Contents

Chapter 1

Real Analysis

(1.5)
$OrderTrichotomy(<, S) := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$
$OrderTransitivity(<, S) := \forall_{x,y,z \in S}\big((x < y \wedge y < z) \implies x < z\big)$
$Order(<, S) := OrderTrichotomy(<, S) \wedge OrderTransitivity(<, S)$

(1.7)
$BoundedAbove(E, S, <) := Order(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(x \leq \beta)$
$BoundedBelow(E, S, <) := Order(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(\beta \leq x)$
$UpperBound(\beta, E, S, <) := Order(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(x \leq \beta)$
$LowerBound(\beta, E, S, <) := Order(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(\beta \leq x)$

(1.8)
$LUB(\alpha, E, S, <) := UpperBound(\alpha, E, S, <) \wedge \forall_{\gamma}\big(\gamma < \alpha \implies \neg UpperBound(\gamma, E, S, <)\big)$
$GLB(\alpha, E, S, <) := LowerBound(\alpha, E, S, <) \wedge \forall_{\beta}\big(\alpha < \beta \implies \neg LowerBound(\beta, E, S, <)\big)$

(1.10)
$LUBProperty(S, <) := \forall_E\Big(\big(\emptyset \neq E \subset S \wedge BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$
$GLBProperty(S, <) := \forall_E\Big(\big(\emptyset \neq E \subset S \wedge BoundedBelow(E, S, <)\big) \implies \exists_{\alpha \in S}\big(GLB(\alpha, E, S, <)\big)\Big)$

(1.11)
$\boxed{LUBPropertyImpliesGLBProperty}$ $\quad LUBProperty(S, <) \implies GLBProperty(S, <)$

(1)	$LUBProperty(S, <) \implies \ldots$	wts: 2
(1.1)	$\big(\emptyset \neq B \subset S \wedge BoundedBelow(B, S, <)\big) \implies \ldots$	wts: 1.2
(1.1.1)	$Order(<, S) \wedge \exists_{\delta' \in S}\big(LowerBound(\delta', B, S, <)\big)$	from: $BoundedBelow$, 1.1
(1.1.2)	$\lvert B\rvert = 1 \implies \ldots$	wts: 1.1.3
(1.1.2.1)	$\exists_{u'}(u' \in B) \;\blacksquare\; u := choice(\{u' \mid u' \in B\}) \;\blacksquare\; B = \{u\}$	from: 1.1.2
(1.1.2.2)	$GLB(u, B, S, <) \;\blacksquare\; \exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$	
(1.1.3)	$\lvert B\rvert = 1 \implies \exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$	
(1.1.4)	$\lvert B\rvert \neq 1 \implies \ldots$	wts: 1.1.5
(1.1.4.1)	$\forall_E\Big(\big(\emptyset \neq E \subset S \wedge BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$	from: $LUBProperty$, 1
(1.1.4.2)	$L := \{s \in S \mid LowerBound(s, B, S, <)\}$	
(1.1.4.3)	$\lvert B\rvert > 1 \wedge OrderTrichotomy(<, S) \;\blacksquare\; \exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$	from: $Order$, 1.1.1 wts: 1.1.4.7
(1.1.4.4)	$b_1 := choice\Big(\{b_1' \in B \mid \exists_{b_0' \in B}(b_0' < b_1')\}\Big) \;\blacksquare\; \neg LowerBound(b_1, B, S, <)$	from: 1.1.4.2
(1.1.4.5)	$b_1 \notin L \;\blacksquare\; L \subset S$	
(1.1.4.6)	$\delta := choice\big(\{\delta' \in S \mid LowerBound(\delta', B, S, <)\}\big) \;\blacksquare\; \delta \in L \;\blacksquare\; \emptyset \neq L$	from: 1.1.1
(1.1.4.7)	$\emptyset \neq L \subset S$	from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y\in L}\big(\mathit{LowerBound}(y,B,S,<)\big) \blacksquare \forall_{y\in L}\forall_{x\in B}(y\le x)$ from: *LowerBound*, 1.1.4.2 wts: 1.1.4.10

(1.1.4.9) $\forall_{x\in B}\Big(x\in S\wedge\forall_{y\in L}(y\le x)\Big) \blacksquare \forall_{x\in B}\big(\mathit{UpperBound}(x,L,S,<)\big)$ from: *UpperBound*

(1.1.4.10) $\exists_{x\in S}\big(\mathit{UpperBound}(x,L,S,<)\big) \blacksquare \mathit{BoundedAbove}(L,S,<)$

(1.1.4.11) $\emptyset\ne L\subset S\wedge\mathit{BoundedAbove}(L,S,<)$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha'\in S}\big(\mathit{LUB}(\alpha',L,S,<)\big) \blacksquare \alpha := choice\Big(\{\alpha'\in S|\big(\mathit{LUB}(\alpha',L,S,<)\big)\}\Big)$ from: 1.1.4.1 wts: 1.1.4.21

(1.1.4.13) $\forall_x\big(x\in B\implies\mathit{UpperBound}(x,L,S,<)\big)$ from: 1.1.4.9 wts: 1.1.4.17

(1.1.4.14) $\forall_x\big(\neg\mathit{UpperBound}(x,L,S,<)\implies x\notin B\big)$

(1.1.4.15) $\gamma<\alpha\implies\dots$ wts: 1.1.4.16

(1.1.4.15.1) $\neg\mathit{UpperBound}(\gamma,L,S,<) \blacksquare \gamma\notin B$ from: *LUB*, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma<\alpha\implies\gamma\notin B \blacksquare \gamma\in B\implies\gamma\ge\alpha$

