Contents

Chapter 1

Real Analysis

(1.5)
$OrderTrichotomy(<, S) := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$
$OrderTransitivity(<, S) := \forall_{x,y,z \in S}\big((x < y \wedge y < z) \implies x < z\big)$
$Order(<, S) := OrderTrichotomy(<, S) \wedge OrderTransitivity(<, S)$

(1.7)
$BoundedAbove(E, S, <) := Order(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(x \leq \beta)$
$BoundedBelow(E, S, <) := Order(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(\beta \leq x)$
$UpperBound(\beta, E, S, <) := Order(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(x \leq \beta)$
$LowerBound(\beta, E, S, <) := Order(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(\beta \leq x)$

(1.8)
$LUB(\alpha, E, S, <) := UpperBound(\alpha, E, S, <) \wedge \forall_{\gamma}\big(\gamma < \alpha \implies \neg UpperBound(\gamma, E, S, <)\big)$
$GLB(\alpha, E, S, <) := LowerBound(\alpha, E, S, <) \wedge \forall_{\beta}\big(\alpha < \beta \implies \neg LowerBound(\beta, E, S, <)\big)$

(1.10)
$LUBProperty(S, <) := \forall_E\Big(\big(\emptyset \neq E \subset S \wedge BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$

$GLBProperty(S, <) := \forall_E\Big(\big(\emptyset \neq E \subset S \wedge BoundedBelow(E, S, <)\big) \implies \exists_{\alpha \in S}\big(GLB(\alpha, E, S, <)\big)\Big)$

(1.11)

$\boxed{LUBPropertyImpliesGLBProperty}$ $LUBProperty(S, <) \implies GLBProperty(S, <)$

(1)	$LUBProperty(S, <) \implies \ldots$	wts: 2
(1.1)	$\big(\emptyset \neq B \subset S \wedge BoundedBelow(B, S, <)\big) \implies \ldots$	wts: 1.2
(1.1.1)	$Order(<, S) \wedge \exists_{\delta' \in S}\big(LowerBound(\delta', B, S, <)\big)$	from: $BoundedBelow$, 1.1
(1.1.2)	$|B| = 1 \implies \ldots$	wts: 1.1.3
(1.1.2.1)	$\exists_{u'}(u' \in B)$ ▮ $u := choice(\{u'|u' \in B\})$ ▮ $B = \{u\}$	from: 1.1.2
(1.1.2.2)	$GLB(u, B, S, <)$ ▮ $\exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$	
(1.1.3)	$|B| = 1 \implies \exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$	
(1.1.4)	$|B| \neq 1 \implies \ldots$	wts: 1.1.5
(1.1.4.1)	$\forall_E\Big(\big(\emptyset \neq E \subset S \wedge BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$	from: $LUBProperty$, 1
(1.1.4.2)	$L := \{s \in S | LowerBound(s, B, S, <)\}$	
(1.1.4.3)	$|B| > 1 \wedge OrderTrichotomy(<, S)$ ▮ $\exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$	from: $Order$, 1.1.1 wts: 1.1.4.7
(1.1.4.4)	$b_1 := choice\Big(\{b_1' \in B | \exists_{b_0' \in B}(b_0' < b_1')\}\Big)$ ▮ $\neg LowerBound(b_1, B, S, <)$	from: 1.1.4.2
(1.1.4.5)	$b_1 \notin L$ ▮ $L \subset S$	
(1.1.4.6)	$\delta := choice\big(\{\delta' \in S | LowerBound(\delta', B, S, <)\}\big)$ ▮ $\delta \in L$ ▮ $\emptyset \neq L$	from: 1.1.1
(1.1.4.7)	$\emptyset \neq L \subset S$	from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y\in L}\Big(LowerBound(y, B, S, <)\Big)$ ▌ $\forall_{y\in L}\forall_{x\in B}(y \leq x)$
<div align="right" style="font-size:small">from: *LowerBound*, 1.1.4.2
wts: 1.1.4.10</div>

(1.1.4.9) $\forall_{x\in B}\Big(x \in S \wedge \forall_{y\in L}(y \leq x)\Big)$ ▌ $\forall_{x\in B}\Big(UpperBound(x, L, S, <)\Big)$
<div align="right" style="font-size:small">from: *UpperBound*</div>

(1.1.4.10) $\exists_{x\in S}\Big(UpperBound(x, L, S, <)\Big)$ ▌ $BoundedAbove(L, S, <)$

(1.1.4.11) $\emptyset \neq L \subset S \wedge BoundedAbove(L, S, <)$
<div align="right" style="font-size:small">from: 1.1.4.7, 1.1.4.10</div>

(1.1.4.12) $\exists_{\alpha'\in S}\Big(LUB(\alpha', L, S, <)\Big)$ ▌ $\alpha := choice\Big(\{\alpha' \in S \,|\, (LUB(\alpha', L, S, <))\}\Big)$
<div align="right" style="font-size:small">from: 1.1.4.1
wts: 1.1.4.21</div>

(1.1.4.13) $\forall_x\Big(x \in B \implies UpperBound(x, L, S, <)\Big)$
<div align="right" style="font-size:small">from: 1.1.4.9
wts: 1.1.4.17</div>

(1.1.4.14) $\forall_x\Big(\neg UpperBound(x, L, S, <) \implies x \notin B\Big)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$
<div align="right" style="font-size:small">wts: 1.1.4.16</div>

(1.1.4.15.1) $\neg UpperBound(\gamma, L, S, <)$ ▌ $\gamma \notin B$
<div align="right" style="font-size:small">from: *LUB*, 1.1.4.12, 1.1.4.14</div>

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B$ ▌ $\gamma \in B \implies \gamma \geq \alpha$

(1.1.4.17) $\forall_{\gamma\in B}(\alpha \leq \gamma)$ ▌ $LowerBound(\alpha, B, S, <)$
<div align="right" style="font-size:small">from: *LowerBound*</div>

(1.1.4.18) $\alpha < \beta \implies \ldots$
<div align="right" style="font-size:small">wts: 1.1.4.19</div>

(1.1.4.18.1) $\forall_{y\in L}(y \leq \alpha < \beta)$ ▌ $\forall_{y\in L}(y \neq \beta)$
<div align="right" style="font-size:small">from: *LUB*, 1.1.4.12, 1.1.4.18</div>

(1.1.4.18.2) $\beta \notin L$ ▌ $\neg LowerBound(\beta, B, S, <)$
<div align="right" style="font-size:small">from: 1.1.4.2</div>

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound(\beta, B, S, <)$ ▌ $\forall_{\beta\in S}\Big(\alpha < \beta \implies \neg LowerBound(\beta, B, S, <)\Big)$

(1.1.4.20) $LowerBound(\alpha, B, S, <) \wedge \forall_{\beta\in S}\Big(\alpha < \beta \implies \neg LowerBound(\beta, B, S, <)\Big)$
<div align="right" style="font-size:small">from: 1.1.4.17, 1.1.4.19</div>

