Contents

Chapter 1

Real Analysis

(1.5)

$\mathit{OrderTrichotomy}(<, S) := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$

$\mathit{OrderTransitivity}(<, S) := \forall_{x,y,z \in S}\big((x < y \wedge y < z) \implies x < z\big)$

$\mathit{Order}(<, S) := \mathit{OrderTrichotomy}(<, S) \wedge \mathit{OrderTransitivity}(<, S)$

(1.7)

$\mathit{BoundedAbove}(E, S, <) := \mathit{Order}(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(x \leq \beta)$

$\mathit{BoundedBelow}(E, S, <) := \mathit{Order}(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(\beta \leq x)$

$\mathit{UpperBound}(\beta, E, S, <) := \mathit{Order}(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(x \leq \beta)$

$\mathit{LowerBound}(\beta, E, S, <) := \mathit{Order}(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(\beta \leq x)$

(1.8)

$\mathit{LUB}(\alpha, E, S, <) := \mathit{UpperBound}(\alpha, E, S, <) \wedge \forall_{\gamma}\big(\gamma < \alpha \implies \neg \mathit{UpperBound}(\gamma, E, S, <)\big)$

$\mathit{GLB}(\alpha, E, S, <) := \mathit{LowerBound}(\alpha, E, S, <) \wedge \forall_{\beta}\big(\alpha < \beta \implies \neg \mathit{LowerBound}(\beta, E, S, <)\big)$

(1.10)

$\mathit{LUBProperty}(S, <) := \forall_{E}\Big(\big(\emptyset \neq E \subset S \wedge \mathit{BoundedAbove}(E, S, <)\big) \implies \exists_{\alpha \in S}\big(\mathit{LUB}(\alpha, E, S, <)\big)\Big)$

$\mathit{GLBProperty}(S, <) := \forall_{E}\Big(\big(\emptyset \neq E \subset S \wedge \mathit{BoundedBelow}(E, S, <)\big) \implies \exists_{\alpha \in S}\big(\mathit{GLB}(\alpha, E, S, <)\big)\Big)$

(1.11)

$\mathit{LUBPropertyImpliesGLBProperty} := \mathit{LUBProperty}(S, <) \implies \mathit{GLBProperty}(S, <)$

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(1) $\mathit{LUBProperty}(S, <) \implies \ldots$ — wts: 2

(1.1) $\big(\emptyset \neq B \subset S \wedge \mathit{BoundedBelow}(B, S, <)\big) \implies \ldots$ — wts: 1.2

(1.1.1) $\mathit{Order}(<, S) \wedge \exists_{\delta' \in S}\big(\mathit{LowerBound}(\delta', B, S, <)\big)$ — from: *BoundedBelow*, 1.1

(1.1.2) $|B| = 1 \implies \ldots$ — wts: 1.1.3

(1.1.2.1) $\exists_{u'}(u' \in B) \;\blacksquare\; u := \mathit{choice}(\{u' | u' \in B\}) \;\blacksquare\; B = \{u\}$ — from: 1.1.2

(1.1.2.2) $\mathit{GLB}(u, B, S, <) \;\blacksquare\; \exists_{\epsilon_0 \in S}\big(\mathit{GLB}(\epsilon_0, B, S, <)\big)$

(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S}\big(\mathit{GLB}(\epsilon_0, B, S, <)\big)$

(1.1.4) $|B| \neq 1 \implies \ldots$ — wts: 1.1.5

(1.1.4.1) $\forall_{E}\Big(\big(\emptyset \neq E \subset S \wedge \mathit{BoundedAbove}(E, S, <)\big) \implies \exists_{\alpha \in S}\big(\mathit{LUB}(\alpha, E, S, <)\big)\Big)$ — from: *LUBProperty*, 1

(1.1.4.2) $L := \{s \in S | \mathit{LowerBound}(s, B, S, <)\}$

(1.1.4.3) $|B| > 1 \wedge \mathit{OrderTrichotomy}(<, S) \;\blacksquare\; \exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$ — from: *Order*, 1.1.1; wts: 1.1.4.7

(1.1.4.4) $b_1 := \mathit{choice}\Big(\{b_1' \in B | \exists_{b_0' \in B}(b_0' < b_1')\}\Big) \;\blacksquare\; \neg \mathit{LowerBound}(b_1, B, S, <)$ — from: 1.1.4.2

(1.1.4.5) $b_1 \notin L \;\blacksquare\; L \subset S$

(1.1.4.6) $\delta := \mathit{choice}\big(\{\delta' \in S | \mathit{LowerBound}(\delta', B, S, <)\}\big) \;\blacksquare\; \delta \in L \;\blacksquare\; \emptyset \neq L$ — from: 1.1.1

(1.1.4.7) $\emptyset \neq L \subset S$ — from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y \in L}\big(\mathit{LowerBound}(y_0, B, S, <)\big) \;\blacksquare\; \forall_{y \in L}\forall_{x \in B}(y_0 \leq x)$ — from: *LowerBound*, 1.1.4.2; wts: 1.1.4.10

(1.1.4.9) $\forall_{x\in B}\Big(x\in S\land\forall_{y\in L}(y_0\leq x)\Big)\ \blacksquare\ \forall_{x\in B}\big(UpperBound(x,L,S,<)\big)$ from: $UpperBound$

(1.1.4.10) $\exists_{x\in S}\big(UpperBound(x,L,S,<)\big)\ \blacksquare\ BoundedAbove(L,S,<)$

(1.1.4.11) $\emptyset\neq L\subset S\land BoundedAbove(L,S,<)$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha'\in S}\big(LUB(\alpha',L,S,<)\big)\ \blacksquare\ \alpha:=choice\Big(\{\alpha'\in S|\big(LUB(\alpha',L,S,<)\big)\}\Big)$ from: 1.1.4.1 wts: 1.1.4.21

(1.1.4.13) $\forall_x\big(x\in B\implies UpperBound(x,L,S,<)\big)$ from: 1.1.4.9 wts: 1.1.4.17

(1.1.4.14) $\forall_x\big(\neg UpperBound(x,L,S,<)\implies x\notin B\big)$

(1.1.4.15) $\gamma<\alpha\implies\dots$ wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound(\gamma,L,S,<)\ \blacksquare\ \gamma\notin B$ from: LUB, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma<\alpha\implies\gamma\notin B\ \blacksquare\ \gamma\in B\implies\gamma\geq\alpha$

(1.1.4.17) $\forall_{\gamma\in B}(\alpha\leq\gamma)\ \blacksquare\ LowerBound(\alpha,B,S,<)$ from: $LowerBound$

(1.1.4.18) $\alpha<\beta\implies\dots$ wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y\in L}(y_0\leq\alpha<\beta)\ \blacksquare\ \forall_{y\in L}(y_0\neq\beta)$ from: LUB, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta\notin L\ \blacksquare\ \neg LowerBound(\beta,B,S,<)$ from: 1.1.4.2

(1.1.4.19) $\alpha<\beta\implies\neg LowerBound(\beta,B,S,<)\ \blacksquare\ \forall_{\beta\in S}\big(\alpha<\beta\implies\neg LowerBound(\beta,B,S,<)\big)$

(1.1.4.20) $LowerBound(\alpha,B,S,<)\land\forall_{\beta\in S}\big(\alpha<\beta\implies\neg LowerBound(\beta,B,S,<)\big)$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB(\alpha,B,S,<)\ \blacksquare\ \exists_{\epsilon_1\in S}\big(GLB(\epsilon_1,B,S,<)\big)$

(1.1.5) $|B|\neq 1\implies\exists_{\epsilon_1\in S}\big(GLB(\epsilon_1,B,S,<)\big)$

(1.1.6) $\Big(|B|=1\implies\exists_{\epsilon_0\in S}\big(GLB(\epsilon_0,B,S,<)\big)\Big)\land\Big(|B|\neq 1\implies\exists_{\epsilon_1\in S}\big(GLB(\epsilon_1,B,S,<)\big)\Big)$ from: 1.1.3, 1.1.5

(1.1.7) $(|B|=1\lor|B|\neq 1)\implies\exists_{\epsilon\in S}\big(GLB(\epsilon,B,S,<)\big)\ \blacksquare\ \exists_{\epsilon\in S}\big(GLB(\epsilon,B,S,<)\big)$

