

Contents

Chapter 1

Real Analysis

(1.5)

$$\begin{aligned} \text{OrderTrichotomy}[\prec, S] &:= \forall_{x,y \in S} (x < y \vee x = y \vee y < x) \\ \text{OrderTransitivity}[\prec, S] &:= \forall_{x,y,z \in S} ((x < y \wedge y < z) \implies x < z) \\ \text{Order}[\prec, S] &:= (\text{OrderTrichotomy}[\prec, S]) \wedge (\text{OrderTransitivity}[\prec, S]) \end{aligned}$$

(1.7)

$$\begin{aligned} \text{BoundedAbove}[E, S, \prec] &:= (\text{Order}[\prec, S]) \wedge (E \subset S) \wedge \left(\exists_{\beta \in S} \forall_{x \in E} (x \leq \beta) \right) \\ \text{BoundedBelow}[E, S, \prec] &:= (\text{Order}[\prec, S]) \wedge (E \subset S) \wedge \left(\exists_{\beta \in S} \forall_{x \in E} (\beta \leq x) \right) \\ \text{UpperBound}[\beta, E, S, \prec] &:= (\text{Order}[\prec, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E} (x \leq \beta)) \\ \text{LowerBound}[\beta, E, S, \prec] &:= (\text{Order}[\prec, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E} (\beta \leq x)) \end{aligned}$$

(1.8)

$$\begin{aligned} \text{LUB}[\alpha, E, S, \prec] &:= (\text{UpperBound}[\alpha, E, S, \prec]) \wedge \left(\forall_{\gamma} (\gamma < \alpha \implies \neg \text{UpperBound}[\gamma, E, S, \prec]) \right) \\ \text{GLB}[\alpha, E, S, \prec] &:= (\text{LowerBound}[\alpha, E, S, \prec]) \wedge \left(\forall_{\beta} (\alpha < \beta \implies \neg \text{LowerBound}[\beta, E, S, \prec]) \right) \end{aligned}$$

(1.10)

$$\begin{aligned} \text{LUBProperty}[S, \prec] &:= \forall_E \left(((\emptyset \neq E \subset S) \wedge (\text{BoundedAbove}[E, S, \prec]) \implies \exists_{\alpha \in S} (\text{LUB}[\alpha, E, S, \prec])) \right) \\ \text{GLBProperty}[S, \prec] &:= \forall_E \left(((\emptyset \neq E \subset S) \wedge (\text{BoundedBelow}[E, S, \prec]) \implies \exists_{\alpha \in S} (\text{GLB}[\alpha, E, S, \prec])) \right) \end{aligned}$$

(1.11)

$$\text{LUBPropertyImpliesGLBProperty} := \text{LUBProperty}[S, \prec] \implies \text{GLBProperty}[S, \prec]$$

(1) $\text{LUBProperty}[S, \prec] \implies \dots$

wts: 2

(1.1) $(\emptyset \neq B \subset S \wedge \text{BoundedBelow}[B, S, \prec]) \implies \dots$

wts: 1.2

(1.1.1) $\text{Order}[\prec, S] \wedge \exists_{\delta' \in S} (\text{LowerBound}[\delta', B, S, \prec])$

from: [BoundedBelow](#), 1.1

(1.1.2) $|B| = 1 \implies \dots$

wts: 1.1.3

(1.1.2.1) $\exists_{u'} (u' \in B) \blacksquare u := \text{choice}(\{u' : u' \in B\}) \blacksquare B = \{u\}$

from: 1.1.2

(1.1.2.2) $\text{GLB}[u, B, S, \prec] \blacksquare \exists_{\epsilon_0 \in S} (\text{GLB}[\epsilon_0, B, S, \prec])$

(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S} (\text{GLB}[\epsilon_0, B, S, \prec])$

(1.1.4) $|B| \neq 1 \implies \dots$

wts: 1.1.5

(1.1.4.1) $\forall_E ((\emptyset \neq E \subset S \wedge \text{BoundedAbove}[E, S, \prec]) \implies \exists_{\alpha \in S} (\text{LUB}[\alpha, E, S, \prec]))$

from: [LUBProperty](#), 1

(1.1.4.2) $L := \{s \in S : \text{LowerBound}[s, B, S, \prec]\}$

(1.1.4.3) $|B| > 1 \wedge \text{OrderTrichotomy}[\prec, S] \blacksquare \exists_{b_1' \in B} \exists_{b_0' \in B} (b_0' < b_1')$

from: [Order](#), 1.1.1
wts: 1.1.4.7

(1.1.4.4) $b_1 := \text{choice}(\{b_1' \in B : \exists_{b_0' \in B} (b_0' < b_1')\}) \blacksquare \neg \text{LowerBound}[b_1, B, S, \prec]$

from: 1.1.4.2

(1.1.4.5) $b_1 \notin L \blacksquare L \subset S$

(1.1.4.6) $\delta := \text{choice}(\{\delta' \in S : \text{LowerBound}[\delta', B, S, \prec]\}) \blacksquare \delta \in L \blacksquare \emptyset \neq L$

from: 1.1.1

(1.1.4.7) $\emptyset \neq L \subset S$

from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y \in L} (\text{LowerBound}[y_0, B, S, \prec]) \blacksquare \forall_{y \in L} \forall_{x \in B} (y_0 \leq x)$

from: [LowerBound](#), 1.1.4.2
wts: 1.1.4.10

$$(1.1.4.9) \quad \forall_{x \in B} \left(x \in S \wedge \forall_{y \in L} (y_0 \leq x) \right) \quad \blacksquare \quad \forall_{x \in B} (\text{UpperBound}[x, L, S, <])$$

from: *UpperBound*

$$(1.1.4.10) \quad \exists_{x \in S} (\text{UpperBound}[x, L, S, <]) \quad \blacksquare \quad \text{BoundedAbove}[L, S, <]$$

$$(1.1.4.11) \quad \emptyset \neq L \subset S \wedge \text{BoundedAbove}[L, S, <]$$

from: 1.1.4.7, 1.1.4.10

$$(1.1.4.12) \quad \exists_{\alpha' \in S} (\text{LUB}[\alpha', L, S, <]) \quad \blacksquare \quad \alpha := \text{choice}(\{\alpha' \in S : (\text{LUB}[\alpha', L, S, <])\})$$

from: 1.1.4.1
wts: 1.1.4.21

$$(1.1.4.13) \quad \forall_x (x \in B \implies \text{UpperBound}[x, L, S, <])$$

from: 1.1.4.9
wts: 1.1.4.17

$$(1.1.4.14) \quad \forall_x (\neg \text{UpperBound}[x, L, S, <] \implies x \notin B)$$

$$(1.1.4.15) \quad \gamma < \alpha \implies \dots$$

wts: 1.1.4.16

$$(1.1.4.15.1) \quad \neg \text{UpperBound}[\gamma, L, S, <] \quad \blacksquare \quad \gamma \notin B$$

from: *LUB*, 1.1.4.12, 1.1.4.14

$$(1.1.4.16) \quad \gamma < \alpha \implies \gamma \notin B \quad \blacksquare \quad \gamma \in B \implies \gamma \geq \alpha$$

$$(1.1.4.17) \quad \forall_{\gamma \in B} (\alpha \leq \gamma) \quad \blacksquare \quad \text{LowerBound}[\alpha, B, S, <]$$

from: *LowerBound*

$$(1.1.4.18) \quad \alpha < \beta \implies \dots$$

wts: 1.1.4.19

$$(1.1.4.18.1) \quad \forall_{y \in L} (y_0 \leq \alpha < \beta) \quad \blacksquare \quad \forall_{y \in L} (y_0 \neq \beta)$$

from: *LUB*, 1.1.4.12, 1.1.4.18

$$(1.1.4.18.2) \quad \beta \notin L \quad \blacksquare \quad \neg \text{LowerBound}[\beta, B, S, <]$$

from: 1.1.4.2

$$(1.1.4.19) \quad \alpha < \beta \implies \neg \text{LowerBound}[\beta, B, S, <] \quad \blacksquare \quad \forall_{\beta \in S} (\alpha < \beta \implies \neg \text{LowerBound}[\beta, B, S, <])$$

$$(1.1.4.20) \quad \text{LowerBound}[\alpha, B, S, <] \wedge \forall_{\beta \in S} (\alpha < \beta \implies \neg \text{LowerBound}[\beta, B, S, <])$$

from: 1.1.4.17, 1.1.4.19

$$(1.1.4.21) \quad \text{GLB}[\alpha, B, S, <] \quad \blacksquare \quad \exists_{\epsilon_1 \in S} (\text{GLB}[\epsilon_1, B, S, <])$$

$$(1.1.5) \quad |B| \neq 1 \implies \exists_{\epsilon_1 \in S} (\text{GLB}[\epsilon_1, B, S, <])$$

$$(1.1.6) \quad \left(|B| = 1 \implies \exists_{\epsilon_0 \in S} (\text{GLB}[\epsilon_0, B, S, <]) \right) \wedge \left(|B| \neq 1 \implies \exists_{\epsilon_1 \in S} (\text{GLB}[\epsilon_1, B, S, <]) \right)$$

from: 1.1.3, 1.1.5

$$(1.1.7) \quad (|B| = 1 \vee |B| \neq 1) \implies \exists_{\epsilon \in S} (\text{GLB}[\epsilon, B, S, <]) \quad \blacksquare \quad \exists_{\epsilon \in S} (\text{GLB}[\epsilon, B, S, <])$$

$$(1.2) \quad (\emptyset \neq B \subset S \wedge \text{BoundedBelow}[B, S, <]) \implies \exists_{\epsilon \in S} (\text{GLB}[\epsilon, B, S, <])$$