(1.1.4.21) $GLB(\alpha, B, S, <)$ ▌ $\exists_{\epsilon_1\in S}\Big(GLB(\epsilon_1, B, S, <)\Big)$

(1.1.5) $|B| \neq 1 \implies \exists_{\epsilon_1\in S}\Big(GLB(\epsilon_1, B, S, <)\Big)$

(1.1.6) $\Big(|B| = 1 \implies \exists_{\epsilon_0\in S}\big(GLB(\epsilon_0, B, S, <)\big)\Big) \wedge \Big(|B| \neq 1 \implies \exists_{\epsilon_1\in S}\big(GLB(\epsilon_1, B, S, <)\big)\Big)$
<div align="right" style="font-size:small">from: 1.1.3, 1.1.5</div>

(1.1.7) $(|B| = 1 \vee |B| \neq 1) \implies \exists_{\epsilon\in S}\Big(GLB(\epsilon, B, S, <)\Big)$ ▌ $\exists_{\epsilon\in S}\Big(GLB(\epsilon, B, S, <)\Big)$

(1.2) $\Big(\emptyset \neq B \subset S \wedge BoundedBelow(B, S, <)\Big) \implies \exists_{\epsilon\in S}\Big(GLB(\epsilon, B, S, <)\Big)$

(1.3) $\forall_B\Big(\big(\emptyset \neq B \subset S \wedge BoundedBelow(B, S, <)\big) \implies \exists_{\epsilon\in S}\big(GLB(\epsilon, B, S, <)\big)\Big)$

(1.4) $GLBProperty(S, <)$

(2) $LUBProperty(S, <) \implies GLBProperty(S, <)$

(1.12)

$$Field(F, +, *) := \exists_{0,1\in F}\forall_{x,y,z\in F}\begin{cases} x + y \in F & \wedge & x * y \in F & \wedge \\ x + y = y + x & \wedge & x * y = y * x & \wedge \\ (x + y) + z = x + (y + z) & \wedge & (x * y) * z = x * (y * z) & \wedge \\ 1 \neq 0 & \wedge & x * (y + z) = (x * y) + (x * z) & \wedge \\ 0 + x = x & \wedge & 1 * x = x & \wedge \\ \exists_{-x\in F}\big(x + (-x) = 0\big) \wedge \Big(x \neq 0 \implies \exists_{1/x\in F}\big(x * (1/x) = 1\big)\Big) \end{cases}$$

********************* $\Big(Field(F, +, *) \wedge x, y, z \in F\Big) \implies \ldots$ *********************

(1.14)

| $AdditiveCancellation$ | $(x + y = x + z) \implies y = z$

(1) $y = 0 + y = \big(x + (-x)\big) + y = \big((-x) + x\big) + y = (-x) + (x + y) = \ldots$
<div align="right" style="font-size:small">from: *Field*</div>

(2) $(-x) + (x + z) = \big((-x) + x\big) + z = \big(x + (-x)\big) + z = 0 + z = z$
<div align="right" style="font-size:small">from: *Field*</div>

| $AdditiveIdentityUniqueness$ | $(x + y = x) \implies y = 0$

(1) $x + y = x = 0 + x = x + 0$
<div align="right" style="font-size:small">from: *Field*</div>

(2) $y = 0$
<div align="right" style="font-size:small">from: *AdditiveCancellation*</div>

$\boxed{AdditiveInverseUniqueness}$ $(x + y = \mathbb{0}) \implies y = -x$

(1) $\quad x + y = \mathbb{0} = x + (-x)$

(2) $\quad y = -x$

$\boxed{DoubleNegative}$ $\quad x = -(-x)$

(1) $\quad \mathbb{0} = x + (-x) = (-x) + x \quad\blacksquare\quad \mathbb{0} = (-x) + x$

(2) $\quad x = -(-x)$

(1.15)

$\boxed{MultiplicativeCancellation}$ $\quad (x \neq \mathbb{0} \land x * y = x * z) \implies y = z \qquad$ —

$\boxed{MultiplicativeIdentityUniqueness}$ $\quad (x \neq \mathbb{0} \land x * y = x) \implies y = \mathbb{1} \qquad$ —

$\boxed{MultiplicativeInverseUniqueness}$ $\quad (x \neq \mathbb{0} \land x * y = \mathbb{1}) \implies y = 1/x \qquad$ —

$\boxed{DoubleReciprocal}$ $\quad (x \neq \mathbb{0}) \implies x = 1/(1/x) \qquad$ —

(1.16)

$\boxed{Domination}$ $\quad \mathbb{0} * x = \mathbb{0}$

(1) $\quad \mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x \quad\blacksquare\quad \mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$

(2) $\quad \mathbb{0} * x = \mathbb{0}$

$\boxed{NonDomination}$ $\quad (x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $\quad (x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies \ldots$

(1.1) $\quad (x * y = \mathbb{0}) \implies \ldots$

(1.1.1) $\quad \mathbb{1} = \mathbb{1} * \mathbb{1} = \big(x * (1/x)\big) * \big(y * (1/y)\big) = (x * y) * \big((1/x) * (1/y)\big) = \mathbb{0} * \big((1/x) * (1/y)\big) = \mathbb{0}$

(1.1.2) $\quad \mathbb{1} = \mathbb{0} \land \mathbb{1} \neq \mathbb{0} \quad\blacksquare\quad \bot$

(1.2) $\quad (x * y = \mathbb{0}) \implies \bot \quad\blacksquare\quad x * y \neq \mathbb{0}$

(2) $\quad (x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

$\boxed{NegationCommutativity}$ $\quad (-x) * y = -(x * y) = x * (-y)$

(1) $\quad x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0} \quad\blacksquare\quad x * y + (-x) * y = \mathbb{0}$

(2) $\quad (-x) * y = -(x * y)$

(3) $\quad x * y + x * (-y) = x * (y + -y) = x * \mathbb{0} = \mathbb{0} \quad\blacksquare\quad x * y + x * (-y) = \mathbb{0}$

(4) $\quad x * (-y) = -(x * y)$

(5) $\quad (-x) * y = -(x * y) = x * (-y)$

$\boxed{NegativeMultiplication}$ $\quad (-x) * (-y) = x * y$

(1) $\quad (-x) * (-y) = -\big(x * (-y)\big) = -\big(-(x * y)\big) = x * y$

(1.17)

$OrderedField(F, +, *, <) := \left(\begin{array}{l} Field(F, +, *) \quad \land \quad Order(<, F) \quad \land \\ \forall_{x,y,z \in F}(y < z \implies x + y < x + z) \quad \land \\ \forall_{x,y \in F}\big((x > \mathbb{0} \land y > \mathbb{0}) \implies x * y > \mathbb{0}\big) \end{array} \right)$

************************** $\big(OrderedField(F, +, *, <) \land x, y, z \in F\big) \implies \ldots$ **************************

(1.18)

$\boxed{NegationOnOrder}$ $\quad x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $\quad x > \mathbb{0} \implies \ldots$