(1.2) $\big(\emptyset\neq B\subset S\land BoundedBelow(B,S,<)\big)\implies\exists_{\epsilon\in S}\big(GLB(\epsilon,B,S,<)\big)$

(1.3) $\forall_B\Big(\big(\emptyset\neq B\subset S\land BoundedBelow(B,S,<)\big)\implies\exists_{\epsilon\in S}\big(GLB(\epsilon,B,S,<)\big)\Big)$

(1.4) $GLBProperty(S,<)$

(2) $LUBProperty(S,<)\implies GLBProperty(S,<)$

(1.12)

$$Field(F,+,*):=\exists_{\mathbb{0},\mathbb{1}\in F}\forall_{x,y,z\in F}\left(\begin{array}{l}x+y\in F\quad\land\quad x*y\in F\quad\land\\ x+y=y+x\quad\land\quad x*y=y*x\quad\land\\ (x+y)+z=x+(y_0+z)\quad\land\quad(x*y)*z=x*(y_0*z)\quad\land\\ \mathbb{1}\neq\mathbb{0}\quad\land\quad x*(y_0+z)=(x*y)+(x*z)\quad\land\\ \mathbb{0}+x=x\quad\land\quad\mathbb{1}*x=x\quad\land\\ \exists_{-x\in F}\big(x+(-x)=\mathbb{0}\big)\land\Big(x\neq\mathbb{0}\implies\exists_{1/x\in F}\big(x*(1/x)=\mathbb{1}\big)\Big)\end{array}\right)$$

************************ $\big(Field(F,+,*)\land x,y,z\in F\big)\implies\dots$ ************************

(1.14)
$AdditiveCancellation:=(x+y=x+z)\implies y=z$

(1) $y=\mathbb{0}+y=\big(x+(-x)\big)+y=\big((-x)+x\big)+y=(-x)+(x+y)=\dots$ from: $Field$

(2) $(-x)+(x+z)=\big((-x)+x\big)+z=\big(x+(-x)\big)+z=\mathbb{0}+z=z$ from: $Field$

$AdditiveIdentityUniqueness:=(x+y=x)\implies y=\mathbb{0}$

(1) $x+y=x=\mathbb{0}+x=x+\mathbb{0}$ from: $Field$

(2) $y=0$ from: $AdditiveCancellation$

$AdditiveInverseUniqueness:=(x+y=\mathbb{0})\implies y=-x$

(1) $x+y=\mathbb{0}=x+(-x)$ from: $Field$

(2) $y=-x$ from: $AdditiveCancellation$

$\textcolor{red}{DoubleNegative} := x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x \;\blacksquare\; \mathbb{0} = (-x) + x$ — from: *Field*

(2) $x = -(-x)$ — from: *AdditiveInverseUniqueness*

(1.15)
$\textcolor{red}{MultiplicativeCancellation} := (x \neq 0 \wedge x * y = x * z) \implies y = z$ —
$\textcolor{red}{MultiplicativeIdentityUniqueness} := (x \neq 0 \wedge x * y = x) \implies y = \mathbb{1}$ —
$\textcolor{red}{MultiplicativeInverseUniqueness} := (x \neq 0 \wedge x * y = \mathbb{1}) \implies y = 1/x$ —
$\textcolor{red}{DoubleReciprocal} := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)
$\textcolor{red}{Domination} := \mathbb{0} * x = \mathbb{0}$

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x \;\blacksquare\; \mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$ — from: *Field*

(2) $\mathbb{0} * x = \mathbb{0}$ — from: *AdditiveIdentityUniqueness*

$\textcolor{red}{NonDomination} := (x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies \dots$

(1.1) $(x * y = 0) \implies \dots$

(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = \big(x * (1/x)\big) * \big(y * (1/y)\big) = (x * y) * \big((1/x) * (1/y)\big) = \mathbb{0} * \big((1/x) * (1/y)\big) = \mathbb{0}$ — from: *Field*, *Domination*, 1, 1.1

(1.1.2) $\mathbb{1} = \mathbb{0} \wedge \mathbb{1} \neq \mathbb{0} \;\blacksquare\; \bot$ — from: *Field*

(1.2) $(x * y = 0) \implies \bot \;\blacksquare\; x * y \neq 0$

(2) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq 0$

$\textcolor{red}{NegationCommutativity} := (-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0} \;\blacksquare\; x * y + (-x) * y = \mathbb{0}$ — from: *Field*, *Domination*; wts: 2

(2) $(-x) * y = -(x * y)$ — from: *AdditiveInverseUniqueness*

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * \mathbb{0} = \mathbb{0} \;\blacksquare\; x * y + x * (-y) = \mathbb{0}$ — from: *Field*, *Domination*; wts: 4

(4) $x * (-y) = -(x * y)$ — from: *AdditiveInverseUniqueness*

(5) $(-x) * y = -(x * y) = x * (-y)$ — from: 2, 4

$\textcolor{red}{NegativeMultiplication} := (-x) * (-y) = x * y$

(1) $(-x) * (-y) = -\big(x * (-y)\big) = -\big(-(x * y)\big) = x * y$ — from: *NegationCommutativity*, *DoubleNegative*

(1.17)

$$\textcolor{red}{OrderedField}(F, +, *, <) := \left(\begin{array}{l} \textcolor{cyan}{Field}(F, +, *) \quad \wedge \quad \textcolor{cyan}{Order}(<, F) \quad \wedge \\ \forall_{x,y,z \in F}(y_0 < z \implies x + y < x + z) \;\wedge \\ \forall_{x,y \in F}\big((x > \mathbb{0} \wedge y > \mathbb{0}) \implies x * y > \mathbb{0}\big) \end{array} \right)$$

$$\text{************************} \;\big(\textcolor{cyan}{OrderedField}(F, +, *, <) \wedge x, y, z \in F\big) \implies \dots \;\text{************************}$$

(1.18)
$\textcolor{red}{NegationOnOrder} := x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \dots$

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x \;\blacksquare\; \mathbb{0} > -x \;\blacksquare\; -x < \mathbb{0}$ — from: *OrderedField*

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \dots$

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x \;\blacksquare\; \mathbb{0} < x \;\blacksquare\; x > \mathbb{0}$ — from: *OrderedField*

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \wedge -x < \mathbb{0} \implies x > \mathbb{0} \;\blacksquare\; x > \mathbb{0} \iff -x < \mathbb{0}$ — from: 2, 4

$\textcolor{red}{PositiveFactorPreservesOrder} := (x > \mathbb{0} \wedge y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \wedge y < z) \implies \dots$

(1.1) $(-y) + z > (-y) + y = \mathbb{0} \;\blacksquare\; z + (-y) = \mathbb{0}$ — from: *OrderedField*

(1.2) $x * \big(z + (-y)\big) > \mathbb{0} \;\blacksquare\; x * z + x * (-y) > \mathbb{0}$ — from: $\mathit{OrderedField}$

(1.3) $x * z = \mathbb{0} + x * z = \big(x * y + -(x * y)\big) + x * z = \big(x * y + x * (-y)\big) + x * z = \dots$ — from: $\mathit{Field}, \mathit{NegationCommutativity}$

(1.4) $x * y + \big(x * z + x * (-y)\big) > x * y + \mathbb{0} = x * y$ — from: $\mathit{Field}, 1.2$

(1.5) $x * z > x * y$ — from: 1.3, 1.4

(2) $(x > \mathbb{0} \wedge y < z) \implies x * z > x * y$

$\mathit{NegativeFactorFlipsOrder} := (x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \wedge y < z) \implies \dots$

(1.1) $-x > \mathbb{0}$ — from: $\mathit{NegationOnOrder}$

(1.2) $(-x) * y < (-x) * z \;\blacksquare\; \mathbb{0} = x * y + (-x) * y < x * y + (-x) * z \;\blacksquare\; \mathbb{0} < x * y + (-x) * z$ — from: $\mathit{PositiveFactorPreservesOrder}$

(1.3) $\mathbb{0} < (-x) * (-y + z) \;\blacksquare\; \mathbb{0} > x * (-y + z) \;\blacksquare\; \mathbb{0} > -(x * y) + x * z$ — from: $\mathit{NegationOnOrder}$

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

$\mathit{SquareIsPositive} := (x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ — from: $\mathit{OrderedField}$

