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Chapter 1

Philosopherers

(1.5)

$$\text{OrderTrichotomy}(<, S) := \forall_{x,y \in S} (x < y \vee x = y \vee y < x)$$

$$\text{OrderTransitivity}(<, S) := \forall_{x,y,z \in S} ((x < y \wedge y < z) \implies x < z)$$

$$\text{Order}(<, S) := \text{OrderTrichotomy}(<, S) \wedge \text{OrderTransitivity}(<, S)$$

(1.7)

$$\text{BoundedAbove}(E, S, <) := \text{Order}(<, S) \wedge E \subset S \wedge \exists_{\beta \in S} \forall_{x \in E} (x \leq \beta)$$

$$\text{BoundedBelow}(E, S, <) := \text{Order}(<, S) \wedge E \subset S \wedge \exists_{\beta \in S} \forall_{x \in E} (\beta \leq x)$$

$$\text{UpperBound}(\beta, E, S, <) := \text{Order}(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E} (x \leq \beta)$$

$$\text{LowerBound}(\beta, E, S, <) := \text{Order}(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E} (\beta \leq x)$$

(1.8)

$$\text{LUB}(\alpha, E, S, <) := \text{UpperBound}(\alpha, E, S, <) \wedge \forall_{\gamma} (\gamma < \alpha \implies \neg \text{UpperBound}(\gamma, E, S, <))$$

$$\text{GLB}(\alpha, E, S, <) := \text{LowerBound}(\alpha, E, S, <) \wedge \forall_{\beta} (\alpha < \beta \implies \neg \text{LowerBound}(\beta, E, S, <))$$

(1.10)

$$\text{LUBProperty}(S, <) := \forall_E \left((\emptyset \neq E \subset S \wedge \text{BoundedAbove}(E, S, <)) \implies \exists_{\alpha \in S} (\text{LUB}(\alpha, E, S, <)) \right)$$

$$\text{GLBProperty}(S, <) := \forall_E \left((\emptyset \neq E \subset S \wedge \text{BoundedBelow}(E, S, <)) \implies \exists_{\alpha \in S} (\text{GLB}(\alpha, E, S, <)) \right)$$

(1.11)

$$\boxed{\text{LUBPropertyImpliesGLBProperty}} \quad \text{LUBProperty}(S, <) \implies \text{GLBProperty}(S, <)$$

$$(1) \quad \text{LUBProperty}(S, <) \implies \dots$$

wtS: 2

$$(1.1) \quad (\emptyset \neq B \subset S \wedge \text{BoundedBelow}(B, S, <)) \implies \dots$$

wtS: 1.2

$$(1.1.1) \quad \text{Order}(<, S) \wedge \exists_{\delta' \in S} (\text{LowerBound}(\delta', B, S, <))$$

from: BoundedBelow, 1.1

$$(1.1.2) \quad |B| = 1 \implies \dots$$

wtS: 1.1.3

$$(1.1.2.1) \quad \exists_{u'} (u' \in B) \blacksquare u := \text{choice}(\{u' \mid u' \in B\}) \blacksquare B = \{u\}$$

from: 1.1.2

$$(1.1.2.2) \quad \text{GLB}(u, B, S, <) \blacksquare \exists_{\epsilon_0 \in S} (\text{GLB}(\epsilon_0, B, S, <))$$

$$(1.1.3) \quad |B| = 1 \implies \exists_{\epsilon_0 \in S} (\text{GLB}(\epsilon_0, B, S, <))$$

$$(1.1.4) \quad |B| \neq 1 \implies \dots$$

wtS: 1.1.5

$$(1.1.4.1) \quad \forall_E \left((\emptyset \neq E \subset S \wedge \text{BoundedAbove}(E, S, <)) \implies \exists_{\alpha \in S} (\text{LUB}(\alpha, E, S, <)) \right)$$

from: LUBProperty, 1

$$(1.1.4.2) \quad L := \{s \in S \mid \text{LowerBound}(s, B, S, <)\}$$

$$(1.1.4.3) \quad |B| > 1 \wedge \text{OrderTrichotomy}(<, S) \blacksquare \exists_{b_1' \in B} \exists_{b_0' \in B} (b_0' < b_1')$$

from: Order, 1.1.1
wtS: 1.1.4.7

$$(1.1.4.4) \quad b_1 := \text{choice}(\{b_1' \in B \mid \exists_{b_0' \in B} (b_0' < b_1')\}) \blacksquare \neg \text{LowerBound}(b_1, B, S, <)$$