$$(1.3) \quad \forall_B ((\emptyset \neq B \subset S \wedge \text{BoundedBelow}[B, S, <]) \implies \exists_{\epsilon \in S} (\text{GLB}[\epsilon, B, S, <]))$$

$$(1.4) \quad \text{GLBProperty}[S, <]$$

$$(2) \quad \text{LUBProperty}[S, <] \implies \text{GLBProperty}[S, <]$$

$$(1.12)$$

$$\text{Field}[F, +, *] := \exists_{0, 1 \in F} \forall_{x, y, z \in F} \left(\begin{array}{l} x + y \in F \quad \wedge \quad x * y \in F \quad \wedge \\ x + y = y + x \quad \wedge \quad x * y = y * x \quad \wedge \\ (x + y) + z = x + (y + z) \quad \wedge \quad (x * y) * z = x * (y * z) \quad \wedge \\ 1 \neq 0 \quad \wedge \quad x * (y + z) = (x * y) + (x * z) \quad \wedge \\ 0 + x = x \quad \wedge \quad 1 * x = x \quad \wedge \\ \exists_{-x \in F} (x + (-x) = 0) \wedge \left(x \neq 0 \implies \exists_{1/x \in F} (x * (1/x) = 1) \right) \end{array} \right)$$

$$\text{***** (Field}[F, +, *] \wedge x, y, z \in F) \implies \dots \text{*****}$$

$$(1.14)$$

$$\text{AdditiveCancellation} := (x + y = x + z) \implies y = z$$

$$(1) \quad y = 0 + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \dots$$

from: *Field*

$$(2) \quad (-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = 0 + z = z$$

from: *Field*

$$\text{AdditiveIdentityUniqueness} := (x + y = x) \implies y = 0$$

$$(1) \quad x + y = x = 0 + x = x + 0$$

from: *Field*

$$(2) \quad y = 0$$

from: *AdditiveCancellation*

$$\text{AdditiveInverseUniqueness} := (x + y = 0) \implies y = -x$$

$$(1) \quad x + y = 0 = x + (-x)$$

from: *Field*

$$(2) \quad y = -x$$

from: *AdditiveCancellation*

$$\text{DoubleNegative} := x = -(-x)$$

$$(1) \quad 0 = x + (-x) = (-x) + x \quad \blacksquare \quad 0 = (-x) + x$$

from: *Field*

$$(2) \quad x = -(-x)$$

from: [AdditiveInverseUniqueness](#)

(1.15)

$$\textcolor{red}{\textit{MultiplicativeCancellation}} := (x \neq 0 \wedge x * y = x * z) \implies y = z \quad \text{---}$$

$$\textcolor{red}{\textit{MultiplicativeIdentityUniqueness}} := (x \neq 0 \wedge x * y = x) \implies y = 1 \quad \text{---}$$

$$\textcolor{red}{\textit{MultiplicativeInverseUniqueness}} := (x \neq 0 \wedge x * y = 1) \implies y = 1/x \quad \text{---}$$

$$\textcolor{red}{\textit{DoubleReciprocal}} := (x \neq 0) \implies x = 1/(1/x) \quad \text{---}$$

(1.16)

$$\textcolor{red}{\textit{Domination}} := 0 * x = 0$$

from: [Field](#)

$$(1) \quad 0 * x = (0 + 0) * x = 0 * x + 0 * x \quad \blacksquare \quad 0 * x = 0 * x + 0 * x$$

$$(2) \quad 0 * x = 0$$

from: [AdditiveIdentityUniqueness](#)

$$\textcolor{red}{\textit{NonDomination}} := (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$$

$$(1) \quad (x \neq 0 \wedge y \neq 0) \implies \dots$$

$$(1.1) \quad (x * y = 0) \implies \dots$$

$$(1.1.1) \quad 1 = 1 * 1 = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = 0 * ((1/x) * (1/y)) = 0$$

from: [Field](#), [Domination](#), 1, 1.1

$$(1.1.2) \quad 1 = 0 \wedge 1 \neq 0 \quad \blacksquare \quad \perp$$

from: [Field](#)

$$(1.2) \quad (x * y = 0) \implies \perp \quad \blacksquare \quad x * y \neq 0$$

$$(2) \quad (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$$

$$\textcolor{red}{\textit{NegationCommutativity}} := (-x) * y = -(x * y) = x * (-y)$$

$$(1) \quad x * y + (-x) * y = (x + -x) * y = 0 * y = 0 \quad \blacksquare \quad x * y + (-x) * y = 0$$

from: [Field](#), [Domination](#)
wts: 2

$$(2) \quad (-x) * y = -(x * y)$$

from: [AdditiveInverseUniqueness](#)

$$(3) \quad x * y + x * (-y) = x * (y_0 + -y) = x * 0 = 0 \quad \blacksquare \quad x * y + x * (-y) = 0$$

from: [Field](#), [Domination](#)
wts: 4

$$(4) \quad x * (-y) = -(x * y)$$

from: [AdditiveInverseUniqueness](#)

$$(5) \quad (-x) * y = -(x * y) = x * (-y)$$

from: 2, 4

$$\textcolor{red}{\textit{NegativeMultiplication}} := (-x) * (-y) = x * y$$

$$(1) \quad (-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$$

from: [NegationCommutativity](#), [DoubleNegative](#)

(1.17)

$$\textcolor{red}{\textit{OrderedField}}[F, +, *, <] := \left(\begin{array}{l} \textcolor{blue}{\textit{Field}}[F, +, *] \quad \wedge \quad \textcolor{blue}{\textit{Order}}[<, F] \quad \wedge \\ \forall_{x,y,z \in F} (y_0 < z \implies x + y < x + z) \quad \wedge \\ \forall_{x,y \in F} ((x > 0 \wedge y > 0) \implies x * y > 0) \end{array} \right)$$

$$\textcolor{blue}{\textit{OrderedField}}[F, +, *, <] \wedge x, y, z \in F \implies \dots$$

(1.18)

$$\textcolor{red}{\textit{NegationOnOrder}} := x > 0 \iff -x < 0$$

$$(1) \quad x > 0 \implies \dots$$

$$(1.1) \quad 0 = (-x) + x > (-x) + 0 = -x \quad \blacksquare \quad 0 > -x \quad \blacksquare \quad -x < 0$$

from: [OrderedField](#)

$$(2) \quad x > 0 \implies -x < 0$$

$$(3) \quad -x < 0 \implies \dots$$

$$(3.1) \quad 0 = x + (-x) < x + 0 = x \quad \blacksquare \quad 0 < x \quad \blacksquare \quad x > 0$$

from: [OrderedField](#)

$$(4) \quad -x < 0 \implies x > 0$$

$$(5) \quad x > 0 \implies -x < 0 \wedge -x < 0 \implies x > 0 \quad \blacksquare \quad x > 0 \iff -x < 0$$

from: 2, 4

$$\textcolor{red}{\textit{PositiveFactorPreservesOrder}} := (x > 0 \wedge y < z) \implies x * y < x * z$$

$$(1) \quad (x > 0 \wedge y < z) \implies \dots$$

$$(1.1) \quad (-y) + z > (-y) + y = 0 \quad \blacksquare \quad z + (-y) = 0$$

from: [OrderedField](#)

$$(1.2) \quad x * (z + (-y)) > 0 \quad \blacksquare \quad x * z + x * (-y) > 0$$

from: [OrderedField](#)

$$(1.3) \quad x * z = 0 + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \dots$$

from: *Field, NegationCommutativity*

$$(1.4) \quad x * y + (x * z + x * (-y)) > x * y + 0 = x * y$$

from: *Field, 1.2*

$$(1.5) \quad x * z > x * y$$

from: 1.3, 1.4

$$(2) \quad (x > 0 \wedge y < z) \implies x * z > x * y$$

$$\textcolor{red}{NegativeFactorFlipsOrder} := (x < 0 \wedge y < z) \implies x * y > x * z$$

$$(1) \quad (x < 0 \wedge y < z) \implies \dots$$

$$(1.1) \quad -x > 0$$

from: *NegationOnOrder*

$$(1.2) \quad (-x) * y < (-x) * z \quad \blacksquare \quad 0 = x * y + (-x) * y < x * y + (-x) * z \quad \blacksquare \quad 0 < x * y + (-x) * z$$

from: *PositiveFactorPreservesOrder*

$$(1.3) \quad 0 < (-x) * (-y + z) \quad \blacksquare \quad 0 > x * (-y + z) \quad \blacksquare \quad 0 > -(x * y) + x * z$$

from: *NegationOnOrder*

$$(1.4) \quad x * y > x * z$$

$$(2) \quad (x < 0 \wedge y < z) \implies x * y > x * z$$

$$\textcolor{red}{SquareIsPositive} := (x \neq 0) \implies x * x > 0$$

$$(1) \quad (x > 0) \implies x * x > 0$$

from: *OrderedField*

$$(2) \quad (x < 0) \implies \dots$$

$$(2.1) \quad -x > 0 \quad \blacksquare \quad x * x = (-x) * (-x) > 0 \quad \blacksquare \quad x * x > 0$$

from: *NegationOnOrder, OrderedField, NegativeMultiplication*

$$(3) \quad (x < 0) \implies x * x > 0$$

$$(4) \quad x \neq 0 \implies (x > 0 \vee x < 0) \implies x * x > 0 \quad \blacksquare \quad x \neq 0 \implies x * x > 0$$