(1.1) $0 = (-x) + x > (-x) + 0 = -x$ ▮ $0 > -x$ ▮ $-x < 0$

(2) $x > 0 \implies -x < 0$

(3) $-x < 0 \implies \ldots$

(3.1) $0 = x + (-x) < x + 0 = x$ ▮ $0 < x$ ▮ $x > 0$

(4) $-x < 0 \implies x > 0$

(5) $x > 0 \implies -x < 0 \land -x < 0 \implies x > 0$ ▮ $x > 0 \iff -x < 0$

PositiveFactorPreservesOrder $(x > 0 \land y < z) \implies x * y < x * z$

(1) $(x > 0 \land y < z) \implies \ldots$

(1.1) $(-y) + z > (-y) + y = 0$ ▮ $z + (-y) = 0$

(1.2) $x * (z + (-y)) > 0$ ▮ $x * z + x * (-y) > 0$

(1.3) $x * z = 0 + x * z = \big(x * y + -(x * y)\big) + x * z = \big(x * y + x * (-y)\big) + x * z = \ldots$

(1.4) $x * y + \big(x * z + x * (-y)\big) > x * y + 0 = x * y$

(1.5) $x * z > x * y$

(2) $(x > 0 \land y < z) \implies x * z > x * y$

NegativeFactorFlipsOrder $(x < 0 \land y < z) \implies x * y > x * z$

(1) $(x < 0 \land y < z) \implies \ldots$

(1.1) $-x > 0$

(1.2) $(-x) * y < (-x) * z$ ▮ $0 = x * y + (-x) * y < x * y + (-x) * z$ ▮ $0 < x * y + (-x) * z$

(1.3) $0 < (-x) * (-y + z)$ ▮ $0 > x * (-y + z)$ ▮ $0 > -(x * y) + x * z$

(1.4) $x * y > x * z$

(2) $(x < 0 \land y < z) \implies x * y > x * z$

SquareIsPositive $(x \neq 0) \implies x * x > 0$

(1) $(x > 0) \implies x * x > 0$

(2) $(x < 0) \implies \ldots$

(2.1) $-x > 0$ ▮ $x * x = (-x) * (-x) > 0$ ▮ $x * x > 0$

(3) $(x < 0) \implies x * x > 0$

(4) $x \neq 0 \implies (x > 0 \lor x < 0) \implies x * x > 0$ ▮ $x \neq 0 \implies x * x > 0$

OneIsPositive $1 > 0$

(1) $1 \neq 0$ ▮ $1 = 1 * 1 > 0$

ReciprocationOnOrder $(0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \ldots$

(1.1) $x * (1/x) = 1 > 0$ ▮ $x * (1/x) > 0$

(1.2) $1/x < 0 \implies x * (1/x) < 0 \land x * (1/x) > 0 \implies \bot$ ▮ $1/x > 0$

(1.3) $y * (1/y) = 1 > 0$ ▮ $y * (1/y) > 0$

(1.4) $1/y < 0 \implies y * (1/y) < 0 \land y * (1/y) > 0 \implies \bot$ ▮ $1/y > 0$

(1.5) $(1/x) * (1/y) > 0$

(1.6) $0 < 1/y = \big((1/x) * (1/y)\big) * x < \big((1/x) * (1/y)\big) * y = 1/x$

(1.19)

$\boxed{Ordered\,Field\,Q}$ $Ordered\,Field(\mathbb{Q}, +, *, <)$ —

$Subfield(K, F, +, *) := Field(F, +, *) \wedge K \subset F \wedge Field(K, +, *)$
$Ordered\,Subfield(K, F, +, *, <) := Ordered\,Field(F, +, *, <) \wedge K \subset F \wedge Ordered\,Field(K, +, *, <)$
$Cut\,I(\alpha) := \emptyset \neq \alpha \subset \mathbb{Q}$
$Cut\,II(\alpha) := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha)$
$Cut\,III(\alpha) := \forall_{p \in \alpha} \exists_{r \in \alpha}(p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} | Cut\,I(\alpha) \wedge Cut\,II(\alpha) \wedge Cut\,III(\alpha)\}$

$\boxed{CutCorollaryI}$ $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \ldots$	
(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}}(q' < p' \implies q' \in \alpha)$	from: $CutII, 1$
(1.2) $q < p \implies q \in \alpha$ ▌ $q \notin \alpha \implies q \geq p$	from: 1
(1.3) $(q \notin \alpha) \implies \ldots$	
(1.3.1) $q \geq p$	from: 1.2
(1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp$ ▌ $q \neq p$	from: 1, 1.3
(1.3.3) $q \geq p \wedge q \neq p$ ▌ $p < q$	
(1.4) $q \notin \alpha \implies p < q$ ▌ $p < q$	from: 1
(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$	

$\boxed{CutCorollaryII}$ $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$	
(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}}(r' < s' \implies r' \in \alpha)$	from: $CutII, 1$
(1.2) $s \in \alpha \implies \big(r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)\big)$ ▌ $s \in \alpha \implies r \in \alpha$	from: 1, 1.1
(1.3) $r \notin \alpha \implies s \notin \alpha$ ▌ $s \notin \alpha$	from: 1, 1.2
(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$	

$<_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$\boxed{OrderTrichotomyOf\,R}$ $OrderTrichotomy(\mathbb{R}, <_{\mathbb{R}})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$	
(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$	
(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$	from: $<_{\mathbb{R}}, 1.1$
(1.1.2) $\exists_{p'}(p' \in \alpha \wedge p' \notin \beta)$ ▌ $p := choice(\{p' | p' \in \alpha \wedge p' \notin \beta\})$	
(1.1.3) $q \in \beta \implies \ldots$	
(1.1.3.1) $p, q \in \mathbb{Q}$	
(1.1.3.2) $q < p$	from: $CutCorollaryI$
(1.1.3.3) $q \in \alpha$	from: $CutII$
(1.1.4) $q \in \beta \implies q \in \alpha$	
(1.1.5) $\forall_{q \in \beta}(q \in \alpha)$ ▌ $\beta \subseteq \alpha$	
(1.1.6) $\beta \subset \alpha$ ▌ $\beta <_{\mathbb{R}} \alpha$	
(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$	
(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$ ▌ $(\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$	
(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$	
(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$	
(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$	
(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$	
(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$	

(3) $\forall_{\alpha,\beta\in\mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) *OrderTrichotomy*$(\mathbb{R}, <_{\mathbb{R}})$

OrderTransitivityOfR   *OrderTransitivity*$(\mathbb{R}, <_{\mathbb{R}})$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

  (1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

    (1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

    (1.1.2) $\forall_{a\in\alpha}(a \in \beta) \wedge \forall_{b\in\beta}(b \in \gamma)$

    (1.1.3) $\forall_{a\in\alpha}(\alpha \in \gamma)$ ▮ $\alpha \subset \gamma$ ▮ $\alpha <_{\mathbb{R}} \gamma$

  (1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(3) $\forall_{\alpha,\beta,\gamma\in\mathbb{R}}\big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(4) *OrderTransitivity*$(\mathbb{R}, <_{\mathbb{R}})$