(1.1.4.17) $\forall_{\gamma\in B}(\alpha\le\gamma) \blacksquare \mathit{LowerBound}(\alpha,B,S,<)$ from: *LowerBound*

(1.1.4.18) $\alpha<\beta\implies\dots$ wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y\in L}(y\le\alpha<\beta) \blacksquare \forall_{y\in L}(y\ne\beta)$ from: *LUB*, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta\notin L \blacksquare \neg\mathit{LowerBound}(\beta,B,S,<)$ from: 1.1.4.2

(1.1.4.19) $\alpha<\beta\implies\neg\mathit{LowerBound}(\beta,B,S,<) \blacksquare \forall_{\beta\in S}\big(\alpha<\beta\implies\neg\mathit{LowerBound}(\beta,B,S,<)\big)$

(1.1.4.20) $\mathit{LowerBound}(\alpha,B,S,<)\wedge\forall_{\beta\in S}\big(\alpha<\beta\implies\neg\mathit{LowerBound}(\beta,B,S,<)\big)$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $\mathit{GLB}(\alpha,B,S,<) \blacksquare \exists_{\epsilon_1\in S}\big(\mathit{GLB}(\epsilon_1,B,S,<)\big)$

(1.1.5) $|B|\ne1\implies\exists_{\epsilon_1\in S}\big(\mathit{GLB}(\epsilon_1,B,S,<)\big)$

(1.1.6) $\Big(|B|=1\implies\exists_{\epsilon_0\in S}\big(\mathit{GLB}(\epsilon_0,B,S,<)\big)\Big)\wedge\Big(|B|\ne1\implies\exists_{\epsilon_1\in S}\big(\mathit{GLB}(\epsilon_1,B,S,<)\big)\Big)$ from: 1.1.3, 1.1.5

(1.1.7) $(|B|=1\vee|B|\ne1)\implies\exists_{\epsilon\in S}\big(\mathit{GLB}(\epsilon,B,S,<)\big) \blacksquare \exists_{\epsilon\in S}\big(\mathit{GLB}(\epsilon,B,S,<)\big)$

(1.2) $(\emptyset\ne B\subset S\wedge\mathit{BoundedBelow}(B,S,<))\implies\exists_{\epsilon\in S}\big(\mathit{GLB}(\epsilon,B,S,<)\big)$

(1.3) $\forall_B\Big((\emptyset\ne B\subset S\wedge\mathit{BoundedBelow}(B,S,<))\implies\exists_{\epsilon\in S}\big(\mathit{GLB}(\epsilon,B,S,<)\big)\Big)$

(1.4) $\mathit{GLBProperty}(S,<)$

(2) $\mathit{LUBProperty}(S,<)\implies\mathit{GLBProperty}(S,<)$

(1.12)

$$\mathit{Field}(F,+,*) := \exists_{\mathbb{0},\mathbb{1}\in F}\forall_{x,y,z\in F}\left(\begin{array}{l} x+y\in F \quad\wedge\quad x*y\in F \quad\wedge\\ x+y=y+x \quad\wedge\quad x*y=y*x \quad\wedge\\ (x+y)+z=x+(y+z) \quad\wedge\quad (x*y)*z=x*(y*z) \quad\wedge\\ \mathbb{1}\ne\mathbb{0} \quad\wedge\quad x*(y+z)=(x*y)+(x*z) \quad\wedge\\ \mathbb{0}+x=x \quad\wedge\quad \mathbb{1}*x=x \quad\wedge\\ \exists_{-x\in F}\big(x+(-x)=\mathbb{0}\big)\wedge\Big(x\ne\mathbb{0}\implies\exists_{1/x\in F}\big(x*(1/x)=\mathbb{1}\big)\Big)\end{array}\right)$$

*********************** $\big(\mathit{Field}(F,+,*)\wedge x,y,z\in F\big)\implies\dots$ ***********************

(1.14)

$\boxed{\mathit{AdditiveCancellation}}$ $(x+y=x+z)\implies y=z$

(1) $y=\mathbb{0}+y=\big(x+(-x)\big)+y=\big((-x)+x\big)+y=(-x)+(x+y)=\dots$ from: *Field*

(2) $(-x)+(x+z)=\big((-x)+x\big)+z=\big(x+(-x)\big)+z=\mathbb{0}+z=z$ from: *Field*

$\boxed{\mathit{AdditiveIdentityUniqueness}}$ $(x+y=x)\implies y=\mathbb{0}$

(1) $x+y=x=\mathbb{0}+x=x+\mathbb{0}$ from: *Field*

(2) $y=0$ from: *AdditiveCancellation*

$\boxed{AdditiveInverseUniqueness}$ $(x+y=\mathbb{0}) \implies y=-x$

(1) $x+y=\mathbb{0}=x+(-x)$ — from: *Field*

(2) $y=-x$ — from: *AdditiveCancellation*

$\boxed{DoubleNegative}$ $x=-(-x)$

(1) $\mathbb{0}=x+(-x)=(-x)+x$ ▌ $\mathbb{0}=(-x)+x$ — from: *Field*

(2) $x=-(-x)$ — from: *AdditiveInverseUniqueness*

(1.15)

$\boxed{MultiplicativeCancellation}$ $(x\neq 0 \wedge x*y=x*z) \implies y=z$ —

$\boxed{MultiplicativeIdentityUniqueness}$ $(x\neq 0 \wedge x*y=x) \implies y=\mathbb{1}$ —

