Convergent Sequences

Part 2

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Given a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, and a function $\mathbb{N}\to\mathbb{N}$ denoted by $i\mapsto N_i$ such that

$$i<j \implies N_i<N_j, \tag{1}$$

we call $(a_{N_i})_{i\in\mathbb{N}}$ a *subsequence* of $(a_n)_{n\in\mathbb{N}}$.
Given a subsequence $(a_{N_i})_{i\in\mathbb{N}}$ of a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, by the Trichotomy Law, the condition $i\neq j$ means that either $i<j$ or $i>j$. Then by (1), we have either $N_i<N_j$ or $N_i>N_j$, which implies $N_i\neq N_j$. We have thus shown that $i\neq j$ implies $N_i\neq N_j$, and by contraposition,

$$N_i=N_j \implies i=j. \tag{2}$$

Therefore, $i\mapsto N_i$ is injective. The converse

$$i=j \implies N_i=N_j, \tag{3}$$

of (2) is true because $i\mapsto N_i$ is a function. Also, if $N_i<N_j$, then $N_i\neq N_j$, and by the contrapositive of (3), we have $i\neq j$.

Some notes on subsequences

If $i > j$, then we get, from (1), the contradiction $N_i > N_j$, and so the only possiblity is $i < j$. That is,

$$N_i < N_j \implies i < j. \tag{4}$$

From (1)–(4), we obtain

$$i \leq j \iff N_i \leq N_j. \tag{5}$$

Using an elementary proof, the equivalence (5) can be used to prove that the conditions

$$\forall \varepsilon > 0 \quad \exists N_I \in \mathbb{N} \quad \forall N_i \geq N_I \quad |a_{N_i} - a| < \varepsilon, \tag{6}$$

$$\forall \varepsilon > 0 \quad \exists I \in \mathbb{N} \quad \forall i \geq I \quad |a_{N_i} - a| < \varepsilon, \tag{7}$$

are equivalent. Hence, if the subsequence $(a_{N_i})_{i\in\mathbb{N}}$ converges to some $a \in \mathbb{R}$, both notations $\lim_{N_i\to\infty} a_{N_i}$ and $\lim_{i\to\infty} a_{N_i}$ are valid, and furthermore,

$$\lim_{N_i\to\infty} a_{N_i} = a \iff \lim_{i\to\infty} a_{N_i} = a.$$

i.e., The limiting process for the convergent subsequence $(a_{N_i})_{i\in\mathbb{N}}$ is the same regardless of whether we view this limiting process in terms of the original indices, as in $N_i \to \infty$, or in terms of the 'secondary' indices, as in $i \to \infty$.
Another important property of a subsequence $(a_{N_i})_{i\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ is that

$$\forall i \in \mathbb{N} \;\; [i \leq N_i]\,. \tag{8}$$

If $i = 1$, then by the fact that $N_i \in \mathbb{N}$, we have $N_i \geq 1 = i$.
Suppose $i \leq N_i$ for some $i \in \mathbb{N}$. Tending towards a contradiction, suppose $i + 1 > N_{i+1}$. Since both $i + 1$ and N_i are integers, we further have $i \geq N_{i+1}$. By the inductive hypothesis, $N_i \geq i \geq N_{i+1}$.
But this contradicts $N_i < N_{i+1}$ because of (1) and $i < i + 1$.
Therefore, $i + 1 \leq N_{i+1}$, and we have proven (8) by induction.

Proposition 1

If $(a_n)_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$, then any convergent subsequence of $(a_n)_{n\in\mathbb{N}}$ also converges to a.