from: *OrderTrichotomy, 1.3*

$$\textcolor{red}{OneIsPositive} := 1 > 0$$

$$(1) \quad 1 \neq 0 \quad \blacksquare \quad 1 = 1 * 1 > 0$$

from: *Field, SquareIsPositive*

$$\textcolor{red}{ReciprocationOnOrder} := (0 < x < y) \implies 0 < 1/y < 1/x$$

$$(1) \quad (0 < x < y) \implies \dots$$

$$(1.1) \quad x * (1/x) = 1 > 0 \quad \blacksquare \quad x * (1/x) > 0$$

from: *Field, OneIsPositive*

$$(1.2) \quad 1/x < 0 \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp \quad \blacksquare \quad 1/x > 0$$

from: *NegativeFactorFlipsOrder, 1*

$$(1.3) \quad y * (1/y) = 1 > 0 \quad \blacksquare \quad y * (1/y) > 0$$

from: *Field, OneIsPositive*

$$(1.4) \quad 1/y < 0 \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp \quad \blacksquare \quad 1/y > 0$$

from: *NegativeFactorFlipsOrder, 1*

$$(1.5) \quad (1/x) * (1/y) > 0$$

from: *OrderedField*

$$(1.6) \quad 0 < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$$

from: *OrderedField, 1, 1.4, 1.5*

(1.19)

$$\textcolor{red}{OrderedFieldQ} := \textcolor{teal}{OrderedField}[\mathbb{Q}, +, *, <] \quad \text{---}$$

$$\textcolor{red}{Subfield}[K, F, +, *] := \textcolor{teal}{Field}[F, +, *] \wedge K \subset F \wedge \textcolor{teal}{Field}[K, +, *]$$

$$\textcolor{red}{OrderedSubfield}[K, F, +, *, <] := \textcolor{teal}{OrderedField}[F, +, *, <] \wedge K \subset F \wedge \textcolor{teal}{OrderedField}[K, +, *, <]$$

$$\textcolor{red}{CutI}[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$$

$$\textcolor{red}{CutII}[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$$

$$\textcolor{red}{CutIII}[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$$

$$\mathbb{R} := \{\alpha \in \mathbb{Q} : \textcolor{teal}{CutI}[\alpha] \wedge \textcolor{teal}{CutII}[\alpha] \wedge \textcolor{teal}{CutIII}[\alpha]\}$$

$$\textcolor{red}{CutCorollaryI} := (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$$

$$(1) \quad (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \dots$$

$$(1.1) \quad \forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$$

from: *CutII, 1*

$$(1.2) \quad q < p \implies q \in \alpha \quad \blacksquare \quad q \notin \alpha \implies q \geq p$$

from: 1

$$(1.3) \quad (q \notin \alpha) \implies \dots$$

$$(1.3.1) \quad q \geq p$$

from: 1.2

$$(1.3.2) \quad (q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp \quad \blacksquare \quad q \neq p$$

from: 1, 1.3

(1.3.3)	$q \geq p \wedge q \neq p \blacksquare p < q$	
(1.4)	$q \notin \alpha \implies p < q \blacksquare p < q$	from: 1
(2)	$(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$	

CutCorollaryI $:= (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1)	$(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \dots$	
(1.1)	$\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$	from: CutII, 1
(1.2)	$s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)) \blacksquare s \in \alpha \implies r \in \alpha$	from: 1, 1.1
(1.3)	$r \notin \alpha \implies s \notin \alpha \blacksquare s \notin \alpha$	from: 1, 1.2
(2)	$(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$	

$<_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

OrderTrichotomyOfR $:= \text{OrderTrichotomy}[\mathbb{R}, <_{\mathbb{R}}]$

(1)	$(\alpha, \beta \in \mathbb{R}) \implies \dots$	
(1.1)	$\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \dots$	
(1.1.1)	$\alpha \not\subset \beta \wedge \alpha \neq \beta$	from: $<_{\mathbb{R}}$, 1.1
(1.1.2)	$\exists_{p'} (p' \in \alpha \wedge p' \notin \beta) \blacksquare p := \text{choice}(\{p' : p' \in \alpha \wedge p' \notin \beta\})$	
(1.1.3)	$q \in \beta \implies \dots$	
(1.1.3.1)	$p, q \in \mathbb{Q}$	
(1.1.3.2)	$q < p$	from: CutCorollaryI
(1.1.3.3)	$q \in \alpha$	from: CutII
(1.1.4)	$q \in \beta \implies q \in \alpha$	
(1.1.5)	$\forall_{q \in \beta} (q \in \alpha) \blacksquare \beta \subseteq \alpha$	
(1.1.6)	$\beta \subset \alpha \blacksquare \beta <_{\mathbb{R}} \alpha$	
(1.2)	$\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$	
(1.3)	$\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \blacksquare (\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$	
(1.4)	$\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$	
(1.5)	$\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$	
(1.6)	$\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$	
(1.7)	$\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta$	
(2)	$(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$	
(3)	$\forall_{\alpha, \beta \in \mathbb{R}} (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$	
(4)	OrderTrichotomy $[\mathbb{R}, <_{\mathbb{R}}]$	

OrderTransitivityOfR $:= \text{OrderTransitivity}[\mathbb{R}, <_{\mathbb{R}}]$

(1)	$(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$	
(1.1)	$(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \dots$	
(1.1.1)	$\alpha \subset \beta \wedge \beta \subset \gamma$	
(1.1.2)	$\forall_{a \in \alpha} (a \in \beta) \wedge \forall_{b \in \beta} (b \in \gamma)$	
(1.1.3)	$\forall_{a \in \alpha} (\alpha \in \gamma) \blacksquare \alpha \subset \gamma \blacksquare \alpha <_{\mathbb{R}} \gamma$	
(1.2)	$(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$	
(2)	$(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$	
(3)	$\forall_{\alpha, \beta, \gamma \in \mathbb{R}} ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$	
(4)	OrderTransitivity $[\mathbb{R}, <_{\mathbb{R}}]$	

OrderOfR $:= \text{Order}[<_{\mathbb{R}}, \mathbb{R}]$

LUBPropertyOfR $:= \text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}]$

(1)	$(\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \dots$	from: OrderTrichotomyR , OrderTransitivityR wts:
(1.1)	$\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A} (p \in \alpha)\}$	

(1.2)	$A \neq \emptyset \quad \blacksquare \quad \exists_{\alpha}(\alpha \in A) \quad \blacksquare \quad \alpha_0 := \text{choice}(\{\alpha : \alpha \in A\})$
(1.3)	$\alpha_0 \neq \emptyset \quad \blacksquare \quad \exists_a(a \in \alpha_0) \quad \blacksquare \quad a_0 := \text{choice}(\{a : a \in \alpha_0\}) \quad \blacksquare \quad a_0 \in \gamma \quad \blacksquare \quad \gamma \neq \emptyset$
(1.4)	$\text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}] \quad \blacksquare \quad \exists_{\beta}(\text{UpperBound}[\beta, A, \mathbb{R}, <_{\mathbb{R}}])$
(1.5)	$\beta_0 := \text{choice}(\{\beta : \text{UpperBound}[\beta, A, \mathbb{R}, <_{\mathbb{R}}]\})$
(1.6)	$\text{UpperBound}[\beta_0, A, \mathbb{R}, <_{\mathbb{R}}] \quad \blacksquare \quad \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \beta_0) \quad \blacksquare \quad \forall_{\alpha \in A}(\alpha \subseteq \beta_0) \quad \blacksquare \quad \forall_{\alpha \in A} \forall_{a \in \alpha}(a \in \beta_0)$
(1.7)	$(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \quad \blacksquare \quad \forall_{a \in \gamma}(a \in \beta_0) \quad \blacksquare \quad \gamma \subseteq \beta_0$
(1.8)	$\beta_0 \subset \mathbb{Q} \quad \blacksquare \quad \gamma \subseteq \beta_0 \subset \mathbb{Q} \quad \blacksquare \quad \gamma \subset \mathbb{Q}$
(1.9)	$\emptyset \neq \gamma \subset \mathbb{Q} \quad \blacksquare \quad \text{CutI}[\gamma]$
(1.10)	$(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
(1.10.1)	$p \in \gamma \quad \blacksquare \quad \exists_{\alpha \in A}(p \in \alpha) \quad \blacksquare \quad \alpha_1 := \text{choice}(\{\alpha \in A : p \in \alpha\})$
(1.10.2)	$p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \quad \blacksquare \quad q \in \alpha_1 \quad \blacksquare \quad q \in \gamma$
(1.11)	$(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \quad \blacksquare \quad \forall_{p \in \gamma} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma) \quad \blacksquare \quad \text{CutII}[\gamma]$
(1.12)	$p \in \gamma \implies \dots$
(1.12.1)	$\exists_{\alpha \in A}(p \in \alpha) \quad \blacksquare \quad \alpha_2 := \text{choice}(\{\alpha \in A : p \in \alpha\})$
(1.12.2)	$\alpha_2 \in \mathbb{R} \quad \blacksquare \quad \text{CutII}[\alpha_2] \quad \blacksquare \quad \exists_{r \in \alpha_2}(p < r) \quad \blacksquare \quad r_0 := \text{choice}(\{r \in \alpha_2 : p < r\})$
(1.12.3)	$r_0 \in \alpha_2 \quad \blacksquare \quad r_0 \in \gamma$
(1.12.4)	$p < r_0 \quad \blacksquare \quad p < r_0 \wedge r_0 \in \gamma \quad \blacksquare \quad \exists_{r \in \gamma}(p < r)$
(1.13)	$p \in \gamma \implies \exists_{r \in \gamma}(p < r) \quad \blacksquare \quad \forall_{p \in \gamma} \exists_{r \in \gamma}(p < r) \quad \blacksquare \quad \text{CutIII}[\gamma]$
(1.14)	$\text{CutI}[\gamma] \wedge \text{CutII}[\gamma] \wedge \text{CutIII}[\gamma] \quad \blacksquare \quad \gamma \in \mathbb{R}$
(1.15)	$\forall_{\alpha \in A}(\alpha \subseteq \gamma) \quad \blacksquare \quad \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma)$
(1.16)	$\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R} \quad \blacksquare \quad \text{UpperBound}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$
(1.17)	$\delta <_{\mathbb{R}} \gamma \implies \dots$
(1.17.1)	$\delta \subset \gamma \quad \blacksquare \quad \exists_s(s \in \gamma \wedge s \notin \delta) \quad \blacksquare \quad s_0 := \text{choice}(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$
(1.17.2)	$s_0 \in \gamma \quad \blacksquare \quad \exists_{\alpha \in A}(s_0 \in \alpha) \quad \blacksquare \quad \alpha_3 := \text{choice}(\{\alpha \in A : s_0 \in \alpha\})$
(1.17.3)	$s_0 \in \alpha_3 \wedge s_0 \notin \delta \quad \blacksquare \quad \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$
(1.17.4)	$\delta \geq_{\mathbb{R}} \alpha_3 \implies \dots$
(1.17.4.1)	$\alpha_3 \subseteq \delta \quad \blacksquare \quad \forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \quad \blacksquare \quad \neg \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$
(1.17.4.2)	$\neg \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \quad \blacksquare \quad \perp$
(1.17.5)	$\delta \geq_{\mathbb{R}} \alpha_3 \implies \perp \quad \blacksquare \quad \delta <_{\mathbb{R}} \alpha_3 \quad \blacksquare \quad \exists_{\alpha \in A}(\delta <_{\mathbb{R}} \alpha) \quad \blacksquare \quad \exists_{\alpha \in A}(\neg(\alpha \leq_{\mathbb{R}} \delta))$
(1.17.6)	$\neg \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \delta) \quad \blacksquare \quad \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$
(1.18)	$\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}] \quad \blacksquare \quad \forall_{\delta}(\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$
(1.19)	$\text{UpperBound}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \wedge \forall_{\delta}(\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$
(1.20)	$\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \quad \blacksquare \quad \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$
(2)	$(\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$
(3)	$\forall_A \left((\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]) \right) \quad \blacksquare \quad \text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}]$

