If $\mathbb{S}_{\mathscr{F}} = \{\mathscr{F}\}$ then $\mathscr{F}$ is an ultrafilter. Suppose $\mathbb{S}_{\mathscr{F}}$ contains other filters. Now $\mathscr{F}$ can be the base element of many chains in $\mathbb{S}_{\mathscr{F}}$. In fact, $\mathbb{S}_{\mathscr{F}}$

[1] Zorn's lemma is proven to be equivalent to the Axiom of choice. A proof appears in R. André, *Axioms and set theory*

can be viewed as the union of a family of filter-chains, $\{\mathscr{C}_\alpha : \alpha \in J\}$, each with base element, $\mathscr{F}$. Pick an arbitrary filter-chain, $\mathscr{C}_\gamma = \{\mathscr{F}_i : i \in I\}$ in $\mathbb{S}_\mathscr{F}$. It looks like,

$$\mathscr{F} \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \mathscr{F}_3 \subseteq \cdots$$

Let $\mathscr{T} = \cup\{\mathscr{F}_i : \mathscr{F}_i \in \mathscr{C}_\gamma\}$. Quite clearly, $\mathscr{F}_i \subseteq \mathscr{T}$ for each i. We claim that the family of sets, $\mathscr{T}$, is itself a filter in the chain $\mathscr{C}_\gamma$. If U and V are elements of $\mathscr{T}$, then U and V are elements of some $\mathscr{F}_j$ in the chain, $\mathscr{C}_\gamma$, so $U \cap V$ is an element of $\mathscr{F}_j$; so $U \cap V$ is also an element of $\mathscr{T}$. Then $\mathscr{T}$ is a filter base of sets which is a maximal element in the chain $\mathscr{C}_\gamma$.

So every chain, $\mathscr{C}_\alpha$, in $\mathbb{S}_\mathscr{F}$ has a maximal element. We can invoke Zorn's lemma, to conclude from this that $\mathbb{S}_\mathscr{F}$, must also have a maximal filter element, $\mathscr{U}$. Since, $\mathbb{S}_\mathscr{F}$ contains every filter which contains $\mathscr{F}$, then $\mathscr{U}$ is the maximal filter which contains $\mathscr{F}$. Then, by definition, $\mathscr{U}$ is the unique ultrafilter which contains $\mathscr{F}$.

Example 8. Show that an ultrafilter of sets in a Hausdorff space can have, at most, one accumulation point.

Solution : If $\mathscr{U}$ is a free ultrafilter then, by definition of free, it has no accumulation point. Suppose u and p are two accumulation points. If u is a limit point of $\mathscr{U}$, then p is a non-limit accumulation point (since, in Hausdorff spaces, the limit of a filter is unique). Then, by theorem 13.10, there exists a filter, $\mathscr{H}^*$, which contains $\mathscr{U}$ and converges to p. Since $\mathscr{U}$ is an ultrafilter this is not possible. So there can be no other accumulation point then u.

13.11 Convergence of filters on product spaces.

Suppose $\{S_i : i \in I\}$ is a family of topological spaces and $S = \prod_{i\in I} S_i$ is a product space.[1] Suppose there is a filter, $\mathscr{F}$, in S which converges to a point, $\{x_i\}$. Then, for each $i \in I$,

$$\pi_i[\mathscr{F}] = \{\pi_i[F] : F \in \mathscr{F}^*\} = \mathscr{F}_i$$

is a filter base in S_i. By continuity of π_i, each filter, $\mathscr{F}_i$. will converge to $\pi_i(\{x_i\}) = x_i$ (by theorem 13.13, page 219). The following theorem shows that the converse holds true.

[1] We are assuming that the product space is non-empty. This is a consequence of Axiom of choice.

Theorem 13.19 Let $\{S_i : i \in I\}$ be a family of topological spaces and $p = \{x_i : i \in I\}$ be a point in the product space, $S = \prod_{i\in I} S_i$. Suppose that $\mathscr{F}^*$ is a filter of sets in S such that, for each $i \in I$, the filter,

$$\pi_i[\mathscr{F}^*] = \{f[F] : F \in \mathscr{F}^*\} = \mathscr{F}_i$$

converges to x_i. Then $\mathscr{F}$ converges to the point, $p = \{x_i : i \in I\}$.[2]

Proof: Given: A filter, $\mathscr{F}^*$, in the product space, $S = \prod_{i\in I} S_i$, and that $p = \{x_i : i \in I\}$ is a point in S. For each i, the filter, $\pi_i[\mathscr{F}^*] = \mathscr{F}_i$, converges to x_i. We are required to show that $\mathscr{F}^*$ converges to p.

Let F be a finite subset of I. For each $i \in F$, let U_i be an open neighbourhood of $x_i \in S_i$. Then

$$B = \cap\{\pi_i^{\leftarrow}[U_i] : i \in F\}$$

forms a *basic* open neighbourhood of $p = \{x_i\} \in S$. To show that $\mathscr{F}^*$ converges to p we must find an element, A of $\mathscr{F}^*$ which is contained in B.

Since $\mathscr{F}_i$ converges to x_i, $U_i \in \mathscr{F}_i$ (since $\mathscr{F}_i$ is a filter), so $\pi_i^{\leftarrow}[U_i] \in \mathscr{F}$. For each $i \in F$, since π_i is continuous there exists an open neighbourhood, A_i, of p, in $\mathscr{F}$ such that $x_i \in \pi_i[A_i] \subseteq U_i$. By the filter base property, there exists $A \in \mathscr{F}$ such that $p \in A \subseteq \cap\{A_i : i \in F\}$. Now, for each $i \in F$,

$$\begin{aligned} \pi_i[A] &\subseteq \pi_i[\cap\{A_i\}] \\ &\subseteq \cap\{\pi_i[A_i]\} \\ &\subseteq \pi_i[A_i] \\ &\subseteq U_i \end{aligned}$$

So we have found $A \in \mathscr{F}$ such that $p \in A \subseteq \cap\{\pi_i^{\leftarrow}[U_i] : i \in F\}$. So $\mathscr{F}$ converges to $p = \{x_i\}$, as required.

Corollary 13.20 Let $\{S_j : j \in J\}$ be a family of topological spaces and $S = \prod_{j\in J} S_j$ be a product space. Suppose that S contains a sequence, $B = \{\beta(i) : i \in \mathbb{N}\}$, and that, for each $j \in J$, $\pi_j[\{\beta(i) : i \in \mathbb{N}\}] = \{\alpha_j(i) : i \in \mathbb{N}\} \subseteq S_j$. Suppose that, for each $j \in J$, the sequence $\{\alpha_j(i) : i \in \mathbb{N}\}$ in S_j converges to $p_j \in S_j$. Then $\{\beta(i) : i \in \mathbb{N}\} \rightarrow \{p_j : j \in I\} \in S$.

[2]This result will be useful to show that the product of compact spaces is compact in the next section.

Proof: We are given the product space, $S = \prod_{j \in J} S_j$, and a sequence, $B = \{\beta(i) : i \in \mathbb{N}\}$ in S.

Let $\mathscr{F}$ be the filter base generated by the sequence B and, for each $j \in J$, $\mathscr{F}_j = \pi_j[\mathscr{F}]$.

For each $j \in J$, let

$$\pi_j[\{\beta(i) : i \in \mathbb{N}\}] = \{\alpha_j(i) : i \in \mathbb{N}\}$$

and suppose that $\{\alpha_j(i) : i \in \mathbb{N}\} \to p_j \in S_j$. Then, by theorem 13.8, the filter, $\mathscr{F}_j$, in S_j generated by the sequence $\{\alpha_j(i) : i \in \mathbb{N}\}$ converges to p_j. By the theorem 13.19, the filter $\mathscr{F}$ converges to $p = \{p_j : j \in J\} \in \prod_{j \in J} S_j$. Then the sequence, $\{\beta(i) : i \in \mathbb{N}\}$, in $\prod_{j \in J} S_j$, converges to $p = \{p_j : j \in J\}$, as required.

13.12 Topic: Existence of non-constant ultranets in S.

Recall, from the definition 13.15, that "ultranets are those nets, $\{x_i\}$, such that, for each $B \in \mathscr{P}(S)$, either B or $S \backslash B$ contains a tail end of $\{x_i\}$".

Example 9. Let S be a topological space. Suppose $A = \{f(i) : i \in D\}$ is a net in S and $\mathscr{F}$ is a filter of sets in S.

a) Show that, if A is an ultranet and $\mathscr{F}$ is the filter generated by A, then $\mathscr{F}$ is an ultrafilter.
b) Show that, if $x \in S$ is the limit point of the ultranet A, then a tail end of A is constant. So a convergent ultranet must have a constant tail end.
c) Show that, if $\mathscr{F}$ is an ultrafilter and A is a net determined by $\mathscr{F}$, then A is an ultranet.
d) Show that a non-constant ultranet exists.

Solution : We are given that S is a topological space and $A = \{f(i) : i \in D\}$ is a net in S. Also, $\mathscr{F}$ is a filter of sets in S.

a) Suppose A is an ultranet and $\mathscr{F} = \{f[T_u] : u \in D\}$, where $T_u = \{i \in D : i \geq u\}$, is the filter generated by the net A. We are required to show that $\mathscr{F}$ is an ultrafilter.

   Let $B \in \mathscr{P}(S) \backslash \varnothing$. By theorem 13.17, either B or $S \backslash B$ belongs to an ultrafilter of sets. Suppose $S \backslash B$ does not belong to $\mathscr{F}$. Then $S \backslash B$ does not contain a tail end, say $f[T_j]$, of A (for, if it did, then $S \backslash B$ would belong to $\mathscr{F}$). Then, by definition of "ultranet", $f[T_j] \subseteq B$. Since $\mathscr{F}$ is a filter, then $B \in \mathscr{F}$. So $\mathscr{F}$ is an ultrafilter.

b) We are given that the ultranet, A, converges to x and that $\mathscr{F} = \{f[T_u] : u \in D\}$ is the ultrafilter generated by A. Then $\mathscr{F}$ must also converge to x.

We are required to show that some tail end of A is constant. Suppose no tail end of A is constant. Then, for $k \in D$, there exists two non-empty non-intersecting cofinal sets, C and E of D, such that $f[D_k] = C \cup E$ and both C and E converge to the net limit point x. Since $\mathscr{F}$ is an ultrafilter, either C or $S\backslash C$ belongs to $\mathscr{F}$. Suppose $C \in \mathscr{F}$. Then $S\backslash C$ does not belong to $\mathscr{F}$. Since $\mathscr{F}$ is an ultrafilter, $S\backslash C$ does not intersect some element F of $\mathscr{F}$. Then, since $S\backslash C$ contains E, the cofinal set, E, does not intersect some $F \in \mathscr{F}$, that is some tail end of the ultranet, A. But this can't be since E is cofinal in A. So A must be constant on a tail end. Since x is a limit point of A then the constant is x.

c) We are given that $\mathscr{F}$ is an ultrafilter and $A = \{f(y, F) : F \in \mathscr{F}$ and $y \in F\}$ is the net determined by $\mathscr{F}$. Recall that $(u, F) \leq (v, H)$ if and only if $v \in H \subseteq F$ and $f(u, F) = u$. We are required to show that A is an ultranet.

Let $B \in \mathscr{P}(S)\setminus\{0\}$. Then $B \in \mathscr{F}$ or $S\setminus B \in \mathscr{F}$. Suppose $s \in B \in \mathscr{F}$. Suppose $(u, H) \in D_{\mathscr{F}}$ such that $(u, H) \geq (s, B)$. Then $u \in H \subseteq B$. Then $f(u, H) = u \in B$. Then the tail end, $\{f(y, F) : (y, F) \geq (s, B)\}$, of A belongs to B. By definition, A is an ultranet.

d) Since convergent ultranets have a constant tail end, we will only investigate non-converging ultranets. We begin with a free ultrafilter. Let $\mathscr{F}$ be a *free* ultrafilter of sets in S. Let $c : \mathscr{F} \to S$ be a choice function defined as $c(F) = x_F$, where $x_F \in F$. Let $A = \{f(x_F, F) : F \in \mathscr{F}\}$ be the free ultranet determined by $\mathscr{F}$. Since A is free it cannot be constant on any tail end, otherwise it would converge to that constant element. So A is a free ultranet with no constant tail end. As required.

Theorem 13.21 Let $A = \{f(i) : i \in D\}$ be a net in a topological space, S. Then A has a subnet U which is an ultranet.

Proof: We are given that $A = \{f(i) : i \in D\}$ is a net in a space, S. Let $\mathscr{F} = \{f[T_u] : u \in D\}$ be the filter generated by the net A, where $T_u = \{i \in D : i \geq u\}$ and $f[T_u] = \{f(i) : i \in T_u\}$. Then there exists and ultrafilter, $\mathscr{U}$, which contains the filter $\mathscr{F}$.

We now construct a net derived from the ultrafilter, $\mathscr{U}$. Let

$$(D_{\mathscr{U}}, \leq) = \{(i, F) \in D \times \mathscr{U} : x_i \in F\}$$

where $(i, F) \leq (j, H)$ if and only if $j \geq i$ and $x_j \in H \subseteq F$. So $D_{\mathscr{U}}$ is a set directed by $\leq$.

We define the function, $g : D_{\mathscr{U}} \to D$, as $g(j, F) = j$ if $x_j \in F$.

Then $(f \circ g)(j, F) = f(j) \in A$ where $i \geq j$ implies $(f \circ g)(i, F) = f(i) \geq f(j)$. So $(f \circ g)$ maps $D_{\mathscr{U}}$ in A, respecting the order, and $A_{\mathscr{U}} = (f \circ g)[D_{\mathscr{U}}]$ is cofinal. We have shown above that the net,

$$A_{\mathscr{U}} = \{(f \circ g)(u, F) : (u, F) \in D_{\mathscr{U}}\}$$

generated by an ultrafilter is an ultranet. So $A_{\mathscr{U}}$ is an ultranet which is a subnet of A.

Concepts review:

1. Define filter base and filter.
2. How does a filter base generate a filter?
3. What does it mean to say that a filter base is fixed? When is it free?
4. If A is a net describe a filter base of sets determined by the net, A.
5. What does it mean to say that a filter base of sets converges to a point x?
6. How do we define an accumulation point of a filter base of sets?
7. Given a filter base, $\mathscr{F}$, how do we define the adherence of $\mathscr{F}$?
8. In what kind of topological space do *convergent* filter bases have precisely one limit point?
9. If a filter base, $\mathscr{F}$, has a non-limit accumulation point, p, describe a filter which contains $\mathscr{F}$ and converges to p.
10. How do we recognize the points in the closure, $\text{cl}_S F$, of a set, F, in terms of filters?
11. Give a characterization of a continuous function, $f : S \to T$, at a point in terms of filters.
12. What does it mean to say that the filter, $\mathscr{F}$, is finer than the filter, $\mathscr{H}$?
13. Define an ultrafilter of sets, in $\mathscr{P}(S)$.

14. Give two characterizations of an ultrafilter of sets.
15. If S is a Hausdorff space, how many accumulation points can an ultrafilter in $\mathscr{P}(S)$ have?
16. Define an ultranet.

EXERCISES

1. Suppose $\mathscr{H}$ is a filter base of sets which is finer than the filter base $\mathscr{F}$. If $\mathscr{F}$ converges to p does $\mathscr{H}$ necessarily converge to p?
2. Suppose p is an accumulation point of the ultrafilter of sets, $\mathscr{U}$. Does $\mathscr{U}$ necessarily converge to p?
3. Let $f : S \to T$ be a function mapping the topological space S onto the topological space T. If $\mathscr{F}$ is an ultrafilter of sets in $\mathscr{P}(S)$ and $f[\mathscr{F}] = \{f[F] : F \in \mathscr{F}\}$ is $f[\mathscr{F}]$ necessarily a filter?
4. If V is a non-empty open subset of the topological space S and the filter of sets, $\mathscr{F}$, converges to some point $p \in V$, does V necessarily belong to $\mathscr{F}$?
5. Suppose U is a non-empty subset of the space S. Suppose U belongs to every filter, $\mathscr{F}$, which converges to a point p of U, is U necessarily open in S?
6. Let u be a point in a topological space S. Let $\mathscr{F} = \{F \in \mathscr{P}(S) : u \in F \text{ and } S\backslash F \text{ is finite}\}$.
   a) Show that $\mathscr{F}$ is a filter of sets in $\mathscr{P}(S)$.
   b) Show that $\mathscr{F}$ is a neighbourhood system of u.
7. Let S be a topological space and $\mathscr{F}$ and $\mathscr{H}$ be two filters in $\mathscr{P}(S)$. Let
$$\mathscr{F} \times \mathscr{H} = \{F \times H : F \in \mathscr{F} \text{ and } H \in \mathscr{H}\}$$
   Is $\mathscr{F} \times \mathscr{H}$ a filter base?
8. Let S be a topological space an $\mathscr{F}$ is a filter of sets in $\mathscr{P}(S)$. Let $a(\mathscr{F})$ be the adherence set of $\mathscr{F}$. Show that $a(\mathscr{F})$ is a closed subset of S.
9. Suppose $\mathscr{U}$ is an ultrafilter in $\mathscr{P}(S)$. If U and V are disjoint non-empty subsets of S such that $U \cup V$ belongs to $\mathscr{U}$ show that either U or V belongs to $\mathscr{U}$.

Part V

Compact spaces and relatives

14 / Compactness: Definition and basic properties.

Summary. *The property of compactness is important in general topology as well as in other fields of study which call upon topological techniques to solve various problems. In this section, we give the most general definition of compactness in terms of covers and subcovers. The few characterizations given for this property will simplify the task of determining when a space is compact. These characterizations also provide a deeper understanding of what it means for a set to be "compact". We also show that the compact property is carried over from one topological space to another by continuous functions. Furthermore, the compact property carries over from the factors of a product space to the product space itself. Subsets of a compact space, S, will be seen to be closed only if S is Hausdorff. But closed subsets of any compact space are always compact.*

14.1 Introduction.

Many readers may already be familiar with the notion of "compactness". It is often encountered early in a course of real analysis, in a form that is somewhat different from the one we will encounter here. Initially, some readers will wonder whether we are talking about the same property. A possible definition would have looked something like this: "A subset, F, of $\mathbb{R}^n$ is compact if and only if every sequence in F has a convergent subsequence with its limit point inside F". Or, in the context of normed vector spaces, the reader may have encountered a theorem which states: "If V is a finite dimensional vector space, the compact subsets of V are precisely the closed and bounded subsets". One of many reasons the notion of compactness would have been introduced in a previous course is to access a statement commonly called the "Extreme value theorem" (EVT) which says: "Given that $f : V \to \mathbb{R}$ is a continuous function mapping a normed vector into $\mathbb{R}$, if F is a compact subset of V, then f attains a maximum value and a minimum value on F". We will be studying precisely the same notion of compactness in the more general context of a topological space. Compact spaces are encountered in numerous other fields of mathematics. In general, most students will feel that spaces which are both compact and Hausdorff are more intuitive and are seen to be "well-behaved" since they come with many useful tools that can be used to solve various problems.

In what follows, we will be referring to an "open cover" of a space, S. We say that $\mathscr{U} = \{U_i : i \in I\}$ is an *open cover of* S if each U_i is an open subset of S and $S = \cup\{U_i : i \in I\}$. A finite subcover, $\{U_i : i \in F\}$, is a finite subset of an open cover, $\mathscr{U}$, of S.

Definition 14.1 Let F be any non-empty subspace of the topological space, S. We say that F is *compact* if and only if every open cover of F has a finite subcover.

Since the reader may have encountered other definitions of "compact" we will refer to the one above as the *topological definition of compactness*. We provide a few examples of compact and non-compact spaces.

Example 1. Note that the cover, $\mathscr{V} = \{(i-2/3, i+2/3) : i \in \mathbb{Z}\}$, forms an open cover of each of the four spaces, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{J}$, (the irrationals) $\mathbb{Z}$ with the usual topology. But removing just one set from $\mathscr{V}$ will leave a point which is not covered by the other sets in $\mathscr{V}$. So $\mathscr{V}$ has no finite subcover. So these four spaces are *not* compact.

Example 2. Consider the set, $S = \{\, 1/n : n = 1, 2, 3, \dots\} \cup \{0\}$ with the subspace topology inherited from $\mathbb{R}$. Let $\mathscr{V}$ be an open cover of S. There exists $U \in \mathscr{V}$ such that $0 \in U$. Then there are at most finitely many points in $S \backslash U$. For each of these points we can choose precisely one set from $\mathscr{V}$ which contains it. The set U along with each one of these sets covers S. So S is compact.

It will be useful to remember the particular technique exhibited in the following example, since it will be called upon to help solve various questions in the next few chapters.

Example 3. Consider the bounded closed interval, $S = [3, 7]$, viewed as a subspace of $\mathbb{R}$. Let $\mathscr{V} = \{V_i : i \in I\}$ be an open covering of S. Let

$$\mathscr{U} = \{u \in [3,7] : \text{ where } [3,u] \text{ is covered by finitely many sets from } \mathscr{V}\}$$

Let $k = \sup\{u : u \in \mathscr{U}\}$. Then $3 \leq k \leq 7$ and $[3,3] \subseteq [3,k] \subseteq [3,7]$.
There exists $V_j \in \mathscr{V}$ such that $k \in V_j$.

Suppose $k < 7$. Then there exists $\varepsilon > 0$ such that $k \in (k-\varepsilon, k+\varepsilon) \subseteq V_j \subseteq [3,7]$. By definition of k, $[3, k-\frac{\varepsilon}{2}]$ has a finite subcover, say $\mathscr{V}_F$. Then $\mathscr{V}_F \cup \{V_j\}$ is a finite subcover of $[3,k]$. Also, $[3, k+\frac{\varepsilon}{2}]$ has a finite subcover, $\mathscr{V}_F \cup \{V_j\}$. This contradicts the definition of k. The source of our contradiction is our supposition that $k < 7$. So $k = 7$. Hence S has a finite subcover so S is compact.

14.2 Characterizations of the compact property.

We will be needing a few efficient ways of determining whether a subspace is compact (or not). The following characterizations will be useful.

But first we should develop some familiarity with the following concept. It is often abbreviated by the acronym, FIP.

> A family, $\mathscr{F}$, of sets is said to satisfy the *finite intersection property* if "every finite subfamily of $\mathscr{F}$ has non-empty intersection".

A "filter of sets" is a prime example of family of sets which satisfies the finite intersection property .

We present the following various ways of recognizing a compact set.

Theorem 14.2 Let S be a topological space. The following are equivalent.

a) The space S is compact.

b) Any family, $\mathscr{F}$, of closed subsets of S which satisfies the finite intersection property has non-empty intersection.

c) Every filter of sets in S has an accumulation point.

d) Every ultrafilter of sets in S has a limit point.

e) Every net has an accumulation point.

f) Every ultranet in S has a limit point.

Proof : We are given that S is a topological space.

(a $\Rightarrow$ b) Suppose the family, $\mathscr{F}$, of closed subsets of S satisfies the finite intersection property. That is, for every finite subset, $\mathscr{W}$, of $\mathscr{F}$, $\cap\{F : F \in \mathscr{W}\} \neq \varnothing$.

We are required to show that $\cap\{F : F \in \mathscr{F}\} \neq \varnothing$.

Suppose $\cap\{F : F \in \mathscr{F}\} = \varnothing$. Then, $\{S \backslash F : F \in \mathscr{F}\}$ is an open cover of S. Then, by hypothesis, there exists a finite subset, $\mathscr{W}$, of $\mathscr{F}$, such that $S = \cup\{S \backslash F : F \in \mathscr{W}\}$. Equivalently, $\cap\{F : F \in \mathscr{W}\} = \varnothing$, contradicting the given fact that $\mathscr{F}$ satisfies the finite intersection property. So, $\cap\{F : F \in \mathscr{F}\} \neq \varnothing$, as required.

(b $\Rightarrow$ a) Suppose that, whenever a family, $\mathscr{F}$, of closed sets satisfies the finite intersection property, then $\cap\{F : F \in \mathscr{F}\} \neq \varnothing$. Let $\mathscr{U}$ be an open cover of S. If, for any finite $\mathscr{W}$ of $\mathscr{U}$, $\cup\{F : F \in \mathscr{W}\} \neq S$, then the set $\{S \backslash F : F \in \mathscr{U}\}$ satisfies the finite intersection property, and so $\cap\{S \backslash F : F \in \mathscr{U}\} \neq \varnothing$. This can only mean that, $\mathscr{U}$ is not an open cover of S, a contradiction. So $\mathscr{U}$ has a finite subcover of S.

(b $\Rightarrow$ c) We are given that if a family of closed sets satisfies the FIP then it has non-empty intersection. Let $\mathscr{F}$ be a filter base of subsets in S. We are required to show that $\mathscr{F}$ has an accumulation point. That is, there is a point x such that $x \in \mathrm{cl}_S F$ for all $F \in \mathscr{F}$. Then, by definition of filter base, $\mathscr{F}$ satisfies the finite intersection property. Then, for any finite subset, $\mathscr{G}$ of $\mathscr{F}$, $\cap\{F : F \in \mathscr{G}\} \neq \varnothing$. Then $\cap\{\mathrm{cl}_S F : F \in \mathscr{G}\} \neq \varnothing$. By hypothesis, there exists $x \in \cap\{\mathrm{cl}_S F : F \in \mathscr{F}\} \neq \varnothing$. Then $\mathscr{F}$ has an accumulation point.

(c $\Rightarrow$ d) Suppose every filter of sets has an accumulation point. Let $\mathscr{U}$ be an ultrafilter of sets in $\mathscr{P}(S)$. We must show that $\mathscr{U}$ has a limit point. By hypothesis, $\mathscr{U}$ must have an accumulation point, say p. By theorem 13.10, p is a limit point of some filter, $\mathscr{H}^*$ containing $\mathscr{U}$. Since $\mathscr{U}$ is a maximal filter, then $\mathscr{U} = \mathscr{H}^*$. So p is a limit point of $\mathscr{U}$.

(d $\Rightarrow$ c) We are given that every ultrafilter has a limit point. We must show that every filter of sets has an accumulation point. Let $\mathscr{F}$ be a filter base. Then, by theorem 13.18, $\mathscr{F}$ is contained in some ultrafilter, $\mathscr{U}$. Let p be the limit point of $\mathscr{U}$. Then, for any open neighbourhood, $U \in \mathscr{B}_p$, of p, there is $F \in \mathscr{U}$ such that $F \subseteq U$. This means every open neighbourhood of p intersects every $F \in \mathscr{U}$. Then $p \in \cap\{\mathrm{cl}_S F : F \in \mathscr{F}\}$. So p is an accumulation pont of $\mathscr{F}$

(c $\Rightarrow$ b) We are given that every filter base in $\mathscr{P}(S)$ has an accumulation point.

Suppose the family, $\mathscr{F}$, of closed subsets of S satisfies the finite intersection property. By hypothesis, $\mathscr{F}$ has an accumulation point, say p.

We are required to show that $\cap\{F : F \in \mathscr{F}\} \neq \varnothing$.

Note that, $\mathscr{F}$, satisfies the main filter base property and so $\mathscr{F}$ a "filter base of closed sets" which has an accumulation point, p. Then, by definition, $p \in \cap\{\mathrm{cl}_S F : F \in \mathscr{F}\} \neq \varnothing$. Since each F is closed in S, $p \in \cap\{F : F \in \mathscr{F}\} \neq \varnothing$, as required.

(f $\Leftrightarrow$ d) By theorem 13.8 and the example on page 226, an ultrafilter has limit point p if and only if its corresponding ultranet has limit p.

(e $\Leftrightarrow$ c) By theorem 13.8 a filter has accumulation point p if and only if its corresponding net has accumulation limit p. By theorem 12.4, if a net has an accumulation point p then it has a subnet which converges to p.

The reader will soon discover that, compact spaces are richer in interesting properties when equipped with the Hausdorff separation axiom. It is often assumed that "compact sets are always closed" since, in many books, to remove clutter from the main body of the text, a blanket assumption is initially made that all spaces are hypothesized to be Hausdorff. But a compact space may not be closed if the space is not Hausdorff. When referring to various texts, readers should keep this in mind, and check at the begin of the text to see if any blanket assumptions are made.

14.3 Properties of compact subsets.

We now verify whether the compactness property is carried over by continuous functions from its domain to its codomain; also, closed subspaces of compact spaces are compact. These results will be familiar to readers with some experience in real analysis. The proofs confirm that these properties generalize nicely, from $\mathbb{R}^n$ to topological spaces.

Theorem 14.3 Let S and T be topological spaces.

a) Suppose $f : S \to T$ is a continuous function mapping S into T. If F is a compact subset of S then $f[F]$ is compact in T. So continuous images of compact sets are compact.

b) Suppose S is compact. If H is a closed subset of S, then H is compact.

c) Suppose S is Hausdorff.
   i) If H is a compact subset of S then H is closed in S.
   ii) If S is also compact then S is regular.
   iii) If S is also compact then S is normal.

Proof: We are given that S is a topological space.

a) We are given that the function, $f : S \to T$, is continuous and that F is a compact subset of S.

   Let $\mathscr{U}$ be an open cover of $f[F]$. Then $\{f^{\leftarrow}[U] \cap F : U \in \mathscr{U}\}$ is an open cover of F. Since F is compact, there is a finite subcover, $\{f^{\leftarrow}[U] \cap F : U \in \mathscr{F}\}$, of open subsets of F. Since,

$$F \subseteq \cup\{f^{\leftarrow}[U] \cap F : U \in \mathscr{F}\} \Rightarrow f[F] \subseteq f\left[\cup\{f^{\leftarrow}[U] \cap F : U \in \mathscr{F}\}\right]$$

   then,

$$\begin{aligned} f[F] &\subseteq \cup\{f\left[\{f^{\leftarrow}[U] \cap F\right] : U \in \mathscr{F}\} \\ &\subseteq \cup\{f[\{f^{\leftarrow}[U]] \cap f[F] : U \in \mathscr{F}\} \\ &= \cup\{U \cap f[F] : U \in \mathscr{F}\} \end{aligned}$$

   So $\{U \cap f[F] : U \in \mathscr{F}\}$ is a finite subcover of $f[F]$. So continuous images of compact sets are compact.

b) Suppose S is a compact space. We are given that H is a closed subset of S. Let $\mathscr{F} = \{F_i : i \in I\}$ be a family of closed subsets in the subspace, H, such that $\mathscr{F}$ satisfies the finite intersection property. Note that, since H is closed, then $\text{cl}_S F_i = F_i$ for all $i \in I$. So that $\mathscr{F}$ is a family of closed subsets of S such that $\mathscr{F}$ satisfies the FIP. Since S is compact, there is a point $p \in \cap\{F : F \in \mathscr{F}\}$. Since $p \in F \subseteq H$, $\cap\{F : F \in \mathscr{F}\}$ is a non-empty subset of H. So H is compact.[1]

---

[1]Note that the Hausdorff property for S is not required for this to hold true. That is, "*Closed subsets of compact spaces are always compact*".

c) For what follows, suppose S is Hausdorff.

i) Let H be a compact subset of S (where S need not be compact). We are required to show that H is closed in S. It will suffice to show that $S \setminus H$ is open in S. Let $u \in S \setminus H$. Since S is Hausdorff, then for each $x \in H$, there exists disjoint open neighbourhoods, U_x and V_u^x, of x and u respectively. Since H is compact, the open cover, $\{U_x : x \in H\}$, will have a finite subcover, say $\{U_{x_i} : i \in F\}$ and finite open neighbourhoods of u, say $\{V_{x_i}^u : i \in F\}$, where $U_{x_i} \cap V_{x_i}^u = \varnothing$, for $i \in F$. Then, for each $i \in F$,

$$U_{x_i} \cap [\cap\{V_{x_i}^u : i \in F\}] = \varnothing$$

Let $W_u = \cup\{U_{x_i} : i \in F\}$ and $M_u = \cap\{V_{x_i}^u : i \in F\}$.
Then $W_u \cap M_u = \varnothing$. So $M_u \cap H = \varnothing$. Then $M_u \cap H = \varnothing$ holds true for any choice of $u \in S \setminus H$. We have then shown that every point, u, in $S \setminus H$ has an open neighbourhood, M_u, which has empty intersection with H. So $S \setminus H$ is open; hence H is closed. (Note how the Hausdorff property plays a role in the proof.)

ii) We are given that S is compact Hausdorff. We are required to show that S is regular. Let H be closed and $u \in S \setminus H$. Then, since S is compact, H is compact. In part ii) we constructed disjoint open neighbourhoods, W_u and M_u, of H and $\{u\}$, respectively. So S is regular.

iii) We are given that S is compact Hausdorff. Let H and K be disjoint closed subsets of S. Then, since S is compact, H and K are compact. Let $u \in K$. We showed in part ii) that S is regular and so there exists disjoint open neighbourhoods, W_u and M_u of H and $\{u\}$, respectively. Then $\{M_u : u \in K\}$ forms an open cover of K which has a finite subcover, say $\{M_{u_i} : i \in F\}$. Then $\cap\{W_{u_i} : i \in F\}$ and $\cap\{M_{u_i} : i \in F\}$ form disjoint open neighbourhoods of H and K. So S is normal.

Some readers sometimes find it useful to "think" of compact subsets of Hausdorff spaces as "points".

Example 4. Let F be a compact subset of a topological space S. Suppose $f : S \to T$ is a continuous one-to-one function mapping F into a Hausdorff space, T. Show that $f[F]$ is a homeomorphic copy of F.

Solution : By theorem 6.9, it will suffice to show that $f : F \to T$ is a closed function on F. Let K be a closed subset of F. Then, by the above theorem, K is compact. Then $f[K]$ is compact in T. Since T is Hausdorff, then $f[F]$ is Hausdorff. Since $f[K]$ is a compact subset of $f[F]$, then it is closed. Then f is a closed mapping on F. So $f : F \to f[F]$ is homeomorphism.

The following important theorem titled *The Tychonoff theorem* will often be referred to in the proofs that will follow. It is surprising how often this theorem is invoked in proofs of topology and real analysis. It shows that the Cartesian product of any number of compact spaces is compact.

Theorem 14.4 *The Tychonoff theorem* . Let $\{S_i : i \in I\}$ be a family of topological spaces. Then the product space, $S = \prod_{i\in I} S_i$, is compact if and only if S_i is compact for each $i \in I$.

Proof: Let $\{S_i : i \in I\}$ be a family of topological spaces.

($\Rightarrow$) We are given that $S = \prod_{i\in I} S_i$ is compact. Then, since the projection map, $\pi_i : S \to S_i$, is continuous for each $i \in I$, and continuous images of compact sets are compact, then $\pi_i[S] = S_i$ is compact, for each $i \in I$.

($\Leftarrow$) We are given that S_i is compact for each $i \in I$. Let $\mathscr{U}$ be an ultrafilter in $S = \prod_{i\in I} S_i$. By theorem 14.2, it will suffice to show that $\mathscr{U}$ has a limit point. Consider the filter base, $\mathscr{U}_i = \pi_i[\mathscr{U}]$ in S_i. Since $\mathscr{U}$ is maximal then $\mathscr{U}_i$ must be maximal for each $i \in I$. Since S_i is compact each $\mathscr{U}_i$ converges to a point x_i. By theorem 13.19, $\mathscr{U}$ converges to $\{x_i\}$. So the product space S is compact.

We will now see that, in Euclidean spaces $\mathbb{R}^n$, the compact subsets are precisely the "closed and bounded" subsets.

Theorem 14.5 Let $\mathbb{R}^n$ be the topological space equipped with the product topology. We will say that a subset, U, is *bounded in* $\mathbb{R}^n$ if there exists $k \in \mathbb{R}^+$ such that $U \subseteq [-k, k]^n \subseteq \mathbb{R}^n$. The non-empty subset, T of $\mathbb{R}^n$, is a compact subset if and only if T is both closed and bounded in $\mathbb{R}^n$. So, in $\mathbb{R}^n$, the compact property is equivalent the closed and bounded properties combined.

Proof: The proof is straightforward and so is assigned as an exercise question. (You may invoke the Tychonoff theorem).

Theorem 14.6 Let F be a compact subset of a topological space, S. If f is a continuous real-valued function on F then f attains its maximum and minimum values inside F.

Proof: We are given that F is compact in S and $f[F]$ is the continuous image of F in $\mathbb{R}$. Then $f[F]$ is compact in $\mathbb{R}$ and so is closed and bounded. Then boundedness of $f[F]$ guarantees that $f[F] \subseteq [-k, k]$, for some k. Since $f[F]$ is closed in $\mathbb{R}$ then $\sup f[F]$ and $\inf f[F]$ must both belong to $f[F]$. So f attains both its maximum and minimum values on F.

Example 5. Suppose S and T are topological spaces. We know that projection maps on product spaces are open maps. (See theorem on page 105)

Show that, in the case where the space S is Hausdorff and the space T is compact then the projection map, $\pi_1 : S \times T \to S$, is a closed map.

Solution : Let K be a closed subset of $S \times T$. We are required to show that $\pi_1[K]$ is closed. It then suffices to show that $S \backslash \pi_1[K]$ is open in S.

Let $u \in S \backslash \pi_1[K]$. Then $(\{u\} \times T) \cap K = \varnothing$. Since T is compact then we easily see that $\{u\} \times T$ is compact. Since S is Hausdorff, for each $(u, x) \in \{u\} \times T$ there is an open neighbourhood, $V_u^x \times U_x$, which does not meet K. In this way we obtain an open cover

$$\{V_u^x \times U_x : x \in T\}$$

of $\{u\} \times T$ which then has a finite subcover

$$\{V_u^{x_i} \times U_{x_i} : x_i \in F \subseteq T\}$$

Then $\cap\{V_u^{x_i} \times U_{x_i} : x_i \in F \subseteq T\}$ forms an open neighbourhood of u which does not intersect $\pi_1[K]$. So $S \backslash \pi_1[K]$ is open in S, as claimed. We conclude that $\pi_1 : S \times T \to S$ is a closed projection map.

Example 6. For the Hausdorff topological space, S, and the compact space, T, let $f : S \to T$ be a function mapping S into T. The set, G, will represent the graph of f, defined as

$$G = \{(x, f(x)) : x \in S\} \subseteq S \times T$$

Show that, if G is a closed subspace of the product space, $S \times T$, then $f : S \to T$ must be a continuous function.

Solution : Given: S is Hausdorff and T is compact. Also, we are given that $G = \{(x, f(x)) : x \in S\}$ is a closed subset of $S \times T$. Let K be a closed subset of T. To show continuity of f, it will suffice to show that $f^{\leftarrow}[K]$ is a closed subset of S.

By the result stated in the example immediately above, $\pi_1 : S \times T \to S$ is a closed map. See that $\pi_2^{\leftarrow}[K] \cap G$ is the intersection of two closed sets and so is a closed subset of $S \times T$. Since

$$\pi_2^{\leftarrow}[K] \cap G = \{(x, f(x)) : f(x) \in K\}$$

then $\pi_1[\pi_2^{\leftarrow}[K] \cap G] = f^{\leftarrow}[K]$.

Since π_1 is a closed map, $f^{\leftarrow}[K]$ is closed. We have shown that f pulls back closed sets to closed sets; so f is continuous.

14.4 Topic : The Embedding theorem III.

Recall the statement of the *Embedding theorem II* (theorem 10.15) which says that a space S is completely regular if and only if the evaluation map with respect to $C^*(S)$ embeds S inside a cube, $T = \prod_{i \in I}[a_i, b_i]$. Now we know something else about cubes: Since they are the product of compact sets, the Tychonoff theorem guarantees that they are compact. Since $\prod_{i \in I}[a_i, b_i]$ and $\prod_{i \in I}[0, 1]$ are homeomorphic then S can be embedded in the compact space, $\prod_{i \in I}[0, 1]$. With this in mind, we raise the Embedding theorem II to the higher level, Embedding theorem III, in the form of the following theorem.

Theorem 14.7 *Embedding theorem III.* If a topological space S is completely regular then S can be densely embedded in a compact Hausdorff space.

Proof : We are given a completely regular topological space S.

Then, by the theorem 10.15, the evaluation map with respect to $C^*(S)$ embeds S into a cube, $T = \prod_{i \in I}[a_i, b_i]$. Each factor of the product T is Hausdorff. Since a product of Hausdorff spaces was shown to be Hausdorff, T is Hausdorff. By the Tychonoff theorem, T is a compact Hausdorff space.

Since $\text{cl}_T e[S]$ is a closed subset of a Hausdorff compact space T, it is Hausdorff compact. Then the homeomorphic image, $e[S]$, of S is dense in $\text{cl}_T e[S]$, a Hausdorff compact set, as required.

Theorem 14.8 Let S be a topological space. Then S is completely regular if and only if S is a subspace of a Hausdorff compact space.

Proof : We are given that S is any topological space.

($\Rightarrow$) Suppose S is a subspace of a compact Hausdorff space, T. By Urysohn's lemma, there is a continuous bounded function which separates any two closed subsets (so one that separates a point and a closed set). So T is completely regular. By theorem 10.5, since S is a subspace of a completely regular space, it is completely regular, as required.

($\Leftarrow$) Suppose S is completely regular. Then, by theorem 14.7, S is densely embedded in a compact Hausdorff space T. That is, a topological copy of S is a subspace of T. So S is a subspace T.

14.5 Topic : Completely normal spaces.

Given the some results involving compact space proven above, we are now able to delve a bit further on the topic of complete separation of sets. We will begin by recalling a few previously proven results.

- The Cartesian product (equipped with the product topology) of compact spaces is compact. (Tychonoff theorem 14.4)
- Any compact Hausdorff space is normal. (Theorem 14.3 c))
- Metrizable spaces are normal. (Theorem 9.18)
- Subspaces of metrizable spaces are metrizable (Example on page 79)

The above results make it more easy to see that there can be normal spaces with non-normal subspaces. We have shown in the Embedding theorem III (Theorem 14.7) that every completely regular space can be (densely) embedded in a compact Hausdorff space. Hence every completely regular space can be embedded in a normal space. But we know that there are completely regular spaces which are not normal (See the Moore plane examples on page 177). The Moore plane is such an example. Then...

... there exist normal spaces with non-normal subspaces.

Even if it is incorrect to say that "a subspace of a normal space is normal" there are certain types of normal spaces, S, for which every subspace of S is normal.
This short preamble leads us to the following definition.

Definition 14.9 A *completely normal space* is a space in which every subspace is normal.

Then every metrizable space is completely normal. While non-normal completely regular spaces can be embedded in a non-completely-normal compact space.

By definition we have

$$\text{completely normal} \Rightarrow \text{normal}$$

Recall that a *perfectly normal space* is a normal space whose closed subsets are G_δ's. How does "completely normal" compare with "perfectly normal"?

Suppose S is perfectly normal and T is a subspace of S. We claim that T is normal. A closed subset F of T is a G_δ since F is the intersection of a G_δ of S with T. So T is perfectly normal and so is a normal subspace of S. So every subspace of S is normal, as claimed. We can then say that a perfectly normal space, S, is completely normal. So we have the chain,

$$\text{perfectly normal} \Rightarrow \text{completely normal} \Rightarrow \text{normal} \Rightarrow \text{completely regular}$$

We have the following characterization of a completely normal space.

Example 7. Show that S is completely normal if and only if for any pair of subsets A and B such that $A \cap \text{cl}_S B = \varnothing = \text{cl}_S A \cap B$ there exists disjoint open subsets U and V such that $A \subseteq U$ and $B \subseteq V$.[1]

Solution: ($\Leftarrow$) Let S be a topological space. Suppose that for any pair of subsets A and B such that $A \cap \text{cl}_S B = \varnothing = \text{cl}_S A \cap B$ there exists disjoint open subsets U and V such that $A \subseteq U$ and $B \subseteq V$. Let T be a subspace of S. We claim that T is normal. Suppose F and K are disjoint non-empty closed subsets of T and F^* and K^* are closed subsets of S such that $F = F^* \cap T$ and $K = K^* \cap T$. Then $F^* \cap \text{cl}_S K^* = \varnothing = \text{cl}_S F^* \cap K^*$. By hypothesis, there exists disjoint open subsets of S, U^* and V^*, such that $F^* \subseteq U^*$ and $K^* \subseteq V^*$. Then $F \subseteq U^* \cap T$ and $K \subseteq V^* \cap T$. Then T must be normal. We can conclude that, by definition, S is completely normal.

The direction ($\Rightarrow$) is left as an exercise.

The following example illustrates a completely normal space which is not perfectly normal.

Example 8. *A normal space which is not perfectly normal.* Let S where S is uncountable and $p \in S$ and suppose S has a topology defined as follows:

$$\tau = \{T \subseteq S : S \backslash T \text{ is finite or } p \notin T\}$$

Show that S is completely normal but not perfectly normal.

Solution : We claim that S is T_1. The set $\{p\}$ is closed and if $x \neq p$ then $S \backslash \{x\}$ is open so $\{x\}$ is closed. So S is T_1, as claimed.

[1] This is an exercise question which appears in 15B of S. Willard's Topology.

We claim that S is completely normal. Suppose F and K are disjoint closed subsets of S. If $p \notin K \cup F$ then K and F are both open. So F and K are clopen and so $F \cap \mathrm{cl}_S K = \varnothing = \mathrm{cl}_S F \cap K$. Suppose $p \in F$. Then K is clopen and finite. That is $K = \mathrm{cl}_S K$ is open. Now $S \backslash K$ is open since K is finite. So $\mathrm{cl}_S F$ and K are contained disjoint open sets, $S \backslash K$ and K as well as F and $\mathrm{cl}_S K$. So, by the statement proven in example 8 above, S is completely normal, as claimed.

We claim that S is not perfect. Let $\{U_i : i \in \mathbb{N}\}$ be a countable family of sets in S each containing the point p. Then each U_i is closed in S and $p \in \cap\{U_i : i \in \mathbb{N}\}$. If each U_i is also open $S \backslash U_i$ if finite. So each U_i is uncountable. Then $S \backslash \cap\{U_i : i \in \mathbb{N}\} = \cup\{S \backslash U_i : i \in \mathbb{N}\}$ a countable subset of S. So $\cap\{U_i : i \in \mathbb{N}\}$ is uncountable. So $p \neq \cap\{U_i : i \in \mathbb{N}\}$. So p is not a G_δ. So S is not perfectly normal, as claimed.

So we have the result,

$$\text{completely normal} \not\Rightarrow \text{perfectly normal}$$

14.6 Topic : Compactness in terms of z-filters.

In what follows (S, τ) is a topological space. The symbols $C(S)$ and $C^*(S)$ denote the set of all real-valued continuous functions and the set of real-valued continuous bounded functions, respectively, on S. We summarize concepts seen in the chapter on separation of closed subsets with functions.

- If $f \in C(S)$ then $Z(f) = f^{\leftarrow}(0)$ is a zero-set in S; while $Z[S] = \{Z(f) : f \in C(S)\}$.
- Recall that zero-sets are closed G_δ's.
- On the other hand. Given a closed G_δ, F, of S, F need not be a zero-set.
- However, if S is a normal space, a closed G_δ is a zero-set.

Even though $Z[S]$ is just a particular type of subset of $\mathscr{P}(S)$, the expression $Z[S]$ makes sense only if S is a known topological space. After all, by definition, $Z(f)$ is the zero-set associated to a *continuous* real-valued function on S.

We would like to discuss subfamilies of $Z[S]$ which are "filters". To distinguish a filter in $Z[S]$ from a filter of sets in $\mathscr{P}(S)$ we refer to these as *z-filters*. Even though the definition of a z-filter is simply the analogue of the definition of a filter of sets, for the sake of completeness, we include the following definition.

Definition 14.10 Let S be a topological space. Let $\mathscr{F} \subseteq Z[S]$.
If $\mathscr{F}$ satisfies the two properties:

1) $\varnothing \notin \mathscr{F}$,

2) For every $U, V \in \mathscr{F}$, there exists $F \in \mathscr{F}$ such that $F \subseteq U \cap V$.

then we will say that $\mathscr{F}$ is a *z-filter base*.

If $\mathscr{F}$ is a z-filter base which also satisfies the condition:

3) Whenever $U \in \mathscr{F}$ and there is $V \in Z[S]\backslash\{\varnothing\}$ such that $U \subseteq V$, then $V \in \mathscr{F}$.

Then $\mathscr{F}$ is called a *z-filter*.

4) If $\mathscr{F}$ is a z-filter which is not contained in any other strictly larger z-filter then we say that $\mathscr{F}$ is a *maximal z-filter* or, more commonly, a *z-ultrafilter*.

A quick way of describing a z-filter is to say: "A z-filter is a subset of $Z[S]\backslash\{\varnothing\}$ which is closed under finite intersections and supersets both in $Z[S]$."

If a subset $\mathscr{F}$ is not a z-filter base in $Z[S]$ then, if $Z(f)$ and $Z(g)$ belong to $\mathscr{F}$, we can always recruit $Z(f^2+g^2) = Z(f) \cap Z(g)$ from $Z[S]$ and add it to $\mathscr{F}$. To build up a z-filter we can recruit $Z(fg) = Z(f) \cup Z(g)$ from $Z[S]$. So $Z[S]$ is such that even a single zero-set $Z(f)$ can be a seed used to generate a z-filter in $Z[S]$.

A z-filter is, itself, a z-filter base, while a z-filter base, if not a z-filter base, can be completed to become a z-filter. Given a z-filter base, $\mathscr{F}$, we define the larger set, $\mathscr{F}^*$, as follows:

$$\mathscr{F}^* = \{U \in Z[S] : F \subseteq U, \text{ for some } F \in \mathscr{F}\}$$

We have simply adjoined to $\mathscr{F}$ all its supersets which belong to $Z[S]$. It is easily verified that $\mathscr{F}^*$ is a filter. We will say that the z-filter base, $\mathscr{F}$, *generates the z-filter $\mathscr{F}^*$*.

In the event that every subset is a zero-set (such as a discrete space, for example) a z-filter is simply a filter of sets.

The use of the notation $Z(f)$ for the zero-set generated by f allows us to view Z as a function

$$Z : C(S) \to Z[S]$$

where $C(S)$ can be viewed as the ring $(C(S),\ +,\ \cdot)$ of continuous functions on S, $Z[S]$ is a subset of $\mathscr{P}(S)$. So Z maps f to $Z(f) = f^{\leftarrow}(0)$. A subset $(U,\ +,\ \cdot)$ of $C(S)$ which is also a ring is a subring.

Those subrings, $(U,\ +,\ \cdot)$, of $C(S)$ satisfying the property,

"For every $f \in U$, $gf \in U$, for all $g \in C(S)$"

are called *ideals*[1]. For obvious reasons we tend to represent an ideal in $\mathscr{P}(C(S))$ by I. If J is an ideal in $C(S)$ such that for any ideal I, $J \subseteq I$ implies $I = J$ we say that J is a *maximal ideal* in $C(S)$. Ideals in $C(S)$ are relevant in our study since the image $Z[I]$ of an ideal I under the map $Z : C(S) \to Z[S]$, will be seen to be z-filters. Also the function $Z^{\leftarrow} : Z[S] \to C(S)$ will pull back a z-filter, U, in $Z[S]$ to a corresponding ideal, $I = Z^{\leftarrow}[U]$.[2]

Theorem 14.11 Suppose S is a topological space. Let I be an ideal in $C(S)$ and $\mathscr{F}$ be a z-filter in $Z[S]$.

a) Then $Z[I]$ is a z-filter.

b) Then $Z^{\leftarrow}[\mathscr{F}]$ is an ideal in $C(S)$.

c) If J is a maximal ideal then $Z[J]$ is a z-ultrafilter. If $\mathscr{F}$ is a z-ultrafilter then $Z^{\leftarrow}[\mathscr{F}]$ is maximal ideal.

d) If $\mathscr{M}$ is the set of all maximal ideals in $\mathscr{P}(C(S))$ and $\mathscr{Z}$ is the set of all z-ultrafilters in $\mathscr{P}(Z[S])$, then $Z : \mathscr{M} \to \mathscr{Z}$ is one-to-one and onto.

e) If $\mathscr{Z} = Z[\mathscr{M}]$ is a z-ultrafilter in $\mathscr{P}(Z[S])$ and $Z(f)$ is a zero-set which meets every member of $\mathscr{Z}$ then $Z(f) \in \mathscr{Z}$.

Proof: We are given that I is a (proper) ideal in $C(S)$ and $\mathscr{F}$ is a z-filter in $Z[S]$.

a) Since I is proper then unit function 1 is not an element of I. Then the empty-set in the form of $Z(1)$ is not in $Z[I]$. Suppose $Z(f)$ and $Z(g)$ belong to $Z[I]$. Then $f^2 + g^2 \in I$ and so
$$Z(f^2 + g^2) = Z(f) \cap Z(g) \in Z[I]$$
Suppose $f \in I$ and $Z(g) \in Z[S]$ such that $Z(f) \subseteq Z(g)$. We claim that $Z(g) \in Z[I]$. Then $gf \in I$. Simply see that
$$Z(g) = Z(f) \cup Z(g) = Z(fg) \in Z[I]$$

b) Given a z-filter $\mathscr{F}$ consider $Z^{\leftarrow}[\mathscr{F}]$. Since $\varnothing \notin \mathscr{F}$, the unit function 1 is not in $Z^{\leftarrow}[\mathscr{F}]$. Suppose f and g belong to $Z^{\leftarrow}[\mathscr{F}]$. Then $Z(f)$ and $Z(g)$ belong to $\mathscr{F}$ hence, $Z(f) \cap Z(g) \in \mathscr{F}$. Since $Z(f) \cap Z(g) \subseteq Z(f - g)$, then $f - g \in Z^{\leftarrow}[\mathscr{F}]$ so $f + g \in Z^{\leftarrow}[\mathscr{F}]$. Since $Z(f) \subseteq Z(fk)$, then $fk \in Z^{\leftarrow}[\mathscr{F}]$. So $Z^{\leftarrow}[\mathscr{F}]$ is an ideal.

[1] Note that $C(S)$ is an ideal, but we will agree that when we say "ideal" we mean a proper ideal.

[2] Note that $(C^*(S), +, \cdot)$ also constitutes a ring, and so we can speak of *ideals* and *maximal ideals* in $C^*(S)$ also.

c) If J is a maximal ideal then $Z[J]$ is a z-filter. Suppose $\mathscr{F}$ is z-filter such that $Z[J] \subseteq \mathscr{F}$. We claim that $\mathscr{F} = Z[J]$. Then $J \subseteq Z^{\leftarrow}[Z[J]] \subseteq Z^{\leftarrow}[\mathscr{F}]$. Then $Z^{\leftarrow}[\mathscr{F}] = J$. So $\mathscr{F} = Z[Z^{\leftarrow}[\mathscr{F}]] = Z[J]$. So $Z[J]$ is a z-ultrafilter.

   On the other hand, if $\mathscr{F}$ is a z-ultrafilter, then $Z^{\leftarrow}[\mathscr{F}]$ is an ideal. Suppose K is an ideal such that $Z^{\leftarrow}[\mathscr{F}] \subseteq K$. See that $\mathscr{F} = Z[Z^{\leftarrow}[\mathscr{F}]] \subseteq Z[K]$. Then $Z[K] = \mathscr{F}$. So $K \subseteq Z^{\leftarrow}[Z[K]] = Z^{\leftarrow}[\mathscr{F}]$. So $Z^{\leftarrow}[\mathscr{F}]$ is a maximal ideal.

d) The result follows immediately from the definition of "maximal".

e) The proof is analogous to the similar statement referring to an ultrafilter of sets proven previously in the text.

Theorem 14.12 Let S be a topological space and $(\mathcal{Z}, \subseteq)$, denote the set of all z-filters in $\mathscr{P}(S)$, partially ordered by inclusion. If $\mathscr{F}$ is any z-filter belonging to the family, $\mathcal{Z}$, of all z-filters, then there exists a z-ultrafilter, $\mathscr{U}$, in $\mathcal{Z}$ which contains $\mathscr{F}$.

Proof: We are given that $\mathscr{F} \in \mathcal{Z}$. We wish to prove the existence of a z-ultrafilter which contains $\mathscr{F}$. We set up the problem so that Zorn's lemma applies.

For a given $\mathscr{F} \in \mathcal{Z}$, consider the set $\mathbb{S}_{\mathscr{F}} = \{\mathscr{H} \in \mathcal{Z} : \mathscr{F} \subseteq \mathscr{H}\}$ of all z-filters which contain $\mathscr{F}$. We are required to show that some z-filter in $\mathbb{S}_{\mathscr{F}}$ is a z-ultrafilter (i.e., $\mathbb{S}_{\mathscr{F}}$ has a maximal element). If $\mathbb{S}_{\mathscr{F}} = \{\mathscr{F}\}$ then $\mathscr{F}$ is a z-ultrafilter. Suppose $\mathbb{S}_{\mathscr{F}}$ contains other filters. Now $\mathscr{F}$ can be the base element of many chains in $\mathbb{S}_{\mathscr{F}}$. In fact, $\mathbb{S}_{\mathscr{F}}$ can be viewed as the union of a family of filter-chains, $\{\mathscr{C}_\alpha : \alpha \in J\}$, each with base element, $\mathscr{F}$. Pick an arbitrary filter-chain, $\mathscr{C}_\gamma = \{\mathscr{F}_i : i \in I\}$ in $\mathbb{S}_{\mathscr{F}}$. It looks like,

$$\mathscr{F} \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \mathscr{F}_3 \subseteq \cdots$$

Let $\mathscr{T} = \cup\{\mathscr{F}_i : \mathscr{F}_i \in \mathscr{C}_\gamma\}$. Quite clearly, $\mathscr{F}_i \subseteq \mathscr{T}$ for each i. We claim that the family of sets, $\mathscr{T}$, is itself a z-filter in the chain $\mathscr{C}_\gamma$. If U and V are elements of $\mathscr{T}$, then U and V are elements of some $\mathscr{F}_j$ in the chain, $\mathscr{C}_\gamma$, so $U \cap V$ is an element of $\mathscr{F}_j$; so $U \cap V$ is also an element of $\mathscr{T}$. Then $\mathscr{T}$ is a z-filter base of sets which is a maximal element in the chain $\mathscr{C}_\gamma$.

So every chain, $\mathscr{C}_\alpha$, in $\mathbb{S}_{\mathscr{F}}$ has a maximal element. We can invoke Zorn's lemma, to conclude from this that $\mathbb{S}_{\mathscr{F}}$, must also have a maximal z-filter element, $\mathscr{U}$. Since, $\mathbb{S}_{\mathscr{F}}$ contains every z-filter which contains $\mathscr{F}$, then $\mathscr{U}$ is the maximal z-filter which contains $\mathscr{F}$. Then, by definition, $\mathscr{U}$ is the unique z-ultrafilter which contains $\mathscr{F}$.

Definition 14.13 Let S be a topological space. Let I be an ideal in $C(S)$ or $C^*(S)$ and $\mathscr{F} = Z[I]$ be the corresponding z-filter.

a) If $\cap\{Z(f) : f \in I\} \neq \varnothing$ then we will say that $\mathscr{F}$ is a *fixed* z-filter and that I is a *fixed* ideal.

b) Otherwise, we will say that $\mathscr{F}$ is a *free* z-filter and I is a *free* ideal.

Theorem 14.14 Suppose S is a completely regular topological space. Then the following are equivalent.

a) The space S is a compact space.

b) Every z-filter in $Z[S]$ is fixed. Every ideal in $C(S)$ is fixed. Every ideal in $C^*(S)$ is fixed.

c) Every z-ultrafilter is fixed. Every maximal ideal in $C(S)$ is fixed. Every maximal ideal in $C^*(S)$ is fixed.

Proof: We are given that S is a completely regular topological space.

a) $\Rightarrow$ b) Suppose S is compact. Since S is compact then every function in $C(S)$ is bounded so $C(S) = C^*(S)$. Suppose $\mathscr{F} = \{Z(f) : f \in I\}$ is a z-filter in $Z[S]$ corresponding to an ideal I in $C(S)$. Since $\mathscr{F}$ satisfies the finite intersection property then $\cap\{Z(f) : f \in I\}$ is non-empty. So $\mathscr{F}$ is fixed. Equivalently I is a fixed ideal in $C(S)$.

b) $\Rightarrow$ a) Suppose every z-filter in $Z[S]$ is fixed. Also suppose $\mathscr{F}$ is a family of closed subsets with the FIP. We claim that $\mathscr{F}$ has non-empty intersection. If $F \in \mathscr{F}$ and $p \in S \backslash F$ then, since S is completely regular, there exists $f \in C(S)$ such that $F \subseteq Z(f)$ and $p \in f^{\leftarrow}(1)$. Then the set $F = \cap\{Z(f) : f \in U\}$, the intersection of zero-sets. Then $\cap\{F : F \in \mathscr{F}\} = \cap\{Z : Z \in \mathscr{G} \subseteq Z[S]\}$, the intersection of zero-sets in $Z[S]$. Then since $\{Z : Z \in \mathscr{G} \subseteq Z[S]\}$ is a z-filter, then $\cap\{F : F \in \mathscr{F}\}$ is non-empty so S is compact.

b) $\Rightarrow$ c) Suppose every z-filter in $Z[S]$ is fixed. Then every z-ultrafilter in $Z[S]$ is fixed. As well every ideal is fixed and so every maximal ideal is fixed.

c) $\Rightarrow$ b) Suppose every z-ultrafilter in $Z[S]$ is fixed. Then every maximal ideal is fixed. Suppose $\mathscr{F}$ is a z-filter in $Z[S]$. Then, by theorem 14.12, $\mathscr{F}$ is contained some z-ultrafilter, $\mathscr{G}$. By hypothesis, there exists $p \in \cap\{F : F \in \mathscr{G}\}$. Then $p \in \cap\{F : F \in \mathscr{F}\}$. So $\mathscr{F}$ is fixed. Also every corresponding ideal in $C(S)$ is fixed.

Concepts review:

1. Define an open cover of a space. What does it mean to say that it has a finite subcover?

2. State the formal topological definition of compact subspace.
3. What does it mean to say that a family of sets satisfies the “finite intersection property”?
4. State a characterization of compact space involving the “finite intersection property”.
5. What kind of properties do nets in a compact space satisfy?
6. What kind of properties do filters in a compact space satisfy?
7. What can we say about ultrafilters of sets in a compact set?
8. What can we say about continuous images of compact sets?
9. What can we say about closed subsets of compact sets?
10. Under what conditions on S does the statement “Compact subsets of S are closed” hold true, in general?
11. Complete the statement: A one-to-one continuous function, $f : F \to T$, mapping a compact space, F, into a Hausdorff space, T, is....
12. Suppose $S = \prod_{i \in I} S_i$ is a product space with compact factors, S_i. Is this statement true or false? “The product space S maybe compact but it depends on the size of the index I”.
13. What separation axioms will compact Hausdorff spaces satisfy?
14. Define the separation property referred to as *completely normal*.
15. Give a characterization of “completely normal”.
16. How does a completely normal space compare with a normal space and a perfectly normal space.
17. Define a z-filter base and a z-filter.
18. Define a z-ultrafilter.
19. Define an ideal and a maximal ideal in $C^*(S)$.
20. Describe a relationship between ideals in $C^*(S)$ and z-filters.
21. Define a fixed filter and a free filter. How are they recognized?
22. Give a characterization of a compact space in terms of z-filters.

EXERCISES

1. Show that a closed and bounded subset of $\mathbb{R}$ is compact.
2. Show that disjoint compact sets in a Hausdorff space are respectively contained in disjoint open subsets.
3. Let F and H be be disjoint closed subsets of $[0, 5]^n$. Show that they are contained in disjoint open neighbourhoods.
4. Suppose $g : S \to T$ is a function mapping the topological space S into the compact Hausdorff space T. Show that g is continuous on S if and only if the graph of g, $\{(x, g(x)) : x \in S\}$ is a closed subset of $S \times T$.
5. Show that the finite union of compact subsets of a topological space is compact.
6. The set T is a compact subset of $\mathbb{R}^n$ if and only if T is both closed and bounded in $\mathbb{R}^n$.
7. Suppose $\{F_i : i \in I\}$ is a family of compact sets in a Hausdorff space. Show that $\cap\{F_i : i \in I\}$ is compact.

15 / Countably compact spaces.

Summary. *We discuss a slightly weaker version of the compactness property. It is called "countable compactness". After formally defining it and discussing its properties, we show that, in some topological spaces (such as metric spaces) countably compact and compact represent the same property, while for others, they are easily distinguished.*

15.1 The countably compact property.

The compact property imposes strong restrictions on a topological space. Sometimes, when a topological space is not too large, we may not require the full strength version of its definition, "Every open cover has a finite subcover". A slightly weakened version, such as "Every countable open cover has a finite subcover", may suffice. Since this version appears to be such a close relative of the compact property, it is worth pausing for a bit to verify how it will modify the basic compactness properties or examine which spaces satisfy it.

Definition 15.1 Let F be any non-empty subset of the topological space, S. We say that F is *countably compact* if and only if every countable open cover of F has a finite subcover.

Just as for compact subsets, it will be useful to have various ways of recognizing the countably compact sets. It will also allow us to better see how it distinguishes itself from the compact property.

Theorem 15.2 Let S be a topological space and T be a non-empty subset of S. Then the following are equivalent.

a) The set T is a countably compact set.

b) If T is countably infinite, then it has at least one cluster point inside T.

c) If $A = \{x_i\}$ is a sequence in T, then A has at least one accumulation point in T.

d) A countable family, $\mathscr{F} = \{F_i : i \in \mathbb{N}\}$, of closed sets in T satisfying the finite intersection property has non-empty intersection in T.

Proof : Given: S is a topological space and T is a non-empty subset of S.

($a \Rightarrow b$) We are given that T is countably infinite and countably compact. Suppose T has *no* cluster point. Then T would have to be closed in S for, if not, T would have to be a proper subset of S which has a boundary point, p, in $\text{cl}_S T \cap S \setminus T$. This would mean that every open neighbourhood of p intersects T; this would make of p a cluster point. If no cluster point belongs to T, then every singleton set, $\{x\}$, of T would be open. Then $\{\{x\} : x \in T\}$ would be an open cover of T with no finite subcover. A contradiction. So T has a cluster point.

($b \Rightarrow c$) We are given that, in T, countably infinite subsets have a cluster point in T. We can construct a sequence, $A = \{f(i) : i \in \mathbb{N}\}$, in T. We are required to show that the sequence, A, has an accumulation point in T.