Proof of Proposition 1

Suppose $(a_{N_i})_{i\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$ that converges to $b \in \mathbb{R}$, and let $\varepsilon > 0$. The conditions $a = \lim_{n\to\infty} a_n$ and $b = \lim_{N_i\to\infty} a_{N_i}$ imply that there exist $N, N_I \in \mathbb{N}$ such that

$$n \geq N \implies |a - a_n| = |a_n - a| < \frac{\varepsilon}{2}, \tag{9}$$

$$N_i \geq N_I \implies |a_{N_i} - b| < \frac{\varepsilon}{2}. \tag{10}$$

Let us consider those indices N_i such that $i > \max\{N, N_I\}$. Using (8), we have $N_i \geq i > N$, so the conclusion of (9) is true for $n = N_i$. Also using (8), we have $N_i \geq i > N_I$, so the conclusion of (10) is also true. By the triangle inequality,

$$|a - b| = |(a - a_{N_i}) + (a_{N_i} - b)| \leq |a - a_{N_i}| + |a_{N_i} - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is $|a - b| < \varepsilon$ for an arbitrary $\varepsilon > 0$. Therefore, $a = b$. $\square$

The limit superior of a sequence

Let us return our attention to an arbitrary sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Given $n \in \mathbb{N}$, let us collect the terms of the sequence "at index n and beyond" in the following set:

$$\{a_k \ : \ k \geq n\} = \{a_n, a_{n+1}, a_{n+2}, \ldots\}. \tag{11}$$

If the set (11) has an upper bound $M \in \mathbb{R}$, then its supremum

$$\sup_{k\geq n} a_k := \sup\{a_k \ : \ k \geq n\} = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}, \tag{12}$$

exists as an element of $\mathbb{R}$. Otherwise, we define $\sup_{k\geq n} a_k$ as ∞. Note that the number $\sup_{k\geq n} a_k$ depends on n, and so we now have a new sequence

$$\sup_{k\geq 1} a_k, \quad \sup_{k\geq 2} a_k, \quad \sup_{k\geq 3} a_k, \quad \ldots, \quad \sup_{k\geq n} a_k, \quad \ldots \tag{13}$$

of extended real numbers, where in the subscripts after the "$k \geq$" we find the indices of the terms of the sequence (13).

Observe that the supremum (12) of (11) need not be one of the terms in (11), and so it is important to note here that (13) is not necessarily a subsequence of $(a_n)_{n\in\mathbb{N}}$. If the set of all terms in the sequence (13) has a lower bound $M' \in \mathbb{R}$, then the infimum

$$\begin{aligned} \limsup_{n\to\infty} a_n &= \inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k \quad := \quad \inf\left\{\sup_{k\geq n} a_k : \ n\in\mathbb{N}\right\}, \\ &= \inf\left\{\sup_{k\geq 1} a_k, \quad \sup_{k\geq 2} a_k, \quad \dots\right\}, \end{aligned}$$

of the set of all terms of (13) exists as an element of $\mathbb{R}$. Otherwise, we define $\limsup_{n\to\infty} a_n$ as $-\infty$. We call the number $\limsup_{n\to\infty} a_n$ the *limit superior* or *upper limit* of the sequence $(a_n)_{n\in\mathbb{N}}$.

Lemma 2

Let $M \in \mathbb{R}$, and consider a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. If $a_n \leq M$ for any $n \in \mathbb{N}$, then $\limsup_{n\to\infty} a_n \leq M$.

Proof of Lemma 2

Since $a_n \leq M$ for any index n, in particular, given $k \in \mathbb{N}$, we have $a_n \leq M$ 'at index k and beyond.' That is,

$$k \geq n \quad \Longrightarrow \quad a_k \leq M,$$

which means that M is an upper bound of $\{a_k \ : \ k \geq n\}$, and the relationship of this upper bound to the supremum is

$$\sup_{k \geq n} a_k \quad \leq \quad M.$$

But since $\inf_{n \in \mathbb{N}} \sup_{k \geq n} a_k$ is a lower bound of $\left\{\sup_{k \geq n} a_k \ : \ n \in \mathbb{N}\right\}$, we further have

$$\inf_{n \in \mathbb{N}} \sup_{k \geq n} a_k \leq \sup_{k \geq n} a_k \quad \leq \quad M.$$

