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Chapter 1

Real Analysis

(1.5)
$\mathit{OrderTrichotomy}[<, S] := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$
$\mathit{OrderTransitivity}[<, S] := \forall_{x,y,z \in S}((x < y \wedge y < z) \implies x < z)$
$\mathit{Order}[<, S] := (\mathit{OrderTrichotomy}[<, S]) \wedge (\mathit{OrderTransitivity}[<, S])$

(1.7)
$\mathit{BoundedAbove}[E, S, <] := (\mathit{Order}[<, S]) \wedge (E \subset S) \wedge (\exists_{\beta \in S} \forall_{x \in E}(x \leq \beta))$
$\mathit{BoundedBelow}[E, S, <] := (\mathit{Order}[<, S]) \wedge (E \subset S) \wedge (\exists_{\beta \in S} \forall_{x \in E}(\beta \leq x))$
$\mathit{UpperBound}[\beta, E, S, <] := (\mathit{Order}[<, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E}(x \leq \beta))$
$\mathit{LowerBound}[\beta, E, S, <] := (\mathit{Order}[<, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E}(\beta \leq x))$

(1.8)
$\mathit{LUB}[\alpha, E, S, <] := (\mathit{UpperBound}[\alpha, E, S, <]) \wedge (\forall_\gamma(\gamma < \alpha \implies \neg \mathit{UpperBound}[\gamma, E, S, <]))$
$\mathit{GLB}[\alpha, E, S, <] := (\mathit{LowerBound}[\alpha, E, S, <]) \wedge (\forall_\beta(\alpha < \beta \implies \neg \mathit{LowerBound}[\beta, E, S, <]))$

(1.10)
$\mathit{LUBProperty}[S, <] := \forall_E(((\emptyset \neq E \subset S) \wedge (\mathit{BoundedAbove}[E, S, <]) \implies \exists_{\alpha \in S}(\mathit{LUB}[\alpha, E, S, <])))$
$\mathit{GLBProperty}[S, <] := \forall_E(((\emptyset \neq E \subset S) \wedge (\mathit{BoundedBelow}[E, S, <]) \implies \exists_{\alpha \in S}(\mathit{GLB}[\alpha, E, S, <])))$

(1.11)
$\mathit{LUBPropertyImpliesGLBProperty} := \mathit{LUBProperty}[S, <] \implies \mathit{GLBProperty}[S, <]$

(1) $\mathit{LUBProperty}[S, <] \implies \ldots$ — wts: 2

(1.1) $(\emptyset \neq B \subset S \wedge \mathit{BoundedBelow}[B, S, <]) \implies \ldots$ — wts: 1.2

(1.1.1) $\mathit{Order}[<, S] \wedge \exists_{\delta' \in S}(\mathit{LowerBound}[\delta', B, S, <])$ — from: *BoundedBelow*, 1.1

(1.1.2) $|B| = 1 \implies \ldots$ — wts: 1.1.3

(1.1.2.1) $\exists_{u'}(u' \in B)$ ▌ $u := choice(\{u' | u' \in B\})$ ▌ $B = \{u\}$ — from: 1.1.2

(1.1.2.2) $\mathit{GLB}[u, B, S, <]$ ▌ $\exists_{\epsilon_0 \in S}(\mathit{GLB}[\epsilon_0, B, S, <])$

(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S}(\mathit{GLB}[\epsilon_0, B, S, <])$

(1.1.4) $|B| \neq 1 \implies \ldots$ — wts: 1.1.5

(1.1.4.1) $\forall_E((\emptyset \neq E \subset S \wedge \mathit{BoundedAbove}[E, S, <]) \implies \exists_{\alpha \in S}(\mathit{LUB}[\alpha, E, S, <]))$ — from: *LUBProperty*, 1

(1.1.4.2) $L := \{s \in S | \mathit{LowerBound}[s, B, S, <]\}$

(1.1.4.3) $|B| > 1 \wedge \mathit{OrderTrichotomy}[<, S]$ ▌ $\exists_{b_1' \in B} \exists_{b_0' \in B}(b_0' < b_1')$ — from: *Order*, 1.1.1; wts: 1.1.4.7

(1.1.4.4) $b_1 := choice(\{b_1' \in B | \exists_{b_0' \in B}(b_0' < b_1')\})$ ▌ $\neg \mathit{LowerBound}[b_1, B, S, <]$ — from: 1.1.4.2

(1.1.4.5) $b_1 \notin L$ ▌ $L \subset S$

(1.1.4.6) $\delta := choice(\{\delta' \in S | \mathit{LowerBound}[\delta', B, S, <]\})$ ▌ $\delta \in L$ ▌ $\emptyset \neq L$ — from: 1.1.1

(1.1.4.7) $\emptyset \neq L \subset S$ — from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y \in L}(\mathit{LowerBound}[y_0, B, S, <])$ ▌ $\forall_{y \in L} \forall_{x \in B}(y_0 \leq x)$ — from: *LowerBound*, 1.1.4.2; wts: 1.1.4.10

(1.1.4.9) $\forall_{x \in B}(x \in S \wedge \forall_{y \in L}(y_0 \leq x))$ ▌ $\forall_{x \in B}(\mathit{UpperBound}[x, L, S, <])$ — from: *UpperBound*

(1.1.4.10) $\exists_{x \in S}(\mathit{UpperBound}[x, L, S, <])$ ▌ $\mathit{BoundedAbove}[L, S, <]$

(1.1.4.11) $\emptyset \neq L \subset S \wedge BoundedAbove[L, S, <]$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha' \in S}(LUB[\alpha', L, S, <])$ ▌ $\alpha := choice(\{\alpha' \in S | (LUB[\alpha', L, S, <])\})$ from: 1.1.4.1 wts: 1.1.4.21

(1.1.4.13) $\forall_x(x \in B \implies UpperBound[x, L, S, <])$ from: 1.1.4.9 wts: 1.1.4.17

(1.1.4.14) $\forall_x(\neg UpperBound[x, L, S, <] \implies x \notin B)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$ wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound[\gamma, L, S, <]$ ▌ $\gamma \notin B$ from: LUB, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B$ ▌ $\gamma \in B \implies \gamma \geq \alpha$

(1.1.4.17) $\forall_{\gamma \in B}(\alpha \leq \gamma)$ ▌ $LowerBound[\alpha, B, S, <]$ from: LowerBound

(1.1.4.18) $\alpha < \beta \implies \ldots$ wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y \in L}(y_0 \leq \alpha < \beta)$ ▌ $\forall_{y \in L}(y_0 \neq \beta)$ from: LUB, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta \notin L$ ▌ $\neg LowerBound[\beta, B, S, <]$ from: 1.1.4.2

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound[\beta, B, S, <]$ ▌ $\forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.20) $LowerBound[\alpha, B, S, <] \wedge \forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB[\alpha, B, S, <]$ ▌ $\exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.5) $|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.6) $(|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])) \wedge (|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <]))$ from: 1.1.3, 1.1.5

(1.1.7) $(|B| = 1 \vee |B| \neq 1) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$ ▌ $\exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.2) $(\emptyset \neq B \subset S \wedge BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.3) $\forall_B((\emptyset \neq B \subset S \wedge BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <]))$