$$+_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$$

$$0_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 0\}$$

$$0InR := 0_{\mathbb{R}} \in \mathbb{R}$$

(1)	$-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}} \quad \blacksquare \quad \emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q} \quad \blacksquare \quad \text{CutI}[0_{\mathbb{R}}]$
(2)	$(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}} \quad \blacksquare \quad \forall_{x \in 0_{\mathbb{R}}} \forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}}) \quad \blacksquare \quad \text{CutII}[0_{\mathbb{R}}]$
(3)	$y := x/2 \quad \blacksquare \quad (x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y) \quad \blacksquare \quad \forall_{x \in 0_{\mathbb{R}}} \exists_{y \in 0_{\mathbb{R}}}(x < y) \quad \blacksquare \quad \text{CutIII}[0_{\mathbb{R}}]$
(4)	$\text{CutI}[0_{\mathbb{R}}] \wedge \text{CutII}[0_{\mathbb{R}}] \wedge \text{CutIII}[0_{\mathbb{R}}] \quad \blacksquare \quad 0_{\mathbb{R}} \in \mathbb{R}$

$$\text{FieldAdditionClosureOf } R := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$$

(1)	$(\alpha, \beta \in \mathbb{R}) \implies \dots$
(1.1)	$(\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$
(1.2)	$\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3)	$\exists_a(a \in \alpha) ; \exists_b(b \in \beta) \blacksquare a_0 := choice(\{a : a \in \alpha\}) ; b_0 := choice(\{b : b \in \beta\}) \blacksquare a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$
(1.4)	$\exists_x(x \notin \alpha) ; \exists_y(y_0 \notin \beta) \blacksquare x_0 := choice(\{x : x \notin \alpha\}) ; y_0 := choice(\{y : y \notin \beta\})$
(1.5)	$\forall_{r \in \alpha}(r < x_0) ; \forall_{s \in \beta}(s < y_0) \blacksquare \forall_{r \in \alpha} \forall_{s \in \beta}(r + s < x_0 + y_0) \blacksquare x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$
(1.6)	$\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q} \blacksquare \textcolor{blue}{CutI}[\alpha +_{\mathbb{R}} \beta]$
(1.7)	$(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
(1.7.1)	$\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s) \blacksquare (r_0, s_0) := choice((r, s) \in \alpha \times \beta : p = r + s)$
(1.7.2)	$q < p = r_0 + s_0 \blacksquare (q - s_0) < r_0 \blacksquare (q - s_0) \in \alpha$
(1.7.3)	$s_0 \in \beta \blacksquare q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta \blacksquare q \in \alpha +_{\mathbb{R}} \beta$
(1.8)	$(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta) \blacksquare \textcolor{blue}{CutII}[\alpha +_{\mathbb{R}} \beta]$
(1.9)	$p \in \alpha \implies \dots$
(1.9.1)	$\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s) \blacksquare (r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta : p = r + s\})$
(1.9.2)	$r_1 \in \alpha \blacksquare \exists_{t \in \alpha}(r_1 < t) \blacksquare t_0 := choice(\{t \in \alpha : r_1 < t\})$
(1.9.3)	$s_1 \in \beta \blacksquare t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1 \blacksquare \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$
(1.10)	$p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r) \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} \beta} \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r) \blacksquare \textcolor{blue}{CutIII}[\alpha +_{\mathbb{R}} \beta]$
(1.11)	$\textcolor{blue}{CutI}[\alpha +_{\mathbb{R}} \beta] \wedge \textcolor{blue}{CutII}[\alpha +_{\mathbb{R}} \beta] \wedge \textcolor{blue}{CutIII}[\alpha +_{\mathbb{R}} \beta] \blacksquare \alpha +_{\mathbb{R}} \beta \in \mathbb{R}$
(2)	$(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

FieldAdditionCommutativityOf \mathbb{R} $:= (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s : r \in \alpha \wedge s \in \beta\} = \{s + r : s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

FieldAdditionAssociativityOf \mathbb{R} $:= (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma))$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \dots$

(1.2) $\{a + (b + c) : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

FieldAdditionIdentityOf \mathbb{R} $:= (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \dots$

(1.1) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \dots$

(1.1.1) $s < 0 \blacksquare r + s < r + 0 = r \blacksquare r + s < r \blacksquare r + s \in \alpha$

(1.2) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha \blacksquare \forall_{r \in \alpha} \forall_{s \in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}}(p \in \alpha) \blacksquare \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \dots$

(1.4.1) $\exists_{r \in \alpha}(p < r) \blacksquare r_2 := choice(\{r \in \alpha : p < r\})$

(1.4.2) $p < r_2 \blacksquare p - r_2 < r_2 - r_2 = 0 \blacksquare (p - r_2) < 0 \blacksquare (p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha \blacksquare p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare \forall_{p \in \alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

FieldAdditionInverseOf \mathbb{R} $:= (\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(1) $\alpha \in \mathbb{R} \implies \dots$

(1.1) $\beta := \{p \in \mathbb{Q} : \exists_{r > 0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q} \blacksquare \exists_{s \in \mathbb{Q}}(s \notin \alpha) \blacksquare s_0 := choice(\{s : s \notin \alpha\}) \blacksquare p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha \blacksquare -p_0 - 1 \notin \alpha \blacksquare \exists_{r > 0}(-p_0 - r \notin \alpha) \blacksquare p_0 \in \beta$

(1.4) $\emptyset \neq \alpha \blacksquare \exists_{q \in \alpha} \blacksquare q_0 := choice(\{q \in \mathbb{Q} : q \in \alpha\})$

(1.5) $r > 0 \implies \dots$

(1.5.1) $q_0 \in \alpha \blacksquare -(-q_0) - r = q_0 - r < q_0 \blacksquare -(-q_0) - r < q_0 \blacksquare -(-q_0) - r \in \alpha$

(1.6) $\forall_{r > 0}(-(-q_0) - r \in \alpha) \blacksquare \neg \exists_{r > 0}(-(-q_0) - r \notin \alpha) \blacksquare -q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q} \blacksquare \textcolor{blue}{CutI}[\beta]$