Therefore, $\limsup_{n \to \infty} a_n \leq M$. $\square$

Lemma 3

If $-\infty < \limsup_{n\to\infty} a_n < \infty$*, then there exists a subsequence* $(a_{N_i})_{i\in\mathbb{N}}$ *of* $(a_n)_{n\in\mathbb{N}}$ *such that for any* $i \in \mathbb{N}$*,*

$$\left| a_{N_i} - \limsup_{n\to\infty} a_n \right| < \frac{1}{i}. \tag{14}$$

Proof of Lemma 3

Suppose $-\infty < \limsup_{n\to\infty} a_n < \infty$, and let $i \in \mathbb{N}$. Since $\frac{1}{i} > 0$, the number

$$\frac{1}{i} + \limsup_{n\to\infty} a_n \tag{15}$$

exceeds the infimum of

$$\left\{ \sup_{k\geq 1} a_k, \quad \sup_{k\geq 2} a_k, \quad \sup_{k\geq 3} a_k, \quad \ldots, \quad \sup_{k\geq n} a_k, \quad \ldots \right\} \tag{16}$$

and is hence not a lower bound of (16). That is, the set (16) has an element not bounded below by ($\not\geq$) the number (15). This element has an index M_i that appears after the "$k \geq$" and so we have

Proof of Lemma 3

$$\sup_{k \geq M_i} a_k < \frac{1}{i} + \limsup_{n\to\infty} a_n. \tag{17}$$

We note here that (17) cannot be possible if $\limsup_{n\to\infty} a_n = -\infty$, in which case there shall be no number below $\frac{1}{i} + \limsup_{n\to\infty} a_n = \frac{1}{i} - \infty = -\infty$. Hence, the assumption $\limsup_{n\to\infty} a_n > -\infty$ is important. Since $-\frac{1}{i} < 0$, the number

$$-\frac{1}{i} + \sup_{k \geq M_i} a_k \tag{18}$$

is less than the supremum of

$$\{a_{M_i}, a_{M_i+1}, a_{M_i+2}, \ldots\} \tag{19}$$

which means that (18) is not an upper bound of (19), and so (19) has an element not bounded above by ($\not\leq$) the number (18).

Proof of Lemma 3

This element has an index N_i which is one of the indices $M_i, M_i + 1, \ldots$, which means $N_i \geq M_i$. We now have the inequality

$$a_{N_i} > -\frac{1}{i} + \sup_{k \geq M_i} a_k. \tag{20}$$

Since $\sup_{k \geq M_i} a_k$ is in (16) and $\limsup_{n\to\infty} a_n$ is a lower bound of (16), we can further extend the inequality (20) as

$$a_{N_i} > -\frac{1}{i} + \sup_{k \geq M_i} a_k \geq -\frac{1}{i} + \limsup_{n\to\infty} a_n. \tag{21}$$

The strict inequality in (21) would not be possible for the case $\limsup_{n\to\infty} a_n = \infty$, because in such a case, there would be no number above $-\frac{1}{i} + \limsup_{n\to\infty} a_n = -\frac{1}{i} + \infty = \infty$, and this tells us that the assumption $\limsup_{n\to\infty} a_n < \infty$ is important.

Recall earlier that $N_i \geq M_i$, so a_{N_i} is in (19), and since $\sup\limits_{k \geq M_i} a_k$ is an upper bound of (19), the inequality (17) can be extended as

$$a_{N_i} \leq \sup_{k \geq M_i} a_k < \frac{1}{i} + \limsup_{n \to \infty} a_n. \tag{22}$$

From (21) and (22), we get

$$\begin{aligned} -\frac{1}{i} + \limsup_{n \to \infty} a_n &< a_{N_i} < \frac{1}{i} + \limsup_{n \to \infty} a_n, \\ -\frac{1}{i} &< a_{N_i} - \limsup_{n \to \infty} a_n < \frac{1}{i}, \end{aligned} \tag{23}$$

from which we obtain (14). $\square$

Given a real number $M > 0$, we say that a sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ is *bounded by* M if $|x_n| \leq M$ for all $n \in \mathbb{N}$. Any sequence bounded by some positive real number is a *bounded* sequence.