(1.4) $GLBProperty[S, <]$

(2) $LUBProperty[S, <] \implies GLBProperty[S, <]$

(1.12)

$$Field[F, +, *] := \exists_{0,1 \in F}\forall_{x,y,z \in F}\left(\begin{array}{l} x + y \in F \quad \wedge \quad x * y \in F \quad \wedge \\ x + y = y + x \quad \wedge \quad x * y = y * x \quad \wedge \\ (x + y) + z = x + (y_0 + z) \ \wedge \ (x * y) * z = x * (y_0 * z) \ \wedge \\ \mathbb{1} \neq \mathbb{0} \quad \wedge \quad x * (y_0 + z) = (x * y) + (x * z) \quad \wedge \\ \mathbb{0} + x = x \quad \wedge \quad \mathbb{1} * x = x \quad \wedge \\ \exists_{-x \in F}(x + (-x) = \mathbb{0}) \wedge (x \neq \mathbb{0} \implies \exists_{1/x \in F}(x * (1/x) = \mathbb{1})) \end{array}\right)$$

*********************** $(Field[F, +, *] \wedge x, y, z \in F) \implies \ldots$ ***********************

(1.14)

$AdditiveCancellation := (x + y = x + z) \implies y = z$

(1) $y = \mathbb{0} + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \ldots$ from: Field

(2) $(-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = \mathbb{0} + z = z$ from: Field

$AdditiveIdentityUniqueness := (x + y = x) \implies y = \mathbb{0}$

(1) $x + y = x = \mathbb{0} + x = x + \mathbb{0}$ from: Field

(2) $y = 0$ from: AdditiveCancellation

$AdditiveInverseUniqueness := (x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$ from: Field

(2) $y = -x$ from: AdditiveCancellation

$DoubleNegative := x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x$ ▌ $\mathbb{0} = (-x) + x$ from: Field

(2) $x = -(-x)$ from: AdditiveInverseUniqueness

(1.15)

$\textit{MultiplicativeCancellation} := (x \neq 0 \land x * y = x * z) \implies y = z$ —
$\textit{MultiplicativeIdentityUniqueness} := (x \neq 0 \land x * y = x) \implies y = \mathbb{1}$ —
$\textit{MultiplicativeInverseUniqueness} := (x \neq 0 \land x * y = \mathbb{1}) \implies y = 1/x$ —
$\textit{DoubleReciprocal} := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)
$\textit{Domination} := \mathbb{0} * x = \mathbb{0}$

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x \;\blacksquare\; \mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$ — from: *Field*

(2) $\mathbb{0} * x = \mathbb{0}$ — from: *AdditiveIdentityUniqueness*

$\textit{NonDomination} := (x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies \ldots$

(1.1) $(x * y = 0) \implies \ldots$

(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = \mathbb{0} * ((1/x) * (1/y)) = \mathbb{0}$ — from: *Field*, *Domination*, 1, 1.1

(1.1.2) $\mathbb{1} = \mathbb{0} \land \mathbb{1} \neq \mathbb{0} \;\blacksquare\; \bot$ — from: *Field*

(1.2) $(x * y = 0) \implies \bot \;\blacksquare\; x * y \neq 0$

(2) $(x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq 0$

$\textit{NegationCommutativity} := (-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0} \;\blacksquare\; x * y + (-x) * y = \mathbb{0}$ — from: *Field*, *Domination*; wts: 2

(2) $(-x) * y = -(x * y)$ — from: *AdditiveInverseUniqueness*

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * \mathbb{0} = \mathbb{0} \;\blacksquare\; x * y + x * (-y) = \mathbb{0}$ — from: *Field*, *Domination*; wts: 4

(4) $x * (-y) = -(x * y)$ — from: *AdditiveInverseUniqueness*

(5) $(-x) * y = -(x * y) = x * (-y)$ — from: 2, 4

$\textit{NegativeMultiplication} := (-x) * (-y) = x * y$

(1) $(-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$ — from: *NegationCommutativity*, *DoubleNegative*

(1.17)

$$\textit{OrderedField}[F, +, *, <] := \begin{pmatrix} \textit{Field}[F, +, *] \quad \land \quad \textit{Order}[<, F] \quad \land \\ \forall_{x,y,z \in F}(y_0 < z \implies x + y < x + z) \land \\ \forall_{x,y \in F}((x > \mathbb{0} \land y > \mathbb{0}) \implies x * y > \mathbb{0}) \end{pmatrix}$$

*********************** $(\textit{OrderedField}[F, +, *, <] \land x, y, z \in F) \implies \ldots$ ***********************

(1.18)
$\textit{NegationOnOrder} := x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \ldots$

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x \;\blacksquare\; \mathbb{0} > -x \;\blacksquare\; -x < \mathbb{0}$ — from: *OrderedField*

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \ldots$

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x \;\blacksquare\; \mathbb{0} < x \;\blacksquare\; x > \mathbb{0}$ — from: *OrderedField*

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \land -x < \mathbb{0} \implies x > \mathbb{0} \;\blacksquare\; x > \mathbb{0} \iff -x < \mathbb{0}$ — from: 2, 4

$\textit{PositiveFactorPreservesOrder} := (x > \mathbb{0} \land y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \land y < z) \implies \ldots$

(1.1) $(-y) + z > (-y) + y = \mathbb{0} \;\blacksquare\; z + (-y) = \mathbb{0}$ — from: *OrderedField*

(1.2) $x * (z + (-y)) > \mathbb{0} \;\blacksquare\; x * z + x * (-y) > \mathbb{0}$ — from: *OrderedField*

(1.3) $x * z = \mathbb{0} + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \ldots$ — from: *Field*, *NegationCommutativity*

(1.4) $x * y + (x * z + x * (-y)) > x * y + \mathbb{0} = x * y$ — from: *Field*, 1.2

(1.5) $x * z > x * y$ — from: 1.3, 1.4

(2) $(x > \mathbb{0} \land y < z) \implies x * z > x * y$

$NegativeFactorFlipsOrder := (x < \mathbb{0} \land y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \land y < z) \implies \ldots$

(1.1) $-x > \mathbb{0}$ from: $NegationOnOrder$

(1.2) $(-x) * y < (-x) * z \blacksquare \mathbb{0} = x * y + (-x) * y < x * y + (-x) * z \blacksquare \mathbb{0} < x * y + (-x) * z$ from: $PositiveFactorPreservesOrder$

(1.3) $\mathbb{0} < (-x) * (-y + z) \blacksquare \mathbb{0} > x * (-y + z) \blacksquare \mathbb{0} > -(x * y) + x * z$ from: $NegationOnOrder$

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \land y < z) \implies x * y > x * z$

$SquareIsPositive := (x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ from: $OrderedField$

(2) $(x < \mathbb{0}) \implies \ldots$

(2.1) $-x > \mathbb{0} \blacksquare x * x = (-x) * (-x) > \mathbb{0} \blacksquare x * x > \mathbb{0}$ from: $NegationOnOrder, OrderedField, NegativeMultiplication$

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \lor x < \mathbb{0}) \implies x * x > \mathbb{0} \blacksquare x \neq \mathbb{0} \implies x * x > \mathbb{0}$ from: $OrderTrichotomy, 1, 3$

$OneIsPositive := \mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0} \blacksquare \mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ from: $Field, SquareIsPositive$

$ReciprocationOnOrder := (0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \ldots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0} \blacksquare x * (1/x) > 0$ from: $Field, OneIsPositive$

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \land x * (1/x) > 0 \implies \perp \blacksquare 1/x > \mathbb{0}$ from: $NegativeFactorFlipsOrder, 1$

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0} \blacksquare y * (1/y) > 0$ from: $Field, OneIsPositive$

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \land y * (1/y) > 0 \implies \perp \blacksquare 1/y > \mathbb{0}$ from: $NegativeFactorFlipsOrder, 1$

(1.5) $(1/x) * (1/y) > \mathbb{0}$ from: $OrderedField$

(1.6) $\mathbb{0} < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$ from: $OrderedField, 1, 1.4, 1.5$