If A has only finitely many distinct points then $f(i) = k$ for infinitely many values of i. So k is an accumulation point of A. Suppose, on the other hand that, A has infinitely many distinct points. Then, by hypothesis, A has a cluster point, say z, in T. Then, for any neighbourhood, B_z, of z, $B_z \cap A$ contains infinitely many distinct points. So $B_z \cap \{f(i) : i > q\}$ is non-empty for all $q \in \mathbb{N}$. This means that any neighbourhood of z intersects a tail end of A. So, by definition, z is an accumulation point of A in T.

($c \Rightarrow a$) We are given that any sequence in T has an accumulation point. We are required to show that T is countably compact. Let $\mathscr{U} = \{U_i : i \in \mathbb{N}\}$ be a countable family of open sets which covers T. Suppose $\mathscr{U}$ has no finite subcover. Then, for every n, it is possible to choose, $f(n) \in T \setminus \cup\{U_i : i = 1 \text{ to } n\}$. Consider the sequence $\{f(n) : n \in \mathbb{N}\setminus\{0\}\}$. Then for each k,

$$f[\,[k, \infty)\,] \subseteq T \setminus \cup\{U_i : i = 1 \text{ to } k\}$$

So no element of $\mathscr{U}$ can intersect a tail end of $\{f(n)\}$, contradicting our hypothesis stating that T has an accumulation point. So $\mathscr{U}$ must have a finite subcover.

($a \Rightarrow d$) We are given that T is countably compact. Let $\mathscr{F} = \{F_i : i \in \mathbb{N}\}$ be countable family of closed sets with the finite intersection property. We are required to show that $\cap\{F_i : i \in \mathbb{N}\} = \varnothing$. Suppose $\cap\{F_i : i \in \mathbb{N}\}$ is empty. Then $\{T \setminus F_i : i \in \mathbb{N}\}$ forms an open cover with no finite subcover, contradicting the property of countable compactness on T.

($d \Rightarrow a$) We are given that, if $\mathscr{F}$ is a countable family of closed sets with the finite intersection property, then $\cap\mathscr{F}$ is not empty. Suppose $\mathscr{U} = \{U_i : i \in \mathbb{N}\}$ is an open cover of T. We are required to show that $\mathscr{U}$ contains a finite subcover of T. Suppose $\mathscr{U}$ has no finite subcover. Then $\{T \setminus U_i : i \in \mathbb{N}\}$ forms a family of closed sets satisfying the finite intersection property which has empty intersection. This contradicts our hypothesis. So T is countably compact.

15.2 Bolzano-Weierstrass property

We will look, more closely, at the countable compactness characterization on T, in part c), which states that, in countably compact spaces,...

> "Every sequence, $A = \{x_i : i \in \mathbb{N}\}$, in T has an accumulation point $p \in T$."

We see that it slightly weakens a characterization of the compactness property by replacing the word "net" with the word "sequence". This is not unexpected, since, by definition, countable compactness is indeed weaker than compactness (in the sense that "compact $\Rightarrow$ countably compact" but "countably compact $\not\Rightarrow$ compact"). By theorem 12.4, this characterization of countable compactness is equivalent to saying

> "Every sequence $A = \{x_i : i \in \mathbb{N}\}$ in T has a subnet which converges to some accumulation point, p, of A."

To some, this may be reminiscent of a real analysis statement called the *Bolzano-Weierstrass theorem*. It states that

> "Every bounded sequence in $\mathbb{R}$ has a convergent subsequence."

There is, however, a fundamental difference between the statement of the Bolzano-Weierstrass theorem (BWT) and this particular characterization of the countable compactness. We see that the BWT refers only to $\mathbb{R}$, not an arbitrary topological space, and only to those sequences which are "bounded". (Note that the property of "boundedness" has not yet been defined for arbitrary topological spaces.) Because of similarities, we have a particular name for this particular countable compactness property. We will formally define it.

Definition 15.3 A topological space, S, is said to satisfy the *Bolzano-Weierstrass property* if and only if every sequence, $A = \{x_i : i \in \mathbb{N}\}$, in S has an accumulation point in S.

Both the BWT and the "Bolzano-Weierstrass property", refer to some condition that will guarantee that some subnet or subsequence of a sequence will converge to some point. In the case of $\mathbb{R}$ only the bounded sequences will satisfy this property. By the characterization theorem above, any sequence of a countably compact topological space will satisfy this property.

15.3 Properties of countably compact spaces.

Before presenting some examples, it will help to list a few properties of countably compact spaces. The first statement says that a countable compact space is "closed-hereditary".

Theorem 15.4 Closed subspaces of countably compact spaces are countably compact.

Proof: Given: The space, S, is a countably compact topological space and F is a non-empty closed subset of S.

Suppose $\mathscr{F}$ is a countable family of closed subsets of F which satisfies the finite intersection property. It suffices to show that $\mathscr{F}$ has non-empty intersection. Then every set in $\mathscr{F}$ is also a closed subset in S. Since S is countably compact the family of sets in $\mathscr{F}$ has non-empty intersection inside F. So F is also countably compact.

We know that continuous images of compact sets are compact. The following theorem shows that an analogous statement for countably compact spaces holds true.

Theorem 15.5 Suppose S is countably compact and the continuous function, $f : S \to T$, maps S into the topological space, T. Then $f[S]$ is a countably compact subspace of T.

Proof: Given: The space S is countably compact and T is a topological spaces. Suppose $f : S \to T$ is continuous.

Let $\mathscr{U} = \{U_i : i \in \mathbb{N}\}$ be a countable open cover of $f[S]$. Then $\mathscr{V} = \{f^{\leftarrow}[U_i] : i \in \mathbb{N}\}$ is a countable open cover of S. Since S is countably compact, $\mathscr{V}$ has a finite subcover $\mathscr{V}_F = \{f^{\leftarrow}[U_i] : i \in F\}$ of S. Then

$$\begin{aligned} f[S] &\subseteq f[\cup\{f^{\leftarrow}[U_i] : i \in F\}] \\ &= \cup\{f[f^{\leftarrow}[U_i]] : i \in F]\} \\ &= \{U_i : i \in F\} \end{aligned}$$

So $\mathscr{U}$ has a finite subcover $\mathscr{U}_F = \{U_i : i \in F\}$ of $f[S]$. So $f[S]$ is also countably compact.

We now consider in invariability of the countable compactness property over products.

Theorem 15.6 Let $\{S_j : j = 1, 2, 3, \ldots,\}$ be a family of topological spaces. We consider the corresponding product space, $S = \prod_{j\in\mathbb{N}\setminus\{0\}} S_j$.

a) If the product space, S, is countably compact then so is each one of its S_j factors.

b) If each S_j factor of the product space, S, is, simultaneously countably compact and first countable then the infinite product space, S, is countably compact.

Proof: Given: The set, $S = \prod_{j\in\mathbb{N}\setminus\{0\}} S_j$, is a product space.

a) Suppose the product space, S, is countably compact. Since each projection map, π_j, is continuous, then, for each factor S_j, $S_j = \pi_j[S]$, the continuous image of a countably compact space. So each factor, S_j, is countably compact. We are done with part a).

b) We are given that each S_j is both first countable and countably compact. We are required to show that $S = \prod_{j\in\mathbb{N}\setminus\{0\}} S_j$ is countably compact.

Let $B = \{\beta(i) : i = 1, 2, 3, \ldots,\}$ be a sequence in S. To show that S is countably compact it suffices to show that B has an accumulation point.

Then $\pi_1[B] = \{\alpha_1(i) : i = 1, 2, 3, \ldots,\}$, is a sequence in S_1. Since S_1 is countably compact, then, by theorem 15.2, $\pi_1[B]$ has an accumulation point, say p_1. By theorem 11.5, since S_1 is first countable, $\pi_1[B]$ has a subsequence,

$$\{\alpha_1(g_1(i)) : i = 1, 2, 3, \ldots,\}$$

converging to p_1.

Then

$$\{\beta_1(i) : i = 1, 2, 3, \ldots,\} = \{\beta(g_1(i)) : i = 1, 2, 3, \ldots,\}$$

defines a subsequence of B in S. Similarly, $\pi_2[B] = \{\alpha_2(i) : i = 1, 2, 3, \ldots,\}$ has a subsequence $\{\alpha_2(g_1(i)) : i = 1, 2, 3, \ldots,\}$ which has an accumulation point, say p_2, and so, itself, has a subsequence, $\{\alpha_2(g_2(g_1(i))) : i = 1, 2, 3, \ldots,\}$, which converges to p_2. Again,

$$\{\beta_2(i) : i = 1, 2, 3, \ldots,\} = \{\beta_1(g_2(g_1(i))) : i = 1, 2, 3, \ldots,\}$$

is a subsequence of B.

Eventually we obtain a sequence $\{\alpha_n(g_n(\cdots(g(i)))\cdots) : i = 1, 2, 3, \ldots,\}$ converging to p_n in S_n and, from this, the subsequence

$$\{\beta_n(i) : i = 1, 2, 3, \ldots,\} = \{\beta_{n-1}(g_n(\cdots(g(i)))\cdots)\}$$

of B.

Let $\gamma : \mathbb{N}\backslash\{0\} \to B$ be defined as,

$$\begin{array}{rcl}\gamma(1) & = & \beta_1(1)\\ \gamma(2) & = & \beta_2(2)\\ \gamma(3) & = & \beta_3(3)\\ & \vdots & \\ \gamma(i) & = & \beta_i(i)\\ & \vdots & \end{array}$$

Then $A = \{\gamma(i) : i = 1, 2, 3, \ldots, \}$ is a subsequence of B. We claim that the subsequence, A, converges.

For any $m \in \mathbb{N}$, $\gamma[T_m] \subseteq \beta_m[T_m]$. So $\pi_m(\gamma(m)) = \pi_m(\beta_m(m)) = p_m$. This means that $\{\pi_j(\gamma(i) : i = 1, 2, 3, \ldots, \}$ converges for each $j \in \mathbb{N}$. By corollary 13.20, $\{\gamma(i)\} \to \{p_j : j \in \mathbb{N}\}$. Since $B = \{\beta(i) : i = 1, 2, 3, \ldots, \}$ has a converging subsequence, $A = \{\gamma(i) : i = 1, 2, 3, \ldots, \}$, then B has an accumulation point and so the product space, S, is countably compact.

Example 1. Consider the set $S = [0, \omega_1]$, of all countable ordinals union the singleton set, $\{\omega_1\}$, where ω_1 is the first uncountable ordinal. The Hausdorff topology on S is generated by the base, $\mathscr{B}$, for open sets defined as, $\mathscr{B} = \{(\alpha, \beta] : \alpha < \beta\}$.

a) Show that S is compact.

b) Show that the subspace, $T = [0, \omega_1)$, of S is not compact but is, nevertheless, countably compact.

Solution : Given: $S = [0, \omega_1]$ and $T = [0, \omega_1)$.

a) We will use the formal definition of compactness. The technique is similar as the one used to show that closed and bounded intervals of $\mathbb{R}$ are compact. Let $\mathscr{U} = \{B_i : i \in I\}$ be an open cover of S. We are required to show that S has a finite subcover.

Let $V = \{u : [0, u]$ has a finite subcover$\}$.[1] Suppose $\omega_1 \notin V$. Since the ordinal space, S, is well-ordered every non-empty subset has a least element. Let k be the least element in $S\backslash V$. Then $[0, k]$ does not have a finite subcover. There is some element, $B_j \in \mathscr{U}$, which contains k. Then there exists some $m \in S$ such that the basic open neighbourhood of k, $(m, k] \subseteq B_j$. But $[0, m]$ has a finite subcover, say

[1] The set S is well-ordered by $\leq$.

$\mathscr{U}_F \subset \mathscr{U}$. Then $\mathscr{U}_F \cup \{B_j\}$ is a finite subcover of $[0, k]$. This is a contradiction. So ω_1 belongs to V. Hence $S = [0, \omega_1]$ is compact.

b) Consider $T = [0, \omega_1)$, a subspace of compact S. It is not closed seeing that it does not contain the boundary point, ω_1 (since every basic open neighbourhood of ω_1 intersects T). Since compact subsets of Hausdorff spaces are closed, then $T = [0, \omega_1)$ is *not compact*.

We claim that T is countably compact. Suppose $A = \{x_i : i = 1, 2, 3, \ldots, \}$ is a sequence in T. By a characterization of "countably compact" it suffices to show that the sequence has an accumulation point in T.

Since no countably infinite subset has a sequence which reaches the first uncountable ordinal, ω_1, there must exist $k \in S$ such that $A \subseteq [0, k]$. Since $[0, k]$ is a closed subset of the compact space, $[0, \omega_1]$, then $[0, k]$ is itself compact. We know that, since $[0, k]$ is compact, every net in $[0, k]$ has an accumulation point in $[0, k]$ (this is a characterization of the compact property). Then the sequence A must have an accumulation point, say p in $[0, k]$. Since $p \in [0, k] \subset [0, \omega_1)$, then every sequence in T has an accumulation point in T. Since this is characterization of the countable compactness property, T is countably compact.

15.4 Countably compact versus compact

In the following results we see how the countably compact property compares with the compact property in certain spaces.

Theorem 15.7 Suppose S is a second countable topological space. Then S is compact if and only if S is countably compact.

Proof: Given: The space S is a second countable topological space.

($\Rightarrow$) It is always true that, if S is compact then S is countably compact.

($\Leftarrow$) Suppose S is countably compact. Let $\mathscr{U}$ be an open cover of S. Since S is second countable then S has a countable open base, $\mathscr{B}$.

Let x be a point in S. Since $\mathscr{U}$ is an open cover, there exist $U_x \in \mathscr{U}$ which contains x. Since $\mathscr{B}$ is a base for open sets, there exists in $\mathscr{B}$, at least one B_x such that $x \in B_x \subseteq U_x$. For each $p \in S$, let

$$\mathscr{A}_p = \{B_p \in \mathscr{B} : p \in B_p \subseteq U_p\}$$

Let $\mathscr{A} = \cup\{\mathscr{A}_p : p \in S\}$.

Since S is second countable, then $\mathscr{A}$ is a countable set, say $\{B_i : i \in \mathbb{N}\backslash\{0\}\}$. For each $i \in \mathbb{N}\backslash\{0\}$, $p \in B_i \subseteq U_p$, for some p. For each i, choose exactly one U_{p_i} such that

$B_i \subseteq U_{p_i}$. Then $\mathscr{V} = \{U_{p_i} : i \in \mathbb{N}\backslash\{0\}\}$ forms a countable subcover of S.

Since S is countably compact, then $\mathscr{V}$ contains a finite subcover $\mathscr{W} = \{U_{p_i} : i \in F\}$ of S. We have shown that $\mathscr{U}$ contains a finite subcover, $\mathscr{W}$, of S. So the second countable property combined with the countably compact property on S implies S is compact.

Theorem 15.8 Those metric spaces which are countably compact are compact metric spaces.

Proof: Suppose (S, ρ) is a countably compact metric space. It suffices to show that S is separable since, by theorem 5.11, this implies S is second countable and, by theorem 15.7, second countable countably compact spaces are compact.

Let $\varepsilon > 0$. Recall that since S is countably compact every infinite subset V has a cluster point.

Claim 1: There exists a finite set $\{x_i : i \in F\} \subseteq S$ such that $S \subseteq \cup\{B_\varepsilon(x_i) : i \in F\}$.

Proof of claim: Suppose not. Then there is some $\delta > 0$ such that $S \not\subseteq \cup\{B_\delta(x_i) : i \in F\}$ for all finite sets F. Suppose $y_0 \in S$. If $B_\delta(y_0)$ does not cover S, then there is $y_1 \in S$ such that $\rho(y_0, y_1) \geq \delta$. Inductively, suppose $\{y_i : i = 1, 2, 3, \ldots, k\}$ are such that $S \not\subseteq \cup\{B_\delta(y_i) : i = 0, 1, 2, 3, \ldots, k\}$. then there exists

$$y_{k+1} \in S\backslash\cup\{B_\delta(y_i) : i = 0, 1, 2, 3, \ldots, k\}$$

Then, we can construct in this way, the infinite subset, $V = \{y_i : i \in \mathbb{N}\}$, of S. Then V has no cluster point, contradicting the countable compactness property. This establishes the claim.

Claim 2: The metric space (S, ρ) is separable.

Proof of claim: Let $n \in \mathbb{N}\backslash\{0\}$. By claim 1, for this value, n, we can construct, $\{y_{(i,n)} : i = 1, 2, 3, \ldots, k_n\}$ such that, if $S \subseteq B_n = \cup\{B_{1/n}(y_{(i,n)}) : i = 1, 2, 3, \ldots, k_n\}$. By constructing in this way the sets, $\{B_n : n \in \mathbb{N}\backslash\{0\}\}$, we extract the subset,

$$D = \{y_{(i,n)} : i = 1, 2, 3, \ldots k_n, n = 1, 2, 3, \ldots\}$$

where $S \subseteq B_n$ for each $n = 1, 2, 3, \ldots$. Then D is easily seen to be dense in S. So S is separable, as claimed.

By theorems 15.7 and 5.10, S is compact.

Example 2. Show that the space $[0, \omega_1)$ is non-metrizable.

Solution : We have shown in the previous example that $[0, \omega_1)$ is countably compact but not compact. The above theorem then guarantees that $[0, \omega_1)$ is a non-metrizable topological space.

Many of the results above allow us to confirm that the real line, equipped with the usual topology, is not countably compact. One of the characterization states that any sequence will have an accumulation point. We already mentioned that the sequence constructed from the natural numbers has no accumulation point. One could also raise the argument that, since $\mathbb{R}$ is known to be second countable, then, if $\mathbb{R}$ was countably compact, it would have to be compact. In the theorem below we show that continuous real-valued functions on countably compact spaces are bounded functions. If $\mathbb{R}$ was countably compact, then every real-valued continuous function on $\mathbb{R}$ would have to be bounded, which is not the case.

Theorem 15.9 Let S be a countably compact topological space. If f is a real-valued continuous function on S, then $f[S]$ is compact.[1]

Proof : Given: The space S is a countably compact topological space and $f[S]$ is the continuous image of S in $\mathbb{R}$, equipped with the usual topology.

Then, by theorem 15.5, $f[S]$ is countably compact in $\mathbb{R}$. In the example on page 77, it is shown that the real line, $\mathbb{R}$, equipped with the usual topology is second countable. In theorem 5.13, it is shown that the second countable property is hereditary. So $f[S]$ is second countable in $\mathbb{R}$. Then by theorem 15.7, $f[S]$ is compact.

Concepts review:

1. Define a countably compact space.
2. Give three characterizations of the countable compactness property.
3. State the Bolzano-Weierstrass property.
4. Are closed subsets of countably compact spaces necessarily countably compact?
5. What can we say about continuous images of countably compact spaces?

[1] Later in the text we will refer to those spaces, S, in which all continuous real-valued functions are bounded on S as being *pseudocompact spaces*. This theorem can then correctly be paraphrased as "All countably compact Hausdorff spaces are pseudocompact".

6. Suppose S is the product of countably compact spaces. If we want some guarantee that S be countably compact, what conditions must be satisfied?

7. Give an example of a space which is countably compact but not compact.

8. Identify topological spaces in which the compact property is equivalent to the countable compactness property.

9. What can we say about continuous real-valued functions on a countably compact space.

EXERCISES

1. Show that a space S is countably compact if and only if whenever a family, $\mathscr{F} = \{F_i : i \in \mathbb{N}\}$ of closed non-empty sets in S, is such that $F_{i+1} \subseteq F_i$ for all i, then $\cap\{F_i : i \in \mathbb{N}\}$ is non-empty.

2. Suppose $f : S \to T$ is a one-to-one continuous function mapping a countably compact space, S, onto a first countable space, T. Show that T is a homeomorphic image of S.

---

16 / Lindelöf spaces.

Summary. *We will now investigate a property which can be seen as a weak relative of the compact property. It is called the Lindelöf property. We will formally define it and study its characterizations and basic properties. We compare its characteristics to those of the countable compactness property. Finally, we will study some examples.*

16.1 Introduction.

There are various ways to weaken the "open cover" property used to described compactness of a space. The "countably compact" property applied to those spaces in which every *countable* open cover has a finite subcover. As we have seen, for many large spaces, the difference between countable compactness and compactness can be quite significant. There is another approach we can use to weaken the compactness property. We can require that arbitrarily large open covers only have a countable subcover (rather than the stricter "finite subcover"). This property is called *Lindelöf.* Some authors would hesitate referring to the Lindelöf property as a form of compactness, since, when most of us think the compact property, we visualize some confined closed topological space which seems tightly wound-up in various place. The Lindelöf property suggest a space which is much "looser". We included it in this part of the text because it often appears in the literature along with other weaker compact properties. Furthermore, it shares with the compact property a property by which we can reduce the size of arbitrarily large open covers while all other properties of the space remain the same. We will see that this property may be of interest or can serve as a useful tool to solve other questions.

16.2 The definition.

The following definition applies to arbitrary topological spaces. Some text define the *Lindelöf property* only for regular spaces, possibly because it is in such spaces that the Lindelöf property is most commonly studied.

Definition 16.1 A topological space, S, is said to be *Lindelöf*, or satisfies the *Lindelöf property*, if every open cover of S has a countable subcover.[1]

[1]Named after the Finnish mathematician Ernst Lindelöf (1870 - 1946). He made contributions to the fields of real analysis, complex analysis and general topology.

Obviously, since finite sets are by definition countable, if an open cover has a finite subcover then this cover has a countable subcover; so we can say that, ...

Every compact space is Lindelöf.

Also, by combining the Lindelöf property with the countably compac property, we see that the Lindelöf property reduces an open cover to a countable one, and the countable compact property reduces the countable open cover to a finite one. So, ...

Every countably compact Lindelöf space is a compact space.

There are other properties (other than the Lindolöf property itself) which guarantee that an open cover will have a countable subcover.

Theorem 16.2 If S is second countable then S must be Lindelöf.

Proof : Given: That S is a second countable space and $\mathscr{U}$ is an open cover on S.

If $p \in S$, there exist $V_p \in \mathscr{U}$ which contains p. By hypothesis, S has a countable open base, $\mathscr{B}$, for S. Then, we can choose, from $\mathscr{B}$, at least one B_p such that $p \in B_p \subseteq V_p$. Let

$$\mathscr{A}_p = \{B_p \in \mathscr{B} : p \in B_p \subseteq V_p\}$$

Let $\mathscr{A} = \cup\{\mathscr{A}_p : p \in S\}$.

Since S is second countable, then $\mathscr{A}$ can be reduced to a countable subfamily, $\mathscr{A}^*$, say $\mathscr{A}^* = \{B_i : i \in \mathbb{N}\backslash\{0\}\}$, without sacrificing its open base property. For each $i \in \mathbb{N}\backslash\{0\}$, $p \in B_i \subseteq V_p$, for some p. For each i, choose exactly one V_{p_i} such that $B_i \subseteq V_{p_i}$. Then $\mathscr{V} = \{V_{p_i} : i \in \mathbb{N}\backslash\{0\}\}$ forms a countable subcover of S. So, if S is second countable, then S must be Lindelöf.

16.3 Characterizations.

Those who only work with metric spaces or metrizable topological spaces will notice, in the following characterizations, that the introduction of the Lindelöf property does not provide anything new in their corner of the topological universe, since, as we shall see, in metric spaces, the Lindelöf property is equivalent to the second countable property as well as the separable property. Also, notice how, in the first characterization, the word segment, "finite intersection property", found in the characterization of compactness is simply replaced with the word segment, "countable intersection property".

We say that the family, $\mathscr{F} = \{F_i : i \in I\}$, satisfies the *countable intersection property* (CIP) if, for any countable subfamily, $\mathscr{F}_{\mathbb{N}} = \{F_{i_j} : j \in \mathbb{N}\}$, of $\mathscr{F}$,

$$\cap\{F : F \in \mathscr{F}_{\mathbb{N}}\} \neq \varnothing$$

Theorem 16.3 Let S be a topological space.

a) The space, S, is Lindelöf if and only if each filter of closed sets with the *countable intersection property* has non-empty intersection.

b) Suppose (S, ρ) is a metric space. Then the following are equivalent.

   i. The space S is Lindelöf.
   ii. The space S is second countable.
   iii. The space S is separable.

c) Suppose S is a regular space. Then the following are equivalent.

   i. The space S is Lindelöf.
   ii. If each open cover, $\mathscr{U} = \{U_i : i \in I\}$, of S has a subfamily, $\{U_{i_j} : j \in \mathbb{N}\}$, such that $\{\text{cl}_S U_{i_j} : j \in \mathbb{N}\}$ covers S, then $\{U_{i_j} : j \in \mathbb{N}\}$ covers S.
   iii. Every filter of open sets in S with the countable intersection property has an accumulation point.

Proof: We are given that S is a topological space.

a) We are given that $\mathscr{F} = \{F_i : i \in I\}$ represents a filter of closed sets which satisfies the countable intersection property. That is,

$$\cap\{F_{i_j} : j \in \mathbb{N}\} \neq \varnothing \qquad (*)$$

for any countable subfamily, $\{F_{i_j} : j \in \mathbb{N}\} \subseteq \mathscr{F}$.

$(\Rightarrow)$ Suppose S is Lindelöf and $\mathscr{F}$ satisfies the CIP. We are required to show that $\cap\{F_i : i \in I\} \neq \varnothing$.

Suppose not. That is, suppose $\cap\{F_i : i \in I\} = \varnothing$. Then $\{S \setminus F_i : i \in I\}$ is an open cover of S. Since S is Lindelöf, $\{S \setminus F_{i_j} : j \in \mathbb{N}\}$ forms a countable subcover. This means, $\cap\{F_{i_j} : j \in \mathbb{N}\} = \varnothing$. This contradicts $(*)$. So $\cap\{F_i : i \in I\} \neq \varnothing$, as required.

$(\Leftarrow)$ Suppose $\mathscr{U} = \{U_i : i \in I\}$ is an open cover of S. Let $\mathscr{F} = \{S \setminus U_i : i \in I\}$. We are required to show that $\mathscr{U}$ has a countable subcover.
Suppose not. That is, suppose that, for any countable subfamily, $\{U_{i_j} : j \in \mathbb{N}\}$, $\cap\{S \setminus U_{i_j} : j \in \mathbb{N}\} \neq \varnothing$. Then $\mathscr{F}$ satisfies the CIP. Then $\cap\{S \setminus U_i : i \in I\} \neq \varnothing$. This contradicts that $\mathscr{U}$ is an open cover. So $\mathscr{U}$ has a countable subcover. This means that S is Lindelöf.

b) We are given that S a metric space.

(i $\Leftrightarrow$ ii) For $\Leftarrow$, by theorem 16.2, second countable spaces are Lindelöf spaces, always. We are done with $\Leftarrow$.

For $\Rightarrow$, suppose S is a Lindelöf space. We are required to show that S has a countable base for open sets.

For each $i \in \mathbb{N}\backslash\{0\}$, we construct an open cover, $\mathscr{U}_i = \{B_{1/i}(u) : u \in S\}$, of S. By hypothesis, each $\mathscr{U}_i$ has a countable subcover, say

$$\mathscr{B}_i = \{B_{1/i}(u_{i_j}) : u_{i_j} \in S,\ j \in \mathbb{N}\}$$

We claim that the countable family of open sets, $\mathscr{B} = \cup\{\mathscr{B}_i : i \in \mathbb{N}\backslash\{0\}\}$, is a base for open sets of S. That is, every open subset is the union of elements from $\mathscr{B}$.

Proof of claim : Suppose U is an open neighbourhood of $p \in S$. We are required to show that p belongs to some element, B, of $\mathscr{B}$ such that $B \subseteq U$. Then there exists some natural number, k, such that $B_{1/k}(p) \subseteq U$. Now, $\mathscr{B}_{3k}$ has been hypothesized to be an open cover of S. So there is some $v \in S$, such that $p \in B_{1/3k}(v)$. If $x \in B_{1/3k}(v)$, $\rho(x,p) < \frac{2}{3k} < \frac{1}{k}$. So $x \in B_{1/k}(p) \subseteq U$. Then $p \in B_{1/3k}(v) \subseteq U$ where $B_{1/3k}(v)$ is an element of $\mathscr{B}$. Then $\mathscr{B}$ is a countable base for open sets, as claimed.

So S is second countable, as required. We are done with $\Rightarrow$.

(ii $\Leftrightarrow$ iii) For $\Leftarrow$, if S a separable metric space, by theorem 5.11, S is second countable.

For $\Rightarrow$, suppose S is second countable. Then, by theorem 5.10, S is separable, always.

c) We are given that S is a regular topological space.

(i $\Leftrightarrow$ ii) For $\Rightarrow$, suppose S is Lindelöf. Then, trivially, statement ii) holds true.
For $\Leftarrow$, we are given that, if $\mathscr{V} = \{U_i : i \in I\}$ is an open cover, then $\{\mathrm{cl}_S U_{i_j} : j \in \mathbb{N}\}$ covers S for some countable subfamily, $\{U_{i_j} : j \in \mathbb{N}\}$, of $\mathscr{V}$.

We claim that $\{U_{i_j} : j \in \mathbb{N}\}$ must also be a subcover.

Proof of claim: Let $y \in S$. Then there exist U_y in the open cover, $\mathscr{V}$, which contains y. By regularity, there exists an open neighbourhood, B_y, of y such that $y \in B_y \subseteq \mathrm{cl}_S B_y \subseteq U_y$. Then $\{B_y : y \in S\}$ forms an open cover of S. By hypothesis, we have $\{\mathrm{cl}_S B_{y_j} : j \in \mathbb{N}\}$ covers S. Since $\mathrm{cl}_S B_{y_j} \subseteq U_{y_j}$ for each j, then $\{U_{y_j} : j \in \mathbb{N}\}$ covers S, as claimed.

We conclude that S is Lindelöf. We are done for $\Leftarrow$.

(i $\Leftrightarrow$ iii) For $\Rightarrow$, suppose S is Lindelöf.

Let $\mathscr{U} = \{U_i : i \in I\}$ be a filter base of open sets in S which satisfies the countable intersection property.

We are required to show that $\cap\{\text{cl}_S U_i : i \in I\}$ is non-empty.

Consider the family $\mathscr{V} = \{\text{cl}_S U_i : i \in I\}$. Since $\mathscr{U}$ satisfies the countable intersection property then so does $\mathscr{V}$. Then $\mathscr{V}$ is a filter of closed sets in S. Since S is Lindelöf, by part a), $\cap\{\text{cl}_S U_i : i \in I\}$ is non-empty. If $p \in \cap\{\text{cl}_S U_i : i \in I\}$, p is an accumulation point of the filter of open sets, $\mathscr{U}$.

For $\Leftarrow$, we are given that filters of open sets in S which satisfy the countable intersection property have an accumulation point.

Suppose $\mathscr{U} = \{U_i : i \in I\}$ is an open cover of S. We are required to show that $\mathscr{U}$ has a countable subcover, $\{U_{i_j} : j \in \mathbb{N}\}$.

To obtain a contradiction, let us suppose $\mathscr{U}$ does not have a countable subcover.

Let $x \in S$. Then there exists U_x in $\mathscr{U}$ such that $x \in U_x$. By regularity, there exists W_x such that $x \in W_x \subseteq \text{cl}_S W_x \subseteq U_x$. Then $\mathscr{W} = \{W_x : x \in S\}$ as well as

$$\mathscr{W}^* = \{\text{cl}_S W_x : x \in S\}$$

cover S. Then, for any countable subfamily, $\{\text{cl}_S W_{x_j} : j \in \mathbb{N}\}$, of $\mathscr{W}^*$,

$$\cap\{S \backslash \text{cl}_S W_{x_j} : j \in \mathbb{N}\}$$

is non-empty. Then, $\{S \backslash \text{cl}_S W_x : x \in S\}$ is a family of open sets which satisfies the countable intersection property. By hypothesis, it must have an accumulation point p. This means $p \in \cap\{\text{cl}_S[S \backslash \text{cl}_S W_x] : x \in S\}$. But $\text{cl}_S[S \backslash \text{cl}_S W_x] = S \backslash W_x$. Then $p \in \cap\{S \backslash W_x] : x \in S\}$. Then p is not in any W_x, contradicting that $\{W_x : x \in S\}$ is an open cover of S. The source of the contradiction is the assumption that $\mathscr{U}$ does not have a countable subcover.

So $\mathscr{U}$ has a countable subcover. We conclude that S is Lindelöf.

Example 1. Since we have shown in chapter five that the set of all real numbers, equipped with the usual topology, is both second countable and separable, by part b) of the above characterization theorem, given that $\mathbb{R}$ is metrizable, we can conclude that $\mathbb{R}$ is a Lindelöf space. That is, any open cover of $\mathbb{R}$, has a countable subcover.

Example 2. The real numbers, $(\mathbb{R}, \tau_S)$, equipped with the upper limit topology (Sorgenfrey line) has been shown to not have a countable open base and so is not second countable. However, as we will soon see, it is Lindelöf.

We claim that $(\mathbb{R}, \tau_S)$ is Lindelöf. To show this, we will use the same technique as the one used to show that closed bounded intervals in $\mathbb{R}$ are compact.

Let $\mathscr{U}$ be an open cover of $\mathbb{R}$. We will fix $n \in \mathbb{N}$. Let $S_n = \{[u, n] : [u, n]$ has a countable subcover$\}$.

Let $k_n = \inf\{u : [u, n] \in S_n\}$. We claim that $k_n \notin \mathbb{R}$. Suppose $k_n \in \mathbb{R}$. Then $[k_n, n]$ has a countable subcover, $\mathscr{U}_n = \{U_i : i \in \mathbb{N}\}$. This means that $k_n \in U_j$, for some $j \in \mathbb{N}$. Since U_j is open, there is some $\varepsilon > 0$ such that $k_n \in (k_n - \varepsilon, k_n] \subseteq U_j$. Then $[k_n - \varepsilon/2, n]$ has a countable subcover contradicting the definition of k_n. So, $(-\infty, n]$ has a countable subcover, $\mathscr{U}_n$, as claimed. This is independent of the value of n. Then $\cup\{\mathscr{U}_n : n \in \mathbb{N}\}$ is a countable subcover of $\cup\{(-\infty, n] : n \in \mathbb{N}\} = \mathbb{R}$. So $\mathbb{R}$ is Lindelöf.

Notice that $(\mathbb{R}, \tau_S)$ has been found to be Lindelöf in spite of it not being second countable. Does this contradict theorem 16.3, part b)? The answer is "No it doesn't". It simply guarantees that $(\mathbb{R}, \tau_S)$ is not metrizable.

Example 3. In the example on page 256, we showed that the ordinal space, $S = [0, \omega_1)$, is countably compact but not compact. Then it cannot be Lindelöf, for, if it was both countably compact and Lindelöf, it would have to be compact.

16.4 Two invariance theorems for the Lindelöf property.

Arbitrary subspaces generally do not inherit the Lindelöf property from its superset. Also, products of Lindelöf spaces need not be Lindelöf. But closed subspaces do inherit the Lindelöf property from its superset.

We will also show that the Lindelöf property is carried over by continuous functions.

Theorem 16.4 If S is Lindelöf and F is a closed subset of S then F is Lindelöf.

Proof : Given: That S is Lindelöf and F is closed in S.

Let $\mathscr{U} = \{U_i : i \in I\}$ be a family of open sets in S which covers F. Then $\mathscr{U}^* = \mathscr{U} \cup \{S \setminus F\}$ covers S. Since S is Lindelöf then $\mathscr{U}^*$ has a countable subcover $\{U_{i_j} : j \in \mathbb{N}\} \cup \{S \setminus F\}$. Then $\{U_{i_j} : j \in \mathbb{N}\}$ is a countable subcover of F. So F is Lindelöf.

Theorem 16.5 If S is Lindelöf and $f : S \to T$ is continuous and onto T, then T is Lindelöf.

Proof : Given: That S is Lindelöf and $f : S \to T$ is continuous and onto T.

Let $\mathscr{U} = \{U_i : i \in I\}$ be a family of open sets which covers T. Then $\mathscr{V} = \{f^{\leftarrow}[U_i] : i \in I\}$ covers S. Then $\mathscr{V}$ has a countable subcover, $\{f^{\leftarrow}[U_{i_j}] : j \in \mathbb{N}\}$, of S. Then $\{f[f^{\leftarrow}[U_{i_j}]] : j \in \mathbb{N}\} = \{U_{i_j} : j \in \mathbb{N}\}$ is a countable subcover of T. So T is Lindelöf.

16.5 Another Lindelöf space property.

We now show that regular spaces that are equipped with the Lindelöf property are in fact normal spaces.

Theorem 16.6 If S is both regular and Lindelöf then it is normal.

Proof: Given: That S is regular and Lindelöf.

Let F and K be disjoint closed sets. If $x \in F$ and $y \in K$, then, by regularity, there exists open B_x such that $\mathrm{cl}_S B_x \cap K = \varnothing$ and open D_y such that $\mathrm{cl}_S D_y \cap F = \varnothing$. Then

$$\begin{aligned} \mathscr{B}_F &= \{B_x : x \in F\} \\ \mathscr{D}_K &= \{D_y : y \in K\} \end{aligned}$$

form open covers of F and K, respectively. Since S is Lindelöf then so are the closed subsets F and K. So we can obtain

$$\begin{aligned} \mathscr{B}_F{}^* &= \{B_{x_i} : i \in \mathbb{N}\} \\ \mathscr{D}_K{}^* &= \{D_{y_i} : i \in \mathbb{N}\} \end{aligned}$$

as countable subcovers of F and K, respectively.

We now inductively construct disjoint open neighbourhoods of F and K:

For	$F_1 = B_{x_1}$	let $K_1 = D_{y_1} \backslash \mathrm{cl}_S F_1$
For $i \in \mathbb{N}$ if	$F_i = B_{x_i} \backslash \mathrm{cl}_S(K_1 \cup \cdots \cup K_{i-1})$	let $K_i = D_{y_i} \backslash \mathrm{cl}_S(F_1 \cup \cdots \cup F_i)$

Then $\cup\{F_i : i \in \mathbb{N}\}$ and $\cup\{K_i : i \in \mathbb{N}\}$ form disjoint open neighbourhoods of F and K, respectively. So S is normal.

Concepts review:

1. Define the Lindelöf property on a topological space.
2. What kind of space is obtained if we combine countable compactness with the Lindelöf property?

3. How does a space with the second countable property compare with a Lindelöf space?
4. Give a characterization of the Lindelöf property in terms of a filter of closed sets.
5. In a metric space, state two properties each of which is equivalent to the Lindelöf property.
6. In a regular space, state two properties which are equivalent to the Lindelöf property.
7. Is there a quick argument you can use to conclude that $\mathbb{R}$ is Lindelöf, based on the theory developed in the chapter.
8. What can you say about subspaces of Lindelöf spaces?
9. What about continuous images of Lindelöf spaces.
10. In what way does the Lindelöf property enhance a regular space?
11. Provide examples of Lindelöf and non-Lindelöf spaces.

EXERCISES

1. Suppose that the space S is such that it can be expressed as a countable union of compact spaces. Show that S is Lindelöf.
2. Suppose S is a topological space. We say that the open cover, $\mathscr{U} = \{U_i : i \in I\}$, refines (or is a refinement of) the open cover, $\mathscr{V} = \{V_i : i \in I\}$, if for every V_i there is some U_j such that $U_j \subseteq V_i$. Show that S is Lindelöf if and only if every open cover of S has a countable refinement.
3. Recall that a T_1-space, S, is perfectly normal if and only if, for any pair of non-empty disjoint close subsets, F and K, there exists a continuous function $f : S \to [0, 1]$ such that $F = f^{\leftarrow}(0)$ and $K = f^{\leftarrow}(1)$. Suppose S is a regular space such that each of its open subsets is Lindelöf. Show that S is perfectly normal.

---

17 / Sequentially compact and pseudocompact spaces.

Summary. *In this section we define the sequentially compact property and determine in which precise circumstances it is just another way of referring to the compactness property. At the same time we outline those conditions under which a compact space is not sequentially compact. To illustrate this we provide various examples. We also introduce the concept of a pseudocompact space and discuss its properties.*

17.1 Sequentially compact spaces

For many readers sequentially compact is just another way of saying "compact" by using two words instead of one. In many cases, this is fine. As long as we keep in mind that it would be impossible to produce an entirely correct proof of the two statements "sequentially compact spaces are compact" or "compact spaces are sequentially compact" as stated. There are counterexamples for each. We will begin by a formal definition and then examine cases where the sequential compact spaces are equivalent to compact ones. Based on our experience with countably compact spaces we expect that countability statements will, somehow, be involved.

Definition 17.1 Let S be a topological space. We say that a subspace, T, of S is *sequentially compact* if and only if every sequence in T has a subsequence which converges in T.

Theorem 17.2 Let S be a topological space.

a) If S is second countable, then the following are equivalent.

   i. The space S is compact.
   ii. The space S is countably compact.
   iii. The space S is sequentially compact

b) If S is metric, then the following are equivalent.

   i. The space S is compact.
   ii. The space S is countably compact.
   iii. The space S is sequentially compact

c) If $T \subseteq S = \mathbb{R}^n$ the following are equivalent.

i. The subspace T is compact.

ii. The subspace T is closed and bounded.

iii. The subspace T is sequentially compact

Proof: We are given that S is a topological space.

a) We are given that S is second countable.
(i $\Leftrightarrow$ ii) In second countable spaces, the equivalence of compactness and countable compactness is proved in theorem 15.7.
(ii $\Leftrightarrow$ iii) Recall that second countable spaces are first countable, always.

$$T \text{ ctbly cpct} \quad \Leftrightarrow \quad \text{every seq in } T \text{ has an accum'tion pt. } {\scriptstyle (\text{Thm } 15.2)}$$

In a first countable space T,

$$p = \text{accum'tion point of a seq} \quad \Leftrightarrow \quad p = \text{limit point of a subseq. } {\scriptstyle (\text{Thm } 11.5)}$$

So, in a second countable space,

$$T \rightarrow \text{sequentially compact} \quad \Leftrightarrow \quad T \rightarrow \text{countably compact}$$

b) We are given that S is a metric space.
(i $\Leftrightarrow$ ii) In theorem 15.8, it is shown that in metric space, compactness and countable compactness are equivalent.
(ii $\Leftrightarrow$ iii) Recall that metric spaces are first countable. As shown in (ii $\Leftrightarrow$ iii) of part a), in first countable spaces, countable compactness and sequential compactness are equivalent.

c) We are given that $T \subseteq S = \mathbb{R}^n$ with the usual topology.

(i $\Leftrightarrow$ iii) Since T is a metric subspace, T is compact if and only if T is sequentially compact. (By part b))

(i $\Leftrightarrow$ ii) For $\Rightarrow$. We are given that T is compact. Since S is Hausdorff, compact subspaces are closed in S, so T is closed in S.

We are left to show boundedness of T. For $u \in T$, let $f : T \to \mathbb{R}$ be defined as, $f(x) = \text{distance}(u, x)$ (known to be continuous). Since f is continuous on the compact set T, f attains a maximum value, k, at some element $v \in T$, so $k = \text{dist}(u, v) \geq \text{dist}(u, x)$ independently of the choice of x in T. So, for any $a, b \in T$, $\text{dist}(a, b) \leq \text{dist}(a, u) + \text{dist}(u, b) \leq k + k = 2k$. So T is bounded, as required.

For $\Leftarrow$. We are given that T is closed and bounded in $\mathbb{R}^n$. Since T is bounded in $\mathbb{R}^n$ there exists $k \in \mathbb{R}^+$ such that $T \subseteq [-k, k]^n$. Since $[-k, k]$ is compact then so is its product, $\prod_{i=1}^{n}[-k, k]_i$. Then, since T is a closed subset of a compact space, T is compact.

Many readers may already be familiar with the following result about compact subsets of metric spaces.

Theorem 17.3 If S is a metric space and T is compact in S, then T is closed and bounded.

Proof : The proof is left as an exercise for the reader.

Note: The converse fails. Consider an infinite discrete space with metric $\rho(a, b) = 1$ if $a \neq b$.

Theorem 17.4 If S is a sequentially compact space then it is countably compact.

Proof : We are given that S is a sequentially compact space. We are required to show that it is countably compact. Suppose S is not countably compact. Then S has a countable open cover, $\mathscr{U} = \{U_i : i \in \mathbb{N}\}$, with no finite subcover. For each $n \in \mathbb{N}$, we can then choose $p_n \in S \backslash \cup \{U_i : i = 0, 1, 2, 3, \ldots, n\}$. Then the sequence, $\{p_n : n \in \mathbb{N}\}$, thus constructed, has no accumulation point, contradicting a characterization of countable compactness. So spaces which are sequentially compact are countably compact.

We have seen that, in second countable spaces as well as in metric spaces, sequential compactness is equivalent to countable compactness. But it is true in general, as we just showed, that a sequential compact is always countably compact. So any space which is not countably compact cannot be sequentially compact.

We now verify that sequential compactness is preserved by continuous functions.

Theorem 17.5 Let S be a topological space and T be a Hausdorff topological space. If $f : S \to T$ is a continuous function and S is sequentially compact then $f[S]$ is sequentially compact in T.

Proof : We are given that S is a sequentially compact space, T is Hausdorff and $f : S \to T$ is continuous. We are required to show that $f[T]$ is sequentially compact.

The proof follows directly from the definition of sequentially compact. The details are left as an exercise.

17.2 Example of a compact space which not sequentially compact.

Based on the proofs presented above, we cannot help but notice the close relationship between the sequential compact property and the countable compact property; they are even identical in some spaces we often work with. The characterizations tell us how they are different.

We studied convergence properties of $S = [0,1]^{[0,1]}$ on page 219. In that example, we saw that S is a compact space in which a sequence has an accumulation point but no subsequence converging to that point. We will revisit the example, $S = [0,1]^{[0,1]}$, to show that a compact space need not be sequentially compact.

Example 1. Let $S = [0,1]^{[0,1]}$ be equipped with the product topology. That is, we view S as $\prod_{i\in[0,1]}[0,1]_i$.

a) Show that S, when equipped with the product topology, is not first countable.
b) Show that S is compact, hence countably compact.
c) Show that, in spite of its compactness, S is not sequentially compact.

Solution : We are given that the space $S = [0,1]^{[0,1]}$, the family of all functions mapping $[0,1]$ into $[0,1]$, is equipped with the product topology. If $f \in S$, $f = \{f(i) : i \in [0,1]\} = \{x_i\}$.

a) Suppose S is first countable. That is, suppose it has a countable neighbourhood base, $\mathscr{B} = \{B_n : n \in \mathbb{N}\}$, at some point, say $f = \{x_i\} \in S$. Then each B_n is of the form
$$\cap\{\pi_i^{\leftarrow}[U_i] : i \in F_n \subseteq [0,1], \quad U_i \subseteq [0,1]\}$$
where F_n is finite and U_i is open. Then $\cup\{F_n : \mathbb{N}\}$ is a proper subset of $[0,1]$. There must exist $k \in [0,1]\backslash\cup\{F_n : \mathbb{N}\}$ such that $\pi_k[B_n] = [0,1]$ for all $n \in \mathbb{N}$. Let V be a proper open subset of $[0,1]$ which contains $x_k \in f = \{x_i\}$. Then $x_k \in \pi_k[B_n] = [0,1] \nsubseteq V$ for all n. So $\pi_k^{\leftarrow}[V]$ is an open neighbourhood of f which does not contain any B_n in $\mathscr{B}$. This contradicts that $\mathscr{B}$ is a countable neighbourhood base of $\{x_i\}$. So S is not first countable.

b) Since $[0,1]$ is compact and arbitrary products of compact sets are compact then $S = \prod_{i\in[0,1]}[0,1]_i$ is compact. Any compact set is countably compact.

c) We will consider sequences in $S = [0,1]^{[0,1]}$. Suppose $\{f_n\}$ is a sequence where, $f_n : [0,1] \to \{0,1\}$ is defined as $f_n(x)$ is the n^{th} digit in the binary expansion of

x. Suppose $\{f_n\}$ converges to f. Since convergence with respect to the product topology is pointwise, then the sequence $\{f_n\}$ converges to f if and only if $\{f_n(x) : n \in \mathbb{N}\}$ converges to $f(x)$ for each $x \in [0, 1]$.

We can choose $y \in [0, 1]$ such that $f_1(y) = 1$, $f_2(y) = 0$, $f_3(y) = 1$, $f_4(y) = 0$ and more generally $f_n(y) = 1$ if n is odd and $f_n(y) = 0$ if n is even. By pointwise convergence we must have

$$[\,\{f_n\} \to f\,] \;\Rightarrow\; [\,\{f_n(y)\} \to f(y)\,] \;\Rightarrow\; [\,\{1, 0, 1, 0, 1, 0, \cdots, \} \to f(y)\,]$$

a contradiction. So, by altering the indices we will see that, the sequence $\{f_n\}$ can have no convergent subsequence. So the compact space, $S = [0, 1]^{[0,1]}$, is not sequentially compact.

17.3 Pseudocompact spaces

We know that the continuous image of a compact space is always compact. This means that continuous real-valued functions on a compact space are always bounded. However, a space S in which the property "all real-valued continuous functions are bounded on S" need not be compact. A space which satisfies this property is said to be *pseudocompact*. We formally define this concept and briefly discuss its properties.

Definition 17.6 Let S be a topological space. If every continuous real-valued function on S is bounded in $\mathbb{R}$ then S is *pseudocompact*.

Using the newly introduced terminology above, compact spaces are pseudocompact spaces. The set of all real numbers, $\mathbb{R}$, is, of course, not pseudocompact (as witnessed by $f(x) = x^3$ on $\mathbb{R}$). We have seen in theorem 15.9, that every continuous real-valued function on a countably compact space, S, is bounded on S. So countably compact space are pseudocompact. What about sequentially compact spaces? Consider a continuous real-valued function, $f : S \to \mathbb{R}$, on a sequentially compact space S. Since S is also countably compact then f must be bounded on S. So sequentially compact spaces are pseudocompact.

We have already encountered an example of a space which is pseudocompact but not compact. We saw that the ordinal space,

$$S = [0, \omega_1)$$

is countably compact and not compact. So $S = [0, \omega_1)$ is pseudocompact but not compact.

Concepts review:

1. Define a sequentially compact space.
2. If S is a second countable or metrizable topological spaces which satisfies the sequentially compact property, name two other properties which are equivalent properties in such spaces.
3. Describe a topological space in which sequentially compact subsets are precisely the closed and bounded ones.
4. Is it true that, in all metric spaces, the compact subsets are alway closed and bounded?
5. Is it true that, in all metric spaces, the the closed and bounded subsets are always compact?
6. Given a sequentially compact property and the countably compact property, one always implies the other. Which one? Can you provide a counter example for the one that doesn't.
7. Define a "pseudocompact space"?
8. Is a sequentially compact space pseudocompact?
9. Is a countably compact space necessarily pseudocompact?
10. Are there any spaces which are pseudocompact and not compact? If so, give an example.
11. Are there any spaces which are compact and not pseudocompact? If so, give an example.

EXERCISES

1. Show that, if F is a non-empty closed subset of a sequentially compact space, then F is sequentially compact.

2. Let $f : S \to T$ denote a function mapping S into a Hausdorff space T. Show that, if f is continuous on S and S is a sequentially compact space, then $f[S]$ is sequentially compact.

3. Suppose $f : S \to \mathbb{R}$ is a continuous function mapping the sequentially compact space into $\mathbb{R}$, equipped with the usual topology. Can the function f be "onto" $\mathbb{R}$?

---

18 / Locally compact spaces.

Summary. *We will discuss another way of weakening the compact property without sacrificing many of the properties we find desirable in a topological space. It is called the locally compact property. Rather than have the compact property assigned to the whole space, we assign it on a neighbourhood of each point. After defining it formally we examine what are its main properties. Even though local compactness is usually seen along with the Hausdorff property our formal definition will be in its most general form. We have seen that all Hausdorff compact spaces are normal; however, locally compact Hausdorff spaces are simply completely regular. All metrizable spaces will also be seen to be locally compact and Hausdorff. We will also set the stage to study the topic of compactifications of locally compact Hausdorff spaces covered in depth later on.*

18.1 The locally compact property

We present a formal definition of the locally compact property. We define this property for arbitrary topological spaces. We alert the reader to the fact that many authors define local compactness for Hausdorff spaces only, probably because it simplifies its definition, and, more often then not, it is referred to in the context of Hausdorff spaces. It is a preference of this writer (who has occasionally been led astray by unwritten assumptions of a theorem statement) to define it in its most general form.

Definition 18.1 Let S be a topological space. We say that S is *locally compact* if, for every point, x, in S there is a compact set, K, such that $x \in \text{int}_S K \subseteq K$.

When a particular topological property is applied only to some unspecified neighbourhood of each point of a space rather than on the whole space, we simply precede the name of this property with the adverb "locally". In this case we refer to a compact neighbourhood.

18.2 Characterization.

Note that in the above definition, in cases where the space S is *not* Hausdorff, it may occur that the compact set is not a closed subset of S. The following characterization referring specifically to locally compact spaces which are also Hausdorff allows us to say, "there is an open set U with compact closure such that $x \in U \subseteq \text{cl}_S U$".

Theorem 18.2 Let S be a Hausdorff topological space. The space, S, is locally compact if and only if each point, x, in S has an open neighbourhood base, $\mathscr{B}_x = \{B_i : i \in I\}$, such that $\mathrm{cl}_S B_i$ is compact.

Proof : We are given that S is a topological space.

For $\Leftarrow$: If B is an open neighbourhood of a point, x in S, such that $\mathrm{cl}_S B$ is compact then, by definition, x has a compact neighbourhood. So S is locally compact.

For $\Rightarrow$: We are given that S is locally compact and Hausdorff. Let $p \in S$ and U be an open neighbourhood of p and K be any compact neighbourhood of p. Then $p \in (\mathrm{int}_S K) \cap U \subseteq K$. Since K is compact and S is Hausdorff then, by theorem 14.3 b) iii), K is a regular subspace. This means that there exists an open set V such that

$$p \in V \subseteq \mathrm{cl}_K V \subseteq \ (\mathrm{int}_S K) \cap U \subseteq K$$

Since $\mathrm{cl}_K V$ is closed in the compact set K, it must be compact. So $p \in \mathrm{int}_S \mathrm{cl}_K V \subseteq \mathrm{cl}_K V \subseteq U$. We can conclude that each point of S has an open neighbourhood base whose elements have a compact closure.

In the proof of the above theorem, note why the Hausdorff property on S is required.

The above characterization of local compactness on Hausdorff spaces allows us to quickly see that every compact Hausdorff space, S, is locally compact: Since compact Hausdorff spaces are regular we can construct a compact neighbourhood of a point, p, inside any open neighbourhood of p.

Example 1. Consider the space $\mathbb{R}$ with its usual topology. Show that $\mathbb{R}$ is locally compact.

Solution : Let $\mathscr{B}_p = \{(a, b) : a < p < b\}$ be an open neighbourhood base for the point p. For each $(a, b) \in \mathscr{B}_p$, there exists $\varepsilon > 0$, such that,

$$p \in \left(p - \frac{\varepsilon}{3},\, p + \frac{\varepsilon}{3}\right) \subseteq \left[p - \frac{\varepsilon}{3},\, p + \frac{\varepsilon}{3}\right] \subseteq (p - \varepsilon,\, p + \varepsilon) \subseteq (a, b)$$

Since the closed interval in this chain of containments is the closure of the first interval and is compact, then $\mathbb{R}$, with the usual topology, is locally compact.

Example 2. Consider the space, $\mathbb{Z}$, of all integers with the discrete topology. Show that $\mathbb{Z}$ is locally compact.

Solution : Then, if $n \in \mathbb{Z}$, and U is an open set containing n, $\text{cl}_{\mathbb{Z}}\{n\}$ is a compact neighbourhood of n, so,

$$n \in \{n\} = \text{ int}_{\mathbb{Z}}\{n\} = \text{cl}_{\mathbb{Z}}\{n\} = \{n\} \subseteq U$$

Then $\mathbb{Z}$, with the discrete topology, is locally compact. In fact, every discrete space is locally compact.

18.3 Some basic properties of local compactness.

We now examine some invariance properties. A subset of a locally compact space does not always inherit the locally compact property from its superset, as we shall soon see. The locally compact property is carried over from the domain a continuous function to its range provided the *function is an open mapping.* For a product space, the locally compact property is carried over from the factors to the product, provided all factors are locally compact and *at most* finitely many of those factors are *not* compact. We now immediately prove these facts.

Theorem 18.3 Let S be a locally compact Hausdorff topological space.

a) If F is a closed subspace of S, then F is locally compact. So a *closed* subspace inherits the locally compact property from its Hausdorff superset.
b) If U is an open subspace of S, then U is locally compact. So an *open* subspace inherits the locally compact property from its Hausdorff superset.
c) In any Hausdorff topological space, the intersection of two locally compact subspaces of S is locally compact. In particular, if S is locally compact and Hausdorff, any subset which is the intersection of an open set and a closed one is a locally compact subspace.
d) Let S be any Hausdorff topological space which contains a non-empty subspace, W. If W is locally compact, then W is the intersection of an open subset with a closed subset of S.

Proof : We are given that S is a topological space.

a) Let F be a closed subset of the locally compact Hausdorff space S. Let $x \in F$ and V be an open neighbourhood of x in S. Since S is locally compact and Hausdorff, there exist an open neighbourhood U of x such that $x \in U \subseteq \text{cl}_S U \subseteq V$.
Then $x \in U \cap F \subseteq \text{cl}_S U \cap F \subseteq V \cap F$, where $\text{cl}_S U \cap F$ is a closed subset of the compact set $\text{cl}_S U$, hence is a compact neighbourhood of x in F. So F is locally compact.
b) Let V be an open subset of the locally compact Hausdorff space S and U be an open neighbourhood of $x \in V$. Then $U \cap V$ is an open neighbourhood of x in S. Since S is locally compact and Hausdorff, there exist an open neighbourhood, W, such that $x \in W \subseteq \text{cl}_S W \subseteq U \cap V$, where $\text{cl}_S W$ is compact in V. So V is locally compact, as required.

c) We are given that S is a Hausdorff space containing the two locally compact subspaces U and V with non-empty intersection. Since S is Hausdorff so are U, V and $U \cap V$. Let $x \in U \cap V$ and Z be an open subset of S such that $x \in Z$. Then there exists an open subset, A, in U such that

$$x \in A \subseteq \mathrm{cl}_U A \subseteq Z \cap U \subseteq U$$

and an open subset, B, in V such that

$$x \in B \subseteq \mathrm{cl}_V B \subseteq Z \cap V \subseteq V$$

where $\mathrm{cl}_U A$ and $\mathrm{cl}_V B$ are compact; because of the Hausdorff property, they are closed. Then $U \cap V$ is locally compact, as required.

Combining parts a, b, and c the second part of the statement quickly follows.

d) Suppose W is a subset of a Hausdorff topological space. Suppose W is locally compact. We are required to show that W is the intersection of an open set and a closed set in S. If we can show that W is open in S then $W = W \cap \mathrm{cl}_S W$, an intersection of an open and closed subset of S, and we will be done. To show W is open this we will prove that every point in W belongs to an open subset of S, contained in W.

Let $x \in W$. Since W is locally compact and Hausdorff, S contains an open neighbourhood, A, of x such that

$$x \in A \cap W \subseteq \mathrm{cl}_W(A \cap W) = \mathrm{cl}_S(A \cap W) \cap W$$

where $\mathrm{cl}_W(A \cap W)$ is a closed and compact set in W and so $\mathrm{cl}_S(A \cap W) \cap W$ is also closed and compact in S. Since $A \cap W$ is contained in the closed set, $\mathrm{cl}_S(A \cap W) \cap W$, it must follow that $\mathrm{cl}_S(A \cap W) \subseteq \mathrm{cl}_S(A \cap W) \cap W$, and so, $\mathrm{cl}_S(A \cap W) \subseteq W$. So $x \in \mathrm{int}_S(\mathrm{cl}_S(A \cap W)) \subseteq W$. Then each point of W is contained in an open neighbourhood of S and W is open. So $W = W \cap \mathrm{cl}_S W$, as required.

Corollary 18.4 Let S be a compact Hausdorff topological space. A dense subset, D, of S is locally compact if and only if D is an open subset of S.

Proof : We are given that S is a compact Hausdorff topological space.

$(\Rightarrow)$ Suppose D is a locally compact subspace of S. Then, by the above theorem 18.3, D is an open subset of $\mathrm{cl}_S D = S$. So D is open in S.

$(\Leftarrow)$ Since S is compact it is locally compact. Suppose D is a an open dense subset of S. As shown in theorem 18.3, open subsets of locally compact sets are locally compact. So D is locally compact. We are done.

Example 3. Determine whether the set, $\mathbb{Q}$, of all rationals is locally compact or not.

Solution : The set $\mathbb{Q}$ is dense in the locally compact $\mathbb{R}$. If $\mathbb{Q}$ was locally compact in $\mathbb{R}$ it would have to be open. Knowing that every interval containing an element of $\mathbb{Q}$ contains an irrational, $\mathbb{Q}$ cannot be locally compact.

We know that the compact property is carried over from the domain to the codomain of a continuous function. A similar result holds for the locally compact property but only if the function is both continuous and open.

Theorem 18.5 Let S be a locally compact topological space and suppose T is any topological space. If $f : S \to T$ is a continuous open function mapping S into T then $f[S]$ is locally compact.

Proof: We are given that S is a locally compact topological space and $f : S \to T$ is a continuous open function.

We are required to show that $f[S]$ is locally compact. Let $u \in f[S]$ and U be any open neighbourhood of u. Suppose $v \in f^{\leftarrow}[\{u\}] \subseteq f^{\leftarrow}[U]$. Then, by hypothesis, v has a compact neighbourhood, say K, such that $v \in \text{int}_S K \subseteq K \subseteq f^{\leftarrow}[U]$. Since f is open and continuous and

$$u = f(v) \in f[\text{int}_S K] \subseteq f[K] \subseteq f[f^{\leftarrow}[U]] = U$$

where $f[\text{int}_S K]$ is open and $f[K]$ is compact, then every point in $f[S]$ has a compact neighbourhood. So $f[S]$ is locally compact.

Note that the Hausdorff property is not required in the above statement.

Theorem 18.6 Let $\{S_i : i \in I\}$ be a family of topological spaces and $S = \prod_{i \in I} S_i$ be a product space. Then S is locally compact if and only if every factor, S_i, is locally compact and at most, finitely many of the factors are *not* compact.

Proof: We are given that $\{S_i : i \in I\}$ is a family of topological spaces and $S = \prod_{i \in I} S_i$ is a product space.

($\Rightarrow$) Suppose $S = \prod_{i\in I} S_i$ is locally compact. Since every projection map is open (see page 105) then, by theorem 18.5, every factor is locally compact. We now claim that, at most, finitely many of the factors are not compact spaces.

If $u \in S$, then, since S is locally compact, there exists a compact set K in S, such that $u \in \mathrm{int}_S K \subseteq K$. Then there exists a finite subset, F, of I such that

$$u \in U = \cap\{\pi_i^{\leftarrow}[U_i] : i \in F\} \subseteq \ \mathrm{int}_S K$$

where U_i is an open subset of S_i. We claim that, if $i \notin F$ then S_i is compact. Consider $i \notin F$: Then $\pi_i[U] = S_i \subseteq \pi_i[K]$ (since $U \subseteq K$). So $S_i = \pi_i[K]$. Since π_i is continuous and K is compact then S_i is compact, for all $i \notin F$, as claimed. Done!

($\Leftarrow$) Suppose every factor, S_i, is locally compact and at most finitely many are not compact. We are required to show that the product space, S, is locally compact.

Let $\{x_i\} \in S = \prod_{i\in I} S_i$ and $U = \cap\{\pi_i^{\leftarrow}[U_i] : i \in F\}$ be a basic open neighbourhood of $\{x_i\}$. Suppose F_c is the largest subset of I, such that, if $i \in F_c$, S_i is not compact. Then, by hypothesis, F_c is finite. Let $F_1 = F \cup F_c$. Then F_1 is finite. Let $U_1 = \cap\{\pi_i^{\leftarrow}[U_i] : i \in F_1\} \subseteq U$.

We claim that $\{x_i\}$ has a compact neighbourhood, K, which is contained in U_1.

Given that, for each $i \in F_1$, S_i is locally compact, there must be a compact set, K_i, such that $x_i \in \mathrm{int}_{S_i} K_i \subseteq K_i \subseteq U_i$.

Then

$$\{x_i\} \ \in \ \cap\{\pi_i^{\leftarrow}[\mathrm{int}_S K_i] : i \in F_1\} \ \subseteq \ \cap\{\pi_i^{\leftarrow}[K_i] : i \in F_1\} \ = \ K \subseteq U_1$$

where K is a neighbourhood of $\{x_i\}$. See that, for each $i \notin F_1$, X_i is a compact factor of K, and, for each $i \in F$, K_i, is a compact factor of K. So K is a compact neighbourhood of $\{x_i\}$, as claimed. So S is locally compact.

18.4 A characterization of the Hausdorff locally compact property.

From the Embedded theorem part III, we arrived at the conclusion that "... any completely regular space can be densely embedded in a Hausdorff compact space". We would like to state a similar result involving Hausdorff locally compact spaces. That is, we will show that any locally compact Hausdorff space, S, can be embedded in a Hausdorff compact space. We will denote this specific compact space by S_ω. This compact space is constructed by adding a single point to S and then defining an appropriate topology on it.

Theorem 18.7 Let (S, τ) be a Hausdorff topological space and let ω be a point which does not belong to S. The set S_ω is defined as

$$S_\omega = S \cup \{\omega\}$$

The topology, τ_ω, on S_ω is defined as follows: Firstly, $\tau \subseteq \tau_\omega$. Secondly, if $\omega \in U \subseteq S_\omega$ and $S_\omega \backslash U$ is a compact subset of S, then $U \in \tau_\omega$.

a) Then the family of sets, τ_ω, thus defined, is a valid topology on S_ω.

b) The topological space, (S_ω, τ_ω), is a compact space.

c) The compact space, (S_ω, τ_ω), densely contains a homeomorphic copy of S.

d) The compact space, (S_ω, τ_ω), is Hausdorff if and only if S is locally compact.