Lemma 4

If $c \in \mathbb{R}$ and if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded sequences in $\mathbb{R}$, then the sequences

$$(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}\,, \qquad c\,(a_n)_{n\in\mathbb{N}}\,, \qquad (a_n)_{n\in\mathbb{N}}\,(b_n)_{n\in\mathbb{N}}\,,$$

are also bounded.

Suppose $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded by M and N, respectively. By routine computations using the properties of inequalities in $\mathbb{R}$, we find that the sequences $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}$, $c\,(a_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}\,(b_n)_{n\in\mathbb{N}}$ are bounded by $M+N$, $|c|\cdot M$ and MN, respectively. $\square$

Corollary 5

The set $\ell^\infty(\mathbb{R})$ of all bounded sequences in $\mathbb{R}$ is an associative algebra over $\mathbb{R}$ that is unital and commutative.

Proof of Corollary 5

All the algebraic properties, except closure, of the three operations—addition of sequences as vector addition, left-multiplication by a constant as scalar multiplication, and multiplication of sequences as vector multiplication—that were discussed in the previous lecture are valid for all sequences, and, in particular, for all the sequences in $\ell^\infty(\mathbb{R})$. The closure of $\ell^\infty(\mathbb{R})$ under the said three operations is asserted in Lemma 4.

Lemma 6

If $(a_n)_{n\in\mathbb{N}}$ *is bounded, then* $-\infty < \limsup_{n\to\infty} a_n < \infty$.

Proof of Lemma 6

If for any $n \in \mathbb{N}$, we have $|a_n| \leq M$, or equivalently

$$-M \leq a_n \leq M, \tag{24}$$

then, in particular, $a_n \leq M$, and by Lemma 2, we have

$$\limsup_{n\to\infty} a_n \leq M < \infty. \tag{25}$$

By (24), we have $-M \leq a_n$ for any index n, and in particular for any index $k \geq n$. Thus means that $-M$ is a lower bound of $\{a_k \ : \ k \in \mathbb{N}\}$, but since $\sup_{k\geq n} a_k$ is an upper bound of $\{a_k \ : \ k \in \mathbb{N}\}$, we have

$$-M \leq \sup_{k\geq n} a_k. \tag{26}$$

Since (26) holds for any $n \in \mathbb{N}$, we find that $-M$ is a lower bound of $\left\{\sup_{k\geq n} a_k \ : \ n \in \mathbb{N}\right\}$, and is thus less than or equal to the infimum of $\left\{\sup_{k\geq n} a_k \ : \ n \in \mathbb{N}\right\}$. That is,

$$-M \leq \inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k = \limsup_{n\to\infty} a_n,$$

which, in conjunction with (25), gives us $-\infty < M \leq \limsup_{n\to\infty} a_n < \infty$. $\square$

We summarize in the following the logical relationship between the notions of boundedness and convergence of a sequence in $\mathbb{R}$.

Theorem 7

1. *A convergent sequence in $\mathbb{R}$ is bounded.*
2. *A bounded sequence in $\mathbb{R}$ is not necessarily convergent.*
3. *[The Bolzano-Weierstrass Theorem.] A bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof of Theorem 7(i)

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, and suppose $a = \lim_{n\to\infty} a_n$ for some $a \in \mathbb{R}$. In symbols,

$$\forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \;\left[|a_n - a| < \varepsilon\right]. \tag{27}$$

The trick is to instantiate (27) at the value $\varepsilon = 1$. That is, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - a| < 1. \tag{28}$$