(1.19)
$OrderedFieldQ := OrderedField[\mathbb{Q}, +, *, <]$ —

$Subfield[K, F, +, *] := Field[F, +, *] \land K \subset F \land Field[K, +, *]$
$OrderedSubfield[K, F, +, *, <] := OrderedField[F, +, *, <] \land K \subset F \land OrderedField[K, +, *, <]$
$CutI[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$
$CutII[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
$CutIII[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} | CutI[\alpha] \land CutII[\alpha] \land CutIII[\alpha]\}$

$CutCorollaryI := (\alpha \in \mathbb{R} \land p \in \alpha \land q \in \mathbb{Q} \land q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \land p \in \alpha \land q \in \mathbb{Q} \land q \notin \alpha) \implies \ldots$

(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ from: $CutII, 1$

(1.2) $q < p \implies q \in \alpha \blacksquare q \notin \alpha \implies q \geq p$ from: 1

(1.3) $(q \notin \alpha) \implies \ldots$

(1.3.1) $q \geq p$ from: 1.2

(1.3.2) $(q = p) \implies (p \in \alpha \land p \notin \alpha) \implies \perp \blacksquare q \neq p$ from: 1, 1.3

(1.3.3) $q \geq p \land q \neq p \blacksquare p < q$

(1.4) $q \notin \alpha \implies p < q \blacksquare p < q$ from: 1

(2) $(\alpha \in \mathbb{R} \land p \in \alpha \land q \in \mathbb{Q} \land q \notin \alpha) \implies p < q$

$CutCorollaryII := (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$

(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ — from: $CutII$, 1

(1.2) $s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha))$ ▮ $s \in \alpha \implies r \in \alpha$ — from: 1, 1.1

(1.3) $r \notin \alpha \implies s \notin \alpha$ ▮ $s \notin \alpha$ — from: 1, 1.2

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$OrderTrichotomyOfR := OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$

(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$ — from: $<_{\mathbb{R}}$, 1.1

(1.1.2) $\exists_{p'}(p' \in \alpha \wedge p' \notin \beta)$ ▮ $p := choice(\{p' | p' \in \alpha \wedge p' \notin \beta\})$

(1.1.3) $q \in \beta \implies \ldots$

(1.1.3.1) $p, q \in \mathbb{Q}$

(1.1.3.2) $q < p$ — from: $CutCorollaryI$

(1.1.3.3) $q \in \alpha$ — from: $CutII$

(1.1.4) $q \in \beta \implies q \in \alpha$

(1.1.5) $\forall_{q \in \beta}(q \in \alpha)$ ▮ $\beta \subseteq \alpha$

(1.1.6) $\beta \subset \alpha$ ▮ $\beta <_{\mathbb{R}} \alpha$

(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$ ▮ $(\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

$OrderTransitivityOfR := OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a \in \alpha}(a \in \beta) \wedge \forall_{b \in \beta}(b \in \gamma)$

(1.1.3) $\forall_{a \in \alpha}(\alpha \in \gamma)$ ▮ $\alpha \subset \gamma$ ▮ $\alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}}((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(4) $OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

$OrderOfR := Order[<_{\mathbb{R}}, \mathbb{R}]$ — from: $OrderTrichotomyR, OrderTransitivityR$

wts:

$LUBPropertyOfR := LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} | \exists_{\alpha \in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset$ ▮ $\exists_{\alpha}(\alpha \in A)$ ▮ $\alpha_0 := choice(\{\alpha | \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset$ ▮ $\exists_a(a \in \alpha_0)$ ▮ $a_0 := choice(\{a | a \in \alpha_0\})$ ▮ $a_0 \in \gamma$ ▮ $\gamma \neq \emptyset$

(1.4) $BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]$ ▮ $\exists_{\beta}(UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.5) $\beta_0 := choice(\{\beta | UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}]\})$

(1.6) $UpperBound[\beta_0, A, \mathbb{R}, <_{\mathbb{R}}] \;\blacksquare\; \forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \beta_0) \;\blacksquare\; \forall_{\alpha\in A}(\alpha \subseteq \beta_0) \;\blacksquare\; \forall_{\alpha\in A}\forall_{a\in\alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \;\blacksquare\; \forall_{a\in\gamma}(a \in \beta_0) \;\blacksquare\; \gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q} \;\blacksquare\; \gamma \subseteq \beta_0 \subset \mathbb{Q} \;\blacksquare\; \gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q} \;\blacksquare\; CutI[\gamma]$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.10.1) $p \in \gamma \;\blacksquare\; \exists_{\alpha\in A}(p \in \alpha) \;\blacksquare\; \alpha_1 := choice(\{\alpha \in A | p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \;\blacksquare\; q \in \alpha_1 \;\blacksquare\; q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \;\blacksquare\; \forall_{p\in\gamma}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \gamma) \;\blacksquare\; CutII[\gamma]$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha\in A}(p \in \alpha) \;\blacksquare\; \alpha_2 := choice(\{\alpha \in A | p \in \alpha\})$

(1.12.2) $\alpha_2 \in \mathbb{R} \;\blacksquare\; CutII[\alpha_2] \;\blacksquare\; \exists_{r\in\alpha_2}(p < r) \;\blacksquare\; r_0 := choice(\{r \in \alpha_2 | p < r\})$

(1.12.3) $r_0 \in \alpha_2 \;\blacksquare\; r_0 \in \gamma$

(1.12.4) $p < r_0 \;\blacksquare\; p < r_0 \wedge r_0 \in \gamma \;\blacksquare\; \exists_{r\in\gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r\in\gamma}(p < r) \;\blacksquare\; \forall_{p\in\gamma}\exists_{r\in\gamma}(p < r) \;\blacksquare\; CutIII[\gamma]$

(1.14) $CutI[\gamma] \wedge CutII[\gamma] \wedge CutIII[\gamma] \;\blacksquare\; \gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha\in A}(\alpha \subseteq \gamma) \;\blacksquare\; \forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma)$

(1.16) $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R} \;\blacksquare\; UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.17) $\delta <_{\mathbb{R}} \gamma \implies \ldots$

(1.17.1) $\delta \subset \gamma \;\blacksquare\; \exists_s(s \in \gamma \wedge s \notin \delta) \;\blacksquare\; s_0 := choice(\{s \in \mathbb{Q} | s \in \gamma \wedge s \notin \delta\})$

(1.17.2) $s_0 \in \gamma \;\blacksquare\; \exists_{\alpha\in A}(s_0 \in \alpha) \;\blacksquare\; \alpha_3 := choice(\{\alpha \in A | s_0 \in \alpha\})$

(1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta \;\blacksquare\; \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \ldots$

(1.17.4.1) $\alpha_3 \subseteq \delta \;\blacksquare\; \forall_{s\in\mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \;\blacksquare\; \neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4.2) $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \;\blacksquare\; \bot$

(1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \bot \;\blacksquare\; \delta <_{\mathbb{R}} \alpha_3 \;\blacksquare\; \exists_{\alpha\in A}(\delta <_{\mathbb{R}} \alpha) \;\blacksquare\; \exists_{\alpha\in A}(\neg(\alpha \leq_{\mathbb{R}} \delta))$

(1.17.6) $\neg\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \delta) \;\blacksquare\; \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}]) \;\blacksquare\; \forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.19) $UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \wedge \forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.20) $LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \;\blacksquare\; \exists_{\gamma\in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma\in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(3) $\forall_A((\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma\in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])) \;\blacksquare\; LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