Proof : We are given that (S, τ) is a Hausdorff topological space and that ω is a point which does not belong to S. Also, $S_\omega = S \cup \{\omega\}$.

a) The proof is routine and so is left as an exercise.

b) We are given (S_ω, τ_ω). Let $\mathscr{U} = \{U_i : i \in I\}$ be an open cover of S_ω.

   By hypothesis, there exists, U_k, such that $\omega \in U_k$ and $S \backslash U_k$ is compact. Then $S \backslash U_k$ has a finite subcover, $\{U_i : i \in F\}$. Then $\{U_i : i \in F\} \cup \{U_k\}$ is a finite subcover of S_ω and so S_ω is compact.

c) The identity map, $i : S \to S \cup \{\omega\}$, is easily seen to be a continuous one-to-one open map onto $i[S] = S$.

d) We are given that S_ω is compact.

   ($\Rightarrow$) Suppose S_ω is Hausdorff. Then S is a Hausdorff subspace (since the Hausdorff property is hereditary). Also, for any $x \in S$, there exists disjoint open neighbourhoods U and V such that $\omega \in U$ and $x \in V$. Then S is open and dense in S_ω. By corollary 18.4, S is locally compact.

   ($\Leftarrow$) We are given that S is a Hausdorff locally compact dense subspace of the compact space, S_ω. We are required to show that S_ω is Hausdorff. It follows immediately that distinct pairs of points in S are contained disjoint open neighbourhoods of S_ω. Suppose $p \in S$. Since S is Hausdorff and locally compact there exists an open neighbourhood U of p such that $p \in U \subseteq \mathrm{cl}_S U \subseteq S$, where $\mathrm{cl}_S U$ is a compact neighbourhood of p. By hypothesis, $S \backslash \mathrm{cl}_S U$ is an open neighbourhood of ω disjoint from U. So S_ω is Hausdorff, as required.

We now show that the local compactness property on a space S, when combined with the Hausdorff property, promotes this separation axiom to a completely regular one.

Theorem 18.8 A locally compact Hausdorff space is completely regular.

Proof: We are given that S is a Hausdorff locally compact topological space. Then S is embedded in the compact Hausdorff topological space $S_\omega = S \cup \{\omega\}$. Then by theorem 14.3, S_ω is normal and so is completely regular. The completely regular property is hereditary, so S is completely regular.

From the above result we again argue that if S is locally compact and Hausdorff it can be densely embedded in at least two essentially different Hausdorff compact spaces. We now know that S is embedded in $S_\omega = S \cup \{\omega\}$; secondly, from the Embedding theorem 14.7, we know that...

> *A locally compact Hausdorff space, S, can be densely embedded in a compact Hausdorff subspace of a cube.*

18.5 Sigma-compact spaces.

We know that the union of a countably infinite number of compact sets need not be compact. But such sets that are have properties that are worth discussing. In particular, a locally compact Hausdorff space which is a countable union of compact sets is another way of describing a Lindelöf Hausdorff space.

Definition 18.9 Let S be a topological space. We say that S is σ-compact if S is the union of countably many compact sets.

Theorem 18.10 Suppose S is a locally compact Hausdorff space. Then S is σ-compact if and only if S is Lindelöf.

Proof: We are given that S is locally compact Hausdorff topological space.

($\Leftarrow$) Suppose S is Lindelöf. Since S is locally compact and Hausdorff, we can construct an open covering of $\mathscr{B} = \{U_x : x \in S\}$ of sets with compact closures, $\{\text{cl}_S U_x : x \in S\}$. Let $\{U_{x_i} : i \in \mathbb{N}\}$ be a countable subcover of $\mathscr{B}$. Then $S = \cup\{U_{x_i} : i \in \mathbb{N}\} \subseteq \{\text{cl}_S U_{x_i} : i \in \mathbb{N}\}$. So S is σ-compact.

($\Rightarrow$) Suppose S is σ-compact. Then $S = \cup\{C_i : i \in \mathbb{N},\ C_i \text{ is compact }\}$. Since S is locally compact and Hausdorff and C_0 is compact, then there is an open neighbourhood, U_0, with compact closure, $\text{cl}_S U_0$, such that $C_0 \subseteq U_0 \subseteq \text{cl}_S U_0$. (Justifying the details are left to reader.) Inductively, and the case where each closure, $\mathrm{cl}_S U_i$'s, is compact,

$$\begin{array}{ccccc} C_1 \cup \mathrm{cl}_S U_0 & \subseteq & U_1 & \subseteq & \mathrm{cl}_S U_1 \\ C_2 \cup \mathrm{cl}_S U_1 & \subseteq & U_2 & \subseteq & \mathrm{cl}_S U_2 \\ \vdots & & \vdots & & \\ C_i \cup \mathrm{cl}_S U_{i-1} & \subseteq & U_i & \subseteq & \mathrm{cl}_S U_i \\ \vdots & & \vdots & & \end{array}$$

and setting $K_i = \mathrm{cl}_S U_i \subseteq U_{i+1} \subseteq \mathrm{cl}_S U_{i+1} = K_{i+1}$, we obtain a family of compact sets, $\{K_i : i \in \mathbb{N}\}$ where $K_{i-1} \subseteq K_i$ and $S = \cup\{K_i : i \in \mathbb{N}\}$.

Suppose $\{V_j : j \in J\}$ is an open cover of S. For each $i \in \mathbb{N}$, there exist a finite subset, F_i of J such that $K_i \subseteq \cup\{V_{j_k} : k \in F_i\}$. Then $\{V_{j_k} : k \in F_i, i \in \mathbb{N}\}$ is a countable subcover of S. So S is Lindelöf, as required.

Concepts review:

1. Define the locally compact property on a topological space.
2. State a characterization of Hausdorff locally compact spaces.
3. If S is locally compact and Hausdorff what kind of subsets are guaranteed to inherit the locally compact property?
4. If T is a locally compact subspace of the Hausdorff space, S, what property does T satisfy?
5. If D is a dense subset of a compact Hausdorff space, S, what can we say about D?
6. Provide an easy example of a non-locally compact space.
7. Are continuous images of locally compact spaces necessarily locally compact? Explain.
8. If S is a product space which is locally compact. What can say about its factors?
9. What conditions must be satisfied if we want the locally compact factors to be carried over to their product.
10. Describe the Hausdorff compact space, S_ω, which contains a dense homeomorphic copy of a locally compact Hausdorff space, S.
11. If S is a Hausdorff locally compact space, what other separation axiom does it satisfy?

12. If S is a Hausdorff locally compact space, is there another compact space, other than S_ω, which contains a dense copy of S?

13. Define a σ-compact space.

14. If S is locally compact and Hausdorff. In such a case provide a characterization of the σ-compact property.

EXERCISES

1. Suppose $f : S \to T$ is a continuous open function mapping a locally compact space, S, onto a space T. Show that any compact subspace, K, of T is the image under f of some compact subspace, F, of S.

2. Show that the real numbers equipped with the upper limit topology (Sorgenfrey line) is not locally compact.

3. A *perfect function*, $f : S \to T$, is a continuous closed and onto function which pulls back each point in T to a compact set in S. Suppose S is Hausdorff and T is locally compact. Show that $f : S \to T$ is a perfect function if and only if f pulls back compact sets in T to compact sets in S.

4. Let S be a locally compact regular space and K be a closed and compact subset of S. Suppose K is a subset of an open set, U, in S. Show that there is some compact set V, such that $F \subseteq \text{int}_S V \subseteq V \subseteq U$.

5. Show that the Moore plane is not locally compact.

6. Suppose S is a Hausdorff space which contains a dense subset, D. Suppose $x \in D$. Show that, if V is compact neighbourhood of x in D, then V is a neighbourhood of x in S.

---

19 / Paracompact topological spaces.

Summary. *We will develop in this section some familiarity with the paracompact property. After giving a formal definition we provide a few examples and discuss its invariance properties.*

19.1 Paracompact topological spaces

We now introduce a last important class of topological spaces closely related to the family of compact spaces. Its importance was discovered in the role it plays in characterizations of metrizable spaces. We begin by introducing some new terminology.

Definition 19.1 Let S be a topological space.

a) Let $\mathscr{U} = \{U_i : i \in I\}$ be a family of subsets of S. We say that $\mathscr{U}$ is a *locally finite family of sets* if each point, p in S has at least one open neighbourhood which intersects each element of a finite subfamily, $\{U_i : i \in F\}$, of $\mathscr{U}$.[1]

b) We say that $\mathscr{V} = \{V_j : j \in J\}$ *is a refinement of*, (or refines) the family, $\mathscr{U} = \{U_i : i \in I\}$, if, for every $j \in J$, there is some $i \in I$ such that $V_j \subseteq U_i$. If the elements of $\mathscr{V}$ are open, then we would more specifically say that $\mathscr{V}$ *is an open refinement* of $\mathscr{U}$.

c) Let $\mathscr{U} = \{U_i : i \in I\}$ be a family of subsets of the topological space S. We say that $\mathscr{U}$ *has a locally finite open refinement* if there is a family, $\mathscr{V} = \{V_j : j \in J\}$, of open subsets in S which both refines $\mathscr{U}$ and satisfies the locally finite property in S.

With these definitions in mind we will practice applying this terminology to familiar classes of topological spaces.

Example 1. Consider the statement, "If the space, S, is such that every open cover of S has a finite open refinement which covers S, then S is compact. " Is this true? If so prove it. If not say why.

Solution : We will try to prove this statement and see if anything can go wrong. Suppose $\mathscr{U} = \{U_i : i \in I\}$ is a family of open subsets which covers the topological space S. We are given that $\mathscr{U}$ has a finite open refinement which covers S. This means we can associate to $\mathscr{U}$ a family, $\mathscr{V} = \{V_j : j \in J\}$ of open subsets, such that

[1] The word "neighbourhood-finite" is sometimes used instead of "locally finite".

for each, $j \in J$, $V_j \subseteq U_{f(j)}$ for some $f(j) \in I$. We are also given that there exists a finite subset F of J such that $S = \cup\{V_j : j \in F\}$. Then,

$$S = \cup\{V_j : j \in F\} \subseteq \cup\{U_{f(j)} : j \in F\}$$

Then $\mathscr{U}$ has a finite subcover, $\{U_{f(j)} : j \in F\}$, of S. So S is compact in the "usual" sense. The statement is expressed in an unconventional way but is true.

Example 2. Consider the statement "If S is compact then every open cover, $\mathscr{U}$, of S, has a locally finite open refinement, $\mathscr{V}$." Is this true? Is so prove it. If not say why.

Solution : We will try to prove this statement and see if anything can go wrong. Suppose $\mathscr{U} = \{U_i : i \in I\}$ is a family of open subsets which covers the topological space S. Then, by hypothesis, $\mathscr{U}$ has a finite open subfamily, $\mathscr{U}_F = \{U_i : i \in F\}$, such that $S = \cup\{U_i : i \in F\}$. This means that for each element $p \in S$, $p \in U_i$ for $i \in K \subseteq F$ for some $K \subseteq F$. Then each element, p, of S has a neighbourhood which intersects, at most, finitely many elements of $\mathscr{U}_F$, and so, at most, finitely many elements of $\mathscr{U}$. So $\mathscr{U}$ is locally finite family of open subsets. Since, for each $U_i \in \mathscr{U}$, $U_i \subseteq U_i$, then $\mathscr{U}$ is an open refinement of $\mathscr{U}$. So every open cover, $\mathscr{U} = \{U_i : i \in I\}$ of S, has a locally finite open refinement of $\mathscr{U}$. The statement is true.

We now define the main subject of this section.

Definition 19.2 Let S be a topological space. The space S is said to be a *paracompact* space if any open cover $\mathscr{U}$ of S, has a locally finite open refinement of $\mathscr{U}$.

We alert the reader to the fact, in some books, the Hausdorff property is incorporated into the formal definition of the paracompact property, in the sense that, for these authors, all paracompact spaces are hypothesized to be Hausdorff. In this book, a paracompact space is Hausdorff only when we explicitly state it as such.

Theorem 19.3 Any compact space is a paracompact space.

Proof : This statement has been proven in the example 2 above.

Since all compact spaces are paracompact we already know of many topological spaces which are paracompact. We consider the following example of a non-compact paracompact space.

Example 3. Let S be an infinite space equipped with the discrete topology. Since $\mathscr{V} = \{\{x\} : x \in S\}$ is an open cover with no subcover, the space, S, is not compact. Show that S is paracompact.

Solution : Let $\mathscr{U} = \{U_i : i \in I\}$ be an arbitrary open cover of S. Then the family $\mathscr{V} = \{\{x\} : x \in S\}$ is a refinement of $\mathscr{U}$, since every one of its elements, $\{x\}$, is a subset of some set, U_i, in $\mathscr{U}$. The family, $\mathscr{V}$, is locally finite since, for any $x \in S$, there is a neighbourhood, $\{x\}$ of x, which intersects finitely many elements of $\mathscr{V}$. So $\mathscr{V}$ is a locally finite open refinement of $\mathscr{U}$. We can then conclude that the discrete space, S, is paracompact.

Example 4. Consider the space $S = \mathbb{R}^n$ equipped with the usual topology. We know that this metrizable space is not compact. Show that S is paracompact.

Solution : Suppose we have an open cover, $\mathscr{U} = \{U_i : i \in I\}$, of S. We are required to construct an open refinement of $\mathscr{U}$ which is locally finite.

For each $n \in \mathbb{N}\backslash\{0\}$, let $B_n(0)$ denote an open ball of radius n with center 0. Since, for each n, $\text{cl}_S B_n(0)$ is closed and bounded it is a compact subset of S. So $\text{cl}_S B_n(0)$ has a finite open cover, $\mathscr{U}_n = \{U_i : i \in F_n\} \subseteq \mathscr{U}$. For each $i \in F_n$, let

$$V_i = U_i \cap [S\backslash \text{cl}_S B_{n-1}(0)]$$

and let

$$\mathscr{V}_n = \{V_i : i \in F_n\}$$

Then, for each n, $\mathscr{V}_n$ is a refinement of $\mathscr{U}_n$ and covers $\text{cl}_S B_n(0)\backslash \text{cl}_S B_{n-1}(0)$. Furthermore,

$$\mathscr{V} = \cup\{\mathscr{V}_n : n \in \mathbb{N}\backslash\{0\}\}$$

is an open refinement of $\mathscr{U}$ which also covers S. If $p \in S$, then $p \in S\backslash \text{cl}_S B_m(0)$, for some m, and so p has a neighbourhood which intersects at most finitely elements of $\mathscr{V}_n \subseteq \mathscr{V}$. We have constructed the family, $\mathscr{V}$, which is both an open refinement of $\mathscr{U}$ which covers S and is locally finite. So $S = \mathbb{R}^n$ is paracompact.

We know that Hausdorff compact spaces are normal. We prove that Hausdorff paracompact spaces satisfy the same separation axiom.

Theorem 19.4 Let S be a Hausdorff paracompact topological space.

a) The space, S, is a regular space.

b) The space, S, is a normal space.

Proof : We are given that S is a Hausdorff paracompact topological space.

a) We are required to show that S is regular. Let H be a closed subset of S and $u \in S \backslash H$ and $x \in H$. Since S is Hausdorff, then for each $x \in H$, there exists an open neighbourhood, U_x such that $u \notin \mathrm{cl}_S U_x$. Now, $\mathscr{U} = \{U_x : x \in H\} \cup \{S \backslash H\}$ forms an open covering of S. By invoking the paracompactness property we obtain a locally finite open refinement, $\mathscr{V} = \{V_i : i \in I\} \cup \{V\}$, of $\mathscr{U}$, a cover of S, where each V_i intersects H and V refines the open set $S \backslash H$. That is, $V \subseteq S \backslash H$. Then

$$V_i \subseteq \mathrm{cl}_S V_i \subseteq \mathrm{cl}_S U_i \subseteq S \backslash \{u\}$$

Claim: $\cup\{\mathrm{cl}_S V_i : i \in I\} = \mathrm{cl}_S(\cup\{V_i : i \in I\})$.

Now, $\cup\{\mathrm{cl}_S V_i : i \in I\} \subseteq \mathrm{cl}_S(\cup\{V_i : i \in I\})$ is always true. Let $y \in \mathrm{cl}_S(\cup\{V_i : i \in I\})$. See that y has a neighbourhood which intersects at most finitely many V_i's, say $\{V_i : i \in F_y\}$. Suppose $y \in S \backslash \cup\{\mathrm{cl}_S V_i : i \in F_y\}$. Then $y \notin \mathrm{cl}_S(\cup\{V_i : i \in I\})$, a contradiction. So $\mathrm{cl}_S(\cup\{V_i : i \in I\}) \subseteq \cup\{\mathrm{cl}_S V_i : i \in I\}$, hence $\cup\{\mathrm{cl}_S V_i : i \in I\} = \mathrm{cl}_S(\cup\{V_i : i \in I\})$, as claimed.

Since $\mathrm{cl}_S(\cup\{V_i : i \in I\}) = \cup\{\mathrm{cl}_S V_i : i \in I\} \subseteq (\cup\{\mathrm{cl}_S U_i : i \in I\}) \subseteq S \backslash \{u\}$, S is regular, the property we wanted to obtain.

b) We are now required to prove that S is a normal space. Let H be a closed set in S. To prove the normal property holds replace the point u in the proof of part a) and the Hausdorff property by the regular property and mimick the steps in the proof above. The details are left to the reader.

If the compact property is carried over from the domain to the codomain of arbitrary continuous functions, this is not the case for the paracompact property. But if the function is closed and continuous it does, as the following result shows.

Theorem 19.5 Let S be a paracompact topological space and T be any space. If $f : S \to T$ is a closed and continuous function then $f[S]$ is a paracompact subspace of T.

Proof : The proof is lengthy and involved. A reasonably complete proof is found in [Eng].

Theorem 19.6 Let S be a paracompact topological space and F be a closed subset of S. Then F inherits the paracompact property from S.

Proof: We are given that S be a paracompact topological space and F is closed in S. We are required to show that F is also paracompact.

Let $\mathscr{U} = \{U_i : i \in I\}$ is an open cover of F such that each i, $U_i = V_i \cap F$, for some open subset V_i of S. We then obtain an open cover, $\mathscr{V} = \{V_i : i \in I\} \cup \{S \backslash F\}$ of S. Since S is paracompact, then there is a locally finite open refinement, $\mathscr{W} = \{W_i : i \in I\}$, of $\mathscr{V}$ which also covers S.

The family $\{W_i \cap F : W_i \cap F \neq \varnothing\}$ forms an open cover of F which is locally finite and a refinement of $\mathscr{U}$. Then F is paracompact.

Example 5. A standard, non-trivial, example of a non-paracompact space is the ordinal space, $S = [0, \omega_1)$, where ω_1 is the first uncountable ordinal. We have shown in an example on page 256 that S is countably compact but not compact and so, by theorem 15.8, S is not metrizable. Show that S is not paracompact.

Solution . Recall that the sets of all intervals of the form, $(a, b]$, is a base for the open sets in S. For each $i \in S$, let $\mathscr{V} = \{[0, i) : i \in S\}$ be an open covering of S. Suppose $\mathscr{U} = \{U_j : j \in J\}$ is an open refinement of $\mathscr{V}$. Then for each j, $U_j \subseteq [\,0, i\,)$ for some $i \in S$. Since each U_i is open there is an interval, $(\,g(i), i\,] \subseteq U_i$.

Let $I = \{i : i \in S = [\,0, \omega_1\,)\}$ be a cofinal[1] subset of $[\,0, \omega_1\,)$ and consider a corresponding family of basic open neighbourhoods, $\{(\,g(i), i\,] : i \in I\}$ where, for all $i \in I$, $g(i) < i$, and $(\,g(i), i\,] \subseteq U_i$.

We claim that, if J is a countably infinite subset of I, there exists some ordinal, z which belongs to $\cap\{(\,g(i), i\,] : i \in J\}$. If so, it would imply that every neighbourhood of z would intersect infinitely many elements of the open refinement, $\mathscr{U}$, and so $\mathscr{U}$ could not be locally finite.

Proof of claim . Suppose $\cap\{(\,g(i), i\,] : i \in J\}$ is empty.

Then, for any $z \in S$, there must be at least one $k \in J$ such that, for the interval, $(\,g(k), k\,]$, $\;z \leq g(k)$. Then, for each z, the set

$$N_z = \{i \in J : (\,g(i), i\,] \Rightarrow z \leq g(i)\,\}$$

is non-empty.

[1]The subset I is cofinal in S if for every $j \in S$, there is a $i \in I$ such that $i \geq j$.

Since S is a well-ordered set, for each z, N_z, *must* have a least element, say $i_z = i(z)$. That is, $(\, g(i_z), i_z\,] \Rightarrow z \leq g(i_z)$ and i_z is the least ordinal satisfying this property. We construct the following sequence by recursively using the function, $i : J \to J$.

$$\begin{array}{rcll}
i_0 & = & i(0) & \text{is least in } J \text{ such that } (\, g(i_0), i_0\,] \Rightarrow 0 \leq g(i_0) \\
i_1 & = & i(i_0) & \text{is least in } J \text{ such that } (\, g(i_1), i_1\,] \Rightarrow i_0 \leq g(i_1) \\
i_2 & = & i(i_1) & \text{is least in } J \text{ such that } (\, g(i_2), i_2\,] \Rightarrow i_1 \leq g(i_2) \\
i_3 & = & i(i_2) & \qquad\qquad \vdots \\
 & \vdots & & \qquad\qquad \vdots \\
i_{n-1} & = & i(i_{n-2}) & \\
i_n & = & i(i_{n-1}) & \text{is least in } J \text{ such that } (\, g(i_n), i_n\,] \Rightarrow i_{n-1} \leq g(i_n) \\
 & \vdots & &
\end{array}$$

It is important to note that, for each n, $i_{n-1} \leq g(i_n) < i_n$.

Let $q = \sup\{i_n : n \in \mathbb{N}\backslash\{0\}\}$. Then, since $i_{n-1} \leq g(i_n) < i_n$ for all n, we obtain,

$$q \leq g(q) \leq q \text{ which implies that } g(q) = q$$

Since J is a countably infinite subset of the cofinal set I, then then q must belong to I. But for all $i \in I$ it must be that $g(i) < i$. The equality $q = g(q)$ contradicts this. So we have obtained the desired contradiction.
So the open refinement $\mathscr{U}$ cannot be locally finite.
Hence $[0, \omega_1)$ cannot be paracompact.

Concepts review:

1. What does it mean to say that $\mathscr{U}$ is a *locally finite* family of subsets of a space S?
2. What does it mean to say that the family of subsets, $\mathscr{V}$, is an *open refinement* of the family $\mathscr{U}$?
3. What does it mean to say that the family of subsets, $\mathscr{V}$, is a *locally finite open refinement* of the family $\mathscr{U}$?
4. Define the paracompact property of a topological space.
5. What class of topological spaces has elements which are guaranteed to be paracompact?

6. Provide an example of a paracompact space and briefly summarize arguments which confirms your answer.

7. Provide an example of a non-paracompact space.

8. If a paracompact space is Hausdorff what other separation axioms does it satisfy?

9. If $f : S \to T$ is a function mapping the paracompact space S into T, what properties must be satisfied by f if we want to guarantee that $f[S]$ is paracompact?

10. If S is a paracompact space what kind of subsets of S will share this property?

EXERCISES

1. Show that, if a paracompact space S is countably compact, then S is compact.

2. Suppose S is a Lindelöf space and $\mathscr{U}$ is a family of subsets of S. Show that $\mathscr{U}$ is a countable family of subsets.

3. Suppose S is a regular space and $\mathscr{U} = \{F_i : i \in I\}$ is a family of paracompact closed subsets of S. Show that if $\mathscr{U}$ is locally finite then $\cup\{F_i : i \in I\}$ is paracompact.

4. Let S be a perfectly normal paracompact space. Show that any non-empty subspace is also paracompact.

Part VI

The connected property

20 / Connected spaces and properties

Summary. *In this section we formally define the topological property of connectedness while providing a few examples of connected spaces, and of spaces which are not. Continuous images of connected spaces are proven to be connected. Arbitrary unions of families of connected sets are proven to be connected provided they have at least one point in common. If the set A is connected and is dense in a set B then B is shown to be connected. In particular, closures of connected sets are connected. The connected property will be seen to be invariant over arbitrary products of connected spaces, without conditions on the number of factors. A connected component is defined as being the largest connected subspace containing a given point. Components will be seen to be closed, but not necessarily open. Properties of spaces that have an open base of connected neighbourhoods will be briefly discussed. We will also define the pathwise connected property; we will see that sets which satisfy this property are connected. But the converse does not hold true. Finally we will introduce the totally disconnected property.*

20.1 Definition.

The property of connectedness is one that is, more or less, independent of the topological properties that we have studied until now. By this, we mean that we cannot obtain this property with some combination of the others. We will define this property and develop ways of recognizing whether a space is connected or not, providing along the way, enough examples to develop some familiarity with this concept. It is fairly easy to construct mental images of connected spaces. A good way to start is to think of a single string and ask yourself "Is this string connected?" to which you would answer, "...of course!". Then take a pair of scissors and cut it, and ask, "...what about now?" to which you would intuitively answer "... well no, not anymore". Someone watching might add "Both represent the same string, but the second one differs topologically from the the first". And you would agree by adding "...yes, by their connectedness." All this is stated based on an intuitive understanding of the word "connected". Of course, intuition is just a guiding tool whose use precedes the process we call *rigourously defining a mathematical concept.* That is, producing a definition which can only be interpreted in one way and which satisfies our intuitive understanding of this property. A rigourous definition should allow us to answer more difficult questions about objects which possess this property but whose complexity defies our imagination. Once we understand the mathematical notion of connectedness we might wonder about what are the properties which are specific to connectedness.

If a space is *not* connected when considered in its entirety, we might wonder if it is "locally connected" or "pathwise connected", both concepts which we will soon define.

Definition 20.1 A topological space S is said to be *connected* if it is **not** the disjoint union of two non-empty open sets.

Recall that, in this book, those subsets of a space that are both open and closed are referred to as being *clopen*. So, if S contains a non-empty proper clopen subspace, U, then $S\backslash U$ is also a proper clopen subset. So we can say that a space ...

"S is *not* connected if and only if S contains a non-empty clopen proper subset".

Example 1. The topological space, $\mathbb{R}$, with the usual topology is connected since no non-empty proper subset in $\mathbb{R}$ is clopen. However, the subspace of all rationals, $\mathbb{Q}$, is not connected since $\mathbb{Q}$ the disjoint union of the two open sets, $U = (-\infty, \pi) \cap \mathbb{Q}$ and $\mathbb{Q}\backslash U = (\pi, \infty) \cap \mathbb{Q}$. So the connected property is not hereditary.

It is not difficult to prove that the only connected subspaces of $\mathbb{R}$ are those subspaces which contain a single element, or those subspaces which are intervals (either open, closed or half open).

Example 2. The topological space $\mathbb{R}$ with the upper limit topology is not connected since $\mathbb{R}$ is the disjoint union of $U = (-\infty, 3]$ and $\mathbb{R}\backslash U = (3, \infty)$ both of which are open in this topology. (Verify this.)

20.2 Properties of connected spaces.

We now investigate some invariance properties for connectedness.

Theorem 20.2 The continuous image of a connected space is connected.

Proof: We are given that S and T are topological spaces, that S is connected and the function, $f : S \to T$, is continuous. We are required to show that $f[S]$ is connected.

Suppose $f[S]$ was not connected. Then $f[S]$ contains a non-empty proper clopen set U. Let $h : f[S] \to \{0, 1\}$ be defined as $h[U] = \{0\}$ and $h[\ f[S]\backslash U\ ] = \{1\}$. Then h is continuous on $f[S]$, hence $h{\circ}f : S \to \{0, 1\}$ is continuous on S. Now $\{0\}$ and $\{1\}$ are clopen subsets of $(h{\circ}f)[S] = \{0, 1\}$, hence $(h{\circ}f)^{\leftarrow}[\{0\}]$ is a proper non-empty clopen subset of S. Since S is connected we have a contradiction. So the continuous image, $f[S]$, of S must be connected.

Example 3. Suppose $(\mathbb{R}, \tau)$ is the set of reals equipped with the usual topology and $(\mathbb{R}, \tau_s)$ is the set of reals with the upper limit topology. Show that the identity map $i : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_s)$ is not continuous.

Solution : We have seen that $(\mathbb{R}, \tau)$ is connected and that $(\mathbb{R}, \tau_s)$ is not. Since the continuous image of a connected set is connected then $i : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_s)$ cannot be continuous, as required.

Does the union of connected sets preserve the connected property? Our instinct states that they must, a least, have non-empty intersection. We should check this out. Suppose $S = \cap\{U_i : i \in I\}$ where each U_i is connected in S. Suppose $q \in \cap\{U_i : i \in I\}$. If there exists a function, $h : \cup\{U_i : i \in I\} = \{0, 1\}$, which is continuous on S then h must be constant on each U_i. There is no other option. It must then follow that, for each i, $h(q)$ and $h[U_i]$ have the same value. So h must be constant on the union of these sets. So $\cup\{U_i : i \in I\}$ cannot be the union of two non-empty clopen sets and so must be connected.

It is worth recording this as a theorem, for future reference.

Theorem 20.3 The arbitrary union of a family of connected sets which have at least one point in common is connected.

Proof : The statement is proved in the paragraph above.

Example 4. Suppose $S = \mathbb{R}^2$ equipped with the usual topology. Show that S is a connected space.

Solution : For each point $x \in S$, let L_x denote the infinite line containing both x and 0.
Then $S = \cup\{L_x : x \in S\}$. For each $x \in S, L_x$ is homeomorphic to $\mathbb{R}$ (previously shown to be connected) and contains $(0, 0)$; hence S is the union of a family of connected spaces which contain one point in common. Then S is connected.

In the following theorem we see that the closure operation preserves connectedness. We then verify that arbitrarily large products preserve connectedness, provided every factor is connected. Also that connected product spaces must have connected factors.

Theorem 20.4 The closure of a connected set is connected. Furthermore, any space, S, which has a connected dense subset is connected.

Proof: We are given that T is a connected subspace of the topological space S.

We are required to show that $\mathrm{cl}_S T$ is connected. Suppose $h : \mathrm{cl}_S T \to \{0, 1\}$ is a continuous function. It suffices to show that h is constant on $\mathrm{cl}_S T$. Since h is continuous, then h must be constant on the connected set T. Suppose $q \in \mathrm{cl}_S T \backslash T$. If $h(q) \neq h[T]$, then $h^{\leftarrow}[h(q)]$ is the pull-back of a clopen set in $\{0, 1\}$ and so must intersect T, contradicting the fact that h is constant on T. So h must be constant on $\mathrm{cl}_S T$ and so is connected.

By applying a similar reasoning, if S has a connected dense subset then S is connected.

Theorem 20.5 Let $\{S_i : i \in I\}$ be a family of topological spaces and $S = \prod_{i\in I} S_i$ be a product space. Then S is connected if and only if each S_i is connected.

Proof: We are given that $S = \prod_{i\in I} S_i$ is a product space.

($\Rightarrow$) If the product space, S, is connected then each factor, S_i, is the continuous image of S under the projection map, $\pi_i : S \to S_i$, and so is connected.

($\Leftarrow$) Suppose S_i is connected for each $i \in I$.

Let's first consider the case where S has only two connected factors, S_1 and S_2.

We *claim* that $S_1 \times S_2$ is connected. Suppose (q_1, q_2) is a fixed point in $S_1 \times S_2$ and let (x, y) be any other point. Then $(q_1, q_2) \in S_1 \times \{q_2\}$ and $(x, y) \in \{x\} \times S_2$. Then the set,

$$M_{(x,y)} = (S_1 \times \{q_2\}) \ \cup \ (\{x\} \times S_2)$$

is the union of two connected. Hence $M_{(x,y)}$ is connected and contains the fixed point (q_1, q_2), for any choice of (x, y). More generally, every element of $\{M_{(x,y)} : (x, y) \in S_1 \times S_2\}$ is a connected set and contains (q_1, q_2). So $\cup\{M_{(x,y)} : (x, y) \in S_1 \times S_2\}$ is connected. Since $S_1 \times S_2 = \cup\{M_{(x,y)} : (x, y) \in S_1 \times S_2\}$, then $S_1 \times S_2$ is connected, as claimed.

By finite induction, the product space of finitely many connected factors is connected.

Let $\mathscr{F}$ denote the family of all finite subsets of the index set, I. Suppose $q =< q_i >_{i\in I}$ is an element of the product space, S, and let $F \in \mathscr{F}$. Let

$$\begin{aligned} T_i &= S_i \quad \text{if } i \in F \\ T_i &= \{q_i\} \quad \text{otherwise} \end{aligned}$$

Let $K_F = \prod_{i \in I} T_i$. By the claim established above, K_F is a connected subspace of the product space, S. Hence $\{K_F : F \in \mathscr{F}\}$ is a family of connected subspaces of S, where

$$\{q\} = \{ < q_i >_{i \in I} \} = \cap\{K_F : F \in \mathscr{F}\}$$

By theorem 20.3, $V = \cup\{K_F : F \in \mathscr{F}\}$ is a connected subset of S.

We claim that V is dense in S. Let $u = < u_i >_{i \in I} \in S$. Let $U = \prod_{i \in I} U_i$ be a basic open neighbourhood of u. Then there is a finite subset, F^* of I, such that, $U_i = \pi_i^{\leftarrow}[U]$ for $i \in F^*$ and equals S_i, otherwise. It suffices to show that, $U \cap V \neq \varnothing$. See that $K_{F^*} \cap U = \prod_{i \in I} V_i$ such that, $V_i = U_i$ for $i \in F^*$ and $V_i = \{q_i\}$, otherwise. So $K_{F^*} \cap U$ is non-empty. Hence $V \cap U$ is non-empty. So V is dense in S. Since V is dense and connected then, by theorem 20.4, so is S.

20.3 The connected components of a topological space.

A topological space, S, which is not connected can always be partitioned in a family of subspaces which are connected. A trivial example could be the partitioning of $S = \{\{x\} : x \in S\}$, into singleton sets, since each singleton set is connected. However, a point x may be contained in a larger connected subspace of S which contains it. One might want to partition S as

$$S = \cup\{C_x : x \in S\}$$

where C_x denotes "the largest connected subspace of S which contains x". To remove any ambiguity about what we mean by "... C_x is the largest connected set in S, ..." we might prefer to say "C_x is the union of all the connected subspaces of S which contain x". This leads us to the following definition.

Definition 20.6 If S is a topological space and $u \in S$, then we define the *connected component of u in S* as being the union of all connected subspaces of S that contain the point u. When there can be, by context, no confusion we often drop the adjective "connected" and only say "component" .

Clearly, the only component of a connected space is the space itself. The definition confirms our perception that the component of a point u in S, is the *unique* largest connected subset of S which contains the point u. Also, two components, C_1 and C_2, in S cannot intersect, for if $z \in C_1 \cap C_2$, then $C_1 \cup C_2$ is connected; hence neither C_1 nor C_2 could be components of z. Since every point of a space S belongs to precisely one connected component, then the family of all components partitions the space, S.

Example 5. Consider the subspace, $T = B_1(0,0) \cup B_1(0,2)$, the union of two open balls of radius one in $\mathbb{R}^2$ with center $(0,0)$ and $(0,2)$ respectively. Since T is the union of two open disjoint subsets of T, then, by definition, T is not a connected space. The space, T, has precisely two components, $B_1(0,0)$ and $B_1(0,2)$.

Example 6. Let $U = B_1(0,0)$, $V = B_1(0,2)$ in $\mathbb{R}^2$. Consider the subspace,

$$T^* = U \cup V \cup \{(0,1)\}$$

of $\mathbb{R}^2$. It is clearly the case that T^* is the union of three non-intersecting connected subsets, the two connected sets U and V and the connected set $\{(0,1)\}$. But we cannot conclude from this that T^* is not connected. We can view T^* as the union of the two connected sets, $U \cup \{(0,1\}$ and $V \cup \{0,1)\}$ with non-empty intersection, $\{(0,1)\}$. So T^* is a connected set with itself as the only component.

Theorem 20.7 Suppose C is a component of a space, S. Then C is a closed subset of S.

Proof: We are given that C is a component of the topological space S.

Since C is, by definition, connected, then $\text{cl}_S C$ is also connected. Since C is the largest connected set containing one of its points, then $\text{cl}_S C = C$. So C is closed.

The definition of the "connected space" may lead one to surmise that components in S must also be open in S. If the reader is tempted by this affirmation, the following example will convince otherwise.

Example 7. Consider the subspace, $\mathbb{Q}$ with the subspace topology. The subset $\{1/2\}$ is a connected closed subset of $\mathbb{Q}$. If u is any other element in $\mathbb{Q}$ then the set $\{u, 1/2\}$ is the union of two open subsets of $\mathbb{Q}$ and so is not connected in $\mathbb{Q}$. So $\{1/2\}$ is a component in $\mathbb{Q}$. It is however not open in $\mathbb{Q}$, since it is the limit point of a sequence of rationals in $\mathbb{Q}\backslash\{1/2\}$, hence its complement cannot be closed. So in spite of being a connected component, $\{1/2\}$ is not open in $\mathbb{Q}$.

Example 8. Let $\{S_i : i \in I\}$ be a family of topological spaces and $S = \prod_{i \in I} S_i$ be the corresponding product space. Suppose that $u = < u_i >_{i \in I} \in S$. For each

$i \in I$, let C_i be the unique connected component in S_i which contains the point u_i. Suppose $C = \prod_{i\in I} C_i$. Show that C is the component in S which contains the point u.

Solution : If $u_i \in C_i$ for each $i \in I$, then $u \in C = \prod_{i\in I} C_i$. By theorem 20.5, C is a connected subset of S containing u. We claim that C is a component in S. Suppose not. Suppose D is the component in S which contains u and suppose $v =< v_i >_{i\in I} \in D \backslash C$. Then there must be some $j \in I$ such that $v_j \notin C_j$. Since $\pi_j : S \to S_j$ is continuous, it must be that $\pi_j[D]$ is a connected subset of S_j where $C_j \cup \{v_j\} \subseteq \pi_j[D]$. Since C_j is the largest connected set in S_j which contains u_j, then $\pi_j[D]$ cannot be connected. We have a contradiction due to our supposition that $D\backslash C$ is non-empty. So C is the component in S which contains u, as claimed.

The previous example shows that. . . ,

. . . the product of connected components is a connected component.

Example 9. Let $\{S_i : i \in I\}$ be an infinite family of discrete topological spaces, none of which contain only one point. Then the connected components of each S_i must be singleton sets. Let $S = \prod_{i\in I} S_i$ be the corresponding product space. Show that the connected components of S are singleton sets. Is the product space, S, a discrete space? Why?

Solution : Let $u =< u_i >_{i\in I} \in S$. Since S_i is discrete, $C_i = \{u_i\}$ is a connected component, for each i. Then, by the example above (where it is shown that the product of connected components is a connected component),

$$C = \prod_{i\in I} C_i = \prod_{i\in I}\{u_i\} = \{u\}$$

is the connected component in S containing u. So the connected components of S are singleton sets.

We claim that, even though S is the disjoint union of components each of which is a singleton set, it is not discrete. Simply see that the basic open neighbourhood, $\cap\{\pi_i^{\leftarrow}(u_i) : i \in F\}$, (where F is finite in I) cannot be contained in $\{u\}$. So the set $\{u\}$ is not an open subset of S. So the product space, S, is not a discrete space in spite of the fact that each of its factors is discrete.

Example 10. Show that the components of the Cantor set are all singleton sets.

Solution : Recall that, in example 2 of page 119 and theorem 7.15, the Cantor set was defined as being the image of the infinite product, $\prod_{n\in\mathbb{Z}^+}\{0, 2\}$, of the discrete space, $\{0, 2\}$, under the homeomorphism function $\varphi : \prod_{n\in\mathbb{Z}^+}\{0, 2\} \to [0, 1]$. By the above example, the components of $\prod_{n\in\mathbb{Z}^+}\{0, 2\}$ are singleton sets. The homeomorphic image of a component must be a component. So the components of the Cantor set are singleton sets.

Example 11. *Decomposition of a space by components.* Let S be a compact Hausdorff topological space which is not connected. We will partition the space S by its connected components. That is,

$$\mathscr{D}_S = \{C_x : x \in S,\, C_x \text{ a component containing } x\}$$

denotes the family of all connected components of S where $x \in C_x$. Let $\theta : S \to \mathscr{D}_S$ be defined such that $\theta(u) = C_u$ if $u \in C_u$ in $\mathscr{D}_S$. Then we obtain a decomposition space $(\mathscr{D}_S, \tau)$ where U is open in $\mathscr{D}_S$ if and only if $\theta^{\leftarrow}[U]$ is the union of connected components of S which is open in S. Show that the only connected components of $\mathscr{D}_S$ are points.[1]

Solution : We are given that S is compact Hausdorff and the function

$$\theta : S \to \mathscr{D}_S$$

defined as

$$\theta(u) = C_u \text{ if } u \in C_u$$

Let $p, q \in S$ such they are contained in distinct components of S. Then $\theta(p)$ and $\theta(q)$ are distinct points in $\mathscr{D}_S$.

We claim that there exists a clopen subset, M, of $\mathscr{D}_S$, such that $\theta(q) \in M \subseteq S\backslash\{\theta(p)\}$. If so, then no two points in $\mathscr{D}_S$ belong to the same component and so we will be able to conclude that the singleton set, $\{\theta(p)\}$ in $\mathscr{D}_S$, is a connected component containing $\theta(p)$.

Proof of claim. Each connected component, $\theta^{\leftarrow}(C_x)$, of S has been shown to be closed in S. Since S is compact then $\theta^{\leftarrow}(C_x)$ is compact. Now "compact and Hausdorff" $\Rightarrow$ "normal", so, for each $x \neq p$, there exists a pair of disjoint open sets, V_x and P_x, containing the components $\theta^{\leftarrow}(C_x)$ and $\theta^{\leftarrow}(C_p)$, respectively. Then the family open sets,

$$\mathscr{V} = \{V_x : x \in S, x \neq p\} \cup \{P_x : x \in S, x \neq p\}$$

forms an open cover of S. Since S is compact, then $\mathscr{V}$ has a finite subcover (of S)

$$\mathscr{F} = \{V_{x_i} : x_i \in S, x_i \neq p, i \in F\} \cup \{P_{x_i} : x_i \in S, x_i \neq p, i \in F\}$$

where F is a finite indexing set and $V_{x_i} \cap P_{x_i} = \varnothing$. Then S is the disjoint union of the two open subsets,

$$\begin{aligned} V &= \cup\{V_{x_i} : x_i \in S, x_i \neq p, i \in F\} \\ P &= \cap\{P_{x_i} : x_i \in S, x_i \neq p, i \in F\} \end{aligned}$$

containing $\theta^{\leftarrow}(C_q)$ and $\theta^{\leftarrow}(C_p)$, respectively. Since P and V are both open and disjoint such that $S = P \cup V$, then V and $S\backslash V = P$ are clopen in S and so are both

[1] We will later refer to such spaces as ones which are *totally disconnected*.

compact. So $\mathscr{D}_S$ is the disjoint union of the compact sets $\theta[V]$ and $\theta[P]$. Each of these must then be clopen in $\mathscr{D}_S$. Given that the set, $\theta[V]$, is clopen, then

$$\theta(q) = C_q \subseteq \theta[V] \subseteq S\backslash\theta(p)$$

as claimed.

It follows that no two points belong to the same connected component and so the only connected components of $\mathscr{D}_S$ are its points.

20.4 Locally connected: Spaces with an open base of connected sets.

Even when a space is not connected it may have a base for open sets whose elements are connected subspaces. For example, any discrete space, S, can easily be seen to have an open base of connected subspaces, $\mathscr{B} = \{\{x\} : x \in S\}$. So does the connected space of real numbers, $\mathbb{R}$, with the usual topology since it has an open base, $\mathscr{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$.

We formally define this particular notion.

Definition 20.8 If S is a topological space. We say that the space S is *locally connected* if it has an open base, $\mathscr{B} = \{B_i : i \in I\}$, where each element, B_i, is a connected subspace.

What would a non-locally connected space look like? We construct a connected subspace of $\mathbb{R}^2$ which is not locally connected.

Example 12. For each $n \in \mathbb{N}\backslash\{0\}$, let

$$g_n(x) = \left(\frac{\sin\left(\frac{\pi}{4n}\right)}{\cos\left(\frac{\pi}{4n}\right)}\right)x$$

For each n, let L_n denote the line,

$$L_n = \{(x, y_n) : y_n = g_n(x)\}$$

Let

$$T = \cup\{L_n : n \in \mathbb{N}\backslash\{0\}\} \cup \{(1, 0)\}$$

be a subspace of $\mathbb{R}^2$ with the usual topology. Show that T is a connected subspace which is not locally connected.

Solution : For each $n \in \mathbb{N}\backslash\{0\}$, L_n is a line in $\mathbb{R}^2$ which is homeomorphic to $\mathbb{R}$ and so is connected. Since each L_n contains the point, $(0, 0)$, then $\cup\{L_n : n \in \mathbb{N}\backslash\{0\}\}$ is connected. The connected set $\cup\{L_n : n \in \mathbb{N}\backslash\{0\}\}$ is dense in T and so T is connected. Consider the open neighbourhood base,

$$\mathscr{B} = \{B_{1/k}(1,0) \cap T \,:\, k \geq 2\}$$

of the point $(1,0) \in T$. For each k, the open "ball", $B = B_{1/k}(1,0) \cap T$, in T contains countably many line segments (belonging to the L_n's). If $j \neq m$ there is a line L^* whose slope is strictly in between the slopes of L_j and L_m, so the line of L^* does not appear in T. Then B is disconnected at the line L^*. So T is not locally connected at the point (1, 0).

20.5 Which spaces have clopen connected components?

Suppose U is an open subspace of a locally connected space S (possibly equal to the open space, S, itself). This means that S has an open base, $\mathscr{B} = \{B_i : i \in I\}$, of connected subspaces. Let $u \in U$ and C_u be a connected component in U which contains u. We claim that C_u must be an open subset in U. See that there is an element, B_j, in $\mathscr{B}$ containing u which is entirely contained in the open subset U of S. Then B_j is a connected open subspace of U. Then $u \in B_j \cap C_u$. If $y \in B_j\backslash C_u$ then there exists a clopen subset V of U such that $y \in V \subseteq U\backslash C_u$. This contradicts the fact that B_j is connected in U. So $u \in B_j \subseteq C_u$. This means that C_u must be an open subset of U, as claimed. So, the components of an open subspace, U, of a locally connected space S, are clopen.

This leads to an interesting characterization of the locally connected property on a space.

Theorem 20.9 Suppose S is a topological space.

a) The space S is a locally connected space if and only if every open subspace of S (including S itself) has clopen components.

b) If the space S is both locally connected and compact then it has at most finitely many connected components.

Proof: The direction ($\Rightarrow$) of part a) is proven in the paragraph above.

We prove the direction ($\Leftarrow$): Suppose that every open subspace of S has clopen components. We are required to show that S has an open base of connected sets. Let V be an open neighbourhood of a point x in S. Let C_x be a component of V which contains x. By hypothesis, C_x is an open subset of V. So C_x is an open neighbourhood of x in S.

We claim that C_x is connected in S. If not, then there is a clopen subset, U, of S which intersects only a part of C_x. Then $U \cap V$ is a clopen subset of V which intersects only a part of C_x, contradicting the fact that C_x is a connected subset of V. So

C_x is a connected neighbourhood of x in S which is contained in V. So every open neighbourhood V of x contains an open connected set C_x in S such that $x \in C_x \subseteq V$. This implies that S is a locally connected space.

For part b), if S is compact and locally connected and it has infinitely many components then its components will form an open cover of S with no finite subcover, a contradiction. So a locally connected space can have at most finitely many components.

Example 13. Consider the space T described in the previous example and consider the space $S = T\backslash\{(0,0),(1,0)\}$. Given any two distinct lines L_j and L_k of slope s_j and s_k, respectively, in T, there is a line, L^*, of slope s^* in between s_j and s_k which does not belong to T and so disconnects L_j and L_k. This implies that L_j and L_k belong to distinct components of S. Since any line L in S can be separated from all other lines by two lines L^* and L^{**} which don't belong to S then each line is an open component in S. So every component of S is clopen. Since S has infinitely many clopen components then S is non-compact. Showing that every open subspace of S has clopen components is left as an exercise.

Theorem 20.10 Let $\{S_i : i \in I\}$ be a family of topological spaces and $S = \prod_{i\in I} S_i$ be the corresponding product space. Then S is locally connected if and only if every factor, S_i, is locally connected and, at most, finitely many of the factors are *not* connected.

Proof: We are given that $\{S_i : i \in I\}$ is a family of topological spaces and $S = \prod_{i\in I} S_i$ is a product space.

($\Rightarrow$) Suppose $S = \prod_{i\in I} S_i$ is locally connected.

Claim #1. That all but finitely many of the factors are connected spaces. Let $u \in S$. Since S is locally connected, then, by theorem 20.9, there exists an *open* connected component, C in S, such that $u \in C$. Let U be a basic open neighbourhood of u contained in C. That is, for a finite subset, F, of I,

$$u \in U = \cap\{\pi_i^{\leftarrow}[U_i] : i \in F\} \subseteq C$$

where U_i is an open subset of S_i. Consider $i \notin F$. Then $\pi_i[U] = S_i \subseteq \pi_i[C]$ (since $U \subseteq C$). So $\pi_i[U] = S_i = \pi_i[C]$. Since C is a connected component of S and π_i is continuous, S_i is connected for all i's except possibly the ones in F. This establishes the claim #1.

Claim #2. That each S_i is locally connected. For $i \in I$ let $v_i \in S_i$. Let W be an open neighbourhood of v_i in S_i and $v \in S$ such that $\pi_i(v) = v_i$. Since S is locally connected, and $\pi_i^{\leftarrow}[W]$ is an open neighbourhood of v in S, there exists an open connected subspace, V, in S such that $v \in V \subseteq \pi_i^{\leftarrow}[W]$. Since π_i is both continuous and open then $\pi_i[V]$ is a connected open neighbourhood of v_i contained in W. So v_i has a connected neighbourhood base. Then S_i is locally connected, as claimed. We are done with this direction.

($\Leftarrow$) Suppose every factor, S_i, is locally connected and at most finitely many of these factors are not connected. We are required to show that the product space, S, is locally connected.

Suppose F_c is the largest subset of the indexing set, I, such that, if $i \in F_c$, S_i is not connected. Then, by hypothesis, F_c is finite.

Let $x = < x_i >_{i \in I} \in S = \prod_{i \in I} S_i$ and let $V = \cap\{\pi_i^{\leftarrow}[U_i] : i \in F\}$ be an open base element containing x. Let $F_1 = F \cup F_c$ and

$$U_1 = \cap\{\pi_i^{\leftarrow}[U_i] : i \in F_1\}$$

Clearly, since $F \subseteq F_1$, $U_1 \subseteq V$.

We *claim* that the point, x, has a connected neighbourhood, W, which is contained in U_1.

Given that every S_i is locally connected, for each $i \in F_1$, there must be an open connected set, C_i in S_i, such that $x_i \in C_i \subseteq U_i$.

Then

$$x \in \cap\{\pi_i^{\leftarrow}[C_i] : i \in F_1\} = W \subseteq U_1 \subseteq V$$

exhibits an open neighbourhood, W, of x in S which is contained in V. If $j \notin F_1$, the j^{th} factor of W is the connected space, S_j. If $j \in F_1$, the j^{th} factor of W is the connected, C_j. By theorem 20.5, W a connected subspace in S.

So x has a connected open neighbourhood, W, contained in U_1, as claimed. Since $U_1 \subseteq V$, S is locally connected, as required.

20.6 Pathwise connected spaces

Different persons may describe the property of connectedness differently depending on the way they perceive connected mathematical objects in the field they study. The formal definition of the *connected space* given earlier was " ... a space that is not the union of two disjoint open sets." appeals to its proponents because of its simplicity. It is a clear and unambiguous definition. However, some critics might have argued that this definitions sounds to much like "... a connected space is one that is not disconnected; and we all know what *disconnected* means." The formal definition, along with a few examples, and some discussion of the properties derived from it, allowed users to develop a better understanding of what a connected space is. But, in spite of this, the space, T, described in example 9, provided on page 303, may not appear to be connected, by some participants of a random survey. We define below, the "pathwise connected property" a slightly stronger definition than that of the *connected property* in the sense that *pathwise connected spaces* are always connected, but not conversely. To some, it is a more appealing form of connectedness even though it is sightly more difficult to work with.

Definition 20.11 Let S be a topological space. A *path* in S is a *continuous* function, $f : [0, 1] \to S$, mapping the closed unit interval into S. If, for a path, $f : [0, 1] \to S$, $a = f(0)$ and $b = f(1)$, we say that...

"*f is a path from a to b in S*"

where a is referred to as being the *initial point of the path f* and b is called the *terminal point of the path f*. These terms prescribe a *direction* for the image, $f[0, 1]$, of f. The space, S, is said to be *pathwise connected* if, for any pair of points, a and b in S, there is a path, $f : [0, 1] \to S$, joining $a = f(0)$ to $b = f(1)$.[1]

Since $[0, 1]$ is connected in $\mathbb{R}$ and f is continuous on $[0, 11$ then the image $f[0, 1]$ is a connected subspace of S. The set of all real numbers, $\mathbb{R}$, is of course pathwise connected since for any a and b in $\mathbb{R}$

$$f(x) = (b - a)x + a$$

is a path connecting $a = f(0)$ to $b = f(1)$.

This particular perception of "connected" is also quite intuitive. It suggests that if one can draw a curve joining any two points in S, without lifting the pencil off the page, then the space is connected. The following theorem guarantees that the pathwise connected spaces are connected as described by the formal definition.

Theorem 20.12 If the topological space, S, is pathwise connected then it is connected.

[1]If the continuous function, f, from $a = f(0)$ to $b = f(1)$ is a *homeomorphism* then f is called an *arc*. If every path on S is a homeomorphism then the space is said to be *arcwise connected*.

Proof: Suppose S is pathwise connected and $a \in S$. Then, for any $b \in S$, there is a path, $f_b : [0,1] \to S$, such that a is its initial point, $f_b(0)$, and b is its terminal point, $f_b(1)$. Since f_b is continuous, $f_b[0,1]$ is connected. Then $\{f_b[0,1] : b \in S, b \neq a\}$ is a family of connected sets which entirely covers S where all paths have the element a in common. Then S is connected.

It is worth noting that, if there is a path, f, from the point a to the point b in a space S, then there is a path, $g : [0,1] \to S$, defined as $g(x) = f(1-x)$ which goes in the opposite direction, from b to a.

Example 14. We reconsider the example presented earlier described as follows: For each $n \in \mathbb{N}\backslash\{0\}$, let L_n denote the set,

$$\{(x, y_n) : y_n = g_n(x) = \left(\frac{\sin\left(\frac{\pi}{4n}\right)}{\cos\left(\frac{\pi}{4n}\right)}\right) x$$

Let

$$T = \cup\{L_n : n \in \mathbb{N}\backslash\{0\}\} \cup \{(1,0)\}$$

be a subspace of $\mathbb{R}^2$ with the usual topology. We have shown, in example 9, that T is a connected subspace but is not locally connected. Show that T is not pathwise connected.

Solution : Suppose there is a path joining the point $a = (1,0)$ in T to a point, $b = (u,v) \in L_q$, for some $q \in \mathbb{N}$, $b \neq a$. That is, suppose there is a continuous function $f : [0,1] \to T$, such that $f(0) = (1,0)$ and $f(1) = b$.

Let $\varepsilon \in (0, 1/10)$. Then there exists $\delta > 0$ such that $f[0,\delta) \subseteq B_\varepsilon(1,0)$. Then $f[0,\delta)$ does not contain the point $(0,0)$. Suppose $f(\delta/z) \in L_d$ for some $d \in \mathbb{N}$. Then there exists $y > z$ such that $f(\delta/y)$ in L_k, where $k > d$. So $f[\delta/y, \delta/z]$ must be a connected subset in T. But since $f[\delta/y, \delta/z]$ has points on distinct lines L_d and L_k and does not contain $(0,0)$, it cannot be connected. Then there can be no path joining a to b. We must conclude that T is not pathwise connected.

Not surprisingly, pathwise connectedness is an invariant with respect to continuous functions. We verify this now.

Theorem 20.13 If $g : S \to T$ is a continuous function mapping a pathwise connected space S to a space T, then $g[S]$ is a pathwise connected subspace of T.

Proof: Suppose S is a pathwise connected space. Then if u and v are distinct points in S there is a continuous function $f : [0,1] \to S$ such that $f(0) = u$ and $f(1) = v$.

Suppose now that $g : S \to T$ is a continuous function mapping S into T. We claim that $g[S]$ is pathwise connected. Let a and b be distinct points in $g[S]$. Then we can choose disstinct points, $x \in g^{\leftarrow}(a)$ and $y \in g^{\leftarrow}(b)$. Since S is pathwise connected there is a continuous function $f : [0,1] \to S$ such that $f(0) = x$ and $f(1) = y$. Then the continuous function, $g{\circ}f : [0,1] \to g[S]$, maps 0 to a and 1 to b. So $g[S]$ is pathwise connected.

We now consider invariance of the pathwise connected property with respect to products.

Theorem 20.14 Let $\{S_i : i \in I\}$ be family of topological spaces. Then the product space, $S = \prod_{i\in I} S_i$, is pathwise connected if and only if every factor, S_i, is pathwise connected.

Proof: We are given that $\{S_i : i \in I\}$ is a family of topological spaces and that $S = \prod_{i\in I} S_i$ is the corresponding product space.

For ($\Rightarrow$), if S is pathwise connected, since each projection map, π_i, is continuous, then each S_i is pathwise connected.

For ($\Leftarrow$), suppose S_i is pathwise connected for each $i \in I$.

Let $< a_i >_{i\in I}$ and $< b_i >_{i\in I}$ be distinct points in S. For each $i \in I$, there is a continuous function, $f_i : [0,1] \to S_i$, such that $f_i(0) = a_i$ and $f_i(1) = b_i$. Let $f : [0,1] \to S$ be defined as

$$f(u) = < f_i(u) >_{i\in I}$$

Then $f_i(u) = (\pi_i{\circ}f)(u)$. By lemma 7.6, the function, $f : [0,1] \to S$, is continuous on $[\,0,1\,]$ if and only if each f_i is continuous on $[\,0,1\,]$. Since each f_i maps 0 to a_i and 1 to b_i then

$$\begin{aligned} f(0) &= < f_i(0) >_{\imath\in I} = < a_i >_{i\in I} \\ f(1) &= < f_i(1) >_{i\in I} = < b_i >_{i\in I} \end{aligned}$$

So S is pathwise continuous.

Example 15. It follows from the previous theorem that the space, $S = \mathbb{R}^{\mathbb{R}}$, equipped with the product topology is pathwise connected since each factor, $\mathbb{R}$, is pathwise connected.

Example 16. Let u be a point in a topological space, S. Show that S is pathwise connected if and only if there is a path joining each point, $x \in S$, to u.

Solution : For the direction ($\Rightarrow$) the statement is obviously true.

For the direction ($\Leftarrow$), we are given that every point in S can be joined by a path to u. Suppose a and b are distinct points in S. By hypothesis, there are two paths, $f_a : I \to S$ and $f_b : I \to S$, such that $f_a(0) = a$ and $f_a(1) = u$ and $f_b(1) = b$ and $f_b(0) = u$. Consider the function $g(x) = f_a(2x)$ on the interval $[0, 1/2]$ and $g(x) = f_b(2x - 1)$ on the interval $[1/2, 1]$. Then g continuously joins a to u on $[0, 1/2]$ and joins u to b on $[1/2, 1]$ and forms a path from a to b. Then S is pathwise connected.

Theorem 20.15 Let $\{S_i : i \in I\}$ be a family of pathwise connected topological spaces which have the point u in common. Then space $S = \cup\{S_i : i \in I\}$, is pathwise connected.

Proof: The statement follows immediately from the argument presented in the previous example.

Definition 20.16 If S is a topological space and $u \in S$, then we define the *pathwise component of* u as being the union of all pathwise connected spaces that contain the point u. Hence, if C is the pathwise component containing u, it is the largest subspace containing u which is pathwise connected.

Example 17. Consider the space T presented in the example above. We showed that T is connected but no pathwise connected. Show that a pathwise component in T need not be closed.

Solution : Consider the subspace, $S = T\setminus\{(1, 0)\}$. It is the infinite union of straight lines in $\mathbb{R}^2$ each of which is pathwise connected and contains the point $(0, 0)$; this makes it pathwise connected. It can also be seen to be a pathwise component of T. But since S is missing the limit point $(1, 0)$ in T, it is not closed in T. So a pathwise components of a space need not be closed.

Under particular conditions it does occur that a pathwise component is a clopen subset. This condition is described in the following theorem.

Theorem 20.17 Let S be a topological space. The pathwise components of S are clopen if and only if every point in S has a pathwise connected neighbourhood.

Proof : We are given that S is a topological space.

($\Rightarrow$) Suppose each pathwise component is clopen. Let $x \in S$. We are required to show that x has a pathwise connected neighbourhood. Let C be a pathwise component containing x. By hypothesis, C is open in S, so x has a clopen neighbourhood which is pathwise connected. We are done with this direction.

($\Leftarrow$) Let $x \in S$ and C be the (unique) pathwise component containing x. By hypothesis, the point, x, has a pathwise connected neighbourhood, say U.

We are required to show that C is clopen.

Then since C is the largest pathwise connected neighbourhood which contains x, then $x \in \text{int}_S U \subseteq C$. So C is open. We claim it is also closed: Since the components partition the space S, then C is the complement of the union of all other open components of S and so is also closed. So all pathwise components are clopen.

The equivalent conditions, "pathwise components are clopen" and "every point in S has a pathwise connected neighbourhood" described in the above theorem, when combined to the connected property characterizes pathwise connectedness, as shown in the next theorem.

Theorem 20.18 Let S be a topological space. The space, S, is pathwise connected if and only if S is both connected and each of its points, has a pathwise connected neighbourhood.

Proof : We are given that S is a topological space.

($\Rightarrow$) We are given that S is a pathwise connected space. The fact that S is a connected space follows from theorem 20.12. That every point has a pathwise connected neighbourhood follows from the the fact that the pathwise connected space is a neighbourhood of every point in S.

($\Leftarrow$) Suppose S is connected and every one of its points has a pathwise connected neighbourhood. We are required to show that S is pathwise connected. Let $x \in S$. By theorem 20.17, the (unique) pathwise component, C, of x is clopen. Since S is connected, the clopen set, C, cannot be a proper subset. So the pathwise component, C, must be S itself. So S is pathwise connected.

In $\mathbb{R}^n$, any connected open subspace is a pathwise connected subspace as the following theorem shows.

Theorem 20.19 Let $\mathbb{R}^n$ be equipped with the usual topology. An open subspace, U, of $\mathbb{R}^n$ is connected if and only if U is a pathwise connected subspace of $\mathbb{R}^n$.

Proof: The space $\mathbb{R}^n$ be equipped with the usual topology and U is an open subspace of $\mathbb{R}^n$.

($\Rightarrow$) Suppose U is a connected subspace of $\mathbb{R}^n$. We are required to show that U is a pathwise connected subspace. By theorem 20.18 it suffices to show that each point in U has a pathwise connected neighbourhood

Well, let $x \in U$. Since U is open in $\mathbb{R}^n$ there is an $\varepsilon > 0$ such that $x \in B_\varepsilon(x) \subseteq U$. Let $a =< a_i >_{i=1..n}$ and $b =< b_i >_{i=1..n}$ be distinct points in $B_\varepsilon(x)$. If $f(x) = (b-a)(x) + a$ then $f[\,0,1\,]$ is a line segment entirely contained in $B_\varepsilon(x)$. So $B_\varepsilon(x)$ is pathwise connected. From this we can conclude that every point in U has a pathwise connected neighbourhood, so by the above theorem, U is pathwise connected, as required.

($\Leftarrow$) Suppose U is pathwise connected. By theorem 20.12, U is a connected subspace. Done.

20.7 Totally disconnected spaces

For some spaces, the only connected subspaces are singleton sets. We briefly discuss some basic properties of these types of spaces.

Definition 20.20 A topological space, S, is said to be *totally disconnected* if and only if the only connected components of S are its points.

Discrete spaces, the subspace $\mathbb{Q}$, the Cantor set (viewed as a subspace of $[0,1]$) and $\mathbb{R}\backslash\mathbb{Q}$ all previously discussed sets in this chapter are standard examples of such spaces. We verify that the totally disconnected property is preserved over arbitrary products.

Theorem 20.21 Products of totally disconnected spaces are themselves totally disconnected.

Proof: Let $\{S_i : i \in I\}$ be a family of totally disconnected spaces and $S = \prod_{i\in I} S_i$ be the corresponding Cartesian product space. Let C be a connected component in S. Suppose $a = < a_i >_{i\in I}$ and $b = < b_i >_{i\in I}$ are points in C.

We claim that $a = b$. Suppose a and b are distinct. Then there is some $j \in I$ such that $a_j \neq b_j$. Since both $\{a_j\}$ and $\{b_j\}$ are connected components of S_j then there is a clopen set, U_j, in S_j such that $a_j \in U_j \subseteq S_j \backslash \{b_j\}$. Then $\pi_j{}^{\leftarrow}[U_j]$ and $\pi_j{}^{\leftarrow}[S\backslash U_j]$ are disjoint basic clopen neighbourhoods of a and b respectively. Then a and b cannot both belong to the same component of S. Then $a = b$, as claimed.

So components of S can only contain a single point. We conclude that the product space, S, is totally disconnected.[1]

Example 18. Show that if S is totally disconnected and $f : S \to T$ is continuous then $f[S]$ need not be totally disconnected.

Solution : In theorem 7.16, we showed that there is a function $\psi : C \to [0, 1]$ which maps the totally disconnected Cantor set C onto the connected set, $[0, 1]$. So the continuous image of a totally disconnected set need not by totally disconnected.

Total disconnectedness versus the zero-dimensional property.

Recall from definition 5.16 that zero-dimensional spaces are those spaces that have an open base of clopen sets. Equivalently, each point has a neighbourhood base of clopen sets. We know that a non-empty clopen set allows us to express a space as a union of two open sets. So we surmise that zero-dimensional spaces and totally disconnected spaces are similar in many ways. For certain types of topological spaces they may even be equivalent, as we shall soon see.