(1.8)	$(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$	
(1.8.1)	$p \in \beta \blacksquare \exists_{r>0}(-p-r \notin \alpha) \blacksquare r_0 := \text{choice}(\{r > 0 : -p-r \notin \alpha\})$	
(1.8.2)	$q < p \blacksquare -p-r < -q-r$	
(1.8.3)	$-q-r \notin \alpha \blacksquare q \in \beta$	
(1.9)	$(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta \blacksquare \forall_{p \in \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \beta) \blacksquare \text{CutII}[\beta]$	
(1.10)	$p \in \beta \implies \dots$	
(1.10.1)	$p \in \beta \blacksquare \exists_{r>0}(-p-r \notin \alpha) \blacksquare r_1 := \text{choice}(\{r > 0 : -p-r \notin \alpha\})$	
(1.10.2)	$t_0 := p + (r_1/2)$	
(1.10.3)	$r_1 > 0 \blacksquare r_1/2 > 0$	
(1.10.4)	$t_0 > t_0 - (r_1/2) = p \blacksquare t_0 > p$	
(1.10.5)	$-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$	
(1.10.6)	$-p - r_1 \notin \alpha \blacksquare -t_0 - (r_1/2) \notin \alpha \blacksquare \exists_{r>0}(-t_0 - r \notin \alpha) \blacksquare t_0 \in \beta$	
(1.10.7)	$t_0 > p \wedge t_0 \in \beta \blacksquare \exists_{t \in \beta}(p < t)$	
(1.11)	$p \in \beta \implies \exists_{t \in \beta}(p < t) \blacksquare \forall_{p \in \beta} \exists_{t \in \beta}(p < t) \blacksquare \text{CutIII}[\beta]$	
(1.12)	$\text{CutI}[\beta] \wedge \text{CutII}[\beta] \wedge \text{CutIII}[\beta] \blacksquare \beta \in \mathbb{R}$	
(1.13)	$(r \in \alpha \wedge s \in \beta) \implies \dots$	
(1.13.1)	$s \in \beta \blacksquare \exists_{t>0}(-s-t \notin \alpha) \blacksquare t_1 := \text{choice}(\{t > 0 : -s-t \notin \alpha\}) \blacksquare -s - t_1 < -s$	
(1.13.2)	$\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha \blacksquare -s \notin \alpha$	
(1.13.3)	$\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha \blacksquare r < -s \blacksquare r + s < 0 \blacksquare r + s \in 0_{\mathbb{R}}$	
(1.14)	$(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}} \blacksquare \forall_{(r,s) \in \alpha \times \beta}(r + s \in 0_{\mathbb{R}}) \blacksquare \alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$	
(1.15)	$v \in 0_{\mathbb{R}} \implies \dots$	
(1.15.1)	$v < 0 \blacksquare w_0 := -v/2 \blacksquare w > 0$	
(1.15.2)	$\exists_{n \in \mathbb{Z}}(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha) \blacksquare n_0 := \text{choice}(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$	from: ARCHIMEDEANPROPERTYOFQ + LUB???
(1.15.3)	$p_0 := -(n_0 + 2)w_0 \blacksquare -p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha \blacksquare -p_0 - w_0 \notin \alpha \blacksquare p_0 \in \beta$	
(1.15.4)	$n_0 w_0 \in \alpha \wedge p_0 \in \beta \blacksquare n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$	
(1.16)	$v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{v \in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta) \blacksquare 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$	
(1.17)	$\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta \blacksquare \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$	
(1.18)	$\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}} \blacksquare \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$	
(2)	$\alpha \in \mathbb{R} \implies \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$	

$*_{\mathbb{R}}[\alpha, \beta] := \text{---}$
 $1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$11s\text{Nor}0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}} \text{---}$
 $11nR := 1_{\mathbb{R}} \in \mathbb{R} \text{---}$
 $\text{Field Multiplication Closure Of } \mathbb{R} := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta) \in \mathbb{R} \text{---}$
 $\text{Field Multiplication Commutativity Of } \mathbb{R} := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha) \text{---}$
 $\text{Field Multiplication Associativity Of } \mathbb{R} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)) \text{---}$
 $\text{Field Multiplication Identity Of } \mathbb{R} := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha \text{---}$
 $\text{Field Multiplication Inverse Of } \mathbb{R} := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha \in \mathbb{R}}(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}) \text{---}$

$\text{Field Distributativity Of } \mathbb{R} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta \text{---}$

$\text{Field With } \mathbb{R} := \text{Field}[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}] \text{---}$
 $\text{Ordered Field With } \mathbb{R} := \text{OrderedField}[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}] \text{---}$

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$
 $\text{QROrderedSubfieldOf } \mathbb{R} := \text{OrderedSubfield}[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}] \text{---}$
 $\text{QIsomorphicToQR} := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q} \text{---}$
 $\text{Completeness Of } \mathbb{R} := \exists_{\mathbb{R}}(\text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}] \wedge \text{OrderedSubfield}[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]) \text{---}$

(1.20)
 $\text{ArchimedeanPropertyOf } \mathbb{R} := \forall_{x,y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \dots$

(1.1) $A := \{nx : n \in \mathbb{N}^+\} \blacksquare (\emptyset \neq A \subset \mathbb{R}) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$

(1.2) $\neg \exists_{n \in \mathbb{N}^+}(nx > y) \implies \dots$

(1.2.1) $\neg \exists_{n \in \mathbb{N}^+}(nx > y) \blacksquare \forall_{n \in \mathbb{N}^+}(nx \leq y) \blacksquare \text{UpperBound}[y_0, A, \mathbb{R}, <] \blacksquare \text{BoundedAbove}[A, \mathbb{R}, <]$

(1.2.2) $\text{CompletenessOf } \mathbb{R} \blacksquare \text{LUBProperty}[\mathbb{R}, <]$

(1.2.3) $(\text{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (\text{BoundedAbove}[A, \mathbb{R}, <]) \blacksquare \exists_{\alpha \in \mathbb{R}}(\text{LUB}[\alpha, A, \mathbb{R}, <]) \dots$

(1.2.4) $\dots \alpha_0 := \text{choice}(\{\alpha \in \mathbb{R} : \text{LUB}[\alpha, A, \mathbb{R}, <]\}) \blacksquare \text{LUB}[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0 \blacksquare \alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge (\text{LUB}[\alpha_0, A, \mathbb{R}, <]) \blacksquare \neg \text{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg \text{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <] \blacksquare \exists_{c \in A}(\alpha_0 - x < c) \dots$

(1.2.8) $\dots c_0 := \text{choice}(\{c \in A : \alpha_0 - x < c\}) \blacksquare (c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)) \blacksquare \exists_{m \in \mathbb{N}^+}(mx = c_0) \dots$

(1.2.10) $\dots m_0 := \text{choice}(\{m \in \mathbb{N}^+ : mx = c_0\}) \blacksquare (m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0) \blacksquare \alpha_0 - x < c_0 = m_0 x \blacksquare \alpha_0 < m_0 x + x \blacksquare \alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+ \blacksquare m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)) \blacksquare (m_0 + 1)x \in A$

(1.2.14) $(\alpha_0 < (m_0 + 1)x) \wedge ((m_0 + 1)x \in A) \blacksquare \exists_{c \in A}(\alpha_0 < c)$

(1.2.15) $\text{LUB}[\alpha_0, A, \mathbb{R}, <] \blacksquare \text{UpperBound}[\alpha_0, A, \mathbb{R}, <] \blacksquare \forall_{c \in A}(c \leq \alpha_0) \blacksquare \neg \exists_{c \in A}(c > \alpha_0) \blacksquare \neg \exists_{c \in A}(\alpha_0 < c)$

(1.2.16) $(\exists_{c \in A}(\alpha_0 < c)) \wedge (\neg \exists_{c \in A}(\alpha_0 < c)) \blacksquare \perp$

(1.3) $\neg \exists_{n \in \mathbb{N}^+}(nx > y) \implies \perp \blacksquare \exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y) \blacksquare \forall_{x, y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

QDenseInR := $\forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \dots$

(1.1) $x < y \blacksquare (0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $\text{ArchimedeanPropertyOf } \mathbb{R} \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R}) \blacksquare \exists_{n \in \mathbb{N}^+}(n(y - x) > 1) \dots$

(1.3) $\dots n_0 := \text{choice}(\{n \in \mathbb{N}^+ : n(y - x) > 1\}) \blacksquare (n_0 \in \mathbb{N}^+) \wedge (n_0(y - x) > 1)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R}) \blacksquare n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $\text{ArchimedeanPropertyOf } \mathbb{R} \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R}) \blacksquare \exists_{m \in \mathbb{N}^+}(m(1) > n_0 x) \dots$

(1.6) $\dots m_1 := \text{choice}(\{m \in \mathbb{N}^+ : m(1) > n_0 x\}) \blacksquare (m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $\text{ArchimedeanPropertyOf } \mathbb{R} \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R}) \blacksquare \exists_{m \in \mathbb{N}^+}(m(1) > -n_0 x) \dots$

(1.8) $\dots m_2 := \text{choice}(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\}) \blacksquare (m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x) \blacksquare -m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+ \blacksquare |m_1 - (-m_2)| \geq 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge (|m_1 - (-m_2)| \geq 2) \blacksquare \exists_{m \in \mathbb{Z}}((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)) \dots$

(1.12) $\dots m_0 := \text{choice}(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\}) \blacksquare (-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$

(1.13) $(n_0(y - x) > 1) \wedge (m_0 - 1 \leq n_0 x < m_0) \blacksquare n_0 x < m_0 \leq 1 + n_0 x < n_0 y \blacksquare n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y) \blacksquare x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z} \blacksquare m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y) \blacksquare \exists_{p \in \mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y) \blacksquare \forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1.21)