$+_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$

$0_{\mathbb{R}} := \{x \in \mathbb{Q} | x < 0\}$

$0InR := 0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}} \;\blacksquare\; \emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q} \;\blacksquare\; CutI[0_{\mathbb{R}}]$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}} \;\blacksquare\; \forall_{x\in 0_{\mathbb{R}}}\forall_{y\in\mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}}) \;\blacksquare\; CutII[0_{\mathbb{R}}]$

(3) $y := x/2 \;\blacksquare\; (x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y\in 0_{\mathbb{R}}}(x < y) \;\blacksquare\; \forall_{x\in 0_{\mathbb{R}}}\exists_{y\in 0_{\mathbb{R}}}(x < y) \;\blacksquare\; CutIII[0_{\mathbb{R}}]$

(4) $CutI[0_{\mathbb{R}}] \wedge CutII[0_{\mathbb{R}}] \wedge CutIII[0_{\mathbb{R}}] \;\blacksquare\; 0_{\mathbb{R}} \in \mathbb{R}$

$FieldAdditionClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha) ; \exists_b(b \in \beta) \;\blacksquare\; a_0 := choice(\{a | a \in \alpha\}) ; b_0 := choice(\{b | b \in \beta\}) \;\blacksquare\; a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha) ; \exists_y(y_0 \notin \beta) \;\blacksquare\; x_0 := choice(\{x | x \notin \alpha\}) ; y_0 := choice(\{y | y \notin \beta\})$

(1.5) $\forall_{r\in\alpha}(r < x_0) ; \forall_{s\in\beta}(s < y_0) \;\blacksquare\; \forall_{r\in\alpha}\forall_{s\in\beta}(r + s < x_0 + y_0) \;\blacksquare\; x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q} \;\blacksquare\; CutI[\alpha +_{\mathbb{R}} \beta]$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.7.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s)$ ▮ $(r_0, s_0) := choice((r, s) \in \alpha \times \beta | p = r + s)$

(1.7.2) $q < p = r_0 + s_0$ ▮ $(q - s_0) < r_0$ ▮ $(q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta$ ▮ $q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta$ ▮ $q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta$ ▮ $\forall_{p\in\alpha+_{\mathbb{R}}\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta)$ ▮ $CutII[\alpha +_{\mathbb{R}} \beta]$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s)$ ▮ $(r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta | p = r + s\})$

(1.9.2) $r_1 \in \alpha$ ▮ $\exists_{t\in\alpha}(r_1 < t)$ ▮ $t_0 := choice(\{t \in \alpha | r_1 < t\})$

(1.9.3) $s_1 \in \beta$ ▮ $t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1$ ▮ $\exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$ ▮ $\forall_{p\in\alpha+_{\mathbb{R}}\beta}\exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$ ▮ $CutIII[\alpha +_{\mathbb{R}} \beta]$

(1.11) $CutI[\alpha +_{\mathbb{R}} \beta] \wedge CutII[\alpha +_{\mathbb{R}} \beta] \wedge CutIII[\alpha +_{\mathbb{R}} \beta]$ ▮ $\alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

$FieldAdditionCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s | r \in \alpha \wedge s \in \beta\} = \{s + r | s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

$FieldAdditionAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma))$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

$FieldAdditionIdentityOfR := (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \ldots$

(1.1.1) $s < 0$ ▮ $r + s < r + 0 = r$ ▮ $r + s < r$ ▮ $r + s \in \alpha$

(1.2) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha$ ▮ $\forall_{r\in\alpha}\forall_{s\in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▮ $\forall_{p\in\alpha+_{\mathbb{R}}0_{\mathbb{R}}}(p \in \alpha)$ ▮ $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r\in\alpha}(p < r)$ ▮ $r_2 := choice(\{r \in \alpha | p < r\})$

(1.4.2) $p < r_2$ ▮ $p - r_2 < r_2 - r_2 = 0$ ▮ $(p - r_2) < 0$ ▮ $(p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha$ ▮ $p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▮ $p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▮ $\forall_{p\in\alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▮ $\alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▮ $0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

$FieldAdditionInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} | \exists_{r>0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q}$ ▮ $\exists_{s\in\mathbb{Q}}(s \notin \alpha)$ ▮ $s_0 := choice(\{s | s \notin \alpha\})$ ▮ $p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha$ ▮ $-p_0 - 1 \notin \alpha$ ▮ $\exists_{r>0}(-p_0 - r \notin \alpha)$ ▮ $p_0 \in \beta$

(1.4) $\emptyset \neq \alpha$ ▮ $\exists_{q\in\alpha}$ ▮ $q_0 := choice(\{q \in \mathbb{Q} | q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha$ ▮ $-(-q_0) - r = q_0 - r < q_0$ ▮ $-(-q_0) - r < q_0$ ▮ $-(-q_0) - r \in \alpha$

(1.6) $\forall_{r>0}(-(-q_0) - r \in \alpha)$ ▮ $\neg\exists_{r>0}(-(-q_0) - r \notin \alpha)$ ▮ $-q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q}$ ▮ $CutI[\beta]$

(1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.8.1) $p \in \beta$ ▮ $\exists_{r>0}(-p - r \notin \alpha)$ ▮ $r_0 := choice(\{r > 0 | -p - r \notin \alpha\})$

(1.8.2) $q < p$ ▮ $-p - r < -q - r$

(1.8.3) $-q - r \notin \alpha$ ▮ $q \in \beta$

(1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta$ ▮ $\forall_{p\in\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \beta)$ ▮ $CutII[\beta]$

(1.10) $p \in \beta \implies \ldots$

(1.10.1) $p \in \beta \;\blacksquare\; \exists_{r>0}(-p-r \notin \alpha) \;\blacksquare\; r_1 := choice(\{r > 0| -p-r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0 \;\blacksquare\; r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p \;\blacksquare\; t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha \;\blacksquare\; -t_0 - (r_1/2) \notin \alpha \;\blacksquare\; \exists_{r>0}(-t_0 - r \notin \alpha) \;\blacksquare\; t_0 \in \beta$

(1.10.7) $t_0 > p \wedge t_0 \in \beta \;\blacksquare\; \exists_{t\in\beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t\in\beta}(p < t) \;\blacksquare\; \forall_{p\in\beta}\exists_{t\in\beta}(p < t) \;\blacksquare\; CutIII[\beta]$

(1.12) $CutI[\beta] \wedge CutII[\beta] \wedge CutIII[\beta] \;\blacksquare\; \beta \in \mathbb{R}$

(1.13) $(r \in \alpha \wedge s \in \beta) \implies \ldots$

(1.13.1) $s \in \beta \;\blacksquare\; \exists_{t>0}(-s-t \notin \alpha) \;\blacksquare\; t_1 := choice(\{t > 0| -s-t \notin \alpha\}) \;\blacksquare\; -s-t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s-t_1 < -s \wedge -s-t_1 \notin \alpha \;\blacksquare\; -s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha \;\blacksquare\; r < -s \;\blacksquare\; r+s < 0 \;\blacksquare\; r+s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \wedge s \in \beta) \implies r+s \in 0_{\mathbb{R}} \;\blacksquare\; \forall_{(r,s)\in\alpha\times\beta}(r+s \in 0_{\mathbb{R}}) \;\blacksquare\; \alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \ldots$

(1.15.1) $v < 0 \;\blacksquare\; w_0 := -v/2 \;\blacksquare\; w > 0$

(1.15.2) $\exists_{n\in\mathbb{Z}}(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha) \;\blacksquare\; n_0 := choice(\{n \in \mathbb{Z}|nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$ from: $ARCHIMEDEANPROPERTYOFQ + LUB$???