from: 1.1.4.2

$$(1.1.4.5) \quad b_1 \notin L \blacksquare L \subset S$$

$$(1.1.4.6) \quad \delta := \text{choice}(\{\delta' \in S \mid \text{LowerBound}(\delta', B, S, <)\}) \blacksquare \delta \in L \blacksquare \emptyset \neq L$$

from: 1.1.1

$$(1.1.4.7) \quad \emptyset \neq L \subset S$$

from: 1.1.4.5, 1.1.4.6

(1.1.4.8)	$\forall_{y \in L} (\text{LowerBound}(y, B, S, <)) \blacksquare \forall_{y \in L} \forall_{x \in B} (y \leq x)$	from: LowerBound, 1.1.4.2 wts: 1.1.4.10
(1.1.4.9)	$\forall_{x \in B} (x \in S \wedge \forall_{y \in L} (y \leq x)) \blacksquare \forall_{x \in B} (\text{UpperBound}(x, L, S, <))$	from: UpperBound
(1.1.4.10)	$\exists_{x \in S} (\text{UpperBound}(x, L, S, <)) \blacksquare \text{BoundedAbove}(L, S, <)$	
(1.1.4.11)	$\emptyset \neq L \subset S \wedge \text{BoundedAbove}(L, S, <)$	from: 1.1.4.7, 1.1.4.10
(1.1.4.12)	$\exists_{\alpha' \in S} (\text{LUB}(\alpha', L, S, <)) \blacksquare \alpha := \text{choice}(\{\alpha' \in S \mid (\text{LUB}(\alpha', L, S, <))\})$	from: 1.1.4.1 wts: 1.1.4.21
(1.1.4.13)	$\forall_x (x \in B \implies \text{UpperBound}(x, L, S, <))$	from: 1.1.4.9 wts: 1.1.4.17
(1.1.4.14)	$\forall_x (\neg \text{UpperBound}(x, L, S, <) \implies x \notin B)$	
(1.1.4.15)	$\gamma < \alpha \implies \dots$	wts: 1.1.4.16
(1.1.4.15.1)	$\neg \text{UpperBound}(\gamma, L, S, <) \blacksquare \gamma \notin B$	from: LUB, 1.1.4.12, 1.1.4.14
(1.1.4.16)	$\gamma < \alpha \implies \gamma \notin B \blacksquare \gamma \in B \implies \gamma \geq \alpha$	
(1.1.4.17)	$\forall_{\gamma \in B} (\alpha \leq \gamma) \blacksquare \text{LowerBound}(\alpha, B, S, <)$	from: LowerBound
(1.1.4.18)	$\alpha < \beta \implies \dots$	wts: 1.1.4.19
(1.1.4.18.1)	$\forall_{y \in L} (y \leq \alpha < \beta) \blacksquare \forall_{y \in L} (y \neq \beta)$	from: LUB, 1.1.4.12, 1.1.4.18
(1.1.4.18.2)	$\beta \notin L \blacksquare \neg \text{LowerBound}(\beta, B, S, <)$	from: 1.1.4.2
(1.1.4.19)	$\alpha < \beta \implies \neg \text{LowerBound}(\beta, B, S, <) \blacksquare \forall_{\beta \in S} (\alpha < \beta \implies \neg \text{LowerBound}(\beta, B, S, <))$	
(1.1.4.20)	$\text{LowerBound}(\alpha, B, S, <) \wedge \forall_{\beta \in S} (\alpha < \beta \implies \neg \text{LowerBound}(\beta, B, S, <))$	from: 1.1.4.17, 1.1.4.19
(1.1.4.21)	$\text{GLB}(\alpha, B, S, <) \blacksquare \exists_{\epsilon_1 \in S} (\text{GLB}(\epsilon_1, B, S, <))$	
(1.1.5)	$ B \neq 1 \implies \exists_{\epsilon_1 \in S} (\text{GLB}(\epsilon_1, B, S, <))$	
(1.1.6)	$(B = 1 \implies \exists_{\epsilon_0 \in S} (\text{GLB}(\epsilon_0, B, S, <))) \wedge (B \neq 1 \implies \exists_{\epsilon_1 \in S} (\text{GLB}(\epsilon_1, B, S, <)))$	from: 1.1.3, 1.1.5
(1.1.7)	$(B = 1 \vee B \neq 1) \implies \exists_{\epsilon \in S} (\text{GLB}(\epsilon, B, S, <)) \blacksquare \exists_{\epsilon \in S} (\text{GLB}(\epsilon, B, S, <))$	
(1.2)	$(\emptyset \neq B \subset S \wedge \text{BoundedBelow}(B, S, <)) \implies \exists_{\epsilon \in S} (\text{GLB}(\epsilon, B, S, <))$	
(1.3)	$\forall_B ((\emptyset \neq B \subset S \wedge \text{BoundedBelow}(B, S, <)) \implies \exists_{\epsilon \in S} (\text{GLB}(\epsilon, B, S, <)))$	
(1.4)	$\text{GLBProperty}(S, <)$	
(2)	$\text{LUBProperty}(S, <) \implies \text{GLBProperty}(S, <)$	