The next trick is to use the reverse triangle inequality in the conclusion of (28). If $n \geq N$, then

$$\begin{array}{rcrcl} & & |a_n - a| & < & 1, \\ ||a_n| - |a|| & \leq & |a_n - a| & < & 1, \\ & & ||a_n| - |a|| & < & 1, \\ -1 & < & |a_n| - |a| & < & 1, \\ & & |a_n| - |a| & < & 1, \\ & & |a_n| & < & 1 + |a|, \end{array}$$

and so (28) becomes

$$n \geq N \implies |a_n| < 1 + |a|. \tag{29}$$

Recall that our goal here is to find some $M \in \mathbb{R}$ such that every term of $(a_n)_{n\in\mathbb{N}}$ has absolute value less than or equal to M. The inequality in (29) tells us that all terms 'at index N and beyond' already have an absolute value less than $1 + |a|$. The only terms not covered are those with index $N - 1$ and below. Thus, we let

$$M := \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a|\}. \tag{30}$$

If $n \geq N$, then by (29), $|a_n| < 1 + |a| \leq M$, and if $n < N$, then by (30), $|a_n| \leq M$. Combining these two cases, we have $|a_n| \leq M$ for all $n \in \mathbb{N}$. Therefore, $(a_n)_{n\in\mathbb{N}}$ is bounded.

Proof of Theorem 7(ii)

Our goal here is to exhibit a sequence in $\mathbb{R}$ that is both bounded and not convergent. For any $n \in \mathbb{N}$, let $a_n := (-1)^n$. That is $a_n = 1$ if n is even, and $a_n = -1$ if n is odd. Hence, $|a_n| = 1$, and consequently, $|a_n| \leq 1$ for any $n \in \mathbb{N}$, which means that $(a_n)_{n\in\mathbb{N}}$ is bounded. To show $(a_n)_{n\in\mathbb{N}}$ is not convergent, let $a \in \mathbb{R}$. We produce a value of ε by cases depending on the value of $|a-1| \geq 0$. If $|a-1| = 0$, then we set $\varepsilon = \frac{1}{2} > 0$, and if $|a-1| > 0$, we set $\varepsilon = |a-1| > 0$. Let $N \in \mathbb{N}$. If $|a-1| = 0$, then $a = 1$, and we choose any odd $n \geq N$, for which $|a_n - a| = |-1-1| = 2 \geq \varepsilon$. If $|a-1| > 0$, then we choose any even $n \geq N$, for which $|a_n - 1| = |1-1| = 0$, and by the triangle inequality,

$$\varepsilon = |a-1| \leq |a-a_n| + |a_n - 1| = |a - a_n| + 0,$$

and we still have $|a_n - a| \geq \varepsilon$. We have thus shown

$$\exists \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geq N \ [|a_n - a| \geq \varepsilon],$$

with $a \in \mathbb{R}$ arbitrary. Therefore, $(a_n)_{n\in\mathbb{N}}$ does not converge to any element of $\mathbb{R}$.

Proof of Theorem 7(iii)

If $(a_n)_{n\in\mathbb{N}}$ is bounded, then by Lemma 6, we have $-\infty < \limsup_{n\to\infty} a_n < \infty$, which by Lemma 3 implies that there exists a subsequence $(a_{N_i})_{i\in\mathbb{N}}$ such that

$$\left| a_{N_i} - \limsup_{n\to\infty} a_n \right| < \frac{1}{i},$$

for any $i \in \mathbb{N}$. If $\varepsilon > 0$, then there exists [an integer] $I > \frac{1}{\varepsilon}$, and for any $i \geq I$,

$$\left| a_{N_i} - \limsup_{n\to\infty} a_n \right| < \frac{1}{i} \leq \frac{1}{I} < \varepsilon.$$

Therefore, $(a_{N_i})_{i\in\mathbb{N}}$ is convergent. $\square$