$\boxed{MultiplicativeInverseUniqueness}$ $(x\neq 0 \wedge x*y=\mathbb{1}) \implies y=1/x$ —

$\boxed{DoubleReciprocal}$ $(x\neq 0) \implies x=1/(1/x)$ —

(1.16)

$\boxed{Domination}$ $\mathbb{0}*x=\mathbb{0}$

(1) $\mathbb{0}*x=(\mathbb{0}+\mathbb{0})*x=\mathbb{0}*x+\mathbb{0}*x$ ▌ $\mathbb{0}*x=\mathbb{0}*x+\mathbb{0}*x$ — from: *Field*

(2) $\mathbb{0}*x=\mathbb{0}$ — from: *AdditiveIdentityUniqueness*

$\boxed{NonDomination}$ $(x\neq\mathbb{0}\wedge y\neq\mathbb{0}) \implies x*y\neq\mathbb{0}$

(1) $(x\neq\mathbb{0}\wedge y\neq\mathbb{0}) \implies \ldots$

(1.1) $(x*y=0) \implies \ldots$

(1.1.1) $\mathbb{1}=\mathbb{1}*\mathbb{1}=\big(x*(1/x)\big)*\big(y*(1/y)\big)=(x*y)*\big((1/x)*(1/y)\big)=\mathbb{0}*\big((1/x)*(1/y)\big)=\mathbb{0}$ — from: *Field*, *Domination*, 1, 1.1

(1.1.2) $\mathbb{1}=\mathbb{0}\wedge\mathbb{1}\neq\mathbb{0}$ ▌ $\perp$ — from: *Field*

(1.2) $(x*y=0) \implies \perp$ ▌ $x*y\neq 0$

(2) $(x\neq\mathbb{0}\wedge y\neq\mathbb{0}) \implies x*y\neq 0$

$\boxed{NegationCommutativity}$ $(-x)*y=-(x*y)=x*(-y)$

(1) $x*y+(-x)*y=(x+-x)*y=\mathbb{0}*y=\mathbb{0}$ ▌ $x*y+(-x)*y=\mathbb{0}$ — from: *Field*, *Domination*; wts: 2

(2) $(-x)*y=-(x*y)$ — from: *AdditiveInverseUniqueness*

(3) $x*y+x*(-y)=x*(y+-y)=x*\mathbb{0}=\mathbb{0}$ ▌ $x*y+x*(-y)=\mathbb{0}$ — from: *Field*, *Domination*; wts: 4

(4) $x*(-y)=-(x*y)$ — from: *AdditiveInverseUniqueness*

(5) $(-x)*y=-(x*y)=x*(-y)$ — from: 2, 4

$\boxed{NegativeMultiplication}$ $(-x)*(-y)=x*y$

(1) $(-x)*(-y)=-\big(x*(-y)\big)=-\big(-(x*y)\big)=x*y$ — from: *NegationCommutativity*, *DoubleNegative*

(1.17)

$$OrderedField(F,+,*,<) := \begin{pmatrix} Field(F,+,*) \quad \wedge \quad Order(<,F) \quad \wedge \\ \forall_{x,y,z\in F}(y<z \implies x+y<x+z) \quad \wedge \\ \forall_{x,y\in F}\big((x>\mathbb{0}\wedge y>\mathbb{0}) \implies x*y>\mathbb{0}\big) \end{pmatrix}$$

*********************** $\big(OrderedField(F,+,*,<)\wedge x,y,z\in F\big) \implies \ldots$ ***********************

(1.18)

$\boxed{NegationOnOrder}$ $x>\mathbb{0} \iff -x<\mathbb{0}$

(1) $x>\mathbb{0} \implies \ldots$

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x$ ▮ $\mathbb{0} > -x$ ▮ $-x < \mathbb{0}$ — from: *Ordered Field*

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \dots$

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x$ ▮ $\mathbb{0} < x$ ▮ $x > \mathbb{0}$ — from: *Ordered Field*

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \land -x < \mathbb{0} \implies x > \mathbb{0}$ ▮ $x > \mathbb{0} \iff -x < \mathbb{0}$ — from: 2, 4

PositiveFactorPreservesOrder $(x > \mathbb{0} \land y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \land y < z) \implies \dots$

(1.1) $(-y) + z > (-y) + y = \mathbb{0}$ ▮ $z + (-y) = \mathbb{0}$ — from: *Ordered Field*

(1.2) $x * (z + (-y)) > \mathbb{0}$ ▮ $x * z + x * (-y) > \mathbb{0}$ — from: *Ordered Field*

(1.3) $x * z = \mathbb{0} + x * z = \big(x * y + -(x * y)\big) + x * z = \big(x * y + x * (-y)\big) + x * z = \dots$ — from: *Field*, *NegationCommutativity*

(1.4) $x * y + \big(x * z + x * (-y)\big) > x * y + \mathbb{0} = x * y$ — from: *Field*, 1.2

(1.5) $x * z > x * y$ — from: 1.3, 1.4

(2) $(x > \mathbb{0} \land y < z) \implies x * z > x * y$

NegativeFactorFlipsOrder $(x < \mathbb{0} \land y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \land y < z) \implies \dots$

(1.1) $-x > \mathbb{0}$ — from: *NegationOnOrder*

(1.2) $(-x) * y < (-x) * z$ ▮ $\mathbb{0} = x * y + (-x) * y < x * y + (-x) * z$ ▮ $\mathbb{0} < x * y + (-x) * z$ — from: *PositiveFactorPreservesOrder*

