Solutions Manual to Walter Rudin's *Principles of Mathematical Analysis*

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Chapter 1

The Real and Complex Number Systems

Exercise 1.1 If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution. If r and $r + x$ were both rational, then $x = r + x - r$ would also be rational. Similarly if rx were rational, then $x = \frac{rx}{r}$ would also be rational.

Exercise 1.2 Prove that there is no rational number whose square is 12.

First Solution. Since $\sqrt{12} = 2\sqrt{3}$, we can invoke the previous problem and prove that $\sqrt{3}$ is irrational. If m and n are integers having no common factor and such that $m^2 = 3n^2$, then m is divisible by 3 (since if m^2 is divisible by 3, so is m). Let $m = 3k$. Then $m^2 = 9k^2$, and we have $3k^2 = n^2$. It then follows that n is also divisible by 3 contradicting the assumption that m and n have no common factor.

Second Solution. Suppose $m^2 = 12n^2$, where m and n have no common factor. It follows that m must be even, and therefore n must be odd. Let $m = 2r$. Then we have $r^2 = 3n^2$, so that r is also odd. Let $r = 2s + 1$ and $n = 2t + 1$. Then

$$4s^2 + 4s + 1 = 3(4t^2 + 4t + 1) = 12t^2 + 12t + 3,$$

so that

$$4(s^2 + s - 3t^2 - 3t) = 2.$$

But this is absurd, since 2 cannot be a multiple of 4.

Exercise 1.3 Prove Proposition 1.15, i.e., prove the following statements:

(*a*) *If* $x \neq 0$ *and* $xy = xz$, *then* $y = z$.
(*b*) *If* $x \neq 0$ *and* $xy = x$, *then* $y = 1$.
(*c*) *If* $x \neq 0$ *and* $xy = 1$, *then* $y = 1/x$.
(*d*) *If* $x \neq 0$, *then* $1/(1/x) = x$.

Solution. (*a*) Suppose $x \neq 0$ and $xy = xz$. By Axiom (M5) there exists an element $1/x$ such that $1/x = 1$. By (M3) and (M4) we have $(1/x)(xy) = ((1/x)x)y = 1y = y$, and similarly $(1/x)(xz) = z$. Hence $y = z$.

(*b*) Apply (*a*) with $z = 1$.

(*c*) Apply (*a*) with $z = 1/x$.

(*d*) Apply (*a*) with x replaced by $1/x$, $y = 1/(1/x)$, and $z = x$.

Exercise 1.4 Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E, and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution. Since E is nonempty, there exists $x \in E$. Then by definition of lower and upper bounds we have $\alpha \leq x \leq \beta$, and hence by property *ii* in the definition of an ordering, we have $\alpha < \beta$ unless $\alpha = x = \beta$.

Exercise 1.5 Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Solution: We need to prove that $-\sup(-A)$ is the greatest lower bound of A. For brevity, let $\alpha = -\sup(-A)$. We need to show that $\alpha \leq x$ for all $x \in A$ and $\alpha \geq \beta$ if β is any lower bound of A.

Suppose $x \in A$. Then, $-x \in -A$, and, hence $-x \leq \sup(-A)$. It follows that $x \geq -\sup(-A)$, i.e., $\alpha \leq x$. Thus α is a lower bound of A.

Now let β be any lower bound of A. This means $\beta \leq x$ for all x in A. Hence $-x \leq -\beta$ for all $x \in A$, which says $y \leq -\beta$ for all $y \in -A$. This means $-\beta$ is an upper bound of $-A$. Hence $-\beta \geq \sup(-A)$ by definition of sup, i.e., $\beta \leq -\sup(-A)$, and so $-\sup(-A)$ is the greatest lower bound of A.

Exercise 1.6 Fix $b > 1$.

(*a*) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

(*b*) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Solution. (a) Let $k = mq = np$. Since there is only one positive real number c such that $c^{nq} = b^k$ (Theorem 1.21), if we prove that both $(b^m)^{1/n}$ and $(b^p)^{1/q}$ have this property, it will follow that they are equal. The proof is then a routine computation: $\left((b^m)^{1/n}\right)^{nq} = (b^m)^q = b^{mq} = b^k$, and similarly for $(b^p)^{1/q}$.