Example 19. Show that a T_1 zero-dimensional space, S, is totally disconnected.

Solution : Since S is T_1, any point in S is closed. Given distinct points p and y the set $S\backslash\{y\}$ is an open set containing p. Since p has a neighbourhood base of clopen sets then there exists a clopen set U such that $p \in U \subset S\backslash\{y\}$. So p and y cannot belong to the same component. So the only connected component which contains p

[1] Some may argue that since $\pi_j[C]$ is a singleton set it immediately follows that C is a single set. This argument is not quite "immediate" enough and may even mislead the casual reader.

is $\{p\}$. This means that S is totally disconnected.

Example 20. Suppose S is a compact T_2 space satisfying the property: For distinct pairs of points, p and y, there is a clopen set U such that $p \in U \subset S\backslash\{y\}$. Show that S must then be zero-dimensional.

Solution : Suppose V is a proper open subset containing the point p. Then $S\backslash V$ is closed and so is a compact subset of S. By hypothesis, $S\backslash V$ has an open cover, $\mathscr{U} = \{U_x : x \in S\backslash V\}$, of clopen sets such that $x \in U_x \subseteq S\backslash\{p\}$. Let $\{U_{x_i} : i \in F\}$ be a finite subcover. Then $M = \cup\{U_{x_i} : i \in F\}$ is a clopen set containing $S\backslash V$. Then $p \in S\backslash M \subseteq V$. Since $S\backslash M$ is clopen p has a clopen neighbourhood which is contained in V. We deduce from this that p has a neighbourhood base of clopen sets. So S is zero-dimensional, as required.

Theorem 20.22 A locally compact Hausdorff space is totally disconnected if and only if it is zero-dimensional.

Proof: ($\Leftarrow$) We have shown in the example above that T_1 zero-dimensional spaces are totally disconnected. Since Hausdorff spaces are T_1 we are done.

($\Rightarrow$) Suppose S is a locally compact Hausdorff totally disconnected space. Suppose V is an open subset of S containing the point p. It suffices to show the existence of a clopen neighbourhood, W, such that $p \in W \subseteq V$.

Since S is locally compact there is an open subset M whose closure, $\text{cl}_S M$, is compact and $p \in M \subseteq \text{cl}_S M \subseteq V$. Since the boundary, $\text{bd}_S M$, of M is closed in the compact set, $\text{cl}_S M$, then it is itself compact in $\text{cl}_S M$.

Since S is totally disconnected no point on $\text{bd}_S M$ is in a component containing p. We can then find clopen sets, $\{U_x : x \in \text{bd}_S M\}$, such that $p \in U_x$ in $\text{cl}_S M$ and $x \notin U_x$. So $\{S\backslash U_x : x \in \text{bd}_S M\}$ forms an open cover of $\text{bd}_S M$.

Then there is a finite subcover, $\{S\backslash U_{x_i} : i \in F\}$ of $\text{bd}_S M$. Then $p \in W = \cap\{U_{x_i} : i \in F\}$, a clopen neighbourhood of p. Since $S\backslash W$ contains $\text{bd}_S M$, $p \in W \subseteq \text{cl}_S M \subseteq V$. We deduce that S is zero-dimensional.

Example 21. Show that subspaces of totally disconnected spaces are also totally disconnected

Solution : Suppose T is a subspace of a totally disconnected locally compact Hausdorff space S. Then S is zero-dimensional. Let C be a connected component of T. If a and b are points in C then there exists a clopen neighbourhood U (in S) of a which does not contain b. So the clopen subset $U \cap T$ (of T) separates a and b and so, if $a \neq b$, C cannot be connected. So $a = b$. Then connected components of T are singleton sets. So T is totally disconnected.

Example 22. Let S be a compact Hausdorff topological space which is not connected. We showed in an example on page 302, that if we collapse the connected components of S to points by the quotient map $\theta : S \to \theta[S]$ we obtain a decomposition space $\theta[S]$ whose points are connected components. This means that the compact space $\theta[S]$ is totally disconnected and hence zero-dimensional.

Concepts review:

1. What does it mean to say that a topological space is connected?
2. Is the connected property invariant with respect to the continuous functions?
3. Under what conditions, if any, are unions of connected sets connected?
4. If U is a connected dense subset of V what can say about V?
5. Under what conditions, if any, is the product space, $S = \prod_{i \in I} S_i$, of connected, S_i's, connected?
6. Define a connected component of a topological space.
7. Briefly argue that a connected component of a topological space is closed.
8. Provide a simple example which illustrates that a connected component need not be open.
9. Describe the connected components of a discrete space.
10. Describe the connected components of an infinite product of discrete spaces all of which have more than one point.
11. Is an infinite product of discrete spaces (all of which have more than one point) discrete?
12. Define a locally connected space.
13. What kind of locally connected spaces are guaranteed to have clopen connected components.
14. Describe the conditions under which the locally connected property carries over on a product space, in both directions.

15. Suppose a locally connected space is shown to be compact. What can we say about its connected components?
16. If a person speaks of a path joining a point, a, to a point, b, in a space S, what is this person talking about?
17. If you are familiar with the notion of a "curve" in a space S, is there a subtle difference between a curve and a path?
18. What does it mean to say that a space S is pathwise connected?
19. When is a connected space a pathwise connected space?
20. Are continuous images of pathwise connected spaces necessarily pathwise connected?
21. Under what conditions is the pathwise connected property carried over products in both directions?
22. Under what conditions, if any, are unions of pathwise connected spaces pathwise connected?
23. Define the pathwise component of a point in a space S.
24. Are pathwise components necessarily closed?
25. Describe a particular condition under which a pathwise component is clopen.
26. Describe a condition under which connected spaces are pathwise connected.
27. Define the property called totally disconnected.
28. Give a characterization of the totally disconnected property and describe the conditions under which it holds true.
29. What can we say about the product of totally disconnected spaces?
30. What can we say about the subspace of a totally disconnected space?

EXERCISES

1. Suppose U is subset of $\mathbb{R}$ which is not connected and $V \subseteq U$. Is it possible for V to be connected?
2. Let (S, τ_S) and (T, τ_T) be two topological spaces where T is connected. If T has a strictly stronger topology then S, determine whether S is necessarily connected.

3. Consider the subset

$$T = \{(x \sin(1/x)) : x \in [0, 2\pi]\} \cup \{(0,0)\}$$

of $\mathbb{R}^2$. Is T a connected set?

4. Is the set T described in the previous question locally connected?

5. Is the set T described in the previous question pathwise connected?

6. Let $\{S_i : i \in I\}$ be an infinite family of connected sets. Suppose that, if $i \neq k$, then $S_i \cap S_k \neq \varnothing$. Determine whether the space $S = \cup\{S_i : i \in I\}$ is connected or not.

7. Let S be an infinite set equipped with the cofinite topology (i.e., $\{\varnothing\}$ union the family of all sets with a finite complement). Determine whether S is a connected set or not.

8. Suppose the space (S, τ) is the topological space where

$$\tau = \{\{\varnothing\} \cup \{U : S \backslash U \text{ is finite }\}\}$$

Let the function, $f : [0,1] \to S$, be continuous. Show that $f[\,[0,1]\,]$ in S, is a singleton set.

9. Let $S = \mathbb{R}^n$ be the space equipped with the usual topology. Suppose U is a non-empty open subspace of S. How many connected components can S have? If the possible cardinality of all components is infinite, is it countable or uncountable?

10. Are open subspaces of locally connected spaces necessarily locally connected?

11. Suppose S is a T_1-space which has, at each point, a neighbourhood base of clopen sets. Show that the connected components of S are all singleton sets.

Part VII

Topics

21 / Compactifications of completely regular spaces

Summary. *In this section, we discuss those spaces, S, which can densely be embedded in compact Hausdorff space. Only completely regular spaces can possess this property. The process by which we determine such a compact space, αS, for S, is called compactifying S. The space, αS, is called the compactification of S. The family of all compactifications of a completely regular space can be partially ordered. The maximal compactification of S with respect to the chosen partial ordering is called the Stone-Čech compactification. We discuss methods for its construction. We will show that only locally compact spaces have a minimal compactification with respect to the chosen partial ordering. It is called the one-point compactification.*

21.1 Compactifying a space

In this section we will briefly talk about methods for "compactifying a space (S, τ_S)". This essentially means adding a set of points, F, to S, to obtain a larger set, $T = S \cup F$, and topologizing T so that (T, τ_T) is a compact Hausdorff space in which a homeomorphic copy of S appears as a dense subspace of T.

With certain bounded subspaces of $\mathbb{R}^n$, this can, sometimes, be quite easy to do. For example, if $S = [-1, 3) \cup (3, 7)$ is equipped with the subspace topology, then simply by adding the points, $\{3, 7\}$, to S we obtain a set $T = S \cup \{3, 7\} = [-1, 7]$ which, when equipped with the subspace topology, is a compact Hausdorff space which densely contains a homeomorphic copy of S. In such a case, we will say that T is a *compactification* of S. If we are given a space such as $\mathbb{N}$ or $\mathbb{Q}$, it is not at all obvious how one would go about compactifying such spaces. We will show techniques which allow us to achieve this objective.

In what follows, recall that the evaluation map $e : S \to \prod_{i \in J}[a_i, b_i]$ induced by $C^*(S)$ (the set of all continuous bounded real-valued functions on S) is defined as $e(x) = \{f_i(x)\} \in \prod_{i \in J}[a_i, b_i]$ where $f_i \in C^*(S)$ and $f_i[S] \subseteq [a_i, b_i]$.

Definition 21.1 Let (S, τ_S) be a topological space and (T, τ_T) be a compact Hausdorff space. We will say that T *is a compactification of* S if S is densely embedded in T.[1]

---

[1] When we say "compactification of S" we always mean a Hausdorff compactification.

If S is a compact Hausdorff space, then S can be viewed as being a compactification of itself. Recall that a compact Hausdorff space is normal, and so is completely regular. Since subspaces of completely regular spaces are completely regular then only a completely regular space can have a compactification.

In theorem 14.7, we showed how any completely regular space can be compactified. In the proof of that theorem, we witness how an evaluation map, $e : S \to T$, induced by $C^*(S)$ embeds S into a cube $T = \prod_{i\in J}[a_i, b_i]$.

There may be slight variations in the description of the compact space in which S is embedded. Since each interval $[a_i, b_i]$ is homeomorphic to $[0, 1]$ then there is a homeomorphism, $h : T \to \prod_{i\in J}[0, 1]$, which maps T onto $P = \prod_{i\in J}[0, 1]$. By Tychonoff's theorem, P is guaranteed to be compact. So the function, $q : S \to P$, defined as, $q = h \circ e$, embeds S into $\prod_{i\in J}[0, 1]$. Hence $\mathrm{cl}_P q[S]$ is a compact subspace of the product space, P, which densely contains the homeomorphic image, $q[S]$, of S.[1] So, even common topological spaces such at $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ have at least the compactification obtained by the method just described.

21.2 The Stone-Čech compactification.

We have described only one of the various ways to obtain a homeomorphic copy of the compactification, $\mathrm{cl}_T e[S]$, of S. This particular compactification has a special name.

Definition 21.2 Let (S, τ_S) be a completely regular topological space. Let

$$e : S \to \prod_{i\in I}[a_i, b_i]$$

be the evaluation map induced by $C^*(S)$ which embeds S in the product space, $T = \prod_{i\in I}[a_i, b_i]$.

The subspace, $\mathrm{cl}_T e[S]$, is called the *Stone-Čech compactification of S*. The Stone-Čech compactification of S is uniquely (and universally) denoted by, βS.

So we see that a non-compact completely regular space, S, always has at least one compactification, namely, it's Stone-Čech compactification, βS. We will soon see that there can be more than one compactification, for the same space.

[1] This is just one small example which shows why Tychonoff theorem deserves to be titled and why it is such an important theorem in topology.

Equivalent compactifications

Suppose we find two compactifications for S, say αS and γS. If there is a homeomorphism

$$h : \alpha S \to \gamma S$$

mapping αS onto γS such that $h(x) = x$ for all $x \in S$, then αS and γS will be considered as being *equivalent compactifications* of S. The intention is that "equivalent compactifications of a space S" be perceived as being the same compactification of S. This equivalence is usually expressed by the symbol, $\alpha S \equiv \gamma S$. But for convenience, it is sometimes expressed by, $\alpha S = \gamma S$, even though αS and γS are not necessarily equal sets (but are, of course, topologically the same).

The outgrowth of a topological space.

Given a topological space, S, and a compactification, αS, we refer to the set $\alpha S \backslash S$ as the *outgrowth of* S. We can see and analyze the space S, but its outgrowth, $\alpha S \backslash S$, may appear at first as a black box with unknown content and properties. But there is one thing we know: The properties of the content are directly related, somehow, and in various ways, to the properties of S. Our main task is to precisely determine as many rules as we can which describe these properties from the properties of S. We will never completely know what is in the box since it cannot be opened. But as we rigourously establish more and more rules of association, we will develop a better insight of what is inside the outgrowth.

21.3 A partial ordering of Hausdorff compactifications of a space.

Suppose the completely regular space, S, has a family of compactifications,

$$\mathscr{C} = \{\alpha_i S : i \in I\}$$

Then we will partially order the family, $\mathscr{C}$, by defining "$\preceq$" as follows: $\alpha_i S \preceq \alpha_j S$, if and only if there is a continuous function $f : \alpha_j S \to \alpha_i S$, mapping $\alpha_j S \backslash S$ onto $\alpha_i S \backslash S$ which fixes the points of S.

Notation. If two compactifications αS and γS are such that $\alpha S \preceq \gamma S$, then by definition, there is a continuous function $f : \gamma S \to \alpha S$, mapping $\gamma S \backslash S$ onto $\alpha S \backslash S$ which fixes the points of S. We will represent this continuous function f as,

$$\pi_{\gamma \to \alpha} : \gamma S \to \alpha S$$

The function $\pi_{\gamma \to \alpha}$ explicitly expresses which compactification is larger than (or equal to) the other and so which is the domain and which is the range.

Note that a pair of compactifications need not necessarily be comparable in the sense that one need not necessarily by "less than" the other.

Suppose αS is any compactification of S possibly distinct from βS. We will now show that, in the partially ordered family, $\mathscr{C}$, of all compactifications of S, $\alpha S \preceq \beta S$. Showing this requires that we produce a continuous function $h : \beta S \to \alpha S$ such that $h(x) = x$, for all $x \in S$. If we can prove this, then we will have shown that

"βS is the unique maximal compactification of a completely regular space, S

Theorem 21.3 If αS is a compactification of S, then $\alpha S \preceq \beta S$.

Proof: We have proven above (theorem 21.6) that the inclusion map $i : S \to \alpha S$, extends to a continuous function $i^* : \beta S \to \alpha S$. Then $S \subseteq i^*[\beta S] \subseteq \alpha S$, where $i^*[\beta S]$ is compact, hence closed in αS. Since S is dense in αS, then the open set $\alpha S \setminus i^*[\beta S]$ must be empty. So the continuous function,

$$i^*[\beta S] = i^*[\mathrm{cl}_{\beta S} S] = \mathrm{cl}_{\alpha S} i^*[S] = \mathrm{cl}_{\alpha S} S = \alpha S$$

maps βS onto αS. So $\alpha S \preceq \beta S$.

Then for any compactification αS, there is the continuous function

$$\pi_{\beta \to \alpha} : \beta S \to \alpha S$$

which maps βS onto αS where $\pi_{\beta \to \alpha}$ fixes the points of S.

21.4 On C^*-embedded subsets.

Given a subset T of a topological space and a continuous bounded real-valued function $f : T \to \mathbb{R}$ it is not guaranteed that there is a bounded continuous function $g : S \to \mathbb{R}$ such that $g|_T = f$ on T. If there is then we will say that g is a continuous extension of f from T to S. This motivates the following definition.

Definition 21.4 Let (S, τ) be a topological space and U be a proper non-empty subset of S. We say that U is C^*-*embedded* in S if every real-valued bounded function, $f \in C^*(U)$, continuously extends to a function, $f^* \in C^*(S)$, in the sense that $f^*|_U = f$.

The notion of "C^*-embedding" is closely related to completely regular spaces and their Stone-Čech compactification. For this reason, we will discuss this property in depth now even though C^*-embeddings may be discussed in other contexts. The following theorem shows that for a completely regular space S, S is C^*-embedded in βS. That is, every function, $f : S \to \mathbb{R}$, in $C^*(S)$ extends continuously to $f^\beta : \beta S \to \mathbb{R}$.

Theorem 21.5 Let S be a completely regular space. Then S is C^*-embedded in βS.

Proof: If $f \in C^*(S)$ and let I_f be the range of f. Let

$$T = \prod_{f\in C^*(S)} \mathrm{cl}_{\mathbb{R}} I_f$$

Then the evaluation map, $e : S \to T$ embeds S in T. Recall that, by definition,

$$\beta S = \mathrm{cl}_T e[S] \subseteq \prod_{f\in C^*(S)} \mathrm{cl}_{\mathbb{R}} I_f$$

Suppose $g \in C^*(S)$. We are required to show that $g : S \to \mathbb{R}$ extends continuously to some function, $g^\beta : \beta S \to \mathbb{R}$.
If π_g is g^{th}-projection map then

$$\pi_g : \prod_{f\in C^*(S)} \mathrm{cl}_{\mathbb{R}} I_f \to \mathrm{cl}_{\mathbb{R}} I_g$$

where $\beta S = \mathrm{cl}_T e[S]$ and so,

$$\pi_g|_{\beta S} : \beta S \to \mathrm{cl}_{\mathbb{R}} I_g$$

maps βS into $\mathrm{cl}_{\mathbb{R}} I_g$. Let $g^\beta = \pi_g|_{\beta S}$. Then $g^\beta[\beta S] = g^\beta[\mathrm{cl}_T e[S]] = \mathrm{cl}_{\mathbb{R}} g[S] \subseteq \mathrm{cl}_{\mathbb{R}} I_g$.

It follows that $g^\beta : \beta S \to \mathrm{cl}_{\mathbb{R}} I_g$ and, since $g[S] \subseteq I_g$, for $x \in S$, $g^\beta|_S(x) = g(x)$.

So g^β is a continuous extension of g from S to βS.

Note that, in the case where S is a compact space, $e[S]$ is a compact space densely embedded in βS and so $\beta S \backslash S = \varnothing$. Then S and βS are homeomorphic.

The above theorem guarantees that every real-valued continuous bounded function, f, on a completely regular space, S, extends to a continuous function $f^\beta : \beta S \to \mathbb{R}$. The function f^β is normally referred to as the *extension of* f from S to βS. Recall that continuity guarantees that two continuous functions which agree on a dense subset D of a Hausdorff space, S, must agree on all of S. So there can only be one extension, f^β, of f from S to βS.

We will soon show that, if αS is a compactification of S and S is C^*-embedded in αS then αS must be the compactification, βS. That is,

...βS is the only compactification in which S is C^*-embedded

A generalization of the extension, $f \to f^\beta$. The above theorem can be generalized a step further. Suppose $C(S, K)$ denotes all continuous functions mapping S into a compact set K. We show that every function f in $C(S, K)$ extends to a function $f^{\beta(K)} \in C(\beta S, K)$. Note that neither f or $f^{\beta(K)}$ need be real-valued. The space, K, represents any compact set which contains the image of S under f.

Theorem 21.6 Let S be a completely regular (non-compact) space and $g : S \to K$ be a continuous function mapping S into a compact Hausdorff space, K. Then g extends uniquely to a continuous function, $g^{\beta(K)} : \beta S \to K$.

Proof: We are given that $g : S \to K$ continuously maps the completely regular space, S, into the compact Hausdorff space K. We are required to show that g extends to $g^{\beta(K)} : \beta S \to K$.

Since K is compact Hausdorff it is completely regular; hence there exists a function (the evaluation map) which embeds the compact set K in $V = \prod_{i \in J}[0, 1]$. Since V contains a topological copy of K let us view K as a subset of V.

Since $g : S \to K$ then, for every $x \in S$, $g(x) = \{g_i(x) : i \in J\} \in K \subseteq [0, 1]^J$.

Since S is C^*-embedded in βS then, for each $i \in J$, $g_i : S \to [0, 1]$ extends to $g_i^\beta : \beta S \to [0, 1]$.

We define the function $g^{\beta(K)} : \beta S \to V$ as

$$g^{\beta(K)}(x) = \{g_i^\beta(x) : i \in J\} \in V = \prod_{i \in J}[0, 1]$$

Since g_i^β is continuous on βS, for each i, then $g^{\beta(K)} : \beta S \to \prod_{i \in J}[0, 1]$ is continuous on βS.

See that $g^{\beta(K)}[\beta S] = g^{\beta(K)}[\mathrm{cl}_{\beta S}(S)] = \mathrm{cl}_V(g[S]) \subseteq \mathrm{cl}_V(K) = K$.

Since $g^{\beta(K)}|_S = g$ then $g : S \to K$ continuously extends to $g^{\beta(K)} : \beta S \to K$ on βS. As required.

While we are considering this topic we present a miscellany of results which will help us more easily recognize C^*-embedded subsets. We will return to our discussion of compactification immediately following this. The simplest example of a C^*-embedded subset is a compact set.

Theorem 21.7 If K is a compact subset of $\mathbb{R}$ then K is C^*-embedded in $\mathbb{R}$.

Proof: Let K be a compact subset of $\mathbb{R}$ and $f : K \to \mathbb{R}$ be a continuous function on K. Since every continuous real-valued function is bounded on a compact subset then $f \in C^*(K)$. Suppose $u = \sup\{K\}$ and $v = \inf\{K\}$. Since K is closed and bounded in $\mathbb{R}$ then u and v belong to K. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a function such that g agrees with f on K, $g(x) = f(u)$ if $x \geq u$ and $g(x) = f(v)$ if $x \leq v$. It is easily verified that g is a continuous extension of f from K to $\mathbb{R}$. Then K is C^*-embedded in $\mathbb{R}$.

Example 1. The set $\mathbb{N}$ is C^*-embedded in $\mathbb{R}$. One way of visualizing this is to plot the points of $\{(n, f(n)) : n \in \mathbb{N}\}$ of a function $f \in C^*(\mathbb{N})$ in the Cartesian plane $\mathbb{R}^2$ and join every pair of successive points $(n, f(n))$ and $(n+1, f(n+1))$ by a straight line. This results in a continuous curve representing a continuous function g on $\mathbb{R}$ which extends f.

The following theorem often referred to as *Urysohn's extension theorem* provides an important and useful tool for recognizing C^*-embedded sets.[1]

Theorem 21.8 *Urysohn's extension theorem.* Let T be a subset of the completely regular space S. Then T is C^*-embedded in S if and only if pairs of sets which can be completely separated by some function in $C^*(T)$ can also be separated by some function in $C^*(S)$.

Proof: In order to maintain our attention on the main flow of ideas in this chapter the proof of this theorem is relegated to the appendix of this book.

Example 2. Use Urysohn's extension lemma to show that any compact subset, K, of a completely regular space, S, is C^*-embedded in S.

Solution: Let U and V be disjoint closed subsets of the compact set K. The set K is normal so U and V are contained in disjoint zero-sets A and B. Since K is compact, A and B are compact in S. Then A and B are completely separated in S. By Urysohn's

[1] The Urysohn's extension theorem should not be confused with the *Urysohn's lemma*. Urysohn's lemma states that "The topological space (S, τ_S) is *normal* if and only if given a pair of disjoint non-empty closed sets, F and W, in S there exists a continuous function $f : S \to [0,1]$ such that, $F \subseteq f^{\leftarrow}[\{0\}]$ and $W \subseteq f^{\leftarrow}[\{1\}]$"

extension lemma, K is C^*-embedded in S.

Uniqueness of βS. We mentioned early that there can be many compactifications of a locally compact Hausdorff space. At this point we have provided an example of only one called the Stone Čech compactification.

We are now able to prove that, up to equivalence, the Stone-Čech compactification of S is the only compactification in which S is C^*-embedded. By this we mean that, if S is C^*-embedded in the compactification, γS, of S, then γS is equivalent to βS. So the symbol, βS, is strictly reserved to represent the Stone-Čech compactification of S.

Theorem 21.9 The completely regular space S is C^*-embedded in the compactification, γS, if and only if $\gamma S \equiv \beta S$.

Proof : We are given that S is completely regular.

($\Leftarrow$) To say that γS and βS are equivalent means that there is a homeomorphism, $h : \gamma S \to \beta S$, mapping γS onto βS such that $h(x) = x$, for all $x \in S$. If $f \in C^*(S)$ then f extends to $f^* : \beta S \to \mathbb{R}$. Then f extends to $f^{\wedge\wedge} : \gamma S \to \mathbb{R}$ via $f^{\wedge\wedge}(x) = f^*{\circ}h(x)$. Then S is C^*-embedded in γS.

($\Rightarrow$) Suppose S is C^*-embedded in γS. Let $i : S \to S$ be the identity map. Then, by theorem 21.6, i extends to $i^* : \beta S \to \gamma S$. Also, just as shown in the proof of theorem 21.6, $i^{\leftarrow}$ extends to $i^{\leftarrow\wedge} : \gamma S \to \beta S$. Then $i^{\leftarrow}{\circ}i$ and $i{\circ}i^{\leftarrow}$ are both identity maps on S and, since S is dense in both βS and γS, respectively, then $i^*{\circ}i^{\leftarrow\wedge}$ and $i^{\leftarrow\wedge}{\circ}i^*$ are identity maps on γS and on βS, respectively. Then both i^* and $i^{\leftarrow\wedge}$ are homeomorphisms. Hence γS and βS are equivalent compactifications.

Example 3. Show that if F is a closed subset of a metric space S then F is C^*-embedded in S.

Solution : Let F be a closed subset of the metric space S. Let A and B be completely separated in F. Then, by definition, there is a function f in $C^*(F)$ such that $A \subseteq Z(f)$ and $B \subseteq Z(f-1)$. Then $\mathrm{cl}_F A \subseteq Z(f)$ and $\mathrm{cl}_F B \subseteq Z(f-1)$. Since F is closed in S then so are the disjoint sets $\mathrm{cl}_F A$ and $\mathrm{cl}_F B$. It is shown on page 174 that in metric spaces closed subsets are the same as zero-sets. So $\mathrm{cl}_F A$ and $\mathrm{cl}_F B$ are disjoint zero-sets in S, say $\mathrm{cl}_F A = Z(f)$ and $\mathrm{cl}_F B = Z(g)$ in S. If

$$h = \frac{|f|}{|f| + |g|}$$

on S, $\mathrm{cl}_F A = Z(h)$ and $\mathrm{cl}_F B = Z(h-1)$ in S. So A and B are completely separated in S. By Urysohn's extension lemma every closed subset of a metric space is C^*-embedded.

Because of this, it is useful to remember that any closed subset of $\mathbb{R}$ is C^*-embedded in $\mathbb{R}$.

21.5 Associating compactifications to subalgebras of $C^*(S)$.

To each compactification αS we can associate a subset, $C_\alpha(S)$, of $C^*(S)$ such that

$$C_\alpha(S) = \{f|_S : f \in C(\alpha S)\}$$

Theorem 21.10 Let αS and γS be two compactifications of S such that $\alpha S \preceq \gamma S$. Let $C^*_\alpha(S)$ denote the set of all real-valued continuous bounded functions on S that extend to αS and $C^*_\gamma(S)$ denote the set of all real-valued continuous bounded functions on S that extend to γS. Then $C^*_\alpha(S) \subseteq C^*_\gamma(S)$.

Proof: We are given that $\alpha S \preceq \gamma S$. Then there is a continuous function $\pi_{\gamma\to\alpha} : \gamma S \to \alpha S$ such that $\pi_{\gamma\to\alpha}(x) = x$ on S. Suppose $t \in C^*_\alpha(S)$ and $t^\alpha : \alpha S \to \mathbb{R}$ is such that $t^\alpha|_S = t$. Define the function $g : \gamma S \to \mathbb{R}$ as $g = t^\alpha \circ \pi_{\gamma\to\alpha}$. Since $\pi_{\gamma\to\alpha} : \gamma S \to \alpha S$ and $t^\alpha : \alpha S \to \mathbb{R}$ are both continuous then g is continuous on γS and $g|_S(x) = (t\circ\pi_{\gamma\to\alpha})(x) = t(x)$. So $t = g|_S \in C^*_\gamma(S)$. Hence $C^*_\alpha(S) \subseteq C^*_\gamma(S)$.

The following theorem shows that there are as many compactifications of S as there are subalgebras of $C^*(S)$ (up to homomorphisms).

Theorem 21.11 Let (S,τ) be a completely regular space and $\mathscr{F} \subseteq C^*(S)$. Then $\mathscr{F}$ is a subalgebra[1] of $C^*(S)$ which contains the constant functions and separates points and closed sets in S if and only if there is a compactification, αS, of S such that

$$\mathscr{F} = C_\alpha(S) = \{f|_S \in C^*(S) : f \in C(\alpha S)\}$$

[1] A subalgebra is a subset of $C^*(S)$ closed under all its operations.

Proof: ($\Rightarrow$) We are given that (S, τ) is a completely regular space. Suppose $\mathscr{F}$ is a subalgebra of $C^*(S)$ which contains the constant functions and separates points and closed sets in S.

Let $T = \prod_{f\in\mathscr{F}} f[S]$. Since $\mathscr{F}$ is a subalgebra which contains the constant functions and separates points and closed sets in S the evaluation map $e_{\mathscr{F}} : S \to \prod_{f\in\mathscr{F}} f[S]$ generated by $\mathscr{F}$ embeds S in $T = \prod_{f\in\mathscr{F}} f[S]$. (See theorem 7.14). Since each function $f \in \mathscr{F}$ extends continuously to f^β on βS, then $e_{\mathscr{F}} : S \to \prod_{f\in\mathscr{F}} \mathbb{R}$ extends continuously to $e^\beta_{\mathscr{F}} : \beta S \to \prod_{f\in\mathscr{F}} \mathbb{R}$. Let $q : \beta S \to T$ be the quotient map induced by $e^\beta_{\mathscr{F}}$ which collapses the fibers[1] of $e^\beta_{\mathscr{F}}$ to points. Then $q[\beta S]$ is compact (since q is continuous on βS). Since $\mathscr{F}$ separates points and closed sets of S then q fixes the points of S, and so $q[\beta S]$ is a compactification, say αS, of S.

If $f \in \mathscr{F}$ then $f^\alpha : \alpha S \to \mathbb{R}$ defined as $f^\alpha(u) = f^\beta(q^\leftarrow(u))$ is a continuous extension of $f : S \to \mathbb{R}$ to αS. So $f^\alpha \in C(\alpha S)$. Then every function in $\mathscr{F}$ extends to αS. Then $\mathscr{F} \subseteq C_\alpha(S)$. Since $C_\alpha(S)$ generates the same compactification $\alpha S = \mathrm{cl}_T e_{\mathscr{F}}[S]$ then $C_\alpha(S) \subseteq \mathscr{F}$. So $\mathscr{F} = C_\alpha(S)$.

($\Leftarrow$) This direction is straightforward and so the proof is left for the reader.

Suppose γS is a compactification of S and $C_\gamma(S) = \{f|_S \in C(S) : f \in C(\gamma S)\}$. That is, $C_\gamma(S)$ is the set of all function, f, in $C^*(S)$ which extend to $f^\gamma : \gamma S \to \mathbb{R}$. We have shown that $\gamma S \preceq \beta S$ and $C_\gamma(S)$ is a subalgebra of $C^*(S)$. We have also seen that there is a continuous map $\pi_{\beta\to\gamma} : \beta S \to \gamma S$ which fixes the points of S.

In the following lemma we express the function $\pi_{\beta\to\gamma} : \beta S \to \gamma S$ in a form which better describes the mechanism behind the function itself.

Lemma 21.12 Let γS be a compactification of the space S. Let $\mathscr{G} = C_\gamma(S)$. Then

$$\pi_{\beta\to\gamma} = {e_{\mathscr{G}}}^{\gamma\leftarrow} \circ {e_{\mathscr{G}}}^{\beta}$$

where $e_{\mathscr{G}}$ is the evaluation map generated by $\mathscr{G}$.

Proof: If $f \in C_\gamma(S)$, for $x \in \beta S$, $f^\beta(x) = (f^\gamma \circ \pi_{\beta\to\gamma})(x)$. Then

$$\begin{aligned} {e_{\mathscr{G}}}^{\beta}(x) &= < f^\beta(x) >_{f\in C_\gamma(S)} \\ &= < (f^\gamma \circ \pi_{\beta\to\gamma})(x) >_{f\in C_\gamma(S)} \\ &= < f^\gamma(\pi_{\beta\to\gamma}(x)) >_{f\in C_\gamma(S)} \\ &= {e_{\mathscr{G}}}^{\gamma}(\pi_{\beta\to\gamma}(x)) \\ &\in ({e_{\mathscr{G}}}^{\gamma} \circ \pi_{\beta\to\gamma})[\beta S] \end{aligned}$$

[1]Reminding the reader that the fibers are those subsets in βS on which $e^\beta_{\mathscr{F}}$ is constant.

Then $e_{\mathscr{G}}{}^{\beta}[\beta S] = (e_{\mathscr{G}}{}^{\gamma} \circ \pi_{\beta\to\gamma})[\beta S]$. Then $\pi_{\beta\to\gamma} = e_{\mathscr{G}}{}^{\gamma\leftarrow} \circ e_{\mathscr{G}}{}^{\beta}$.

21.6 Zero-sets in relation to compactifications.

Suppose $\mathscr{Z} = \{Z(f) : f \in M \subseteq C^*(S)\}$ is a free z-ultrafilter in the locally compact Hausdorff space S, where M is the corresponding free maximal ideal in $C^*(S)$. Then, since βS is compact Hausdorff and the corresponding family of of sets

$$\mathscr{Z}^* = \{\text{cl}_{\beta S} Z(f) : f \in M\}$$

is an ultrafilter of sets and so satisfies the finite intersection property, $\mathscr{Z}^*$ must have non-empty intersection in βS. Then it is fixed and so must have a unique limit point, $p = \cap\{\text{cl}_{\beta S} Z(f) : f \in M\}$ in $\beta S \setminus S$. We clearly have $\text{cl}_{\beta S} Z(f) \subseteq Z(f^\beta)$ for each $f \in M \subseteq C^*(S)$. Since $f^\beta|_{Z(f)}$ agrees with f^β on $\text{cl}_{\beta S} Z(f)$ then its extension to $Z(f^\beta)$ agrees with f^β on $Z(f^\beta)$. So

$$\text{cl}_{\beta S} Z(f) = Z(f^\beta)$$

So for any free z-ultrafilter, $\mathscr{Z} = \{Z(f) : f \in M \subseteq C^*(S)\}$, in $Z[S]$ we can write,

$$\{p\} = \cap\{\text{cl}_{\beta S} Z(f) : f \in M\} = \cap\{Z(f^\beta) : p \in Z(f^\beta)\}$$

where $p \in \beta S\backslash S$. So every point in $\beta S\backslash S$ is the limit of some free z-ultrafilter in $Z[S]$. Of course, if $\mathscr{Z}$ is a fixed z-ultrafilter in $Z[S]$ then

$$\{p\} = \cap\{Z(f) : f \in M\}$$

for some p in S.

Recall that a topological space is said to be *pseudocompact* if every continuous real-valued function on S is bounded. That is, if $C(S) = C^*(S)$. Although the pseudocompact property has a simple (and easily understandable) definition, it turns out that, when not compact, such spaces are not easy to recognize just based on its topological properties. It will be helpful to obtain a few characterizations. We can now present an interesting characterization of pseudcompactness for those spaces which are locally compact and Hausdorff. Note that, when we speak of a zero-set, $Z(f)$, in S, it is in the most general sense where f may be continuous real-valued and *unbounded*. In the following theorem, we show that pseudocompact spaces are precisely those spaces, S, where $\beta S\backslash S$ does not contain a zero-set.

Theorem 21.13 A locally compact Hausdorff space S is pseudocompact if and only if no zero set Z in $Z[\beta S]$ is entirely contained in $\beta S\backslash S$.

Proof: Let S be a locally compact Hausdorff space.

($\Rightarrow$) Suppose S is a pseudocompact space and $Z(f^{\beta}) \in Z[\beta S]$. If $Z(f^{\beta}) \subseteq \beta S\backslash S$ then, since f is not zero in S, the function $g = 1/f$ is a well-defined function and so belongs to $C(S)$. Let $z \in Z(f^{\beta})$. Since z belongs to $\text{cl}_{\beta S}S$ then there is a net $\{x_i\}$ in S which converges to z. By continuity, the corresponding net, $\{f^{\beta}(x_i)\}$ in $\mathbb{R}$, must converge to $f^{\beta}(z) = 0$. So g is unbounded on S, which contradicts the hypothesis. So $Z(f^{\beta})$ must intersect with S.

($\Leftarrow$) Suppose now that, if $Z \in Z[\beta S]$, then $Z \cap S \neq \varnothing$. Let $f \in C(S)$. We are required to show that $f \in C^*(S)$.

Suppose $f \in C(S)\backslash C^*(S)$. We can assume $|f| \geq k > 0$. Then there is a well-defined function, $g = 1/f \in C^*(S)$, such that $Z(g^{\beta}) \cap S = \varnothing$, contradicting our hypothesis. So $f \in C^*(S)$. By definition, S is pseudocompact

---

21.7 The one-point compactification.

It was shown in theorem 18.8, that every locally compact Hausdorff space is completely regular. Hence a locally compact Hausdorff space, S, has at least one compactification, namely, βS. This compactification is maximal when compared to all others in the family, $\mathscr{C}$, of all compactifications. Does the family $\mathscr{C}$ contain a minimal element? That is, does $\mathscr{C}$ have a compactification, γS, such that $\gamma S \preceq \alpha S$, for all compactifications, αS in $\mathscr{C}$? The answer will depend on the space, S. In a previous chapter of the book, we, in fact, provided an answer to this question, as we shall soon see.

In theorem 18.7, we showed that given a locally Hausdorff space (S, τ) and a point $\omega \notin S$, we can construct a larger set,

$$S_{\omega} = S \cup \{\omega\}$$

By first defining, $\mathscr{B}_{\omega} = \{U \cup \{\omega\} : U \in \tau \text{ and } S\backslash U \text{ is compact }\}$, we then define a topology, τ_{ω}, on S_{ω} as follows:

$$\tau_{\omega} = \tau \ \cup \ \mathscr{B}_{\omega}$$

We then showed that $(S_{\omega}, \tau_{\omega})$ is a compact space which densely contains S. Then $(S_{\omega}, \tau_{\omega})$ satisfies the definition of a compactification of S. Furthermore, and quite importantly, we show that S_{ω} is Hausdorff if and only if S is locally compact.

So a non-compact, locally compact Hausdorff space, S, has a compactification, S_{ω}, which may be different from βS. Evidently, it is the smallest compactification a non-compact space, S, can have. We formally define it.

Definition 21.14 Let (S,τ) be a locally compact Hausdorff topological space, ω be a point not in S and $S_\omega = S \cup \{\omega\}$. If

$$\mathscr{B}_\omega = \{U \cup \{\omega\} : U \in \tau \text{ and } S \backslash U \text{ is compact }\}$$

and $\tau_\omega = \tau \cup \mathscr{B}_\omega$ then (S_ω, τ_ω) is called *the one-point compactification* of S.[1] We will, more succinctly, denote the one-compactification of S by, ωS.

It is worth emphasizing the fact that

> "...for a space S to have a one-point compactification which is Hausdorff, S must be locally compact".

We can even say more. Amongst all the the completely regular spaces, S, the only ones that are open in any compactification are the ones where S is locally compact. We will prove this now.

Theorem 21.15 Let S be a completely regular topological space and αS be any compactification of S. Then S is open in αS if and only if S is locally compact.

Proof : We are given that S is completely regular and αS is a compactification of S.

($\Rightarrow$) Suppose S is open in αS. Since the space S is the intersection of the open set S and the closed set αS, by theorem 18.3, S is locally compact.

($\Leftarrow$) Suppose S is locally compact in αS. By theorem 18.3 part d), S is the intersection of an open subset, U, and a closed subset, F. Since $F = \alpha S$ and S is dense in αS, then S is open in αS, as required.

If S is locally compact then the map, $\pi_{\beta\to\omega} : \beta S \to \omega S$, collapses the set $\beta S \backslash S$ down to the singleton set $\{\omega\}$.

---

[1] The one-point compactification of S is also referred to as the *Alexandrov compactification* of S, named after the soviet mathematician, Pavel Alexandrov, (1896-1982).

Theorem 21.16 *Uniqueness of ωS.* Let S be a locally compact completely regular topological space. Suppose αS and γS are both compactifications of S which contain only one point in their compact extension. Then they are equivalent compactifications.

Proof: We are given that S is locally compact completely regular. Suppose $\alpha S \backslash S = \{\omega_\alpha\}$ and $\gamma S \backslash S = \{\omega_\gamma\}$, both singleton sets.

Consider the map, $h : \alpha S \to \gamma S$, where $h(x) = x$ on S and $h(\omega_\alpha) = \omega_\gamma$. Then h is one-to-one and onto. It will suffice to show that h maps open neighbourhoods to open neighbourhoods.

Let U_α be any open neighbourhood of a point in αS. If $U_\alpha \subseteq S$ then U_α is open in S. Then $h[U_\alpha] = U_\alpha$. Since S is open in γS, $U_\alpha = U_\alpha \cap S$ is open in γS.

Suppose $\omega_\alpha \in U_\alpha$. Since $\alpha S \backslash U_\alpha$ is compact and $h|_S$ is continuous, then $h[\alpha S \backslash U_\alpha]$ is a compact. Since h is one-to-one, $h[\alpha S \backslash U_\alpha] = h[\alpha S] \backslash h[U_\alpha] = \gamma S \backslash h[U_\alpha]$, a compact set which doesn't contain ω_γ. So $h[U_\alpha]$ is open. Hence $h : \alpha S \to \gamma S$ is a homeomorphism. We can conclude that $\alpha S \equiv \gamma S$.

From this theorem we can conclude that the one-point compactification is unique, up to equivalence. We now consider a few examples involving compactifications of a space.

Example 4. Suppose that S is locally compact and its one-point compactification, ωS, of S is metrizable. Show that S must be second countable.

Solution : Suppose ωS is metrizable. In the proof of theorem 15.8 it is shown that countably compact metric spaces are separable. By theorem 5.11, a separable metric space is second countable. Since ωS is compact and so is countably compact then, by combining these two results we obtain that ωS is second countable. By theorem, 5.13, subspaces of second countable spaces are second countable. So S is second countable, as required.

The converse of the statement in this example is true. That is, "Locally compact second countable spaces have a metrizable one-point compactification" has been proven. But its proof is fairly involved. So we will not show it here.

Example 5. Consider the set $S = (0, 1]$ equipped with the usual subspace topology. Determine ωS. Is it possible that ωS is equivalent to βS?

Solution : Since [0, 1] is a compact set which densely contains S, and the one-point compactification is unique, then $\omega S = [0, 1]$. Note that, since the function $f(x) = \sin \frac{1}{x}$ is a bounded continuous function on S, then it extends to βS. Since f does not extend continuously to $[0, 1]$ then $[0, 1]$ is not the Stone-Čech compactification of S. But we can at least say that βS maps onto $[0,1]$ while fixing the points of S.

Theorem 21.17 If S is the ordinal space $[0,\omega_1)$ (where ω_1 is the first uncountable ordinal) then $\beta S \backslash S = \{\omega_1\}$ and so $\beta S = \omega S = [0,\omega_1]$, the one-point compactification of S.

Proof: In the example on page 256, it is shown that $S = [0,\omega_1)$ is countably compact. It is of course non-compact since the open cover $\{[0,\gamma) : \gamma \in S\}$ has no finite subcover. In theorem 15.9 it is shown that if $f \in C^*(S)$, then $f[S]$ is compact in $\mathbb{R}$.

If $f : S \to \mathbb{R}$ is bounded and continuous then $f[[0,\omega_1)]$ forms a net,

$$N = \{f(\alpha) : \alpha \in S\} = f[S]$$

in the compact set $f[S]$. Since f maps S into compact $f[S]$ then, by theorem 21.6, f extends to f^β and so $f^\beta[\beta S] = f[S]$. Suppose q and q^* are two elements in $f^\beta[\beta S \backslash S] \subseteq f[S]$.

We claim that $q = q^*$. Both q and q^* are accumulation points of the net $f[S]$. So, for each $n \in \mathbb{N}$, both $B_{1/2n}(q)$ and $B_{1/2n+1}(q^*)$ each contain a cofinal subset of the tail end of the net N. We can then choose, $\{\alpha_n : n \in \mathbb{N}\}$ strictly increasing in S such that $f(\alpha_{2n}) \in B_{1/2n}(q)$ and $f(\alpha_{2n+1}) \in B_{1/2n+1}(q^*)$. If $\sup\{a_n\} = \kappa$ then $\sup\{\alpha_{2n}\} = \kappa = \sup\{\alpha_{2n+1}\}$. Hence

$$\lim_{n\to\infty} f(\alpha_{2n}) = q = f(\kappa) = q^* = \lim_{n\to\infty} f(\alpha_{2n+1})$$

So $q = q^*$ as claimed.

From this we can conclude that, for all $f \in C^*(S)$, f^β is constant on $\beta S \backslash S$. Since βS is completely regular, then $\beta S = [0,\omega_1] = \omega S$, the one-point compactification of S.

21.8 Topic: The Stone-Čech compactification of a product.

We know from theorem 7.10 that if $U_\alpha \subseteq S_\alpha$ and $\prod_{\alpha\in I} U_\alpha \subseteq S$,

$$\mathrm{cl}_S\left(\prod_{\alpha\in I} U_\alpha\right) = \prod_{\alpha\in I}\left(\mathrm{cl}_{S_\alpha} U_\alpha\right)$$

But one should be cautious about $S = \prod_{i\in J} S_i$ when βS is involved.

If we compactify each S_i with $\mathrm{cl}_{\beta S_i} S_i$, then the Tychonoff theorem will impose a compactification on $\prod_{i\in J} S_i$. But this compactification of S need not be βS. It is not true in general that $\beta S = \prod_{i\in J} \beta S_i$. It is true that $\prod_{i\in J} \mathrm{cl}_{\beta S_i} S_i$ is a compact space that contains $S = \prod_{i\in J} S_i$. Equality

$$\beta(\prod_{i\in J} S_i) = \prod_{i\in J} \beta S_i$$

will hold true only in the case where $\prod_{i\in J} S_i$ is pseudocompact.

21.9 Topic: Cardinality of some common Stone-Čech compactifications.

We know the cardinality of the most common sets we encounter (such as $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}^n$). We can sometimes determine the cardinality of sets as abstract as their Stone-Čech compactifications. We know the cardinality, $|\mathbb{N}|$, of the set $\mathbb{N}$ is $\aleph_0$ while $|\mathbb{R}| = c = 2^{\aleph_0}$.

Theorem 21.18 The cardinality, $|\beta\mathbb{N}|$, of the set $\beta\mathbb{N}$ is 2^c.

Proof: We first claim that $|\beta\mathbb{N}| \geq 2^c$.

In theorem 7.5, it is shown that, since $[0,1]$ is separable then the product space, $K = \prod_{i\in\mathbb{R}}[0,1]$, with $|\mathbb{R}| = c$ factors, is separable. This means that K contains a dense countably infinite subset, say, D. Then there exist a function $g : \mathbb{N} \to D$ mapping $\mathbb{N}$ onto D. Since K is compact and g is continuous, g extends continuously to $g^{\beta(K)} : \beta\mathbb{N} \to K$. Then

$$K = \mathrm{cl}_K g^{\beta(K)}[\mathbb{N}] = g^{\beta(K)}[\mathrm{cl}_{\beta\mathbb{N}}\mathbb{N}] = g^{\beta(K)}[\beta\mathbb{N}]$$

Now $|K| = |\prod_{i\in\mathbb{R}}[0,1]| = c^c = 2^c$. (See footnote)[1] Since $\beta\mathbb{N}$ is the domain of the function $g^{\beta(K)}$ (which could possibly not be one-to-one), then

$$|\beta\mathbb{N}| \geq |K| = 2^c$$

as claimed.

We now claim that $|\beta\mathbb{N}| \leq |K| = 2^c$.

We know that each function in $C^*(\mathbb{N})$ can be seen as a sequence $\{x_i : i \in \mathbb{N}\}$ in $\mathbb{R}^\mathbb{N}$. Then

$$I = |C^*(\mathbb{N})| = |\mathbb{R}^\mathbb{N}| = c^{\aleph_0} = 2^c = c \quad \text{(See footnote)}^2$$

That is, $C^*(\mathbb{N})$ contains c distinct functions. Then the product space $T = \prod_{i\in I}[a_i, b_i]$ has cardinality, $|\prod_{i\in I}[a_i, b_i]| = c^c = 2^c$.

---

[1]The proof of $|\prod_{i\in\mathbb{R}}[0,1]| = c^c = 2^c$ is shown in section 24.2 of *Axioms and Set theory*, R. André
[2]The proof of $|\mathbb{R}^\mathbb{N}| = 2^c = c$ is shown in theorem 25.2 of *Axioms and Set theory*, R. André

If $e_{C^*(\mathbb{N})}$ is the evaluation map generated by $C^*(\mathbb{N})$,

$$e_{C^*(\mathbb{N})}(n) \;=\; <f_i(n)>_{f_i \in C^*(\mathbb{N})} \in \; \prod_{i\in I}[a_i, b_i]$$

and so $e_{C^*(\mathbb{N})}$ embeds $\mathbb{N}$ into $T = \prod_{i\in I}[a_i, b_i]$, a compact set of cardinality 2^c.
So $|\beta\mathbb{N}| = |\mathrm{cl}_T e_{C^*(\mathbb{N})}[\mathbb{N}]| \leq 2^c$, as claimed.

So $|\beta\mathbb{N}| \leq 2^c$ and $|\beta\mathbb{N}| \geq 2^c$ implies $|\beta\mathbb{N}| = 2^c$.

Since we can associate to each point in $\beta\mathbb{N}\backslash\mathbb{N}$ a unique free z-ultrafilter in $Z[\mathbb{N}]$ and every free z-ultrafilter has a limit point in $\beta\mathbb{N}\backslash\mathbb{N}$ the above theorem confirms that there are 2^c free z-ultrafilters in $Z[\mathbb{N}]$.

From this result we can determine the cardinalities of $\beta\mathbb{R}$ and $\beta\mathbb{Q}$.

Theorem 21.19 The sets $\beta\mathbb{N}$, $\beta\mathbb{R}$ and $\beta\mathbb{Q}$ each have a cardinality equal to 2^c.

Proof: We have already shown that $|\beta\mathbb{N}| = 2^c$.

Claim 1: $|\beta\mathbb{Q}| \leq 2^c$.

We know $\mathbb{N}$ and $\mathbb{Q}$ are countable so both have cardinality $\aleph_0$. Then there exists a one-to-one function, $f : \mathbb{N} \to \beta\mathbb{Q}$, mapping $\mathbb{N}$ onto $\mathbb{Q}$. Since $\mathbb{N}$ is discrete f is continuous. By theorem 21.6, f extends to

$$f^{\beta(\beta\mathbb{Q})} : \beta\mathbb{N} \to \mathrm{cl}_{\beta\mathbb{Q}} f[\mathbb{N}] = \beta\mathbb{Q}$$

Then

$$f^{\beta(\beta\mathbb{Q})}[\beta\mathbb{N}] = f^{\beta(\beta\mathbb{Q})}[\mathrm{cl}_{\beta\mathbb{N}}\mathbb{N}] = \mathrm{cl}_{\beta\mathbb{Q}} f[\mathbb{N}] = \mathrm{cl}_{\beta\mathbb{Q}}\mathbb{Q} = \beta\mathbb{Q}$$

So $|\beta\mathbb{Q}| \leq |\beta\mathbb{N}| = 2^c$. This establishes claim 1.

Claim 2: $|\beta\mathbb{R}| \leq |\beta\mathbb{Q}|$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ then it is dense in $\beta\mathbb{R}$ so $\mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} = \beta\mathbb{R}$. Consider the continuous inclusion function $i : \mathbb{Q} \to \mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} = \beta\mathbb{R}$. By theorem 21.6, i extends to $i^{\beta(\beta\mathbb{R})} : \beta\mathbb{Q} \to \beta\mathbb{R}$.

Then

$$i^{\beta(\beta\mathbb{R})}[\beta\mathbb{Q}] = i^{\beta(\beta\mathbb{R})}[\mathrm{cl}_{\beta\mathbb{Q}}\mathbb{Q}] = \mathrm{cl}_{\beta\mathbb{Q}} i[\mathbb{Q}] = \mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} = \beta\mathbb{R}$$

So $|\beta\mathbb{R}| \leq |\beta\mathbb{Q}|$. This establishes claim 2.
Up to now we have shown that $|\beta\mathbb{R}| \leq |\beta\mathbb{Q}| \leq |\beta\mathbb{N}| = 2^c$.

Claim 3: $|\beta\mathbb{R}| \geq |\beta\mathbb{N}|$.

We know that $\mathbb{N}$ is C^*-embedded in $\mathbb{R}$. (See example on page 327 or theorem 21.8.) Then, if $i : \mathbb{N} \to \beta\mathbb{N}$ is the continuous inclusion map, since $\mathbb{N}$ is C^*-embedded in $\mathbb{R}$, $i : \mathbb{N} \to \beta\mathbb{N}$ extends to $i^* : \mathbb{R} \to \beta\mathbb{N}$. Also, $i^* : \mathbb{R} \to \beta\mathbb{N}$ extends to $i^{*\beta} : \beta\mathbb{R} \to \beta\mathbb{N}$. Then

$$\begin{aligned} \beta\mathbb{N} &= \mathrm{cl}_{\beta\mathbb{N}}\mathbb{N} \\ &= \mathrm{cl}_{\beta\mathbb{N}}i[\mathbb{N}] \\ &\subseteq \mathrm{cl}_{\beta\mathbb{N}}i^*[\mathbb{R}] \\ &= i^{*\beta}[\mathrm{cl}_{\beta\mathbb{R}}\mathbb{R}] \\ &= i^{*\beta}[\beta\mathbb{R}] \end{aligned}$$

Since $\beta\mathbb{N}$ is in the range of $\beta\mathbb{R}$, then $|\beta\mathbb{R}| \geq |\beta\mathbb{N}|$. This establishes claim 3.

Combining the results in the three claims above we conclude that $|\beta\mathbb{R}| = |\beta\mathbb{Q}| = |\beta\mathbb{N}| = 2^c$ as required.

21.10 About a subset T of $S \subseteq \beta S$

If T is a non-compact subset of S. It is interesting to reflect on how $\mathrm{cl}_{\beta S}T$ compares with βT. Does it make sense to say that $\beta T \subseteq \beta S$? We examine this question in the following example.

Example 6. Let T be a non-empty subspace of a completely regular space, S. Show that

$$\mathrm{cl}_{\beta S}T = \beta T \subseteq \beta S$$

Solution: We are given that $T \subseteq S$. Since subspaces of completely regular spaces are completely regular then T is completely regular. Let $i : T \to \beta S$ be the identity function which embeds T into βS. By theorem 21.6, $i : T \to \beta S$ extends continuously to $i^{\beta(\beta S)} : \beta T \to \beta S$. Then

$$\begin{aligned} \beta T &= i^{\beta(\beta S)}[\beta T] \\ &= i^{\beta(\beta S)}[\mathrm{cl}_{\beta T}T] \\ &= \mathrm{cl}_{\beta S}i[T] \\ &= \mathrm{cl}_{\beta S}T \\ &\subseteq \beta S \end{aligned}$$

Example 7. The compactification, $\beta\mathbb{N}$, is easily seen to be separable ($\mathbb{N}$ is a dense subset of $\beta\mathbb{N}$.). Show that $\beta\mathbb{N}\backslash\mathbb{N}$ is not separable.

Solution: If Z is a zero-set in $\mathbb{N}$ then it is clopen in $\mathbb{N}$. See that Z is a zero-set of a characteristic function, g, on $\mathbb{N}$. Since $g : \mathbb{N} \to \{0,1\}$ extends to $g^{\beta} : \beta\mathbb{N} \to \{0,1\}$, $Z(g^{\beta}) = \text{cl}_{\beta\mathbb{N}}Z$ is a clopen zero-set in $\beta\mathbb{N}$. As well, $\beta\mathbb{N}\backslash Z(g^{\beta})$ is a clopen zero-set in $\beta\mathbb{N}$.

Suppose that $\beta\mathbb{N}\backslash\mathbb{N}$ is separable. Then it has a dense countable subset $D = \{x_i : i \in \mathbb{N}\}$. Since the cardinality of $\beta\mathbb{N}$ is 2^c (shown above), there is a $q \in \beta\mathbb{N}\backslash\mathbb{N}$ not in D and $t \in \mathbb{N}$.

Since $\beta\mathbb{N}$ is normal, for each x_i there is a zero-set, $Y_i = \text{cl}_{\beta\mathbb{N}}Z_i$ such that $x_i \in Y_i$ and $\{q,t\} \subseteq W_i = \beta\mathbb{N}\setminus Y_i$. Then W_i is a zero-set clopen neighbourhood of $\{q,t\}$. Then $\{q,t\} \subseteq W = \cap\{W_i : i \in \mathbb{N}\}$. See that W is a G_δ. In a normal space, G_δ's are zero-sets (by theorem 10.10). So W is a zero-set of $\beta\mathbb{N}$ which intersects $\mathbb{N}$. Then $\text{cl}_{\beta\mathbb{N}}(W \cap \mathbb{N}) \cap \beta\mathbb{N}\backslash\mathbb{N}$ is a clopen zero-set neighbourhood of q in $\beta\mathbb{N}\backslash\mathbb{N}$ which does not intersect the dense subset D. A contradiction. So $\beta\mathbb{N}\backslash\mathbb{N}$ is not separable.

21.11 Topic: C-embedded subsets.

We see that the notion of C^*-embedding is often raised in discussions involving the Stone-Čech compactification. When we speak of a C^*-embedded subset U of a space S, we mean that any bounded real-valued function in $C^*(U)$ can be continuously extended to a function in $C^*(S)$. As a prime example we have the fact "S is C^*-embedded in βS". We now want a generalization from "the extension of a function in $C^*(U)$ to one in $C^*(S)$" to "the extension of a function $f \in C(U)$ to $f^* \in C(S)$". In the following chapters we will require a few facts involving C-embedded sets.

Definition 21.20 A subset U of topological space S is C-embedded in S if every continuous function f in $C(U)$ can be continuously extend to a function f^* in $C(S)$.

It is worth studying properties spaces with C-embedded subsets carefully because because they play an important role in the study of extensions of spaces, such as βS.

First we should mention that, since $C^*(U) \subseteq C(U)$, a set U which is C-embedded in S is C^*-embedded in S. For, if U is C-embedded and $f \in C^*(U)$ such that $|f| < M$ then f extends to $f^* \in C(S)$. Then $g = (f^* \vee -M) \wedge M$ is a continuous bounded function in $C^*(S)$ which agrees with f on U.

It is not true in general that a C^*-embedded set U in S is C-embedded in S. That is, is not true in general that if every function f in $C^*(U)$ extends to a function $f^* \in C^*(S)$ then if every function $f \in C(U)$ extends to a function $f^* \in C(S)$. But when a certain condition is satisfied, it is.

Theorem 21.21 Suppose the subset U of topological space S is C^*-embedded in S. Then U is C-embedded in S if and only if U is completely separated from any disjoint zero-set in S.

Proof: We are given that U is C^*-embedded in S.

($\Leftarrow$) Suppose that U is completely separated from any zero-set in S. We are required to show that U is also a C-embedded subset of S. Let $f : U \to \mathbb{R}$ be a function in $C(U)$. We are required to find a function $t \in C(S)$ such that $t|_U = f$.

We begin by constructing a zero-set, Q, in S which is disjoint from U. We construct the function, $(\arctan \circ f) : U \to (-\pi/2, \pi/2)$. Let $g = \arctan \circ f$. Then $|g| : U \to [0, \pi/2)$ is a function in $C^*(U)$. Since U is C^*-embedded in S, g extends to $g^* : S \to \mathbb{R}$ in $C^*(S)$.

The set $[\frac{\pi}{2}, \infty)$ is a closed G_δ in $\mathbb{R}$ and so is a zero-set in $\mathbb{R}$, say $Z(h)$. We thus obtain the zero-set

$$Q = Z(h \circ g^*) = (h \circ g^*)^{\leftarrow}(0) = g^{*\leftarrow}[h^{\leftarrow}(0)] = g^{*\leftarrow}\left[\frac{\pi}{2}, \infty\right)$$

in S. If $x \in U$, $g^*(x) \notin [\frac{\pi}{2}, \infty)$, so $g^*(x) \notin Q$. So Q is a zero-set in S such that $U \cap Q = \varnothing$. We can now apply the hypothesis.

The hypothesis states that there is a function, $k : S \to \mathbb{R}$ such that $|k| \leq 1$, $Q \subseteq Z(k)$ and $U \subseteq Z(k-1)$. Then $|gk|[S] \subseteq [0, \frac{\pi}{2})$ and $(gk)[U]] = g(1) = (\arctan \circ f)(1)$. So

$$(gk)|_U = (\arctan \circ f)|_U$$

Then $(\tan \circ gk) \in C(S)$ such that

$$(\tan \circ gk)[U] = (\tan \circ \arctan \circ f)[U] = f[U]$$

So $t = (\tan \circ gk)$ is a continuous extension in $C(S)$ such that $t|_U = f$. We are done with this direction.

($\Rightarrow$) Suppose U is C-embedded in S. Suppose that $Z(f)$ is a zero-set of a function $f \in C(S)$ disjoint from U. Let $k = \frac{1}{f}$ on U. Then $k \in C(U)$. Then $k : U \to \mathbb{R}$ extends to $k^* \in C(S)$. Then $k^* f \in C(S)$ where $U \subseteq Z(k^* f - 1)$ and $Z(f) \subseteq Z(k^* f)$. So U and $Z(f)$ are completely separated, as required.

Example 8. Show that every closed subset of $\mathbb{R}$ is C-embedded.

Solution: Let F be a closed subset of $\mathbb{R}$. By the previous example on metric spaces, every closed subset of $\mathbb{R}$ is C^*-embedded. Let $Z(f)$ be a zero-set which does not intersect F. Now recall that $\mathbb{R}$ is normal and so F and $Z(f)$ are completely separated (by 10.1). By the theorem above, F is C-embedded in $\mathbb{R}$.

Theorem 21.22 Suppose U is a subspace of a topological space S. If there exists a function $f \in C(S)$ such that $f|_U$ is a homeomorphism and $f|_U[U]$ is closed in $\mathbb{R}$, then U is C-embedded in S.

Proof: We are given a function $f \in C(S)$ which maps U homeomorphically onto a closed subset, $A = f[U]$, in $\mathbb{R}$. We are required to show that U is C-embedded in S. To do this we choose any h in $C(U)$. We claim there is a function $t \in C(S)$ such that $t|_U = h$.

See that $h{\circ}f|_U{}^{\leftarrow}$ continuously maps A to $h[U]$ in $\mathbb{R}$. So $h{\circ}f|_U \in C(A)$. Since A is a closed subset of $\mathbb{R}$ it is C-embedded in $\mathbb{R}$ (as shown in the example above). This means there is a function $g \in C(S)$ such that $g|_A = h{\circ}f|_U{}^{\leftarrow}$. Let $t = g{\circ}f \in C(S)$. Then

$$\begin{aligned} t|_U &= (g{\circ}f)|_U[U] \\ &= g[f[U]] \\ &= g[A] \\ &= (h{\circ}f|_U{}^{\leftarrow})[A] \\ &= h[U] \end{aligned}$$

So $t|_U = h$. So $h \in C(U)$ extends continuously to $t \in C(S)$. So U is C-embedded in S.

One consequence of the above theorem is the fact that ...

> *... if there is an unbounded real-valued function on S then S contains a C-embedded copy of $\mathbb{N}$.*

We briefly prove this statement: Without loss of generality suppose f is unbounded above.

For each $i \in \mathbb{N}$, choose $m_i \in [2i-1, 2i+1] \cap f[S]$, if non-empty and not equal to m_{i-1}. Since f is unbounded we can choose infinitely many such m_i's to construct the subset $N = \{m_i : i \in \mathbb{N}\}$ in $\mathbb{R}$. See that ...

- N is unbounded,
- Each m_i is clopen in N
- The set N is closed in $\mathbb{R}$: For any $q \in f[S]$ there an open ball $B_\varepsilon(q)$ which contains at most one point of N.

So N is a closed unbounded copy of $\mathbb{N}$ in $f[S]$.

For each $i \in \mathbb{N}$, choose $n_i \in f^{\leftarrow}(m_i)$ in S. If $A = \{n_i : i \in \mathbb{N}\}$, then $f|_A : A \to N \subseteq f[S]$ homeomorphically maps A onto the copy N of $\mathbb{N}$ in $f[S]$.

Since $f[A] = N$ is a homeomorphic copy of a closed subset of $\mathbb{R}$ then, by the theorem 21.22, A is C-embedded in S, as claimed.

It follows from this statement that ...

> *If S does* not *contain a C-embedded copy of* $\mathbb{N}$ *then $C(S)$ cannot contain an unbounded function and so S must be pseudocompact.*

The converse also holds true as we shall see. We present this as a theorem.

Theorem 21.23 The space S is pseudocompact if and only if S does not contain a C-embedded copy of $\mathbb{N}$.

Proof: ($\Leftarrow$) is proven in the paragraph above.

($\Rightarrow$) Suppose S is pseudocompact. Suppose S contains a C-embedded copy, A, of $\mathbb{N}$. Then there exists a function $f \in C(S)$ which maps A onto $\mathbb{N}$ in $\mathbb{R}$. Since such an f is unbounded in $C(S)$ this contradicts the fact that S is pseudocompact. The statement follows.

We present a couple of consequences of the three theorems above which will be useful the following chapters.

Corollary 21.24 Let S be a locally compact Hausdorff space and N be a C-embedded copy of $\mathbb{N}$ in S. Let U be an open neighbourhood of N in S. Then $\mathrm{cl}_{\beta S}(S\backslash U) \cap \mathrm{cl}_{\beta S} N = \varnothing$.