(1.12)

$$\text{Field}(F, +, *) := \exists_{0, 1 \in F} \forall_{x, y, z \in F} \left(\begin{array}{l} x + y \in F \quad \wedge \quad x * y \in F \quad \wedge \\ x + y = y + x \quad \wedge \quad x * y = y * x \quad \wedge \\ (x + y) + z = x + (y + z) \quad \wedge \quad (x * y) * z = x * (y * z) \quad \wedge \\ 1 \neq 0 \quad \wedge \quad x * (y + z) = (x * y) + (x * z) \quad \wedge \\ 0 + x = x \quad \wedge \quad 1 * x = x \quad \wedge \\ \exists_{-x \in F} (x + (-x) = 0) \wedge (x \neq 0 \implies \exists_{1/x \in F} (x * (1/x) = 1)) \end{array} \right)$$

***** $(\text{Field}(F, +, *) \wedge x, y, z \in F) \implies \dots$ *****

(1.14)

$$\boxed{\text{AdditiveCancellation}} \quad (x + y = x + z) \implies y = z$$

$$(1) \quad y = 0 + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \dots$$

$$(2) \quad (-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = 0 + z = z$$

$$\boxed{\text{AdditiveIdentityUniqueness}} \quad (x + y = x) \implies y = 0$$

$$(1) \quad x + y = x = 0 + x = x + 0$$

$$(2) \quad y = 0$$

AdditiveInverseUniqueness $(x + y = 0) \implies y = -x$

$$(1) \quad x + y = 0 = x + (-x)$$

from: [Field](#)

$$(2) \quad y = -x$$

from: [AdditiveCancellation](#)

DoubleNegative $x = -(-x)$

$$(1) \quad 0 = x + (-x) = (-x) + x \quad \blacksquare \quad 0 = (-x) + x$$

from: [Field](#)

$$(2) \quad x = -(-x)$$

from: [AdditiveInverseUniqueness](#)

(1.15)

MultiplicativeCancellation $(x \neq 0 \wedge x * y = x * z) \implies y = z$ —

MultiplicativeIdentityUniqueness $(x \neq 0 \wedge x * y = x) \implies y = 1$ —

MultiplicativeInverseUniqueness $(x \neq 0 \wedge x * y = 1) \implies y = 1/x$ —

DoubleReciprocal $(x \neq 0) \implies x = 1/(1/x)$ —

(1.16)

Domination $0 * x = 0$

$$(1) \quad 0 * x = (0 + 0) * x = 0 * x + 0 * x \quad \blacksquare \quad 0 * x = 0 * x + 0 * x$$

from: [Field](#)

$$(2) \quad 0 * x = 0$$

from: [AdditiveIdentityUniqueness](#)

NonDomination $(x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$

$$(1) \quad (x \neq 0 \wedge y \neq 0) \implies \dots$$

$$(1.1) \quad (x * y = 0) \implies \dots$$

from: [Field](#), [Domination](#), 1, 1.1

$$(1.1.1) \quad 1 = 1 * 1 = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = 0 * ((1/x) * (1/y)) = 0$$

$$(1.1.2) \quad 1 = 0 \wedge 1 \neq 0 \quad \blacksquare \quad \perp$$

from: [Field](#)

$$(1.2) \quad (x * y = 0) \implies \perp \quad \blacksquare \quad x * y \neq 0$$

$$(2) \quad (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$$

NegationCommutativity $(-x) * y = -(x * y) = x * (-y)$

$$(1) \quad x * y + (-x) * y = (x + -x) * y = 0 * y = 0 \quad \blacksquare \quad x * y + (-x) * y = 0$$

from: [Field](#), [Domination](#)
wts: 2

$$(2) \quad (-x) * y = -(x * y)$$

from: [AdditiveInverseUniqueness](#)

$$(3) \quad x * y + x * (-y) = x * (y + -y) = x * 0 = 0 \quad \blacksquare \quad x * y + x * (-y) = 0$$

from: [Field](#), [Domination](#)
wts: 4

$$(4) \quad x * (-y) = -(x * y)$$

from: [AdditiveInverseUniqueness](#)

$$(5) \quad (-x) * y = -(x * y) = x * (-y)$$

from: 2, 4

NegativeMultiplication $(-x) * (-y) = x * y$

$$(1) \quad (-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$$

from: [NegationCommutativity](#), [DoubleNegative](#)