(b) Let $r = \frac{m}{n}$ and $s = \frac{v}{w}$. Then $r + s = \frac{mw+vn}{nw}$, and

$$b^{r+s} = (b^{mw+vn})^{1/nw} = \left((b^{mw}b^{vn})\right)^{1/nw},$$

by the laws of exponents for integer exponents. By the corollary to Theorem 1.21 we then have

$$b^{r+s} = (b^{mw})^{1/nw}(b^{nv})^{1/nw} = b^r b^s,$$

where the last equality follows from part (a).

(c) It will simplify things later on if we amend the definition of $B(x)$ slightly, by defining it as $\{b^t : t \text{ rational}, t < x\}$. It is then slightly more difficult to prove that $b^r = \sup B(r)$ if r is rational, but the technique of Problem 7 comes to our rescue. Here is how: It is obvious that b^r is an upper bound of $B(r)$. We need to show that it is the least upper bound. The inequality $b^{1/n} < t$ if $n > (b-1)/(t-1)$ is proved just as in Problem 7 below. It follows that if $0 < x < b^r$, there exists an integer n with $b^{1/n} < b^r/x$, i.e., $x < b^{r-1/n} \in B(r)$. Hence x is not an upper bound of $B(r)$, and so b^r is the least upper bound.

(d) By definition $b^{x+y} = \sup B(x+y)$, where $B(x+y)$ is the set of all numbers b^t with t rational and $t < x + y$. Now any rational number t that is less than $x + y$ can be written as $r + s$, where r and s are rational, $r < x$, and $s < y$. To do this, let r be any rational number satisfying $t - y < r < x$, and let $s = t - r$. Conversely any pair of rational numbers r, s with $r < x$, $s < y$ gives a rational sum $t = r + s < x + y$. Hence $B(x+y)$ can be described as the set of all numbers $b^r b^s$ with $r < x$, $s < y$, and r and s rational, i.e., $B(x+y)$ is the set of all products uv, where $u \in B(x)$ and $v \in B(y)$.

Since any such product is less than $\sup B(x) \sup B(y)$, we see that the number $M = \sup B(x) \sup B(y)$ is an upper bound for $B(x+y)$. On the other hand, suppose $0 < c < \sup B(x) \sup B(y)$. Then $c/(\sup B(x)) < \sup B(y)$. Let $m = (1/2)\left(c/(\sup B(x)) + \sup B(y)\right)$. Then $c/\sup B(x) < m < \sup B(y)$, and there exist $u \in B(x)$, $v \in B(y)$ such that $c/m < u$ and $m < v$. Hence we have

$c = (c/m)m < uv \in B(x+y)$, and so c is not an upper bound for $B(x+y)$. It follows that $\sup B(x) \sup B(y)$ is the least upper bound of $B(x+y)$, i.e.,

$$b^{x+y} = b^x b^y,$$

as required.

Exercise 1.7 Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b*.)

(*a*) For any positive integer n, $b^n - 1 \geq n(b-1)$.

(*b*) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(*c*) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

(*d*) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this apply part (*c*) with $t = y \cdot b^{-w}$.

(*e*) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

(*f*) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^w = y$.

(*g*) Prove that this x is unique.

Solution. (*a*) The inequality $b^n - 1 \geq n(b-1)$ is equality if $n = 1$. Then, by induction $b^{n+1} - 1 = b^{n+1} - b + (b-1) = b(b^n - 1) + (b-1) \geq bn(b-1) + (b-1) = (bn+1)(b-1) \geq (n+1)(b-1)$.

(*b*) Replace b by $b^{1/n}$ in part (*a*).

(*c*) The inequality $n > (b-1)/(t-1)$ can be rewritten as $n(t-1) > (b-1)$, and since $b - 1 \geq n(b^{1/n} - 1)$, we have $n(t-1) > n(b^{1/n} - 1)$, which implies $t > b^{1/n}$.

(*d*) The application of part (*c*) with $t = y \cdot b^{-w} > 1$ is immediate.

(*e*) The application of part (*c*) with $t = b^w \cdot (1/y)$ yields the result, as in part (*d*) above.

(*f*) There are only three possibilities for the number $x = \sup A$: 1) $b^x < y$; 2) $b^x > y$; 3) $b^x = y$. The first assumption, by part (*d*), implies that $x + (1/n) \in A$ for large n, contradicting the assumption that x is an upper bound for A. The second, by part (*e*), implies that $x - (1/n)$ is an upper bound for A if n is large, contradicting the assumption that x is the smallest upper bound. Hence the only remaining possibility is that $b^x = y$.