(2) $(x < \mathbb{0}) \implies \dots$

(2.1) $-x > \mathbb{0} \;\blacksquare\; x * x = (-x) * (-x) > \mathbb{0} \;\blacksquare\; x * x > \mathbb{0}$ — from: $\mathit{NegationOnOrder}, \mathit{OrderedField}, \mathit{NegativeMultiplication}$

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \vee x < \mathbb{0}) \implies x * x > \mathbb{0} \;\blacksquare\; x \neq \mathbb{0} \implies x * x > \mathbb{0}$ — from: $\mathit{OrderTrichotomy}, 1, 3$

$\mathit{OneIsPositive} := \mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0} \;\blacksquare\; \mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ — from: $\mathit{Field}, \mathit{SquareIsPositive}$

$\mathit{ReciprocationOnOrder} := (0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \dots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0} \;\blacksquare\; x * (1/x) > 0$ — from: $\mathit{Field}, \mathit{OneIsPositive}$

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \bot \;\blacksquare\; 1/x > \mathbb{0}$ — from: $\mathit{NegativeFactorFlipsOrder}, 1$

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0} \;\blacksquare\; y * (1/y) > 0$ — from: $\mathit{Field}, \mathit{OneIsPositive}$

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \bot \;\blacksquare\; 1/y > \mathbb{0}$ — from: $\mathit{NegativeFactorFlipsOrder}, 1$

(1.5) $(1/x) * (1/y) > \mathbb{0}$ — from: $\mathit{OrderedField}$

(1.6) $\mathbb{0} < 1/y = \big((1/x) * (1/y)\big) * x < \big((1/x) * (1/y)\big) * y = 1/x$ — from: $\mathit{OrderedField}, 1, 1.4, 1.5$

(1.19)

$\mathit{OrderedFieldQ} := \mathit{OrderedField}(\mathbb{Q}, +, *, <)$ —

$\mathit{Subfield}(K, F, +, *) := \mathit{Field}(F, +, *) \wedge K \subset F \wedge \mathit{Field}(K, +, *)$

$\mathit{OrderedSubfield}(K, F, +, *, <) := \mathit{OrderedField}(F, +, *, <) \wedge K \subset F \wedge \mathit{OrderedField}(K, +, *, <)$

$\mathit{CutI}(\alpha) := \emptyset \neq \alpha \subset \mathbb{Q}$

$\mathit{CutII}(\alpha) := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$

$\mathit{CutIII}(\alpha) := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$

$\mathbb{R} := \{\alpha \in \mathbb{Q} \mid \mathit{CutI}(\alpha) \wedge \mathit{CutII}(\alpha) \wedge \mathit{CutIII}(\alpha)\}$

$\mathit{CutCorollaryI} := (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \dots$

(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ — from: $\mathit{CutII}, 1$

(1.2) $q < p \implies q \in \alpha \;\blacksquare\; q \notin \alpha \implies q \geq p$ — from: 1

(1.3) $(q \notin \alpha) \implies \dots$

(1.3.1) $q \geq p$ — from: 1.2

(1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \bot \;\blacksquare\; q \neq p$ — from: 1, 1.3

(1.3.3) $q \geq p \wedge q \neq p \;\blacksquare\; p < q$

(1.4) $q \notin \alpha \implies p < q \;\blacksquare\; p < q$ — from: 1

(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

$\mathit{CutCorollaryII} := (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$

(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ — from: CutII, 1

(1.2) $s \in \alpha \implies \big(r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)\big) \;\blacksquare\; s \in \alpha \implies r \in \alpha$ — from: 1, 1.1

(1.3) $r \notin \alpha \implies s \notin \alpha \;\blacksquare\; s \notin \alpha$ — from: 1, 1.2

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$\mathit{OrderTrichotomyOfR} := \mathit{OrderTrichotomy}(\mathbb{R}, <_{\mathbb{R}})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$

(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$ — from: $<_{\mathbb{R}}$, 1.1

(1.1.2) $\exists_{p'}(p' \in \alpha \wedge p' \notin \beta) \;\blacksquare\; p := \mathit{choice}(\{p' | p' \in \alpha \wedge p' \notin \beta\})$

(1.1.3) $q \in \beta \implies \ldots$

(1.1.3.1) $p, q \in \mathbb{Q}$

(1.1.3.2) $q < p$ — from: $\mathit{CutCorollaryI}$

(1.1.3.3) $q \in \alpha$ — from: CutII

(1.1.4) $q \in \beta \implies q \in \alpha$

(1.1.5) $\forall_{q \in \beta}(q \in \alpha) \;\blacksquare\; \beta \subseteq \alpha$

(1.1.6) $\beta \subset \alpha \;\blacksquare\; \beta <_{\mathbb{R}} \alpha$

(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \;\blacksquare\; (\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $\mathit{OrderTrichotomy}(\mathbb{R}, <_{\mathbb{R}})$

$\mathit{OrderTransitivityOfR} := \mathit{OrderTransitivity}(\mathbb{R}, <_{\mathbb{R}})$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a \in \alpha}(a \in \beta) \wedge \forall_{b \in \beta}(b \in \gamma)$

(1.1.3) $\forall_{a \in \alpha}(\alpha \in \gamma) \;\blacksquare\; \alpha \subset \gamma \;\blacksquare\; \alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}}\big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(4) $\mathit{OrderTransitivity}(\mathbb{R}, <_{\mathbb{R}})$

$\mathit{OrderOfR} := \mathit{Order}(<_{\mathbb{R}}, \mathbb{R})$ — from: $\mathit{OrderTrichotomyR}, \mathit{OrderTransitivityR}$

$\mathit{LUBPropertyOfR} := \mathit{LUBProperty}(\mathbb{R}, <_{\mathbb{R}})$ — wts:

(1) $\big(\emptyset \neq A \subset \mathbb{R} \wedge \mathit{BoundedAbove}(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} | \exists_{\alpha \in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset \blacksquare \exists_{\alpha}(\alpha \in A) \blacksquare \alpha_0 := choice(\{\alpha | \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset \blacksquare \exists_a(a \in \alpha_0) \blacksquare a_0 := choice(\{a | a \in \alpha_0\}) \blacksquare a_0 \in \gamma \blacksquare \gamma \neq \emptyset$

(1.4) $BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}}) \blacksquare \exists_{\beta}\big(UpperBound(\beta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.5) $\beta_0 := choice\big(\{\beta | UpperBound(\beta, A, \mathbb{R}, <_{\mathbb{R}})\}\big)$

(1.6) $UpperBound(\beta_0, A, \mathbb{R}, <_{\mathbb{R}}) \blacksquare \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \beta_0) \blacksquare \forall_{\alpha \in A}(\alpha \subseteq \beta_0) \blacksquare \forall_{\alpha \in A}\forall_{a \in \alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \blacksquare \forall_{a \in \gamma}(a \in \beta_0) \blacksquare \gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q} \blacksquare \gamma \subseteq \beta_0 \subset \mathbb{Q} \blacksquare \gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q} \blacksquare CutI(\gamma)$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.10.1) $p \in \gamma \blacksquare \exists_{\alpha \in A}(p \in \alpha) \blacksquare \alpha_1 := choice(\{\alpha \in A | p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \blacksquare q \in \alpha_1 \blacksquare q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \blacksquare \forall_{p \in \gamma}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma) \blacksquare CutII(\gamma)$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha \in A}(p \in \alpha) \blacksquare \alpha_2 := choice(\{\alpha \in A | p \in \alpha\})$

(1.12.2) $\alpha_2 \in \mathbb{R} \blacksquare CutII(\alpha_2) \blacksquare \exists_{r \in \alpha_2}(p < r) \blacksquare r_0 := choice(\{r \in \alpha_2 | p < r\})$

(1.12.3) $r_0 \in \alpha_2 \blacksquare r_0 \in \gamma$

(1.12.4) $p < r_0 \blacksquare p < r_0 \wedge r_0 \in \gamma \blacksquare \exists_{r \in \gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r \in \gamma}(p < r) \blacksquare \forall_{p \in \gamma}\exists_{r \in \gamma}(p < r) \blacksquare CutIII(\gamma)$