(1.15.3) $p_0 := -(n_0+2)w_0 \;\blacksquare\; -p_0 - w_0 = (n_0+2)w_0 - w_0 = (n_0+1)w_0 \notin \alpha \;\blacksquare\; -p_0 - w_0 \notin \alpha \;\blacksquare\; p_0 \in \beta$

(1.15.4) $n_0w_0 \in \alpha \wedge p_0 \in \beta \;\blacksquare\; n_0w_0 + p_0 = n_0(-v/2) + -(n_0+2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta \;\blacksquare\; \forall_{v\in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta) \;\blacksquare\; 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta \;\blacksquare\; \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}} \;\blacksquare\; \exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

$*_{\mathbb{R}}[\alpha, \beta] :=$ —

$1_{\mathbb{R}} := \{x \in \mathbb{Q}|x < 1\}$

$1IsNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ —

$1InR := 1_{\mathbb{R}} \in \mathbb{R}$ —

$FieldMultiplicationClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R})$ —

$FieldMultiplicationCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ —

$FieldMultiplicationAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma))$ —

$FieldMultiplicationIdentityOfR := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ —

$FieldMultiplicationInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha\in\mathbb{R}}(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}})$ —

$FieldDistributativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ —

$FieldWithR := Field[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}]$ —

$OrderedFieldWithR := OrderedField[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q}|r < q\}|q \in \mathbb{Q}\}$

$QROrderedSubfieldOfR := OrderedSubfield[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$QIsomorphicToQR := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ —

$CompletenessOfR := \exists_{\mathbb{R}}(LUBProperty[\mathbb{R}, <_{\mathbb{R}}] \wedge OrderedSubfield[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}])$ —

(1.20)

$ArchimedeanPropertyOfR := \forall_{x,y\in\mathbb{R}}(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y))$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx|n \in \mathbb{N}^+\} \;\blacksquare\; (\emptyset \neq A \subset \mathbb{R}) \wedge (a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a))$

(1.2) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \;\blacksquare\; \forall_{n\in\mathbb{N}^+}(nx \leq y) \;\blacksquare\; UpperBound[y_0, A, \mathbb{R}, <] \;\blacksquare\; BoundedAbove[A, \mathbb{R}, <]$

(1.2.2) $CompletenessOfR \;\blacksquare\; LUBProperty[\mathbb{R}, <]$

(1.2.3) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (BoundedAbove[A, \mathbb{R}, <]) \;\blacksquare\; \exists_{\alpha\in\mathbb{R}}(LUB[\alpha, A, \mathbb{R}, <])$ ...

(1.2.4) $\ldots \alpha_0 := choice(\{\alpha \in \mathbb{R} | LUB[\alpha, A, \mathbb{R}, <]\})$ ▮ $LUB[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0$ ▮ $\alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge (LUB[\alpha_0, A, \mathbb{R}, <])$ ▮ $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$ ▮ $\exists_{c \in A}(\alpha_0 - x < c)\ \ldots$

(1.2.8) $\ldots c_0 := choice(\{c \in A | \alpha_0 - x < c\})$ ▮ $(c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$ ▮ $\exists_{m \in \mathbb{N}^+}(mx = c_0)\ \ldots$

(1.2.10) $\ldots m_0 := choice(\{m \in \mathbb{N}^+ | mx = c_0\})$ ▮ $(m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0)$ ▮ $\alpha_0 - x < c_0 = m_0 x$ ▮ $\alpha_0 < m_0 x + x$ ▮ $\alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+$ ▮ $m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$ ▮ $(m_0 + 1)x \in A$

(1.2.14) $(\alpha_0 < (m_0 + 1)x) \wedge ((m_0 + 1)x \in A)$ ▮ $\exists_{c \in A}(\alpha_0 < c)$

(1.2.15) $LUB[\alpha_0, A, \mathbb{R}, <]$ ▮ $UpperBound[\alpha_0, A, \mathbb{R}, <]$ ▮ $\forall_{c \in A}(c \leq \alpha_0)$ ▮ $\neg\exists_{c \in A}(c > \alpha_0)$ ▮ $\neg\exists_{c \in A}(\alpha_0 < c)$

(1.2.16) $(\exists_{c \in A}(\alpha_0 < c)) \wedge (\neg\exists_{c \in A}(\alpha_0 < c))$ ▮ $\bot$

(1.3) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \bot$ ▮ $\exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y)$ ▮ $\forall_{x, y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

$QDenseInR := \forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y$ ▮ $(0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $ArchimedeanPropertyOfR \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R})$ ▮ $\exists_{n \in \mathbb{N}^+}(n(y - x) > 1)\ \ldots$

(1.3) $\ldots n_0 := choice(\{n \in \mathbb{N}^+ | n(y - x) > 1\})$ ▮ $(n_0 \in \mathbb{N}^+) \wedge (n_0(y - x) > 1)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R})$ ▮ $n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R})$ ▮ $\exists_{m \in \mathbb{N}^+}(m(1) > n_0 x)\ \ldots$

(1.6) $\ldots m_1 := choice(\{m \in \mathbb{N}^+ | m(1) > n_0 x\})$ ▮ $(m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R})$ ▮ $\exists_{m \in \mathbb{N}^+}(m(1) > -n_0 x)\ \ldots$

(1.8) $\ldots m_2 := choice(\{m \in \mathbb{N}^+ | m(1) > -n_0 x\})$ ▮ $(m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x)$ ▮ $-m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+$ ▮ $|m_1 - (-m_2)| \geq 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge (|m_1 - (-m_2)| \geq 2)$ ▮ $\exists_{m \in \mathbb{Z}}((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m))\ \ldots$

(1.12) $\ldots m_0 := choice(\{m \in \mathbb{Z} | (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\})$ ▮ $(-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$

(1.13) $(n_0(y - x) > 1) \wedge (m_0 - 1 \leq n_0 x < m_0)$ ▮ $n_0 x < m_0 \leq 1 + n_0 x < n_0 y$ ▮ $n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y)$ ▮ $x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z}$ ▮ $m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y)$ ▮ $\exists_{p \in \mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y)$ ▮ $\forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1.21)
$RootLemma := (0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b$ ▮ $b/a > 1$

(1.3) $b/a > 1$ ▮ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq \sum_{i=1}^{n}(b^{n-i}a^{i-1}(b/a)^{i-1}) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$ ▮ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq (b - a)nb^{n-1}$ ▮ $b^n - a^n \leq (b - a)nb^{n-1}$

(2) $(0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

$RootExistenceInR := \forall_{0 < x \in \mathbb{R}}\forall_{0 < n \in \mathbb{Z}}\exists!_{0 < y \in \mathbb{R}}({y_0}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} | t > 0 \wedge t^n < x\}$ ▮ $t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$

(1.2) $t_0 := x/(1 + x)$ ▮ $(t_0 = x/(1 + x)) \wedge (t_0 \in \mathbb{R})$

(1.3) $0 < x$ ▮ $0 < x < 1 + x$ ▮ $t_0 = x/(1 + x) > 0$ ▮ $t_0 > 0$

(1.4) $1 = (1+x)/(1+x) > x/(1+x) = t_0 \blacksquare 1 > t_0$

(1.5) $(t_0 > 0) \wedge (1 > t_0) \blacksquare 0 < t_0 < 1$

(1.6) $(0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1) \blacksquare {t_0}^n \leq t_0$

(1.7) $0 < x \blacksquare x > x/(1+x) = t_0 \blacksquare x > t_0$

(1.8) $({t_0}^n \leq t_0) \wedge (x > t_0) \blacksquare {t_0}^n < x$

(1.9) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge ({t_0}^n < x) \blacksquare t_0 \in E \blacksquare \emptyset \neq E$