(1.17)

$$\textbf{OrderedField}(F, +, *, <) := \left(\begin{array}{l} \textbf{Field}(F, +, *) \quad \wedge \quad \textbf{Order}(<, F) \quad \wedge \\ \forall_{x,y,z \in F} (y < z \implies x + y < x + z) \quad \wedge \\ \forall_{x,y \in F} ((x > 0 \wedge y > 0) \implies x * y > 0) \end{array} \right)$$

$$\textbf{OrderedField}(F, +, *, <) \wedge x, y, z \in F \implies \dots$$

(1.18)

NegationOnOrder $x > 0 \iff -x < 0$

$$(1) \quad x > 0 \implies \dots$$

$$(1.1) \quad 0 = (-x) + x > (-x) + 0 = -x \quad \blacksquare \quad 0 > -x \quad \blacksquare \quad -x < 0$$

from: OrderedField

$$(2) \quad x > 0 \implies -x < 0$$

$$(3) \quad -x < 0 \implies \dots$$

$$(3.1) \quad 0 = x + (-x) < x + 0 = x \quad \blacksquare \quad 0 < x \quad \blacksquare \quad x > 0$$

from: OrderedField

$$(4) \quad -x < 0 \implies x > 0$$

$$(5) \quad x > 0 \implies -x < 0 \wedge -x < 0 \implies x > 0 \quad \blacksquare \quad x > 0 \iff -x < 0$$

from: 2, 4

$$\text{PositiveFactorPreservesOrder} \quad (x > 0 \wedge y < z) \implies x * y < x * z$$

$$(1) \quad (x > 0 \wedge y < z) \implies \dots$$

$$(1.1) \quad (-y) + z > (-y) + y = 0 \quad \blacksquare \quad z + (-y) = 0$$

from: OrderedField

$$(1.2) \quad x * (z + (-y)) > 0 \quad \blacksquare \quad x * z + x * (-y) > 0$$

from: OrderedField

$$(1.3) \quad x * z = 0 + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \dots$$

from: Field, NegationCommutativity

$$(1.4) \quad x * y + (x * z + x * (-y)) > x * y + 0 = x * y$$

from: Field, 1.2

$$(1.5) \quad x * z > x * y$$

from: 1.3, 1.4

$$(2) \quad (x > 0 \wedge y < z) \implies x * z > x * y$$

$$\text{NegativeFactorFlipsOrder} \quad (x < 0 \wedge y < z) \implies x * y > x * z$$

$$(1) \quad (x < 0 \wedge y < z) \implies \dots$$

$$(1.1) \quad -x > 0$$

from: NegationOnOrder

$$(1.2) \quad (-x) * y < (-x) * z \quad \blacksquare \quad 0 = x * y + (-x) * y < x * y + (-x) * z \quad \blacksquare \quad 0 < x * y + (-x) * z$$

from: PositiveFactorPreservesOrder

$$(1.3) \quad 0 < (-x) * (-y + z) \quad \blacksquare \quad 0 > x * (-y + z) \quad \blacksquare \quad 0 > -(x * y) + x * z$$

from: NegationOnOrder

$$(1.4) \quad x * y > x * z$$

$$(2) \quad (x < 0 \wedge y < z) \implies x * y > x * z$$

$$\text{SquareIsPositive} \quad (x \neq 0) \implies x * x > 0$$

$$(1) \quad (x > 0) \implies x * x > 0$$

from: OrderedField

$$(2) \quad (x < 0) \implies \dots$$

$$(2.1) \quad -x > 0 \quad \blacksquare \quad x * x = (-x) * (-x) > 0 \quad \blacksquare \quad x * x > 0$$

from: NegationOnOrder, OrderedField, NegativeMultiplication

$$(3) \quad (x < 0) \implies x * x > 0$$

$$(4) \quad x \neq 0 \implies (x > 0 \vee x < 0) \implies x * x > 0 \quad \blacksquare \quad x \neq 0 \implies x * x > 0$$