Proof: We are given that $N = \{n_i : i \in \mathbb{N}\}$ is a C-embedded copy of $\mathbb{N}$ in S and U be an open neighbourhood of N in S.

By local compactness, S is open in βS, so we can construct the family, $\{\mathrm{cl}_S V_i : i \in \mathbb{N}\}$, of pairwise disjoint compact sets such that V_i is an open neighbourhood of n_i and $\mathrm{cl}_S V_i \subseteq U$.

Then $S\backslash U \subseteq S\backslash \mathrm{cl}_S V_i \subseteq S\backslash V_i$, for each i. Since S is completely regular, for each i there a function $g_i \in C^*(S)$ such that

$$\begin{array}{rcl} n_i & \in & Z(g_i - 1) \\ S\backslash V_i & \subseteq & Z(g_i) \end{array}$$

Then

$$S\backslash U \subseteq Z = \cap\{Z(g_i) : i \in \mathbb{N}\}$$

where Z is a countable intersection of zero-sets such that $n_i \notin Z$ for all i, so $Z \cap N = \varnothing$. We have already explained on page 171 that countable intersections of zero-sets are zero-sets. Say, $Z = Z(t)$. So

$$Z(t) \cap N = \varnothing$$

Since N is C-embedded, by theorem 21.21, N is completely separated from $Z(t)$.

Then there is a continuous function $h : S \to [0,1]$ such that

$$\begin{array}{rcl} N & \subseteq & Z(h) \\ Z(t) & \subseteq & Z(h-1) \end{array}$$

It follows that,

$$\begin{array}{rcl} \mathrm{cl}_{\beta S} N & \subseteq & \mathrm{cl}_{\beta S} Z(h) = Z(h^\beta) \\ \mathrm{cl}_{\beta S}(S\backslash U) & \subseteq & \mathrm{cl}_{\beta S} Z(h-1) = Z(h^\beta - 1) \end{array}$$

Since $Z(h^\beta - 1) \cap Z(h^\beta) = \varnothing$ then $\mathrm{cl}_{\beta S}(S\backslash U) \cap \mathrm{cl}_{\beta S} N = \varnothing$, as required.

Corollary 21.25 Let S be a locally compact Hausdorff space and N be a C-embedded copy of $\mathbb{N}$ in S. Suppose Z be a zero-set in $\beta S \backslash S$ which contains $(\mathrm{cl}_{\beta S} N)\backslash N$. Then $(\mathrm{cl}_{\beta S} N)\backslash N \subseteq \mathrm{int}_{\beta S\backslash S} Z$.

Proof: We are given a C-embedded copy, $N = \{n_i : i \in \mathbb{N}\}$, of $\mathbb{N}$. We are also given that $(\mathrm{cl}_{\beta S} N)\backslash N \subseteq Z$ where Z is a zero-set in $\beta S\backslash S$. We are required to show that $(\mathrm{cl}_{\beta S} N)\backslash N \subseteq \mathrm{int}_{\beta S\backslash S} Z$.

There exists a function $g \in C^*(\beta S\backslash S)$ such that $Z = Z(g)$. Since $\beta S\backslash S$ is compact, $g : \beta S\backslash S \to \mathbb{R}$ extends to $f \in C(\beta S)$. Then

$$(\mathrm{cl}_{\beta S} N)\backslash N \subseteq Z(f)$$

Since $f[(\mathrm{cl}_{\beta S} N)\backslash N] = \{0\}$,

$$\{f(n_i) : i \in \mathbb{N}\} \longrightarrow 0$$

We construct an open neighbourhood, U, of N: By continuity of f, for each $i \in \mathbb{N}$, there exists an open neighbourhood, V_i, of n_i, in S such that $\{\mathrm{cl}_S V_i : i \in \mathbb{N}\}$ forms a family of pairwise disjoint compact subsets of S such that

$$f[V_i] \subseteq B_{1/i+1}(f(n_i))$$

Then, if $U = \cup\{V_i : i \in \mathbb{N}\}$, we have the C-embedded set N which has an open neighbourhood U in S.

By the corollary 21.24 above,

$$\mathrm{cl}_{\beta S}(S\backslash U) \cap \mathrm{cl}_{\beta S} N = \varnothing$$

Fact # 1: If B is an open interval of 0, then $f^{\leftarrow}[B]$ intersects infinitely many V_i's. To see this, for each $i \in \mathbb{N}$, choose $x_i \in V_i$.

Then,

$$\begin{array}{rcl} |f(x_i)| & = & |f(x_i) - f(n_i) + f(n_i)| \\ & \leq & |f(x_i)| - f(n_i)(x)| + |f(n_i)| \\ & < & 1/(i+1) + |f(n_i)| \end{array}$$

so $\{f(x_i) : i \in \mathbb{N}\} \longrightarrow 0$. We conclude that, if B is an open interval of 0, then $f^{\leftarrow}[B]$ intersects infinitely many V_i's, as stated.

Fact # 2: There exists a function, $h \in C(\beta S)$ such that $\mathrm{cl}_{\beta S} N \subseteq Z(h-1)$ and $\mathrm{cl}_{\beta S}(S\backslash U) \subseteq Z(h)$. This follows from the fact that βS is normal and $\mathrm{cl}_{\beta S}(S\backslash U) \cap \mathrm{cl}_{\beta S} N = \varnothing$. Then $\mathrm{cl}_{\beta S}(S\backslash U)$ and $\mathrm{cl}_{\beta S} N$ are completely separated so there exists $h \in C(\beta S)$ such that $\mathrm{cl}_{\beta S} N \subseteq Z(h-1)$ and $\mathrm{cl}_{\beta S}(S\backslash U) \subseteq Z(h)$, which confirms fact # 2.

So

$$\mathrm{cl}_{\beta S} N \ \subseteq \ \mathrm{coz}(h) = S\backslash Z(h) \ \subseteq \ S\backslash \mathrm{cl}_{\beta S}(S\backslash U). \quad (*)$$

To conclude we claim : $\beta S\backslash S \cap \mathrm{coz}(h) \subseteq \beta S\backslash S \ \cap Z(f)$.

Proof of claim: Let $p \in \beta S\backslash S \cap \mathrm{coz}(h)$. It suffices to show that $f(p) = 0$.

By $(*)$, $\mathrm{coz}(h) \subseteq S\backslash \mathrm{cl}_{\beta S}(S\backslash U)$, so p has a neighbourhood V such that $p \in V \subseteq S\backslash \mathrm{cl}_{\beta S}(S\backslash U)$.

Then we can write,

$$\begin{aligned} p &\in \mathrm{coz}(h) \\ &\subseteq S\backslash \mathrm{cl}_{\beta S}(S\backslash U) \\ &= \mathrm{int}_{\beta S}[\, S\backslash(S\backslash \cup\{V_i : i \in \mathbb{N}\})\,] \\ &= \mathrm{int}_{\beta S}[\, \cup\{V_i : i \in \mathbb{N}\}\,] \\ &\subseteq \mathrm{int}_{\beta S}[\, \cup\{\mathrm{cl}_{\beta S} V_i : i \in \mathbb{N}\}\,] \\ &= \mathrm{int}_{\beta S}[\, \cup\{\mathrm{cl}_S V_i : i \in \mathbb{N}\}\,] \end{aligned}$$

Then $p \in \mathrm{int}_{\beta S}[\, \cup\{\mathrm{cl}_S V_i : i \in \mathbb{N}\}\,]$.

Fact # 3: Any βS-open neighbourhood, V_p, of p must intersect infinitely many of the compact $\mathrm{cl}_S V_i$'s and so must meet infinitely many V_i's. This follows from the fact, $p \in \mathrm{int}_{\beta S}[\, \cup\{\mathrm{cl}_S V_i : i \in \mathbb{N}\}\,]$ (see what occurs if V_p intersects only finitely many $\mathrm{cl}_S V_i$'s). This confirms fact # 3.

Then, for each $j = 1, 2, 3, \ldots,\quad p \in f^{\leftarrow}[\,(f(p) - 1/j, f(p) + 1/j)\,]$. From Fact # 3, there exists infinitely many u_j's such that

$$u_j \in f^{\leftarrow}[(f(p) - 1/j, f(p) + 1/j)] \cap V_j$$

By fact # 1, $\{f(u_j) : j \in \mathbb{N}\} \longrightarrow 0$. Since $\{f(u_j) : j \in \mathbb{N}\} \longrightarrow f(p)$ then $f(p) = 0$. So $\beta S\backslash S \cap \mathrm{coz}(h) \subseteq \beta S\backslash S\ \cap Z(f)$. This establishes the claim.

Since $\mathrm{cl}_{\beta S} N \backslash N \subseteq \mathrm{coz}(h) \subseteq Z(f)$, then $\mathrm{cl}_{\beta S} N \backslash N$ belongs to the interior of every zero-set in $\beta S\backslash S$ which contains $\mathrm{cl}_{\beta S} N \backslash N$.

Concepts review:

1. Suppose S is a topological space and T is a compact Hausdorff space. What does it mean to say that T is a compactification of S?

2. If S has a compactification, αS, what separation axiom is guaranteed to be satisfied by S?

3. Given a completely regular space S let $e : S \to \pi_{i\in I}[a_i, b_i]$ be the evaluation map on S induced by $C^*(S)$. Give a definition of the Stone-Čech compactification of S which involves this evaluation map.

4. What does it mean to say that the two compactifications of S, αS and γS, are equivalent compactifications?

5. If $\mathscr{C} = \{\alpha_i S : i \in I\}$ denotes the family of all compactifications of S. Define a partial ordering of $\mathscr{C}$.

6. If U is a subset of the topological space S, what does it mean to say that U is C^*-embedded in S?

7. If S is C^*-embedded in the compactification, αS, of S what can we say about αS?

8. Suppose S is completely regular and $g : S \to K$ is a continuous function mapping S into a compact Hausdorff space K. For which compactifications, αS, does the following statement hold true: "the function g extends to a continuous function $g^* : \alpha S \to K$"?

9. Suppose S is locally compact and Hausdorff. Define the one-point compactification, ωS, of S.

10. What can we says about those subspaces of a compactification, αS, which are locally compact? What can we says about those subspaces of a compactification, αS, which are open in αS?

11. What is the Stone-Čech compactification of the ordinal space $[0, \omega_1)$?

12. What is the cardinality of $\beta\mathbb{N}$, $\beta\mathbb{Q}$ and $\beta\mathbb{R}$?

13. Define a C-embedded subset of a space S.

14. Describe a property which characterizes those C^*-embedded which are C-embedded.

15. Show that closed subsets of $\mathbb{R}$ are C-embedded.

16. State a characterization of the pseudocompact property proven in this chapter.

22 / Singular sets and singular compactifications.

Summary. *In this chapter we introduce an alternate method to construct a compactification of a locally compact Hausdorff space. We define the notion of the singular set, $S(f)$, of a function, $f : S \to T$. We will show that we can always use $S(f)$ to construct a compactification, $\alpha S = S \cup S(f)$, by applying the right topology on αS. If the singular set, $S(f)$, contains the image, $f[S]$, of f, we refer to f as a singular map. When f is singular the resulting compactification, $S \cup_f S(f)$, is called a singular compactification.*

22.1 Singular compactifications: definitions.

We begin by formally defining a "singular set" of a continuous function on a locally compact non-compact Hausdorff space. We alse define what we mean when a function is called a "singular map".

Definition 22.1 Let (S, τ) be a locally compact non-compact Hausdorff topological space and $f : S \to T$ be a continuous function mapping S into some *compact* space T. We define the *singular set,* $S(f)$, of f as follows:

$$S(f) = \{x \in \mathrm{cl}_T f[S] : \mathrm{cl}_S f^{\leftarrow}[U] \text{ is not compact for any neighbourhood } U \text{ of } x\}$$

If $S(f) = \mathrm{cl}_T f[S]$ then $f : S \to T$ is said to be a *singular function* or *singular map.*

We make the following few remarks about the two concepts we have just introduced.

1. If S is a non-compact locally compact Hausdorff, the singular set $S(f)$ is never empty in T (whether f is a singular map or not). To see this recall that, by theorem 21.6, $f : S \to T$, extends to $f^{\beta(T)} : \beta S \to T$. Suppose $u \in \beta S \setminus S$ and

$$x = f^{\beta(T)}(u) \in f^{\beta(T)}[\beta S \setminus S]$$

If U is an open neighbourhood of x in T, then $u \in f^{\beta(T)\leftarrow}[U]$ an open subset of βS. If $\mathrm{cl}_S f^{\leftarrow}[U]$ is a compact subset of S, then $f^{\beta(T)\leftarrow}[U] \setminus \mathrm{cl}_S f^{\leftarrow}[U]$ is an open neighbourhood of u contained in $\beta S \setminus S$, a contradiction. So $\mathrm{cl}_S f^{\leftarrow}[U]$ is not compact. By definition, $x \in S(f)$. So $S(f)$ is non-empty.

2. The singular set $S(f)$ is always closed, and hence compact, in T (whether f is a singular map or not). To see this suppose $x \in T \backslash S(f)$. Then there exists an open neighbourhood U of x in T, such that $\mathrm{cl}_S f^{\leftarrow}[U]$ is compact in S. Then every point $p \in U$ also belongs to $T \backslash S(f)$. So $x \in U \subseteq T \backslash S(f)$. Hence $S(f)$ is a closed (and so is a compact) subset of the compact space T.

3. If $f : S \to T$ is a singular map, then $f[S]$ is a dense subset of $S(f)$ (since, by definition of singular function, $S(f) = \mathrm{cl}_T f[S]$).

Definition 22.2 Let (S, τ) be a locally compact non-compact Hausdorff topological space and $f : S \to T$ be a continuous function mapping S into a compact space T. Then $S(f)$ denotes its singular set. If f is a singular map then, by definition, $S(f) = \mathrm{cl}_T f[S] \subseteq T$. We construct a new set by adjoining $S(f)$ to S to obtain a larger set,

$$S \cup_f S(f)$$

The basic open neighbourhoods of points in S will be the same as the ones in S when viewed as a topological space on its own.[1] If $x \in S(f)$, F is a compact subset of S and U an open neighbourhood of x in $S(f)$, we define $U \cup f^{\leftarrow}[U] \backslash F$ as a basic open neighbourhood of x. This defines a topology on $S \cup_f S(f)$ easily seen to be a Hausdorff compactification of S. We will refer to $S \cup_f S(f)$ as the *singular compactification* induced by the singular map $f : S \to T$.

So, when given a singular compactification, we normally would like to know which function induces it. It will be useful to present a characterization of a singular compactification at this point. We remind the reader that, for a topological space S, ...

> ... *if $A \subset S$ and $r : S \to A$ is a* continuous *function which fixes the points of A, then r is referred to as a "retraction" of S onto A. In such a case A is called a "retract" of S.*

Theorem 22.3 Let S be locally compact and Hausdorff. Let αS be a Hausdorff compactification of S. Then αS is a singular compactification of S if and only if $\alpha S \backslash S$ is a retract of αS.

[1] Remember that, if S is locally compact Hausdorff, then S is open in any compactification of S.

Proof : We are given that αS is a Hausdorff compactification of S.

($\Rightarrow$) Suppose αS is a singular compactification of S induced by the continuous function, $f : S \to S(f) = \alpha S \setminus S$.

By definition, $f[S]$ is dense in $S(f)$. Let $f^\alpha : \alpha S \to S(f)$ be a function which agrees with f on S and fixes the points of $S(f)$. We claim that f^α is continuous on αS. Let $x \in S(f)$ and U be an open neighbourhood of x. Then $f^{\alpha\leftarrow}[U] = U \cup f^{\leftarrow}[U]$ which is, by definition of singular compactification, a basic open neighbourhood in αS. So f^α is continuous as claimed. Hence $\alpha S \setminus S$ is a retract of αS, as required.

($\Leftarrow$) Suppose $\alpha S \setminus S$ is a retract of αS. Then there is a continuous function $r : \alpha S \to \alpha S \setminus S$ which fixes the points of $\alpha S \setminus S$. Then $r[S] \subseteq \alpha S \setminus S$. Let $x \in \alpha S \setminus S$ and U be an open neighbourhood of x in $\alpha S \setminus S$. Now, U must intersect $r|_S[S]$, for if not, $r^{\leftarrow}[U] = U \subseteq \alpha S \setminus S$ which is not open in αS. So $r^{\leftarrow}[U] = U \cup (r^{\leftarrow}[U] \cap S)$. Then, since $\mathrm{cl}_S r|_S{}^{\leftarrow}[S]$ is not compact in S, then $\alpha S \setminus S = S(r|_S)$ is the singular set of $r|_S$, and so $\alpha S = S \cup_r S(r|_S)$, a singular compactification induced by a singular map $r|_S$.

We now have another way of recognizing a singular compactification: They are those, αS, whose outgrowth, $\alpha S \setminus S$, is a retract of the whole space.

In the following example we verify that, if we have one singular compactification of S, then every compactification "below" it in the partially ordered family of all compactifications will also be a singular compactification.

Example 1. Show that if αS is a singular compactification and γS is another compactification such that $\gamma S \preceq \alpha S$ then γS is also a singular compactification.

Solution : Suppose αS is a singular compactification and γS is another compactification such that $\gamma S \preceq \alpha S$. Then $\alpha S = S \cup_f S(f)$ where $f : S \to T$ is continuous and $f[S] \subseteq S(f) = \mathrm{cl}_T f[S]$. Also, there exists $\pi_{\alpha\to\gamma} : \alpha S \to \gamma S$ such that $\pi_{\alpha\to\gamma}$ is continuous and onto and fixes the points of S. If $g = \pi_{\alpha\to\gamma}\circ f$ then

$$\mathrm{cl}_{\gamma S}\, g[S] = \mathrm{cl}_{\gamma S}(\pi_{\alpha\to\gamma}\circ f)[S] = \pi_{\alpha\to\gamma}[\mathrm{cl}_{\alpha S}(f[S])] = \pi_{\alpha\to\gamma}[\alpha S\setminus S] = \gamma S\setminus S$$

If U is an open subset of $\gamma S\setminus S$

$$\begin{aligned} \mathrm{cl}_S g^{\leftarrow}[U] &= \mathrm{cl}_S(\pi_{\alpha\to\gamma}\circ f)^{\leftarrow}[U] \\ &= \mathrm{cl}_S f^{\leftarrow}(\pi^{\leftarrow}_{\alpha\to\gamma}[U]) \\ &= \mathrm{cl}_S f^{\leftarrow}[V] \quad {\scriptstyle (V \text{ open in } \alpha S\setminus S)} \\ &\quad \text{a non-compact set in } S. \end{aligned}$$

So $\gamma S\setminus S = S(g)$. This means that $\gamma S\setminus S$ is a retract of γS and so γS is a singular compactification.

22.3 Compactifications induced by non-singular functions.

Suppose $f : S \to T$ is a continuous function mapping a non-compact locally compact Hausdorff space S into a compact space T. Now, f may or may not be a singular map. Suppose f is *not* a singular map on S. Then $S(f)$ does not satisfy the conditions required to construct a singular compactification. Despite this, we can still adjoin the singular set $S(f)$ to S to form a larger set, $K = S \cup S(f)$.

We define a topology on K as follows: The basic open neighbourhoods of points in S will be the same as the ones in S when viewed as a topological space on its own. For $x \in S(f)$, F a compact subset of S and U an open neighbourhood of x in T, we define

$$(U \cap S(f)) \cup f^{\leftarrow}[U] \backslash F$$

as a basic open neighbourhood of x. This defines a topology on $K = S \cup S(f)$ easily seen to be a Hausdorff compactification of S. Verifying this fact is routine. In this case, when f is not singular, the only difference is that the set $S(f)$ will not be a retract of the compactification $K = S \cup S(f)$.

Notation 22.4 Suppose $S(f)$ is a singular set of a function $f : S \to T$ where T is compact. If f is not a singular map we will represent the compactification K induced by f as

$$S \cup S(f)$$

while, in the case where f is a singular map we will use the representation for the above defined compactification

$$S \cup_f S(f)$$

Remarks. Suppose $f : S \to T$ is continuous (not necessarily singular) where T is compact and $\alpha S = S \cup S(f)$ is the compactification of S induced by f. Then f extends to $f^{\alpha} : \alpha S \to \mathrm{cl}_T f[S]$ where

$$f^{\alpha}[\alpha S] = [\mathrm{cl}_{\alpha S} S] = \mathrm{cl}_T f[S]$$

We can also write,

$$\begin{aligned} f^{\alpha}[\alpha S] &= f^{\alpha}[S \cup S(f)] \\ &= f^{\alpha}[S(f)] \cup f[S] \\ &= S(f) \cup f[S] \\ &= \mathrm{cl}_T f[S] \end{aligned}$$

where $f^{\alpha}(x) = x$ on $S(f)$. So

$$\mathrm{cl}_T f[S] = f[S] \cup S(f)$$

can be viewed as a compactification, say $\gamma(f[S])$, of $f[S]$ where

$$S(f) = \gamma(f[S]) \backslash f[S]$$

Now if f is a singular map then $f^{\alpha}[\alpha S] = \mathrm{cl}_T f[S] = f[S] \cup S(f) = S(f)$. Otherwise

$$f^{\alpha}[\alpha S] = f[S] \cup S(f)$$

may or may not be a disjoint union of $f[S]$ and $S(f)$.

22.3 A few examples.

The above definition shows that any continuous function, $f : S \to T$ from S into a compact space, T, can be used to construct a compactification of a locally compact Hausdorff space S. We consider a few examples to better see how this is done.

Example 2. Consider the space $\mathbb{R}$ equipped with the usual topology. The space $\mathbb{R}$ is known to be locally compact non-compact Hausdorff. Consider the continuous functions $\sin : \mathbb{R} \to [-1, 1]$ and $\cos : \mathbb{R} \to [-1, 1]$ both mapping $\mathbb{R}$ into the compact subspace $[-1, 1]$. Show that sin and cos are both singular maps on $\mathbb{R}$ and so both induce a singular compactification.

Solution : Case sine: We see that for any open interval, U, in $[-1, 1]$ then $\sin^{\leftarrow}[U]$ is unbounded and so its closure, $\mathrm{cl}_{\mathbb{R}} \sin^{\leftarrow}[U]$, in $\mathbb{R}$ is not compact. Then

$$S(\sin) = \mathrm{cl}_{\mathbb{R}}[\sin[\mathbb{R}]] = [-1, 1]$$

and so $\sin : \mathbb{R} \to [-1, 1]$ is a singular map. We can then use the sine function to construct the singular compactification

$$\mathbb{R} \cup_{\sin} S(\sin) = \mathbb{R} \cup_{\sin} [-1, 1]$$

of $\mathbb{R}$ with outgrowth $[-1, 1]$.

Case cosine: Proceed similarly to show that $\mathbb{R} \cup_{\cos} S(\cos) = \mathbb{R} \cup_{\cos} [-1, 1]$ is also a singular compactification of $\mathbb{R}$.

The above example produces two compactifications of $\mathbb{R}$ with identical outgrowth. It may be tempting to conclude that they are equivalent. We will later verify whether this is actually the case.

Example 3. Consider the space $\mathbb{R}$ equipped with the usual topology. The space S is known to be a locally compact non-compact Hausdorff. Let $T = [-\pi/2, \pi/2]$. Show that $\mathbb{R} \cup S(\arctan)$ is *not* a singular compactification of $\mathbb{R}$. Then find the compactification induced by arctan.

Solution : We consider the function, $\arctan : \mathbb{R} \to T$. We then verify which points in $\mathrm{cl}_T[\arctan[\mathbb{R}]] = [-\pi/2, \pi/2]$ belong to the singular set $S(\arctan)$. We see that arctan, pulls back open intervals of the form (a, b) in $[-\pi/2, \pi/2]$, to intervals whose closure is compact. Then $\arctan[\mathbb{R}] = (-\pi/2, \pi/2) \not\subseteq S(\arctan)$. So arctan is not a singular map. Thus $\mathbb{R} \cup S(\arctan)$ is *not* a singular compactification of $\mathbb{R}$

We now determine $S(\arctan)$. Since the curve of $y = \arctan(x)$ is asymptotic to the horizontal lines $y = -\pi/2$ and $y = \pi/2$, the function arctan pulls backs open intervals of the $(a, \pi/2]$ and $[-\pi/2, b)$ to unbounded sets and so the "pull backs" of these have non-compact closures. So $S(\arctan) = \{-\pi/2, \pi/2\}$. So, even though arctan is not a singular map on $\mathbb{R}$ it does induce a "two-point compactification"

$$\mathbb{R} \cup S(\arctan) = \mathbb{R} \cup \{-\pi/2, \pi/2\}$$

of $\mathbb{R}$.

Example 4. Show that $\beta\mathbb{R}$ is not a singular compactification of $\mathbb{R}$.

Solution : We have shown that $\mathbb{R} \cup S(\arctan) = \mathbb{R} \cup \{-\pi/2, \pi/2\}$ is a compactification of $\mathbb{R}$ but not a singular one. Since $\mathbb{R} \cup S(\arctan) \preceq \beta\mathbb{R}$ then by the example above $\beta\mathbb{R}$ cannot be singular.

The following example is of interest since it shows that two compactifications of the same set with the same outgrowth need not be equivalent compactifications.

Example 5. Given the two singular compactifications,

$$\begin{aligned} \alpha\mathbb{R} &= \mathbb{R} \cup_{\cos} S(\cos) \\ \gamma\mathbb{R} &= \mathbb{R} \cup_{\sin} S(\sin) \end{aligned}$$

show that, in spite of $S(\sin) = [-1, 1] = S(\cos)$, $\alpha\mathbb{R}$ and $\gamma\mathbb{R}$ are *not* equivalent compactifications.

Solution : Suppose $\alpha\mathbb{R}$ and $\gamma\mathbb{R}$ are equivalent compactifications. We will show that this will lead to a contradiction.

Then, by definition of "equivalent compactifications", there exists a homeomorphism, $\pi_{\gamma\to\alpha} : \gamma\mathbb{R} \to \alpha\mathbb{R}$ such that for $x \in \mathbb{R}$, $\pi_{\gamma\to\alpha}(x) = x$.

The function, $\pi_{\gamma\to\alpha}|_{\gamma\mathbb{R}\setminus\mathbb{R}}$, is a homeomorphism mapping $[-1, 1]$ onto $[-1.1]$.
This means that $\pi_{\gamma\to\alpha}|_{\gamma\mathbb{R}\setminus\mathbb{R}}$ is monotone and maps endpoints to endpoints. Suppose, without loss of generality, that $\pi_{\gamma\to\alpha}|_{\gamma\mathbb{R}\setminus\mathbb{R}}(-1) = -1$. Then $\pi_{\gamma\to\alpha}|_{\gamma\mathbb{R}\setminus\mathbb{R}}(1) = 1$.

If $U = (a, 1] \subseteq S(\sin)$ and $V = \pi_{\gamma\to\alpha}|_{\gamma\mathbb{R}\backslash\mathbb{R}} U \subseteq S(\cos)$ see that can choose a small enough so that $\sin^{\leftarrow}[U] \cap \cos^{\leftarrow}[V] \cap [-\pi, \pi]$ is empty. Then $\sin^{\leftarrow}[U] \cap \cos^{\leftarrow}[V] = \varnothing$.

Now $\cos : \mathbb{R} \to [-1, 1]$ extends to $\cos^{\alpha} : \alpha\mathbb{R} \to [-1, 1]$.

Then $U \cup \sin^{\leftarrow}[U]$ is open in $\gamma\mathbb{R}$. Then by continuity, the two sets

$$\begin{aligned} \pi_{\gamma\to\alpha}[U \cup \sin^{\leftarrow}[U]\,] &= V \cup \sin^{\leftarrow}[U] \\ \cos^{\alpha\,\leftarrow}[V] &= V \cup \cos^{\leftarrow}[V] \end{aligned}$$

are both open subsets of $\alpha\mathbb{R}$.

So $(V \cup \sin^{\leftarrow}[U]) \cap (V \cup \cos^{\leftarrow}[V]) = V$ a non-empty open subset of $\alpha\mathbb{R}$ which is entirely contained in $\alpha\mathbb{R}\backslash\mathbb{R}$. Since $\mathbb{R}$ is dense in $\alpha\mathbb{R}$ this is impossible. The source of our contradiction is our supposition that $\alpha\mathbb{R}$ and $\gamma\mathbb{R}$ are equivalent. We are done.

Example 6. Determine whether $f(x) = \sin(1/x)$ with domain $\mathbb{R}\setminus\{0\}$ and range $T = [-1, 1]$ induces a singular compactification. If not determine the non-singular compactification it induces.

Solution : We are given $f(x) = \sin(1/x)$ where $f : S \to T$, maps $S = \mathbb{R}\backslash\{0\}$ onto the compact set $T = [-1, 1]$. Suppose $a \in [-1, 0) \cup (0, 1]$ and $y \in M = [-1, a) \cup (-a, 1]$. Then, for an open interval U in M, $f^{\leftarrow}[U]$ produces a bounded sequence of open intervals converging to zero. If we take the closure, $\mathrm{cl}_S f^{\leftarrow}[U]$, we obtain a bounded sequence of closed intervals converging to zero. Since $0 \notin S$ then $\mathrm{cl}_S f^{\leftarrow}[U]$ is not compact. If $y \in U = (-a, a)$ then $f^{\leftarrow}[U]$ is unbounded on both ends of $\mathbb{R}$ so $\mathrm{cl}_S f^{\leftarrow}[U]$ is not compact. So $f : S \to T$ is a singular map and, for $S(f) = [-1, 1]$,

$$S \ \cup_f \ S(f) = \mathbb{R}\backslash\{0\} \ \cup_f \ [-1, 1]$$

is a singular compactification of $\mathbb{R}\backslash\{0\}$ induced by $f(x) = \sin(1/x)$.

22.4 More on equivalent singular compactifications.

We will now produce a characterization of pairs of singular compactifications which are equivalent. But first we must present a few lemmas involving singular sets $S(f)$.

Lemma 22.5 Let $f : S \to K$ be a continuous function mapping a locally compact Hausdorff space into a compact Hausdorff space, K, and $Y = \mathrm{cl}_K f[S]$. Then

$$S(f) = \cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\}$$

Proof: We are given that $f : S \to K$ is a continuous function mapping S into the compact space K and $Y = \mathrm{cl}_K f[S]$. In the proof, F will always represent a compact set in S.

We claim that $S(f) \subseteq \cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\}$. Let $p \in Y\backslash \mathrm{cl}_Y f[S\backslash F]$. Then there exists an open neighbourhood U of p in Y such that $f^{\leftarrow}[U] \subseteq F$. Then $\mathrm{cl}_S f^{\leftarrow}[U]$ is compact so $p \notin S(f)$. This means that $Y\backslash \mathrm{cl}_Y f[S\backslash F] \cap S(f) = \varnothing$. So $S(f) \subseteq \mathrm{cl}_Y f[S\backslash F]$. We can deduce that $S(f) \subseteq \cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\}$, as claimed.

We now claim that $\cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\} \subseteq S(f)$. Let $p \in \cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\}$. Suppose $p \notin S(f)$. Then there is an open neighbourhood U_1 of p in $Y = \mathrm{cl}_K f[S]$ such that $\mathrm{cl}_S f^{\leftarrow}[U_1]$ is compact. But

$$\begin{array}{rcl} p & \in & \cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\} \\ & \subseteq & \mathrm{cl}_Y f[S\backslash \mathrm{cl}_S f^{\leftarrow}[U_1]] \\ & \subseteq & \mathrm{cl}_Y f[S\backslash f^{\leftarrow}[U_1]] \\ & \subseteq & \mathrm{cl}_Y f{\circ}f^{\leftarrow}[Y\backslash U_1]] \\ & = & Y\backslash U \end{array}$$

The statement "$p \in Y\backslash U$" contradicts the fact that U is a neighbourhood of p. Consequently, $\cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\} \subseteq S(f)$ as claimed.

So $S(f) = \cap\{\mathrm{cl}_Y f[S\backslash F] : F \text{ is compact in } S\}$.

Lemma 22.6 Let αS be a compactification of S and $f : S \to K$ be a continuous function mapping the locally compact Hausdorff space S into a compact Hausdorff space, K. If f extends to $f^{\alpha} : \alpha S \to K$, then $f^{\alpha}[\alpha S\backslash S] = S(f)$.

Proof: If F is compact in S then $\alpha S\backslash S \subseteq \mathrm{cl}_{\alpha S}(S\backslash F)$ and so

$$f^{\alpha}[\alpha S\backslash S] \subseteq f^{\alpha}[\mathrm{cl}_{\alpha S}(S\backslash F)] \subseteq \mathrm{cl}_{\mathrm{cl}_K f[S]} f[S\backslash F]$$

Then $f^{\alpha}[\alpha S\backslash S] \subseteq \cap\{\mathrm{cl}_{\mathrm{cl}_K f[S]} f[S\backslash F] : F \text{ is compact in } S\}$. By the previous lemma $f^{\alpha}[\alpha S\backslash S] \subseteq S(f)$. On the other hand, if $p \notin f^{\alpha}[\alpha S\backslash S]$ and U is an open neighbourhood of p in K such that $\mathrm{cl}_K U$ misses $f^{\alpha}[\alpha S\backslash S]$ then $\mathrm{cl}_S f^{\leftarrow}[U] \subseteq f^{\leftarrow}[\mathrm{cl}_S U]$, a compact subset of S. Hence $\mathrm{cl}_S f^{\leftarrow}[U]$ is compact and so $p \notin S(f)$. So $S(f) \subseteq f^{\alpha}[\alpha S\backslash S]$.

We conclude that, if f extends to $f^{\alpha} : \alpha S \to K$, then $f^{\alpha}[\alpha S\backslash S] = S(f)$

Lemma 22.7 Let αS be a compactification of S and $f : S \to K$ be a continuous function mapping the locally compact Hausdorff space S into a compact Hausdorff space, K. If f extends to $f^{\alpha} : \alpha S \to K$ so that $f^{\alpha} : \alpha S \to K$ separates the points of $\alpha S\backslash S$ then αS is equivalent to $S \cup S(f)$ (not necessarily a singular compactification).

Proof: Let $S(f)$ be the singular set of $f : S \to K$ (not necessarily a singular map) and $S \cup S(f)$ be the compactification of S which is induced by f (not necessarily a singular compactification). We are given that f extends continuously to $f^\alpha : \alpha S \to K$ in such a way that f^α separates the points on $\alpha S \backslash S$.

By lemma 22.6, $f^\alpha[\alpha S \backslash S] = S(f)$. We define $\pi : \alpha S \to S \cup S(f)$ as follows:

$$\pi(x) = \begin{cases} f^\alpha(x) & \text{if } x \in \alpha S \backslash S \\ x & \text{if } x \in S \end{cases}$$

We claim that π is continuous on αS. It suffices to show that π pulls back open neighbourhoods in $S \cup S(f)$ to open sets in αS. Suppose $p \in S \cup S(f)$. Clearly if $p \in S$ and U is open in S then $\pi^{\leftarrow}[U]$ is open in αS. If $p \in S(f)$ then an open neighbourhood of p is, by definition, of the form $[\, U \cap S(f)\,] \cup f^{\leftarrow}[U] \backslash F$ where U is an open subset of K and F some compact set in S. See that,

$$\begin{aligned}
\pi^{\leftarrow}[\, [\, U \cap S(f)\,] \cup f^{\leftarrow}[U] \backslash F\,] &= \pi^{\leftarrow}[\, U \cap S(f)\,] \cup \pi^{\leftarrow}[\, f^{\leftarrow}[U] \backslash F]\,] \\
&= \pi^{\leftarrow}[U] \cap \pi^{\leftarrow}[S(f)] \cup f^{\leftarrow}[U] \backslash F \\
&= (f^{\alpha\leftarrow}[U] \cap f^{\alpha\leftarrow}[S(f)]) \cup f^{\leftarrow}[U] \backslash F \\
&= (f^{\alpha\leftarrow}[U \cap S(f)] \cup f^{\leftarrow}[U] \backslash F \\
&= f^{\alpha\leftarrow}[U] \backslash F \\
&= f^{\alpha\leftarrow}[U] \cap S \backslash F
\end{aligned}$$

We see that π pulls back open neighbourhoods of points in $S(f)$ to open sets. So $\pi : \alpha S \to S \cup S(f)$ is continuous. By definition, αS and $S \cup S(f)$ are equivalent compactifications.

Theorem 22.8 Let S be a completely regular topological space. Suppose $f : S \to K$ and $g : S \to K$ are two continuous *singular* functions mapping S into a compact space K. Suppose $S(f)$ and $S(g)$ are homeomorphic. Then the two induced singular compactifications,

$$\begin{aligned}
\alpha S &= S \cup_f S(f) \\
\gamma S &= S \cup_g S(g)
\end{aligned}$$

are equivalent if and only if the singular function $f : S \to S(f)$ extends continuously to $f^\gamma : \gamma S \to S(g)$ such that f^γ separates the points of $\gamma S \backslash S = S(g)$.

Proof: We are given that $f : S \to K$ and $g : S \to K$ are two continuous *singular* functions mapping S into a compact space K inducing the two singular compactifications, $\alpha S = S \cup_f S(f)$ and $\gamma S = S \cup_g S(g)$. Also $S(f)$ and $S(g)$ are seen to be homeomorphic.

($\Rightarrow$) Suppose $\alpha S = S \cup_f S(f)$ and $\gamma S = S \cup_g S(g)$ are equivalent. We are required to show that the singular function $f : S \to S(f)$ extends continuously to $f^\gamma : \gamma S \to S(g)$ such that f^γ separates the points of $\gamma S \backslash S = S(g)$.

Recall that $f^\alpha : \alpha S \to \alpha S \backslash S$ acts as the identity map on $\alpha S \backslash S$. Also, since αS and γS are equivalent, then there is a continuous map $\pi_{\gamma\to\alpha} : \gamma S \to \alpha S$ such that $\pi_{\gamma\to\alpha}(x) = x$ on S and $\pi_{\gamma\to\alpha}$ maps $\gamma S \backslash S$ homeomorphically onto $\alpha S \backslash S$.

Let $f^\gamma : \gamma S \to S(f)$ be defined as follows: $f^\gamma = f^\alpha \circ \pi_{\gamma\to\alpha}$. Then f^γ is continuous, $f^\gamma|_S = f$ and $f^\gamma|_{S(g)} = \pi_{\gamma\to\alpha}|_{S(g)}$. This shows that $f : S \to S(f)$ extends continuously to the function $f^\gamma : S \cup_g S(g) \to S(f)$. Since $\pi_{\gamma\to\alpha}$ is a homeomorphism on $S(g)$ and f^α is the identity function on $S(f)$ then f^γ separates points of $S(g)$, as required.

($\Leftarrow$) We are given that both f and g are singular maps on S and $\alpha S = S \cup_f S(f)$ and $\gamma S = S \cup_g S(g)$. Suppose that the singular function $f : S \to S(f)$ extends continuously to $f^\gamma : S \cup_g S(g) \to S(g)$ such that f^γ separates the points of $S(g)$.

Then by lemma 22.7, $S \cup_g S(g)$ is a compactification which is equivalent to $S \cup S(f)$.

Since f is singular, $S \cup S(f) = S \cup_f S(f)$. So $S \cup_f S(f)$ and $S \cup_g S(g)$ are equivalent.

Theorem 22.9 Two continuous functions, $f : S \to K$ and $g : S \to K$, will be said to be *homeomorphically related* if there exist a homeomorphic function $h : cl_K g[S] \to cl_K h[S]$ such that $h(g(x)) = f(x)$ for all $x \in S$. Suppose that $f : S \to K$ and $g : S \to K$ are two singular maps such that $S(f) = S(g)$. If f and g are homeomorphically related then $S \cup_f S(f)$ and $S \cup_g S(g)$ are equivalent compactifications.

Proof: We are given two continuous functions, $f : S \to K$ and $g : S \to K$, mapping S into the compact space K.

Suppose f and g are homeomorphically related singular maps which induce the singular compactifications

$$\begin{aligned} \alpha S &= S \cup_f S(f) \\ \gamma S &= S \cup_g S(g) \end{aligned}$$

where $S(f) = S(g)$. By definition, there exists a homeomorphism $h : S(g) \to S(f)$ such that $h(g(x)) = f(x)$.

See that $g : S \to S(g)$ extends continuously to $g^\gamma : S \cup_g S(g) \to S(g)$ where g^γ is the identity map when restricted to $S(g)$. Then $h \circ g : S \to S(f)$ extends to

$$(h \circ g)^\gamma : S \cup_g S(g) \to S(f)$$

where $(h \circ g^\gamma)|_{S(g)}(x) = h(x)$. Since $(h \circ g)(x) = f(x)$ on S, f extends to $(h \circ g)^\gamma = f^\gamma$ where

$$f^\gamma : S \cup_g S(g) \to S(f)$$

and $f^\gamma = h$ on $S(g)$. So f^γ separates points of $S(g)$.

By theorem 22.8, $S \cup_f S(f)$ and $S \cup_g S(g)$ are equivalent, as required.

We can now present an example where this particular concept plays a key role.

Example 7. The two compactifications $\mathbb{R} \cup_{\sin^2} S(\sin^2)$ and $\mathbb{R} \cup_{\cos^2} S(\cos^2)$ are easily seen to be singular compactifications with outgrowth $[0, 1]$. Show that they are equivalent compactifications.

Solution : Consider the function $h(x) = 1 - x$ mapping $[0, 1]$ onto $[0, 1]$. It is a one-to-one continuous function. Also note that

$$h(\sin^2(x)) = 1 - \sin^2(x) = (1 - \sin^2)(x) = \cos^2(x)$$

on $[0, 1]$. Then $\sin^2$ and $\cos^2$ are homeomorphically related. By the above theorem $\mathbb{R} \cup_{\sin^2} S(\sin^2)$ and $\mathbb{R} \cup_{\cos^2} S(\cos^2)$ are equivalent compactifications.

There can be various ways of showing that ...

> ... *the compactification* $\beta\mathbb{N}$ *is not a singular compactification.*

In the next example we propose one method.

Example 8. Show that $\beta\mathbb{N}$ cannot be a singular compactification.

Solution : Suppose $\beta\mathbb{N}$ is a singular compactification, Then there is a retraction function $r : \beta\mathbb{N} \to \beta\mathbb{N}\backslash\mathbb{N}$ which maps $\beta\mathbb{N}$ onto $\beta\mathbb{N}\backslash\mathbb{N}$. We know that $\beta\mathbb{N}$ is separable, but on page 339, we showed that $\beta\mathbb{N}\backslash\mathbb{N}$ is not a separable space. By theorem 6.11 we know that continuous images of separable spaces are separable. So $\beta\mathbb{N}\backslash\mathbb{N}$ cannot be the continuous image of $\beta\mathbb{N}$. So $\beta\mathbb{N}$ is not a singular compactification.

22.5 What kind of space has only singular compactifications?

We have already shown a compactification "less than" a singular compactification must be singular. So, if βS is a singular compactification of S, then all compactifications of S are singular. We wonder what class of topological spaces satisfies this property?

Theorem 22.10 Let S be locally compact, non-compact and Hausdorff. If βS is a singular compactification then S is pseudocompact.

Proof: We are given that S is locally compact and Hausdorff and that βS is a singular compactification.

Since βS is singular there exists a continuous function $r : \beta S \to \beta S \backslash S$ where $r[S] \subseteq r[\beta S \backslash S] = \beta S \backslash S$ and $r(x) = x$ for $x \in \beta S \backslash S$.
So

$$r[\beta S] = r[S] \cup r[\beta S \backslash S] = r[\beta S \backslash S]$$

We are required to show that S is pseudocompact.

Suppose S is not pseudocompact. By theorem 21.23, "pseudocompact spaces are precisely those spaces which do not contain a C-embedded copy of $\mathbb{N}$". So, if the space S is not pseudocompact, it must contain a C-embedded copy, N, of $\mathbb{N}$. Meaning, there is a homeomorphism, $f : N \to \mathbb{N}$ which map N onto $\mathbb{N}$.

From the example found on page 338, we know that

$$\mathrm{cl}_{\beta S} N = \beta N \subseteq \beta S$$

Consider the function $r|_N : N \to \mathrm{cl}_{\beta S} r[N] = r[\beta N]$. Since βN is separable, then

$$r[\beta N] = r[N] \cup r[\beta N \backslash N] = r[N] \cup \beta N \backslash N$$

must be separable (being the continuous image of a separable set). Since $\beta N \backslash N$ was shown to be non-separable $T = r[\beta N] \backslash (\beta N \backslash N)$ must contain countably infinite elements say,

$$T = \{t_i : i \in \mathbb{N}\}$$

(see why finitely many elements won't do). For each $i \in \mathbb{N}$, choose $m_i \in r^{\leftarrow}(t_i)$, obtain the subset,

$$M = \{m_i : i \in \mathbb{N}\}$$

of N. By hypothesis, N is C-embedded in S so M is C-embedded in S. Again $\mathrm{cl}_{\beta S} M = \beta M$.

Claim # 1: $(\mathrm{cl}_{\beta S} T) \backslash T = (\mathrm{cl}_{\beta S} M) \backslash M$.

Proof of claim. First see we have two representations of $r[\mathrm{cl}_{\beta S} M]$:

$$\begin{aligned} r[\mathrm{cl}_{\beta S} M] &= \mathrm{cl}_{\beta S \backslash S} r[M] = \mathrm{cl}_{\beta S \backslash S} T \quad (*) \\ r[\mathrm{cl}_{\beta S} M] &= (\mathrm{cl}_{\beta S} M) \backslash M \cup r[M] = (\mathrm{cl}_{\beta S} M) \backslash M \cup T \quad (**) \end{aligned}$$

Combining * and ** and $T \cap \text{cl}_{\beta S \setminus S} M \setminus M = \varnothing$,

$$(\text{cl}_{\beta S \setminus S} T) \setminus T = (\text{cl}_{\beta S} M) \setminus M$$

This establishes the first claim.

We now define the function $g : \text{cl}_{\beta S \setminus S} T \to \mathbb{R}$ as

$$\begin{aligned} g(t_i) &= \frac{1}{i+1} \\ g[\,(\text{cl}_{\beta S \setminus S} T) \setminus T\,] &= \{0\} \end{aligned}$$

The function g is easily verified to be continuous on the compact domain, $\text{cl}_{\beta S \setminus S} T$. Then g extends to $g^\beta : \beta S \to \mathbb{R}$, where

$$g^\beta[\,(\text{cl}_{\beta S \setminus S} T) \setminus T\,] = \{0\} = g^\beta[\,(\text{cl}_{\beta S} M) \setminus M\,]$$

So

$$(\text{cl}_{\beta S} M) \setminus M \subseteq Z(g^\beta) \cap \beta S \setminus S$$

By corollary 21.25,

$$(\text{cl}_{\beta S} M) \setminus M \subseteq \text{int}_{\beta S \setminus S}[\, Z(g^\beta) \cap \beta S \setminus S\,]$$

Since $(\text{cl}_{\beta S \setminus S} T) \setminus T = (\text{cl}_{\beta S} M) \setminus M$,

$$(\text{cl}_{\beta S} T) \setminus T \subseteq \text{int}_{\beta S \setminus S}[\, Z(g^\beta) \cap \beta S \setminus S\,]$$

Since T is dense in $\text{cl}_{\beta S \setminus S} T$ then $T \cap \text{int}_{\beta S \setminus S}[\, Z(g^\beta) \cap \beta S \setminus S\,] \neq \varnothing$. But this is impossible, since we defined g so that $g(t_i) = 1/(i+1) \neq 0$.

So S cannot contain a C-embedded copy of $\mathbb{N}$. So S is pseudocompact, as required.

Concepts review:

1. Given a continuous function $f : S \to T$ from the completely regular set S into the compact set T, define the singular set $S(f)$.

2. Given a continuous function $f : S \to T$ from the completely regular set S into the compact set T, what does it mean to say that f is a singular map?

3. Given a continuous function $f : S \to T$ from the completely regular set S into the compact set T, define a compactification induced by f.

4. Given a continuous function $f : S \to T$ from the completely regular set S into the compact set T, define a singular compactification induced by f.

5. Show that arctan: $\mathbb{R} \to [-\pi/2, \pi/2]$ is not a singular map. Find a compactification of $\mathbb{R}$ induced by arctan.

6. Produce an example of a pair of singular compactifications with the same singular sets but which are not equivalent.

7. Produce an example of a Stone-Čech compactification which is not singular.

8. State one way of recognizing a pair of singular compactifications which are equivalent.

23 / Realcompact spaces

Summary. *In this section we introduce the set of all points in βS called the real points of a space S. The set of all real points of S is denoted by υS. The set υS is also referred to as the Hewitt-Nachbin realcompactification of S. If $\upsilon S = S$ then S is said to be realcompact. We provide a characterization as well as a few examples of realcompact spaces.*

23.1 Realcompact space: Definitions and characterizations.

Suppose S is a locally compact Hausdorff topological space and $f : S \to \mathbb{R}$ is a function in the set, $C(S)$, of all real-valued continuous functions, including unbounded ones, on S. Let $i : \mathbb{R} \to \omega\mathbb{R}$ be the inclusion function (identity map) which embeds $\mathbb{R}$ into the one-point compactification,

$$\omega\mathbb{R} = \mathbb{R} \cup \{\infty\}$$

defined as, $i(x) = x$ (where ∞ represents a point not in $\mathbb{R}$). We can combine the two functions f and i to define the function $f_+ = i \circ f$, where $f_+ : S \to \omega\mathbb{R}$ maps S into the compact set $\omega\mathbb{R}$. In this case, we have practically identical functions, f_+ and f, except that the range of f_+ is $\omega\mathbb{R}$. We have shown in theorem 21.6 that any continuous function, $g : S \to K$, mapping a locally compact Hausdorff space, S, into a compact space K extends to a function $g^{\beta(K)} : \beta S \to K$. We will denote the extension of $f_+ : S \to \omega\mathbb{R}$ to βS as

$$f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$$

In the case where f is bounded, the extension $f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$ maps the compact space βS onto the compact space, $\mathrm{cl}_{\omega\mathbb{R}} f[S] = \mathrm{cl}_{\mathbb{R}} f[S]$. So, if f is bounded, $f^{\beta(\omega)}$ and the usual $f^\beta : \beta S \to \mathbb{R}$ represent the same function. To see this, for a bounded f, we have $f^{\beta(\omega)}[\beta S] = f^{\beta(\omega)}[\mathrm{cl}_{\beta S} S] = \mathrm{cl}_{\omega\mathbb{R}} f[S] = \mathrm{cl}_{\mathbb{R}} f[S]$, a compact set in $\mathbb{R} \subset \omega\mathbb{R}$. Of course, if $f : S \to \mathbb{R}$ is unbounded,

$$\begin{aligned} f^{\beta(\omega)}[\beta S] &= f^{\beta(\omega)}[\beta S \backslash S \ \cup \ S] \\ &= f^{\beta(\omega)}[\beta S \backslash S] \cup f[S] \\ &\subseteq f^{\beta(\omega)}[\beta S \backslash S] \ \cup \ \mathbb{R} \end{aligned}$$

and, since $f[S]$ is unbounded, it is non-compact. So $f^{\beta(\omega)}$ must map at least one point in $\beta S \backslash S$ to ∞. We stress *at least* one point, since $f^{\beta(\omega)\leftarrow}(\infty)$ need not be a singleton set in βS. We have that $\beta S = f^{\beta(\omega)\leftarrow}[\mathbb{R}] \cup f^{\beta(\omega)\leftarrow}(\infty)$, a disjoint union.

We will want to distinguish those points in βS which belong to $f^{\beta(\omega)\leftarrow}(\infty)$ from those which belong to $f^{\beta(\omega)\leftarrow}[\mathbb{R}]$. For a given $f \in C(S)$ we then define $\upsilon_f S$ [1] as,

$$\upsilon_f S = \beta S \backslash f^{\beta(\omega)\leftarrow}(\infty) = f^{\beta(\omega)\leftarrow}[\mathbb{R}]$$

Clearly, $S \subseteq \upsilon_f S \subseteq \beta S$, for all f.

For a bounded function $f \in C^*(S)$,

$$\upsilon_f S = f^{\beta(\omega)\leftarrow}[\mathbb{R}] = f^{\beta(\omega)\leftarrow}[\mathrm{cl}_{\mathbb{R}} f[S]] = f^{\beta\leftarrow}[\mathrm{cl}_{\mathbb{R}} f[S]] = \beta S$$

The function $f : S \to \mathbb{R}$ in $C(S)$ is one which continuously extends to a real-valued function, $f^{\beta(\omega)} : \upsilon_f S \to \mathbb{R}$ in $C(\upsilon_f S)$. The points in $\upsilon_f S$ are commonly referred to as being the set of all *real points of* f (in the sense that $f^{\beta(\omega)}$ has a real number value at each of point in $\upsilon_f S$). This is just associated to a single function, f. To consider the real point associated to all functions $f \in C(S)$ we define.

$$\upsilon S = \cap\{\upsilon_f S : f \in C(S)\} = \cap\{f^{\beta(\omega)\leftarrow}[\mathbb{R}] : f \in C(S)\}$$

Note that every function $f \in C(S)$ extends continuously a real-valued function, $f^{\beta(\omega)}|_{\upsilon S}$, on υS. More specifically, each function in $C(S)$ is associated to a unique function in $C(\upsilon S)$.

We now formally define the following related concepts.

Definition 23.1 Let S be locally compact and Hausdorff.

a) If $f \in C(S)$, the function $f^{\beta(\omega)} : S \to \omega\mathbb{R}$ represents the extension of f to βS with range $\omega\mathbb{R}$. We define the subset, $\upsilon_f S$, of βS as

$$\upsilon_f S = f^{\beta(\omega)\leftarrow}[\mathbb{R}]$$

If $p \in \beta S \backslash \upsilon_f S$, then $f^{\beta(\omega)}(p) = \infty$. The points in $\upsilon_f S$ are referred to as *real points of* f. We define, the subspace, υS of βS, as

$$\upsilon S = \cap\{\upsilon_f S : f \in C(S)\} = \cap\{f^{\beta(\omega)\leftarrow}[\mathbb{R}] : f \in C(S)\}$$

b) The points $p \in \upsilon S \subseteq \beta S$ are referred to as *real points of* S. If p in βS is not "a real point of S" then there is at least one function $f \in C(S)$ such that $f^{\beta(\omega)}(p) = \infty$.

c) If the space S is such that $S = \upsilon S$, then we refer to S as being a *realcompact space.*

[1] The Greek letter, υ, is pronounced *upsilon*.

Based on the above definition, for any locally compact Hausdorff space S, $S \subseteq \upsilon S \subseteq \beta S$. Furthermore, if f is any function in $C(S)$, then f extends to a continuous function $f^{\beta(\omega)}|_{\upsilon S} : \upsilon S \to \omega\mathbb{R}$, where $S \subseteq \upsilon S \subseteq \beta S$.

We will more succinctly denote the function $f^{\beta(\omega)}|_{\upsilon S}$ by

$$f^{\upsilon} = f^{\beta(\omega)}|_{\upsilon S}$$

We will require the following concept for our first characterization of a "realcompact space". It refers to a slightly stronger condition on z-ultrafilters.

Definition 23.2 Let S be locally compact and Hausdorff. A *real z-ultrafilter*, $\mathscr{X}$, in $Z[S]$ is a z-ultrafilter which is closed under countable intersections. That is, countable subfamilies of $\mathscr{X}$ have non-empty intersection.[1]

The following characterizations of realcompact spaces will be useful as tools used to recognize those spaces which are realcompact.

Theorem 23.3 Let S be a locally compact Hausdorff space. Then the following three statements are equivalent.

a) The space S is realcompact. That is, $S = \upsilon S$.

b) For any $p \in \beta S \setminus S$, there is a zero-set, $Z(h^{\beta(\omega)}) \in Z[\beta S]$ such that $p \in Z(h^{\beta(\omega)}) \subseteq \beta S \backslash S$.

c) Every *real z-ultrafilter* in $Z[S]$ has a limit point p which belongs to S.

Proof : We are given that S is a locally compact Hausdorff space.

(a $\Rightarrow$ b) Suppose S is realcompact and let $p \in \beta S \backslash S$. We are required to show that $p \in Z(f) \subseteq \beta S \backslash S$ for some $f \in C(\beta S)$.

By definition, $S = \upsilon S$. Since $p \in \beta S \backslash \upsilon S$ then p is not a real point of S, in the sense that, for some function, say $g \in C(S)$, $g^{\beta(\omega)}(p) = \infty$. Then g is unbounded on S (for

[1] In some books the author may denote this property by the anagram CIP for "countable intersection property".

if g was bounded $g[\beta S] = \mathrm{cl}_{\mathbb{R}} g[S] \subseteq \mathbb{R}$). If $f = g^2 \vee 1$ then f is positive, unbounded and greater than or equal to 1 on S.

Let $h = 1/f$. Then h is a well-defined continuous function on S such that $h[S] \subseteq (0, 1]$.

The function $h : S \to (0, 1]$ then extends to $h^{\beta} : \beta S \to \mathrm{cl}_{\mathbb{R}}(0, 1] = [0, 1]$. By continuity of h^{β}, for any sequence (net) $\{x_i\}$ in S converging to $p \in \beta S\backslash S$, $\{h^{\beta}(x_i)\}$ converges to $h^{\beta}(\lim_{i\to\infty} x_i) = h^{\beta}(p) \notin (0, 1]$. So $h^{\beta}(p) = 0$.

So $p \in Z(h^{\beta}) \subseteq \beta S\backslash S$. So, if S is realcompact, every point p in $\beta S\backslash S$ is contained in a zero-set $Z(h^{\beta})$ which is itself contained in $\beta S\backslash S$, as required.

(b $\Rightarrow$ a) We are given that every point in $\beta S\backslash S$ belongs to a zero-set entirely contained in $\beta S\backslash S$. Let $p \in \beta S\backslash S$. It suffices to show that p is not a real point. By hypothesis, $p \in Z(f^{\beta(\omega)}) \subseteq \beta S\backslash S$ for some $f \in C(S)$. Then $f(x) \neq 0$ on S.

Let $h : S \to \omega\mathbb{R}$ be a function defined as $h = 1/f$. Then h extends to $h^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$. Suppose $\{x_i\}$ is a sequence (net) in S which converges to p. Then $\{f(x_i)\}$ approaches zero as x_i approaches p. Then, by continuity of $h^{\beta(\omega)}$,

$$h^{\beta(\omega)}(p) = h^{\beta(\omega)}(\lim_{i\to\infty}\{x_i\}) = \lim_{i\to\infty}\{h(x_i)\} = \lim_{i\to\infty}\{1/f(x_i)\} = \infty$$

So p is not a real point. So real points can only belong to S. So $S = \upsilon S$.

(b $\Rightarrow$ c) We are given that every point in $\beta S\backslash S$ belongs to a zero-set entirely contained in $\beta S\backslash S$. If $p \in \beta S\backslash S$ then, by hypothesis, $p \in Z(h^{\beta(\omega)}) \subseteq \beta S\backslash S$, for some $h \in C(S)$. Since $p \notin S$ it corresponds to a free z-ultrafilter, $\mathscr{X}$, in $Z[S]$. It suffices to prove that, $\mathscr{X}$ is not a real z-ultrafilter. To do this we will show that $\mathscr{X}$ does not satisfy the "countable intersection property".

See that,

$$Z(h^{\beta(\omega)}) = \cap\{h^{\beta(\omega)\leftarrow}[-1/n, 1/n] : n \in \mathbb{N}\backslash\{0\}\}$$

where each $h^{\beta(\omega)\leftarrow}[-1/n, 1/n]$ is a zero-set.[1] So $Z(h^{\beta(\omega)})$ is a countable intersection of zero sets. Then, by the paragraph titled "On zero-sets and z-ultrafilters in βS" on page 331, we can write, $h^{\beta(\omega)\leftarrow}[-1/n, 1/n] = \mathrm{cl}_{\beta S} h^{\leftarrow}[-1/n, 1/n]$ so

$$Z(h^{\beta(\omega)}) = \cap\{\mathrm{cl}_{\beta S} h^{\leftarrow}[-1/n, 1/n] : n \in \mathbb{N}\backslash\{0\}\}$$

Since $\cap\{h^{\leftarrow}[-1/n, 1/n] : n \in \mathbb{N}\backslash\{0\}\} \subseteq \beta S\backslash S$ it is empty in S. By definition, $\mathscr{X}$ does not satisfy the countable intersection property and so it is not real. So there are no free real z-ultrafilters in $Z[S]$. Then real z-ultrafilters in $Z[S]$ must be fixed, as required.

[1] Recall that βS is a normal space. By theorem 10.10, "In a normal space, a closed set F is a G_δ if and only if F is a zero-set". See that $h^{\beta(\omega)\leftarrow}[\,[-a, b]\,] = \cap\{h^{\beta(\omega)\leftarrow}[\,(a - 1/n, b + 1/n)\,]\}$ is a G_δ. Then $h^{\beta(\omega)\leftarrow}[-1/n, 1/n]$ is a zero-set, for each n.

(c $\Rightarrow$ b) We are given that every real z-ultrafilter is fixed. Let $p \in \beta S \backslash S$. We are required to produce a zero-set Z containing p where $Z \subset \beta S \backslash S$.

Suppose $\mathscr{Z}^\beta = \{Z(f^\beta) : f \in M \subseteq C(S)\}$ is the unique z-ultrafilter in $Z[\beta S]$ which converges to p. By hypothesis, the corresponding *free* z-ultrafilter, $\mathscr{Z} = \{Z(f) : f \in M \subseteq C(S)\}$, in $Z[S]$ is not real. Then, by definition of real z-ultrafilter, $\mathscr{Z}$ contains a countable subfamily $\{Z(f_n) : f_n \in D \subseteq M\}$ such that $\cap\{Z(f_n) : f_n \in D \subseteq M\} = \varnothing$. So p belongs to $W = \cap\{Z(f_n^{\beta(\omega)}) : f_n \in D \subseteq M\} \subseteq \beta S \backslash S$.

Since W is a countable intersection of zero-sets, then as explained on page 171, W is a zero-set.

So $\beta S \backslash S$ contains a zero-set W. We are done.

We present still another characterization of realcompact spaces in corollary 24.6.

Example 1. Show that $\mathbb{R}$ equipped with the usual topology is a realcompact space.

Solution : Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as, $f(x) = 1/(1 + |x|)$. The function f is nowhere zero on $\mathbb{R}$. Recall from theorem 10.10 that ...

"*..., closed G_δ's in a normal space are zero-sets.*"

Then, since $\mathbb{R}$ is normal, for each $n \in \mathbb{N}\backslash\{0\}$, $f^{\beta(\omega)\leftarrow}[0, 1/n]$ is a zero-set neighbourhood of $\beta\mathbb{R}\backslash\mathbb{R}$.Then $\beta\mathbb{R}\backslash\mathbb{R} = \cap\{f^{\beta(\omega)\leftarrow}[0, 1/n] : n > 0\}$. Being a countable intersection of G_δ's in $\beta\mathbb{R}$, $\beta\mathbb{R}\backslash\mathbb{R}$ is a $\beta\mathbb{R}$ zero-set disjoint from $\mathbb{R}$. By theorem 23.3, $\mathbb{R}$ is realcompact.[1]

23.2 Extending functions to υS.

Let S and T both be locally compact and Hausdorff. Then, being completely regular, S and T can be densely embedded into βS and βT, respectively. Suppose $f : S \to T$ is a continuous function which maps S onto T. By theorem 21.6, the function $f : S \to \beta T$ extends to $f^{\beta(\beta T)} : \beta S \to \beta T$. We know that υS is a subspace of βS, hence the restriction, $f^{\beta(\beta T)}|_{\upsilon S} : \upsilon S \to \beta T$, maps υS into βT.

In the following theorem we will show a little bit more. We show that $f^\upsilon[\upsilon S] \subseteq \upsilon T$. That is, if $f : S \to T$ is any continuous function mapping S to T, (both locally compact and Hausdorff) f will extend to a continuous function f^υ which maps υS *into* υT.

---

[1] In theorem 24.7 of the next chapter where better tools are available, we show that any product of $\mathbb{R}$'s is realcompact.

Theorem 23.4 If S and T are locally compact Hausdorff spaces and $f : S \to T$ is a continuous function then f extends continuously to a unique continuous function $f^{\upsilon} : \upsilon S \to \upsilon T$.

Proof: We are given that S and T are locally compact Hausdorff spaces and $f : S \to T$ is a continuous function. Since f extends to $f^{\beta} : \beta S \to \beta T$ and $S \subseteq \upsilon S \subseteq \beta S$, then f extends to $f^{\upsilon} : \upsilon S \to \beta T$. We are required to show that $f^{\upsilon}[\upsilon S] \subseteq \upsilon T$.

Suppose $h \in C(T)$. Then, if $\omega\mathbb{R} = \mathbb{R} \cup \{\infty\}$,

$$h : T \to \mathbb{R} \text{ will continuously extend to } h^{\omega} : \beta T \to \omega\mathbb{R}$$

Furthermore,

$$(h \circ f) : S \to \mathbb{R} \text{ will continuously extend to } (h \circ f)^{\omega} : \beta S \to \omega\mathbb{R}$$

For $x \in S$, $(h \circ f)^{\omega}|_S(x) = h(f(x)) = (h^{\omega} \circ f^{\beta})|_S(x)$. Since S is dense in βS, then

$$\begin{array}{rcl} (h \circ f)^{\omega} : \beta S & \to & \omega\mathbb{R} \\ (h^{\omega} \circ f^{\beta}) : \beta S & \to & \omega\mathbb{R} \end{array}$$

are equal functions on βS.

Let $r \in \upsilon S$. By definition, r is a real point of the function $h \circ f : S \to \omega\mathbb{R}$. This means that $(h \circ f)^{\omega}(r) = h^{\omega}[f^{\beta}(r)] \neq \infty$. Then $f^{\beta}(r)$ is a real point of h. That is,

$$f^{\upsilon}(r) \in \upsilon_h T$$

But h was an arbitrarily chosen function in $C(T)$. The point r was also arbitrarily chosen in υS. We can then conclude that

$$f^{\upsilon}[\upsilon S] \subseteq \cap\{\upsilon_h T : h \in C(T)\} = \upsilon T$$

So $f^{\upsilon}[\upsilon S] \subseteq \upsilon T$. This is what we were required to prove.