Root Lemma := $(0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

(1) $(0 < a < b) \implies \dots$

(1.1) $b^n - a^n = (b - a) \sum_{i=1}^n (b^{n-i} a^{i-1})$

(1.2) $0 < a < b \blacksquare b/a > 1$

$$(1.3) \quad b/a > 1 \quad \blacksquare \quad \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq \sum_{i=1}^n \left(b^{n-i} a^{i-1} (b/a)^{i-1} \right) = \sum_{i=1}^n (b^{n-1}) = nb^{n-1} \quad \blacksquare \quad \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq \sum_{i=1}^n (b^{n-1}) = nb^{n-1}$$

$$(1.4) \quad b^n - a^n = (b - a) \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq (b - a) nb^{n-1} \quad \blacksquare \quad b^n - a^n \leq (b - a) nb^{n-1}$$

$$(2) \quad (0 < a < b) \implies \left(b^n - a^n \leq (b - a) nb^{n-1} \right)$$

$$\text{Root Existence In } \mathbb{R} := \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}} (y_0^n = x)$$

$$(1) \quad (0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \dots$$

$$(1.1) \quad E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\} \quad \blacksquare \quad t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$$

$$(1.2) \quad t_0 := x/(1+x) \quad \blacksquare \quad (t_0 = x/(1+x)) \wedge (t_0 \in \mathbb{R})$$

$$(1.3) \quad 0 < x \quad \blacksquare \quad 0 < x < 1+x \quad \blacksquare \quad t_0 = x/(1+x) > 0 \quad \blacksquare \quad t_0 > 0$$

$$(1.4) \quad 1 = (1+x)/(1+x) > x/(1+x) = t_0 \quad \blacksquare \quad 1 > t_0$$

$$(1.5) \quad (t_0 > 0) \wedge (1 > t_0) \quad \blacksquare \quad 0 < t_0 < 1$$

$$(1.6) \quad (0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1) \quad \blacksquare \quad t_0^n \leq t_0$$

$$(1.7) \quad 0 < x \quad \blacksquare \quad x > x/(1+x) = t_0 \quad \blacksquare \quad x > t_0$$

$$(1.8) \quad (t_0^n \leq t_0) \wedge (x > t_0) \quad \blacksquare \quad t_0^n < x$$

$$(1.9) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge (t_0^n < x) \quad \blacksquare \quad t_0 \in E \quad \blacksquare \quad \emptyset \neq E$$

$$(1.10) \quad t_1 := \text{choice}(\{t \in \mathbb{R} : t > 1+x\}) \quad \blacksquare \quad (t_1 \in \mathbb{R}) \wedge (t_1 > 1+x)$$

$$(1.11) \quad x > 0 \quad \blacksquare \quad t_1 > 1+x > 1 \quad \blacksquare \quad t_1 > 1 \quad \blacksquare \quad t_1^n \geq t_1$$

$$(1.12) \quad (t_1^n \geq t_1) \wedge (t_1 > 1+x) \wedge (1 > 0) \quad \blacksquare \quad t_1^n \geq t_1 > 1+x > x \quad \blacksquare \quad t_1^n > x$$

$$(1.13) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_1^n > x) \quad \blacksquare \quad t_1 \notin E \quad \blacksquare \quad E \subset \mathbb{R}$$

$$(1.14) \quad (\emptyset \neq E) \wedge (E \subset \mathbb{R}) \quad \blacksquare \quad \emptyset \neq E \subset \mathbb{R}$$

$$(1.15) \quad t \in E \implies \dots$$

$$(1.15.1) \quad (t \in E) \wedge (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \quad \blacksquare \quad t^n < x$$

$$(1.15.2) \quad (t_1^n > x) \wedge (t^n < x) \quad \blacksquare \quad t^n < x < t_1^n \quad \blacksquare \quad t < t_1$$

$$(1.16) \quad t \in E \implies t < t_1 \quad \blacksquare \quad \forall_{t \in E} (t \leq t_1) \quad \blacksquare \quad \text{Upper Bound}[t_1, E, \mathbb{R}, <] \quad \blacksquare \quad \text{Bounded Above}[E, \mathbb{R}, <]$$

$$(1.17) \quad \text{Completeness Of } \mathbb{R} \quad \blacksquare \quad \text{LUB Property}[\mathbb{R}, <]$$

$$(1.18) \quad (\text{LUB Property}[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (\text{Bounded Above}[E, \mathbb{R}, <]) \quad \blacksquare \quad \exists_{y \in \mathbb{R}} (\text{LUB}[y, E, \mathbb{R}, <]) \quad \dots$$

$$(1.19) \quad \dots y_0 := \text{choice}(\{y \in \mathbb{R} : \text{LUB}[y, E, \mathbb{R}, <]\}) \quad \blacksquare \quad \text{LUB}[y_0, E, \mathbb{R}, <]$$

$$(1.20) \quad (\text{LUB}[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0) \quad \blacksquare \quad 0 < t_0 \leq y_0 \in \mathbb{R} \quad \blacksquare \quad 0 < y_0 \in \mathbb{R}$$

$$(1.21) \quad y_0^n < x \implies \dots$$

$$(1.21.1) \quad k_0 := \frac{x - y_0^n}{n(y_0 + 1)^{n-1}} \quad \blacksquare \quad k_0 \in \mathbb{R}$$

$$(1.21.2) \quad y_0^n < x \quad \blacksquare \quad 0 < x - y_0^n$$

$$(1.21.3) \quad (n > 0) \wedge (y_0 > 0) \quad \blacksquare \quad 0 < n(y_0 + 1)^{n-1}$$

$$(1.21.4) \quad (0 < x - y_0^n) \wedge \left(0 < n(y_0 + 1)^{n-1} \right) \quad \blacksquare \quad 0 < \frac{x - y_0^n}{n(y_0 + 1)^{n-1}} = k_0 \quad \blacksquare \quad 0 < k_0$$

$$(1.21.5) \quad (0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R}) \quad \blacksquare \quad 0 < \min(1, k_0) \in \mathbb{R}$$

$$(1.21.6) \quad \text{QDense In } \mathbb{R} \wedge (0, \min(1, k_0)) \in \mathbb{R} \wedge (0 < \min(1, k_0)) \quad \blacksquare \quad \exists_{h \in \mathbb{Q}} (0 < h < \min(1, k_0)) \quad \dots$$

$$(1.21.7) \quad \dots h_0 := \text{choice}(\{h \in \mathbb{Q} : 0 < h < \min(1, k_0)\}) \quad \blacksquare \quad (0 < h_0 < 1) \wedge \left(h_0 < k_0 = \frac{x - y_0^n}{n(y_0 + 1)^{n-1}} \right)$$

$$(1.21.8) \quad (y_0 > 0) \wedge (h_0 > 0) \quad \blacksquare \quad 0 < y_0 < y_0 + h_0$$

$$(1.21.9) \quad \text{Root Lemma} \wedge (0 < y_0 < y_0 + h_0) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < h_0 n(y_0 + h_0)^{n-1}$$

$$(1.21.10) \quad h_0 < 1 \quad \blacksquare \quad h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$$

$$(1.21.11) \quad \left((y_0 + h_0)^n - y_0^n < h_0 n(y_0 + h_0)^{n-1} \right) \wedge \left(h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1} \right) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < h_0 n(y_0 + 1)^{n-1}$$

$$(1.21.12) \quad \left(0 < n(y_0 + 1)^{n-1} \right) \wedge \left(h_0 < k_0 = \frac{x - y_0^n}{n(y_0 + 1)^{n-1}} \right) \quad \blacksquare \quad h_0 n(y_0 + 1)^{n-1} < x - y_0^n$$

$$(1.21.13) \quad \left((y_0 + h_0)^n - y_0^n < h_0 n(y_0 + 1)^{n-1} \right) \wedge \left(h_0 n(y_0 + 1)^{n-1} < x - y_0^n \right) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < x - y_0^n \quad \blacksquare \quad (y_0 + h_0)^n < x$$

$$(1.21.14) \quad (y_0 + h_0)^n - y_0^n < x - y_0^n \quad \blacksquare \quad (y_0 + h_0)^n < x$$

$$(1.21.15) \quad (0 < y_0 \in \mathbb{R}) \wedge (0 < h_0 \in \mathbb{R}) \quad \blacksquare \quad 0 < y_0 < y_0 + h_0 \in \mathbb{R}$$

$$(1.21.16) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ((y_0 + h_0)^n < x) \wedge (0 < y_0 + h_0 \in \mathbb{R}) \quad \blacksquare \quad (y_0 + h_0)^n \in E$$

