0.1 Problem Set 1

1.) Prove Proposition 6: For any $a, b \in \mathbb{R}$, $\begin{pmatrix} a \vee b = \frac{1}{2}(a+b+|a-b|) \\ a \wedge b = \frac{1}{2}(a+b-|a-b|) \end{pmatrix}$.

Let $a, b \in \mathbb{R}$. Suppose $a \geq b$. Clearly, $a \vee b = a$ and $a \wedge b = b$. Furthermore, $|a-b| = a-b$ since $a-b \geq 0$. Thus,

$$a \vee b = a = \frac{1}{2}\big(a+b+(a-b)\big) = \frac{1}{2}(a+b+|a-b|) \tag{1}$$

$$a \wedge b = b = \frac{1}{2}\big(a+b-(a-b)\big) = \frac{1}{2}(a+b-|a-b|). \tag{2}$$

Suppose $a < b$. Clearly, $a \vee b = b$ and $a \wedge b = a$. Furthermore, $|a-b| = -(a-b)$ since $a-b < 0$. Thus,

$$a \vee b = b = \frac{1}{2}\Big(a+b+\big(-(a-b)\big)\Big) = \frac{1}{2}(a+b+|a-b|) \tag{3}$$

$$a \wedge b = a = \frac{1}{2}\Big(a+b-\big(-(a-b)\big)\Big) = \frac{1}{2}(a+b-|a-b|). \tag{4}$$

In either case, $a \vee b = \frac{1}{2}(a+b+|a-b|)$ and $a \wedge b = \frac{1}{2}(a+b-|a-b|)$ holds. ■

2.) Prove Proposition 7: For any $a, b, r \in \mathbb{R}$, $\begin{pmatrix} a \vee b = b \vee a \\ a \wedge b = b \wedge a \\ (a \wedge b \leq r \leq a \vee b) \implies \big((|r-a| \leq |a-b|) \wedge (r-b \leq |a-b|)\big) \end{pmatrix}$.

Let $a, b, r \in \mathbb{R}$. The first two statements immediately follow by applying the commutativity of real numbers and $|a-b| = |-(a-b)| = |b-a|$ to Proposition 6.

Suppose $a \wedge b \leq r \leq a \vee b$. Without loss of generality, let $a \geq b$. Thus,

$$b \leq r \leq a \tag{5}$$

$$r - a \leq 0 \tag{6}$$

$$b - r \leq 0 \tag{7}$$

$$b - a \leq 0 \tag{8}$$

From (5), $r-a \geq b-a$. This along with (6) and (8) implies $|r-a| = -(r-a) \leq -(b-a) = |b-a|$.
From (5), $r-b \leq a-b$. This along with (7) and (8) implies $|r-b| = r-b \leq a-b = |a-b|$. ■