(*g*) Suppose $z \neq x$, say $z > x$. Then $b^z = b^{x+(z-x)} = b^x b^{z-x} > b^x = y$. Hence x is unique. (It is easy to see that $b^w > 1$ if $w > 0$, since there is a positive rational number $r = \frac{m}{n}$ with $0 < r < w$, and $b^r = (b^m)^{1/n}$. Then $b^m > 1$ since $b > 1$, and $(b^m)^{1/n} > 1$ since $1^n = 1 < b^m$.)

Exercise 1.8 Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

Solution. By Part (a) of Proposition 1.18, either i or $-i$ must be positive. Hence $-1 = i^2 = (-i)^2$ must be positive. But then $1 = (-1)^2$, must also be positive, and this contradicts Part (a) of Proposition 1.18, since 1 and -1 cannot both be positive.

Exercise 1.9 Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least upper bound property?

Solution. We need to show that either $z < w$ or $z = w$, or $w < z$. Now since the *real* numbers are ordered, we have $a < c$ or $a = c$, or $c < a$. In the first case $z < w$; in the third case $w < z$. Now consider the second case. We must have $b < d$ or $b = d$ or $d < b$. In the first of these cases $z < w$, in the third case $w < z$, and in the second case $z = w$.

We also need to show that if $z < w$ and $w < u$, then $z < u$. Let $u = e + fi$. Since $z < w$, we have either $a < c$ or $a = c$ and $b < d$. Since $w < u$ we have either $c < f$ or $c = f$ and $d < g$. Hence there are four possible cases:

Case 1: $a < c$ and $c < f$. Then $a < f$ and so $z < u$, as required.
Case 2: $a < c$ and $c = f$ and $d < g$. Again $a < f$, and $z < u$.
Case 3: $a = c$ and $b < d$ and $c < f$. Once again $a < f$ and so $z < u$.
Case 4: $a = c$ and $b < d$ and $c = f$, and $d < g$. Then $a = f$ and $b < g$, and so $z < u$.

Exercise 1.10 Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \quad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception) has two complex square roots.

Solution.

$$z^2 = (a + bi)^2 = (a^2 - b^2) + 2abi.$$

Now

$$a^2 - b^2 = \frac{|w| + u}{2} - \frac{|w| - u}{2} = u,$$

and, since $(xy)^{1/2} = x^{1/2}y^{1/2}$,

$$2ab = 2\left(\frac{|w| + u}{2}\frac{|w| - u}{2}\right)^{1/2} = 2\left(\frac{|w|^2 - u^2}{4}\right)^{1/2}.$$

Hence

$$2ab = 2\left(\left(\frac{v}{2}\right)^2\right)^{1/2}$$

Now $(x^2)^{1/2} = x$ if $x \geq 0$ and $(x^2)^{1/2} = -x$ if $x \leq 0$. We conclude that $2ab = v$ if $v \geq 0$ and $2ab = -v$ if $v \leq 0$. Hence $z^2 = w$ if $v \geq 0$. Replacing b by $-b$, we find that $(\bar{z})^2 = w$ if $v \leq 0$.

Hence every non-zero complex number has (at least) two complex square roots.

Exercise 1.11 If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?

Solution. If $z = 0$, we take $r = 0$, $w = 1$. (In this case w is not unique.) Otherwise we take $r = |z|$ and $w = z/|z|$, and these choices are unique, since if $z = rw$, we must have $r = r|w| = |rw| = |z|$, z/r.

Exercise 1.12 If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Solution. The case $n = 2$ is Part (e) of Theorem 1.33. We can then apply this result and induction on n to get

$$\begin{aligned} |z_1 + z_2 + \cdots z_n| &= |(z_1 + z_2 + \cdots + z_{n-1}) + z_n| \\ &\leq |z_1 + z_2 + \cdots + z_{n-1}| + |z_n| \\ &\leq |z_1| + |z_2| + \cdots + |z_{n-1}| + |z_n|. \end{aligned}$$

Exercise 1.13 If x, y are complex, prove that

$$\big||x| - |y|\big| \leq |x - y|.$$

Solution. Since $x = x - y + y$, the triangle inequality gives

$$|x| \leq |x - y| + |y|,$$

so that $|x| - |y| \leq |x - y|$. Similarly $|y| - |x| \leq |x - y|$. Since $|x| - |y|$ is a real number we have either $\big||x| - |y|\big| = |x| - |y|$ or $\big||x| - |y|\big| = |y| - |x|$. In either case, we have shown that $\big||x| - |y|\big| \leq |x - y|$.