(1.3) $\mathbb{0} < (-x) * (-y + z)$ ▮ $\mathbb{0} > x * (-y + z)$ ▮ $\mathbb{0} > -(x * y) + x * z$ — from: *NegationOnOrder*

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \land y < z) \implies x * y > x * z$

SquareIsPositive $(x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ — from: *Ordered Field*

(2) $(x < \mathbb{0}) \implies \dots$

(2.1) $-x > \mathbb{0}$ ▮ $x * x = (-x) * (-x) > \mathbb{0}$ ▮ $x * x > \mathbb{0}$ — from: *NegationOnOrder*, *Ordered Field*, *NegativeMultiplication*

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \lor x < \mathbb{0}) \implies x * x > \mathbb{0}$ ▮ $x \neq \mathbb{0} \implies x * x > \mathbb{0}$ — from: *OrderTrichotomy*, 1, 3

OneIsPositive $\mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0}$ ▮ $\mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ — from: *Field*, *SquareIsPositive*

ReciprocationOnOrder $(0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \dots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0}$ ▮ $x * (1/x) > 0$ — from: *Field*, *OneIsPositive*

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \land x * (1/x) > 0 \implies \bot$ ▮ $1/x > \mathbb{0}$ — from: *NegativeFactorFlipsOrder*, 1

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0}$ ▮ $y * (1/y) > 0$ — from: *Field*, *OneIsPositive*

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \land y * (1/y) > 0 \implies \bot$ ▮ $1/y > \mathbb{0}$ — from: *NegativeFactorFlipsOrder*, 1

(1.5) $(1/x) * (1/y) > \mathbb{0}$ — from: *Ordered Field*

(1.6) $\mathbb{0} < 1/y = \big((1/x) * (1/y)\big) * x < \big((1/x) * (1/y)\big) * y = 1/x$ — from: *Ordered Field*, 1, 1.4, 1.5

(1.19)
$\boxed{OrderedFieldQ}$ $OrderedField(\mathbb{Q},+,*,<)$ —

$Subfield(K,F,+,*) := Field(F,+,*) \land K \subset F \land Field(K,+,*)$
$OrderedSubfield(K,F,+,*,<) := OrderedField(F,+,*,<) \land K \subset F \land OrderedField(K,+,*,<)$
$CutI(\alpha) := \emptyset \neq \alpha \subset \mathbb{Q}$
$CutII(\alpha) := \forall_{p\in\alpha}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \alpha)$
$CutIII(\alpha) := \forall_{p\in\alpha}\exists_{r\in\alpha}(p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} | CutI(\alpha) \land CutII(\alpha) \land CutIII(\alpha)\}$

$\boxed{CutCorollaryI}$ $(\alpha \in \mathbb{R} \land p \in \alpha \land q \in \mathbb{Q} \land q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \land p \in \alpha \land q \in \mathbb{Q} \land q \notin \alpha) \implies \dots$
(1.1) $\forall_{p'\in\alpha}\forall_{q'\in\mathbb{Q}}(q' < p' \implies q' \in \alpha)$ from: $CutII$, 1
(1.2) $q < p \implies q \in \alpha$ ▌ $q \notin \alpha \implies q \geq p$ from: 1
(1.3) $(q \notin \alpha) \implies \dots$
(1.3.1) $q \geq p$ from: 1.2
(1.3.2) $(q = p) \implies (p \in \alpha \land p \notin \alpha) \implies \bot$ ▌ $q \neq p$ from: 1, 1.3
(1.3.3) $q \geq p \land q \neq p$ ▌ $p < q$
(1.4) $q \notin \alpha \implies p < q$ ▌ $p < q$ from: 1
(2) $(\alpha \in \mathbb{R} \land p \in \alpha \land q \in \mathbb{Q} \land q \notin \alpha) \implies p < q$

$\boxed{CutCorollaryII}$ $(\alpha \in \mathbb{R} \land r, s \in \mathbb{Q} \land r < s \land r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \land r, s \in \mathbb{Q} \land r < s \land r \notin \alpha) \implies \dots$
(1.1) $\forall_{s'\in\alpha}\forall_{r'\in\mathbb{Q}}(r' < s' \implies r' \in \alpha)$ from: $CutII$, 1
(1.2) $s \in \alpha \implies \big(r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)\big)$ ▌ $s \in \alpha \implies r \in \alpha$ from: 1, 1.1
(1.3) $r \notin \alpha \implies s \notin \alpha$ ▌ $s \notin \alpha$ from: 1, 1.2
(2) $(\alpha \in \mathbb{R} \land r, s \in \mathbb{Q} \land r < s \land r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \land \alpha \subset \beta$

$\boxed{OrderTrichotomyOfR}$ $OrderTrichotomy(\mathbb{R}, <_{\mathbb{R}})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \dots$
(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \lor \alpha = \beta) \implies \dots$
(1.1.1) $\alpha \not\subset \beta \land \alpha \neq \beta$ from: $<_{\mathbb{R}}$, 1.1
(1.1.2) $\exists_{p'}(p' \in \alpha \land p' \notin \beta)$ ▌ $p := choice(\{p' | p' \in \alpha \land p' \notin \beta\})$
(1.1.3) $q \in \beta \implies \dots$
(1.1.3.1) $p, q \in \mathbb{Q}$
(1.1.3.2) $q < p$ from: $CutCorollaryI$
(1.1.3.3) $q \in \alpha$ from: $CutII$
(1.1.4) $q \in \beta \implies q \in \alpha$
(1.1.5) $\forall_{q\in\beta}(q \in \alpha)$ ▌ $\beta \subseteq \alpha$
(1.1.6) $\beta \subset \alpha$ ▌ $\beta <_{\mathbb{R}} \alpha$
(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \lor \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$
(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \lor \alpha = \beta) \lor (\alpha <_{\mathbb{R}} \beta \lor \alpha = \beta)$ ▌ $(\beta <_{\mathbb{R}} \alpha) \lor (\alpha <_{\mathbb{R}} \beta \lor \alpha = \beta)$
(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \lor \beta <_{\mathbb{R}} \alpha)$
(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \lor \beta <_{\mathbb{R}} \alpha)$
(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \lor \alpha <_{\mathbb{R}} \beta)$
(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$
(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha,\beta\in\mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $OrderTrichotomy(\mathbb{R}, <_{\mathbb{R}})$