OrderOfR   *Order*$(<_{\mathbb{R}}, \mathbb{R})$

LUBPropertyOfR   *LUBProperty*$(\mathbb{R}, <_{\mathbb{R}})$

(1) $\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \ldots$

  (1.1) $\gamma := \{p \in \mathbb{Q} | \exists_{\alpha\in A}(p \in \alpha)\}$

  (1.2) $A \neq \emptyset$ ▮ $\exists_{\alpha}(\alpha \in A)$ ▮ $\alpha_0 := choice(\{\alpha | \alpha \in A\})$

  (1.3) $\alpha_0 \neq \emptyset$ ▮ $\exists_a(a \in \alpha_0)$ ▮ $a_0 := choice(\{a | a \in \alpha_0\})$ ▮ $a_0 \in \gamma$ ▮ $\gamma \neq \emptyset$

  (1.4) *BoundedAbove*$(A, \mathbb{R}, <_{\mathbb{R}})$ ▮ $\exists_{\beta}\big(UpperBound(\beta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

  (1.5) $\beta_0 := choice(\{\beta | UpperBound(\beta, A, \mathbb{R}, <_{\mathbb{R}})\})$

  (1.6) *UpperBound*$(\beta_0, A, \mathbb{R}, <_{\mathbb{R}})$ ▮ $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \beta_0)$ ▮ $\forall_{\alpha\in A}(\alpha \subseteq \beta_0)$ ▮ $\forall_{\alpha\in A}\forall_{a\in\alpha}(a \in \beta_0)$

  (1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma$ ▮ $\forall_{a\in\gamma}(a \in \beta_0)$ ▮ $\gamma \subseteq \beta_0$

  (1.8) $\beta_0 \subset \mathbb{Q}$ ▮ $\gamma \subseteq \beta_0 \subset \mathbb{Q}$ ▮ $\gamma \subset \mathbb{Q}$

  (1.9) $\emptyset \neq \gamma \subset \mathbb{Q}$ ▮ *CutI*(γ)

  (1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

    (1.10.1) $p \in \gamma$ ▮ $\exists_{\alpha\in A}(p \in \alpha)$ ▮ $\alpha_1 := choice(\{\alpha \in A | p \in \alpha\})$

    (1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p$ ▮ $q \in \alpha_1$ ▮ $q \in \gamma$

  (1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma$ ▮ $\forall_{p\in\gamma}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \gamma)$ ▮ *CutII*(γ)

  (1.12) $p \in \gamma \implies \ldots$

    (1.12.1) $\exists_{\alpha\in A}(p \in \alpha)$ ▮ $\alpha_2 := choice(\{\alpha \in A | p \in \alpha\})$

    (1.12.2) $\alpha_2 \in \mathbb{R}$ ▮ *CutII*(α_2) ▮ $\exists_{r\in\alpha_2}(p < r)$ ▮ $r_0 := choice(\{r \in \alpha_2 | p < r\})$

    (1.12.3) $r_0 \in \alpha_2$ ▮ $r_0 \in \gamma$

    (1.12.4) $p < r_0$ ▮ $p < r_0 \wedge r_0 \in \gamma$ ▮ $\exists_{r\in\gamma}(p < r)$

  (1.13) $p \in \gamma \implies \exists_{r\in\gamma}(p < r)$ ▮ $\forall_{p\in\gamma}\exists_{r\in\gamma}(p < r)$ ▮ *CutIII*(γ)

  (1.14) *CutI*$(\gamma) \wedge CutII(\gamma) \wedge CutIII(\gamma)$ ▮ $\gamma \in \mathbb{R}$

  (1.15) $\forall_{\alpha\in A}(\alpha \subseteq \gamma)$ ▮ $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma)$

  (1.16) $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R}$ ▮ *UpperBound*$(\gamma, A, \mathbb{R}, <_{\mathbb{R}})$

  (1.17) $\delta <_{\mathbb{R}} \gamma \implies \ldots$

    (1.17.1) $\delta \subset \gamma$ ▮ $\exists_s(s \in \gamma \wedge s \notin \delta)$ ▮ $s_0 := choice(\{s \in \mathbb{Q} | s \in \gamma \wedge s \notin \delta\})$

    (1.17.2) $s_0 \in \gamma$ ▮ $\exists_{\alpha\in A}(s_0 \in \alpha)$ ▮ $\alpha_3 := choice(\{\alpha \in A | s_0 \in \alpha\})$

    (1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta$ ▮ $\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

    (1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \ldots$

      (1.17.4.1) $\alpha_3 \subseteq \delta$ ▮ $\forall_{s\in\mathbb{Q}}(s \in \alpha_3 \implies s \in \delta)$ ▮ $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

      (1.17.4.2) $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$ ▮ $\bot$

    (1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \bot$ ▮ $\delta <_{\mathbb{R}} \alpha_3$ ▮ $\exists_{\alpha\in A}(\delta <_{\mathbb{R}} \alpha)$ ▮ $\exists_{\alpha\in A}\big(\neg(\alpha \leq_{\mathbb{R}} \delta)\big)$

    (1.17.6) $\neg\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \delta)$ ▮ $\neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})$

(1.18) $\delta <_\mathbb{R} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_\mathbb{R})$ ▌ $\forall_\delta (\delta <_\mathbb{R} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_\mathbb{R}))$

(1.19) $UpperBound(\gamma, A, \mathbb{R}, <_\mathbb{R}) \wedge \forall_\delta (\delta <_\mathbb{R} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_\mathbb{R}))$

(1.20) $LUB(\gamma, A, \mathbb{R}, <_\mathbb{R})$ ▌ $\exists_{\gamma \in S} (LUB(\gamma, A, \mathbb{R}, <_\mathbb{R}))$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_\mathbb{R})) \implies \exists_{\gamma \in S} (LUB(\gamma, A, \mathbb{R}, <_\mathbb{R}))$

(3) $\forall_A \Big((\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_\mathbb{R})) \implies \exists_{\gamma \in S} (LUB(\gamma, A, \mathbb{R}, <_\mathbb{R})) \Big)$ ▌ $LUBProperty(\mathbb{R}, <_\mathbb{R})$

$+_\mathbb{R}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_\mathbb{R} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$
$0_\mathbb{R} := \{x \in \mathbb{Q} | x < 0\}$

$\boxed{0InR}$ $0_\mathbb{R} \in \mathbb{R}$

(1) $-1 \in 0_\mathbb{R} \wedge 1 \notin 0_\mathbb{R}$ ▌ $\emptyset \neq 0_\mathbb{R} \subseteq \mathbb{Q}$ ▌ $CutI(0_\mathbb{R})$

(2) $(x \in 0_\mathbb{R} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_\mathbb{R}$ ▌ $\forall_{x \in 0_\mathbb{R}} \forall_{y \in \mathbb{Q}} (y < x \implies y \in 0_\mathbb{R})$ ▌ $CutII(0_\mathbb{R})$