Exercise 1.14 If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1+z|^2 + |1-z|^2.$$

Solution. $|1+z|^2 = (1+z)(1+\bar{z}) = 1 + \bar{z} + z + z\bar{z} = 2 + z + \bar{z}$. Similarly $|1-z|^2 = (1-z)(1-\bar{z}) = 1 - z - \bar{z} + z\bar{z} = 2 - z - \bar{z}$. Hence

$$|1+z|^2 + |1-z|^2 = 4.$$

Exercise 1.15 Under what conditions does equality hold in the Schwarz inequality?

Solution. The proof of Theorem 1.35 shows that equality can hold if $B = 0$ or if $Ba_j - Cb_j = 0$ for all j, i.e., the numbers a_j are proportional to the numbers b_j. (In terms of linear algebra this means the vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ in complex n-dimensional space are linearly dependent. Conversely, if these vectors are linearly independent, then strict inequality holds.)

Exercise 1.16 Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in R^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

(a) If $2r > d$, there are infinitely many $\mathbf{z} \in R^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

(b) If $2r = d$, there is exactly one such $\mathbf{z}$.

(c) If $2r < d$, there is no such $\mathbf{z}$.

How must these statements be modified if k is 2 or 1?

Solution. (a) Let $\mathbf{w}$ be any vector satisfying the following two equations:

$$\begin{aligned} \mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) &= 0, \\ |\mathbf{w}|^2 &= r^2 - \frac{d^2}{4}. \end{aligned}$$

From linear algebra it is known that all but one of the components of a solution $\mathbf{w}$ of the first equation can be arbitrary. The remaining component is then uniquely determined. Also, if $\mathbf{w}$ is any non-zero solution of the first equation, there is a unique positive number t such that $t\mathbf{w}$ satisfies both equations. (For example, if $x_1 \neq y_1$, the first equation is satisfied whenever

$$z_1 = \frac{z_2(x_2 - y_2) + \cdots + z_k(x_k - y_k)}{y_1 - x_1}.$$

If $(z_1, z_2, \ldots, z_k)$ satisfies this equation, so does $(tz_1, tz_2, \ldots, tz_k)$ for any real number t.) Since at least two of these components can vary independently, we can find a solution with these components having any prescribed ratio. This ratio does not change when we multiply by the positive number t to obtain a solution of both equations. Since there are infinitely many ratios, there are infinitely many distinct solutions. For each such solution $\mathbf{w}$ the vector $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} + \mathbf{w}$ is a solution of the required equation. For

$$\begin{aligned}
|\mathbf{z}-\mathbf{x}|^2 &= \left|\frac{\mathbf{y}-\mathbf{x}}{2} + \mathbf{w}\right|^2 \\
&= \left|\frac{\mathbf{y}-\mathbf{x}}{2}\right|^2 + 2\mathbf{w}\cdot\frac{\mathbf{x}-\mathbf{y}}{2} + |\mathbf{w}|^2 \\
&= \frac{d^2}{4} + 0 + r^2 - \frac{d^2}{4} \\
&= r^2,
\end{aligned}$$

and a similar relation holds for $|\mathbf{z}-\mathbf{y}|^2$.

(b) The proof of the triangle inequality shows that equality can hold in this inequality only if it holds in the Schwarz inequality, i.e., one of the two vectors is a scalar multiple of the other. Further examination of the proof shows that the scalar must be nonnegative. Now the conditions of this part of the problem show that

$$|\mathbf{x}-\mathbf{y}| = d = |\mathbf{x}-\mathbf{z}| + |\mathbf{z}-\mathbf{y}|.$$

Hence it follows that there is a nonnegative scalar t such that

$$\mathbf{x}-\mathbf{z} = t(\mathbf{z}-\mathbf{y}).$$

However, the hypothesis also shows immediately that $t = 1$, and so $\mathbf{z}$ is uniquely determined as

$$\mathbf{z} = \frac{\mathbf{x}+\mathbf{y}}{2}.$$

(c) If $\mathbf{z}$ were to satisfy this condition, the triangle inequality would be violated, i.e., we would have

$$|\mathbf{x}-\mathbf{y}| = d > 2r = |\mathbf{x}-\mathbf{z}| + |\mathbf{z}-\mathbf{y}|.$$

When $k = 2$, there are precisely 2 solutions in case (a). When $k = 1$, there are no solutions in case (a). The conclusions in cases (b) and (c) do not require modification.