(1.10) $t_1 := choice(\{t \in \mathbb{R} | t > 1+x\}) \blacksquare (t_1 \in \mathbb{R}) \wedge (t_1 > 1+x)$

(1.11) $x > 0 \blacksquare t_1 > 1+x > 1 \blacksquare t_1 > 1 \blacksquare {t_1}^n \geq t_1$

(1.12) $({t_1}^n \geq t_1) \wedge (t_1 > 1+x) \wedge (1 > 0) \blacksquare {t_1}^n \geq t_1 > 1+x > x \blacksquare {t_1}^n > x$

(1.13) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ({t_1}^n > x) \blacksquare t_1 \notin E \blacksquare E \subset \mathbb{R}$

(1.14) $(\emptyset \neq E) \wedge (E \subset \mathbb{R}) \blacksquare \emptyset \neq E \subset \mathbb{R}$

(1.15) $t \in E \implies \ldots$

(1.15.1) $(t \in E) \wedge (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \blacksquare t^n < x$

(1.15.2) $({t_1}^n > x) \wedge (t^n < x) \blacksquare t^n < x < {t_1}^n \blacksquare t < t_1$

(1.16) $t \in E \implies t < t_1 \blacksquare \forall_{t \in E}(t \leq t_1) \blacksquare UpperBound[t_1, E, \mathbb{R}, <] \blacksquare BoundedAbove[E, \mathbb{R}, <]$

(1.17) $CompletenessOfR \blacksquare LUBProperty[\mathbb{R}, <]$

(1.18) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (BoundedAbove[E, \mathbb{R}, <]) \blacksquare \exists_{y \in \mathbb{R}}(LUB[y, E, \mathbb{R}, <]) \ \ldots$

(1.19) $\ldots y_0 := choice(\{y \in \mathbb{R} | LUB[y, E, \mathbb{R}, <]\}) \blacksquare LUB[y_0, E, \mathbb{R}, <]$

(1.20) $(LUB[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0) \blacksquare 0 < t_0 \leq y_0 \in \mathbb{R} \blacksquare 0 < y_0 \in \mathbb{R}$

(1.21) ${y_0}^n < x \implies \ldots$

(1.21.1) $k_0 := \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} \blacksquare k_0 \in \mathbb{R}$

(1.21.2) ${y_0}^n < x \blacksquare 0 < x - {y_0}^n$

(1.21.3) $(n > 0) \wedge (y_0 > 0) \blacksquare 0 < n(y_0+1)^{n-1}$

(1.21.4) $(0 < x - {y_0}^n) \wedge (0 < n(y_0+1)^{n-1}) \blacksquare 0 < \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} = k_0 \blacksquare 0 < k_0$

(1.21.5) $(0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R}) \blacksquare 0 < min(1, k_0) \in \mathbb{R}$

(1.21.6) $QDenseInR \wedge (0, min(1, k_0) \in \mathbb{R}) \wedge (0 < min(1, k_0)) \blacksquare \exists_{h \in \mathbb{Q}}(0 < h < min(1, k_0)) \ \ldots$

(1.21.7) $\ldots h_0 := choice(\{h \in \mathbb{Q} | 0 < h < min(1, k_0)\}) \blacksquare (0 < h_0 < 1) \wedge (h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}})$

(1.21.8) $(y_0 > 0) \wedge (h_0 > 0) \blacksquare 0 < y_0 < y_0 + h_0$

(1.21.9) $RootLemma \wedge (0 < y_0 < y_0 + h_0) \blacksquare (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}$

(1.21.10) $h_0 < 1 \blacksquare h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$

(1.21.11) $((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}) \wedge (h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}) \blacksquare (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}$

(1.21.12) $(0 < n(y_0+1)^{n-1}) \wedge (h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}) \blacksquare h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n$

(1.21.13) $((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}) \wedge (h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n) \blacksquare (y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \blacksquare (y_0 + h_0)^n < x$

(1.21.14) $(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \blacksquare (y_0 + h_0)^n < x$

(1.21.15) $(0 < y_0 \mathbb{R}) \wedge (0 < h_0 < \mathbb{R}) \blacksquare 0 < y_0 < y_0 + h_0 \in \mathbb{R}$

(1.21.16) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ((y_0 + h_0)^n < x) \wedge (0 < y_0 + h_0 \in \mathbb{R}) \blacksquare (y_0 + h_0)^n \in E$

(1.21.17) $((y_0 + h_0)^n \in E) \wedge (y_0 < y_0 + h_0) \blacksquare \exists_{e \in E}(y_0 < e)$

(1.21.18) $LUB[y_0, E, \mathbb{R}, <] \blacksquare UpperBound[y_0, E, \mathbb{R}, <] \blacksquare \forall_{e \in E}(e \leq y_0) \blacksquare \neg\exists_{e \in E}(e > y_0)$

(1.21.19) $(\exists_{e \in E}(e > y_0)) \wedge (\neg\exists_{e \in E}(e > y_0)) \blacksquare \bot$

(1.22) ${y_0}^n < x \implies \bot \blacksquare {y_0}^n \geq x$

(1.23) ${y_0}^n > x \implies \ldots$

(1.23.1) $k_1 := \frac{{y_0}^n - x}{n{y_0}^{n-1}} \blacksquare (k_1 \in \mathbb{R}) \wedge (k_1 n{y_0}^{n-1} = {y_0}^n - x)$

(1.23.2) $(0 < x) \wedge (0 < n \in \mathbb{Z}) \blacksquare {y_0}^n - x < {y_0}^n \leq n{y_0}^n \blacksquare {y_0}^n - x < n{y_0}^n$

(1.23.3) ${y_0}^n - x < n{y_0}^n \blacksquare k_1 = \frac{{y_0}^n - x}{n{y_0}^{n-1}} < \frac{n{y_0}^n}{n{y_0}^{n-1}} = y_0 \blacksquare k_1 < y_0$

(1.23.4) ${y_0}^n > x \blacksquare 0 < {y_0}^n - x$

(1.23.5) $(n > 0) \wedge (y_0 > 0) \blacksquare 0 < n{y_0}^{n-1}$

(1.23.6) $(0 < {y_0}^n - x) \wedge 0 < (n{y_0}^{n-1}) \blacksquare 0 < \frac{{y_0}^n - x}{n{y_0}^{n-1}} = k_1 \blacksquare 0 < k_1$

(1.23.7) $(k_1 < y_0) \wedge (0 < k_1) \blacksquare (0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$

(1.23.8) $t \geq y_0 - k_1 \implies \ldots$

(1.23.8.1) $t \geq y_0 - k_1 \blacksquare t^n \geq (y_0 - k_1)^n \blacksquare -t^n \leq -(y_0 - k_1)^n \blacksquare {y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n$

(1.23.8.2) $RootLemma \wedge (0 < y_0 - k_1 < y_0) \blacksquare {y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}$

(1.23.8.3) $({y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n) \wedge ({y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}) \blacksquare {y_0}^n - t^n < k_1 n {y_0}^{n-1}$

(1.23.8.4) $(k_1 n {y_0}^{n-1} = {y_0}^n - x) \wedge ({y_0}^n - t^n < k_1 n {y_0}^{n-1}) \blacksquare {y_0}^n - t^n < {y_0}^n - x \blacksquare -t^n < -x \blacksquare t^n > x$

(1.23.8.5) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t^n > x) \blacksquare t \notin E$

(1.23.9) $t \geq y_0 - k_1 \implies t \notin E \blacksquare t \in E \implies t < y_0 - k_1 \blacksquare \forall_{t \in E}(t \leq y_0 - k_1) \blacksquare UpperBound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.10) $(LUB[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)) \blacksquare \neg UpperBound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.11) $(UpperBound[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg UpperBound[y_0 - k_1, E, \mathbb{R}, <]) \blacksquare \perp$