(3) $y := x/2$ ▌ $(x \in 0_\mathbb{R}) \implies (x < y < 0) \implies \exists_{y \in 0_\mathbb{R}} (x < y)$ ▌ $\forall_{x \in 0_\mathbb{R}} \exists_{y \in 0_\mathbb{R}} (x < y)$ ▌ $CutIII(0_\mathbb{R})$

(4) $CutI(0_\mathbb{R}) \wedge CutII(0_\mathbb{R}) \wedge CutIII(0_\mathbb{R})$ ▌ $0_\mathbb{R} \in \mathbb{R}$

$\boxed{FieldAdditionClosureOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_\mathbb{R} \beta) \in \mathbb{R})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \dots$

 (1.1) $(\alpha +_\mathbb{R} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$

 (1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

 (1.3) $\exists_a (a \in \alpha) ; \exists_b (b \in \beta)$ ▌ $a_0 := choice(\{a | a \in \alpha\}) ; b_0 := choice(\{b | b \in \beta\})$ ▌ $a_0 + b_0 \in \alpha +_\mathbb{R} \beta$

 (1.4) $\exists_x (x \notin \alpha) ; \exists_y (y \notin \beta)$ ▌ $x_0 := choice(\{x | x \notin \alpha\}) ; y_0 := choice(\{y | y \notin \beta\})$

 (1.5) $\forall_{r \in \alpha} (r < x_0) ; \forall_{s \in \beta} (s < y_0)$ ▌ $\forall_{r \in \alpha} \forall_{s \in \beta} (r + s < x_0 + y_0)$ ▌ $x_0 + y_0 \notin \alpha +_\mathbb{R} \beta$

 (1.6) $\emptyset \neq \alpha +_\mathbb{R} \beta \subset \mathbb{Q}$ ▌ $CutI(\alpha +_\mathbb{R} \beta)$

 (1.7) $(p \in \alpha +_\mathbb{R} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$

   (1.7.1) $\exists_{r \in \alpha} \exists_{s \in \beta} (p = r + s)$ ▌ $(r_0, s_0) := choice((r, s) \in \alpha \times \beta | p = r + s)$

   (1.7.2) $q < p = r_0 + s_0$ ▌ $(q - s_0) < r_0$ ▌ $(q - s_0) \in \alpha$

   (1.7.3) $s_0 \in \beta$ ▌ $q = (q - s_0) + s_0 \in \alpha +_\mathbb{R} \beta$ ▌ $q \in \alpha +_\mathbb{R} \beta$

 (1.8) $(p \in \alpha +_\mathbb{R} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_\mathbb{R} \beta$ ▌ $\forall_{p \in \alpha +_\mathbb{R} \beta} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha +_\mathbb{R} \beta)$ ▌ $CutII(\alpha +_\mathbb{R} \beta)$

 (1.9) $p \in \alpha \implies \dots$

   (1.9.1) $\exists_{r \in \alpha} \exists_{s \in \beta} (p = r + s)$ ▌ $(r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta | p = r + s\})$

   (1.9.2) $r_1 \in \alpha$ ▌ $\exists_{t \in \alpha} (r_1 < t)$ ▌ $t_0 := choice(\{t \in \alpha | r_1 < t\})$

   (1.9.3) $s_1 \in \beta$ ▌ $t + s_1 \in \alpha +_\mathbb{R} \beta \wedge p = r_1 + s_1 < t + s_1$ ▌ $\exists_{r \in \alpha +_\mathbb{R} \beta} (p < r)$

 (1.10) $p \in \alpha \implies \exists_{r \in \alpha +_\mathbb{R} \beta} (p < r)$ ▌ $\forall_{p \in \alpha +_\mathbb{R} \beta} \exists_{r \in \alpha +_\mathbb{R} \beta} (p < r)$ ▌ $CutIII(\alpha +_\mathbb{R} \beta)$

 (1.11) $CutI(\alpha +_\mathbb{R} \beta) \wedge CutII(\alpha +_\mathbb{R} \beta) \wedge CutIII(\alpha +_\mathbb{R} \beta)$ ▌ $\alpha +_\mathbb{R} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_\mathbb{R} \beta) \in \mathbb{R})$

$\boxed{FieldAdditionCommutativityOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies (\alpha +_\mathbb{R} \beta = \beta +_\mathbb{R} \alpha)$

(1) $\alpha +_\mathbb{R} \beta = \{r + s | r \in \alpha \wedge s \in \beta\} = \{s + r | s \in \beta \wedge r \in \alpha\} = \beta +_\mathbb{R} \alpha$

$\boxed{FieldAdditionAssociativityOfR}$ $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma))$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$

 (1.1) $(\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \{(a + b) + c | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \dots$

 (1.2) $\{a + (b + c) | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma)$

$\boxed{FieldAdditionIdentityOfR}$ $(\alpha \in \mathbb{R}) \implies 0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

 (1.1) $(r \in \alpha \wedge s \in 0_\mathbb{R}) \implies \ldots$

  (1.1.1) $s < 0$ ▌ $r + s < r + 0 = r$ ▌ $r + s < r$ ▌ $r + s \in \alpha$

 (1.2) $(r \in \alpha \wedge s \in 0_\mathbb{R}) \implies r + s \in \alpha$ ▌ $\forall_{r \in \alpha} \forall_{s \in 0_\mathbb{R}} (r + s \in \alpha)$

 (1.3) $(r \in \alpha \wedge s \in 0_\mathbb{R}) \iff (r + s \in \alpha +_\mathbb{R} 0_\mathbb{R})$ ▌ $\forall_{p \in \alpha +_\mathbb{R} 0_\mathbb{R}} (p \in \alpha)$ ▌ $\alpha +_\mathbb{R} 0_\mathbb{R} \subseteq \alpha$

 (1.4) $p \in \alpha \implies \ldots$

  (1.4.1) $\exists_{r \in \alpha} (p < r)$ ▌ $r_2 := choice(\{r \in \alpha | p < r\})$

  (1.4.2) $p < r_2$ ▌ $p - r_2 < r_2 - r_2 = 0$ ▌ $(p - r_2) < 0$ ▌ $(p - r_2) \in 0_\mathbb{R}$

  (1.4.3) $r_2 \in \alpha$ ▌ $p = r_2 + (p - r_2) \in \alpha +_\mathbb{R} 0_\mathbb{R}$ ▌ $p \in \alpha +_\mathbb{R} 0_\mathbb{R}$

 (1.5) $p \in \alpha \implies p \in \alpha +_\mathbb{R} 0_\mathbb{R}$ ▌ $\forall_{p \in \alpha} (p \in \alpha +_\mathbb{R} 0_\mathbb{R})$ ▌ $\alpha \subseteq \alpha +_\mathbb{R} 0_\mathbb{R}$

 (1.6) $\alpha +_\mathbb{R} 0_\mathbb{R} \subseteq \alpha \wedge \alpha \subseteq \alpha +_\mathbb{R} 0_\mathbb{R}$ ▌ $0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

| $Field\,Addition\,Inverse\,Of\,R$ | $(\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}} \left(\alpha +_\mathbb{R} (-\alpha) = 0_\mathbb{R} \right)$ |