(1.21.17)	$((y_0 + h_0)^n \in E) \wedge (y_0 < y_0 + h_0) \blacksquare \exists_{e \in E} (y_0 < e)$
(1.21.18)	$\textcolor{blue}{LUB}[y_0, E, \mathbb{R}, <] \blacksquare \textcolor{blue}{UpperBound}[y_0, E, \mathbb{R}, <] \blacksquare \forall_{e \in E} (e \leq y_0) \blacksquare \neg \exists_{e \in E} (e > y_0)$
(1.21.19)	$(\exists_{e \in E} (e > y_0)) \wedge (\neg \exists_{e \in E} (e > y_0)) \blacksquare \perp$
(1.22)	$y_0^n < x \implies \perp \blacksquare y_0^n \geq x$
(1.23)	$y_0^n > x \implies \dots$
(1.23.1)	$k_1 := \frac{y_0^n - x}{ny_0^{n-1}} \blacksquare (k_1 \in \mathbb{R}) \wedge (k_1 ny_0^{n-1} = y_0^n - x)$
(1.23.2)	$(0 < x) \wedge (0 < n \in \mathbb{Z}) \blacksquare y_0^n - x < y_0^n \leq ny_0^n \blacksquare y_0^n - x < ny_0^n$
(1.23.3)	$y_0^n - x < ny_0^n \blacksquare k_1 = \frac{y_0^n - x}{ny_0^{n-1}} < \frac{ny_0^n}{ny_0^{n-1}} = y_0 \blacksquare k_1 < y_0$
(1.23.4)	$y_0^n > x \blacksquare 0 < y_0^n - x$
(1.23.5)	$(n > 0) \wedge (y_0 > 0) \blacksquare 0 < ny_0^{n-1}$
(1.23.6)	$(0 < y_0^n - x) \wedge 0 < (ny_0^{n-1}) \blacksquare 0 < \frac{y_0^n - x}{ny_0^{n-1}} = k_1 \blacksquare 0 < k_1$
(1.23.7)	$(k_1 < y_0) \wedge (0 < k_1) \blacksquare (0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$
(1.23.8)	$t \geq y_0 - k_1 \implies \dots$
(1.23.8.1)	$t \geq y_0 - k_1 \blacksquare t^n \geq (y_0 - k_1)^n \blacksquare -t^n \leq -(y_0 - k_1)^n \blacksquare y_0^n - t^n \leq y_0^n - (y_0 - k_1)^n$
(1.23.8.2)	$\textcolor{blue}{RootLemma} \wedge (0 < y_0 - k_1 < y_0) \blacksquare y_0^n - (y_0 - k_1)^n < k_1 ny_0^{n-1}$
(1.23.8.3)	$(y_0^n - t^n \leq y_0^n - (y_0 - k_1)^n) \wedge (y_0^n - (y_0 - k_1)^n < k_1 ny_0^{n-1}) \blacksquare y_0^n - t^n < k_1 ny_0^{n-1}$
(1.23.8.4)	$(k_1 ny_0^{n-1} = y_0^n - x) \wedge (y_0^n - t^n < k_1 ny_0^{n-1}) \blacksquare y_0^n - t^n < y_0^n - x \blacksquare -t^n < -x \blacksquare t^n > x$
(1.23.8.5)	$(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t^n > x) \blacksquare t \notin E$
(1.23.9)	$t \geq y_0 - k_1 \implies t \notin E \blacksquare t \in E \implies t < y_0 - k_1 \blacksquare \forall_{t \in E} (t \leq y_0 - k_1) \blacksquare \textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$
(1.23.10)	$(\textcolor{blue}{LUB}[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)) \blacksquare \neg \textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$
(1.23.11)	$(\textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg \textcolor{blue}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \blacksquare \perp$
(1.24)	$y_0^n > x \implies \perp \blacksquare y_0^n \leq x$
(1.25)	$\textcolor{blue}{Order}[\mathbb{R}, <] \blacksquare \textcolor{blue}{OrderTrichotomy}[\mathbb{R}, <]$
(1.26)	$(\textcolor{blue}{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_0^n \geq x) \wedge (y_0^n \leq x) \blacksquare y_0^n = x$
(1.27)	$(y_0^n = x) \wedge (y_0 \in \mathbb{R}) \blacksquare \exists_{y \in \mathbb{R}} (y^n = x)$
(1.28)	$y_1, y_2 := \textit{choice}(\{y \in \mathbb{R} : y^n = x\})$
(1.29)	$y_1 \neq y_2 \implies \dots$
(1.29.1)	$(\textcolor{blue}{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_1 \neq y_2) \blacksquare (y_1 < y_2) \vee (y_2 < y_1) \dots$
(1.29.2)	$\dots (x = y_1^n < y_2^n = x) \vee (x = y_2^n < y_1^n = x) \blacksquare (x < x) \vee (x > x) \blacksquare \perp \vee \perp \blacksquare \perp$
(1.30)	$y_1 \neq y_2 \implies \perp \blacksquare y_1 = y_2 \blacksquare \forall_{a,b \in \mathbb{R}} ((a^n = x \wedge b^n = x) \implies a = b)$
(1.31)	$(\exists_{y \in \mathbb{R}} (y^n = x)) \wedge (\forall_{a,b \in \mathbb{R}} ((a^n = x \wedge b^n = x) \implies a = b)) \blacksquare \exists!_{y \in \mathbb{R}} (y^n = x)$
(2)	$(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}} (y^n = x) \blacksquare \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}} (y_0^n = x)$

$$\textcolor{red}{RootExistenceInRCorollary} := \forall_{0 < a \in \mathbb{R}} \forall_{0 < b \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \left((ab)^{1/n} = a^{1/n} b^{1/n} \right) \quad \text{---}$$

$$\textcolor{red}{ExtendedRealSystem}[\bar{\mathbb{R}}, +, *, <] := \left(\begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \quad \wedge \quad x - \infty = -\infty \quad \wedge \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \quad \wedge \\ (x > 0) \implies (x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty) \quad \wedge \\ (x < 0) \implies (x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty) \end{array} \right)$$

$$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$$

$$+_C[\langle a, b \rangle, \langle c, d \rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d \rangle$$

$$*_C[\langle a, b \rangle, \langle c, d \rangle] := \langle a *_R c - b *_R d, a *_R d + b *_R c \rangle$$

$$\textcolor{red}{FieldC} := \textcolor{blue}{Field}[\mathbb{C}, +_C, *_C] \quad \text{---}$$

$$\textcolor{red}{RSubfieldC} := \textcolor{blue}{Subfield}[\mathbb{R}, \mathbb{C}, +, *] \quad \text{---}$$

$$i := \langle 0, 1 \rangle \in \mathbb{C}$$

$$\textcolor{red}{iProperty} := i^2 = -1 \quad \text{---}$$

$$\textcolor{red}{CProperty} := (a, b \in \mathbb{R}) \implies (\langle a, b \rangle = a + bi) \quad \text{---}$$

Chapter 2

Abstract Algebra

$$\begin{aligned} \text{Relation}(f, X) &:= f \subseteq X \\ \text{Function}(f, X, Y) &:= X \neq \emptyset \neq Y \wedge \text{Relation}(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y} ((x, y) \in f) \end{aligned}$$

$$(\text{Function}(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \dots$$

$$(1) \quad \text{Domain}(f) := X; \text{Codomain}(f) := Y$$

$$(2) \quad \text{Image}(f, A) := \{f(a) : a \in A\}; \text{Preimage}(f, B) := \{a : f(a) \in B\}$$

$$(3) \quad \text{Range}(f) := \text{Image}(f, \text{Domain}(f))$$

$$\text{Injective}(f, X, Y) := \text{Function}(f, X, Y) \wedge \forall_{x_1, x_2 \in X} (x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$$

$$\text{Surjective}(f, X, Y) := \text{Function}(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X} (y = f(x))$$

$$\text{Bijective}(f, X, Y) := \text{Injective}(f, X, Y) \wedge \text{Surjective}(f, X, Y)$$

$$\text{SurjectiveEquivalent} := (\text{Range}(f) = \text{Codomain}(f)) \implies \text{Surjective}(f)$$

$$(\text{Function}(f, X, Y) \wedge \text{Function}(g, Y, Z)) \implies (f \circ g)(x) := f(g(x)); \text{Function}(f \circ g, X, Z)$$

$$\text{Properties of Functions} := (\text{Function}(f, A, B) \wedge \text{Function}(g, B, C) \wedge \text{Function}(h, C, D)) \implies \dots$$

$$(1) \quad h \circ (g \circ f) = (h \circ g) \circ f$$

$$(2) \quad (\text{Injective}(f) \wedge \text{Injective}(g)) \implies \text{Injective}(g \circ f)$$

$$(3) \quad (\text{Surjective}(f) \wedge \text{Surjective}(g)) \implies \text{Surjective}(g \circ f)$$

$$(4) \quad (\text{Bijective}(f, A, B)) \implies \exists_{f^{-1}} \left(\text{Function}(f^{-1}, B, A) \wedge \forall_{a \in A} (f^{-1}(f(a)) = a) \wedge \forall_{b \in B} (f(f^{-1}(b)) = b) \right)$$

$$(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}} (b = ac)$$

$$\text{Divisibility Theorems} := (a, b, c, m, x, y \in \mathbb{Z}) \implies \dots$$

$$(1) \quad (a|b) \implies a|bc$$

$$(2) \quad (a|b \wedge b|c) \implies a|c$$

$$(3) \quad (a|b \wedge b|c) \implies a|(bx + cy)$$

$$(4) \quad (a|b \wedge b|a) \implies a = \pm b$$

$$(5) \quad (a|b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$$

$$(6) \quad (a|b) \iff (m \neq 0 \wedge ma|mb)$$

$$\text{Division Algorithm} := (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}} (b = aq + r)$$

$$\text{CD}(a, b, c) := a, b, c \in \mathbb{Z} \wedge a : b \wedge a : c$$

$$\text{GCD}(a, b, c) := \text{CD}(a, b, c) \wedge \forall_d ((d : b \wedge d : c) \implies d : a)$$

$$\text{GCDEquivalent} := 123123$$

Chapter 3

Linear Algebra

$Matrix(A, m, n) := [a_{i,j}]_{m \times n} := m \text{ rows, } n \text{ columns of real numbers}$

$\mathcal{M}_{m,n} := \{A : Matrix(A, m, n)\}$

$+(A, B) := \left((Matrix(A, m, n)) \wedge (Matrix(B, m, n)) \right) \implies (A + B = [a_{i,j} + b_{i,j}]_{m \times n})$

$*(r, A) := \left((r \in \mathbb{R}) \wedge (Matrix(A, m, n)) \right) \implies (r * A = [ra_{i,j}]_{m \times n})$

$*(A, B) := \left((Matrix(A, m, p)) \wedge (Matrix(B, p, n)) \right) \implies \left(A * B = \left[\sum_{k=1}^p (a_{i,k} b_{k,j}) \right]_{m \times n} \right)$

$T(A) := (Matrix(A, m, n)) \implies (A^T = [a_{j,i}]_{n \times m})$

$AddCom := \forall_{A,B \in \mathcal{M}} (A + B = B + A) \quad \blacksquare \Leftarrow A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C \in \mathcal{M}} ((A + B) + C = A + (B + C)) \quad \blacksquare \Leftarrow (A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A \in \mathcal{M}} \exists!_{O \in \mathcal{M}} (A + O = A = O + A) \quad \text{---}$

$AddInv := \forall_{A \in \mathcal{M}} \exists!_{(-A) \in \mathcal{M}} (A + (-A) = O = (-A) + A) \quad \text{---}$