from: OrderTrichotomy, 1, 3

$$\text{OnesPositive} \quad 1 > 0$$

$$(1) \quad 1 \neq 0 \quad \blacksquare \quad 1 = 1 * 1 > 0$$

from: Field, SquaresPositive

$$\text{ReciprocationOnOrder} \quad (0 < x < y) \implies 0 < 1/y < 1/x$$

$$(1) \quad (0 < x < y) \implies \dots$$

$$(1.1) \quad x * (1/x) = 1 > 0 \quad \blacksquare \quad x * (1/x) > 0$$

from: Field, OnesPositive

$$(1.2) \quad 1/x < 0 \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp \quad \blacksquare \quad 1/x > 0$$

from: NegativeFactorFlipsOrder, 1

$$(1.3) \quad y * (1/y) = 1 > 0 \quad \blacksquare \quad y * (1/y) > 0$$

from: Field, OnesPositive

$$(1.4) \quad 1/y < 0 \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp \quad \blacksquare \quad 1/y > 0$$

from: NegativeFactorFlipsOrder, 1

$$(1.5) \quad (1/x) * (1/y) > 0$$

from: OrderedField

$$(1.6) \quad 0 < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$$

from: OrderedField, 1, 1.4, 1.5

(1.19)

FieldQ

 $\text{OrderedField}(\mathbb{Q}, +, *, <)$
 $\text{Subfield}(K, F, +, *) := \text{Field}(F, +, *) \wedge K \subset F \wedge \text{Field}(K, +, *)$
 $\text{OrderedSubfield}(K, F, +, *, <) := \text{OrderedField}(F, +, *, <) \wedge K \subset F \wedge \text{OrderedField}(K, +, *, <)$
 $\text{CutI}(\alpha) := \emptyset \neq \alpha \subset \mathbb{Q}$
 $\text{CutII}(\alpha) := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
 $\text{CutIII}(\alpha) := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
 $\text{SetR} := \mathbb{R} := \{\alpha \in \mathbb{Q} \mid \text{CutI}(\alpha) \wedge \text{CutII}(\alpha) \wedge \text{CutIII}(\alpha)\}$

CutCorollaryI

 $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \dots$

(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$

from: CutII, 1

(1.2) $q < p \implies q \in \alpha \blacksquare q \notin \alpha \implies q \geq p$

from: 1

(1.3) $(q \notin \alpha) \implies \dots$

(1.3.1) $q \geq p$

from: 1.2

(1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp \blacksquare q \neq p$

from: 1, 1.3

(1.3.3) $q \geq p \wedge q \neq p \blacksquare p < q$

(1.4) $q \notin \alpha \implies p < q \blacksquare p < q$

from: 1

(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

CutCorollaryII

 $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \dots$

(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$

from: CutII, 1

(1.2) $s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)) \blacksquare s \in \alpha \implies r \in \alpha$

from: 1, 1.1

(1.3) $r \notin \alpha \implies s \notin \alpha \blacksquare s \notin \alpha$

from: 1, 1.2

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

 $<<(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

OrderTrichotomyR

 $\text{OrderTrichotomy}(\mathbb{R}, <<)$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \dots$

(1.1) $\neg(\alpha << \beta \vee \alpha = \beta) \implies \dots$

(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$

from: *, 1.1

(1.1.2) $\exists_{p'} (p' \in \alpha \wedge p' \notin \beta) \blacksquare p := \text{choice}(\{p' \mid p' \in \alpha \wedge p' \notin \beta\})$

(1.1.3) $q \in \beta \implies \dots$

(1.1.3.1) $p, q \in \mathbb{Q}$

(1.1.3.2) $q < p$

from: CutCorollaryI

(1.1.3.3) $q \in \alpha$

from: CutII

(1.1.4) $q \in \beta \implies q \in \alpha$

(1.1.5) $\forall_{q \in \beta} (q \in \alpha) \blacksquare \beta \subseteq \alpha$

(1.1.6) $\beta \subset \alpha \blacksquare \beta << \alpha$

(1.2) $\neg(\alpha << \beta \vee \alpha = \beta) \implies \beta << \alpha$

(1.3) $\neg(\alpha << \beta \vee \alpha = \beta) \vee (\alpha << \beta \vee \alpha = \beta) \blacksquare (\beta << \alpha) \vee (\alpha << \beta \vee \alpha = \beta)$