(1) $\alpha \in \mathbb{R} \implies \ldots$

 (1.1) $\beta := \{p \in \mathbb{Q} | \exists_{r > 0} (-p - r \notin \alpha)\}$

 (1.2) $\alpha \subset \mathbb{Q}$ ▌ $\exists_{s \in \mathbb{Q}} (s \notin \alpha)$ ▌ $s_0 := choice(\{s | s \notin \alpha\})$ ▌ $p_0 := -s_0 - 1$

 (1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha$ ▌ $-p_0 - 1 \notin \alpha$ ▌ $\exists_{r > 0} (-p_0 - r \notin \alpha)$ ▌ $p_0 \in \beta$

 (1.4) $\emptyset \neq \alpha$ ▌ $\exists_{q \in \alpha}$ ▌ $q_0 := choice(\{q \in \mathbb{Q} | q \in \alpha\})$

 (1.5) $r > 0 \implies \ldots$

  (1.5.1) $q_0 \in \alpha$ ▌ $-(-q_0) - r = q_0 - r < q_0$ ▌ $-(-q_0) - r < q_0$ ▌ $-(-q_0) - r \in \alpha$

 (1.6) $\forall_{r > 0} \left(-(-q_0) - r \in \alpha \right)$ ▌ $\neg \exists_{r > 0} (-(-q_0) - r \notin \alpha)$ ▌ $-q_0 \notin \beta$

 (1.7) $\emptyset \neq \beta \subset \mathbb{Q}$ ▌ $Cut\,I(\beta)$

 (1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

  (1.8.1) $p \in \beta$ ▌ $\exists_{r > 0} (-p - r \notin \alpha)$ ▌ $r_0 := choice(\{r > 0 | -p - r \notin \alpha\})$

  (1.8.2) $q < p$ ▌ $-p - r < -q - r$

  (1.8.3) $-q - r \notin \alpha$ ▌ $q \in \beta$

 (1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta$ ▌ $\forall_{p \in \beta} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \beta)$ ▌ $Cut\,II(\beta)$

 (1.10) $p \in \beta \implies \ldots$

  (1.10.1) $p \in \beta$ ▌ $\exists_{r > 0} (-p - r \notin \alpha)$ ▌ $r_1 := choice(\{r > 0 | -p - r \notin \alpha\})$

  (1.10.2) $t_0 := p + (r_1/2)$

  (1.10.3) $r_1 > 0$ ▌ $r_1/2 > 0$

  (1.10.4) $t_0 > t_0 - (r_1/2) = p$ ▌ $t_0 > p$

  (1.10.5) $-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$

  (1.10.6) $-p - r_1 \notin \alpha$ ▌ $-t_0 - (r_1/2) \notin \alpha$ ▌ $\exists_{r > 0} (-t_0 - r \notin \alpha)$ ▌ $t_0 \in \beta$

  (1.10.7) $t_0 > p \wedge t_0 \in \beta$ ▌ $\exists_{t \in \beta} (p < t)$

 (1.11) $p \in \beta \implies \exists_{t \in \beta} (p < t)$ ▌ $\forall_{p \in \beta} \exists_{t \in \beta} (p < t)$ ▌ $Cut\,III(\beta)$

 (1.12) $Cut\,I(\beta) \wedge Cut\,II(\beta) \wedge Cut\,III(\beta)$ ▌ $\beta \in \mathbb{R}$

 (1.13) $(r \in \alpha \wedge s \in \beta) \implies \ldots$

  (1.13.1) $s \in \beta$ ▌ $\exists_{t > 0} (-s - t \notin \alpha)$ ▌ $t_1 := choice(\{t > 0 | -s - t \notin \alpha\})$ ▌ $-s - t_1 < -s$

  (1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha$ ▌ $-s \notin \alpha$

  (1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha$ ▌ $r < -s$ ▌ $r + s < 0$ ▌ $r + s \in 0_\mathbb{R}$

 (1.14) $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_\mathbb{R}$ ▌ $\forall_{(r,s) \in \alpha \times \beta} (r + s \in 0_\mathbb{R})$ ▌ $\alpha +_\mathbb{R} \beta \subseteq 0_\mathbb{R}$

 (1.15) $v \in 0_\mathbb{R} \implies \ldots$

  (1.15.1) $v < 0$ ▌ $w_0 := -v/2$ ▌ $w > 0$

  (1.15.2) $\exists_{n \in \mathbb{Z}} (n w_0 \in \alpha \wedge (n + 1) w_0 \notin \alpha)$ ▌ $n_0 := choice(\{n \in \mathbb{Z} | n w_0 \in \alpha \wedge (n + 1) w_0 \notin \alpha\})$

  (1.15.3) $p_0 := -(n_0 + 2) w_0$ ▌ $-p_0 - w_0 = (n_0 + 2) w_0 - w_0 = (n_0 + 1) w_0 \notin \alpha$ ▌ $-p_0 - w_0 \notin \alpha$ ▌ $p_0 \in \beta$

  (1.15.4) $n_0 w_0 \in \alpha \wedge p_0 \in \beta$ ▌ $n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_\mathbb{R} \beta$

(1.16) $v \in 0_\mathbb{R} \implies v \in \alpha +_\mathbb{R} \beta$ ▮ $\forall_{v \in 0_\mathbb{R}} (v \in \alpha +_\mathbb{R} \beta)$ ▮ $0_\mathbb{R} \subseteq \alpha +_\mathbb{R} \beta$

(1.17) $\alpha +_\mathbb{R} \beta \subseteq 0_\mathbb{R} \wedge 0_\mathbb{R} \subseteq \alpha +_\mathbb{R} \beta$ ▮ $\alpha +_\mathbb{R} \beta = 0_\mathbb{R}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_\mathbb{R} \beta = 0_\mathbb{R}$ ▮ $\exists_{-\alpha \in \mathbb{R}} (\alpha +_\mathbb{R} (-\alpha) = 0_\mathbb{R})$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha \in \mathbb{R}} (\alpha +_\mathbb{R} (-\alpha) = 0_\mathbb{R})$

$*_\mathbb{R}(\alpha, \beta) :=$ ——
$1_\mathbb{R} := \{x \in \mathbb{Q} | x < 1\}$

$\boxed{1IsNot0}$ $0_\mathbb{R} \neq 1_\mathbb{R}$ ——

$\boxed{1InR}$ $1_\mathbb{R} \in \mathbb{R}$ ——

$\boxed{FieldMultiplicationClosureOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies ((\alpha *_\mathbb{R} \beta) \in \mathbb{R})$ ——

$\boxed{FieldMultiplicationCommutativityOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies (\alpha *_\mathbb{R} \beta = \beta *_\mathbb{R} \alpha)$ ——

$\boxed{FieldMultiplicationAssociativityOfR}$ $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_\mathbb{R} \beta) *_\mathbb{R} \gamma = \alpha *_\mathbb{R} (\beta *_\mathbb{R} \gamma))$ ——