(1.14) $CutI(\gamma) \wedge CutII(\gamma) \wedge CutIII(\gamma) \blacksquare \gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha \in A}(\alpha \subseteq \gamma) \blacksquare \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma)$

(1.16) $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R} \blacksquare UpperBound(\gamma, A, \mathbb{R}, <_{\mathbb{R}})$

(1.17) $\delta <_{\mathbb{R}} \gamma \implies \ldots$

(1.17.1) $\delta \subset \gamma \blacksquare \exists_s(s \in \gamma \wedge s \notin \delta) \blacksquare s_0 := choice(\{s \in \mathbb{Q} | s \in \gamma \wedge s \notin \delta\})$

(1.17.2) $s_0 \in \gamma \blacksquare \exists_{\alpha \in A}(s_0 \in \alpha) \blacksquare \alpha_3 := choice(\{\alpha \in A | s_0 \in \alpha\})$

(1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta \blacksquare \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \ldots$

(1.17.4.1) $\alpha_3 \subseteq \delta \blacksquare \forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \blacksquare \neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4.2) $\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \blacksquare \perp$

(1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \perp \blacksquare \delta <_{\mathbb{R}} \alpha_3 \blacksquare \exists_{\alpha \in A}(\delta <_{\mathbb{R}} \alpha) \blacksquare \exists_{\alpha \in A}\big(\neg(\alpha \leq_{\mathbb{R}} \delta)\big)$

(1.17.6) $\neg\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \delta) \blacksquare \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})) \blacksquare \forall_{\delta}\big(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.19) $UpperBound(\gamma, A, \mathbb{R}, <_{\mathbb{R}}) \wedge \forall_{\delta}\big(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.20) $LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}}) \blacksquare \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(2) $\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(3) $\forall_A\Big(\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)\Big) \blacksquare LUBProperty(\mathbb{R}, <_{\mathbb{R}})$

$+_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$

$0_{\mathbb{R}} := \{x \in \mathbb{Q} | x < 0\}$

$0InR := 0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}} \blacksquare \emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q} \blacksquare CutI(0_{\mathbb{R}})$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}} \blacksquare \forall_{x \in 0_{\mathbb{R}}}\forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}}) \blacksquare CutII(0_{\mathbb{R}})$

(3) $y := x/2 \blacksquare (x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y) \blacksquare \forall_{x \in 0_{\mathbb{R}}}\exists_{y \in 0_{\mathbb{R}}}(x < y) \blacksquare CutIII(0_{\mathbb{R}})$

(4) $CutI(0_{\mathbb{R}}) \wedge CutII(0_{\mathbb{R}}) \wedge CutIII(0_{\mathbb{R}}) \blacksquare 0_{\mathbb{R}} \in \mathbb{R}$

$FieldAdditionClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \land \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha)\,;\, \exists_b(b \in \beta) \;\blacksquare\; a_0 := choice(\{a|a \in \alpha\})\,;\, b_0 := choice(\{b|b \in \beta\}) \;\blacksquare\; a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha)\,;\, \exists_y(y_0 \notin \beta) \;\blacksquare\; x_0 := choice(\{x|x \notin \alpha\})\,;\, y_0 := choice(\{y|y \notin \beta\})$

(1.5) $\forall_{r\in\alpha}(r < x_0)\,;\, \forall_{s\in\beta}(s < y_0) \;\blacksquare\; \forall_{r\in\alpha}\forall_{s\in\beta}(r + s < x_0 + y_0) \;\blacksquare\; x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q} \;\blacksquare\; CutI(\alpha +_{\mathbb{R}} \beta)$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \land q \in \mathbb{Q} \land q < p) \implies \ldots$

(1.7.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s) \;\blacksquare\; (r_0, s_0) := choice\big((r, s) \in \alpha \times \beta | p = r + s\big)$

(1.7.2) $q < p = r_0 + s_0 \;\blacksquare\; (q - s_0) < r_0 \;\blacksquare\; (q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta \;\blacksquare\; q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta \;\blacksquare\; q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \land q \in \mathbb{Q} \land q < p) \implies q \in \alpha +_{\mathbb{R}} \beta \;\blacksquare\; \forall_{p\in\alpha+_{\mathbb{R}}\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta) \;\blacksquare\; CutII(\alpha +_{\mathbb{R}} \beta)$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s) \;\blacksquare\; (r_1, s_1) := choice\big(\{(r, s) \in \alpha \times \beta | p = r + s\}\big)$

(1.9.2) $r_1 \in \alpha \;\blacksquare\; \exists_{t\in\alpha}(r_1 < t) \;\blacksquare\; t_0 := choice(\{t \in \alpha | r_1 < t\})$

(1.9.3) $s_1 \in \beta \;\blacksquare\; t + s_1 \in \alpha +_{\mathbb{R}} \beta \land p = r_1 + s_1 < t + s_1 \;\blacksquare\; \exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r) \;\blacksquare\; \forall_{p\in\alpha+_{\mathbb{R}}\beta}\exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r) \;\blacksquare\; CutIII(\alpha +_{\mathbb{R}} \beta)$

(1.11) $CutI(\alpha +_{\mathbb{R}} \beta) \land CutII(\alpha +_{\mathbb{R}} \beta) \land CutIII(\alpha +_{\mathbb{R}} \beta) \;\blacksquare\; \alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

$FieldAdditionCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s | r \in \alpha \land s \in \beta\} = \{s + r | s \in \beta \land r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

$FieldAdditionAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)\big)$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c | a \in \alpha \land b \in \beta \land c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) | a \in \alpha \land b \in \beta \land c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

$FieldAdditionIdentityOfR := (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \implies \ldots$

(1.1.1) $s < 0 \;\blacksquare\; r + s < r + 0 = r \;\blacksquare\; r + s < r \;\blacksquare\; r + s \in \alpha$

(1.2) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \implies r + s \in \alpha \;\blacksquare\; \forall_{r\in\alpha}\forall_{s\in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \;\blacksquare\; \forall_{p\in\alpha+_{\mathbb{R}}0_{\mathbb{R}}}(p \in \alpha) \;\blacksquare\; \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r\in\alpha}(p < r) \;\blacksquare\; r_2 := choice(\{r \in \alpha | p < r\})$

(1.4.2) $p < r_2 \;\blacksquare\; p - r_2 < r_2 - r_2 = 0 \;\blacksquare\; (p - r_2) < 0 \;\blacksquare\; (p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha \;\blacksquare\; p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; \forall_{p\in\alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \;\blacksquare\; \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \land \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

$FieldAdditionInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} | \exists_{r>0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q} \;\blacksquare\; \exists_{s\in\mathbb{Q}}(s \notin \alpha) \;\blacksquare\; s_0 := choice(\{s | s \notin \alpha\}) \;\blacksquare\; p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha \;\blacksquare\; -p_0 - 1 \notin \alpha \;\blacksquare\; \exists_{r>0}(-p_0 - r \notin \alpha) \;\blacksquare\; p_0 \in \beta$

(1.4) $\emptyset \neq \alpha \;\blacksquare\; \exists_{q\in\alpha} \;\blacksquare\; q_0 := choice(\{q \in \mathbb{Q} | q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha \;\blacksquare\; -(-q_0) - r = q_0 - r < q_0 \;\blacksquare\; -(-q_0) - r < q_0 \;\blacksquare\; -(-q_0) - r \in \alpha$

(1.6) $\forall_{r>0}\big(-(-q_0)-r\in\alpha\big)$ ▮ $\neg\exists_{r>0}\big(-(-q_0)-r\notin\alpha\big)$ ▮ $-q_0\notin\beta$

(1.7) $\emptyset\neq\beta\subset\mathbb{Q}$ ▮ $CutI(\beta)$

(1.8) $(p\in\beta\wedge q\in\mathbb{Q}\wedge q<p)\implies\ldots$

(1.8.1) $p\in\beta$ ▮ $\exists_{r>0}(-p-r\notin\alpha)$ ▮ $r_0:=choice(\{r>0|-p-r\notin\alpha\})$

(1.8.2) $q<p$ ▮ $-p-r<-q-r$

(1.8.3) $-q-r\notin\alpha$ ▮ $q\in\beta$

(1.9) $(p\in\beta\wedge q\in\mathbb{Q}\wedge q<p)\implies q\in\beta$ ▮ $\forall_{p\in\beta}\forall_{q\in\mathbb{Q}}(q<p\implies q\in\beta)$ ▮ $CutII(\beta)$