(1.4) $\alpha = \beta \implies \neg(\alpha << \beta \vee \beta << \alpha)$

(1.5) $\alpha << \beta \implies \neg(\alpha = \beta \vee \beta << \alpha)$

(1.6) $\beta << \alpha \implies \neg(\alpha = \beta \vee \alpha << \beta)$

(1.7) $\alpha << \beta \vee \alpha = \beta \vee \alpha << \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha << \beta \vee \alpha = \beta \vee \alpha << \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}} (\alpha << \beta \vee \alpha = \beta \vee \alpha << \beta)$

(4) *OrderTrichotomy*($\mathbb{R}, <<$)

OrderTransitivityR *OrderTransitivity*($\mathbb{R}, <<$)

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$

(1.1) $(\alpha << \beta \wedge \beta << \gamma) \implies \dots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a \in \alpha} (a \in \beta) \wedge \forall_{b \in \beta} (b \in \gamma)$

(1.1.3) $\forall_{a \in \alpha} (\alpha \in \gamma) \blacksquare \alpha \subset \gamma \blacksquare \alpha << \gamma$

(1.2) $(\alpha << \beta \wedge \beta << \gamma) \implies \alpha << \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha << \beta \wedge \beta << \gamma) \implies \alpha << \gamma)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}} ((\alpha << \beta \wedge \beta << \gamma) \implies \alpha << \gamma)$

(4) *OrderTransitivity*($\mathbb{R}, <<$)

OrderR *Order*($<<, \mathbb{R}$)

from: *OrderTrichotomyR*, *OrderTransitivityR*
wts:

LUBPropertyR *LU BProperty*($\mathbb{R}, <<$)

(1) $(\emptyset \neq A \subset \mathbb{R} \wedge \text{Bounded Above}(A, \mathbb{R}, <<)) \implies \dots$

(1.1) $\gamma := \{p \in \mathbb{Q} \mid \exists_{\alpha \in A} (p \in \alpha)\}$

(1.2) $A \neq \emptyset \blacksquare \exists_{\alpha'} (\alpha' \in A) \blacksquare \alpha := \text{choice}(\{\alpha' \mid \alpha' \in A\})$

(1.3) $\alpha \neq \emptyset \blacksquare \exists_{\alpha'} (\alpha' \in \alpha) \blacksquare a := \text{choice}(\{\alpha' \mid \alpha' \in \alpha\})$

(1.4) $a \in \gamma \blacksquare \gamma \neq \emptyset$

(1.5) \dots

(1.6) $\exists_{\beta} (\text{Upper Bound}(b, A, \mathbb{R}, <<))$

(1.7) \dots

(1.8) $\gamma \subset \mathbb{R}$

(1.9) $\text{Cut I}(\gamma) \wedge \text{Cut II}(\gamma) \wedge \text{Cut III}(\gamma) \blacksquare \gamma \in \mathbb{R}$

(1.10) $\text{Upper Bound}(\gamma, A, \mathbb{R}, <<)$

(1.11) 123123

(1.12) $\forall_{\beta_2} (\beta_2 < \gamma \implies \neg \text{Upper Bound}(\beta_2, A, \mathbb{R}, <<))$

(1.13) $\text{LU B}(\gamma, A, \mathbb{R}, <<)$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge \text{Bounded Above}(A, \mathbb{R}, <<)) \implies \exists_{\gamma \in \mathcal{S}} (\text{LU B}(\gamma, A, \mathbb{R}, <<))$

ExistenceOfR $\exists_{\mathbb{R}} (\text{LU BProperty}(\mathbb{R}, <) \wedge \text{OrderedSubfield}(\mathbb{Q}, \mathbb{R}, +, *, <))$

(1) 123123

TODO: - MORE EXPLICIT MODUS PONENS ON *OrderTrichotomyR* ??? - name all properties - hyperlink all definitions ???

Chapter 2

First Chapter

(1) First

(1.1) Second

(1.2) Third

(2) Fourth

This will be an empty chapter and I will put some text here

$$\sum_{i=0}^{\infty} a_i x^i \tag{2.1}$$

The equation 2.1 shows a sum that is divergent. This formula will later be used in the page ??.

For further references ■ see [Something Linky](#) or go to the next url: <http://www.sharelatex.com> or open the next file [File.txt](#)

It's also possible to link directly any word or any sentence in your document. [supwithit](#)Sup With It Theorem

If you read this text, you will get no information. Really? Is there no information?