Corollary 23.5 Suppose S and T are locally compact Hausdorff spaces. If T is realcompact and $f : S \to T$ is a continuous function then f extends continuously to a unique continuous function $f^{\upsilon} : \upsilon S \to T$. In particular, if $f \in C(S)$, f extends continuously to $f^{\upsilon} \in C(\upsilon S)$.

Proof: This is a special case of the theorem where $T = \upsilon T$. The second part follows from the fact that $\mathbb{R}$ is realcompact as shown in the above example.

From the above corollary, we see that, if S is completely regular, every continuous function $f : S \to \mathbb{R}$ in $C(S)$ extends to $f^{\upsilon} : \upsilon S \to \mathbb{R}$. We know that S is C^*-embedded in βS, but now we can confidently say that

"*S is C-embedded in υS*"

23.3 The Hewitt-Nachbin realcompactification of a space S.

We have seen that $S \subseteq \upsilon S \subseteq \beta S$. We have also seen that, for every function $f \in C(S)$, the function, $f^{\beta(\omega)}|_{\upsilon S}$, continuously maps υS into $\mathbb{R}$. That is, every function $f \in C(S)$ extends to a function, f^{υ}, in $C(\upsilon S)$.

We also make the following observation. Since $\upsilon S \subseteq \beta S$ then $\mathrm{cl}_{\beta S}\upsilon S \subseteq \beta S$. Given that $\mathrm{cl}_{\beta S} S \subseteq \mathrm{cl}_{\beta S}\upsilon S$, then

$$\beta(\upsilon S) = \beta S$$

Theorem 23.6 Suppose S is locally compact and Hausdorff. Then υS is a realcompact subspace of βS.

Proof: To show that υS is realcompact, it suffices to show that $\upsilon(\upsilon S) = \upsilon S$. To do this, it suffices to show that the set of all real points of υS is υS.

First note that $\upsilon S \subseteq \upsilon(\upsilon S)$. Then it suffices to show that $\upsilon(\upsilon S) \subseteq \upsilon S$. Suppose $p \in \upsilon(\upsilon S)$. Then p is a real point of υS. Let $f \in C(S)$. Then there is $g \in C(\upsilon S)$ such that $g = f^{\upsilon}$. Since p is a real point of υS then $g^{\beta(\omega)}(p) \in \mathbb{R}$. See that, since g and f^{υ} agree on υS, $g^{\beta(\omega)} = f^{\beta(\omega)}$ on βS. So $f^{\beta(\omega)}(p) \in \mathbb{R}$. So p is a real point of S. That is $p \in \upsilon S$. We have shown that $\upsilon(\upsilon S) \subseteq \upsilon S$.

So $\upsilon(\upsilon S) = \upsilon S$.

Then we can view υS as a realcompact space which densely contains the locally compact Hausdorff space S.

Given a space S, the topological space, υS, is normally referred to as the

Hewitt-Nachbin realcompactification of S.[1]

So rather than speaking of υS as being the set of all real points of S we will now speak of υS as being the *realcompactification of S*. If a space, S, is realcompact then it is its own realcompactification.

If S is pseudocompact then $C(S) = C^*(S)$ and so every point of βS is a real point of S. Hence βS is both the Stone-Čech compactification and the Hewitt-Nachbin realcompactification of S.

23.4 Another characterization of compactness.

The realcompact property provides us with another characterization of the compact property.

Theorem 23.7 A Hausdorff topological space, S, is compact if and only if S is both realcompact and pseudocompact.

Proof: ($\Rightarrow$) If S is compact then $S = \beta S$; hence every real-valued function on S is bounded. Then, by definition, S is pseudocompact. Since $S \subseteq \upsilon S \subseteq \beta S$, then $S = \upsilon S$, so S is realcompact.

($\Leftarrow$) Suppose S is both pseudocompact and realcompact. We are required to show that S is compact.

Since S is pseudocompact every $f \in C(S)$ is bounded. Then, for such an f,

$$f^{\beta(\omega)}[\beta S] \subseteq \mathrm{cl}_{\mathbb{R}} f[S] \subseteq \mathbb{R}$$

Then

$$\beta S \subseteq \cap\{f^{\beta(\omega)\leftarrow}[\mathbb{R}] : f \in C(S)\} = \upsilon S$$

So $\upsilon S = \beta S$. Since S is realcompact then $S = \upsilon S$. So $S = \beta S$. So S is compact, as required.

Theorem 23.8 Let S be a non-compact space such that $\beta S = \upsilon S$. Then S is pseudocompact.

Proof: Suppose S is not pseudocompact. Then there is an unbounded function, f, in $C(S)$. Then $f : S \to \mathbb{R}$ extends to $f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$. If $f^{\beta(\omega)}[\beta S] \subseteq \mathbb{R}$ then $f^{\beta(\omega)}[\beta S]$ is non-compact since it is unbounded. So there is an $x \in \beta S \backslash S$ such that $f^{\beta(\omega)}(x) = \infty$. In such a case, $\beta S \neq \upsilon S$. So S must be pseudocompact.

23.5 A few examples of realcompact spaces.

In theorem 16.3, we showed that in any Lindelöf space, S, every filter of closed sets in $\mathscr{P}(S)$ which satisfies the countable intersection property has non-empty intersection. In fact, this property characterizes Lindelöf spaces. It follows that, in such spaces, every real z-ultrafilter has non-empty intersection. That is, every real z-ultrafilter is fixed in S. Then, by theorem 23.3, Lindelöf spaces are realcompact. We state this as a formal result.

Theorem 23.9 Lindelöf spaces are realcompact.

Proof : Given in the paragraph above.

Example 2. By definition 18.9, a σ-compact space is the countable union of compact sets. Under what conditions is a σ-compact space guaranteed to be realcompact?

Solution : In theorem 18.10 it is shown that in a locally compact Hausdorff space the σ-compact property is equivalent to the Lindelöf property. So, for any locally compact Hausdorff space, a σ-compact space is realcompact.

Example 3. Show that $\mathbb{N}$ and $\mathbb{Q}$ are realcompact.

Solution : Any countable space is the union of singleton sets and hence is σ-compact. Since both $\mathbb{N}$ and $\mathbb{Q}$ are locally compact Hausdorff countable spaces they are realcompact.

Example 4. Show that any separable metric space is realcompact.

Solution : By definition, a space that has a countable open base is Lindelöf. By theorem 16.3, for metric spaces, the following are equivalent: 1) the second countable property, 2) the separable property and 3) the Lindelöf property. By theorem 23.9, a separable metric space is realcompact.

Example 5. Show that every subspace of $\mathbb{R}^n$ equipped with the usual topology is realcompact.

Solution : The Euclidean space $\mathbb{R}^n$ is a metric space which is a finite product of the separable space $\mathbb{R}$ and hence is separable. Then it is second countable. So every one of its subspaces is a second countable metric space. Then every one of its subspaces is Lindelöf. By theorem 23.9, every one of its subspaces is realcompact.

23.6 Properties of realcompact spaces.

We now examine when a particular space inherits the realcompact property from other spaces known to be realcompact. We show in particular that the realcompact property is closed under arbitrary intersections and arbitrary products. Also show that closed subspaces inherit the realcompact property from its superset.

Theorem 23.10 Let $\{S_\alpha : \alpha \in I\}$ be a family of non-empty completely regular realcompact spaces.

a) If the S_α's are all subspaces of a realcompact space S and $T = \cap\{S_\alpha : \alpha \in I\}$ is non-empty, then T is a realcompact subspace of S.

b) The product space, $\prod_{\alpha\in I} S_\alpha$, is realcompact.

Proof: We are given that $\{S_\alpha : \alpha \in I\}$ is a family of completely regular non-empty *realcompact* spaces.

a) Suppose the S_α's are all subspaces of a realcompact space S and $T = \cap\{S_\alpha : \alpha \in I\}$ is non-empty.

Then, for each α, $S_\alpha = \upsilon S_\alpha$. Since subspaces of completely regular spaces are completely regular, $T = \cap\{S_\alpha : \alpha \in I\}$, a completely regular subspace. As shown in an example on page 338, $\beta T \subseteq \beta S_\alpha$. So $\upsilon T \subseteq \beta S_\alpha$.

We claim that $\upsilon T \subseteq \upsilon S_\alpha$. For each α, let $i_\alpha : T \to S_\alpha$ denote the identity map embedding the subspace T into S_α. Then, by theorem 23.4, for each α, $i_\alpha : T \to S_\alpha$ extends continuously to the unique identity function.

$$i_\alpha^\upsilon : \upsilon T \to \upsilon S_\alpha = S_\alpha$$

embedding υT into υS_α, establishing the claim.

Since $i_\alpha^\upsilon[\upsilon T] = \upsilon T \subseteq \upsilon S_\alpha = S_\alpha$ for all $\alpha \in I$, then $\upsilon T \subseteq \cap\{S_\alpha : \alpha \in I\} = T$. Given that $T \subseteq \upsilon T$, we can only conclude that $T = \upsilon T$ and so T is realcompact.

b) Suppose $S = \prod_{\alpha\in I} S_\alpha$ where each S_α is realcompact. For $\gamma \in I$, the projection map, $\pi_\gamma : S \to S_\gamma$ is continuous and hence extends continuously to

$$\pi_\gamma^\upsilon : \upsilon S \to \upsilon S_\gamma$$

where $S_\gamma = \upsilon S_\gamma$.

Let $\mathscr{G} = \{\pi_\alpha : \alpha \in I\}$ where $\pi_\alpha : S \to S_\alpha$. Then the evaluation map, $e_\mathscr{G} : S \to \prod_{\alpha \in I} S_\alpha$ extends to $e_\mathscr{G}{}^\upsilon : \upsilon S \to \prod_{\alpha \in I} S_\alpha = S$ a function which embeds υS into S.

Then $\upsilon S \subseteq S$. We conclude that $S = \upsilon S$ and so S is realcompact.

The subspace, T, of a realcompact space, S, need not be realcompact unless T is closed in S.

Theorem 23.11 If F is a closed non-empty subspace of a realcompact space S then F is realcompact.

Proof : We are given that F is closed in realcompact S. Then $S = \upsilon S$ and $\mathrm{cl}_{\upsilon S} F = \mathrm{cl}_S F = F$. We are required to show that $F = \upsilon F$.

Let $i : F \to S$ denote the inclusion map embedding F into S. Then $i : F \to F \subseteq S$ extends continuously to $i^\upsilon : \upsilon F \to \upsilon S$ where

$$i^\upsilon[\upsilon F] = \upsilon F \subseteq \upsilon S = S$$

Now F is dense in υF hence

$$\upsilon F \subseteq \mathrm{cl}_{\upsilon S} F = \mathrm{cl}_S F = F$$

Then $\upsilon F \subseteq F$. So $F = \upsilon F$.

23.7 Example of a non-realcompact space.

The above few examples may lead the reader to suspect that non-realcompact spaces are uncommon, and that, if there are any, such spaces are not often witnessed. This motivates us to take the time to try to construct such a space. The following example exhibits two such spaces. The space T which appears in the following example is called the "*deleted Tychonoff plank*".

Example 6. Recall that ω_1 denotes the first uncountable ordinal while ω_0 denotes the first countable infinite ordinal. Let $X = [0, \omega_1)$ and $Y = [0, \omega_0)$ both be equipped with the ordinal topology, and $T = X \times Y$ be the corresponding product space.[1]

In the example on page 256, it is shown that $X = [0, \omega_1)$ is countably compact. By theorem 15.9, since X is countably compact, for any $f \in C(X)$, $f[X]$ is compact in $\mathbb{R}$. This implies that $[0, \omega_1)$ is pseudocompact. But the space X is non-compact since the open cover $\{[0, \gamma) : \gamma \in S\}$ has no finite subcover. By theorem 23.7, pseudocompact spaces with the realcompact property must be compact. So

$[0, \omega_1)$ *is not realcompact*

But the space X, when viewed as a homeomorphic image of the subspace, $[0, \omega_1) \times \{0\}$, of the product, T, is closed in T. By theorem 23.11, closed subspaces of realcompact spaces are realcompact. So

the product space, $[0, \omega_1) \times [0, \omega_0)$*, cannot be realcompact*

Concepts review.

1. If f is a function in $C(S)$ define set, $\upsilon_f S$, of all the real points of f.
2. Define υS in terms of all the sets $\upsilon_f S$.
3. What does it mean to say that the space, S, is realcompact?
4. Define a real z-ultrafilter in $Z[S]$.
5. Give a characterization of the realcompactness property in terms of zero-sets in $Z[\beta S]$.
6. Give a characterization of the realcompactness property in terms of z-ultrafilters of zero-sets in $Z[S]$.
7. If $f : S \to T$ is a continuous function mapping S into T what can be said about a particular continuous extension of f?
8. What does it mean to say that $f \in C(S)$ is C-embedded in υS?
9. What can we say about a space that is both realcompact and pseudocompact?
10. What can we say about Lindelöf spaces in the context discussed in this chapter?

[1]In theorem 21.17, it is shown that $\beta X = [0, \omega_1] = \omega X$, the one-point compactification of X. Recall in an example on page 159 we showed that the spaces $X = [0, \omega_1)$ and $Y = [0, \omega_0)$ equipped with the ordinal topology are both normal but the product space and $T = [0, \omega_1) \times [0, \omega_0)$, is not normal. The space $T = [0, \omega_1) \times [0, \omega_0)$ is referred to as the *deleted Tychonoff plank*.

11. What can we say about arbitrary intersections of realcompact space?
12. What can we say about arbitrary products of realcompact space?
13. What kind of subsets of a realcompact space are guaranteed to be realcompact?
14. Give an example of a realcompact space.
15. Give an example of a space which is not realcompact.

---

24 / Perfect functions

Summary. *In this section we introduce the notion of a perfect function. After providing a formal definition we produce two characterizations. Our brief discussion of perfect functions will refer to notions seen in our study of singular functions and those of the realcompact property.*

24.1 Introduction.

Our study of realcompact spaces started with a brief discussion of a continuous function, $f : S \to \mathbb{R}$ mapping a completely regular space, S, into $\mathbb{R}$. So f could be seen as a map, $f_+ : S \to \omega\mathbb{R}$, continuously mapping S into the one-point compactification $\omega\mathbb{R} = \mathbb{R} \cup \{\infty\}$, where $f_+ = f$ on S. By theorem 21.6, $f_+ : S \to \omega\mathbb{R}$ extends continuously to a function, $f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$. If it was clear that $f^{\beta(\omega)}[S] \subseteq \mathbb{R}$, but even if f was unbounded, there was no reason to assume that $f^{\beta(\omega)}$ would map all of $\beta S \setminus S$ into $\{\infty\}$. In fact, $f^{\beta(\omega)}$ could map some points, x, in $\beta S \setminus S$ to $f^{\beta(\omega)}(x) \in \mathbb{R}$.

We defined $\upsilon_f S$ as

$$\upsilon_f S = \{x \in \beta S : f^{\beta(\omega)}(x) \in \mathbb{R}\} = f^{\beta(\omega)\leftarrow}[\mathbb{R}]$$

The points in $\upsilon_f S$ were referred to as the "real points of f". When $\upsilon_f S$ was not entirely contained in S we described this as some property of S rather than as a property of f. The set

$$\upsilon S = \cap\{\upsilon_f S : f \in C(S)\}$$

was called the set of "all real points of S". When $\upsilon S \cap \beta S \setminus S = \varnothing$, we referred to S as being "realcompact".

In this section we will generalize the procedure used to construct $\upsilon_f S$ and study those functions $f : S \to T$ such that $f_+ : S \to \alpha T$, *for any compactification* αT of T.

24.2 Perfect function: Definition.

Recall that, for a function, $f : S \to T$, the *fibres of* f refers to the elements of the set

$$\{f^{\leftarrow}(y) : y \in f[S]\} \subseteq \mathscr{P}(S)$$

Definition 24.1 Let S and T be completely regular spaces and $f : S \to T$ be a continuous function mapping S into T. We say that f is a *compact mapping* if its fibres are all compact. That is, $f^{\leftarrow}(y)$ is compact for each y in the range of f.

We say that the continuous function, $f : S \to T$, is a *perfect function* if f is both a closed function and a compact mapping.[1]

In our introductory paragraph, we considered a continuous function $f : S \to \mathbb{R}$ to construct the function $f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$. Here we will start with a function, $f : S \to T$, to construct, by using a similar procedure, a function $f^{\beta(\alpha)} : \beta S \to \alpha T$.

Suppose $f : S \to T$ continuously maps S into T (both completely regular). Let $i : T \to \alpha T$ be the identity map embedding T into αT. Let $f_+ : S \to \alpha T$ be defined as

$$f_+(x) = i(f(x))$$

Then, by theorem 21.6, $f_+ : S \to \alpha T$ extends continuously to

$$f^{\beta(\alpha)} : \beta S \to \alpha T$$

With this fact in mind we provide a characterization which relates properties of a perfect function, $f : S \to T$, and the properties of the corresponding function, $f^{\beta(\alpha)} : \beta S \to \alpha T$.

Theorem 24.2 Let $f : S \to T$ be a continuous function mapping a non-compact completely regular space, S, *into* a non-compact completely regular space T. Let αT be any Hausdorff compactification of T. Then the following are equivalent:

a) The function $f : S \to T$ is perfect.

b) If $f : S \to T$, and $f^{\beta(\alpha)} : \beta S \to \mathrm{cl}_{\alpha T} f[S] \subseteq \alpha T$ is its continuous extension to βS, then $f^{\beta(\alpha)}[\beta S \backslash S] \subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$.

Proof: (a $\Rightarrow$ b) We are given that $f : S \to T$ is perfect, αT is any Hausdorff compactification of T and $f^{\beta(\alpha)} : \beta S \to \alpha T$ is the continuous extension of f to βS. We are required to show that $f^{\beta(\alpha)}[\beta S \backslash S] \subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$.

Suppose $f^{\beta(\alpha)}[\beta S \backslash S] \not\subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$. Then $f(u) \in f[S]$ for some $u \in \beta S \backslash S$. Let $M = S \cup \{u\}$, so that S is dense in M. Then, given that $f : S \to T$ is perfect,

[1] Some authors may not require that perfect functions be continuous.

$f^{\leftarrow}(f(u)) \cap S$ is a compact subset of S.

Let $K = f^{\leftarrow}(f(u)) \cap S$. Then K is a closed subset of M.

Then there is an open neighbourhood, U, of M such that $K \subseteq U$ and $u \in M \backslash \mathrm{cl}_M U$. Since $u \in \mathrm{cl}_M(S \backslash U)$. we then have,

$$\begin{array}{rcl} f(u) & \in & f[\mathrm{cl}_M[S \backslash U]] \\ & \subseteq & \mathrm{cl}_T f[S \backslash U] \\ & = & f[S \backslash U] \quad \text{(Since } f \text{ is a closed function.)} \end{array}$$

Then $f(u) = f(t)$ for some point t in $S \backslash U$. Then $t \in f^{\leftarrow}(f(u)) \cap S = K$.
So K intersects $S \backslash U$, a contradiction.

So $f^{\beta(\alpha)}[\beta S \backslash S] \subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$, as required.

(b $\Rightarrow$ a) We are given that $f : S \to T$ is a continuous function which extends to $f^{\beta(\alpha)} : \beta S \to \alpha T$ such that

$$f^{\beta(\alpha)}[\beta S \backslash S] \subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S] \qquad (\dagger)$$

We are required to show that f is both compact and closed.

Let $y \in T$. Then $f^{\beta(\alpha)\leftarrow}(y)$ is closed in βS.

Case 1: $f^{\beta(\alpha)\leftarrow}(y) \subseteq S$. Then it is a closed subset of βS so it is compact.

Case 2: $f^{\beta(\alpha)\leftarrow}(y) \cap \beta S \backslash S \notin \varnothing$. Then, by hypothesis,

$$f^{\beta(\alpha)}[\, f^{\beta(\alpha)\leftarrow}(y) \cap \beta S \backslash S\,] = \{y\} \subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$$

Since $y \in T$, this case cannot occur. So $f^{\beta(\alpha)\leftarrow}(y) \subseteq S$. Then $f^{\beta(\alpha)\leftarrow}(y)$ is compact in S.

We have shown that $f : S \to T$ is a compact function.

We now show that f is a closed function. Let F be a closed subset of S.
See that $\mathrm{cl}_{\beta S} F$ is compact in βS and so $f^{\beta(\alpha)}[\mathrm{cl}_{\beta S} F]$ is a closed subset of αT. Note that

$$\mathrm{cl}_{\beta S} F = (\mathrm{cl}_{\beta S} F \cap \beta S \backslash S) \cup F$$

(a disjoint union).

So

$$\begin{aligned} f^{\beta(\alpha)}[\mathrm{cl}_{\beta S}F] \cap T &= f^{\beta(\alpha)}[\ (\mathrm{cl}_{\beta S}F \cap \beta S\backslash S) \cup F\ ] \cap T \\ &= \Big[\ f^{\beta(\alpha)}[\ (\mathrm{cl}_{\beta S}F \cap \beta S\backslash S)\ ] \cup f^{\beta(\alpha)}[F]\ \Big] \cap T \\ &= \Big[\ f^{\beta(\alpha)}[\ (\mathrm{cl}_{\beta S}F \cap \beta S\backslash S)\ ] \cap T \Big] \cup \Big[\ f^{\beta(\alpha)}[F]\ \cap T \Big] \\ &\subseteq [\ [\mathrm{cl}_{\alpha T} f[S] \backslash f[S]\ ] \cap T\ ] \cup \Big[\ f^{\beta(\alpha)}[F]\ \cap T \Big] \quad \text{(By (†))}. \\ &= \varnothing \cup [\ f[F]\ \cap T\ ] \quad \text{(By our hypothesis)} \\ &= f[F] \end{aligned}$$

Then $f^{\beta(\alpha)}[\mathrm{cl}_{\beta S}F] \cap T = f[F]$, a closed subset of T. So f is a closed map.

By definition, f is a perfect function. We are done with b $\Rightarrow$ a.

24.3 Relating perfect functions to a singular set.

Recall that, if $f : S \to K$ is a continuous function mapping S into the compact space K, then the singular set, $S(f)$, is a subset of K, defined as,

“$S(f) = \{x \in \mathrm{cl}_K f[S] : \mathrm{cl}_S f^{\leftarrow}[U]$ is non-compact, $\forall$ open neighbourhood U of $x\}$”

Also, when suitably topologized,

$$\gamma S = S \cup S(f)$$

is a compactification of S (not necessarily singular).

Recall from the remark on page 351 that

$$f^{\gamma}[\gamma S] = f^{\gamma}[S \cup S(f)] = f[S] \cup S(f) = \mathrm{cl}_K f[S]$$

We now establish a relationship between a perfect function $f : S \to K$ and the singular set, $S(f)$, of f.

Theorem 24.3 Let S and T be non-compact locally compact Hausdorff spaces and $f : S \to T$ be a non-singular continuous function mapping S *into* T. Then f is perfect if and only if

$$f[S] \cap S(f) = \varnothing$$

Proof: We are given that $f : S \to T$ is a non-singular continuous function mapping S *into* T, where S and T are both locally compact and Hausdorff. Let αT be a compactification of T so that f maps S onto the subset $f[S]$ of αT. Let γS represent the compactification $S \cup S(f)$ of S.

By theorem 21.6, the function $f : S \to \alpha T$, extends to $f^{\beta(\alpha)} : \beta S \to \mathrm{cl}_{\alpha T} f[S] \subseteq \alpha T$.

Since $\gamma S \preceq \beta S$ there is a projection map $\pi_{\beta\to\gamma} : \beta S \to \gamma S$ such that $\pi_{\beta\to\gamma}(x) = x$ on S, and $\pi_{\beta\to\gamma}[\beta S \backslash S] = S(f)$. Then, if we define $f^{\gamma(\alpha)} : \gamma S \to \alpha T$, as

$$f^{\gamma(\alpha)}[\pi_{\beta\to\gamma}(x)] = f^{\beta(\alpha)}(x)$$

then, $f^{\gamma(\alpha)}[\pi_{\beta\to\gamma}[\beta S]] = \mathrm{cl}_{\alpha T} f[S] \subseteq \alpha T$ where

$$f^{\gamma(\alpha)} : S \cup S(f) \;\to\; \mathrm{cl}_{\alpha T} f[S] \subseteq \alpha T$$

is a continuous extension of $f : S \to \alpha T$ to all of γS.

Then,

$$\begin{aligned} f^{\gamma(\alpha)}[S \cup S(f)] &= f^{\gamma(\alpha)}[S] \cup f^{\gamma(\alpha)}[S(f)] \\ &= f[S] \cup S(f) \\ &= \mathrm{cl}_{\alpha T} f[S] \end{aligned}$$

$(\Rightarrow)$ Suppose f is perfect. Then $f^{\beta(\alpha)}[\beta S \backslash S] \subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$. Then

$$\begin{aligned} f^{\beta(\alpha)}[\beta S \backslash S] &= f^{\gamma(\alpha)}[\pi_{\beta\to\gamma}[\beta S \backslash S]] \\ &= f^{\gamma(\alpha)}[S(f)] \\ &= S(f) \\ &\subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S] \quad \text{(By hypothesis.)} \end{aligned}$$

So, when $f : S \to T$ is perfect, and $S(f)$ is the singular set of f, since $S(f) \subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$

$$f[S] \cap S(f) = \varnothing$$

$(\Leftarrow)$ Conversely, suppose $f[S] \cap S(f) = \varnothing$.

$$\begin{aligned} f^{\beta(\alpha)}[\beta S \backslash S] &= f^{\gamma(\alpha)}[\pi_{\beta\to\gamma}[\beta S \backslash S]] \\ &= f^{\gamma(\alpha)}[S(f)] \\ &= S(f) \\ &= S(f) \backslash f[S] \quad \text{(Since } f[S] \cap S(f) = \varnothing\text{)} \\ &\subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S] \end{aligned}$$

So $f^{\beta(\alpha)}[\beta S \backslash S] \subseteq \mathrm{cl}_{\alpha T} f[S] \backslash f[S]$. Hence f is perfect, as required.

The above theorem shows that the singular set, $S(f)$, can serve as a useful tool to recognize both perfect functions and singular functions. Since a function $f : S \to T$ is singular when $f[S] \subseteq S(f)$ a singular function can never be perfect. A perfect function is one such that $f[S] \cap S(f)$ is empty.

Example 1. Suppose S is non-compact completely regular and *connected*. Show that every bounded real-valued continuous function on S is perfect.

Solution : We are given that S is connected. Since f is bounded, the function $f : S \to \mathbb{R}$ extends to $f^{\beta} : \beta S \to \mathrm{cl}_{\mathbb{R}} f[S]$ where $f^{\beta}[\beta S] = \mathrm{cl}_{\mathbb{R}} f[S]$. Since S is connected then, by theorem 20.2, so is $f[S]$. This implies $\mathrm{cl}_{\mathbb{R}} f[S] = [a, b]$, for some a and b.

If $b \notin f[S]$ and U is an open neighbourhood of b then $f^{\beta\leftarrow}(b) \cap \mathrm{cl}_{\beta S} S \backslash S \neq \varnothing$. Then

$$f^{\beta\leftarrow}(b) \subseteq \mathrm{cl}_{\beta S} f^{\beta\leftarrow}[U]$$

So $\mathrm{cl}_S f^{\leftarrow}[U]$ is not compact. So $b \in S(f) \backslash f[S]$.

The same holds true for a in the case where $a \notin f[S]$. So $S(f) \subseteq \mathbb{R} \backslash f[S]$. This means that $S(f) \cap f[S] = \varnothing$, so f is perfect.

Example 2. Let $f = \left(\frac{2}{\pi}\right)\arctan$. We then obtain the function $f : \mathbb{R} \to [-1, 1]$. Show that f is perfect.

Solution : The function, f, maps the connected interval $(-\infty, \infty)$ one-to-one onto $T = (-1, 1)$. Since $f : \mathbb{R} \to [-1, 1]$ is real-valued and bounded then, by the statement proven in example 1, f is perfect.

24.4 Realcompact property and perfect evaluation maps.

We will review a few facts related to the realcompact property. Suppose S is completely regular and $\mathscr{F} = C(S)$. If $f \in \mathscr{F}$, we have seen that $f : S \to \mathbb{R}$ extends continuously to $f^{\beta(\omega)} : \beta S \to \omega\mathbb{R}$. Furthermore,

$$\upsilon_f S = \{x \in \beta S : f^{\beta(\omega)}(x) \in \mathbb{R}\} = f^{\beta(\omega)\leftarrow}[\mathbb{R}]$$

represents all the "real points in βS associated to f". If a point p in βS is a real point associated *to all* $f \in C(S)$ then we say that p is a "real point of βS". Then the set

$$\upsilon S = \cap\{f^{\beta(\omega)\leftarrow}[\mathbb{R}] : f \in \mathscr{F}\} = \cap\{\upsilon_f S : f \in \mathscr{F}\}$$

is the set of all real points of βS.

So, if $x \in \beta S \setminus \upsilon S$ there is some $g \in C(S)$ such that $g^{\beta(\omega)}(x) = \infty$. If $\upsilon S = S$ then S is said to be realcompact ; that is, for every point, p, in $\beta S \setminus S$ there is some $f \in C(S)$ such that $f^{\beta(\omega)}(p) = \infty$.

In the special case where $f \in C^*(S)$, then

$$\begin{aligned} f^{\beta(\omega)}[\beta S] &= f^{\beta(\omega)}[\mathrm{cl}_{\beta S} S] \\ &= \mathrm{cl}_{\omega\mathbb{R}} f[S] \\ &= \mathrm{cl}_{\mathbb{R}} \mathbb{R} \\ &\subseteq \mathbb{R} \end{aligned}$$

So, in this case, $\upsilon_f S = f^{\beta(\omega)\leftarrow}[\mathbb{R}] = \beta S$. Then $\cap\{f^{\beta(\omega)\leftarrow}[\mathbb{R}] : f \in C^*(S)\} = \beta S$. When S is not pseudocompact (pseudocompactness being characterized by $C(S) = C^*(S)$), the functions, f, in $C^*(S)$ play no role in distinguishing the real points in $\beta S \setminus S$ from the "non-real points". Only unbounded functions, f, will extend to a function, $f^{\beta(\omega)}$, so that $f^{\beta(\omega)\leftarrow}(\infty)$ is non-empty.

So, if S is *not* pseudocompact we will denote the set of all unbounded real-valued functions on S by $\mathscr{H} = \mathscr{F} \setminus C^*(S)$. Then

$$\upsilon S = \cap\{f^{\beta(\omega)\leftarrow}[\mathbb{R}] : f \in \mathscr{H}\} = \cap\{f^{\beta(\omega)\leftarrow}[\mathbb{R}] : f \in \mathscr{F}\}$$

If $g \in \mathscr{H}$, then $g^{\beta(\omega)}[\mathrm{cl}_{\beta S} S] = \mathrm{cl}_{\omega\mathbb{R}} g[S]$. Since $g[S]$ is unbounded then it is not a compact subset of $\mathbb{R}$, so there must be some point q in $\beta S \setminus S$ such that $g^{\beta(\omega)}(q) = \infty$. In this case, $e_{\mathscr{F}}^{\beta(\alpha)}(q) = < f^{\beta(\omega)}(q) >_{f \in \mathscr{F}}$, where some component $g^{\beta(\omega)}(q) = \infty$, so

$$e_{\mathscr{F}}^{\beta(\alpha)}(q) \notin \textstyle\prod_{f \in \mathscr{F}} \mathbb{R}$$

We will now consider these few notions from a slightly different point of view.

First, some notation. If $\mathscr{F} = C(S)$ and $\mathscr{H} = \mathscr{F} \setminus C^*(S)$,

$$\begin{aligned} H &= \textstyle\prod_{f \in \mathscr{F}} \omega\mathbb{R} \\ K &= \textstyle\prod_{f \in \mathscr{F}} \mathrm{cl}_{\omega\mathbb{R}} f[S] \\ T &= e_{\mathscr{F}}[S] \\ \alpha T &= \mathrm{cl}_K T \end{aligned}$$

By Tychonoff's theorem, H and K are compact with T dense in $\alpha T \subseteq K \subseteq H$.

For each $f : S \to \mathbb{R}$, we obtain $f_+ : S \to \omega\mathbb{R}$ which extends to $f^{\beta(\omega)} : \beta S \to \mathrm{cl}_{\omega\mathbb{R}} f[S]$.

The evaluation function, $e_{\mathscr{F}} : S \to T$, maps S onto T, where

$$e_{\mathscr{F}}(x) \, = < f(x) >_{f \in \mathscr{F}} \in \; T$$

It extends to

$$e^{\beta(\alpha)}{}_{\mathscr{F}} : \beta S \to \alpha T$$

where $e^{\beta(\alpha)}{}_{\mathscr{F}}$ maps βS onto $\mathrm{cl}_K T = \alpha T$.

In the following theorem we establish a fundamental relationship between the realcompact property and properties of $e_{\mathscr{F}}$.

Theorem 24.4 Let S be both a completely regular and non-pseudocompact space and $\mathscr{F} = C(S)$. Suppose S is realcompact. Then the evaluation map,

$$e_{\mathscr{F}} : S \to \prod_{f \in \mathscr{F}} \mathbb{R}$$

is a *perfect function* which maps S onto $e_{\mathscr{F}}[S]$ in $\prod_{f \in \mathscr{F}} \mathbb{R}$.

Proof: We are given that S is both a completely regular and non-pseudocompact space and $\mathscr{F} = C(S)$.

Let $\mathscr{H} = \mathscr{F} \backslash C^*(S)$ and $K_{\mathscr{H}} = \prod_{f \in \mathscr{H}} \mathrm{cl}_{\omega \mathbb{R}} f[S]$.

Since S is non-pseudocompact then $\mathscr{H}$ is non-empty.

Let function, $e_{\mathscr{H}} : S \to e_{\mathscr{H}}[S]$, be the evaluation map generated by $\mathscr{H}$ defined as

$$e_{\mathscr{H}}(x) \, = < f(x) >_{f \in \mathscr{H}} \in \; e_{\mathscr{H}}[S]$$

The function $e_{\mathscr{H}}$ extends to $e^{\beta(\alpha)}{}_{\mathscr{H}} : \beta S \to e^{\beta(\alpha)}{}_{\mathscr{H}}[\beta S] = \mathrm{cl}_{K_{\mathscr{H}}} e_{\mathscr{H}}[S]$ defined as

$$e^{\beta(\alpha)}{}_{\mathscr{H}}(x) \, = < f^{\beta(\omega)}(x) >_{f \in \mathscr{H}} \in \; \mathrm{cl}_{K_{\mathscr{H}}} e_{\mathscr{H}}[S]$$

Suppose S is realcompact.

We are required to show that $e_{\mathscr{F}} : S \to e_{\mathscr{F}}[S] \subseteq \prod_{f \in \mathscr{F}} \mathbb{R}$ is a perfect function.

The realcompact property implies that $\beta S \backslash S = \beta S \backslash \upsilon S$, so for every $x \in \beta S \backslash S$, there is some $f \in \mathscr{H}$ such that $f^{\beta(\omega)}(x) = \infty$.

Then

$$\cap \{ f^{\beta(\omega)\leftarrow}[\mathbb{R}] : f \in \mathscr{H} \} = S$$

Then, for every $x \in \beta S \backslash S$, $e^{\beta(\alpha)}{}_{\mathscr{H}}(x) = < f(x) >_{f \in \mathscr{H}}$ has an entry which is ∞.

Then

$$e^{\beta(\alpha)}{}_{\mathscr{H}}[\beta S\backslash S] \ \cap \ \prod_{f\in\mathscr{H}}\mathbb{R} = \varnothing$$

and so,

$$e^{\beta(\alpha)}{}_{\mathscr{F}}[\beta S\backslash S] \ \cap \ \prod_{f\in\mathscr{F}}\mathbb{R} = \varnothing \qquad (*)$$

Let $T = e_{\mathscr{F}}[S]$. Since $T \subseteq \prod_{f\in\mathscr{F}}\mathbb{R} = \varnothing$ then (*) implies

$$e^{\beta(\alpha)}{}_{\mathscr{F}}[\beta S\backslash S] \cap T = \varnothing \qquad (**)$$

For $K = \prod_{f\in\mathscr{F}}\mathrm{cl}_{\omega\mathbb{R}}f[S]$.

$$\begin{aligned} e^{\beta(\alpha)}{}_{\mathscr{F}}[\beta S] &= e^{\beta(\alpha)}{}_{\mathscr{F}}[\beta S\backslash S] \ \cup \ e_{\mathscr{F}}[S] \\ &= e^{\beta(\alpha)}{}_{\mathscr{F}}[\beta S\backslash S] \ \cup \ T \\ &= \mathrm{cl}_K T \end{aligned}$$

implies

$$e^{\beta(\alpha)}{}_{\mathscr{F}}[\beta S\backslash S] = (\mathrm{cl}_K T)\backslash T$$

So, by 24.2, $e_{\mathscr{F}} : S \to \prod_{f\in\mathscr{F}}f[S]$ is perfect.

Theorem 24.5 Let S be both a completely regular non-pseudocompact space and $\mathscr{F} = C(S)$. If $e_{\mathscr{F}} : S \to \prod_{f\in\mathscr{F}}f[S]$ is a perfect function then S is realcompact.

Proof: Let $R = \prod_{f\in\mathscr{F}}\mathbb{R}$, $K = \prod_{f\in\mathscr{F}}\omega\mathbb{R}$ and $T = e_{\mathscr{F}}[S]$. We are given that $e_{\mathscr{F}} : S \to T$ is perfect. We are required to show that S is realcompact.

To attain this objective it suffices to show that, if $y \in \beta S \setminus S$, then $e^{\beta(\alpha)}{}_{\mathscr{F}}(y)$ has at least one entry which is ∞.

See that R is dense in K and that $\mathrm{cl}_K T \cap R = \mathrm{cl}_R T$. Perfect functions are closed so $e_{\mathscr{F}}$ is a closed function; then T is closed in R. This means that $\mathrm{cl}_R T = T$. So $\mathrm{cl}_K T \cap R = T$. Then

$$\mathrm{cl}_K T\backslash T \ \cap \ R = \varnothing \qquad (\dagger)$$

Since $e_{\mathscr{F}}$ is perfect, by theorem 24.2,

$$e^{\beta(\alpha)}{}_{\mathscr{F}}[\beta S\backslash S] \subseteq \mathrm{cl}_K T\backslash T$$

So, by (†),

$$e^{\beta(\alpha)}{}_{\mathscr{F}}[\beta S\backslash S] \subseteq \left(\prod_{f\in\mathscr{F}}\omega\mathbb{R}\right)\backslash\left(\prod_{f\in\mathscr{F}}\mathbb{R}\right)$$

If $y \in \beta S\backslash S$ then $e^{\beta(\alpha)}{}_{\mathscr{F}}(y) \in K\backslash R$. So $e^{\beta(\alpha)}{}_{\mathscr{F}}(y)$ has at least one entry which is ∞. So y is not a real point of S. So the only real points of βS belong to S. So S is realcompact, as required.

Corollary 24.6 Let S be both a completely regular non-pseudocompact space and $\mathscr{F} = C(S)$. The space S is realcompact if and only if the evaluation map, $e_{\mathscr{F}} : S \rightarrow \prod_{f\in\mathscr{F}} f[S]$, is a perfect function which homeomorphically maps S onto a closed subset, $e_{\mathscr{F}}[S]$, in $\prod_{f\in\mathscr{F}}\mathbb{R}$.

Proof: This simply summarizes the two previous theorems in a single statement.

From the previous results we restate the general statement ...

> *"Realcompact spaces are precisely those spaces, S, such that the evaluation map, $e_{C(S)} : S \rightarrow \prod_{f\in C(S)}\mathbb{R}$, generated by C(S) is perfect.*

We will use the above the techniques illustrated in the above characterization of realcompactness to prove the following statement about arbitrary products of $\mathbb{R}$'s.

Theorem 24.7 Any closed subspace, S, of a product of $\mathbb{R}$'s is realcompact.

Proof: Let $R = \prod_{j\in J}\mathbb{R}$, $K = \prod_{j\in J}\omega\mathbb{R}$ and $\mathscr{F} = C(S)$.

We are required to show that every closed subset of R is realcompact. Since closed subsets of realcompact spaces are realcompact (by theorem 23.11) it will suffice to show that R is realcompact.

Let $\pi_i : R \rightarrow \mathbb{R}$ be a function defined as $\pi_i(< x_j >_{j\in J}) = x_i \in \mathbb{R}$. Essentially π_i is the real-valued continuous i^{th} projection map on R. So we can define the family, $\mathscr{P}$, as,

$$\mathscr{P} = \{\pi_j : j \in J\} \;\subseteq\; C(R)$$

The family $\mathscr{P}$ generates the evaluation map $e_{\mathscr{P}} : R \rightarrow R$ defined as,

$$\begin{aligned} e_{\mathscr{P}}(< x_j >_{j\in J}) &= \;< \pi_j(< x_j >_{j\in J}) >_{j\in J} \\ &= \;< x_j >_{j\in J} \end{aligned}$$

Then $e_{\mathscr{P}}$ turns out to be none other than the identity function mapping R onto R.

We will simplify the notation by representing $e_{\mathscr{P}}$ by $i : R \to R$.

We will rewrite the function $i : R \to R$ as, $i : R \to K$, so that it is now seen as an inclusion map which embeds R into K. Since i pulls back points to points and maps closed sets in R to closed sets in $i[R] = R$, then $i : R \to K$ is a perfect map.

Then i extends to $i^{\beta(K)} : \beta R \to K$. See that

$$\mathrm{cl}_K i[R] = i^{\beta(K)}[\beta R] = i^{\beta(K)}[\beta R \backslash R] \ \cup \ i[R]$$

See that

$$\begin{array}{rcll} i^{\beta(K)}[\beta R \backslash R] & \subseteq & \mathrm{cl}_K i[R] \backslash i[R] & \text{(Since } i : r \to R \text{ is a perfect map)} \\ & \subseteq & K \backslash R & \end{array}$$

If $y \in \beta R \backslash R$ then $i^{\beta(K)}(y) \in K \backslash R$. So

$$i^{\beta(K)}(y) = e_{\mathscr{P}}{}^{\beta(K)}(y)$$

has at least one entry which is ∞. Since $\mathscr{P} \subseteq \mathscr{F} = C(R)$,

$$e_{\mathscr{F}}{}^{\beta(K)}(y)$$

has at least one entry which is ∞.

So y is not a real point of R. So only real points of βR belong to R. So R is realcompact.

Since R is realcompact all of its closed subsets are realcompact, as required.

24.5 On perfect maps and products.

We will briefly discuss a particular relationship between a perfect map and a product space. Suppose

$$S = \prod_{i \in J} S_i$$

is a product space of completely regular spaces, S_i. Then, by theorem 10.6, S is completely regular. As briefly discussed on page 335, we can speak of the compactification, βS, of S. Suppose αS_i is a compactification of each S_i, respectively. If we compactify S with $\beta S = \mathrm{cl}_{\beta S} S$, then for each $i \in J$, $\pi_i[\mathrm{cl}_{\beta S} S] = \mathrm{cl}_{\alpha S_i} S_i$. (Note that $\mathrm{cl}_{\alpha S_i} S_i$ need not necessarily be βS_i. So we can only write $\beta S = \prod_{i\in J} \mathrm{cl}_{\alpha S_i} S_i$, not $\beta S = \prod_{i\in J} \beta S_i$.[1])

Theorem 24.8 Let $S = \prod_{i\in J} S_i$ be a product space of completely regular spaces, S_i. Then the projection map, $\pi_j : S \to S_j$ is perfect if and only if the product space

$$T = \prod_{i\in J\backslash\{j\}} S_i$$

is compact.

Proof: Suppose $S = \prod_{i\in J} S_i$ is non-compact and $j \in J$. Let $T = \prod_{i\in J\backslash\{j\}} S_i$.

To view T as a proper subset of S we redefine T as follows:

$$T = \prod_{i\in J} V_i \to \left\{ \begin{array}{ll} V_i = S_i, & \text{for } i \neq j \\ V_j = \{a_j\} & \text{for some } a_j \in S_j \end{array} \right.$$

Then the redefined T is homeomorphic to the "old" T.

Let

$$S_j^* = \prod_{i\in J} U_i \to \left\{ \begin{array}{l} U_j = S_j \\ U_i = \{a_i\} \text{ for some } a_i \in S_i \text{ if } i \neq j \end{array} \right.$$

Then $S = T \cup S_j^*$ (See product property described on page 103).

Suppose αS_j is a compactification of S_j.

$$\mathrm{cl}_{\beta S_j^*} S_j^* = \prod_{i\in J} W_i \to \left\{ \begin{array}{l} W_j = \alpha S_j = \pi_j[\mathrm{cl}_{\beta S} S] \neq S_j \\ W_i = \{a_i\} \text{ for some } a_i \in S_i \text{ if } i \neq j \end{array} \right.$$

$$\begin{array}{rcl} \mathrm{cl}_{\beta S} S \backslash S & = & \mathrm{cl}_{\beta S}(T \cup S_j^*) \backslash (T \cup S_j^*) \\ & = & (\mathrm{cl}_{\beta S} T \cup \mathrm{cl}_{\beta S}(S_j^*)) \backslash (T \cup S_j^*) \\ & = & (\mathrm{cl}_{\beta S} T \cup \mathrm{cl}_{\beta S}(S_j^*)) \cap \mathrm{cl}_{\beta S} S \backslash (T \cup S_j^*) \\ & = & [\mathrm{cl}_{\beta S} T \cap \mathrm{cl}_{\beta S} S \backslash (T \cup S_j^*)] \;\cup\; (\mathrm{cl}_{\beta S}(S_j^*) \cap \mathrm{cl}_{\beta S} S \backslash (T \cup S_j^*)) \quad (*) \\ & = & A \quad \cup \quad B \end{array}$$

See that

$$T \text{ is compact } \Leftrightarrow A = \varnothing$$

$$T \text{ is compact } \Leftrightarrow \pi_j[\mathrm{cl}_{\beta S} S \backslash S] = \pi_j[A \cup B] = \pi_j[B]$$

---

[1]The equality $\mathrm{cl}_{\beta S} S = \prod_{i\in J} \mathrm{cl}_{\beta S_i} S_i$ holds true only if S is pseudocompact.

We claim that

$$\pi_j[\mathrm{cl}_{\beta S}S\backslash S] = \pi_j[B] = \pi_j[\mathrm{cl}_{\beta S}(S_j^*) \cap \mathrm{cl}_{\beta S}S\backslash(T \cup S_j^*)] \qquad (\dagger)$$

if and only if π_j is perfect.

To prove the claim it suffices to show that $\pi_j[B] \subseteq \mathrm{cl}_{\alpha S_j}(S_j)\backslash S_j$.
Suppose $y = \pi_j(< y_i >_{i\in J}) \in \pi_j[\mathrm{cl}_{\beta S}(S_j^*) \cap (\mathrm{cl}_{\beta S}S\backslash(T \cup S_j^*))]$.

See that $< y_i >_{i\in J} \in \mathrm{cl}_{\beta S}(S_j^*)\backslash S_j^* \cap \mathrm{cl}_{\beta S}S\backslash(T \cup S_j^*)$ implies $y_j \in \mathrm{cl}_{\alpha S_j}S_j\backslash S_j$, otherwise $(y_i)_{i\in J}$ is either in T or S_j^* .
So $y = \pi_j(< y_i >_{i\in J}) \in \mathrm{cl}_{\alpha S_j}(S_j)\backslash S_j$. Then $\pi_j[B] \subseteq \mathrm{cl}_{\alpha S_j}(S_j)\backslash S_j$.
So π_j is perfect if and only if the equation ($\dagger$) holds true.

So T is compact if and only if π_j is perfect.

Concepts review:

1. Define a compact function.
2. Define a perfect function.
3. Provide two characterizations of a perfect function.
4. Provide a characterization of the realcompact property involving a perfect function.
5. Provide a characterization of a perfect function $f : S \to K$ in terms of a singular set.
6. If $e_{C(S)} : S \to \prod_{f\in C(S)} f[S]$ is an evaluation map what can we say about S if $e_{C(S)}$ is perfect?
7. If $S = \prod_{i\in J} S_i$ what property must S satisfy if we want the projection map to be perfect?

---

25 / Perfect and Freudenthal compactifications

Summary. *In this chapter we define the Freudenthal compactification and the perfect compactification. In spite of its name, a perfect compactification has very little to do with a perfect function. A perfect function is one whose fibres are compact, while a perfect compactification, αS, is one for which the fibres of $\pi_{\beta\to\alpha}$ are connected. We produce an algebraic characterization of those compactifications we call "perfect". We also present examples of both a Freudenthal compactification and a perfect compactification. This chapter can be studied immediately after chapter introducing compactifications without loss of continuity.*

25.1 The Freudenthal compactification.

We introduce another type of compactification called the *Freudenthal compactification* for locally compact Hausdorff spaces.[1] First we recall that,

> *. . . a space is said to be zero-dimensional if every point has an open neighbourhood base of clopen sets.*

While,

> *. . . a space is totally disconnected if every connected component is a singleton set.*

By theorem 20.22 a locally compact Hausdorff space is zero-dimensional if and only if it is totally disconnected.

Definition 25.1 If S is a locally compact Hausdorff compactification of S the *Freudenthal compactification*, ϕS, is the maximal compactification whose outgrowth, $\phi S \setminus S$, is totally disconnected − equivalently, is zero-dimensional. That is, if $\mathscr{X} = \{\alpha_i S : i \in I\}$ is the family of all compactifications of S with zero-dimensional remainder, $\phi S \in \mathscr{X}$ and $\alpha_i S \preceq \phi S$, for all i.

---

[1] Hans Freudenthal (1905-1990) was a Jewish-German-born Dutch mathematician at the University of Amsterdam. He was suspended from his duties by the Nazis during the war. After the war he was reinstated to his former position.

There is another way to visualize what the Freudenthal compactification, ϕS, is about. In the following theorem we show that ϕS is the unique compactification of S obtained by collapsing the connected components of $\beta S \backslash S$ to points. This theorem guarantees that the maximal compactification with zero-dimensional outgrowth exists in the partially ordered set of all compactifications. Furthermore, there is only one such compactification (up to equivalence).

Theorem 25.2 Suppose ϕS denotes the Freudenthal compactification of a space, S. Then ϕS can be obtained when $\pi_{\beta\to\phi} : \beta S \backslash S \to \phi S \backslash S$ collapses the connected components of $\beta S \backslash S$ to points of $\phi S \backslash S$.

Proof: Let S be a locally compact Hausdorff space. Suppose the outgrowth, $\alpha S\backslash S$, of αS is zero-dimensional and $p \in \alpha S \backslash S$.

We claim that $\pi_{\beta\to\alpha}{}^{\leftarrow}(p)$ is a union of components in $\beta S \backslash S$.

Let C be a connected component in $\beta S \backslash S$ which intersects $\pi_{\beta\to\alpha}{}^{\leftarrow}(p)$. It suffices to show that $C \subseteq \pi_{\beta\to\alpha}{}^{\leftarrow}(p)$. Suppose $t \in C \backslash \pi_{\beta\to\alpha}{}^{\leftarrow}(p)$. Then there exists $q \neq p$ such that $t \in \pi_{\beta\to\alpha}{}^{\leftarrow}(q)$. Since $\alpha S\backslash S$ is zero-dimensional, there is some clopen neighbourhood U of q in $\alpha S\backslash S$ which does not contain p.

Then $t \in \pi_{\beta\to\alpha}{}^{\leftarrow}(q) \subseteq \pi_{\beta\to\alpha}{}^{\leftarrow}[U]$ where $\pi_{\beta\to\alpha}{}^{\leftarrow}[U] \cap \pi_{\beta\to\alpha}{}^{\leftarrow}(p) = \varnothing$. Since $\pi_{\beta\to\alpha}$ is continuous $\pi_{\beta\to\alpha}{}^{\leftarrow}[U]$ is a clopen set in $\beta S \backslash S$ which only intersects part or C. This contradicts the fact that C is a connected component. So $C\backslash\pi_{\beta\to\alpha}{}^{\leftarrow}(p)$ must be empty. This means that $\pi_{\beta\to\alpha}{}^{\leftarrow}(p)$ is a union of components, as claimed.

In the example on page 302 we explained that the function which collapses the connected components of a compact Hausdorff set to points produces a totally disconnected set (equivalently a zero-dimensional space). Then the function $\pi_{\beta\to\phi}$ which collapses the components of $\beta S\backslash S$ to points and fixes the points of S produces a compactification with zero-dimensional outgrowth. Since $\pi_{\beta\to\phi}{}^{\leftarrow}(p)$ contains only one component then ϕS is indeed the maximal compactification with zero-dimensional outgrowth.

25.2 Perfect compactification definition.

In the proof of the theorem above we see that, if αS has zero-dimensional outgrowth then

"for all $p \in \alpha S \backslash S$, $\pi_{\beta\to\alpha}{}^{\leftarrow}(p)$ is the union of connected components of $\beta S \backslash S$"

We also see that there is only one maximal compactification with zero-dimensional outgrowth. It is the one where each fiber of $\pi_{\beta\to\alpha}|_{\beta S \setminus S}$ is a single connected component of $\beta S\setminus S$. It is called the Freudenthal compactification.

Those compactifications, αS, with zero-dimensional outgrowth must then be less than or equal to the Freudenthal compactification.

There may be compactifications, αS, whose outgrowth, $\alpha S\setminus\alpha$, is not zero-dimensional which satisfy the property,

$$\text{“for all } p \in \alpha S\setminus S,\ \pi_{\beta\to\alpha}{}^{\leftarrow}(p) \text{ is connected”}$$

For such a compactification αS, for each $p \in \alpha S\setminus S$, $\pi_{\beta\to\alpha}{}^{\leftarrow}(p)$ would be a connected subset of at most one component.

Such compactifications are referred to as “perfect” compactifications. We formally define this.

Definition 25.3 Let γS be a Hausdorff compactification of S and $\pi_{\beta\to\gamma} : \beta S \to \gamma S$ be the continuous map which projects βS onto γS. The compactification, γS, is said to be a *perfect compactification* provided $\pi_{\beta\to\gamma}{}^{\leftarrow}(p)$ is connected for every $p \in \gamma S\setminus S$.

The smallest perfect compactification is obviously the Freudenthal compactification, ϕS. So perfect compactifications form a family of compactifications above, ϕS, while the ones with zero-dimensional outgrowth form a family below ϕS.

The main result in this chapter is an algebraic characterization of a perfect compactification.

25.3 Algebraic subrings and maximal stationary sets.

Our algebraic characterization will involve properties of a particular type of subring of $C^*(S)$, called an “algebraic subring”. Setting up the stage for the proof of the characterization first requires a proper understanding of what an algebraic subring is and examining some of its properties. Characterizing the topological property of “perfect compactification”, αS, with an algebraic property of a subring of $C_\alpha(S)$, requires a bit of mathematical prowess. We must first take care a few technical arguments in the form of three lemmas following a definition.

Definition 25.4 Let S be a locally compact Hausdorff space and $C^*(S)$ be the ring of all bounded continuous real-valued functions on S.

a) We will say that $\mathscr{A}$ is an *algebraic subring* of $C^*(S)$ if $\mathscr{A}$ is a subring which contains all of the constant functions and all functions, f, such that $f^2 = f \cdot f \in \mathscr{A}$.

b) Given a subring, $\mathscr{A}$, of $C^*(S)$, a subset T of S on which every function in $\mathscr{A}$ is constant is called a *stationary set of the subring, $\mathscr{A}$*. If for any $p \in S \backslash T$ there is a function $f \in \mathscr{A}$ which is not constant on $T \cup \{p\}$ then we say that T is a *maximal stationary set of $\mathscr{A}$*. Equivalently, if T is a stationary set of $\mathscr{A}$ containing y and $T = \{x \in S : f(x) = f(y), \text{ for all } f \in \mathscr{A}\}$ then T is a *maximal stationary set of $\mathscr{A}$*.

By definition, if T is a maximal stationary set of the subring, $\mathscr{A}$, of $C(S)$ then every function in $\mathscr{A}$ is constant on T. Then, for $f \in \mathscr{A}$, $f[T]$ is some singleton set, say $\{t_f\}$, and so $T \subseteq Z(f - t_f)$. Then $T \subseteq \cap\{Z(f - t_f) : f \in \mathscr{A}\}$. Since T is maximal with respect to this property, then

$$T = \cap\{Z(f - t_f) : f \in \mathscr{A}, \}$$

a closed subset of S.

The following lemma offers yet another way of describing what the maximal stationary set of subring, $\mathscr{F}$, is.

Lemma 25.5 Let S be a compact space. Let $\mathscr{G}$ be a subring of $C(S)$ and $p \in e_{\mathscr{G}}[S]$. Then $e_{\mathscr{G}}{}^{\leftarrow}(p)$ is a maximal stationary set of $\mathscr{G}$.

Proof: Let $\mathscr{G}$ is a subring of $C(S)$ and $p \in e_{\mathscr{G}}[S]$. Then every function in $\mathscr{G}$ is constant on $e_{\mathscr{G}}{}^{\leftarrow}(p)$. Let T be the maximal stationary set which contains $e_{\mathscr{G}}{}^{\leftarrow}(p)$. Let $x \in e_{\mathscr{G}}{}^{\leftarrow}(p)$.

Suppose $t \in S \backslash e_{\mathscr{G}}{}^{\leftarrow}(p)$. Then there exists $h \in \mathscr{G}$ such that $\{h(t)\} \neq \{h(x)\} = h[T]$. Then $t \notin T$. So $e_{\mathscr{G}}{}^{\leftarrow}(p) = T$. Then $e_{\mathscr{G}}{}^{\leftarrow}(p)$ is a maximal stationary set of $\mathscr{G}$.

Lemma 25.6 Let T be a connected subset of a topological space, S, and $\mathscr{G}$ be the subset of $C^*(S)$ which contains only those functions which are constant on T. Then $\mathscr{G}$ is an algebraic subring which is closed with respect to the uniform norm topology.

Proof: We are given that $\mathscr{G}$ contains only those functions which are constant on the connected subspace T.

Clearly, $\mathscr{G}$ is closed under multiplication and addition and so $\mathscr{G}$ is a subring of $C^*(S)$. We now show that $\mathscr{G}$ is algebraic. Suppose $g^2[T] = k$. For $x \in T$, $g(x)$ is k or $-k$.

But S is connected and so the range of g cannot contain two elements. So g must be constant on T and so belongs to $\mathscr{G}$. We deduce that $\mathscr{G}$ is algebraic, as required.

Now for closure. Suppose h is a limit point of $\mathscr{G}$. Then there is a sequence $\{t_i : i \in \mathbb{N}\}$ in $\mathscr{G}$ which converges to h with respect to the uniform norm topology. Since each t_i is constant on T, h is constant on T and so belongs to $\mathscr{G}$. So $\mathscr{G}$ contains it's limit points and so is closed with respect to the uniform norm.

In the next lemma we establish a relationship between a topological property of a maximal stationary set of a subring, $\mathscr{A}$, and an algebraic property of this subring.

Lemma 25.7 If S is a compact Hausdorff space and $\mathscr{A}$ is an algebraic subring of $C^*(S)$ then any maximal stationary set, T, of $\mathscr{A}$ is connected.

Proof: We are given that S is compact and Hausdorff and T is a maximal stationary set of the algebraic subring, $\mathscr{A}$, of $C^*(S)$. For $f \in \mathscr{A}$ let $\{t_f\} = f[T]$. Then

$$T = \cap\{Z(f - t_f) : f \in \mathscr{A}\}$$

If $t = < t_f >_{f\in\mathscr{A}}$ and $e_{\mathscr{A}} : S \to \prod_{f\in\mathscr{A}} \mathbb{R}_f$, is the evaluation map induced by $\mathscr{A}$ then

$$T = e_{\mathscr{A}}{}^{\leftarrow}(t) = \cap\{Z(f - t_f) : f \in \mathscr{A}\}$$

We claim that T is connected.

Suppose T is not connected. That is, suppose $T = F_1 \cup F_2$ where F_1 and F_2 are disjoint closed subsets of T. Since S is compact, they are compact, so there exist disjoint open subsets, U_1 and U_2, of S, respectively containing F_1 and F_2. So $T \subseteq U_1 \cup U_2$. Now $S \setminus (U_1 \cup U_2)$ is closed and hence compact (since S is compact).

Consider a basic open neighbourhood of $t = < t_f >_{f\in\mathscr{A}}$, $U = \prod_{f\in\mathscr{A}} U_f$, where U_f is $\mathbb{R}$ for all but finitely many factors $\{U_{f_i} : i \in F\}$ which would be of the form $U_{f_i} = (t_{f_i} - \varepsilon_i,\ t_{f_i} + \varepsilon_i)$. If $f^{\leftarrow}[t_f,\ t_f + \varepsilon] \cap S \setminus (U_1 \cup U_2) \neq \varnothing$ for all f and ε then, by the FIP compactness property of $S \setminus (U_1 \cup U_2)$, $F_1 \cup F_2 = T = e_{\mathscr{A}}{}^{\leftarrow}[U] \cap S \setminus (U_1 \cup U_2)$ cannot be empty, a contradiction. So there exists at least one function, $h : S \to \mathbb{R}$, in $\mathscr{A}$ and one ε such that

$$e_{\mathscr{A}}{}^{\leftarrow}(t) = F_1 \cup F_2 \subseteq h^{\leftarrow}[t_h,\ t_h + \varepsilon] \subseteq U_1 \cup U_2$$

Let

$$\begin{aligned} W_1 &= h^{\leftarrow}[t_h,\ t_h + \varepsilon] \cap U_1 \quad \text{(Which contains } F_1.) \\ W_2 &= h^{\leftarrow}[t_h,\ t_h + \varepsilon] \cap U_2 \quad \text{(Which contains } F_2.) \end{aligned}$$

Since U_1 and U_2 are disjoint then W_1 and W_2 are disjoint and

$$W_1 \cup W_2 = h^{\leftarrow}[t_h,\ t_h + \varepsilon]$$

So there exists $k \in W_1 \cup W_2$ such that $h(k) = t_h + \varepsilon$.

We define a real-valued function $r : S \to \mathbb{R}$ on S, as follows

$$\begin{array}{rcll} r(x) & = & (t_h + \varepsilon) - h(x) & \text{if } x \in W_1 \\ r(x) & = & h(x) - (t_h + \varepsilon) & \text{if } x \in S \backslash W_1 \end{array}$$

Then $r(x) \geq 0$ on W_1 and $r(x) \leq 0$ on $S \backslash W_1$.

Since $h(k) = t_h + \varepsilon$, $r(k) = 0$.

We claim that r is continuous on S: Clearly, f is continuous both on W_1 and on $S \backslash W_1$. But $r(k) = 0$ and $r(x) \geq 0$ on W_1 and $r(x) \leq 0$ on $S \backslash W_1$, and so k is a boundary point of the piecewise defined function r at which $r(k) = 0$. So $r : S \to \mathbb{R}$ is a continuous function, as claimed.

We now claim that $r \in \mathscr{A}$: Note that $r^2 = ((t_h + \varepsilon) - h(x))^2$. Since $\mathscr{A}$ is a subring then $r^2 \in \mathscr{A}$. Since $\mathscr{A}$ is algebraic and $r^2 \in \mathscr{A}$, then $r \in \mathscr{A}$. As claimed.

Now we have a contradiction, since $r \in \mathscr{A}$ is not constant on T and every function of $\mathscr{A}$ was hypothesized to be constant on T. So T cannot be partitioned by two non-empty closed sets F_1 and F_2. So T is connected. We are done.

25.4 Perfect compactification algebraic characterization.

Having established that (for a compact space S), if the maximal stationary set of an algebraic subring is connected, we are ready to roll out the proof of the following characterization of perfect compactifications.

Theorem 25.8 Let γS be a compactification of S. Then γS is a perfect compactification if and only if $C_\gamma(S)$ is an algebraic subring of $C^*(S)$.

Proof: ($\Leftarrow$) We are given that $\mathscr{G} = C_\gamma(S)$ where $C_\gamma(S)$ is an algebraic subring of $C^*(S)$. We are required to show that, for any $p \in \gamma S$, $\pi_{\beta \to \gamma}{}^{\leftarrow}(p)$ is connected. If $C_\gamma(S)$ is algebraic, it is easily verified, that $C_\gamma(S)^\beta$ must also be an algebraic subring of $C(\beta S)$. By lemma 21.12, $\pi_{\beta\to\gamma} = e_{\mathscr{G}}{}^{\gamma\leftarrow} \circ e_{\mathscr{G}}{}^{\beta}$. So, for $p \in \gamma S \backslash S$,

$$\begin{aligned} \pi_{\beta\to\gamma}{}^{\leftarrow}(p) &= [e_{\mathscr{G}}{}^{\gamma\leftarrow} \circ e_{\mathscr{G}}{}^{\beta}]^{\leftarrow}(p) \\ &= e_{\mathscr{G}}{}^{\beta\leftarrow}(e_{\mathscr{G}}{}^{\gamma}(p)) \\ &= \text{a maximal stationary set of } \mathscr{G}^\beta \text{ in } \beta S \end{aligned}$$

Since $\mathscr{G}^\beta$ is an algebraic subring of $C(\beta S)$, then by lemma 25.7, the maximal stationary set, $\pi_{\beta\to\gamma}{}^{\leftarrow}(p)$, of $\mathscr{G}^\beta$ is connected. So, for any $p \in \gamma S$, $\pi_{\beta\to\gamma}{}^{\leftarrow}(p)$ is connected. By definition, γS is a perfect compactification. As required.

($\Rightarrow$) Suppose γS is a perfect compactification. We are required to show that $C_\gamma(S)$ is an algebraic subring of $C^*(S)$. By definition, $\pi_{\beta\to\gamma}{}^{\leftarrow}(p)$ is connected for every $p \in \gamma S \backslash S$.

Let $\mathscr{G} = C_\gamma(S)$. To show that $\mathscr{G}$ is an algebraic subring of $C^*(S)$ it suffices to show that $\mathscr{G}^\beta$ is an algebraic subring of $C(\beta S)$.

For $p \in \gamma S$, let $\mathscr{F}_p$ denote the set of all functions in $C(\beta S)$ which are constant on the connected set, $\pi_{\beta\to\gamma}{}^{\leftarrow}(p)$. By lemma 25.6, $\mathscr{F}_p$ is an algebraic subring of $C(\beta S)$.