(1.10) $p\in\beta\implies\ldots$

(1.10.1) $p\in\beta$ ▮ $\exists_{r>0}(-p-r\notin\alpha)$ ▮ $r_1:=choice(\{r>0|-p-r\notin\alpha\})$

(1.10.2) $t_0:=p+(r_1/2)$

(1.10.3) $r_1>0$ ▮ $r_1/2>0$

(1.10.4) $t_0>t_0-(r_1/2)=p$ ▮ $t_0>p$

(1.10.5) $-t_0-(r_1/2)=-\big(p+(r_1/2)\big)-(r_1/2)=-p-r_1$

(1.10.6) $-p-r_1\notin\alpha$ ▮ $-t_0-(r_1/2)\notin\alpha$ ▮ $\exists_{r>0}(-t_0-r\notin\alpha)$ ▮ $t_0\in\beta$

(1.10.7) $t_0>p\wedge t_0\in\beta$ ▮ $\exists_{t\in\beta}(p<t)$

(1.11) $p\in\beta\implies\exists_{t\in\beta}(p<t)$ ▮ $\forall_{p\in\beta}\exists_{t\in\beta}(p<t)$ ▮ $CutIII(\beta)$

(1.12) $CutI(\beta)\wedge CutII(\beta)\wedge CutIII(\beta)$ ▮ $\beta\in\mathbb{R}$

(1.13) $(r\in\alpha\wedge s\in\beta)\implies\ldots$

(1.13.1) $s\in\beta$ ▮ $\exists_{t>0}(-s-t\notin\alpha)$ ▮ $t_1:=choice(\{t>0|-s-t\notin\alpha\})$ ▮ $-s-t_1<-s$

(1.13.2) $\alpha\in\mathbb{R}\wedge s,t_1\in\mathbb{Q}\wedge -s-t_1<-s\wedge -s-t_1\notin\alpha$ ▮ $-s\notin\alpha$

(1.13.3) $\alpha\in\mathbb{R}\wedge r\in\alpha\wedge -s\notin\alpha$ ▮ $r<-s$ ▮ $r+s<0$ ▮ $r+s\in 0_{\mathbb{R}}$

(1.14) $(r\in\alpha\wedge s\in\beta)\implies r+s\in 0_{\mathbb{R}}$ ▮ $\forall_{(r,s)\in\alpha\times\beta}(r+s\in 0_{\mathbb{R}})$ ▮ $\alpha+_{\mathbb{R}}\beta\subseteq 0_{\mathbb{R}}$

(1.15) $v\in 0_{\mathbb{R}}\implies\ldots$

(1.15.1) $v<0$ ▮ $w_0:=-v/2$ ▮ $w>0$

(1.15.2) $\exists_{n\in\mathbb{Z}}\big(nw_0\in\alpha\wedge(n+1)w_0\notin\alpha\big)$ ▮ $n_0:=choice\big(\{n\in\mathbb{Z}|nw_0\in\alpha\wedge(n+1)w_0\notin\alpha\}\big)$ from: $ARCHIMEDEANPROPERTYOFQ+LUB$???

(1.15.3) $p_0:=-(n_0+2)w_0$ ▮ $-p_0-w_0=(n_0+2)w_0-w_0=(n_0+1)w_0\notin\alpha$ ▮ $-p_0-w_0\notin\alpha$ ▮ $p_0\in\beta$

(1.15.4) $n_0w_0\in\alpha\wedge p_0\in\beta$ ▮ $n_0w_0+p_0=n_0(-v/2)+-(n_0+2)-v/2=v\in\alpha+_{\mathbb{R}}\beta$

(1.16) $v\in 0_{\mathbb{R}}\implies v\in\alpha+_{\mathbb{R}}\beta$ ▮ $\forall_{v\in 0_{\mathbb{R}}}(v\in\alpha+_{\mathbb{R}}\beta)$ ▮ $0_{\mathbb{R}}\subseteq\alpha+_{\mathbb{R}}\beta$

(1.17) $\alpha+_{\mathbb{R}}\beta\subseteq 0_{\mathbb{R}}\wedge 0_{\mathbb{R}}\subseteq\alpha+_{\mathbb{R}}\beta$ ▮ $\alpha+_{\mathbb{R}}\beta=0_{\mathbb{R}}$

(1.18) $\beta\in\mathbb{R}\wedge\alpha+_{\mathbb{R}}\beta=0_{\mathbb{R}}$ ▮ $\exists_{-\alpha\in\mathbb{R}}\big(\alpha+_{\mathbb{R}}(-\alpha)=0_{\mathbb{R}}\big)$

(2) $\alpha\in\mathbb{R}\implies\exists_{-\alpha\in\mathbb{R}}\big(\alpha+_{\mathbb{R}}(-\alpha)=0_{\mathbb{R}}\big)$

$*_{\mathbb{R}}(\alpha,\beta):=$ —
$1_{\mathbb{R}}:=\{x\in\mathbb{Q}|x<1\}$

$1IsNot0:=0_{\mathbb{R}}\neq 1_{\mathbb{R}}$ —
$1InR:=1_{\mathbb{R}}\in\mathbb{R}$ —
$FieldMultiplicationClosureOfR:=(\alpha,\beta\in\mathbb{R})\implies\big((\alpha *_{\mathbb{R}}\beta)\in\mathbb{R}\big)$ —
$FieldMultiplicationCommutativityOfR:=(\alpha,\beta\in\mathbb{R})\implies(\alpha *_{\mathbb{R}}\beta=\beta *_{\mathbb{R}}\alpha)$ —
$FieldMultiplicationAssociativityOfR:=(\alpha,\beta,\gamma\in\mathbb{R})\implies\big((\alpha *_{\mathbb{R}}\beta)*_{\mathbb{R}}\gamma=\alpha *_{\mathbb{R}}(\beta *_{\mathbb{R}}\gamma)\big)$ —
$FieldMultiplicationIdentityOfR:=(\alpha\in\mathbb{R})\implies 1_{\mathbb{R}}*_{\mathbb{R}}\alpha=\alpha$ —
$FieldMultiplicationInverseOfR:=(\alpha\in\mathbb{R})\implies\exists_{1/\alpha\in\mathbb{R}}\big(\alpha *_{\mathbb{R}}(1/\alpha)=1_{\mathbb{R}}\big)$ —

$FieldDistributativityOfR:=(\alpha,\beta,\gamma\in\mathbb{R})\implies\gamma *_{\mathbb{R}}(\alpha+_{\mathbb{R}}\beta)=\gamma *_{\mathbb{R}}\alpha+\gamma *_{\mathbb{R}}\beta$ —

$FieldWithR:=Field(\mathbb{R},+_{\mathbb{R}},*_{\mathbb{R}})$ —
$OrderedFieldWithR:=OrderedField(\mathbb{R},+_{\mathbb{R}},*_{\mathbb{R}},<_{\mathbb{R}})$ —

$\mathbb{Q}_{\mathbb{R}}:=\{\{r\in\mathbb{Q}|r<q\}|q\in\mathbb{Q}\}$
$QROrderedSubfieldOfR:=OrderedSubfield(\mathbb{Q}_{\mathbb{R}},\mathbb{R},+_{\mathbb{R}},*_{\mathbb{R}},<_{\mathbb{R}})$ —
$QIsomorphicToQR:=\mathbb{Q}_{\mathbb{R}}\simeq\mathbb{Q}$ —
$CompletenessOfR:=\exists_{\mathbb{R}}\big(LUBProperty(\mathbb{R},<_{\mathbb{R}})\wedge OrderedSubfield(\mathbb{Q},\mathbb{R},+_{\mathbb{R}},*_{\mathbb{R}},<_{\mathbb{R}})\big)$ —

(1.20)

$ArchimedeanPropertyOfR := \forall_{x,y\in\mathbb{R}}\big(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y)\big)$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx | n \in \mathbb{N}^+\}$ ▌ $(\emptyset \neq A \subset \mathbb{R}) \wedge \big(a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a)\big)$