(1.24) ${y_0}^n > x \implies \perp \blacksquare {y_0}^n \leq x$

(1.25) $Order[\mathbb{R}, <] \blacksquare OrderTrichotomy[\mathbb{R}, <]$

(1.26) $(OrderTrichotomy[\mathbb{R}, <]) \wedge ({y_0}^n \geq x) \wedge ({y_0}^n \leq x) \blacksquare {y_0}^n = x$

(1.27) $({y_0}^n = x) \wedge (y_0 \in \mathbb{R}) \blacksquare \exists_{y \in \mathbb{R}}(y^n = x)$

(1.28) $y_1, y_2 := choice(\{y \in \mathbb{R} | y^n = x\})$

(1.29) $y_1 \neq y_2 \implies \ldots$

(1.29.1) $(OrderTrichotomy[\mathbb{R}, <]) \wedge (y_1 \neq y_2) \blacksquare (y_1 < y_2) \veebar (y_2 < y_1)\ \ldots$

(1.29.2) $\ldots (x = {y_1}^n < {y_2}^n = x) \veebar (x = {y_2}^n < {y_1}^n = x) \blacksquare (x < x) \veebar (x > x) \blacksquare \perp \veebar \perp \blacksquare \perp$

(1.30) $y_1 \neq y_2 \implies \perp \blacksquare y_1 = y_2 \blacksquare \forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b)$

(1.31) $(\exists_{y \in \mathbb{R}}(y^n = x)) \wedge (\forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b)) \blacksquare \exists!_{y \in \mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}}(y^n = x) \blacksquare \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}}({y_0}^n = x)$

$RootExistenceInRCorollary := \forall_{0<a \in \mathbb{R}} \forall_{0<b \in \mathbb{R}} \forall_{0<n \in \mathbb{Z}}((ab)^{1/n} = a^{1/n} b^{1/n})$ —

$$ExtendedRealSystem[\bar{\mathbb{R}}, +, *, <] := \left(\begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \ \wedge \ x - \infty = -\infty \ \wedge \ \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \ \wedge \\ (x > 0) \implies (x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty) \wedge \\ (x < 0) \implies (x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty) \end{array} \right)$$

$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$
$+_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d \rangle$
$*_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c \rangle$

$FieldC := Field[\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}}]$ —
$RSubfieldC := Subfield[\mathbb{R}, \mathbb{C}, +, *]$ —

$i := \langle 0, 1 \rangle \in \mathbb{C}$

$iProperty := i^2 = -1$ —
$CProperty := (a, b \in \mathbb{R}) \implies (\langle a, b \rangle = a + bi)$ —

$Conjugate[\overline{a + bi}] := a - bi$

$ConjugateProperties := (w, z \in \mathbb{C}) \implies \ldots$ —

(1) $\overline{z + w} = \overline{z} + \overline{w}$

(2) $\overline{z * w} = \overline{z} * \overline{w}$

(3) $Re(z) = (1/2)(z + \overline{z}) \wedge Im(z) = (1/2)(z - \overline{z})$

(4) $0 \leq z * \overline{z} \in \mathbb{R}$

$AbsoluteValueC[|z|] = (z * \overline{z})^{1/2}$
$AbsoluteValueProperties := (z, w \in \mathbb{C}) \implies \ldots$ —

(1) 123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y}((x, y) \in f)$

$(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \ldots$	
(1)	$Domain(f) := X; Codomain(f) := Y$
(2)	$Image(f, A) := \{f(a) \mid a \in A\}; Preimage(f, B) := \{a \mid f(a) \in B\}$
(3)	$Range(f) := Image(Domain(f))$

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X}(x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$
$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X}(y_0 = f(x))$
$Bijective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$
$SurjectiveEquivalent := (Range(f) = Codomain(f)) \implies Surjective(f)$

$(Function(f, X, Y) \wedge Function(g, Y, Z)) \implies (f \circ g)(x) := f(g(x)); Function(f \circ g, X, Z)$

$PropertiesofFunctions := (Function(f, A, B) \wedge Function(g, B, C) \wedge Function(h, C, D)) \implies \ldots$	
(1)	$h \circ (g \circ f) = (h \circ g) \circ f$
(2)	$(Injective(f) \wedge Injective(g)) \implies Injective(g \circ f)$
(3)	$(Surjective(f) \wedge Surjective(g)) \implies Surjective(g \circ f)$
(4)	$(Bijective(f, A, B)) \implies \exists_{f^{-1}}(Function(f^-1, B, A) \wedge \forall_{a \in A}(f^{-1}(f(a)) = a) \wedge \forall_{b \in B}(f(f^{-1}(b)) = b))$

$|(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}}(b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \ldots$	
(1)	$(a \mid b) \implies a \mid bc$
(2)	$(a \mid b \wedge b \mid c) \implies a \mid c$
(3)	$(a \mid b \wedge b \mid c) \implies a \mid (bx + cy)$
(4)	$(a \mid b \wedge b \mid a) \implies a = \pm b$
(5)	$(a \mid b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$
(6)	$(a \mid b) \iff (m \neq 0 \wedge ma \mid mb)$

$DivisionAlgorithm := (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}}(b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \wedge a \mid b \wedge a \mid c$
$GCD(a, b, c) := CD(a, b, c) \wedge \forall_d((d \mid b \wedge d \mid c) \implies d \mid a)$
$GCDEquivalent := 123123$

Chapter 3

Linear Algebra

$Matrix(A, m, n) := [a_{i,j}]_{m\times n}$:= m rows, n columns of real numbers
$+(A, B) := (A + B = [a_{i,j} + b_{i,j}]_{m\times n}) \wedge (Matrix(A, m, n)) \wedge (Matrix(B, m, n))$
$*\,(r, A) := (r * A = [ra_{i,j}]_{m\times n}) \wedge (Matrix(A, m, n))$
$*\,(A, B) := (A * B = \left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]_{m\times n}) \wedge (Matrix(A, m, p)) \wedge (Matrix(B, p, n))$
${}^{T}(A) := (A^T = [a_{j,i}]_{n\times m}) \wedge (Matrix(A, m, n))$

$AddCom := A + B = B + A \;\blacksquare \leftarrow A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$
$AddAssoc := (A + B) + C = A + (B + C) \;\blacksquare \leftarrow (A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$
$AddId := A + O = A = O + A \;\blacksquare \leftarrow A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$
$AddInv := A + (-A) = O = (-A) + A \;\blacksquare \leftarrow A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$

$MulAssoc := (A * B) * C = A * (B * C) \;\blacksquare \leftarrow$

(1) $(A * B) * C = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,j})\right] * C = \left[\sum_{k_2=1}^{p_2}(\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2})c_{k_2,j})\right] = \left[\sum_{k_2=1}^{p_2}\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right] = \ldots$

(2) $\ldots \left[\sum_{k_1=1}^{p_1}\sum_{k_2=1}^{p_2}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right] = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1}\sum_{k_2=1}^{p_2}(b_{k_1,k_2}c_{k_2,j}))\right] = \ldots = A * (B * C)$

$LeftDist := (A + B) * C = A * C + B * C \;\blacksquare \leftarrow$

(1) $(A + B) * C = [a_{i,j} + b_{i,j}] * C = \left[\sum_{k=1}^{p}((a_{i,k} + b_{i,k})c_{k,j})\right] = \ldots$