Since $\pi_{\beta\to\gamma} = e_{\mathscr{G}}{}^{\gamma\leftarrow} \circ e_{\mathscr{G}}{}^{\beta}$ (by lemma 21.12), then $\pi_{\beta\to\gamma}{}^{\leftarrow}(p) = e_{\mathscr{G}}^{\beta\leftarrow}(e_{\mathscr{G}}{}^{\gamma}(p))$. So

$$e_{\mathscr{G}}^{\beta}[\pi_{\beta\to\gamma}{}^{\leftarrow}(p)] = e_{\mathscr{G}}{}^{\gamma}(p)$$

It follows that $\mathscr{G}^\beta \subseteq \mathscr{F}_p$. It is easily verified that the intersection of a family of algebraic subrings is an algebraic subring. So $\cap\{\mathscr{F}_p : p \in \gamma S\}$ is an algebraic subring of $C(\beta S)$. So

$$\mathscr{G}^\beta \subseteq \cap\{\mathscr{F}_p : p \in \gamma S\}$$

If we can show that $\cap\{\mathscr{F}_p : p \in \gamma S\} \subseteq \mathscr{G}^\beta$, then we are done.

Let $f \in \cap\{\mathscr{F}_p : p \in \gamma S\}$. If $g \in C(\gamma S)$ is such that $g(\pi_{\beta\to\gamma}(x)) = f(x)$ for all $x \in \beta S$, then $g|_S = f|_S$ then $f|_S \in C_\gamma(S) = \mathscr{G}$. Hence $\cap\{\mathscr{F}_p : p \in \gamma S\} \subseteq \mathscr{G}^\beta$.

We conclude that $\mathscr{G}^\beta = C_\gamma(S)^\beta = \cap\{\mathscr{F}_p : p \in \gamma S\}$, an algebraic subring of $C(\beta S)$.

So $C_\gamma(S)$ is an algebraic subring of $C^*(S)$. We are done with the direction $\Rightarrow$.

25.5 Example.

Recall that the Freudenthal compactification is the maximal compactification whose outgrowth is zero-dimensional (equivalently, totally disconnected). The Freudenthal compactification can also be seen as the minimal "perfect compactification" of a space.

In the following proposition we provide an example of a space whose Freudenthal compactification is βS. In such a case S has only one perfect compactification, βS.

The proposition involves the space, $\mathbb{N}$, in which any subset A is clopen and so is a zero-set. Then there is a continuous function, $f : \mathbb{N} \to \{0, 1\}$, such that $A = Z(f-1)$. Then f extends continuously to $f^\beta : \mathrm{cl}_{\beta\mathbb{N}} A \to \{0, 1\}$ where

$$f^\beta[\mathrm{cl}_{\beta\mathbb{N}} A] = \mathrm{cl}_{\mathbb{R}} f[A] = \{1\}$$

Then

$$\mathrm{cl}_{\beta\mathbb{N}} A = Z(f^\beta - 1) = \beta\mathbb{N} \backslash Z(f^\beta - 0)$$

is clopen in $\beta\mathbb{N}$. Thus ultrafilters of sets and z-ultrafilters in $Z[\mathbb{N}]$ represent the same families of sets. Every point p in $\beta\mathbb{N}\backslash\mathbb{N}$ is associated to a free ultrafilter of subsets of $\mathbb{N}$.

Proposition 25.9 The Stone-Čech compactification, $\beta\mathbb{N}$, of $\mathbb{N}$ is both a Freudenthal compactification and a perfect compactification.

Proof: To show that $\beta\mathbb{N}$ is the Freudenthal compactification of $\mathbb{N}$, it suffices to show that $\beta\mathbb{N} \backslash \mathbb{N}$ is zero-dimensional, equivalently, totally disconnected. We have shown that subspaces of totally disconnected spaces are themselves totally disconnected. So it will suffice to show that $\beta\mathbb{N}$ is totally disconnected. Suppose p and q are points in a connected component C of $\beta\mathbb{N}$. If $p \neq q$ then they are limits of two distinct free z-ultrafilters $\mathscr{Z}_p$ and $\mathscr{Z}_q$ in $Z[\mathbb{N}]$. Then $\mathscr{Z}_p$ contains a zero-set $Z(f-1)$ which does not belong to $\mathscr{Z}_q$. Then p belongs to the clopen zero-set $Z(f^\beta - 1)$ which does not contain q. Since C is connected then p must equal q. We conclude that $\beta\mathbb{N}$ is totally disconnected. It is the maximal compactification with zero-dimensional outgrowth and so is Freudenthal. It also is the smallest compactification which is perfect.

Since the Freudenthal compactification is the smallest of perfect compactifications, no compactification $\alpha\mathbb{N} \prec \beta\mathbb{N}$ is perfect. So, by the above characterization, no proper subring of $C^*(\mathbb{N})$ is algebraic.

Concepts review:

1. Define the Freudenthal compactification of a space.

2. How can βS be used to construct the Freudenthal compactification ϕS?

3. What does it mean to say that a compact space is a perfect compactification of S?
4. State what we mean when we that $\mathscr{A}$ is an algebraic subring of $C^*(S)$.
5. If $\mathscr{A}$ is an algebraic subring of $C^*(S)$ what is a stationary set of $\mathscr{A}$? What does it mean to say that a subset, T, is a maximal stationary set of $\mathscr{A}$?
6. State an algebraic characterization (involving $C^*(S)$) of a perfect compactification.

Appendix

Appendix A : Applications for Cartesian products.

Here are few interesting results on applications of Cartesian products. This is subject matter which can be appended at end of section 7 on product spaces. The topics covered are as follows:

- Section 7.5.1, Applications: The embedding theorem.
- Section 7.5.2, Applications: Investigating the Cantor set from a topological point of view.
- Section 7.5.3, Applications: A curve which contains every point in a cube.

7.5.1 Applications: The embedding theorem.

Firstly, we will show how a family, $\mathscr{F}$, of continuous functions on a topological space, (S, τ), can be used to embed S in a product space whose factors are the range of the functions in $\mathscr{F}$. This method is exhibited in a theorem titled "The embedding theorem". The *embedding theorem* applies only to topological spaces, S, in which singleton sets are closed. In order to understand its statement we must introduce two very important notions in topology. Firstly, we will define what we mean when we say that *a set of functions separates points and closed sets* of S. Secondly, will define an *evaluation map with respect to a family of functions.*

Definition 7.1 Let (S, τ_S) be a topological space and $\{(X_\alpha, \tau_\alpha) : \alpha \in \Gamma\}$ be an indexed family of non-empty topological spaces. Let $\mathscr{F} = \{f_\alpha : \alpha \in \Gamma\}$ be a set of *continuous functions*, $f_\alpha : S \to X_\alpha$, each one mapping S onto its range $f_\alpha[S] \subseteq X_\alpha$.

a) We say that $\mathscr{F}$ *separates points and closed sets* if, whenever F is a closed subset of S and $x \notin F$, then there exists at least one function $f_\beta \in \mathscr{F}$ such that $f_\beta(x) \notin \text{cl}_{X_\beta} f_\beta[F]$.

b) We define a function, $e : S \to \prod_{\alpha\in\Gamma} X_\alpha$, as follows:

$$e(x) = \{f_\alpha(x)\} \in \prod_{\alpha\in\Gamma} f_\alpha[S] \subseteq \prod_{\alpha\in\Gamma} X_\alpha$$

We refer to the function $e : S \to \prod_{\alpha\in\Gamma} X_\alpha$ as the *evaluation map of S into $\prod_{\alpha\in\Gamma} X_\alpha$ with respect to $\mathscr{F}$.*

Theorem 7.2. *The embedding theorem.* Let (S, τ_S) be a topological space in which every singleton set in S is a closed subset of S. Given an indexed family of non-empty topological spaces, $\{(X_\alpha, \tau_\alpha) : \alpha \in \Gamma\}$, let $\mathscr{F} = \{f_\alpha : \alpha \in \Gamma\}$ be a set of *continuous functions* where each f_α maps S onto its range, $f_\alpha[S]$, inside X_α.

a) If $\mathscr{F}$ *separates points and closed sets* of its domain, S, then the evaluation map,

$$e(x) = \{f_\alpha(x)\} \in \prod_{\alpha\in\Gamma} f_\alpha[S] \subseteq \prod_{\alpha\in\Gamma} X_\alpha$$

with respect to $\mathscr{F}$, is both continuous and one-to-one on S.

b) Furthermore, the function, $e : S \to \prod_{\alpha\in\Gamma} X_\alpha$, maps S homeomorphically onto $e[S] \subseteq \prod_{\alpha\in\Gamma} f_\alpha[S]$ in $\prod_{\alpha\in\Gamma} X_\alpha$. Hence this evaluation map embeds a homeomorphic copy of S into $\prod_{\alpha\in\Gamma} X_\alpha$.

Proof: We are given that (S, τ_S) is a topological space in which all singleton sets, $\{x\}$, are closed in S and a family of topological spaces, $\{X_\alpha : \alpha \in \Gamma\}$. For the set, $\mathscr{F} = \{f_\alpha : S \to X_\alpha\}_{\alpha\in\Gamma}$, of continuous functions on S, we define $e : S \to \prod_{\alpha\in\Gamma} f_\alpha[S] \subseteq \prod_{\alpha\in\Gamma} X_\alpha$ as an evaluation map with respect to $\mathscr{F}$.

a) Note that, for each $\alpha \in \Gamma$ and $x \in S$, $(\pi_{\alpha}\circ e)(x)) = \pi_\alpha(\{f_\alpha(x)\}) = f_\alpha(x)$. Since, for each $\alpha \in \Gamma$, f_α is continuous then so is $\pi_{\alpha}\circ e : S \to f_\alpha[S]$. By lemma 7.6, $e : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ is continuous.

We now show that $e : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ is one-to-one on S. Suppose a and b are distinct points in S. Then, since the single set $\{b\}$ is closed and $\mathscr{F}$ separates points and closed sets, there exists $\beta \in \Gamma$ such that $f_\beta(a) \notin \mathrm{cl}_{X_\beta} f_\beta[\{b\}]$. Then the β^{th} component of $e(a) = \{f_\alpha(a)\}$ and $e(b) = \{f_\alpha(b)\}$ are distinct and so $e(a) \neq e(b)$. We conclude that the evaluation map $e : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ is one-to-one on S.

b) To prove that the evaluation map $e : S \to \prod_{\alpha\in\Gamma} f_\alpha[S]$ embeds S into $\prod_{\alpha\in\Gamma} X_\alpha$, it will suffice to show that it is an open function and then invoke theorem 6.9. Let U be a non-empty open subset of S with the point $u \in U$. Then $F = S \setminus U$ is closed in S. Since $\mathscr{F}$ separates points and closed sets, there exists $\beta \in \Gamma$ such that $f_\beta(u) \notin \mathrm{cl}_{X_\beta} f_\beta[F]$. That means, $f_\beta(u) \in X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]$.

We now show that $e[U]$ is open in $Y = \prod_{\alpha\in\Gamma} X_\alpha$.
Note that

$$\begin{aligned}(\pi_{\beta}\circ e)(u) &= \pi_\beta(e(u)) \\ &= \pi_\beta(\{f_\alpha(u)\}) \\ &= f_\beta(u) \\ &\in X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\end{aligned}$$

Since $e(u) \in \pi_\beta^{\leftarrow}\left[X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right]$ and since π_β is continuous then $\pi_\beta^{\leftarrow}\left[X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right]$ is an open neighbourhood of $e(u)$ in Y. It now suffices to show that

$$\pi_\beta^{\leftarrow}\left[X_\beta \setminus \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right] \subseteq e[U]$$

Suppose $e(a) \in \pi_\beta^{\leftarrow}\left[X_\beta \backslash \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right]$.

$$\begin{aligned}
e(a) \in \pi_\beta^{\leftarrow}\left[X_\beta \backslash \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right] &\Rightarrow (\pi_\beta \circ e)(a) \in \pi_\beta\left[\pi_\beta^{\leftarrow}\left[X_\beta \backslash \mathrm{cl}_{X_\beta} f_\beta[F]\right]\right] \\
&\Rightarrow f_\beta(a) \in \left[X_\beta \backslash \mathrm{cl}_{X_\beta} f_\beta[F]\right] \\
&\Rightarrow f_\beta(a) \notin \mathrm{cl}_{X_\beta} f_\beta[F] \\
&\Rightarrow a \in S \backslash F = S \backslash (S \backslash U) = U \\
&\Rightarrow e(a) \in e[U]
\end{aligned}$$

So $\pi_\beta^{\leftarrow}\left[X_\beta \backslash \left[\mathrm{cl}_{X_\beta} f_\beta[F]\right]\right]$ is an open neighbourhood of $e(u)$ which is entirely contained in $e[U]$. We conclude $e[U]$ is open and so $e : S \to \prod_{\alpha\in\Gamma} S_\alpha$ is a homeomorphism.

7.5.2 Applications: Investigating the Cantor set from a topological point of view.

Recall that, in example 2 of page 119, the Cantor set was defined as being a proper subset, C, of $[0,1]$ which is the image of the one-to-one function, $\varphi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to [0,1]$ where

$$\varphi(\{m_n\}) = \sum_{n=1}^{\infty} \frac{m_n}{3^n}$$

The Cantor set was viewed simply as a set. The topology of sets involved was not discussed in our example and so we couldn't speak of the "continuity" of φ on the product $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. Now that we have decided on a topology on product spaces we can discuss the continuity of φ, or lack thereof, on its domain. We will define topologies of all sets involved in the most natural way. We will equip the set $\{0,2\}$ with the discrete topology, the set $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ with the product topology, and finally, the Cantor set C, with the subspace topology inherited from $\mathbb{R}$. We will now show that, with these topologies, φ maps $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ homeomorphically onto C.

Theorem 7.3. The one-to-one function, $\varphi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to C$, defined as,

$$\varphi(\{m_n\}) = \sum_{n=1}^{\infty} \frac{m_n}{3^n}$$

maps the product space $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ homeomorphically onto the Cantor set C.

Proof: Let $\varepsilon > 0$. First note that, for all $\{m_n\} \in \prod_{n\in\mathbb{Z}^+}\{0,2\}$, $|\varphi(\{m_n\})| \leq \sum_{n=1}^{\infty} \frac{2}{3^n} = 1$. There then exists N such that $\sum_{n=N+1}^{\infty} \frac{2}{3^n} < \varepsilon$.

Let $\{k_n\} \in \prod_{n\in\mathbb{Z}^+}\{0,2\}$. Then $\varphi(\{k_n\}) = \sum_{n=1}^{\infty} \frac{k_n}{3^n} = x \in C$.

Let $B_\varepsilon(x)$ be and open interval in $\mathbb{R}$ with center x and radius ε. Then $B_\varepsilon(x) \cap C$ is an open neighbourhood of x in C. To show continuity of φ at $\{k_n\}$, it will suffice to find an open neighbourhood, U, of $\{k_n\}$ such that $\varphi[U] \subseteq B_\varepsilon(x) \cap C$.

Since the series, $\sum_{n=1}^{\infty} \frac{k_n}{3^n}$, converges to x then the sequence $\left\{x - \sum_{n=1}^{m} \frac{k_n}{3^n}\right\} = \left\{\sum_{n=m+1}^{\infty} \frac{k_n}{3^n}\right\}$ converges to zero as m tends to infinity. Then,

$$\left|\sum_{n=N+1}^{\infty} \frac{k_n}{3^n}\right| \leq \sum_{n=N+1}^{\infty} \frac{2}{3^n} < \varepsilon$$

Let $U = \pi_1^{\leftarrow}(k_1) \cap \cdots \cap \pi_N^{\leftarrow}(k_N)$ be an open base neighbourhood of $\{k_n\}$ in $\prod_{n \in \mathbb{Z}^+} \{0, 2\}$ and suppose $\{b_n\}$ is some element in U. This implies $b_1 = k_1, b_2 = k_2, \ldots, b_N = k_N$. So

$$\begin{aligned}
|x - \varphi(\{b_n\})| &= \left|x - \sum_{n=1}^{\infty} \frac{b_n}{3^n}\right| \\
&= \left|\sum_{n=1}^{\infty} \frac{k_n}{3^n} - \sum_{n=1}^{\infty} \frac{b_n}{3^n}\right| \\
&\leq 0 + \sum_{n=N+1}^{\infty} \frac{|k_n - b_n|}{3^n} \\
&\leq \sum_{n=N+1}^{\infty} \frac{2}{3^n} \\
&< \varepsilon
\end{aligned}$$

Then $\varphi[U] \subseteq B_\varepsilon(x) \cap C$. Then φ is continuous at $\{k_n\}$, and so at all points of $\prod_{n \in \mathbb{Z}^+} \{0, 2\}$.

Now, to prove that φ is a homeomorphism it will suffice to show it is open.

For $y = \{k_n\} \in \prod_{n \in \mathbb{Z}^+} \{0, 2\}$, $\sum_{n=1}^{\infty} \frac{k_n}{3^n}$ converges to $\varphi(y)$.

Let $U = \pi_{j_1}^{\leftarrow}(k_{j_1}) \cap \cdots \cap \pi_{j_m}^{\leftarrow}(k_{j_m})$ be an arbitrary open neighbourhood base element of $\{k_n\}$.

To show that φ is open, it will suffice to find some ε such that $B_\varepsilon(\varphi(y)) \cap C \subseteq \varphi[U]$.

Let $N = \max\{j_1, j_2, \ldots, j_m\}$ and let $\varepsilon = \frac{1}{3^{N+1}}$. (*)

We claim that, if $|\varphi(\{k_n\}) - \varphi(\{z_n\})| < \varepsilon$, then $\varphi(\{z_n\}) \in \varphi[U]$.

$$\begin{aligned}|\varphi(\{k_n\}) - \varphi(\{z_n\})| &= \left|\sum_{n=1}^{\infty}\frac{k_n}{3^n} - \sum_{n=1}^{\infty}\frac{z_n}{3^n}\right| \\ &= \left|\sum_{n=1}^{\infty}\frac{k_n - z_n}{3^n}\right| \\ &= \left|\sum_{n=1}^{N}\frac{k_n - z_n}{3^n} + \sum_{n=N+1}^{\infty}\frac{k_n - z_n}{3^n}\right| < \varepsilon\end{aligned}$$

Then

$$-\varepsilon = \frac{-1}{3^{N+1}} < \sum_{n=1}^{N}\frac{k_n - z_n}{3^n} + \sum_{n=N+1}^{\infty}\frac{k_n - z_n}{3^n} < \frac{1}{3^{N+1}} = \varepsilon$$

$$\frac{-1}{3^{N+1}} - \sum_{n=N+1}^{\infty}\frac{k_n - z_n}{3^n} < \sum_{n=1}^{N}\frac{k_n - z_n}{3^n} < \frac{1}{3^{N+1}} - \sum_{n=N+1}^{\infty}\frac{k_n - z_n}{3^n}$$

After some computation we obtain,

$$\frac{-4}{3^{N+1}} < \sum_{n=1}^{N}\frac{k_n - z_n}{3^n} < \frac{4}{3^{N+1}} \qquad \text{(Verify this!)}$$

To show that $\varphi(\{z_n\}) \in \varphi[U]$, it suffices to show that, $y_n - z_n = 0$ for $n = 1$ to N. For $m = 1$ to N, let

$$S_m = \sum_{n=1}^{m}\frac{k_n - z_n}{3^n}$$

Suppose $k_1 - z_1 = 2$ or -2; then $S_1 = \pm 2/3 \notin (-\frac{4}{3^{N+1}}, \frac{4}{3^{N+1}})$. So $k_1 - z_1 = 0$. Suppose $k_m - z_m = 0$ for $m < N$ and $k_{m+1} - z_{m+1} = 2$ or -2. Then $S_{m+1} = \pm 2/3 \notin (-\frac{4}{3^{N+1}}, \frac{4}{3^{N+1}})$. So $k_{m+1} - z_{m+1} = 0$.

We then have $y_n - z_n = 0$ for $n = 1$ to N.

This is precisely what was needed to show that $\varphi(\{z_n\}) \in \varphi[U]$. This means that, for our choice of $\varepsilon = \frac{1}{3^{N+1}}$ (at (*)), $B_\varepsilon(\varphi(\{k_n\}) \cap C \subseteq \varphi[U]$. So φ is an open map on $\prod_{n\in\mathbb{Z}^+}\{0, 2\}$.

Parts one and two together allow us to conclude that $\varphi : \prod_{n\in\mathbb{Z}^+}\{0, 2\} \to C$ is a homeomorphism, as required

We see that investigating the Cantor set from a topological point of view provides us with a different perspective on the product space $\prod_{n\in\mathbb{Z}^+}\{0,2\}$. Since we have shown it is a homeomorphic copy of C then, topologically speaking, the product space, $\prod_{n\in\mathbb{Z}^+}\{0,2\}$, "is" the Cantor set. The topological point of view certainly provides much more insight on the nature of C as well as those sets that are linked to it via continuous functions.

Theorem 7.4. There is a continuous function, $\delta : C \to [0,1]$, which maps the Cantor set, C, onto the closed interval $[0,1]$.

Proof: Recall that the function, $\varphi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to C$, maps $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ homeomorphically onto C. It is defined as $\varphi(\{m_n\}) = \sum_{n=1}^{\infty}\frac{m_n}{3^n}$. We begin by defining a similar function, $\psi : \prod_{n\in\mathbb{Z}^+}\{0,2\} \to [0,1]$ which maps the same set, $\prod_{n\in\mathbb{Z}^+}\{0,2\}$, onto [0,1]. It is defined as,

$$\psi(\{m_n\}) = \sum_{n=1}^{\infty}\frac{m_n}{2^{n+1}}$$

We claim that ψ is continuous on its domain and onto the closed interval $[0,1]$. To see that ψ is onto $[0,1]$ simply note that the expression

$$\sum_{n=1}^{\infty}\frac{m_n}{2^{n+1}} = \sum_{n=1}^{\infty}\frac{m_n/2}{2^{n}} = \sum_{n=1}^{\infty}\frac{s_n}{2^{n}}$$

where each s_n is 0 or 1 is simply the dyadic expansion for every element in $[0,1]$.[1]

To prove continuity of ψ proceed precisely as for φ in the previous theorem.

We define $\delta : C \to [0,1]$ as $\delta = \psi\circ\varphi^{\leftarrow}$, where $\varphi^{\leftarrow}$ continuously maps C one-to-one and onto $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ and ψ continuously maps $\prod_{n\in\mathbb{Z}^+}\{0,2\}$ onto $[0,1]$.[2] So δ continuously maps C onto $[0,1]$, as required.

About the cardinality of C.

The theorem, immediately above, allows us to conclusively arrive at a surprising conclusion about the cardinality of the Cantor set. Our geometric description of the construction of the Cantor set, C, showed that C was obtained by successively removing open middle-third interval from a previous set, leaving behind the endpoints.

[1]For example, if $y = \{m_n\}$ is the string $\{2,0,2,0,2,0,\ldots,\}$, then $\psi(y) = 0.10101010\ldots$ a point in [0, 1] in dyadic expansion form. The function ψ is easily seen to be onto $[0,1]$. For example, given $x = [0.001001001\ldots,]$, $\psi(\{0,0,2,0,0,2,\ldots\}) = x$.

[2]Note that ψ is not one-to-one since ψ maps the two distinct strings $\{0,0,2,2,2.\ldots,\}$ and $\{0,2,0,0,0.\ldots,\}$ to the same point $0.001111\ldots = 0.01000000\ldots$

The endpoints, all of them of the form, $\frac{m_n}{3_n}$, are rationals and are never removed and so must belong to C. It is impossible to logically deduce from this description of the construction of the Cantor set that C contains anything else but these endpoints of the form $\frac{m_n}{3_n}$ each one left behind in the construction process. So, if we were only to believe our eyes we might conclude that C is countably infinite. But the previous theorem shows that this cannot be so. It states that C can be mapped continuously onto the uncountable set $[0, 1]$. The range of a function can never have more points then the number of points in its domain. So C must be uncountable. Then $|C| = 2^{\aleph_0} = c$. We leave the reader with the more challenging question: If x is a point in C which is not an endpoint of a middle third, what does it look like? How can it be that a "non-endpoint" is left behind?

Is the Cantor topological space discrete?

By this question, we are wondering whether every single point of C is both open and closed. We claim that the subspace C cannot be discrete. We know that $\mathbb{R}$ is second countable and in theorem 5.13, we showed that "second countable" was a hereditary property. So C is second countable. If every point of the uncountable set, C, was open then C could not contain a countable dense subset and so would not be separable. Since "second countable" implies "separable" we would have a contradiction. So C must contain some points which are not open. That means that C is not a discrete subspace of $\mathbb{R}$.

7.5.3 Applications: A curve which contains every point in a cube.

As a final example of an application of product spaces we show the somewhat surprising result which states that a cube

$$[0, 1]^3 = \{(x, y, z) : x, y, z \in [0, 1]\}$$

with the product topology, in $\mathbb{R}^3$ is the continuous image of the closed interval [0, 1] with the usual topology inherited from $\mathbb{R}$. We present this in the form of a solved example question.

Example 5. Find a function which maps the closed interval [0,1] continuously onto the cube $[0, 1]^3$.

Solution : We first summarize a few facts presented up to now.

- There exists a continuous function $\psi : \prod_{n\in\mathbb{Z}^+}\{0, 2\} \to [0, 1]$ mapping $\prod_{n\in\mathbb{Z}^+}\{0, 2\}$ onto [0,1]. (By theorem 7.16.)
- Since $\mathbb{Z}^+ = \mathbb{N}\backslash\{0\}$ is countably infinite it has the same cardinality as $3\mathbb{Z}^+$; we can express as $3\mathbb{Z}^+ = \mathbb{Z}^+_A \cup \mathbb{Z}^+_B \cup \mathbb{Z}^+_C$ (the disjoint union of three copies of $\mathbb{Z}^+$). Then there is a homeomorphism, $h : \prod_{n\in\mathbb{Z}^+}\{0, 2\} \to \prod_{n\in 3\mathbb{Z}^+}\{0, 2\}$. (By theorem 7.9.1.)

– As in proposition 7.11, there is a homeomorphism

$$\theta : \prod_{n \in \mathbb{Z}^+_A \cup \mathbb{Z}^+_B \cup \mathbb{Z}^+_C} \{0,2\} \to \prod_{n \in \mathbb{Z}^+_A} \{0,2\} \times \prod_{n \in \mathbb{Z}^+_B} \{0,2\} \times \prod_{n \in \mathbb{Z}^+_C} \{0,2\}$$

defined as

$$\theta(\{x_\alpha\}_{\alpha \in 3\mathbb{Z}^+}) = \{\theta_A(\{x_\alpha\}_{\alpha \in 3\mathbb{Z}^+}), \theta_B(\{x_\alpha\}_{\alpha \in 3\mathbb{Z}^+}), \theta_C(\{x_\alpha\}_{\alpha \in 3\mathbb{Z}^+})\}$$

where θ_A, θ_B and θ_C are defined appropriately.

– In theorem 7.16, we exhibited a homeomorphism, $\varphi^{\leftarrow} : C \to \prod_{n \in \mathbb{Z}^+} \{0,2\}$, mapping the Cantor set C onto $\prod_{n \in \mathbb{Z}^+} \{0,2\}$.

Each of the three functions $f_A = \pi_A \circ \theta \circ h \circ \varphi^{\leftarrow}$, $f_B = \pi_B \circ \theta \circ h \circ \varphi^{\leftarrow}$ and $f_C = \pi_C \circ \theta \circ h \circ \varphi^{\leftarrow}$ continuously maps the points in C onto the points of $[0,1]$.

Recall that C is a subset of $[0,1]$ with multiple open intervals missing. We can then extend the continuous function $f_A : C \to [0,1]$ to a continuous function $F_A : [0,1] \to [0,1]$ so that, on each missing open interval (a,b) in C, F_A is defined as being linear from $(a, F_A(a)) = (a, f_A(a))$ to $(b, F_A(b)) = (b, f_A(b))$. The continuous function F_A then maps $[0,1]$ continuously onto $[0,1]$. The same holds true for the similarly defined $F_B : [0,1] \to [0,1]$ and $F_C : [0,1] \to [0,1]$.

Then the function $F : [0.1] \to [0,1]^3$ defined as $F(x) = \{F_A(x), F_B(x), F_C(x)\}$ maps the closed interval [0, 1] continuously onto $[0,1]^3$.

Mathematicians specializing in various fields of study refer to the continuous image of the closed interval $[0,1]$ as a *curve*. In this sense, saying that "$[0,1]$ can be mapped continuously onto the cube $[0,1]^3$" has the same meaning as saying "there is a curve that can fill the cube $[0,1]^3$" or "there is a curve that goes through every point of the cube $[0,1]^3$". The reader who follows through the proof carefully will notice that it can be generalized to the statement "For any integer n there is a curve which goes through each point of the cube $[0,1]^n$". The curve is referred to as *Peano's space filling curve*. A quick internet search will lead to many illustrations of curves which fill $[0,1]^2$ or $[0,1]^3$. The figure below illustrates part of a curve gradually filling up a cube.

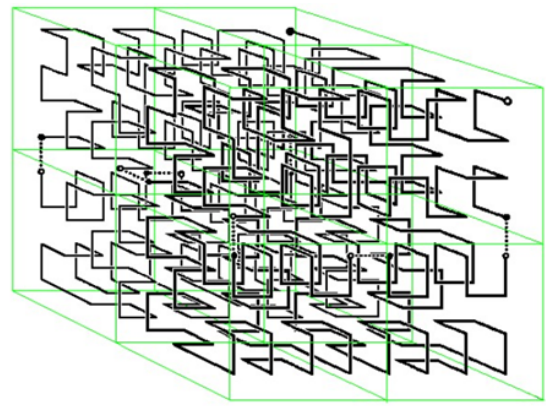


Figure 3: Part of a space filling curve in $[0, 1]^3$.

Appendix B : Axioms and set theory statements.

I Axioms and classes : 1 / Classes, sets and axioms

Axiom A1 (Axiom of extent) : For the classes x, A and B, $[A = B] \Leftrightarrow [x \in A \Leftrightarrow x \in B]$

Axiom A2 (Axiom of class construction): Let $P(x)$ designate a statement about x which can be expressed entirely in terms of the symbols $\in, \vee, \wedge, \neg, \Rightarrow, \forall$, brackets and variables x, y, z, $\dots$, A, B, $\dots$ Then there exists a class C which consists of all the elements x which satisfy $P(x)$.

Axiom A3 (Axiom of pair) : If A and B are sets , then the doubleton $\{A, B\}$ is a set.

Axiom A4 (Axiom of subsets) : If S is a set and ϕ is a formula describing a particular property, then the class of all sets in S which satisfy this property ϕ is a set. More succinctly, every subclass of a set of sets is a set. (Also called the Axiom of comprehension, Axiom of separation or Axiom of specification).

Axiom A5 (Axiom of power set): If A is a set then the power set $P(A)$ is a set.

Axiom A6 (Axiom of union): If $\mathscr{A}$ is a set of sets then $\bigcup_{C\in\mathscr{A}} C$ is a set.

Axiom A7 (Axiom of replacement): Let A be a set. Let $\phi(x, y)$ be a formula which associates to each element x of A a set y in such a way that, whenever both $\phi(x, y)$ and $\phi(x, z)$ hold true, $y = z$. Then there exists a set B which contains all sets y such that $\phi(x, y)$ holds true for some $x \in A$.

Axiom A8 (Axiom of infinity): There exists a non-empty set A that satisfies the condition: "$X \in A$" $\Rightarrow$ "$X \cup \{X\} \in A$". (A set satisfying this condition is called a *successor set* or an *inductive set*.)

Axiom A9 (Axiom of regularity) Every non-empty set A contains an element x whose intersection with A is empty.

Axiom of choice : For every set $\mathscr{A}$ of non-empty sets there is a rule f which associates to every set A in $\mathscr{A}$ an element $a \in A$.

I Axioms and classes : 2 / Constructing classes and sets

Theorem 2.1 For any class C, $C = C$. If it is not true that $A = B$ we will write $A \neq B$.

Definition 2.2 If A and B are classes (sets) we define $A \subseteq B$ to mean that every element of A is an element of B. That is, $A \subseteq B$ iff $x \in A \Rightarrow x \in B$ If $A \subseteq B$ we will say that A is a *subclass* (*subset*) of B. If $A \subseteq B$ and $A \neq B$ we will say that A is a *proper subclass* (*proper subset*) of B and write $A \subset B$ when we explicitly want to say $A \neq B$.

Theorem 2.3 If C, D, and E are classes (sets) then:

a) $C = C$.
b) $C = D \Rightarrow D = C$.
c) $C = D$ and $D = E \Rightarrow C = D$.
d) $C \subseteq D$ and $D \subseteq C \Rightarrow C = D$.
e) $C \subseteq D$ and $D \subseteq E \Rightarrow C \subseteq E$.

Theorem 2.4 There exists a class which is not an *element*.

Definition 2.5 The Axiom 2 states that $C = \{x : x \neq x\}$ is a class. It contains no elements. We will call the class with no elements the *empty class* and denote it by $\varnothing$.

Theorem 2.6 For any class C, $\varnothing \subseteq C$.

Theorem 2.7 Let S be a set. Then:

a) $\varnothing \subseteq S$ and so $\varnothing$ is a set.
b) The set S is an element. Hence all sets are elements.

Definition 2.8 If A is a set then we define the *power set of* A as being the class $\mathscr{P}(A)$ of all subsets of A. It can be described as follows: $\mathscr{P}(A) = \{X : X \subseteq A\}$.

II Class operations 3 / Operations on classes and sets

Definition 3.1 Let A and B be classes (sets). We define the *union* $A \cup B$ of the class A and the class B as

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$

That is, the element $x \in A \cup B$ iff $x \in A$ or $x \in B$. If $\mathscr{A}$ is a non-empty class of classes then we define the *union of all classes in* $\mathscr{A}$ as

$$\bigcup_{C \in \mathscr{A}} C = \{x : x \in C \text{ for some } C \in \mathscr{A}\}$$

That is, the element $x \in \bigcup_{C \in \mathscr{A}} C$ iff there exists $C \in \mathscr{A}$ such that $x \in C$.

Definition 3.2 Let A and B be classes (sets). We define the *intersection* $A \cap B$ of the class A and the class B as

$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$

That is, the element $x \in A \cap B$ iff $x \in A$ and $x \in B$. If $\mathscr{A}$ is a non-empty class of classes then we define the *intersection of all classes in* $\mathscr{A}$ as

$$\bigcap_{C \in \mathscr{A}} C = \{x : x \in C \text{ for all } C \in \mathscr{A}\}$$

That is, the element $x \in \bigcap_{C \in \mathscr{A}} C$ iff $x \in C$ for every class C in $\mathscr{A}$.

Definition 3.3 We will say that two classes (sets) C and D are *disjoint* if the two classes have no elements in common. That is, the classes C and D are disjoint if and only if $C \cap D = \varnothing$.

Definition 3.4 The *complement*, C', of a class (set) C is the class of all elements which are not in C. That is, if C is a class, then

$$C' = \{x : x \notin C\}$$

Hence $x \in C'$ iff $x \notin C$. Given two classes (sets) C and D, the *difference* $C - D$, of C and D, is the class

$$C - D = C \cap D'$$

The *symmetric difference*, $C \triangle D$, is the class

$$C \triangle D = (C - D) \cup (D - C)$$

Theorem 3.5 Let C and D be classes (sets). Then,

a) $C \subseteq C \cup D$
b) $C \cap D \subseteq C$

Theorem 3.6 Let C and D be classes (sets). Then,

a) $C \cup (C \cap D) = C$
b) $C \cap (C \cup D) = C$

Theorem 3.7 Let C be a class (a set). Then $(C')' = C$.

Theorem 3.8 *DeMorgan's laws.* Let C and D be classes (sets). Then,

a) $(C \cup D)' = C' \cap D'$
b) $(C \cap D)' = C' \cup D'$

Theorem 3.9 Let C, D and E be classes (sets). Then,

a) $C \cup D = D \cup C$ and $C \cap D = D \cap C$ (*Commutative laws*)
b) $C \cup C = C$ and $C \cap C = C$ (*Idempotent laws*)
c) $C \cup (D \cup E) = (C \cup D) \cup E$ and $C \cap (D \cap E) = (C \cap D) \cap E$ (*Associative laws*)
d) $C \cup (D \cap E) = (C \cup D) \cap (C \cup E)$ and $C \cap (D \cup E) = (C \cap D) \cup (C \cap E)$ (*Distribution*)

Theorem 3.10 Let A be a class (a set) and $\mathscr{U}$ denote the class of all elements.

a) $\mathscr{U} \cup A = \mathscr{U}$
b) $A \cap \mathscr{U} = A$
c) $\mathscr{U}' = \varnothing$
d) $\varnothing' = \mathscr{U}$
e) $A \cup A' = \mathscr{U}$

Theorem 3.11 Let $\mathscr{A}$ be a non-empty class (set).

a) $\left(\bigcup_{C\in\mathscr{A}} C\right)' = \bigcap_{C\in\mathscr{A}} C'$
b) $\left(\bigcap_{C\in\mathscr{A}} C\right)' = \bigcup_{C\in\mathscr{A}} C'$

Theorem 3.12 Let D be a class and $\mathscr{A}$ be a non-empty class (set) of classes.

a) $D \cap \left(\bigcup_{C\in\mathscr{A}} C\right) = \bigcup_{C\in\mathscr{A}} (D\cap C)$
b) $D \cup \left(\bigcap_{C\in\mathscr{A}} C\right) = \bigcap_{C\in\mathscr{A}} (D\cup C)$

Theorem 3.13 Let $\{B_{(i,j)} : i = 1, 2, 3, \ldots,\ j = 1, 2, 3, \ldots\}$ be a set of sets Then $\cup_{i=1}^{\infty}(\cap_{j=1}^{\infty} B_{(i,j)}) = \cap_{j=1}^{\infty}(\cup_{i=1}^{\infty} B_{(i,j)})$.

II Class operations 4 / Cartesian products

Definition 4.1 Let c and d be elements. We define the *ordered pair* (c, d) as $(c, d) = \{\{c\}, \{c, d\}\}$.

Theorem 4.2 Let a, b, c and d be classes (which are elements). Then $(a, b) = (c, d)$ iff $a = c$ and $b = d$.

Alternate definition 4.3 If c and d are classes define (c, d) as follows: $(c, d) = \{\ \{c, \varnothing\}, \{d, \{\varnothing\}\}\ \}$.

Definition 4.4 Let C and D be two classes (sets). We define the *Cartesian product*, $C\times D$, as follows: $C \times D = \{(c, d) : c \in C \text{ and } d \in D\}$.

Lemma 4.5 Let C and D be two classes (sets). Then the *Cartesian product*, $C \times D$, of C and D satisfies the property: $C \times D \subseteq \mathscr{P}(\mathscr{P}(C \cup D))$.

Corollary 4.6 If C and D are classes, then the *Cartesian product*, $C \times D$, is a class. If C and D are sets, then $C \times D$ is a set.

Theorem 4.7 Let C, D, E and F be a classes. Then

a) $C \times (D \cap E) = (C \times D) \cap (C \times E)$
b) $C \times (D \cup E) = (C \times D) \cup (C \times E)$
c) $(C \cap E) \times D = (C \times D) \cap (E \times D)$
d) $(C \cup E) \times D = (C \times D) \cup (E \times D)$
e) $(C \cup D) \times (E \cup F) = (C \times E) \cup (D \times E) \cup (C \times F) \cup (D \times F)$
f) $(C \cap D) \times (E \cap F) = (C \times E) \cap (D \times E) \cap (C \times F) \cap (D \times F)$

Theorem 4.8 If $C \subseteq D$ and $E \subseteq F$, then $C \times E \subseteq D \times F$.

Theorem 4.9 Given three classes (sets) S, U and V there is a one-to-one correspondence between the two classes (sets) $S \times (U \times V)$ and $(S \times U) \times V$.

Theorem 4.10 For classes c, d, e and f, if $(c,d) = \{\{c,\varnothing\},\{d,\{\varnothing\}\}\}$ and $(e,f) = \{\{e,\varnothing\},\{f,\{\varnothing\}\}\}$ then $(c,d) = (e,f)$ iff $c = e$ and $d = f$.

III Relations 5 / Relations on a class or set

Definition 5.1 a) We will call any subset R of ordered pairs in $\mathscr{U} \times \mathscr{U}$ a *binary relation.*

b) We will say that R is *binary relation on a class* C if R is a subclass (subset) of $C \times C$. In such cases we will simply say that R is a *relation in* C or *on* C.

c) If A and B are classes (sets) and R is a subclass (subset) of $A \times B$ then R can be viewed as a relation on $A \cup B$.

Definitions 5.2 Let C be a class (a set). a) The relation $\in_C = \{(x,y) : x \in C, y \in C, x \in y\}$ is called the *membership relation on* C. b) The relation

$$Id_C = \{(x,y) : x \in C, y \in C, x = y\}$$

is called the *identity relation on* C.

Definitions 5.3 Let R be a relation on a class (set) C. The *domain of* R is the class, $\operatorname{dom} R = \{x : x \in C \text{ and } (x,y) \in R \text{ for some } y \in C\}$. The *image of* R is the class, $\operatorname{im} R = \{y : y \in C \text{ and } (x,y) \in R \text{ for some } x \in C\}$. The word *range of* R is often used instead of "the image of R". If $R \subseteq A \times B$ is viewed as a relation on $A \cup B$, then $\operatorname{dom} R \subseteq A$ and $\operatorname{im} R \subseteq B$.

Definition 5.4 Let C be a class (a set) and let R be a relation defined in C. The inverse, R^{-1}, of the relation R is defined as follows:

$$R^{-1} = \{(x,y) : (y,x) \in R\}$$

Definition 5.5 Let C be a class (a set) and let R and T be two relations in C. We define the relation $T \circ R$ as follows: $T \circ R = \{(x,y) : \text{there exists some } z \in \operatorname{im} R \text{ such that } (x,z) \in R \text{ and } (z,y) \in T\}$

III Relations 6 / Equivalence relations and order relations.

Definition 6.1 Let S be a class and R be a relation on S.

a) We say that R is a *reflexive* relation on S if, for every $x \in S$, $(x,x) \in R$.

b) We say that R is a *symmetric* relation on S if, whenever $(x,y) \in R$ then $(y,x) \in R$.

c) We say that R is an *anti-symmetric* relation on S if, whenever $(x,y) \in R$ and $(y,x) \in R$ then $x = y$.

d) We say that R is an *asymmetric* relation on S if, whenever $(x,y) \in R$ then $(y,x) \notin R$.

e) We say that R is a *transitive* relation on S if, whenever $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$.

f) We say that R is an *irreflexive* relation on S if, for every $x \in S$, $(x, x) \notin R$.

g) We say that R satisfies the property of *comparability* on S if, for every $x, y \in S$ where $x \neq y$, either $(x, y) \in R$ or $(y, x) \in R$.

Definition 6.2 Let S be a class and R be a relation on S. We say that R is an *equivalence relation* on S if R is simultaneously 1) reflexive, 2) symmetric and 3) transitive on S.

Definition 6.3 Let S be a class.

a) *Non-strict order relation.* The relation R is a non-strict order relation on S if it is simultaneously reflexive (aRa holds true for any a in S), antisymmetric (if aRb and bRa then $a = b$) and transitive (aRb and bRc implies aRc) on S. A non-strict order relation, R, on S is said to be a *non-strict linear order relation* if, for every pair of elements a and b in S, either $(a, b) \in R$, $(b, a) \in R$ or $a = b$. That is, every pair of elements are comparable under R.[2] A non-strict ordering, R, on S which is not linear is said to be a *non-strict partial ordering relation* on S.[3]

b) *Strict order relation.* The relation R is a strict order relation if it is simultaneously irreflexive ($(a, a) \notin R$), asymmetric ($(a, b) \in R \Rightarrow (b, a) \notin R$) and transitive on S. If every pair of distinct elements, a and b, in S are comparable under a strict order relation, R, then R is a *strict linear ordering* on S. Those strict orderings which are not linear are called *strict non-linear orderings* or, more commonly, *strict partial ordering relation.*
A non-strict partial order R on S always induces a strict partial order R^* by defining $aR^*b \Rightarrow [aRb$ and $a \neq b]$. Similarly, a strict partial order R on S always induces a non-strict partial order $R^\dagger$ by defining $aR^\dagger b \Rightarrow [aRb$ or $a = b]$.

Definition 6.4 Let S be a class and R be is a partial ordering or a strict ordering relation on S. If R is a partial ordering relation $(a, b) \in R$ is represented as $a \leq b$ and if R is a strict ordering relation $(a, b) \in R$ is represented as $a < b$. If $a \leq b$ and $a \neq b$, we will simply write $a < b$.

a) A subset of S which is linearly ordered by R is called a *chain* in S. If R linearly orders S then S is a linearly ordered subset of itself and so is a chain.

b) An element a of S is called a *maximal* element of S if there does not exist an element b in S such that $a < b$. An element a of S is called a *minimal* element of S if there does not exist an element b in S such that $b < a$.

c) An element m in S is called the *minimum element of* S if $m \leq a$ ($m < a$) for all $a \in S$. An element M in S is called the *maximum element of* S if $a \leq M$

[2] A class on which is defined a linear ordering R is also said to be *fully ordered* or *totally ordered* by R. In certain branches of mathematics "linearly ordered set" is abbreviated as *l.o.set* or simply called *loset.*

[3] In certain branches of mathematics "partially ordered set" is abbreviated as *p.o.set* or simply called a *poset*

$(a < M)$ for all $a \in S$.

III Relations 7 / The partition of a set induced by an equivalence relation.

Notation 7.1 Let R be an equivalence relation on a set S and let $x \in S$. Then the set S_x is defined as follows: $S_x = \{y : (x, y) \in R\}$. That is, S_x is the set of all elements y in S such that y is related to x under R.

Theorem 7.2 Let R be an equivalence relation on a set S. Let x and y be two elements in S which are not related under R. Then any element z in S which is related to x cannot be related to y.

Theorem 7.3 Let R be an equivalence relation on a set S. Let x and y be two elements in S which are not related under R. Then $S_x \cap S_y = \varnothing$.

Theorem 7.4 Let R be an equivalence relation on a set S. Let x and y be two elements in S which are related under R. Then $S_x = S_y$.

Theorem 7.5 Let R be an equivalence relation on a set S. For every $x \in S$ there exists some $y \in S$ such that $x \in S_y$.

Theorem 7.6 Let R be an equivalence relation on a set S. Then $\bigcup_{x \in S} S_x = S$.

III Relations 8 / On partitions and quotient sets of a set.

Definition 8.1 Let S be a set. We say that a set $\mathscr{C} \subseteq \mathscr{P}(S)$ forms a *partition of* S if $\mathscr{C}$ satisfies the 3 properties:

1) $\bigcup_{A \in \mathscr{C}} A = S$
2) If A and $B \in \mathscr{C}$ and $A \neq B$ then $A \cap B = \varnothing$.
3) $A \neq \varnothing$ for all $A \in \mathscr{C}$.

Definition 8.2 Let S be a set on which an equivalence relation R is defined.

a) Each element S_x of $\mathscr{S}_R = \{S_x : x \in S\}$ is called an *equivalence class of* x *under* R or an *equivalence class induced by the relation* R.
b) The set $\mathscr{S}_R = \{S_x : x \in S\}$ of all equivalence classes induced by the relation R is called the *quotient set of* S *induced by* R. The set $\mathscr{S}_R$ is more commonly represented by the symbol S/R. So $S/R = \{S_x : x \in S\}$. From here on we will use the more common notation, S/R.

Theorem 8.3 Let S be a set and $\mathscr{C}$ be a partition of S. Let $R_{\mathscr{C}}$ be the relation such that $(x, y) \in R_{\mathscr{C}}$ iff $\{x, y\} \subseteq S$ for some S in $\mathscr{C}$. Then $R_{\mathscr{C}}$ is an equivalence relation on S.

IV Functions 9 / Functions: A set-theoretic definition.

Definition 9.1 A function from A to B is a triple $\langle f, A, B\rangle$ satisfying the following properties:

1) A and B are classes and $f \subseteq A \times B$
2) For every $a \in A$ there exists $b \in B$ such that $(a, b) \in f$.
3) If $(a, b) \in f$ and $(a, c) \in f$ then $a = c$.

Definition 9.2 If $f : A \to B$ is a function and $D \subseteq A$ then we say that the function $f : D \to C$ is *a restriction of f to D*. In this case we will use the symbol $f|_D$ to represent the restriction of f to D. Note that, if $D \subseteq A$, then we can write $f|_D \subseteq f$ since $f|_D = \{(x, y) : x \in D \text{ and } (x, y) \in f\} \subseteq f$.

Theorem 9.3 Let $f : A \to B$ be a function and suppose $A = C \cup D$. Then $f = f|_c \cup f|_D$.

Theorem 9.4 Two functions $f : A \to B$ and $g : A \to B$ are equal if and only if $f(x) = g(x)$ for all $x \in A$.

Definitions 9.5 Let $f : A \to B$ be a function.

a) We say that "f maps A *onto* B" if $\operatorname{im} f = B$. We often use the expression "$f : A \to B$ is *surjective*" instead of the words *onto* B.
b) We say that "f maps A *one-to-one into* B" if whenever $f(x) = f(y)$ then $x = y$. We often use the expression "$f : A \to B$ is *injective*" instead of the words *one-to-one into* B.
c) If the function $f : A \to B$ is both one-to-one and onto B then we can simply say that f is "one-to-one and onto". Another way of conveying this is to say that f is *bijective*, or f is a *bijection*. So "injective + surjective $\Rightarrow$ bijective".
d) Two classes (or sets) A and B for which there exists some bijective function $f : A \to B$ are said to be in *one-to-one correspondence*.

IV Functions 10 / Compositions of function.

Definition 10.1 Suppose $f : A \to B$ and $g : B \to C$ are two functions such that the image of the function f is contained in the domain of the function g. Let $h = \{(x, z) : y = f(x) \text{ and } z = g(y) = g(f(x))\ \}$. Thus $(x, z) \in h$ if and only if $(x, z) = (x, g(f(x))$. We will call h the *composition of g and f*, and denote it as $g \circ f$.

Theorem 10.2 Let $f : A \to B$ and $g : B \to C$ be two functions such that the image of the function f is contained in the domain of the function g. Then the composition of g and f, $(g \circ f) : A \to C$, is a function.

Theorem 10.3 Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be three functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

Theorem 10.4 Let $f : A \to B$. Then $I_B \circ f = f$ and $f \circ I_A = f$.

Definition 10.5 Let $f : A \to B$. If $g : B \to A$ is a function satisfying $g \circ f = I_A$ then we will call g an "*inverse of f*" and denote it as f^{-1}.

Theorem 10.6 Let $f : A \to B$ be a one-to-one onto function.

a) An inverse function $f^{-1} : B \to A$ of f exists.
b) The function f^{-1} is one-to-one and onto.
c) The function $f^{-1} : B \to A$ satisfies the property $f \circ f^{-1} = I_B$.
d) The inverse function, f^{-1}, of f is unique.

Definition 10.7 A function $f : A \to B$ which is one-to-one and onto is called an *invertible function.*

Theorem 10.8 Let $f : A \to B$ and $g : B \to C$ be two onto-to-one and onto functions.

a) The function $g \circ f$ is also one-to-one and onto.
b) The inverse, $g \circ f$, is $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

IV Functions 11 / Images and inverse images of sets.

Definition 11.1 Let A and B be sets and suppose $f : A \to B$ is a function acting on A. If S is a subset of $A = \operatorname{dom} f$ we define the expression $f[S]$ as follows: $f[S] = \{y : y = f(x) \text{ for some } x \in S\}$. We will say that $f[S]$ is the *image of the set S under f.*

Definitions 11.2 Let $f : A \to B$ be a function where A and B are sets. We define $f^{\leftarrow} : \mathscr{P}(B) \to \mathscr{P}(A)$ as $f^{\leftarrow}(X) = Y$ iff $Y = \{y : y \in A, f(y) \in X\}$. In particular, $f^{\leftarrow}(\{x\}) = Y$ iff $Y = \{y : y \in A, f(y) = x\}$. We will refer to it as the *set-valued inverse function $f^{\leftarrow}$.*

Theorem 11.3 Let $f : A \to B$ be a function. Then $f^{\leftarrow} : \mathscr{P}(\operatorname{im} f) \to \mathscr{P}(A)$ is a one-to-one function on its domain $\mathscr{P}(\operatorname{im} f)$.

Theorem 11.4 Let $f : A \to B$ be a function mapping the set A to the set B. Let $\mathscr{A}$ be a set of subsets of A and $\mathscr{B}$ be a set of subsets of B. Let $D \subseteq A$ and $E \subseteq B$. Then:

a) $f\left[\bigcup_{S \in \mathscr{A}} S\right] = \bigcup_{S \in \mathscr{A}} f[S]$
b) $f\left[\bigcap_{S \in \mathscr{A}} S\right] \subseteq \bigcap_{S \in \mathscr{A}} f[S]$ with equality only if f is one-to-one.
c) $f[A - D] \subseteq B - f[D]$ with equality only if f is one-to-one and onto B.
d) $f^{\leftarrow}\left(\bigcup_{S \in \mathscr{B}} S\right) = \bigcup_{S \in \mathscr{B}} f^{\leftarrow}(S)$
e) $f^{\leftarrow}\left(\bigcap_{S \in \mathscr{B}} S\right) = \bigcap_{S \in \mathscr{B}} f^{\leftarrow}(S)$
f) $f^{\leftarrow}(B - E) = A - f^{\leftarrow}(E)$

IV Functions 12 / Equivalence relations defined by functions.

Definition 12.1 Let $f : A \to B$ be a function which maps a set A into a set B. We define an equivalence relation R_f on A as follows: Two elements a and b are related under R_f if and only if $\{a, b\} \subseteq f^{\leftarrow}(x)$ for some x in im f. The quotient set of A induced by R_f is then $A/R_f = \mathscr{A}_{R_f} = \{f^{\leftarrow}(\{x\}) : x \in f[A]\}$ We will refer to R_f as the *equivalence relation determined by* f and A/R_f (or $\mathscr{A}_{R_f}$) as the *quotient set of* A *determined by f.*

Theorem 12.2 Let $f : S \to T$ be a function where S and T are sets. There exists an onto function $g_f : S \to S/R_f$ and a one-to-one function $h_f : S/R_f \to T$ such that $h_f \circ g_f = f$. The function, $h_f \circ g_f = f$, is called the *canonical decomposition of f.*

V From sets to numbers 13 / The natural numbers.

Definition 13.1 For any set x, we define the *successor* x^+, of x as $x^+ = x \cup \{x\}$.

Definition 13.2 If x is a set then $x^+ = x \cup \{x\}$. A set A is called an *inductive set* if it satisfies the following two properties:

a) $\varnothing \in A$.
b) $x \in A \Rightarrow x^+ \in A$.

Definition 13.3 We define the *set of all natural numbers*, $\mathbb{N}$, as the intersection of all inductive sets. That is $\mathbb{N} = \{x : x \in I$ for any inductive set $I\}$.

Theorem 13.4 Let A be a subset of $\mathbb{N}$. If A satisfies the two properties:

a) $0 \in A$
b) $m \in A \Rightarrow m^+ \in A$

then $A = \mathbb{N}$.

Corollary 13.5 (The Principle of mathematical induction.) Let P denote a particular set property. Suppose $P(n)$ means "the property P is satisfied depending on the value of the natural number n". Let

$$A = \{n \in \mathbb{N} : P(n) \text{ holds true }\}$$

If A satisfies the two properties:

a) $0 \in A$. That is $P(0)$ holds true,
b) $(n \in A) \Rightarrow (n^+ \in A)$. That is, $P(n)$ holds true $\Rightarrow P(n^+)$ holds true.

then $A = \mathbb{N}$. That is, $P(n)$ holds true for all natural numbers n.

Definition 13.6 A set S which satisfies the property "$x \in S \Rightarrow x \subset S$" is called a *transitive set.*

Theorem 13.7 The non-empty set S is a transitive set if and only if the property "$x \in y$ and $y \in S$" $\Rightarrow$ "$x \in S$".

Theorem 13.8 The set $\mathbb{N}$ of natural numbers is a transitive set.

Theorem 13.9 a) For natural numbers n, m, $m \in n \Rightarrow m \subseteq n$. Hence every natural number is a transitive set.

b) For any natural number n, $n \neq n^+$.

c) For any natural number n, $n \notin n$.

d) For any distinct natural numbers n, m, $m \subset n \Rightarrow m \in n$.

Theorem 13.10 Let m and n be distinct natural numbers.

a) If $m \subset n$ then $m^+ \subseteq n$.

b) *All natural numbers are comparable.* Either $m \subset n$ or $n \subset m$. Equivalently, $m \in n$ or $n \in m$. Hence both "$\subset$" and "$\in$" linearly order $\mathbb{N}$.

c) There is no natural number m such that $n \subset m \subset n^+$.

Theorem 13.11 Every natural number has an immediate predecessor. If k and n are natural numbers such that $k^+ = n$ then k is called an *immediate predecessor* of n. For any non-zero natural number n, $k = \bigcup_{m \subset n} m$ is an immediate predecessor of n.

Theorem 3.12 *Unique immediate predecessors.* Any non-zero natural number has a *unique* immediate predecessor.

Theorem 3.13 (The Principle of mathematical induction: second version.) Let P denote a particular property. Suppose $P(n)$ means "the property P is satisfied depending on the value of the natural number n". Let

$$A = \{n \in \mathbb{N} : P(n) \text{ holds true }\}$$

Suppose that, for any natural number n,

$$P(k) \text{ is true for all } k < n \Rightarrow P(n) \text{ is true}$$

Then $P(n)$ holds true for all natural numbers n.

V From sets to numbers 14 / The natural numbers as a well-ordered set.

Notation 14.1 We define the relation "$\in_=$" on $\mathbb{N}$ as follows:

$$m \in_= n \text{ if and only if } m = n \text{ or } m \in n$$

If $m \in_= n$ and we want to state explicitly that $m \neq n$ we write $m \in n$.

Theorem 14.2 Let $(S, \leq)$ be a linearly ordered set. Suppose $T \subseteq S$. The element q is *a least element of T with respect to "$\leq$"* if $q \in T$ and $q \leq m$ for all $m \in T$. If S is equipped with a strict linear ordering "$<$" a *least element of T with respect to* $<$ is an element $q \in T$ such that $q < m$ for all $m \in T$ where $m \neq q$. The set $(S, \leq)$ is said to be *well-ordered* with respect to "$\leq$" if every non-empty subset T of S contains its least element with respect to $\leq$. Similarly, the set $(S, <)$ is said to be *well-ordered* with respect to "$<$" if every non-empty subset T of S contains its least element with respect to $<$.

Theorem 14.3 The natural numbers $\mathbb{N}$ is a strict $\in$-well-ordered set.

Corollary 14.4 Every natural numbers n is a $\in$-well-ordered set.

Theorem 14.5 Any bounded non-empty subset of $(\mathbb{N}, \in)$ has a maximal element.

Definition 14.6 Consider the set $\{1, 2\}^{\mathbb{N}}$ of all functions mapping natural numbers to 1 or 2. We define the *lexicographic order* "$<$" on $\{1, 2\}^{\mathbb{N}}$ as follows: For any two elements $f = \{a_0, a_1, a_2, a_3, \ldots,\}$ and $g = \{b_0, b_1, b_2, b_3, \ldots,\}$ in $\{1, 2\}^{\mathbb{N}}$, $f = g$ if and only if $a_i = b_i$ for all $i \in \mathbb{N}$ and $f < g$ if and only if for the first two unequal corresponding terms a_i and b_i, $a_i \in b_i$. A lexicographic ordering can similarly be defined on $S^{\mathbb{N}}$ where S is any subset of $\mathbb{N}$.

V From sets to numbers 15 / Arithmetic of the natural numbers.

Definition 15.1 Let m be a fixed natural number. Addition of a natural number n with m is defined as the function $r_m : \mathbb{N} \to \mathbb{N}$ satisfying the two conditions

$$\begin{aligned} r_m(0) &= m \\ r_m(n^+) &= [r_m(n)]^+ \end{aligned}$$

The expression $m + n$ as simply another way of writing $r_m(n)$. Thus

$$r_m(0) = m \quad \Leftrightarrow \quad m + 0 = m \tag{1}$$
$$r_m(n^+) = [r_m(n)]^+ \quad \Leftrightarrow \quad m + n^+ = (m + n)^+ \tag{2}$$

Theorem 15.2 Let m be a fixed natural number and let $r_m : \mathbb{N} \to \mathbb{N}$ be a function satisfying the two properties

$$\begin{cases} r_m(0) & = \quad m \\ r_m(n^+) & = \quad [r_m(n)]^+ \end{cases}$$

Then r_m is a well-defined function on $\mathbb{N}$.

Definition 15.3 For any natural number m, multiplication with the natural number m is defined as the function $s_m : \mathbb{N} \to \mathbb{N}$ satisfying the two conditions

$$\begin{array}{rcl} s_m(0) & = & 0 \\ s_m(n^+) & = & s_m(n) + m \end{array}$$

We define the expression mn and $m \times n$ as alternate ways of writing $s_m(n)$. Thus

$$s_m(0) = 0 \quad \Leftrightarrow \quad m0 = m \times 0 = 0 \tag{3}$$

$$s_m(n^+) = s_m(n) + m \quad \Leftrightarrow \quad mn^+ = mn + m = m \times n + m \tag{4}$$

Theorem 15.4 Let m be a fixed natural number and let $s_m : \mathbb{N} \to \mathbb{N}$ be a function satisfying the two properties

$$\begin{cases} s_m(0) & = \quad 0 \\ s_m(n^+) & = \quad s_m(n) + m \end{cases}$$

Then s_m is a well-defined function on $\mathbb{N}$.

Theorem 15.5 For any two natural numbers m and n, $m \in_= n$ if and only if there exists a *unique* natural number k such that $n = m + k$.

Definition 15.6 For any two natural numbers m and n such that $m \leq n$, the *unique* natural number k satisfying $n = m + k$ is called the *difference between n and m* and is denoted by $n - m$. The operation "$-$" is called *subtraction.*

V From sets to numbers 16 / The integers $\mathbb{Z}$ and the rationals $\mathbb{Q}$.

Theorem 16.1 Let $Z = \mathbb{N} \times \mathbb{N}$. Let R_z be a relation on Z that is defined as follows: $(a, b)R_z(c, d)$ if and only if $a + d = b + c$. Then R_z is an equivalence relation on Z.

Corollary 16.2 Let $Z = \mathbb{N} \times \mathbb{N}$ be equipped with the equivalence relation R_z defined as:

$$(a, b)R_z(c, d) \Leftrightarrow a + d = b + c$$

For each $n \in \mathbb{N}$ let $[(0, n)]$ and $[(n, 0)]$ denote the R_z-equivalence classes containing the elements $(0, n)$ and $(n, 0)$ respectively. Then the quotient set induced by R_z can be expressed as $Z/R_z = \{[(0, n)] : n \in \mathbb{N}\} \cup \{[(n, 0)] : n \in \mathbb{N}\}$

Definitions 16.3 The set of *integers*, $\mathbb{Z}$, is defined as:

$$\mathbb{Z} = Z/R_z = \{[(a, b)] : a, b \in \mathbb{N}\} = \{[(0, n)] : n \in \mathbb{N}\} \cup \{[(n, 0)] : n \in \mathbb{N}\}$$

a) *Negative integers*: The set of *negative integers* is defined as being the set

$$\mathbb{Z}^- = \{[(0, n)] : n \in \mathbb{N}\}$$

Positive integers: The set of *positive integers* is defined as being the set

$$\mathbb{Z}^+ = \{[(n, 0)] : n \in \mathbb{N}\}$$

The elements of the form $[(0, n)]$ can be represented by $-n = [(0, n)]$ while the elements of the form $[(n, 0)]$ can be represented as $n = [(n, 0)]$.

b) *Order relation on* $\mathbb{Z}$: We define a relation $\leq_z$ on $\mathbb{Z}$ as follows: $[(a, b)] \leq_z [(c, d)]$ if and only if $a + d \leq b + c$. It is a routine exercise to show that $\leq_z$ is a linear ordering of $\mathbb{Z}$.

c) *Addition on* $\mathbb{Z}$: We must sometimes distinguish between addition of natural numbers and addition of integers. Where there is a risk of confusion we will use the following notation: "$+_n$" means addition of natural numbers while "$+_z$" means addition of integers.
Addition $+_z$ on $\mathbb{Z}$ is defined as:

$$[(a, b)] +_z [(c, d)] = [(a +_n c,\ b +_n d)]$$

d) *Opposites of integers*: The *opposite* $-[(a, b)]$ of $[(a, b)]$ is defined as

$$-[(a, b)] = [(b, a)]^1$$

e) *Subtraction on integers*: Subtraction "$-_z$" on $\mathbb{Z}$ is defined as:

$$[(a, b)] -_z [(c, d)] = [(a, b)] + (-[(c, d)])^2$$

f) *Multiplication of integers*: Multiplication $\times_z$ on $\mathbb{Z}$ is defined as

$$[(a, b)] \times_z [(c, d)] = [(ac + bd, ad + bc)].^3$$

In particular, $[(0, n)] \times_z [(m, 0)] = [(0 + 0, 0 + nm)] = [(0, nm)] = -[(nm, 0)]$ and $[(n, 0)] \times_z [(m, 0)] = [(nm, 0)]$.

g) *Absolute value of an integer*: The absolute value, $|n|$, of an integer n is defined as

$$|n| = \begin{cases} n & \text{if} \quad 0 \leq_z n \\ -n & \text{if} \quad n <_z 0 \end{cases}$$

[1] Note that $-[(n, 0)] = [(0, n)] = -n$.

[2] When there is no risk of confusion with subtraction of other types of numbers we will simply use "$-$".