$\boxed{OrderTransitivityOfR}$ $OrderTransitivity(\mathbb{R}, <_{\mathbb{R}})$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a\in\alpha}(a \in \beta) \wedge \forall_{b\in\beta}(b \in \gamma)$

(1.1.3) $\forall_{a\in\alpha}(a \in \gamma)$ ▌ $\alpha \subset \gamma$ ▌ $\alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(3) $\forall_{\alpha,\beta,\gamma\in\mathbb{R}}\big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(4) $OrderTransitivity(\mathbb{R}, <_{\mathbb{R}})$

from: $OrderTrichotomyR$, $OrderTransitivityR$
wts:

$\boxed{OrderOfR}$ $Order(<_{\mathbb{R}}, \mathbb{R})$

$\boxed{LUBPropertyOfR}$ $LUBProperty(\mathbb{R}, <_{\mathbb{R}})$

(1) $\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} | \exists_{\alpha\in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset$ ▌ $\exists_{\alpha}(\alpha \in A)$ ▌ $\alpha_0 := choice(\{\alpha | \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset$ ▌ $\exists_a(a \in \alpha_0)$ ▌ $a_0 := choice(\{a | a \in \alpha_0\})$ ▌ $a_0 \in \gamma$ ▌ $\gamma \neq \emptyset$

(1.4) $BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})$ ▌ $\exists_{\beta}\big(UpperBound(\beta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.5) $\beta_0 := choice(\{\beta | UpperBound(\beta, A, \mathbb{R}, <_{\mathbb{R}})\})$

(1.6) $UpperBound(\beta_0, A, \mathbb{R}, <_{\mathbb{R}})$ ▌ $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \beta_0)$ ▌ $\forall_{\alpha\in A}(\alpha \subseteq \beta_0)$ ▌ $\forall_{\alpha\in A}\forall_{a\in\alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma$ ▌ $\forall_{a\in\gamma}(a \in \beta_0)$ ▌ $\gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q}$ ▌ $\gamma \subseteq \beta_0 \subset \mathbb{Q}$ ▌ $\gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q}$ ▌ $CutI(\gamma)$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.10.1) $p \in \gamma$ ▌ $\exists_{\alpha\in A}(p \in \alpha)$ ▌ $\alpha_1 := choice(\{\alpha \in A | p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p$ ▌ $q \in \alpha_1$ ▌ $q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma$ ▌ $\forall_{p\in\gamma}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \gamma)$ ▌ $CutII(\gamma)$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha\in A}(p \in \alpha)$ ▌ $\alpha_2 := choice(\{\alpha \in A | p \in \alpha\})$

(1.12.2) $\alpha_2 \in \mathbb{R}$ ▌ $CutII(\alpha_2)$ ▌ $\exists_{r\in\alpha_2}(p < r)$ ▌ $r_0 := choice(\{r \in \alpha_2 | p < r\})$

(1.12.3) $r_0 \in \alpha_2$ ▌ $r_0 \in \gamma$

(1.12.4) $p < r_0$ ▌ $p < r_0 \wedge r_0 \in \gamma$ ▌ $\exists_{r\in\gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r\in\gamma}(p < r)$ ▌ $\forall_{p\in\gamma}\exists_{r\in\gamma}(p < r)$ ▌ $CutIII(\gamma)$

(1.14) $CutI(\gamma) \wedge CutII(\gamma) \wedge CutIII(\gamma)$ ▌ $\gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha\in A}(\alpha \subseteq \gamma)$ ▌ $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma)$

(1.16) $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R}$ ▌ $UpperBound(\gamma, A, \mathbb{R}, <_{\mathbb{R}})$

(1.17) $\delta <_{\mathbb{R}} \gamma \implies \ldots$

(1.17.1) $\delta \subset \gamma$ ▌ $\exists_s(s \in \gamma \wedge s \notin \delta)$ ▌ $s_0 := choice(\{s \in \mathbb{Q} | s \in \gamma \wedge s \notin \delta\})$

(1.17.2) $s_0 \in \gamma$ ▌ $\exists_{\alpha\in A}(s_0 \in \alpha)$ ▌ $\alpha_3 := choice(\{\alpha \in A | s_0 \in \alpha\})$

(1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta$ ▌ $\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \ldots$

(1.17.4.1) $\alpha_3 \subseteq \delta$ ▌ $\forall_{s\in\mathbb{Q}}(s \in \alpha_3 \implies s \in \delta)$ ▌ $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4.2) $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$ ▌ $\perp$

(1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \perp$ ▌ $\delta <_{\mathbb{R}} \alpha_3$ ▌ $\exists_{\alpha\in A}(\delta <_{\mathbb{R}} \alpha)$ ▌ $\exists_{\alpha\in A}\big(\neg(\alpha \leq_{\mathbb{R}} \delta)\big)$