(1.2) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n\in\mathbb{N}^+}(nx > y)$ ▌ $\forall_{n\in\mathbb{N}^+}(nx \leq y)$ ▌ $UpperBound(y_0, A, \mathbb{R}, <)$ ▌ $BoundedAbove(A, \mathbb{R}, <)$

(1.2.2) $CompletenessOfR$ ▌ $LUBProperty(\mathbb{R}, <)$

(1.2.3) $\big(LUBProperty(\mathbb{R}, <)\big) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge \big(BoundedAbove(A, \mathbb{R}, <)\big)$ ▌ $\exists_{\alpha\in\mathbb{R}}\big(LUB(\alpha, A, \mathbb{R}, <)\big)\ \ldots$

(1.2.4) $\ldots\alpha_0 := choice\big(\{\alpha \in \mathbb{R} | LUB(\alpha, A, \mathbb{R}, <)\}\big)$ ▌ $LUB(\alpha_0, A, \mathbb{R}, <)$

(1.2.5) $x > 0$ ▌ $\alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge \big(LUB(\alpha_0, A, \mathbb{R}, <)\big)$ ▌ $\neg UpperBound(\alpha_0 - x, A, \mathbb{R}, <)$

(1.2.7) $\neg UpperBound(\alpha_0 - x, A, \mathbb{R}, <)$ ▌ $\exists_{c\in A}(\alpha_0 - x < c)\ \ldots$

(1.2.8) $\ldots c_0 := choice(\{c \in A | \alpha_0 - x < c\})$ ▌ $(c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge \big(a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a)\big)$ ▌ $\exists_{m\in\mathbb{N}^+}(mx = c_0)\ \ldots$

(1.2.10) $\ldots m_0 := choice(\{m \in \mathbb{N}^+ | mx = c_0\})$ ▌ $(m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0)$ ▌ $\alpha_0 - x < c_0 = m_0 x$ ▌ $\alpha_0 < m_0 x + x$ ▌ $\alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+$ ▌ $m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge \big(a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a)\big)$ ▌ $(m_0 + 1)x \in A$

(1.2.14) $\big(\alpha_0 < (m_0 + 1)x\big) \wedge \big((m_0 + 1)x \in A\big)$ ▌ $\exists_{c\in A}(\alpha_0 < c)$

(1.2.15) $LUB(\alpha_0, A, \mathbb{R}, <)$ ▌ $UpperBound(\alpha_0, A, \mathbb{R}, <)$ ▌ $\forall_{c\in A}(c \leq \alpha_0)$ ▌ $\neg\exists_{c\in A}(c > \alpha_0)$ ▌ $\neg\exists_{c\in A}(\alpha_0 < c)$

(1.2.16) $\big(\exists_{c\in A}(\alpha_0 < c)\big) \wedge \big(\neg\exists_{c\in A}(\alpha_0 < c)\big)$ ▌ $\perp$

(1.3) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \implies \perp$ ▌ $\exists_{n\in\mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n\in\mathbb{N}^+}(nx > y)$ ▌ $\forall_{x,y\in\mathbb{R}}\big(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y)\big)$

$QDenseInR := \forall_{x,y\in\mathbb{R}}\Big(x < y \implies \exists_{p\in\mathbb{Q}}(x < p < y)\Big)$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y$ ▌ $(0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $ArchimedeanPropertyOfR \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R})$ ▌ $\exists_{n\in\mathbb{N}^+}\big(n(y - x) > 1\big)\ \ldots$

(1.3) $\ldots n_0 := choice\big(\{n \in \mathbb{N}^+ | n(y - x) > 1\}\big)$ ▌ $(n_0 \in \mathbb{N}^+) \wedge \big(n_0(y - x) > 1\big)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R})$ ▌ $n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m\in\mathbb{N}^+}\big(m(1) > n_0 x\big)\ \ldots$

(1.6) $\ldots m_1 := choice\big(\{m \in \mathbb{N}^+ | m(1) > n_0 x\}\big)$ ▌ $(m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m\in\mathbb{N}^+}\big(m(1) > -n_0 x\big)\ \ldots$

(1.8) $\ldots m_2 := choice\big(\{m \in \mathbb{N}^+ | m(1) > -n_0 x\}\big)$ ▌ $(m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x)$ ▌ $-m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+$ ▌ $|m_1 - (-m_2)| \geq 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge \big(|m_1 - (-m_2)| \geq 2\big)$ ▌ $\exists_{m\in\mathbb{Z}}\big((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\big)\ \ldots$

(1.12) $\ldots m_0 := choice\big(\{m \in \mathbb{Z} | (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\}\big)$ ▌ $(-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$

(1.13) $\big(n_0(y - x) > 1\big) \wedge (m_0 - 1 \leq n_0 x < m_0)$ ▌ $n_0 x < m_0 \leq 1 + n_0 x < n_0 y$ ▌ $n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y)$ ▌ $x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z}$ ▌ $m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y)$ ▌ $\exists_{p\in\mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p\in\mathbb{Q}}(x < p < y)$ ▌ $\forall_{x,y\in\mathbb{R}}\Big(x < y \implies \exists_{p\in\mathbb{Q}}(x < p < y)\Big)$

(1.21)

$RootLemma1 := (0 < a < b) \implies \Big(b^n - a^n \leq (b - a)nb^{n-1}\Big)$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b-a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b \;\blacksquare\; b/a > 1$

(1.3) $b/a > 1 \;\blacksquare\; \sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}\left(b^{n-i}a^{i-1}(b/a)^{i-1}\right) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1} \;\blacksquare\; \sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b-a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le (b-a)nb^{n-1} \;\blacksquare\; b^n - a^n \le (b-a)nb^{n-1}$

(2) $(0 < a < b) \implies \left(b^n - a^n \le (b-a)nb^{n-1}\right)$

$RootExistenceInR := \forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}({y_0}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} | t > 0 \wedge t^n < x\} \;\blacksquare\; t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$

(1.2) $t_0 := x/(1+x) \;\blacksquare\; \left(t_0 = x/(1+x)\right) \wedge (t_0 \in \mathbb{R})$

(1.3) $0 < x \;\blacksquare\; 0 < x < 1 + x \;\blacksquare\; t_0 = x/(1+x) > 0 \;\blacksquare\; t_0 > 0$

(1.4) $1 = (1+x)/(1+x) > x/(1+x) = t_0 \;\blacksquare\; 1 > t_0$

(1.5) $(t_0 > 0) \wedge (1 > t_0) \;\blacksquare\; 0 < t_0 < 1$

(1.6) $(0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1) \;\blacksquare\; {t_0}^n \le t_0$

(1.7) $0 < x \;\blacksquare\; x > x/(1+x) = t_0 \;\blacksquare\; x > t_0$

(1.8) $({t_0}^n \le t_0) \wedge (x > t_0) \;\blacksquare\; {t_0}^n < x$

(1.9) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge ({t_0}^n < x) \;\blacksquare\; t_0 \in E \;\blacksquare\; \emptyset \ne E$

(1.10) $t_1 := choice(\{t \in \mathbb{R} | t > 1 + x\}) \;\blacksquare\; (t_1 \in \mathbb{R}) \wedge (t_1 > 1 + x)$

(1.11) $x > 0 \;\blacksquare\; t_1 > 1 + x > 1 \;\blacksquare\; t_1 > 1 \;\blacksquare\; {t_1}^n \ge t_1$

(1.12) $({t_1}^n \ge t_1) \wedge (t_1 > 1 + x) \wedge (1 > 0) \;\blacksquare\; {t_1}^n \ge t_1 > 1 + x > x \;\blacksquare\; {t_1}^n > x$

(1.13) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge ({t_1}^n > x) \;\blacksquare\; t_1 \notin E \;\blacksquare\; E \subset \mathbb{R}$

(1.14) $(\emptyset \ne E) \wedge (E \subset \mathbb{R}) \;\blacksquare\; \emptyset \ne E \subset \mathbb{R}$

(1.15) $t \in E \implies \ldots$

(1.15.1) $(t \in E) \wedge \left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \;\blacksquare\; t^n < x$

(1.15.2) $({t_1}^n > x) \wedge (t^n < x) \;\blacksquare\; t^n < x < {t_1}^n \;\blacksquare\; t < t_1$