[3] Note that the "center dot" can be used instead of the "$\times_z''$ symbol. When there is no risk of confusion with multiplication of other types of numbers we will simply use "$\times$".

h) *Equality of two integers*: If (a, b) and (c, d) are ordered pairs which are equivalent under the relation R_z, then the R_z-equivalence classes $[(a, b)]$ and $[(c, d)]$ are equal sets. To emphasize that they are equal sets under the relation R_z we can write

$$[(a, b)] =_z [(c, d)]$$

i) *Distribution properties*: If $[(a, b)]$, $[(c, d)]$ and $[(e, f)]$ are integers then

$$[(a, b)] \times_z ([(c, d)] +_z [(e, f)]) =_z [(a, b)] \times_z [(c, d)] +_z [(a, b)] \times_z [(e, f)]$$

and

$$([(c, d)] +_z [(e, f)]) \times_z [(a, b)] =_z [(c, d)] \times_z [(a, b)] +_z [(e, f)] \times_z [(a, b)]$$

Theorem 16.4 Let $Q = \mathbb{Z} \times \mathbb{Z}^*$ where $\mathbb{Z}^* = \mathbb{Z} - \{0\}$. Let R_q be a relation on Q defined as follows: $(a, b)R_q(c, d)$ if and only if $a \times_z d = b \times_z c$. Then R_q is an equivalence relation on Q.

Definitions 16.5 The set of *rational numbers*, $\mathbb{Q}$, is defined as:

$$\mathbb{Q} = Q/R_q = \{[(a, b)] : a \in \mathbb{Z}, b \in \mathbb{Z}^*\}^1$$

The expression $[(a, b)]$ is normally written in the form $\frac{a}{b}$.

a) We define a relation $\leq_q$ on $\mathbb{Q}$ as follows: If b and d are both positive, $[(a, b)] \leq_q [(c, d)]$ if and only if $a \times_z d \leq_z b \times_z c$.

b) Addition $+_q$ on $\mathbb{Q}$ is defined as:

$$[(a, b)] +_q [(c, d)] = [(ad +_z bc,\ b \times_z d)]$$

c) Subtraction $-_q$ on $\mathbb{Q}$ is defined as:

$$[(a, b)] -_q [(c, d)] = [(a, b)] +_q [(-c, d)]$$

d) Multiplication $\times_q$ on $\mathbb{Q}$ is defined as

$$[(a, b)] \times_q [(c, d)] = [(a \times_z c, b \times_z d)]$$

e) *Equality of two rational numbers*: If (a, b) and (c, d) are ordered pairs of integers $(b, d \neq 0)$ which are equivalent under the relation R_q, then the R_q-equivalence classes $[(a, b)]$ and $[(c, d)]$ are equal sets. To emphasize that they are equal sets under the relation R_q we can write

$$[(a, b)] =_q [(c, d)]$$

[1] Recall that a and b is shorthand for expressions of the form $[(0, a)]$ or $-[(0, b)]$

f) *Opposites of rational numbers.* If (a, b) is an ordered pair of integers $(b \neq 0)$ and $[(a, b)]$ is its R_q-equivalence class then the opposite of the rational number $[(a, b)]$ is defined as $[(-a, b)]$ and is denoted as $-[(a, b)] =_q [(-a, b)]$.

Theorem 16.6 Suppose a and b are positive integers where $b \neq 0$ and $[(a, b)]$ is an R_q equivalence class. Then

a) $[(-a, -b)] = \frac{-a}{-b} = \frac{a}{b} = [(a, b)]$.

b) $-[(a, b)] =_q [(-a, b)] = \frac{-a}{b} = \frac{a}{-b} = [(a, -b)]$

V From sets to numbers 17 / Dedekind cuts: "Real numbers are us!"

Definition 17.1 For any real number r, let ${}_{\mathbb{R}}S_r$ denote the interval $(-\infty, r)$ in $\mathbb{R}$. It is the subset of all real numbers strictly smaller than r. The subset ${}_{\mathbb{R}}S_r$ is called an *initial segment in* $\mathbb{R}$. For any real number r, the subset ${}_{\mathbb{Q}}S_r = (-\infty, r) \cap \mathbb{Q} = {}_{\mathbb{R}}S_r \cap \mathbb{Q}$ is called an *initial segment in* $\mathbb{Q}$. For each of ${}_{\mathbb{R}}S_r$ and ${}_{\mathbb{Q}}S_r$ the real number r is the least upper bound of $(-\infty, r)$ and $(-\infty, r) \cap \mathbb{Q}$ respectively. Note that r may be a non-rational number even for ${}_{\mathbb{Q}}S_r$ a proper subset of $\mathbb{Q}$.

Definition 17.2 The elements ${}_{\mathbb{Q}}S_r = (-\infty, r) \cap \mathbb{Q}$ of the set $\mathscr{D} = \{{}_{\mathbb{Q}}S_r : r \in \mathbb{R}\}$ are called *Dedekind cuts*

Definition 17.3 The set of all Dedekind cuts $\mathscr{D}$, linearly ordered by inclusion with addition $+$ and multiplication $\times$ as described above is called the *real numbers.* Those Dedekind cuts which do not have a least upper bound in $\mathbb{Q}$ are called *irrational numbers.*

Lemma 17.4 The union of a set of Dedekind cuts is either $\mathbb{Q}$ or a Dedekind cut.

Theorem 17.5 Every non-empty subset of $\mathbb{R}$ which has an upper bound has a least upper bound.

VI Infinite sets 18 / Infinite sets versus finite sets.

Definition 18.1 A set S is said to be an *infinite set* if S has a *proper* subset X such that a function $f : S \to X$ maps S one-to-one onto X. If a set S is not infinite then we will say that it is a *finite set.*

Theorem 18.2 Basic properties of infinite and finite sets.

a) The empty set is a finite set.
b) Any singleton set is a finite set.
c) Any set which has a subset which is infinite must itself be infinite.
d) Any subset of a finite set must be finite.

Theorem 18.3 Let $f : X \to Y$ be a one-to-one function mapping X onto Y. The set Y is infinite if and only if the set X is infinite.

Corollary 18.4 The one-to-one image of a finite set is finite.

Lemma 18.5 Let S be an infinite set and $x \in S$. Then $S - \{x\}$ is an infinite set.

Theorem 18.6 Every natural number n is a finite set.

Corollary 18.7 [AC] A set S is finite if and only if S is empty or in one-to-one correspondence with some natural number n.

Theorem 18.8 *The recursively constructed function theorem.* Let S be a set, $k : \mathscr{P}(S) \to S$ be a well-defined function on $\mathscr{P}(S)$ and $f \subseteq \mathbb{N} \times S$ be a relation. We write $f(n) = a$ if and only if $(n, a) \in f$. Let $m \in S$. Suppose the relation f satisfies the two properties

$$\begin{cases} f(0) = m & \Rightarrow & (0, m) = (0, f(0)) \in f \\ (n, f(n)) \in f & \Rightarrow & (n+1, k(S - \{f(0), f(1), \ldots, f(n)\}) = (n+1, f(n+1)) \in f \end{cases}$$

Then f is a well-defined function on $\mathbb{N}$.

Theorem 18.9 [AC] A set S is an infinite set if and only if it contains a one-to-one image of the $\mathbb{N}$.

Theorem 18.10 If the set S is a finite set and $f : S \to X$ is a function, then $f[S]$ is finite.

Theorem 18.11 If a set S contains n elements then $\mathscr{P}(S)$ contains 2^n elements. Hence, if a set S is a finite set, then the set $\mathscr{P}(S)$ is finite.

VI Infinite sets 19 / Countable and uncountable sets.

Definition 19.1 Two sets A and B are said to be *equipotent* sets if there exists a one-to-one function $f : A \to B$ mapping one onto the other. If A and B are *equipotent* we will say that "A is equipotent to B" or "A is equipotent with B".

Definition 19.2 *Countable sets* are those sets that are either finite or equipotent to $\mathbb{N}$. All infinite sets which are not countable are called *uncountable sets*.

Theorem 19.3 A subset of a countable set is countable.

Theorem 19.4 Any finite product, $\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$, of $\mathbb{N}$ is a countable set.

Lemma 19.5 Suppose f maps an infinite countable set A onto a set $B = f[A]$. Then B is countable.

Theorem 19.6 Let $\{A_i : i \in S \subseteq \mathbb{N}\}$ be a countable set of non-empty countable sets A_i. Then $\bigcup_{i \in S} A_i$ is countable.

Theorem 19.7 The set of all real numbers $\mathbb{R}$ is uncountable.

VI Infinite sets 20 / Properties of the equipotence relation.

Theorem 20.1 Let $\mathscr{S}$ be a class of sets. The equipotence relation R_e on $\mathscr{S}$ is an equivalence relation on $\mathscr{S}$.

Theorem 20.2 Suppose A, B, C and D are sets such that $A \sim_e B$ and $C \sim_e D$ where $A \cap C = \varnothing = B \cap D$. Then $(A \cup C) \sim_e (B \cup D)$.

Theorem 20.3 Suppose A, B, C and D are sets such that $A \sim_e B$ and $C \sim_e D$. Then $A \times C \sim_e B \times D$.

Corollary 20.4 Suppose A and B are infinite sets.

1) If $\{A, B\} \subset [\mathbb{N}]_e$ then $A \times B \in [\mathbb{N}]_e$.
2) If $\{A, B\} \subset [\mathbb{R}]_e$ then $A \times B \in [\mathbb{R}]_e$.

Theorem 20.5 If $\{A_i : i = 0, 1, 2, ..., n\}$ is a set of n non-empty countable sets then $\prod_{i=0}^{n} A_i$ is countable for all n.

Theorem 20.6 Suppose S is and infinite set and T is a countable set such that $S \cap T = \varnothing$. Then $S \sim_e S \cup T$.

Theorem 20.7 If the sets A and B are equipotent, then so are their associated power sets $\mathscr{P}(A)$ and $\mathscr{P}(B)$.

Theorem 20.8 Any non-empty set S is embedded in its power set $\mathscr{P}(S)$. But *no* subset of S is equipotent with $\mathscr{P}(S)$.

Definition 20.9 We will say that the non-empty set A is *properly embedded* in the set B if A is equipotent to a proper subset of B but B *is not equipotent to* A *or any of its subsets*. To represent the relationship "A is properly embedded in B" we will write

$$A \hookrightarrow_e B$$

If A and B are sets such that A is equipotent to a subset of B (where B may, or may not, be embedded in A), we will say that A is *embedded* in B. To represent the relationship "A is embedded in B" we will write

$$A \hookrightarrow_{e\sim} B$$

Definition 20.10 Let $\mathscr{S} = \{S : S \text{ is a set}\}$ and $\mathscr{E} = \{[S]_e : S \in \mathscr{S}\}$. Let $[A]_e$ and $[B]_e$ be elements of $\mathscr{E}$. We write

$$[A]_e <_e [B]_e$$

if and only if $A \hookrightarrow_e B$. We write

$$[A]_e \leq_e [B]_e$$

if and only if $A \hookrightarrow_{e\sim} B$.

Proposition 20.11 Let S be any set. Suppose $\mathscr{P}^0(S) = S$, $\mathscr{P}^1(S) = \mathscr{P}(S)$ and $\mathscr{P}^n(S) = \mathscr{P}(\mathscr{P}^{n-1}(S))$ for all $n \geq 1$. The set

$$\{[S]_e, [\mathscr{P}(S)]_e, [\mathscr{P}^2(S)]_e, [\mathscr{P}^3(S)]_e, \ldots, [\mathscr{P}^n(S)]_e \ldots, \}$$

forms an infinite $<_e$-ordered chain of distinct classes in $\mathscr{E}$.

Theorem 20.12 For every any non-empty set S, $2^S = \{\chi_T : T \in \mathscr{P}(S)\} \sim_e \mathscr{P}(S)$

Theorem 20.13 The set $\mathbb{R}$ is embedded in $\mathscr{P}(\mathbb{N})$.

Theorem 20.14 The set $\mathscr{P}(\mathbb{N})$ is embedded in $\mathbb{R}$.

VI Infinite sets 21 / The Shröder-Bernstein theorem.

Theorem 21.1 (The Schröder-Bernstein theorem) If S and T are infinite subsets where S is embedded in T and T is embedded in S then S and T are equipotent.

Lemma 21.2 Let T be a proper subset of the set S and $f : S \to T$ be a one-to-one function mapping S into T. Then there exists a one-to-one function $h : S \to T$ mapping S **onto** T.

Theorem 21.3 The set $\mathbb{R}$ of all real numbers is equipotent to $\mathscr{P}(\mathbb{N})$.

Theorem 21.4 The sets $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{R}$ are equipotent.

Definition 21.5 If A and B are two sets, then the symbol B^A refers to the set of all functions mapping A into B.

VII Cardinal numbers 22 / An introduction to cardinal numbers.

Postulate 22.1 There exists a class of sets $\mathscr{C}$ which satisfies the following properties:

1. Every natural number n is an element of $\mathscr{C}$.
2. Any set $S \in \mathscr{S}$ is equipotent with precisely one element in $\mathscr{C}$

The sets in $\mathscr{C}$ are called *cardinal numbers*. When we say that a set S *has cardinality* κ we mean that $\kappa \in \mathscr{C}$ and that $S \sim_e \kappa$. If the set S has cardinality κ, we will write $|S| = \kappa$.

Definition 23.2 If S and T are sets and $\kappa = |S|$ and $\lambda = |T|$ then we define *addition*"+", *multiplication* "$\times$" and *exponentiation* of two cardinal numbers as follows:

a) If $S \cap T = \varnothing$,
$$\kappa + \lambda = |S \cup T|$$

b)
$$\kappa \times \lambda = |S \times T|$$

c)
$$\kappa^\lambda = |S^T|$$

where S^T represents the set of all functions mapping T into S (as previously defined). That is, $|S|^{|T|} = |S^T|$. For convenience we define $0^\lambda = 0$ and $\kappa^0 = 1$.

Theorem 22.3 The class $\mathscr{C}$ of all cardinal numbers is a proper class.

VII Cardinal numbers 23 / Arithmetic of cardinal numbers.

Theorem 23.1 Addition on $\mathscr{C}$ is well-defined. That is, if S_1, S_2, T_1 and T_2 are sets such that $\kappa = |S_1| = |S_2|$ and $\lambda = |T_1| = |T_2|$, then $|S_1 \cup T_1| = \kappa + \lambda = |S_2 \cup T_2|$.

Theorem 23.2 Let κ, λ, ϕ and ψ be any four cardinal numbers. Then

a) $\kappa + \lambda = \lambda + \kappa$ (Commutativity of addition)
b) $(\kappa + \lambda) + \phi = \kappa + (\lambda + \phi)$ (Associativity of addition)
c) $\kappa \leq \kappa + \lambda$
d) $\kappa \leq \lambda$ and $\phi \leq \psi \Rightarrow \kappa + \phi \leq \lambda + \psi$.

Theorem 23.3 Multiplication on $\mathscr{C}$ is well-defined. That is, if S_1, S_2, T_1 and T_2 are sets such that $\kappa = |S_1| = |S_2|$ and $\lambda = |T_1| = |T_2|$, then $|S_1 \times T_1| = \kappa \times \lambda = |S_2 \times T_2|$.

Theorem 23.4 Let κ, λ, ϕ and ψ be any three cardinal numbers. Then

a) $\kappa \times \lambda = \lambda \times \kappa$ (Commutativity of multiplication)
b) $(\kappa \times \lambda) \times \phi = \kappa \times (\lambda \times \phi)$ (Associativity of multiplication)
c) $\kappa \times (\lambda + \phi) = (\kappa \times \lambda) + (\kappa \times \phi)$ (Left-hand distributivity)
d) $\lambda > 0 \Rightarrow \kappa \leq (\kappa \times \lambda)$
e) $\kappa \leq \lambda$ and $\phi \leq \psi \Rightarrow \kappa \times \phi \leq \lambda \times \psi$.
f) $\kappa + \kappa = 2 \times \kappa$.
g) $\kappa + \kappa \leq \kappa \times \kappa$ when $\kappa \geq 2$.

VII Cardinal numbers 24 / Exponentiation of cardinal numbers.

Theorem 24.1 Exponentiation on $\mathscr{C}$ is well-defined. That is, if S, S^*, T and T^* are sets such that $|S| = |S^*|$ and $|T| = |T^*|$, then $|S^T| = |S^{*T^*}|$.

Theorem 24.2 Let κ, λ and ϕ be any three cardinal numbers. Then

a) $\kappa^{\lambda+\phi} = \kappa^\lambda \times \kappa^\phi$
b) $(\kappa^\lambda)^\phi = \kappa^{\lambda\times\phi}$
c) $(\kappa \times \lambda)^\phi = \kappa^\phi \times \lambda^\phi$.

Theorem 24.3 Let κ, λ, and α be infinite cardinal numbers. Then

a) $\kappa \leq \kappa^\lambda$
b) $\alpha \leq \kappa \Rightarrow \alpha^\lambda \leq \kappa^\lambda$
c) $\alpha \leq \lambda \Rightarrow \kappa^\alpha \leq \kappa^\lambda$

VII Cardinal numbers 25 / Sets of cardinality c

Theorem 25.1 Let $\mathbb{C}$ denote the set of all complex numbers and $\mathbb{J}$ denote the set of all irrational numbers. Let n denote the cardinality of a non-empty finite set.

a) The cardinality of $\mathbb{R}^n$ is c.
b) The cardinality of $\mathbb{C}$ is c.
c) The cardinality of $\mathbb{J}$ is c.

Theorem 25.2

a) Let $\mathscr{S}_\mathbb{R}$ denote the set of all countably infinite sequences of real numbers. The cardinality of $\mathscr{S}_\mathbb{R}$ is c.
b) Let $\mathscr{S}_\mathbb{N}$ denote the set of all countably infinite sequences of natural numbers. The cardinality of $\mathscr{S}_\mathbb{N}$ is c.
c) Let $\mathbb{N}^\mathbb{N}_{(1-1)}$ denote the set of all one-to-one functions mapping $\mathbb{N}$ to $\mathbb{N}$. The cardinality of $\mathbb{N}^\mathbb{N}_{(1-1)}$ is c.
d) Let $\mathbb{R}^\mathbb{N}_{(1-1)}$ denote the set of all one-to-one functions mapping $\mathbb{N}$ to $\mathbb{R}$. Then the cardinality of $\mathbb{R}^\mathbb{N}_{(1-1)}$ is c.

Proposition 25.3 The Cantor set has cardinality c.

VIII Ordinal numbers 26 / Well-ordered sets.

Theorem 26.1 Let $f : T \to S$ be a one-to-one function mapping T onto S. If T is a well-ordered then T induces a well-ordering on S. In particular, every countable set can be well-ordered.

Definition 26.2 Given a *well-ordered* set $(S, \leq)$, a proper subset U satisfying the property

$$[u \in U \text{ and } x \leq u] \Rightarrow [x \in U]$$

is called an *initial segment* of S. In this definition the strict order relation $<$ can be used instead of $\leq$ without altering the meaning of "initial segment".

Theorem 26.3 If $(S, \leq)$ is a well-ordered set then every initial segment in S is of the form $S_a = \{x \in S : x < a\}$ for some $a \in S$.

Definition 26.4 Let $f : (S, \leq_S) \to (T, \leq_T)$ be a function mapping a well-ordered class, $(S, \leq_S)$, onto a well-ordered class, $(T, \leq_T)$. Note that the symbols $\leq_S$ and $\leq_T$ will allow us to distinguish between the order relations applied to each set S and T.

a) We will say that the function f is *increasing* on $(S, \leq_S)$ if

$$(x \leq_S \ y) \Rightarrow (f(x) \leq_T f(y))$$

b) We will say that the function f is *strictly increasing* on $(S, \leq_S)$ if

$$(x <_S \ y) \Rightarrow (f(x) <_T f(y))$$

A strictly increasing function must be one-to-one.

c) If $f : (S, \leq_S) \to (T, \leq_T)$ is strictly increasing then f is said to be an *order isomorphism* mapping S into T.

If there exists an *onto* order isomorphism between the two well-ordered classes, $(S, \leq_S)$ and $(T, \leq_T)$, we will say that the classes are *order isomorphic*, or that a function maps S *order isomorphically* onto T.

If there exists an *onto* order isomorphism between the two well-ordered classes $(S, \leq_S)$ and $(T, \leq_T)$ we will say that the classes are *order isomorphic* or that a function maps S *order isomorphically* onto T.

Theorem 26.5 Let $(S, \leq_S)$ and $(T, \leq_T)$ be a well-ordered sets.

a) The inverse of an order isomorphism is an order isomorphism.

b) If $f : (S, \leq_S) \to (S, \leq_S)$ is a *strictly increasing* function mapping S into itself then $f(x) \geq x$ for all $x \in S$.

c) The set S cannot be order isomorphic to an initial segment of itself.

d) If $f : (S, \leq_S) \to (S, \leq_S)$ is an order isomorphism then f is the identity function.

e) If $f : (S, \leq_S) \to (T, \leq_T)$ and $g : (S, \leq_S) \to (T, \leq_T)$ are order isomorphisms mapping S onto T then $f = g$.

f) Suppose $f : (S, \leq_S) \to (T, \leq_T)$ is an order isomorphism mapping S onto an initial segment of T. Then S and T cannot be order isomorphic.

Notation 26.6 Let S and T be two well-ordered sets. Then the expression

$$S \sim_{\text{wo}} T$$

means "S and T are order isomorphic". The expression

$$S <_{\text{wo}} T$$

means "$S \sim_{\text{wo}} T_a$" where T_a is some *initial segment* of T. The expression

$$S \leq_{\text{wo}} T$$

means "$S \sim_{\text{wo}} T$ or $S <_{\text{wo}} T$".

Theorem 26.7 Let $(S, \leq_S)$ and $(T, \leq_T)$ be two well-ordered sets. Then either $S \leq_{\text{wo}} T$ or $T \leq_{\text{wo}} S$.

Proposition 26.8 For every natural number n, the lexicographically ordered set $S = \{1, 2, \ldots, n\} \times \mathbb{N}$ is well-ordered.

VIII Ordinal numbers 27 / Ordinal numbers: Definition and properties.

Definition 27.1 Let S be a set. If S satisfies the two properties,

1) S a transitive set,
2) S is strictly $\in$-well-ordered

then S is called an *ordinal number*.

Notation 27.2 When viewed as an ordinal number, $\mathbb{N}$ will be represented by the lower-case Greek letter ω.

Theorem 27.3 If α is an ordinal number then so is its successor $\alpha^+ = \alpha \cup \{\alpha\}$.

Definition 27.4 Suppose the set S is $<$-ordered. We say that an element y in S is an *immediate successor* of the element x if $x < y$ and there does not exist any element z in S such that $x < z < y$. We say that x is an *immediate predecessor* of y if y is an immediate successor of x.

Theorem 27.5 Let α be an ordinal number greater than zero.

a) Every element x of the ordinal α is an initial segment of α.

b) The ordinal α is an initial segment of some ordinal.
c) Every initial segment x in α is an ordinal number.
d) Every element of the ordinal α is an ordinal number.

Proposition 27.6 Any infinite ordinal not equal to ω contains ω.

Proposition 27.7 Let α and β be distinct ordinal numbers. If $\alpha \subset \beta$, then $\alpha \in \beta$.

Lemma 27.8 If the ordinals α and β are order isomorphic, then $\alpha = \beta$.

Theorem 27.9 The relation "$\in$" linearly orders the class of all ordinals.

Definition 27.10 An ordinal α which does not contain a maximal element is called a *limit ordinal.*

Proposition 27.11 If U is a non-empty set of ordinals which contains a maximal element β with respect to $\in$, then the union, $\cup\{\alpha : \alpha \in U\}$, is equal to the maximal ordinal, β, of U.

Theorem 27.12 If U is a set of ordinals which does *not* contain a maximal element with respect to "$\in$", then $\gamma = \cup\{\alpha : \alpha \in U\}$ is a limit ordinal which is not contained in U. Furthermore, γ is the $\in$-least ordinal which contains all elements of U.

Corollary 27.13 Let U be a non-empty set of ordinals which contains no maximal element. If U satisfies the "initial segment property", then U is the limit ordinal $\cup\{\alpha : \alpha \in U\}$.

Definition 27.14 Let T be a non-empty subset of an ordered set $(S, <)$. If u is an upper bound of the set T and, for any other upper bound v of T, $u \leq v$, then we say that u is the *least upper bound* of T. We also abbreviate the expression by writing $u = \text{lub}\, T$ or $u = \text{lub}(T)$.

Theorem 27.15 Let γ be a non-zero ordinal number. The following are equivalent:

1) The ordinal γ is a limit ordinal.
2) The ordinal γ is such that $\gamma = \cup\{\alpha : \alpha \in \gamma\}$.
3) The ordinal γ is such that $\text{lub}(\gamma) = \gamma$.

VIII Ordinal numbers 28 / Properties of the class of ordinal numbers.

Theorem 28.1 The class, $\mathscr{O}$, of ordinal numbers is a strict $\in$-linearly ordered class.

Theorem 28.2 The class $\mathscr{O}$ of all ordinal numbers is $\in$-well-ordered.

Theorem 28.3 A set S is an initial segment of $\mathscr{O}$ if and only if S is an ordinal number.

Theorem 28.4 The class $\mathscr{O}$ of all ordinal numbers is not a set.

Theorem 28.5 Principle of transfinite induction. Let $\{x_\alpha : \alpha \in \mathscr{O}\}$ be a class whose elements are indexed by the ordinals. Let P denote a particular element property. Suppose $P(\alpha)$ means "the element x_α satisfies the property P". Suppose that, for any $\beta \in \mathscr{O}$,

$$\text{“}P(\alpha) \text{ is true } \forall\, \alpha \in \beta\text{” implies “}P(\beta) \text{ is true”}$$

Then $P(\alpha)$ holds true for all ordinals $\alpha \in \mathscr{O}$.

Corollary 28.6 Transfinite induction. Version 2. Let $\{x_\alpha : \alpha \in \mathscr{O}\}$ be a class whose elements are indexed by the ordinals. Let P denote a particular element property. Suppose $P(\alpha)$ means "the element x_α satisfies the property P". Suppose that:

1) $P(0)$ holds true,
2) $P(\alpha)$ holds true implies $P(\alpha + 1)$ holds true,
3) If β is a limit ordinal, "$P(\alpha)$ is true for all $\alpha \in \beta$ implies $P(\beta)$ is true".

Then $P(\alpha)$ holds true for all ordinals α.

Theorem 28.7 Let S be a $<$-well-ordered set. Then S is order isomorphic to some ordinal number $\alpha \in \mathscr{O}$. Furthermore this order isomorphism is unique.

Definition 28.8 Let S be a $<$-well-ordered set. If α is the unique ordinal which is order isomorphic to S then we will say that S is of *order type* α, or of *ordinality* α. of S is α. If S is of ordinality α, we will write ${}^{\text{ord}}S = \alpha$.

Lemma 28.9 Hartogs' lemma. Let S be any set. Then there exists an ordinal α which is not equipotent with S or any of its subsets.

Theorem 28.10 There exists an uncountable ordinal.

Corollary 28.11 The class $\omega_1 = \{\alpha \in \mathscr{O} : \alpha \text{ is a countable ordinal }\}$ is the least uncountable ordinal.

Definition 28.12 Let S be any set. Let

$$U = \{\alpha \in \mathscr{O} :\ \alpha \text{ not equipotent to any subset of } S\}$$

By Hartogs' lemma the class U is non-empty. Since $\mathscr{O}$ is $\in$-well-ordered, U contains a unique least ordinal $h(S)$. We will call the ordinal $h(S)$ the *Hartogs number of* S. Then h can be viewed as a class function which associates each set S in the class of all sets to a unique ordinal number α in the class of all ordinals $\mathscr{O}$.

Theorem 28.13 There exists a strictly increasing class $\{\omega_\alpha : \alpha \in \mathscr{O}\}$ of pairwise non-equipotent infinite ordinals all of which are uncountable except for $\omega_0 = \omega$.

Theorem 28.14 Let $\{\omega_\alpha : \alpha \in \mathscr{O}\}$ be the class of ordinals as defined in the previous theorem.

a) Every element of $\{\omega_\alpha : \alpha \in \mathscr{O}\}$ is a limit ordinal.

b) For every ordinal α, either $\alpha \in \omega_\alpha$ or $\alpha = \omega_\alpha$.

Theorem 28.15 The Transfinite recursion theorem. Let W be a well-ordered class and $f : W \to W$ be a class function mapping W into W. Let $u \in W$. Then there exists a unique class function $g : \mathscr{O} \to W$ which satisfies the following properties:

a) $g(0) = u$

b) $g(\alpha^+) = f(g(\alpha)),\ \forall \alpha \in \mathscr{O}$

c) $g(\beta) = \text{lub}\{g(\alpha) : \alpha \in \beta\},\ \forall$ limit ordinals β

VIII Ordinal numbers 29 / Initial ordinals: "Cardinal numbers are us!"

Definition 29.1 We say that β is an *initial ordinal* if it is the least of all ordinals equipotent with itself. That is, β is an *initial ordinal* if $\alpha \in \beta \Rightarrow \alpha \not\sim_e \beta$.

Lemma 29.2 The class of all initial ordinals is a subclass of $\mathscr{I}$.

Theorem 29.3 The class $\mathscr{I}$ is precisely the class of all initial ordinals.

Theorem 29.4 [AC] The Well-ordering theorem. Every set can be well-ordered.

Theorem 29.5 The class of all initial ordinals $\mathscr{I} = \omega_0 \cup \{\omega_\alpha : \alpha \in \mathscr{O}\}$ satisfies the following properties :

1. Every set S is equipotent to exactly one element in $\mathscr{I}$.
2. Two sets S and T are equipotent if and only if they are equipotent to the same element of $\mathscr{I}$.
3. The class $\mathscr{I}$ is $\in$-linearly ordered.

Definition 29.6 Cardinal numbers. An ordinal is called a *cardinal number* if and only if this ordinal is an initial ordinal. The class, $\mathscr{I}$, is also referred to as the class, $\mathscr{C}$, of all cardinal numbers.

Lemma 29.7 For any infinite cardinal number κ, define a relation $<_*$ on $\kappa \times \kappa$ as follows. For pairs (α, β) and (γ, ψ) of *ordinal* pairs in $\kappa \times \kappa$,

$$(\alpha, \beta) <_* (\gamma, \psi) \begin{cases} \alpha \cup \beta \in \gamma \cup \psi \\ \text{or} \\ \beta \in \psi \text{ when } \alpha \cup \beta = \gamma \cup \psi \\ \text{or} \\ \alpha \in \gamma \text{ when } \alpha \cup \beta = \gamma \cup \psi \text{ and } \beta = \psi \end{cases}$$

Then $<_*$ well-orders $\kappa \times \kappa$.

Theorem 29.8 [AC] For any ordinal α, $\aleph_\alpha \times \aleph_\alpha = \aleph_\alpha$.

Corollary 29.9 Let κ be an infinite cardinal and $\{A_\alpha : \alpha \in \beta\}$ be a set of non-empty sets indexed by the elements of the ordinal $\beta \in_= \kappa$ where $|A_\alpha| \in_= \kappa$ for all $\alpha \in \beta$. Then $|\cup \{A_\alpha : \alpha \in \beta\}| \in_= \kappa$.

Corollary 29.10 For any infinite cardinal number κ, $\kappa^\kappa = 2^\kappa$.

Definition 29.11

a) We say that an infinite cardinal number, $\aleph_\gamma$, is a *successor cardinal* if the index, γ, has an immediate predecessor (i.e., $\gamma = \beta + 1$, for some β). The expression, $\aleph_{\alpha^+} = \aleph_{\alpha+1}$, denotes a successor cardinal. We say that an infinite cardinal number, $\aleph_\gamma$, is a *limit cardinal* if γ is a limit ordinal (i.e., $\gamma = \text{lub}\{\alpha : \alpha \in \gamma\}$).

b) We say that a limit cardinal $\aleph_\gamma$ is a *strong limit cardinal* if $\aleph_\gamma$ is uncountable and $\{2^{\aleph_\alpha} : \alpha \in \gamma\} \subseteq \aleph_\gamma$.

Theorem 29.12 [GCH] Every uncountable limit cardinal is a strong limit cardinal.

Theorem 29.13 [AC] There exists a strong limit cardinal number.

Definition 29.14

a) We say that an infinite cardinal $\aleph_\gamma$ is a *singular cardinal number* if $\aleph_\gamma$ is the least upper bound of a strictly increasing sequence of ordinals, $\{\alpha_\kappa : \kappa \in \beta\}$, indexed by the elements of some ordinal β in $\aleph_\gamma$.

b) An infinite cardinal $\aleph_\gamma$ is said to be a *regular cardinal number* if it is not a singular cardinal number. That is, there does not exist an ordinal, β, in $\aleph_\gamma$ such that $\aleph_\gamma = \text{lub}\{\alpha_\kappa : \kappa \in \beta\}$.

Theorem 29.15 Every infinite successor cardinal, $\aleph_{\alpha^+} = \aleph_{\alpha+1}$, is a regular cardinal.

Theorem 29.16 Let γ be an infinite cardinal number. If γ is a singular cardinal then the cardinal number $\aleph_\gamma$ is a singular cardinal.

Definition 29.17 A regular cardinal number which a limit cardinal is called an *inaccessible cardinal.* A regular cardinal number which is a strong limit cardinal is called a *strongly inaccessible cardinal.*

Definition 29.18 Let $\aleph_\gamma$ be an infinite cardinal. We say the *cofinality of* $\aleph_\gamma$ is β and write $\text{cf}(\aleph_\gamma) = \beta$ if β is the least ordinal in $\aleph_\gamma$ which indexes an increasing set of ordinals $\{\theta_\alpha : \alpha < \beta\}$ such that $\aleph_\gamma = \text{lub}\{\theta_\alpha : \alpha < \beta\}$. If no such β exists in $\aleph_\gamma$ then we say the cofinality $\text{cf}(\aleph_\gamma)$ of is $\aleph_\gamma$ and write $\text{cf}(\aleph_\gamma) = \aleph_\gamma$.

Theorem 29.19 The cofinality $\mathrm{cf}(\aleph_\gamma)$ of an infinite cardinal number $\aleph_\gamma$ is a cardinal number. Hence $\mathrm{cf}(\aleph_\gamma)$ is the smallest cardinality of all sets which are cofinal in $\aleph_\gamma$.

Theorem 29.20 The cofinality $\mathrm{cf}(\varphi)$ of an infinite cardinal number φ is a regular cardinal.

Theorem 29.21 If κ is an infinite cardinal and $\mathrm{cf}(\kappa) \leq \lambda$, then $\kappa < \kappa^\lambda$.

IX More on axioms: Choice, regularity and Martin's axiom 30 / Axiom of choice

Theorem 30.1 Suppose $\mathscr{S}$ is a finite set of non-empty sets whose union is the set M. Then there exists a function $f : \mathscr{S} \to M$ which maps each set to one of its elements.

Theorem 30.2 Let AC* denote the statement:

"For any set $\mathscr{S} = \{S_\alpha : \alpha \in \gamma\}$ of non-empty sets, $\Pi_{\alpha\in\gamma} S_\alpha$ is non-empty."

The Axiom of choice holds true if and only if AC* holds true. The Axiom of choice holds true if and only if AC* holds true.

Theorem 30.3 The statement "Every set is well-orderable" holds true if and only if the Axiom of choice holds true.

Theorem 30.4 [AC] Any infinite set can be expressed as the union of a pairwise disjoint set of infinite countable sets.

Theorem 30.5 [AC] Let $(X, <)$ be a partially ordered set. If every chain of X has an upper bound then X has a maximal element.

Theorem 30.6 Suppose that those partially ordered sets $(X, <)$ in which every chain has an upper bound must have a maximal element. Then given any subset $\mathscr{S} \subseteq \mathscr{P}(S) - \varnothing$ there exists a choice function $f : \mathscr{S} \to S$ which maps each set in $\mathscr{S}$ to one of its elements.

Theorem 30.7 [ZL] Every vector space has a basis.

IX More on axioms: Choice, regularity and Martin's axiom 31 / Axiom of regularity and cumulative hierarchy.

Theorem 31.1 The Axiom of regularity holds true if and only if every non-empty set S contains a *minimal* element with respect to the membership relation "$\in$".

Theorem 31.2 [Axiom of regularity] No set is an element of itself.

Definition 31.3 We say that a class is *well-founded* if it does not contain an infinite descending chain of sets. That is, there does not exist an infinite sequence $\{x_n : n \in \omega\}$ such that $\cdots \in x_4 \in x_3 \in x_2 \in x_1 \in x_0$.

Theorem 31.4 [AC] The *Axiom of regularity* and the statement "*Every set is well-founded*" are equivalent statements.

Definition 31.5 Let x be a set. The transitive closure of x is a set t_x satisfying the following three properties:

1) The set t_x is a transitive set.
2) $x \subseteq t_x$
3) t_x is the $\subseteq$-least transitive set satisfying properties 1 and 2.

Theorem 31.6 Let x be a set. Then there exists a smallest transitive set t_x which contains all elements of x. That is, if s is a transitive set such that $x \subseteq s$, then $x \subseteq t_x \subseteq s$.

Definition 31.7 Define the class function $f : \mathscr{S} \to \mathscr{S}$ as $f(S) = \mathscr{P}(S)$. The elements of the class $\{V_\alpha : \alpha \in \mathscr{O}\}$ belong to the image of the class function $g(\alpha) = V_\alpha$ recursively defined as follows:

$$\begin{array}{rcccccl}
g(0) & = & V_0 & = & \varnothing & = & 0 \\
g(1) & = & V_1 & = & f(V_0) & = & \mathscr{P}(0) = \{\varnothing\} = 1 \\
g(2) & = & V_2 & = & f(V_1) & = & \mathscr{P}(1) = \{\{\varnothing\}, \varnothing\} = 2^1 = 2 \\
g(3) & = & V_3 & = & f(V_2) & = & \mathscr{P}(2) = \{\{\{\{\varnothing\}, \varnothing\}\}, \{\{\varnothing\}\}, \{\varnothing\}, \varnothing\} \\
g(4) & = & V_4 & = & f(V_3) & = & \mathscr{P}(4) \text{ (16 elements)} \\
 & \vdots & & \vdots & & & \\
g(\alpha^+) & = & V_{\alpha^+} & = & f(V_\alpha) & = & \mathscr{P}(V_\alpha) \\
 & \vdots & & \vdots & & & \\
\text{If } \lambda = \text{limit ordinal, } g(\lambda) & = & V_\lambda & = & \cup_{\alpha \in \lambda} V_\alpha & & \\
 & \vdots & & \vdots & & &
\end{array}$$

The class $\{V_\alpha : \alpha \in \mathscr{O}\}$ is called the *Cumulative hierarchy of sets*. We define $V = \cup_{\alpha \in \mathscr{O}} V_\alpha$.

Lemma 31.8 a) Each set V_α is a transitive set.

b) If $\alpha \in \beta$ then $V_\alpha \in V_\beta$. Hence $V_\alpha \subset V_\beta$.

Lemma 31.9 For any non-empty set B, $B \subset V$ implies $B \in V$.

Theorem 31.10 For every set x, $x \in V = \cup_{\alpha \in \mathscr{O}} V_\alpha$.

Theorem 31.11 Let $V = \cup_{\alpha \in \mathscr{O}} V_\alpha$ be the class of sets constructed as described above. If V contains all sets then every set has a $\in$-minimal element.

Theorem 31.12 Let $V = \cup_{\alpha \in \mathscr{O}} V_\alpha$ be the class of sets constructed as described above.

a) The rank of the empty set $\varnothing$ is zero.

b) If $U \in V_\beta$ then $\text{rank}(U) < \beta$; hence $U \notin V_{\text{rank}(U)}$ for all sets U. Conversely, $\text{rank}(U) < \beta \Rightarrow U \in V_\beta$.

c) If U and V are sets such that $U \in V$ then $\text{rank}(U) < \text{rank}(V)$.

d) If γ is an ordinal then $\text{rank}(\gamma) = \gamma$.

Proposition 31.13 Let β be a limit ordinal. If a and b are two elements of V_β. Then $\{a, b\}$ is an element of V_β. That is, V_β satisfies the property described by the Axiom of pair.

Proposition 31.14 Let β be any ordinal. If U is an element of V_β then $\cup\{x : x \in U\} \in V_\beta$. That is, V_β satisfies the property described by the Axiom of union.

Proposition 31.15 Let β be a limit ordinal. If U is an element of V_β then V_β also contains a set $Y = \mathscr{P}(U)$ such that $S \subseteq U$ implies $S \in Y$. That is, V_β satisfies the property described by the Axiom of power set.

Proposition 31.16 Let α be any ordinal. For any two elements x and y of V_α, if for all $z \in V_\alpha(z \in x \Leftrightarrow z \in y)$, then x and y are the same set.

Proposition 31.17 Let α be an ordinal number such that $\alpha > \omega_0$. Then $\omega_0 \in V_\alpha$.

Proposition 31.18 Let α be an ordinal number. Then the Axiom of subsets holds true in V_α.

Proposition 31.19 The set V_{ω_0} satisfies the property described by the Axiom of replacement.

Proposition 31.20 Let γ be a limit ordinal. Then the set V_γ satisfies the property described by the Axiom of choice.

Proposition 31.21 Let α be an ordinal. Then the set V_α satisfies the property described by the Axiom of construction.

IX More on axioms: Choice, regularity and Martin's axiom 32 / Martin's axiom.

Definition 32.1 Let $(P, \leq)$ be a partially ordered set. If P contains no uncountable strong antichain then $(P, \leq)$ is said to satisfy the *countable chain condition*. In this case, we say that $(P, \leq)$ satisfies the *ccc* or that $(P, \leq)$ is a *ccc* partial order.

Definition 32.2 Let $(P, \leq)$ be a partially ordered set. Let D be a subset of P such that for every element p in P there exists an element d in D such that $d \leq p$. A subset D satisfying this property is said to be *dense in the partial ordering* $(P, \leq)$.

Definition 32.3 Let F be a subset of a partially ordered set $(P, \leq)$. If F is non-empty and satisfies the two properties, 1) If x and y belong to F there exists z in F which is less than or equal to both x and y (i.e., F is a *filter base* or *downward directed*), 2) if x belongs to F and x is less than or equal to an element y of P, then y belongs to F (i.e., F is upward closed). A filter in $(P, \leq)$ is a *proper filter* if it is not all of P. If $x \in P$ then the set of all elements above x is called a *principal filter with principal element* x. Such a filter is the smallest filter which contains x.

Theorem 32.4 MA(κ): Let κ be an infinite cardinal and $(P, \leq)$ be a non-empty partially ordered set satisfying the *countable chain condition*. Let $\mathscr{D} = \{D \in \mathscr{P}(P) : D \text{ is dense in } P\}$ such that $|\mathscr{D}| \leq \kappa$. Then there is a proper filter $F \subseteq P$ such that, $F \cap D \neq \varnothing$ for every set $D \in \mathscr{D}$. The statement MA($\aleph_0$) holds true in *ZFC*.

Theorem 32.5 The statement MA($2^{\aleph_0}$) fails in *ZFC*.

Definition 32.6 *Martin's axiom*, MA, is defined as being MA(κ) where κ satisfies $\aleph_0 \leq \kappa < 2^{\aleph_0}$

Theorem 32.7 [MA] Suppose κ is an infinite cardinal such that $\kappa < 2^{\aleph_0}$. If X is a Hausdorff compact space with *ccc* and $\{U_\alpha : \alpha \leq \kappa\}$ is a family of open dense subsets of X then $\cap\{U_\alpha : \alpha\} \neq \varnothing$.

X Ordinal numbers arithmetic 33 / Addition.

Definition 33.1 Let $(S, \leq_S)$ and $(T, \leq_T)$ be two *disjoint* well-ordered sets. We define the relation "$\leq_{S\cup T}$" on $S \cup T$ as follows:

a) $u \leq_{S\cup T} v$ if $\{u, v\} \subseteq S$ and $u \leq_S v$.

b) $u \leq_{S\cup T} v$ if $\{u, v\} \subseteq T$ and $u \leq_T v$.

c) $u \leq_{S\cup T} v$ if $u \in S$, $v \in T$.

Theorem 33.2 Let $(S, \leq_S)$ and $(T, \leq_T)$ be two *disjoint* well-ordered sets. Then the relation $\leq_{S\cup T}$ well-orders the set $S \cup T$.

Definition 33.3 Let α and β be two ordinal numbers. Let $(S, \leq_S)$ and $(T, \leq_T)$ be two *disjoint* well-ordered sets of order type α and β respectively.[1] We define $\alpha + \beta$ as follows:

$$\alpha + \beta = {}^{\mathrm{ord}}(S \cup T,\ \leq_{S\cup T})$$

Theorem 33.4 Let $(S, \leq_S)$, $(T, \leq_T)$ and $(U, \leq_U)$, $(V, \leq_V)$ be two pairs of *disjoint* well-ordered sets such that

[1]Addition can also be defined inductively as follows: For all α and β, a) $\beta + 0 = \beta$, b) $\beta + (\alpha + 1) = (\beta + \alpha) + 1$, c) $\beta + \alpha = \text{lub}\{\beta + \gamma : \gamma < \alpha\}$ whenever α is a limit ordinal.

$$\begin{aligned} {}^{\text{ord}}S &= \alpha = {}^{\text{ord}}U \\ {}^{\text{ord}}T &= \beta = {}^{\text{ord}}V \end{aligned}$$

Then ${}^{\text{ord}}(S \cup T, \leq_{S\cup T}) = \alpha + \beta = {}^{\text{ord}}(U \cup V, \leq_{U\cup V})$. Hence addition of ordinal numbers is well-defined.

Theorem 33.5 Let α, β and γ be three ordinal numbers. Then:

a) $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ (Addition is associative.)
b) For any ordinal $\gamma > 0$, $\alpha < \alpha + \gamma$
c) For any ordinal γ, $\gamma \leq \alpha + \gamma$
d) $\alpha < \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$
e) $\alpha < \beta \Rightarrow \gamma + \alpha < \gamma + \beta$
f) $\alpha + \beta = \alpha + \gamma \Rightarrow \beta = \gamma$ (Left term cancellation is acceptable.)
g) $\alpha + 0 = \alpha$

Theorem 33.6 Let β be a limit ordinal. Then, for any ordinal, α,

$$\alpha + \beta = \sup\{\alpha + \gamma : \gamma < \beta\}$$

X Ordinal numbers arithmetic 34 / Multiplication

Definition 34.1 Let $(S, \leq_S)$ and $(T, \leq_T)$ be two well-ordered sets. We define the *lexicographic ordering* on the Cartesian product $S \times T$ as follows:

$$(s_1, t_1) \leq_{S\times T} (s_2, t_2) \text{ provided } \begin{cases} s_1 <_S s_2 \\ \quad\text{or} \\ s_1 = s_2 \quad \text{and} \quad t_1 \leq_T t_2 \end{cases}$$

Theorem 34.2 Let $(S, \leq_S)$ and $(T, \leq_T)$ be two well-ordered sets. The lexicographic ordering of the Cartesian product $S \times T$ is a well-ordering.

Theorem 34.3 If the well-ordered sets S_1 and S_2 are order isomorphic and the well-ordered sets T_1 and T_2 are order isomorphic then the lexicographically ordered Cartesian products $S_1 \times T_1$ and $S_2 \times T_2$ are order isomorphic.

Definition 34.4 Let α and β be two ordinals with set representatives A and B respectively. We define the multiplication $\alpha \times \beta$ as:

$$\alpha \times \beta = {}^{\text{ord}}(B \times A)$$

The product $\alpha \times \beta$ is equivalently written as $\alpha\beta$, (respecting the order). Note the order of the terms in the Cartesian product $B \times A$ is different from the order $\alpha \times \beta$ of their respective ordinalities.

Theorem 34.5 Let α, β and γ be three ordinal numbers. Then:

a) $(\gamma\beta)\alpha = \gamma(\beta\alpha)$ (Multiplication is associative.)
b) For any $\gamma > 0$, $\alpha < \beta \Rightarrow \gamma\alpha < \gamma\beta$
c) $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$ (Left-hand distribution is acceptable.)
d) For any $\gamma > 0$, $\gamma\alpha = \gamma\beta \Rightarrow \alpha = \beta$ (Left-hand cancellation is acceptable.)
e) $\gamma 0 = 0$
f) For any limit ordinal $\beta \neq 0$, $\alpha\beta = \sup\{\alpha\gamma : \gamma < \beta\}$

Definition 34.6 Let γ be any non-zero ordinal. We define the γ-based exponentiation function $g_\gamma : \mathscr{O} \to \mathscr{O}$ as follows:

1) $g_\gamma(0) = 1$
2) $g_\gamma(\alpha^+) = g_\gamma(\alpha)\gamma$
3) $g_\gamma(\alpha) = \text{lub}\{g_\gamma(\beta) : \beta < \alpha\}$ whenever α is a limit ordinal.

Whenever $\gamma \neq 0$ we represent $g_\gamma(\alpha)$ as γ^α. Then $\gamma^{\alpha+1} = \gamma^\alpha\gamma$. If $\gamma = 0$ we define $\gamma^\alpha = 0^\alpha = 0$.

Theorem 34.7 Let α, β and γ be three ordinal numbers. Then, assuming $\gamma > 1$,

$$\alpha < \beta \Leftrightarrow \gamma^\alpha < \gamma^\beta$$

Theorem 34.8 Let α, β and γ be three ordinal numbers where $\alpha \neq 0$.

a) $\gamma^\beta\gamma^\alpha = \gamma^{\beta+\alpha}$
b) $(\gamma^\beta)^\alpha = \gamma^{\beta\alpha}$

Appendix A / Boolean algebras and Martin's axiom.

Definition 0.1 A partially ordered set $(P, \leq)$ is called a *lattice* if $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ both exist in P for all pairs a, b in P.

Definition 0.2 If B is a subset of a partially ordered set, $(P, \leq)$, $\vee B$ denotes the least upper bound of B and $\wedge B$ denotes the greatest lower bound of B (both with respect to $\leq$). Note that $\vee B$ and $\wedge B$ may or may not be an element of B. A lattice $(P, \leq)$ is said to be a *complete lattice* if for any non-empty subset B of P, both $\vee B$ and $\wedge B$ exist and belong to P.

Definition 0.3 Let X be a topological space. A subset B is said to be *regular open in* X if $B = \text{int}_X(\text{cl}_X(B))$. The set of all regular open subsets of X will be denoted as $\mathscr{Ro}(X)$.

Theorem 0.4 Let X be a topological space. Then $(\mathscr{Ro}(X), \subseteq, \vee, \cap)$ is a complete lattice in $(\tau(X), \subseteq)$.

Definition 0.5 Let $(L, \leq, \vee, \wedge)$ be a lattice. An *L-ultrafilter* is a proper filter $\mathscr{F}$ in L which is not properly contained in any other proper filter in L. If the filter $\mathscr{F}$ is such that $\cap\{F : F \in \mathscr{F}\} \neq \varnothing$ then we say that the filter $\mathscr{F}$ is a *fixed ultrafilter*. Ultrafilters which are not fixed are said to be *free ultrafilters*.

Theorem 0.6 Let X be a topological space.

a) Suppose $\mathscr{F}$ is a proper L-filter where $(L, \subseteq, \vee, \wedge)$ is a lattice in $(\mathscr{P}(X), \subseteq)$. Then $\mathscr{F}$ can be extended to an L-ultrafilter.

b) Suppose $\mathscr{F}$ is an L-filter in $(L, \subseteq, \vee, \wedge)$ a lattice in $(\mathscr{P}(X), \subseteq)$. Then $\mathscr{F}$ is an L-ultrafilter if and only if for every $A \subseteq X$, either $A \in \mathscr{F}$ or $X - A \in \mathscr{F}$.

Theorem 0.7 Let X be a topological space. Then $(\mathscr{Ro}(X), \subseteq, \vee, \cap)$ is a complete lattice in $(\tau(X), \subseteq)$. An $\mathscr{Ro}(X)$-filter, $\mathscr{F}$, is an $\mathscr{Ro}(X)$-ultrafilter if and only if, for any $A \in \mathscr{Ro}(X)$, either A or $X - \mathrm{cl}_X(A)$ belongs to $\mathscr{F}$.

Definition 0.8 A lattice $(L, \vee, \wedge)$ is said to be a *distributive lattice* if, for any x, y, and z in L, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

- The lattice $(L, \vee, \wedge)$ is said to be a *complemented lattice* it has a maximum element, denoted by 1, and a minimum element, denoted by 0, and for every $x \in L$ there exists a unique x' such that $x \vee x' = 1$ and $x \wedge x' = 0$.
- A complemented distributive lattice is referred to as being a *Boolean algebra*. A Boolean algebra is denoted as $(B, \leq, \vee, \wedge, 0, 1, ')$ when we explicitly want to express what the maximum and minimum elements are.

Definition 0.9 Suppose we are given two lattices $(B_1, \leq_1, \vee_1, \wedge_1, 0, 1, ')$ and $(B_2, \leq_2, \vee_2, \wedge_2, 0, 1, ')$ and a function f which maps elements of B_1 to elements of B_2. We say that $f : B_1 \to B_2$ is a *Boolean homomorphism* if, for any $x, y \in B$,

1) $f(x \vee_1 y) = f(x) \vee_2 f(y)$,
2) $f(x \wedge_1 y) = f(x) \wedge_2 f(y)$
3) $f(x') = f(x)'$.

The function $f : B_1 \to B_2$ is a *Boolean isomorphism* if f is a bijection and both f and $f^{\leftarrow}$ are Boolean homomorphisms.

Definition 0.10 Let $(B, \leq, \vee, \wedge, 0, 1, ')$ be a Boolean algebra. Let $\mathscr{S}(B) = \{\mathscr{U} : \mathscr{U}$ is a B-ultrafilter$\}$. We define the function $f_B : B \to \mathscr{P}(\mathscr{S}(B))$ as follows: $f_B(x) = \{\mathscr{F} \in \mathscr{S}(B) : x \in \mathscr{U}\}$.

Theorem 0.11 Let $(B, \leq, \vee, \wedge, 0, 1,')$ be a Boolean algebra. Then the set $\{f_B(x) : x \in B\}$ is a base for the open sets of some topology, $\tau(\mathscr{S}(B))$, on the set $\mathscr{S}(B)$ of all B-ultrafilters.

Theorem 0.12 Let $(B, \leq, \vee, \wedge, 0, 1,')$ be a Boolean algebra.

1) The function $f_B : B \to \mathscr{P}(\mathscr{S}(B))$ is a Boolean homomorphism mapping B into $\mathscr{P}(\mathscr{S}(B))$.
2) For every $x \in B$, $f_B(x)$ is clopen in $\mathscr{S}(B)$. Hence $f_B[B] \subseteq \mathscr{B}(\mathscr{S}(B))$ (the set of all clopen sets in $\mathscr{S}(B)$).
3) The function $f_B : B \to \mathscr{S}(B)$ is a Boolean isomorphism mapping B into $\mathscr{B}(\mathscr{S}(B))$ (the set of all clopen sets in $\mathscr{S}(B)$).
4) The topological space $(\mathscr{S}(B), \tau(\mathscr{S}(B)))$ where $\tau(\mathscr{S}(B))$ is the topology generated by the open base $\{f_B(x) : x \in B\}$ is a compact zero-dimensional Hausdorff topological space.
5) The Boolean isomorphism $f_B : B \to \mathscr{S}(B)$ maps B onto $\mathscr{B}(\mathscr{S}(B))$ (the set of all clopen sets in $\mathscr{S}(B)$).

Theorem 0.13 Let κ be an infinite cardinal number such that $\kappa < 2^{\aleph_0}$. Then the following are equivalent:

1) (Martin's axiom MA) If $(P, \leq)$ is a partially ordered set satisfying *ccc* and $\mathscr{D} = \{D_\alpha : \alpha \leq \kappa\}$ is a family of dense subsets of P, then there exists a filter $\mathscr{F}$ on P such that $\mathscr{F} \cap D_\alpha \neq \varnothing$ for each $\alpha \leq \kappa$.
2) If X is a compact Hausdorff topological space satisfying *ccc* and $\mathscr{D} = \{D_\alpha : \alpha \leq \kappa\}$ is a family of dense open subsets of X, then $\cap\{D_\alpha : \alpha \leq \kappa\} \neq \varnothing$.
3) If $(B, \leq, \vee, \wedge, ')$ is a Boolean algebra with the *ccc* property and $\mathscr{D} = \{D_\alpha : \alpha \leq \kappa\}$ is a family of dense subsets of B, then there exists a filter $\mathscr{F}$ on B such that $\mathscr{F} \cap D_\alpha \neq \varnothing$ for each $\alpha \leq \kappa$.
4) If $(P, \leq)$ is a partially ordered set satisfying *ccc* and $|P| \leq \kappa$ and $\mathscr{D} = \{D_\alpha : \alpha \leq \kappa\}$ is a family of dense subsets of P, then there exist a filter $\mathscr{F}$ on P such that $\mathscr{F} \cap D_\alpha \neq \varnothing$ for each $\alpha \leq \kappa$.

Theorem 0.14 Let κ be a cardinal such that $\aleph_0 \leq \kappa < 2^{\aleph_0}$. Let X be a Hausdorff topological space satisfying *ccc* such that $\{x \in X : x \text{ has a compact neighbourhood}\}$ is dense in X. Suppose that $\mathscr{D} = \{D_\alpha : \alpha \leq \kappa\}$ is a family of dense open subsets of X. Then $\cap\{D_\alpha : \alpha \leq \kappa\}$ is dense in X if and only if Martin's axiom holds true.

Appendix C : Proof of Urysohn's extension theorem.

Theorem. *Urysohn's extension theorem*: Let T be a subset of the completely regular space S. Then T is C^*-embedded in S if and only if pairs of sets which can be completely separated by some function in $C^*(T)$ can also be separated by some function in $C^*(S)$.

Proof: ($\Rightarrow$) Suppose T is C^*-embedded in S and U and V are completely separated subsets of T. Then there exists $f \in C^*(T)$ such that $U \subseteq f^{\leftarrow}(0)$ and $V \subseteq f^{\leftarrow}(1)$. Then by hypothesis f extends to $f^* \in C^*(S)$. Then $U \subseteq f^{\leftarrow *}(0)$ and $V \subseteq f^{\leftarrow *}(1)$. So U and V are completely separated in S.

($\Leftarrow$) Suppose that pairs of sets which can be completely separated by some function in $C^*(T)$ can also be separated by some function in $C^*(S)$.

Let f_1 be a function in $C^*(T)$. We are required to show that there exists a function $g \in C^*(S)$ such that $g|_T = f_1$.

Since f_1 is bounded on T then there exists, $k \in \mathbb{R}$, such that $|f_1(x)| \leq k$ for all $x \in T$. Then $f_1 \leq k = 3r_1 = 3 \cdot \left[\frac{k}{2} \cdot \left(\frac{2}{3}\right)^1\right] = k$ Where $r_1 = \frac{k}{2} \cdot \left(\frac{2}{3}\right)^1$

We now inductively define a sequence of functions $\{f_n\} \subseteq C^*(T)$. For $n \in \mathbb{N}$, there exists $f_n \in C^*(T)$ such that $-3r_n \leq f_n(x) \leq 3r_n$ where,

$$3r_n = 3 \cdot \left[\frac{k}{2} \cdot \left(\frac{2}{3}\right)^n\right] = k \cdot \left(\frac{2}{3}\right)^{n-1} \quad \text{Where } r_n = \tfrac{k}{2} \cdot \left(\tfrac{2}{3}\right)^n$$

For this n, let $U_n = f^{\leftarrow}{}_n[\,[-3r_n, -r_n]\,]$ and $V_n = f^{\leftarrow}{}_n[\,[r_n, 3r_n]\,]$.

We see that U_n and V_n are completely separated in T. [1]

By hypothesis, U_n and V_n are completely separated in S. This means there exists $g_n \in C^*(S)$ such that, $g_n[S] \subseteq [-r_n, r_n]$, $g[U_n] = \{-r_n\}$ and $g[V_n] = \{r_n\}$. So the sequence $\{g_n\}$ is precisely defined in $C^*(S)$.

We now inductively define the sequence $\{h_n\} \subseteq C^*(T)$ initiating the process with $h_1 = f_1$ and continuing with

$$h_{n+1} = h_n - g_n|_T$$

Then for each n,

$$|h_{n+1}| \leq 2r_1 = 2 \cdot \frac{k}{2} \cdot \left(\frac{2}{3}\right)^n = 3 \cdot \frac{k}{2} \cdot \left(\frac{2}{3}\right)^{n+1} = 3r_{n+1}$$

[1]To see this: the function $h_n = (-r_n \vee f_n) \wedge r_n)$ has $U_n \subseteq Z(h_n - (-r_n)\,)$ and $V_n \subseteq Z(h_n - r_n)$.

So $g_n|T = h_n - h_{n+1}$. Define $g : S \to \mathbb{R}$ as the series

$$g(x) = \sum_{n \in \mathbb{N} \setminus \{0\}} g_n(x)$$

We claim that $g(x)$ is continuous on S and that $g|_T = f_1$. (If so then, f_1 extends continuously from T to S and we are done.)
See that,

$$\begin{aligned} g(x) = \lim_{m\to\infty} S_m(x) &= \lim_{m\to\infty} \sum_{n=1}^{m} g_n(x) \\ &= \lim_{m\to\infty} (h_1(x) - h_2(x)) + (h_2(x) - h_3(x)) + \cdots + (h_n(x) - h_{m+1}(x)) \\ &= \lim_{m\to\infty} h_1(x) - h_{m+1}(x) \\ &= h_1(x) = f_1(x) \quad \text{(Since } \lim_{m\to\infty} 3r_{m+1} = 0\text{)} \end{aligned}$$

Since $|g_n(x)| \leq \frac{k}{2}\frac{2}{3}^n$, and $\sum_{n\in\mathbb{N}\setminus\{0\}} \frac{k}{2}\frac{2}{3}^n$ is a converging geometric series then $\sum_{n\in\mathbb{N}\setminus\{0\}} g_n(x)$ converges uniformly to $g(x)$. Since each $g_n(x)$ is continuous on S then $g \in C^*S$. So g is a continuous extension of f_1.

Bibliography

(In alphabetical order based on the first letter of the family name.)

1. André, R., *Axioms and Set Theory*, Online, www.math.uwaterloo.ca/~randre/1aaset_theory_140613.pdf, (2015).

2. André, R., *When does the family of singular compactifications form a complete lattice?*, Rocky Mountain Journal of Mathematics, Volume 27, Number 4, Fall 1997.

3. Bell, M.G. and Ginsburg, J., *First countable Lindelöf extensions of uncountable discrete spaces*, Canad. Math. Bull. 23(1980), 397-399.

4. Caterino, A., Faulkner, G.D., Vipera, M.C., *Two applications of singular sets to the theory of compactifications.* Rendiconti Dell'Institue di Mat. dell'Univ. di Thieste.

5. Chandler, R.E., *Hausdorff compactifications*, Marcel Dekker, New York, 1976.

6. Chandler, R.E, Cain G.L., Faulkner, G.D. *Singular sets and remainders*, Trans. Amer. Math Soc. 268(1981) 161-171.

7. Chandler, R.E., Faulkner, G.D., *Singular compactifications: The order structure*, Proceedings of the American Mathematical Society, 100, number 2 (1987) 377-382

8. Comfort, W.W. and Negropontis, S., *The Theory of Ultrafilters*, Springer-Verlag, New York, 1974.

9. Comfort, W.W., *Retractions and other continuous maps, from βX onto $\beta X \backslash X$*, Trans. Amer. Math. Soc. 114 (1965)843-847

10. Devlin, K.J., *Fundamentals of Contemporary Set Theory*, Springer-Verlag, New York, 1974.

11. Diamond, B., *Algebraic subrings and perfect compactifications.*, Topology and its Applications, 39, (1991), 217-228.

12. Dugundji, J., *Topology*, Allyn and Bacon, Boston, Mass., 1967.

13. Enderton, H.B. *Elements of set theory*, Academic Press, Inc., 1977.

14. Engelking, R., *General Topology*, Polish Scientific Publishers, Warsaw, 1977.

15. Gillman, L., and Jerison, M., *Rings of continuous functions*, Princeton, Van Nostand, 1960.

16. Ginsburg, J and Saks, V., *Some applications of ultrafilters in topology*, Pac. J. Math 57(1975), 403-417.

17. Glicksburg, I., *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc. 90(1959), 369-362.

18. Hall, D.W. and Spencer, G.L., *Elementary topology*, John Wiley and Sons, Inc., New York, 1955.

19. Henrickson, M. and Isbell, J.r., *Some properties of compactifications*, Duke Math. J. 25(1958), 83-106.

20. Hrbacek, K., Jech, T. *Introduction to set theory*, Marcel Dekker, Inc., New York and Basel, 1984.

21. Isbell, J.R., *Zero-dimensional spaces*, Tohoku Math. J., 7(1955), 1-8.

22. Kelley, J.L., *General topology*, Springer-Verlag, New York, 1991.

23. Monk, J.D., *Introduction to set theory*, McGraw-Hill, New York, 1969.

24. Monkres, J.R., *Topology*, Prentice Hall Inc., 2000.

25. Ponomarev, V.I., *On paracompact spaces and their continuous mappings*, Dokl. Aked. Nauk SSSR, 143(1962), 46-49 - Soviet Math. Dokl, 3(1962), 347-350.

26. Porter, J. R., Woods, R. G. *Extensions and absolutes of Hausdorff spaces*, Springer-Verlag, 1988.

27. Rayburn, M.C., *On Hausdorff Compactifications*, Pac. J. Math., 44(1973), 707-714.

28. Roitman, J., *Introduction to modern set theory*, John Wiley & sons, New York, 1990.

29. Rudin, M.E., *The box product of countably many compact metric spaces*, General Topology and its Applications, 2(1972), 293-298.

30. Simmons, G.F. *Introduction to Topology and Modern Analysis*, Robert E. Krieger Publishing Company, Florida, 1983.

31. Steen, L.A., and Seebach, J.A. Jr, *Counterexamples in topology*, Springer-Verlag, 1986.

32. Stephenson, R. M., Jr., *Pseudocompact spaces*, Trans. Amer. Math. Soc., 134(1968), 437-448.

33. Thomas, J., *A regular space not completely regular*, Amer. Math. monthly, 76(1969), 181-182.

34. Thrivikraman, T., *On compactifications of Tychonoff spaces*, Yokohama Math. J., 20(1972), 99-106.

35. Walker, R.C., *The Stone-Čech Compactification*, Springer-Verlag, New York, 1974.

36. Willard, S. *General Topology*, Addison-Wesley Publishing Company, 1968.

37. Weese, M., Winfried, J. *Discovering Modern set theory*, American mathematical society, 1998.

ISBN 978-0-9938485-1-3

Index

ISBN978-0-9938485-1-3

ISBN 978-0-9938485-1-3

9 780993 848513 >