(1.17.6) $\neg\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \delta)$ ▌ $\neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}}))$ ▌ $\forall_{\delta}\big(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.19) $UpperBound(\gamma, A, \mathbb{R}, <_{\mathbb{R}}) \wedge \forall_{\delta}\big(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.20) $LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})$ ▌ $\exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(2) $\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(3) $\forall_A\Big(\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)\Big)$ ▌ $LUBProperty(\mathbb{R}, <_{\mathbb{R}})$

$+_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$
$0_{\mathbb{R}} := \{x \in \mathbb{Q} | x < 0\}$

$\boxed{0InR}$ $0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}}$ ▌ $\emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q}$ ▌ $CutI(0_{\mathbb{R}})$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}}$ ▌ $\forall_{x \in 0_{\mathbb{R}}}\forall_{y \in \mathbb{Q}}(y < x \implies y \in 0_{\mathbb{R}})$ ▌ $CutII(0_{\mathbb{R}})$

(3) $y := x/2$ ▌ $(x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $\forall_{x \in 0_{\mathbb{R}}}\exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $CutIII(0_{\mathbb{R}})$

(4) $CutI(0_{\mathbb{R}}) \wedge CutII(0_{\mathbb{R}}) \wedge CutIII(0_{\mathbb{R}})$ ▌ $0_{\mathbb{R}} \in \mathbb{R}$

$\boxed{FieldAdditionClosureOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha)$; $\exists_b(b \in \beta)$ ▌ $a_0 := choice(\{a | a \in \alpha\})$; $b_0 := choice(\{b | b \in \beta\})$ ▌ $a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha)$; $\exists_y(y \notin \beta)$ ▌ $x_0 := choice(\{x | x \notin \alpha\})$; $y_0 := choice(\{y | y \notin \beta\})$

(1.5) $\forall_{r \in \alpha}(r < x_0)$; $\forall_{s \in \beta}(s < y_0)$ ▌ $\forall_{r \in \alpha}\forall_{s \in \beta}(r + s < x_0 + y_0)$ ▌ $x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q}$ ▌ $CutI(\alpha +_{\mathbb{R}} \beta)$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.7.1) $\exists_{r \in \alpha}\exists_{s \in \beta}(p = r + s)$ ▌ $(r_0, s_0) := choice((r, s) \in \alpha \times \beta | p = r + s)$

(1.7.2) $q < p = r_0 + s_0$ ▌ $(q - s_0) < r_0$ ▌ $(q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta$ ▌ $q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta$ ▌ $q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} \beta}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta)$ ▌ $CutII(\alpha +_{\mathbb{R}} \beta)$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r \in \alpha}\exists_{s \in \beta}(p = r + s)$ ▌ $(r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta | p = r + s\})$

(1.9.2) $r_1 \in \alpha$ ▌ $\exists_{t \in \alpha}(r_1 < t)$ ▌ $t_0 := choice(\{t \in \alpha | r_1 < t\})$

(1.9.3) $s_1 \in \beta$ ▌ $t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1$ ▌ $\exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} \beta}\exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▌ $CutIII(\alpha +_{\mathbb{R}} \beta)$

(1.11) $CutI(\alpha +_{\mathbb{R}} \beta) \wedge CutII(\alpha +_{\mathbb{R}} \beta) \wedge CutIII(\alpha +_{\mathbb{R}} \beta)$ ▌ $\alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

$\boxed{FieldAdditionCommutativityOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s | r \in \alpha \wedge s \in \beta\} = \{s + r | s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

$\boxed{FieldAdditionAssociativityOfR}$ $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)\big)$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

$\boxed{FieldAdditionIdentityOfR}$ $(\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \implies \ldots$

(1.1.1) $s < 0 \;\blacksquare\; r + s < r + 0 = r \;\blacksquare\; r + s < r \;\blacksquare\; r + s \in \alpha$

(1.2) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \implies r + s \in \alpha \;\blacksquare\; \forall_{r\in\alpha}\forall_{s\in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \;\blacksquare\; \forall_{p\in\alpha+_{\mathbb{R}}0_{\mathbb{R}}}(p \in \alpha) \;\blacksquare\; \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r\in\alpha}(p < r) \;\blacksquare\; r_2 := choice(\{r \in \alpha | p < r\})$

(1.4.2) $p < r_2 \;\blacksquare\; p - r_2 < r_2 - r_2 = 0 \;\blacksquare\; (p - r_2) < 0 \;\blacksquare\; (p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha \;\blacksquare\; p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; \forall_{p\in\alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \;\blacksquare\; \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \land \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

$\boxed{FieldAdditionInverseOfR}$ $(\alpha \in \mathbb{R}) \implies \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} | \exists_{r>0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q} \;\blacksquare\; \exists_{s\in\mathbb{Q}}(s \notin \alpha) \;\blacksquare\; s_0 := choice(\{s | s \notin \alpha\}) \;\blacksquare\; p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha \;\blacksquare\; -p_0 - 1 \notin \alpha \;\blacksquare\; \exists_{r>0}(-p_0 - r \notin \alpha) \;\blacksquare\; p_0 \in \beta$

(1.4) $\emptyset \neq \alpha \;\blacksquare\; \exists_{q\in\alpha} \;\blacksquare\; q_0 := choice(\{q \in \mathbb{Q} | q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha \;\blacksquare\; -(-q_0) - r = q_0 - r < q_0 \;\blacksquare\; -(-q_0) - r < q_0 \;\blacksquare\; -(-q_0) - r \in \alpha$