Exercise 1.17 Prove that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$. Interpret this geometrically as a statement about parallelograms.

Solution. The proof is a routine computation, using the relation

$$|\mathbf{x}\pm\mathbf{y}|^2 = (\mathbf{x}\pm\mathbf{y})\cdot(\mathbf{x}\pm\mathbf{y}) = |\mathbf{x}|^2 \pm 2\mathbf{x}\cdot\mathbf{y} + |\mathbf{y}|^2.$$

If $\mathbf{x}$ and $\mathbf{y}$ are the sides of a parallelogram, then $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are its diagonals. Hence this result says that the sum of the squares on the diagonals of a parallelogram equals the sum of the squares on the sides.

Exercise 1.18 If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

Solution. If $\mathbf{x}$ has any components equal to 0, then $\mathbf{y}$ can be taken to have the corresponding components equal to 1 and all others equal to 0. If all the components of $\mathbf{x}$ are nonzero, $\mathbf{y}$ can be taken as $(-x_2, x_1, 0, \ldots, 0)$. This is, of course, not true when $k = 1$, since the product of two nonzero real numbers is nonzero.

Exercise 1.19 Suppose $\mathbf{a} \in R^k$, $\mathbf{b} \in R^k$. Find $\mathbf{c} \in R^k$ and $r > 0$ such that

$$|\mathbf{x}-\mathbf{a}| = 2|\mathbf{x}-\mathbf{b}|$$

if and only if $|\mathbf{x}-\mathbf{c}| = r$. (*Solution:* $3\mathbf{c} = 4\mathbf{b}-\mathbf{a}$, $3r = 2|\mathbf{b}-\mathbf{a}|$.)

Solution. Since the solution is given to us, all we have to do is verify it, i.e., we need to show that the equation

$$|\mathbf{x}-\mathbf{a}| = 2|\mathbf{x}-\mathbf{b}|$$

is equivalent to $|\mathbf{x}-\mathbf{c}| = r$, which says

$$\left|\mathbf{x} - \frac{4}{3}\mathbf{b} + \frac{1}{3}\mathbf{a}\right| = \frac{2}{3}|\mathbf{b}-\mathbf{a}|.$$

If we square both sides of both equations, we an equivalent pair of equations, the first of which reduces to

$$3|\mathbf{x}|^2 + 2\mathbf{a}\cdot\mathbf{x} - 8\mathbf{b}\cdot\mathbf{x} - |\mathbf{a}|^2 + 4|\mathbf{b}|^2 = 0,$$

and the second of which reduces to this equation divided by 3. Hence these equations are indeed equivalent.

Exercise 1.20 With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero element!) but that (A5) fails.

Solution. We are now defining a cut to be a proper subset of the rational numbers that contains, along with each of its elements, all smaller rational numbers. Order is defined by containment. Now given a set A of cuts having an upper bound β, let α be the union of all the cuts in A. Obviously α is properly contained in β, and so is a proper subset of the rationals. It also obviously satisfies the property that if $p \in \alpha$ and $q < p$, then $q \in \alpha$; hence α is a cut. It is further obvious that α contains each elements of A, and so is an upper bound for A. It remains to prove that there is no smaller upper bound.

To that end, suppose, $\gamma < \alpha$, then α contains an element x not in γ. By definition of α, x must belong to some cut δ in A. But then $\gamma < \delta$, and so γ is not an upper bound for A. Thus α is the least upper bound.

The proof given in the text goes over without any change to show that (A1), (A2), and (A3) hold. As for (A4) let $O = \{r : r \leq 0\}$. We claim $O + \alpha = \alpha$. The proof is easy. First, we obviously have $O + \alpha \subseteq \alpha$. For $r + s \leq s$ if $r \leq 0$. Hence $r + s \in \alpha$ if $s \in \alpha$. Conversely $\alpha \subseteq O + \alpha$, since each s in α can be written as $0 + s$.

Unfortunately, if $O' = \{r : r < 0\}$, there is no element α such that $\alpha + O' = O$. For $\alpha + O'$ has no largest element. If $x = r + s \in \alpha + O'$, where $r \in \alpha$ and $s \in O'$, there is an element $t \in O'$ with $t > s$, and so $r+t \in \alpha + O'$ and $r+t > s$. Since O has a largest element (namely 0), these two sets cannot be equal.