(1.16) $t \in E \implies t < t_1 \;\blacksquare\; \forall_{t\in E}(t \le t_1) \;\blacksquare\; UpperBound(t_1, E, \mathbb{R}, <) \;\blacksquare\; BoundedAbove(E, \mathbb{R}, <)$

(1.17) $CompletenessOfR \;\blacksquare\; LUBProperty(\mathbb{R}, <)$

(1.18) $\left(LUBProperty(\mathbb{R}, <)\right) \wedge (\emptyset \ne E \subset \mathbb{R}) \wedge \left(BoundedAbove(E, \mathbb{R}, <)\right) \;\blacksquare\; \exists_{y\in\mathbb{R}}\left(LUB(y, E, \mathbb{R}, <)\right) \;\ldots$

(1.19) $\ldots y_0 := choice\left(\{y \in \mathbb{R} | LUB(y, E, \mathbb{R}, <)\}\right) \;\blacksquare\; LUB(y_0, E, \mathbb{R}, <)$

(1.20) $\left(LUB(y_0, E, \mathbb{R}, <)\right) \wedge (t_0 \in E) \wedge (t_0 > 0) \;\blacksquare\; 0 < t_0 \le y_0 \;\blacksquare\; y_0 > 0$

(1.21) ${y_0}^n < x \implies \ldots$

(1.21.1) $k_0 := \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} \;\blacksquare\; k_0 \in \mathbb{R}$

(1.21.2) ${y_0}^n < x \;\blacksquare\; 0 < x - {y_0}^n$

(1.21.3) $(n > 0) \wedge (y_0 > 0) \;\blacksquare\; 0 < n(y_0 + 1)^{n-1}$

(1.21.4) $(0 < x - {y_0}^n) \wedge \left(0 < n(y_0+1)^{n-1}\right) \;\blacksquare\; 0 < \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} = k_0 \;\blacksquare\; 0 < k_0$

(1.21.5) $(0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R}) \;\blacksquare\; 0 < min(1, k_0) \in \mathbb{R}$

(1.21.6) $QDenseInR \wedge \left(0, min(1, k_0) \in \mathbb{R}\right) \wedge \left(0 < min(1, k_0)\right) \;\blacksquare\; \exists_{h\in\mathbb{Q}}\left(0 < h < min(1, k_0)\right) \;\ldots$

(1.21.7) $\ldots h_0 := choice\left(\{h \in \mathbb{Q} | 0 < h < min(1, k_0)\}\right) \;\blacksquare\; (0 < h_0 < 1) \wedge \left(h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}\right)$

(1.21.8) $(y_0 > 0) \wedge (h_0 > 0) \;\blacksquare\; 0 < y_0 < y_0 + h_0$

(1.21.9) $RootLemma1 \wedge (0 < y_0 < y_0 + h_0) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}$

(1.21.10) $h_0 < 1 \;\blacksquare\; h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$

(1.21.11) $\left((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}\right) \wedge \left(h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}\right) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}$

(1.21.12) $\left(0 < n(y_0 + 1)^{n-1}\right) \wedge \left(h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}\right) \;\blacksquare\; h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n$

(1.21.13) $\left((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}\right) \wedge \left(h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n\right) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \;\blacksquare\; (y_0 + h_0)^n < x$

(1.21.14) $(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \;\blacksquare\; (y_0 + h_0)^n < x$

(1.21.15) $(0 < y_0\mathbb{R}) \wedge (0 < h_0 < \mathbb{R}) \blacksquare\ 0 < y_0 < y_0 + h_0 \in \mathbb{R}$

(1.21.16) $\big(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\big) \wedge \big((y_0 + h_0)^n < x\big) \wedge (0 < y_0 + h_0 \in \mathbb{R}) \blacksquare\ (y_0 + h_0)^n \in E$

(1.21.17) $\big((y_0 + h_0)^n \in E\big) \wedge (y_0 < y_0 + h_0) \blacksquare\ \exists_{e \in E}(y_0 < e)$

(1.21.18) $\textcolor{cyan}{LUB}(y_0, E, \mathbb{R}, <) \blacksquare\ \textcolor{cyan}{UpperBound}(y_0, E, \mathbb{R}, <) \blacksquare\ \forall_{e \in E}(e \leq y_0) \blacksquare\ \neg\exists_{e \in E}(e > y_0)$

(1.21.19) $\big(\exists_{e \in E}(e > y_0)\big) \wedge \big(\neg\exists_{e \in E}(e > y_0)\big) \blacksquare\ \bot$

(1.22) ${y_0}^n < x \implies \bot \blacksquare\ {y_0}^n \geq x$

(1.23) ${y_0}^n > x \implies \ldots$ 123123

(1.23.1) $k_1 := \frac{{y_0}^n - x}{n{y_0}^{n-1}} \blacksquare\ k_1 \in \mathbb{R} \wedge k_1 n{y_0}^{n-1} = {y_0}^n - x$

(1.23.2) $0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z} \blacksquare\ {y_0}^n - x < {y_0}^n \leq n{y_0}^n \blacksquare\ {y_0}^n - x < n{y_0}^n$

(1.23.3) $k_1 = \frac{{y_0}^n - x}{n{y_0}^{n-1}} < \frac{n{y_0}^n}{n{y_0}^{n-1}} = y_0 \blacksquare\ k_1 < y_0$

(1.23.4) ${y_0}^n > x \blacksquare\ {y_0}^n - x > 0$

(1.23.5) $n > 0 \wedge y_0 > 0 \blacksquare\ 0 < n{y_0}^{n-1}$

(1.23.6) $0 < {y_0}^n - x \wedge 0 < n{y_0}^{n-1} \blacksquare\ 0 < \frac{{y_0}^n - x}{n{y_0}^{n-1}} = k_1 \blacksquare\ 0 < k_1$

(1.23.7) $k_1 < y_0 \wedge 0 < k_1 \blacksquare\ 0 < k_1 < y_0$

(1.23.8) $t \geq y_0 - k_1 \implies \ldots$

(1.23.8.1) $t \geq y_0 - k_1 \blacksquare\ t^n \geq (y_0 - k_1)^n \blacksquare\ -t^n \leq -(y_0 - k_1)^n \blacksquare\ {y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n$

(1.23.8.2) ${y_0}^n - (y_0 - k_1)^n < (y_0 - y_0 + k_1)n{y_0}^{n-1} = k_1 n{y_0}^{n-1} = {y_0}^n - x$

(1.23.8.3) ${y_0}^n - t^n < {y_0}^n - x \blacksquare\ -t^n < -x \blacksquare\ t_n > x \blacksquare\ \neg(t_n < x) \blacksquare\ t \notin E$

(1.23.9) $t \geq y_0 - k_1 \implies t \notin E \blacksquare\ t \in E \implies t < y_0 - k_1$

(1.23.10) $\forall_{t \in E}(t \leq y_0 - k_1) \blacksquare\ \textcolor{cyan}{UpperBound}(y_0 - k_1, E, \mathbb{R}, <)$

(1.23.11) $\textcolor{cyan}{LUB}(y_0, E, \mathbb{R}, <) \blacksquare\ \forall_{z \in \mathbb{R}}\big(z < y_0 \implies \neg\textcolor{cyan}{UpperBound}(z, E, \mathbb{R}, <)\big)$

(1.23.12) $k_1 > 0 \blacksquare\ y - k_1 < y_0 \blacksquare\ \neg\textcolor{cyan}{UpperBound}(y_0 - k_1, E, \mathbb{R}, <)$

(1.23.13) $\textcolor{cyan}{UpperBound}(y_0 - k_1, E, \mathbb{R}, <) \wedge \neg\textcolor{cyan}{UpperBound}(y_0 - k_1, E, \mathbb{R}, <) \blacksquare\ \bot$

(1.24) ${y_0}^n > x \implies \bot \blacksquare\ {y_0}^n \leq x$

(1.25) $({y_0}^n < x \veebar {y_0}^n = x \veebar x < {y_0}^n) \wedge ({y_0}^n \geq x) \wedge ({y_0}^n \leq x) \blacksquare\ {y_0}^n = x$

(1.26) ${y_0}^n = x \wedge y_0 \in \mathbb{R} \blacksquare\ \exists_{y \in \mathbb{R}}(y^n = x)$

(1.27) $y_1, y_2 := choice(\{y \in \mathbb{R} | y^n = x\}) \blacksquare\ y_1 \neq y_2 \implies \ldots$