(1.6) $\forall_{r>0}\big(-(-q_0) - r \in \alpha\big) \;\blacksquare\; \neg\exists_{r>0}(-(-q_0) - r \notin \alpha) \;\blacksquare\; -q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q} \;\blacksquare\; CutI(\beta)$

(1.8) $(p \in \beta \land q \in \mathbb{Q} \land q < p) \implies \ldots$

(1.8.1) $p \in \beta \;\blacksquare\; \exists_{r>0}(-p - r \notin \alpha) \;\blacksquare\; r_0 := choice(\{r > 0 | -p - r \notin \alpha\})$

(1.8.2) $q < p \;\blacksquare\; -p - r < -q - r$

(1.8.3) $-q - r \notin \alpha \;\blacksquare\; q \in \beta$

(1.9) $(p \in \beta \land q \in \mathbb{Q} \land q < p) \implies q \in \beta \;\blacksquare\; \forall_{p\in\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \beta) \;\blacksquare\; CutII(\beta)$

(1.10) $p \in \beta \implies \ldots$

(1.10.1) $p \in \beta \;\blacksquare\; \exists_{r>0}(-p - r \notin \alpha) \;\blacksquare\; r_1 := choice(\{r > 0 | -p - r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0 \;\blacksquare\; r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p \;\blacksquare\; t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha \;\blacksquare\; -t_0 - (r_1/2) \notin \alpha \;\blacksquare\; \exists_{r>0}(-t_0 - r \notin \alpha) \;\blacksquare\; t_0 \in \beta$

(1.10.7) $t_0 > p \land t_0 \in \beta \;\blacksquare\; \exists_{t\in\beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t\in\beta}(p < t) \;\blacksquare\; \forall_{p\in\beta}\exists_{t\in\beta}(p < t) \;\blacksquare\; CutIII(\beta)$

(1.12) $CutI(\beta) \land CutII(\beta) \land CutIII(\beta) \;\blacksquare\; \beta \in \mathbb{R}$

(1.13) $(r \in \alpha \land s \in \beta) \implies \ldots$

(1.13.1) $s \in \beta \;\blacksquare\; \exists_{t>0}(-s - t \notin \alpha) \;\blacksquare\; t_1 := choice(\{t > 0 | -s - t \notin \alpha\}) \;\blacksquare\; -s - t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \land s, t_1 \in \mathbb{Q} \land -s - t_1 < -s \land -s - t_1 \notin \alpha \;\blacksquare\; -s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \land r \in \alpha \land -s \notin \alpha \;\blacksquare\; r < -s \;\blacksquare\; r + s < 0 \;\blacksquare\; r + s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \land s \in \beta) \implies r + s \in 0_{\mathbb{R}} \;\blacksquare\; \forall_{(r,s)\in\alpha\times\beta}(r + s \in 0_{\mathbb{R}}) \;\blacksquare\; \alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \ldots$

(1.15.1) $v < 0 \;\blacksquare\; w_0 := -v/2 \;\blacksquare\; w > 0$

(1.15.2) $\exists_{n\in\mathbb{Z}}(nw_0 \in \alpha \land (n + 1)w_0 \notin \alpha) \;\blacksquare\; n_0 := choice(\{n \in \mathbb{Z} | nw_0 \in \alpha \land (n + 1)w_0 \notin \alpha\})$ from: $ARCHIMEDEANPROPERTYOFQ + LUB???$

(1.15.3) $p_0 := -(n_0 + 2)w_0 \;\blacksquare\; -p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha \;\blacksquare\; -p_0 - w_0 \notin \alpha \;\blacksquare\; p_0 \in \beta$

(1.15.4) $n_0 w_0 \in \alpha \land p_0 \in \beta \;\blacksquare\; n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{v \in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta) \blacksquare 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta \blacksquare \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}} \blacksquare \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha \in \mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

$*_{\mathbb{R}}(\alpha, \beta) :=$ —

$1_{\mathbb{R}} := \{x \in \mathbb{Q} | x < 1\}$

$\boxed{1IsNot0}$ $0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ —

$\boxed{1InR}$ $1_{\mathbb{R}} \in \mathbb{R}$ —

$\boxed{FieldMultiplicationClosureOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R}\big)$ —

$\boxed{FieldMultiplicationCommutativityOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ —

$\boxed{FieldMultiplicationAssociativityOfR}$ $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)\big)$ —

$\boxed{FieldMultiplicationIdentityOfR}$ $(\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ —

$\boxed{FieldMultiplicationInverseOfR}$ $(\alpha \in \mathbb{R}) \implies \exists_{1/\alpha \in \mathbb{R}}\big(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}\big)$ —

$\boxed{FieldDistributativityOfR}$ $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ —

$\boxed{FieldWithR}$ $Field(\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}})$ —

$\boxed{OrderedFieldWithR}$ $OrderedField(\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}})$ —

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} | r < q\} | q \in \mathbb{Q}\}$

$\boxed{QROrderedSubfieldOfR}$ $OrderedSubfield(\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}})$ —

$\boxed{QIsomorphicToQR}$ $\mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ —

$\boxed{ExistenceOfR}$ $\exists_{\mathbb{R}}\big(LUBProperty(\mathbb{R}, <_{\mathbb{R}}) \wedge OrderedSubfield(\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}})\big)$ —

(1.20)