$\boxed{FieldMultiplicationIdentityOfR}$ $(\alpha \in \mathbb{R}) \implies 1_\mathbb{R} *_\mathbb{R} \alpha = \alpha$ ——

$\boxed{FieldMultiplicationInverseOfR}$ $(\alpha \in \mathbb{R}) \implies \exists_{1/\alpha \in \mathbb{R}} (\alpha *_\mathbb{R} (1/\alpha) = 1_\mathbb{R})$ ——

$\boxed{FieldDistributativityOfR}$ $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_\mathbb{R} (\alpha +_\mathbb{R} \beta) = \gamma *_\mathbb{R} \alpha + \gamma *_\mathbb{R} \beta$ ——

$\boxed{FieldWithR}$ $Field(\mathbb{R}, +_\mathbb{R}, *_\mathbb{R})$ ——

$\boxed{OrderedFieldWithR}$ $OrderedField(\mathbb{R}, +_\mathbb{R}, *_\mathbb{R}, <_\mathbb{R})$ ——

$\mathbb{Q}_\mathbb{R} := \{\{r \in \mathbb{Q} | r < q\} | q \in \mathbb{Q}\}$
$\boxed{QROrderedSubfieldOfR}$ $OrderedSubfield(\mathbb{Q}_\mathbb{R}, \mathbb{R}, +_\mathbb{R}, *_\mathbb{R}, <_\mathbb{R})$ ——

$\boxed{QIsomorphicToQR}$ $\mathbb{Q}_\mathbb{R} \simeq \mathbb{Q}$ ——

$\boxed{ExistenceOfR}$ $\exists_\mathbb{R} (LUBProperty(\mathbb{R}, <_\mathbb{R}) \wedge OrderedSubfield(\mathbb{Q}, \mathbb{R}, +_\mathbb{R}, *_\mathbb{R}, <_\mathbb{R}))$ ——

(1.20)
$\boxed{ArchimedeanPropertyOfR}$ $\forall_{x, y \in \mathbb{R}} (x > 0 \implies \exists_{n \in \mathbb{N}^+} (nx > y))$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \dots$

(1.1) $A := \{nx | n \in \mathbb{N}^+\}$ ▮ $\emptyset \neq A \subset \mathbb{R}$

(1.2) $\neg \exists_{n \in \mathbb{N}^+} (nx > y) \implies \dots$

(1.2.1) $\neg \exists_{n \in \mathbb{N}^+} (nx > y)$ ▮ $\forall_{n \in \mathbb{N}^+} (nx \leq y)$

(1.2.2) $UpperBound(y, A, \mathbb{R}, <)$ ▮ $BoundedAbove(A, \mathbb{R}, <)$

(1.2.3) $LUBProperty(\mathbb{R}, <) \wedge \emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <)$ ▮ $\exists_{\alpha \in \mathbb{R}} (LUB(\alpha, A, \mathbb{R}, <))$

(1.2.4) $\alpha_0 := choice(\{\alpha \in \mathbb{R} | LUB(\alpha, A, \mathbb{R}, <)\})$

(1.2.5) $x > 0$ ▮ $\alpha_0 - x < \alpha_0$

(1.2.6) $LUB(\alpha_0, A, \mathbb{R}, <) \wedge \alpha_0 - x < \alpha_0$ ▮ $\neg UpperBound(\alpha_0 - x, A, \mathbb{R}, <)$

(1.2.7) $\exists_{c \in A} (\alpha_0 - x < c)$ ▮ $c_0 := choice(\{c \in A | \alpha_0 - x < c\})$

(1.2.8) $c_0 \in A$ ▮ $\exists_{m \in \mathbb{N}^+} (mx = c_0)$ ▮ $m_0 := choice(\{m \in \mathbb{N}^+ | mx = c_0\})$

(1.2.9) $\alpha_0 - x < c_0 = m_0 x$ ▮ $\alpha_0 - x < m_0 x$ ▮ $\alpha_0 < (m_0 + 1)x$

(1.2.10) $m_0 + 1 \in \mathbb{N}^+$ ▮ $(m_0 + 1)x \in A$

(1.2.11) $\alpha_0 < (m_0 + 1)x \wedge (m_0 + 1)x \in A$ ▮ $\exists_{c \in A} (\alpha_0 < c)$

(1.2.12) $LUB(\alpha_0, A, \mathbb{R}, <)$ ▮ $UpperBound(\alpha_0, A, \mathbb{R}, <)$ ▮ $\forall_{c \in A} (c \leq \alpha_0)$ ▮ $\neg \exists_{c \in A} (\alpha_0 < c)$

(1.2.13) $\exists_{c \in A} (\alpha_0 < c) \wedge \neg \exists_{c \in A} (\alpha_0 < c)$ ▮ $\bot$

(1.3) $\neg \exists_{n \in \mathbb{N}^+} (nx > y) \implies \bot$ ▮ $\exists_{n \in \mathbb{N}^+} (nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+} (nx > y)$ ▮ $\forall_{x, y \in \mathbb{R}} (x > 0 \implies \exists_{n \in \mathbb{N}^+} (nx > y))$

$\boxed{QDenseInR}$ $\forall_{x, y \in \mathbb{R}} (x < y \implies \exists_{p \in \mathbb{Q}} (x < p < y))$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \dots$

(1.1) $x < y$ ▮ $y - x > 0$ ▮ $\exists_{n \in \mathbb{N}^+} (n(y - x) > 1)$ ▮ $n_0 := choice(\{n \in \mathbb{N}^+ | n(y - x) > 1\})$ ▮ $n_0(y - x) > 1$

(1.2) $1 > 0$ ▮ $\exists_{m\in\mathbb{N}^+}(m(1) > n_0 x)$ ▮ $m_1 := choice(\{m \in \mathbb{N}^+ | m(1) > n_0 x\})$ ▮ $m_1 > n_0 x$

(1.3) $1 > 0$ ▮ $\exists_{m\in\mathbb{N}^+}(m(1) > -n_0 x)$ ▮ $m_2 := choice(\{m \in \mathbb{N}^+ | m(1) > -n_0 x\})$ ▮ $m_2 > -n_0 x$ ▮ $-m_2 < n_0 x$

(1.4) $-m_2 < n_0 x < m_1$

(1.5) $m_1, m_2 \in \mathbb{N}^+$ ▮ $|m_1 - (-m_2)| \geq 2$

(1.6) $-m_2 < n_0 x < m_1 \wedge |m_1 - (-m_2)| \geq 2$ ▮ $\exists_{m\in\mathbb{Z}}(-m_2 < m < m_1 \wedge m - 1 \leq n_0 x < m)$

(1.7) $m_0 := choice(\{m \in \mathbb{Z}| -m_2 < m < m_1 \wedge m - 1 \leq n_0 x < m\})$ ▮ $-m_2 < m_0 < m_1 \wedge m_0 - 1 \leq n_0 x < m_0$

(1.8) $(n_0(y - x) > 1) \wedge (m_0 - 1 \leq n_0 x < m_0)$ ▮ $n_0 x < m_0 \leq 1 + n_0 x < n_0 y$ ▮ $n_0 x < m_0 < n_0 y$