(1.27.1) $(y_1 < y_2) \veebar (y_2 < y_1) \blacksquare\ (x = {y_1}^n < {y_2}^n = x) \veebar (x = {y_2}^n < {y_1}^n = x) \blacksquare\ (x < x) \veebar (x > x) \blacksquare\ \bot \veebar \bot \blacksquare\ \bot$

(1.28) $y_1 \neq y_2 \implies \bot \blacksquare\ y_1 = y_2 \blacksquare\ \forall_{a,b \in \mathbb{R}}\big((a^n = x \wedge b^n = x) \implies a = b\big)$

(1.29) $\exists_{y \in \mathbb{R}}(y^n = x) \wedge \forall_{a,b \in \mathbb{R}}\big((a^n = x \wedge b^n = x) \implies a = b\big) \blacksquare\ \exists!_{y \in \mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}}(y^n = x) \blacksquare\ \forall_{0 < x \in \mathbb{R}}\forall_{0 < n \in \mathbb{Z}}\exists!_{0 < y \in \mathbb{R}}({y_0}^n = x)$

$\textcolor{red}{RootExistenceInRCorollary} := \forall_{0 < a \in \mathbb{R}}\forall_{0 < b \in \mathbb{R}}\forall_{0 < n \in \mathbb{Z}}\Big((ab)^{1/n} = a^{1/n}b^{1/n}\Big)$ —

$$\textcolor{red}{ExtendedRealSystem}(\bar{\mathbb{R}}, +, *, <) := \left(\begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \ \wedge \ x - \infty = -\infty \ \wedge \ \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \ \wedge \\ (x > 0) \implies \big(x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty\big) \wedge \\ (x < 0) \implies \big(x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty\big) \end{array}\right)$$

$\textcolor{red}{\mathbb{C}} := \{\langle a, b\rangle \in \mathbb{R} \times \mathbb{R}\}$
$\textcolor{red}{+_{\mathbb{C}}}(\langle a, b\rangle, \langle c, d\rangle) := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d\rangle$
$\textcolor{red}{*_{\mathbb{C}}}(\langle a, b\rangle, \langle c, d\rangle) := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c\rangle$

$\textcolor{red}{FieldC} := \textcolor{cyan}{Field}(\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}})$ —
$\textcolor{red}{RSubfieldC} := \textcolor{cyan}{Subfield}(\mathbb{R}, \mathbb{C}, +, *)$ —

$\textcolor{red}{i} := \langle 0, 1\rangle \in \mathbb{C}$

$\textcolor{red}{iProperty} := i^2 = -1$ —
$\textcolor{red}{CProperty} := (a, b \in \mathbb{R}) \implies (\langle a, b\rangle = a + bi)$ —

$\textcolor{red}{Conjugate}(\overline{a + bi}) := a - bi$

$ConjugateProperties := (w, z \in \mathbb{C}) \implies \ldots$ —

(1)	$\overline{z+w} = \overline{z} + \overline{w}$
(2)	$\overline{z * w} = \overline{z} * \overline{w}$
(3)	$Re(z) = (1/2)(z + \overline{z}) \wedge Im(z) = (1/2)(z - \overline{z})$
(4)	$0 \leq z * \overline{z} \in \mathbb{R}$

$AbsoluteValueC(|z|) = (z * \overline{z})^{1/2}$
$AbsoluteValueProperties := (z, w \in \mathbb{C}) \implies \ldots$ —

(1)	123123

TODO: - CALL WFFS DEFINITION BUT ABREVIATING ONLY WFF/RELATIONS AND NOT TERMS OR FUNCTIONS - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \land Relation(f, X \times Y) \land \forall_{x \in X} \exists!_{y \in Y} \big((x, y) \in f\big)$

$\big(Function(f, X, Y) \land A \subseteq X \land B \subseteq Y\big) \implies \ldots$

(1) $Domain(f) := X; Codomain(f) := Y$

(2) $Image(f, A) := \{f(a) | a \in A\}; Preimage(f, B) := \{a | f(a) \in B\}$

(3) $Range(f) := Image\big(Domain(f)\big)$

$Injective(f, X, Y) := Function(f, X, Y) \land \forall_{x_1, x_2 \in X} \big(x_1 \neq x_2 \implies f(x_1) \neq f(x_2)\big)$
$Surjective(f, X, Y) := Function(f, X, Y) \land \forall_{y \in Y} \exists_{x \in X} \big(y_0 = f(x)\big)$
$Bijective(f, X, Y) := Injective(f, X, Y) \land Surjective(f, X, Y)$
$SurjectiveEquivalent := \big(Range(f) = Codomain(f)\big) \implies Surjective(f)$

$\big(Function(f, X, Y) \land Function(g, Y, Z)\big) \implies (f \circ g)(x) := f\big(g(x)\big); Function(f \circ g, X, Z)$

$PropertiesofFunctions := \big(Function(f, A, B) \land Function(g, B, C) \land Function(h, C, D)\big) \implies \ldots$

(1) $h \circ (g \circ f) = (h \circ g) \circ f$

(2) $\big(Injective(f) \land Injective(g)\big) \implies Injective(g \circ f)$

(3) $\big(Surjective(f) \land Surjective(g)\big) \implies Surjective(g \circ f)$

(4) $\big(Bijective(f, A, B)\big) \implies \exists_{f^{-1}} \Bigg(Function(f^{-}1, B, A) \land \forall_{a \in A} \Big(f^{-1}\big(f(a)\big) = a\Big) \land \forall_{b \in B} \Big(f\Big(f^{-1}(b)\Big) = b\Big)\Bigg)$

$|(a, b) := a, b \in \mathbb{Z} \land a \neq 0 \land \exists_{c \in \mathbb{Z}}(b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \ldots$

(1) $(a|b) \implies a|bc$

(2) $(a|b \land b|c) \implies a|c$

(3) $(a|b \land b|c) \implies a|(bx + cy)$

(4) $(a|b \land b|a) \implies a = \pm b$

(5) $(a|b \land a > 0 \land b > 0) \implies (a \leq b)$

(6) $(a|b) \iff (m \neq 0 \land ma|mb)$

$DivisionAlgorithm := (a, b \in \mathbb{Z} \land a > 0) \implies \exists!_{q, r \in \mathbb{Z}}(b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \land a|b \land a|c$
$GCD(a, b, c) := CD(a, b, c) \land \forall_d \big((d|b \land d|c) \implies d|a\big)$
$GCDEquivalent := 123123$

Chapter 3

Linear Algebra

$EquivalentSystem()\ldots$
$(AB)^T = B^T A^T$
$Sym(A) := A^T = A$
$Skew(A) := A^T = -A$

$(B = A + A^T) \implies Sym(B)$
$(B = A - A^T) \implies Skew(B)$
$A = (1/2)(A + A^T) + (1/2)(A - A^T) = Sym(B_1) + Skew(B_2)$

$Invertible(A) := \exists_{A^{-1}}(AA^{-1} = I = A^{-1}A)$

$\big(Invertible(A) \wedge Invertible(B)\big) \implies \Big(Invertible(AB) \wedge (AB)^{-1} = B^{-1}A^{-1}\Big)$
$\big(Invertible(A)\big) \implies \Big(Invertible(A^{-1}) \wedge (A^{-1})^{-1} = A\Big)$
$\big(Invertible(A)\big) \implies \Big(Invertible(A^T) \wedge (A^T)^{-1} = (A^{-1})^T\Big)$

$RREF(A) := (Definition 1.18)$
$ElementaryRowOperation(\phi) := (Definition 1.19)$
$RowEquivalent(A, B) := \exists_\Phi\Big(\forall_{\phi\in\Phi}\big(ElementaryRowOperation(\phi)\big) \wedge |\Phi| \in \mathbb{N} \wedge \Phi(A) = B\Big)$

By Gauss-Jordan Elimination: $NonZero(A) \implies \exists_B\big(RREF(B) \wedge RowEquivalent(A, B)\big)$

$\big(AX = B \wedge CX = D \wedge RowEquivalent([A|B], [C|D])\big) \implies ([AX = B] \equiv [CX = D])$
$\big(RowEquivalent(A, B)\big) \implies ([AX = \mathbb{O}] \equiv [BX = \mathbb{O}])$