(2) $\ldots \left[\sum_{k=1}^{p}(a_{i,k}c_{k,j} + b_{i,k}c_{k,j})\right] = \left[\sum_{k=1}^{p}(a_{i,k}c_{k,j})\right] + \left[\sum_{k=1}^{p}(b_{i,k}c_{k,j})\right] = A * C + B * C$

$RightDist := C * (A + B) * C = C * A + C * B$ —
$Scalar1 := r(sA) = (rs)A = s(rA)$ —
$Scalar2 := A(rb) = r(AB)$ —
$Scalar3 := (r + s)A = rA + rS$ —
$Scalar4 := r(A + B) = rA + rB$ —

$Trans1 := A = (A^T)^T \;\blacksquare \leftarrow A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$
$Trans2 := (A + B)^T = A^T + B^T$ —
$Trans3 := (A * B)^T = B^T * A^T \;\blacksquare \leftarrow (A * B)^T = \left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]^T = \left[\sum_{k=1}^{p}(a_{j,k}b_{k,i})\right] = \left[\sum_{k=1}^{p}(b_{k,i}a_{j,k})\right] = \left[\sum_{k=1}^{p}(b^T_{i,k}a^T_{k,j})\right] = B^T * A^T$

$Sym(A) := A = A^T$; $SkewSym(A) := A = -A^T$
$SymGen := Sym(A + A^T) \;\blacksquare \leftarrow (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$
$SkewSymGem := SkewSym(A - A^T) \;\blacksquare \leftarrow -(A - A^T)^T = -(A^T - (A^T)^T) = -(A^T - A) = (A - A^T)$
$SymDec := A = (1/2) * (A + A^T) + (1/2) * (A - A^T)$ —

$Square(A) := Matrix(A, n, n)$
$Diagonal(A) := (Square(A)) \wedge (i \neq j \implies a_{i,j} = 0)$
$Scalar(A) := (Diagonal(A)) \wedge (a_{i,i} = k)$
$I := (Scalar(I)) \wedge (i_{d,d} = 1)$
$MulId := A * I = A = I * A$ —

$UpperTriangular(A) := (Square(A)) \wedge (i > \implies a_{i,j} = 0)$

$LowerTriangular(A) := (Square(A)) \wedge (i < j \implies a_{i,j} = 0)$

$Invertible(A) := \exists_{A^{-1}}(A * A^{-1} = I = A^{-1} * A)$
$UniqueInverse := ((A * B_1 = I) \wedge (A * B_2 = I)) \implies B_1 = B_2$ —
$Inv1 := (Invertible(A) \wedge Invertible(B)) \implies (A * B)^{-1} = B^{-1} * A^{-1}$ $\blacksquare \leftarrow$

(1)	$(A * B) * (A * B)^{-1} = I$ $\blacksquare$ $B * (A * B)^{-1} = A^{-1}$ $\blacksquare$ $(A * B)^{-1} = B^{-1} * A^{-1}$

$Inv2 := (Invertible(A)) \implies ((Invertible(A^{-1}) \wedge (A = (A^{-1})^{-1}))$ —
$InvTrans := (Invertible(A)) \implies ((Invertible(A^T) \wedge ((A^T)^{-1} = (A^{-1})^T))$ $\blacksquare \leftarrow$

(1)	$A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I$ $\blacksquare$ $(A^{-1})^T = (A^T)^{-1}$

$Sys(A, X, B) := (A * X = B) \wedge (Matrix(A, m, n)) \wedge (Matrix(X, n, 1)) \wedge (Matrix(B, m, 1))$
$Sol(X, A, B) := Sys(A, X, B)$
$TrivSol(X, A, B) := (Sol(X, A, B)) \wedge (a_{i,j} = 0)$

$RREF(A) := (Definition 1.18)$
$ElementaryRowOperation(\phi) := (Definition 1.19)$
$RowEquiv(A, B) := \exists_\Phi(\forall_{\phi \in \Phi}(ElementaryRowOperation(\phi)) \wedge |\Phi| \in \mathbb{N} \wedge \Phi(A) = B)$

$(A \neq O) \implies (\exists_B(RREF(B) \wedge RowEquiv(A, B)))$ (By Gauss-Jordan Elimination) —
$(Sys(A, X_1, B) \wedge Sys(C, X_2, D) \wedge RowEquiv([A|B], [C|D])) \implies (X_1 = X_2)$ (By algebra on systems of equations) —
$(RowEquiv(A, B)) \implies (Sys(A, X_1, O) \wedge Sys(B, X_1, O))$ (Corollary) —
$(m < n) \implies (\exists_X(\neg TrivSol(X, A, O)))$ $\blacksquare$ $(\forall_X(TrivSol(X, A, O)) \implies (m \geq n)$ $\blacksquare \leftarrow$

(1)	Let B = GaussJordan(A) and let r be the number of non-zero rows of B $\blacksquare$ $r \leq m$
(2)	$(r \leq m) \wedge (m < n)$ $\blacksquare$ $r < n$ $\blacksquare$ $0 < n - r$
(3)	The solution of B will have r fixed variables and n - r free variables which is also more then zero
(4)	The solution of B will have at least one free variable that can be non-zero, i.e., non-trivial

$Elem(A) := RowEquiv(A, I)$
$(Elem_i(A)) \implies (Invertible(A) \wedge Elem_i(A^{-1}))$
$(ElementaryRowOperation(\phi) \wedge (B = \phi(A))) \implies (Elem(E) \wedge B = E * A)$
$(RowEquiv(A, B)) \iff (B = \Pi(E_i) * A)$
$(RowEquiv(I, B)) \iff (B = \Pi(E_i))$

$(\forall_X(TrivSol(X, A, O)) \implies (RowEquiv(A, I))$ $\blacksquare \leftarrow$

(1)	Let B = GaussJordan(A) $\blacksquare$ $RowEquiv(A, B)$
(2)	$\forall_X(TrivSol(X, B, O)$ $\blacksquare$ $n \geq n$ $\blacksquare$ $n = n$ $\blacksquare$ $B = I$
(3)	$RowEquiv(A, B)$ $\blacksquare$ $RowEquiv(A, I)$

$(Invertible(A)) \iff (A = \Pi(E_i))$ $\blacksquare \leftarrow$

(1)	$Invertible(A) \implies \ldots$
(1.1)	$AX = O$ $\blacksquare$ $X = A^{-1}O = O$ $\blacksquare$ $\forall_X(TrivSol(X, A, O)$
(1.2)	$\forall_X(TrivSol(X, A, O)$ $\blacksquare$ $RowEquiv(A, I)$ $\blacksquare$ $A = \Pi(E_i)$
(2)	$A = \Pi(E_i) \implies \ldots$
(2.1)	$\Pi(E_{-i}{}^{-1}) = A^{-1}$ $\blacksquare$ $Invertible(A)$
(3)	$(Invertible(A)) \iff (A = \Pi(E_i))$

$(Invertible(A)) \iff (RowEquiv(I, A)))$ $\blacksquare \leftarrow (RowEquiv(I, B) \iff B = \Pi(E_i)) \wedge ((Invertible(A)) \iff (A = \Pi(E_i)))$

$(\forall_X(TrivSol(X, A, O))) \implies (Invertible(A))$ $\blacksquare \leftarrow$

(1)	$((\forall_X(TrivSol(X, A, O)) \implies (RowEquiv(A, I))) \wedge ((Invertible(A)) \iff (RowEquiv(I, A))))$

$(Invertible(A)) \implies (\forall_X(TrivSol(X, A, O)))$ $\blacksquare \leftarrow AX = O$ $\blacksquare$ $X = A^{-1}O = O$ $\blacksquare$ $\forall_X(TrivSol(X, A, O)$