(1.9) $n_0 \in \mathbb{N}^+$ ▮ $n_0 > 0$ ▮ $x < m_0/n_0 < y$

(1.10) $m_0, n_0 \in \mathbb{Z}$ ▮ $m_0/n_0 \in \mathbb{Q}$

(1.11) $m_0/n_0 \in \mathbb{Q} \wedge x < m_0/n_0 < y$ ▮ $\exists_{p\in\mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p\in\mathbb{Q}}(x < p < y)$ ▮ $\forall_{x,y\in\mathbb{R}}\exists_{p\in\mathbb{Q}}(x < p < y)$

(1.21)

▭ *Root Lemma* $(0 < a < b) \implies b^n - a^n \leq (b - a)nb^{n-1}$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b$ ▮ $b/a > 1$

(1.3) $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq \sum_{i=1}^{n}(b^{n-i}a^{i-1}(b/a)^{i-1}) \leq \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq (b - a)nb^{n-1}$ ▮ $b^n - a^n \leq (b - a)nb^{n-1}$

(2) $(0 < a < b) \implies b^n - a^n \leq (b - a)nb^{n-1}$

▭ *Root* $\forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}(y^n = x)$

(1) $0 < y_1 < y_2 \implies 0 < y_1{}^n < y_2{}^n$ UNIQUENESS $\ldots$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(2.1) $E := \{t \in \mathbb{R} | t > 0 \wedge t^n < x\}$

(2.2) $t_0 := x/(1 + x)$ ▮ $0 < t_0 < 1 \wedge t_0 = x/(1 + x) < x$ ▮ $t_0 > 0 \wedge t_0{}^n \leq t_0 < x$ ▮ $t_0 \in E$ ▮ $\emptyset \neq E$

(2.3) $t_1 := choice(\{t \in \mathbb{R} | t > 1 + x\})$ ▮ $t_1{}^n > t_1 > 1 + x > x$ ▮ $t_1{}^n > x$

(2.4) $t_1 \notin E$ ▮ $E \subset \mathbb{R}$ ▮ $\emptyset \neq E \subset \mathbb{R}$

(2.5) $t \in E \implies \ldots$

(2.5.1) $t^n < x$

(2.5.2) $t_1{}^n > x$ ▮ $t^n < x < t_1{}^n$ ▮ $t < t_1$

(2.6) $t \in E \implies t < t_1$ ▮ $\forall_{t\in E}(t \leq t_1)$ ▮ *Upper Bound*$(t_1, E, \mathbb{R}, <)$ ▮ *Bounded Above*$(E, \mathbb{R}, <)$

(3) $\emptyset \neq E \subset \mathbb{R} \wedge$ *Bounded Above*$(E, \mathbb{R}, <)$ ▮ $\exists_{y\in\mathbb{R}}(LUB(y, E, \mathbb{R}, <))$ ▮ $y_0 := choice(\{y \in \mathbb{R} | LUB(y, E, \mathbb{R}, <)\})$

(4) $y^n < x \implies \ldots$

(4.1) $k_0 := \frac{x - y^n}{n(y+1)^{n-1}}$ ▮ $k \in \mathbb{R}$

(4.2) $y^n < x$ ▮ $0 < x - y^n$

(4.3) $n > 0 \wedge y > 0$ ▮ $0 < n(y + 1)^{n-1}$ ▮ $0 < \frac{x - y^n}{n(y+1)^{n-1}} = k_0$ ▮ $0 < k_0$

(4.4) $0, 1, k_0 \in \mathbb{R} \wedge (0 < 1 \wedge 0 < k_0)$ ▮ $\exists_{h\in\mathbb{Q}}(0 < h < min(1, k_0))$ ▮ $h_0 := choice(\{h \in \mathbb{Q} | 0 < h < min(1, k_0)\})$

(4.5) $0 < h_0 < 1 \wedge h_0 < k_0 = \frac{x - y^n}{n(y+1)^{n-1}}$ ▮ $h_0 n(y + 1)^{n-1} < x - y^n$

(4.6) $(y + h_0)^n - y^n < (y + h_0 - y)n(y + h_0)^{n-1} = h_0 n(y + h_0)^{n-1} < h_0 n(y + 1)^{n-1} < x - y^n$

(4.7) $(y + h_0)^n - y^n < x - y^n$ ▮ $(y + h_0)^n < x$

(4.8) $y > 0 \wedge h_0 > 0$ ▮ $(y + h_0) > 0$

(4.9) $(y + h_0) > 0 \wedge (y + h_0)^n < x$ ▮ $(y + h_0)^n \in E \wedge y + h_0 > y$ ▮ $\exists_{e\in E}(e > y)$

(4.10) *LUB*$(y, E, \mathbb{R}, <)$ ▮ *Upper Bound*$(y, E, \mathbb{R}, <)$ ▮ $\forall_{e\in E}(e \leq y)$ ▮ $\neg\exists_{e\in E}(e > y)$

(4.11) $\exists_{e\in E}(e > y) \wedge \neg\exists_{e\in E}(e > y)$ ▮ $\perp$

(5) $y^n < x \implies \perp$ ▮ $y^n \geq x$

(6) $y^n > x \implies \ldots$

(6.1)	123123

(7)	$y^n > x \implies \bot$ ▌ $y^n \leq x$

(8)	$(y^n < x \lor y^n = x \lor x < y^n) \land (y^n \geq x) \land (y^n \leq x)$ ▌ $y^n = x$

(9)	$y^n = x \land y \in \mathbb{R}$ ▌ $\exists_{y \in \mathbb{R}}(y^n = x)$ DONE

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$

$Function(f, X, Y) := X \neq \emptyset \neq Y \land Relation(f, X \times Y) \land \forall_{x \in X} \exists!_{y \in Y}((x, y) \in f)$

$(Function(f, X, Y) \land A \subseteq X \land B \subseteq Y) \implies \dots$

(1)	$Domain(f) := X; Codomain(f) := Y$
(2)	$Image(f, A) := \{f(a) | a \in A\}; Preimage(f, B) := \{a | f(a) \in B\}$
(3)	$Range(f) := Image(Domain(f))$

$Injective(f, X, Y) := Function(f, X, Y) \land \forall_{x_1, x_2 \in X}(x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$

$Surjective(f, X, Y) := Function(f, X, Y) \land \forall_{y \in Y} \exists_{x \in X}(y = f(x))$

$Bijective(f, X, Y) := Injective(f, X, Y) \land Surjective(f, X, Y)$

TODO: Definition properties Surjective Equivalent: $(Range(f) = Codomain(f)) \implies Surjective(f)$

Chapter 3

Linear Algebra

$A_{square} = B_{sym} + C_{skewsym}$ $A + A^T = B_{sym}$ $A - A^T = B_{skewsym}$ $A = (1/2)(A + A^T) + (1/2)(A - A^T)$ $(AB)^{-1} = B^{-1}A^{-1}$
$AB * \ldots \ldots \ldots$
$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
$A^1 23123$
TODO PROOFS