$\boxed{ArchimedeanPropertyOfR}$ $\forall_{x,y \in \mathbb{R}}\big(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y)\big)$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx | n \in \mathbb{N}^+\} \blacksquare \emptyset \neq A \subset \mathbb{R}$

(1.2) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \blacksquare \forall_{n \in \mathbb{N}^+}(nx \leq y)$

(1.2.2) $UpperBound(y, A, \mathbb{R}, <) \blacksquare BoundedAbove(A, \mathbb{R}, <)$

(1.2.3) $LUBProperty(\mathbb{R}, <) \wedge \emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <) \blacksquare \exists_{\alpha \in \mathbb{R}}(LUB(\alpha, A, \mathbb{R}, <))$

(1.2.4) $\alpha_0 := choice(\{\alpha \in \mathbb{R} | LUB(\alpha, A, \mathbb{R}, <)\})$

(1.2.5) $x > 0 \blacksquare \alpha_0 - x < \alpha_0$

(1.2.6) $LUB(\alpha_0, A, \mathbb{R}, <) \wedge \alpha_0 - x < \alpha_0 \blacksquare \neg UpperBound(\alpha_0 - x, A, \mathbb{R}, <)$

(1.2.7) $\exists_{c \in A}(\alpha_0 - x < c) \blacksquare c_0 := choice(\{c \in A | \alpha_0 - x < c\})$

(1.2.8) $c_0 \in A \blacksquare \exists_{m \in \mathbb{N}^+}(mx = c_0) \blacksquare m_0 := choice(\{m \in \mathbb{N}^+ | mx = c_0\})$

(1.2.9) $\alpha_0 - x < c_0 = m_0 x \blacksquare \alpha_0 - x < m_0 x \blacksquare \alpha_0 < (m_0 + 1)x$

(1.2.10) $m_0 + 1 \in \mathbb{N}^+ \blacksquare (m_0 + 1)x \in A$

(1.2.11) $\alpha_0 < (m_0 + 1)x \wedge (m_0 + 1)x \in A \blacksquare \exists_{c \in A}(\alpha_0 < c)$

(1.2.12) $LUB(\alpha_0, A, \mathbb{R}, <) \blacksquare UpperBound(\alpha_0, A, \mathbb{R}, <) \blacksquare \forall_{c \in A}(c \leq \alpha_0) \blacksquare \neg\exists_{c \in A}(\alpha_0 < c)$

(1.2.13) $\exists_{c \in A}(\alpha_0 < c) \wedge \neg\exists_{c \in A}(\alpha_0 < c) \blacksquare \perp$

(1.3) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \perp \blacksquare \exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y) \blacksquare \forall_{x,y \in \mathbb{R}}\big(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y)\big)$

$\boxed{QDenseInR}$ $\forall_{x,y \in \mathbb{R}}\Big(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y)\Big)$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y \blacksquare y - x > 0 \blacksquare \exists_{n \in \mathbb{N}^+}(n(y - x) > 1) \blacksquare n_0 := choice(\{n \in \mathbb{N}^+ | n(y - x) > 1\})$

(1.2)	$1 > 0$ $\blacksquare$ $\exists_{m \in \mathbb{N}^+}(m(1) > n_0 x)$ $\blacksquare$ $m_1 := choice(\{m \in \mathbb{N}^+ \vert m(1) > n_0 x\})$ $\blacksquare$ $m_1 > n_0 x$
(1.3)	$\exists_{m \in \mathbb{N}^+}(m(1) > -n_0 x)$ $\blacksquare$ $m_2 := choice(\{m \in \mathbb{N}^+ \vert m(1) > -n_0 x\})$ $\blacksquare$ $m_2 > -n_0 x$ $\blacksquare$ $-m_2 < n_0 x$
(1.4)	$-m_2 < n_0 x < m_1$ $\blacksquare$ $\vert m_1 - (-m_2) \vert \geq 2$
(1.5)	$\exists_{m \in \mathbb{Z}}(-m_2 < n_0 x < m_1 \wedge m - 1 \leq n_0 x)$ $\blacksquare$ $m_0 := choice(\{m \in \mathbb{Z} \vert - m_2 < n_0 x < m_1 \wedge m - 1 \leq n_0 x\})$
(1.6)	123123
(2)	123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \land Relation(f, X \times Y) \land \forall_{x \in X} \exists!_{y \in Y}((x, y) \in f)$

$(Function(f, X, Y) \land A \subseteq X \land B \subseteq Y) \implies \ldots$
(1) $Domain(f) := X; Codomain(f) := Y$
(2) $Image(f, A) := \{f(a) \mid a \in A\}; Preimage(f, B) := \{a \mid f(a) \in B\}$
(3) $Range(f) := Image(Domain(f))$

$Injective(f, X, Y) := Function(f, X, Y) \land \forall_{x_1, x_2 \in X}(x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$
$Surjective(f, X, Y) := Function(f, X, Y) \land \forall_{y \in Y} \exists_{x \in X}(y = f(x))$
$Bijective(f, X, Y) := Injective(f, X, Y) \land Surjective(f, X, Y)$

TODO: Definition properties Surjective Equivalent: $(Range(f) = Codomain(f)) \implies Surjective(f)$

Chapter 3

Linear Algebra

$A_{square} = B_{sym} + C_{skewsym}$ $A + A^T = B_{sym}$ $A - A^T = B_{skewsym}$ $A = (1/2)(A + A^T) + (1/2)(A - A^T)$ $(AB)^{-1} = B^{-1}A^{-1}$
$AB * \dots \dots \dots$
$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
$A^1 23123$
TODO PROOFS