$MulAssoc := \forall_{A,B,C \in \mathcal{M}} ((A * B) * C = A * (B * C)) \quad \blacksquare \Leftarrow$

$$(1) \quad (A * B) * C = \left[\sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,j}) \right] * C = \left[\sum_{k_2=1}^{p_2} \left(\sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,k_2}) c_{k_2,j} \right) \right] = \left[\sum_{k_2=1}^{p_2} \sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,k_2} c_{k_2,j}) \right] = \dots$$

$$(2) \quad \dots \left[\sum_{k_1=1}^{p_1} \sum_{k_2=1}^{p_2} (a_{i,k_1} b_{k_1,k_2} c_{k_2,j}) \right] = \left[\sum_{k_1=1}^{p_1} \left(a_{i,k_1} \sum_{k_2=1}^{p_2} (b_{k_1,k_2} c_{k_2,j}) \right) \right] = \dots = A * (B * C)$$

$LeftDist := \forall_{A,B,C \in \mathcal{M}} ((A + B) * C = A * C + B * C) \quad \blacksquare \Leftarrow$

$$(1) \quad (A + B) * C = [a_{i,j} + b_{i,j}] * C = \left[\sum_{k=1}^p \left((a_{i,k} + b_{i,k}) c_{k,j} \right) \right] = \dots$$

$$(2) \quad \dots \left[\sum_{k=1}^p (a_{i,k} c_{k,j} + b_{i,k} c_{k,j}) \right] = \left[\sum_{k=1}^p (a_{i,k} c_{k,j}) \right] + \left[\sum_{k=1}^p (b_{i,k} c_{k,j}) \right] = A * C + B * C$$

$RightDist := \forall_{A,B,C \in \mathcal{M}} (C * (A + B) * C = C * A + C * B) \quad \text{---}$

$Scalar1 := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} (r(sA) = (rs)A = s(rA)) \quad \text{---}$

$Scalar2 := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} (A(rB) = r(A * B)) \quad \text{---}$

$Scalar3 := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} ((r + s)A = rA + sA) \quad \text{---}$

$Scalar4 := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} (r(A + B) = rA + rB) \quad \text{---}$

$Trans1 := \forall_{A \in \mathcal{M}} (A = (A^T)^T) \quad \blacksquare \Leftarrow A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$

$Trans2 := \forall_{A,B \in \mathcal{M}} ((A + B)^T = A^T + B^T) \quad \text{---}$

$Trans3 := \forall_{A,B \in \mathcal{M}} ((A * B)^T = B^T * A^T) \quad \blacksquare \Leftarrow$

$$(1) \quad (A * B)^T = \left[\sum_{k=1}^p (a_{i,k} b_{k,j}) \right]^T = \left[\sum_{k=1}^p (a_{j,k} b_{k,i}) \right] = \left[\sum_{k=1}^p (b_{k,i} a_{j,k}) \right] = \left[\sum_{k=1}^p (b_{i,k}^T a_{k,j}^T) \right] = B^T * A^T$$

$Sym(A) := A = A^T$

$SkewSym(A) := A = -A^T$

$SymGen := \forall_{A \in \mathcal{M}} (Sym(A + A^T)) \quad \blacksquare \Leftarrow (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$SkewSymGem := \forall_{A \in \mathcal{M}} (SkewSym(A - A^T)) \quad \blacksquare \Leftarrow -(A - A^T)^T = -(A^T - (A^T)^T) = -(A^T - A) = (A - A^T)$

$$SymDec := \forall_{A \in \mathcal{M}} \exists!_{B: Sym(B)} \exists!_{C: SkewSym(C)} (A = B + C = (1/2) * (A + A^T) + (1/2) * (A - A^T)) \quad \text{—}$$

$$Square(A, n) := Matrix(A, n, n)$$

$$UpperTriangular(A) := (Square(A)) \wedge (i > j \implies a_{i,j} = 0)$$

$$LowerTriangular(A) := (Square(A)) \wedge (i < j \implies a_{i,j} = 0)$$

$$Diagonal(A, n) := (Square(A, n)) \wedge (i \neq j \implies a_{i,j} = 0)$$

$$Scalar(A, n, k) := (Diagonal(A, n)) \wedge (a_{i,i} = k)$$

$$I_n := Scalar(I, n, 1)$$

$$MulId := \forall_{A: Square(A, n)} (A * I_n = A = I_n * A) \quad \text{—}$$

$$Invertible(A) := (Square(A, n)) \wedge \left(\exists_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A) \right)$$

$$Inv1 := \forall_{A: Invertible(A)} \left(\exists!_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A) \right) \quad \text{—}$$

$$Inv2 := \forall_{A: Invertible(A)} \forall_{B: Invertible(B)} \left((A * B)^{-1} = B^{-1} * A^{-1} \right) \blacksquare \Leftarrow$$

$$(1) \quad (A * B) * (A * B)^{-1} = I \blacksquare B * (A * B)^{-1} = A^{-1} \blacksquare (A * B)^{-1} = B^{-1} * A^{-1}$$

$$Inv3 := \forall_{A: Invertible(A)} \left((A^{-1})^{-1} = A \right) \quad \text{—}$$

$$InvTrans := \forall_{A: Invertible(A)} \left((A^T)^{-1} = (A^{-1})^T \right) \blacksquare \Leftarrow$$

$$(1) \quad A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I \blacksquare (A^{-1})^T = (A^T)^{-1}$$

$$Sys(A, X, B) := (A * X = B) \wedge (Matrix(A, m, n)) \wedge (Matrix(X, n, 1)) \wedge (Matrix(B, m, 1))$$

$$Sol(X, A, B) := Sys(A, X, B)$$

$$TrivSol(X, A, B) := (Sol(X, A, B)) \wedge (x_{i,1} = 0)$$

$$RREF(A) := (Definition1.18)$$

$$RowOp(\phi) := (Definition1.19)$$

$$RowEquiv(A, B) := \exists_{\Phi: |\Phi| \in \mathbb{N}} \left(\left(\forall_{\phi \in \Phi} (RowOp(\phi)) \right) \wedge (\Phi(A) = B) \right)$$

$$Elem(A, n) := RowEquiv(A, I_n)$$

$$RowOpIsElem := \forall_{\phi: RowOp(\phi)} \left((E = \phi(I)) \implies (\phi(A) = E * A) \right)$$

$$RowEquiv2 := RowEquiv(A, B) \iff \exists_{E: |E| \in \mathbb{N}} (A = \Pi_{E_i \in E} (E_i) * B)$$

$$(Elem_i(A)) \implies (Invertible(A) \wedge Elem_i(A^{-1}))$$

$$(RowEquiv(A, B)) \iff (B = \Pi(E_i) * A)$$

$$(RowEquiv(I, B)) \iff (B = \Pi(E_i))$$

$$GaussJordan := \forall_A \exists!_B (RREF(B) \wedge RowEquiv(A, B)) \quad \text{—}$$

$$SysAlgebra := \forall_{A, B, C, D, X} \left((RowEquiv([A|B], [C|D])) \implies (Sys(A, X, B) \iff Sys(C, X, D)) \right) \quad \text{—}$$

$$SysAlgebraCorollary := \forall_{A, B, X} \left((RowEquiv(A, B)) \implies (Sys(A, X, O) \iff Sys(B, X, O)) \right) \quad \text{—}$$

$$TrivSolLemma := (m < n) \implies \left(\exists_X (\neg TrivSol(X, A, O)) \right) \blacksquare (\forall_X (TrivSol(X, A, O)) \implies (m \geq n)) \blacksquare \Leftarrow$$

$$(1) \quad \text{Let } B = GaussJordan(A) \text{ and let } r \text{ be the number of non-zero rows of } B \blacksquare r \leq m$$

$$(2) \quad (r \leq m) \wedge (m < n) \blacksquare r < n \blacksquare 0 < n - r$$

$$(3) \quad \text{The solution of } B \text{ will have } r \text{ fixed variables and } n - r \text{ free variables which is also more than zero}$$

$$(4) \quad \text{The solution of } B \text{ will have at least one free variable that can be non-zero, i.e., non-trivial}$$

$$(\forall_X (TrivSol(X, A, O)) \implies (RowEquiv(A, I)) \blacksquare \Leftarrow$$

$$(1) \quad \text{Let } B = GaussJordan(A) \blacksquare RowEquiv(A, B)$$

$$(2) \quad \forall_X (TrivSol(X, B, O)) \blacksquare n \geq n \blacksquare n = n \blacksquare B = I$$

$$(3) \quad RowEquiv(A, B) \blacksquare RowEquiv(A, I)$$

$$(Invertible(A)) \iff (A = \Pi(E_i)) \blacksquare \Leftarrow$$

$$(1) \quad Invertible(A) \implies \dots$$

$$(1.1) \quad AX = O \quad \blacksquare \quad X = A^{-1}O = O \quad \blacksquare \quad \forall_X(TrivSol(X, A, O))$$

$$(1.2) \quad \forall_X(TrivSol(X, A, O) \quad \blacksquare \quad RowEquiv(A, I) \quad \blacksquare \quad A = \Pi(E_i))$$

$$(2) \quad A = \Pi(E_i) \implies \dots$$

$$(2.1) \quad \Pi(E_{-i}^{-1}) = A^{-1} \quad \blacksquare \quad Invertible(A)$$

$$(3) \quad (Invertible(A)) \iff (A = \Pi(E_i))$$

$$(Invertible(A)) \iff (RowEquiv(I, A)) \quad \blacksquare \quad \Leftarrow (RowEquiv(I, B) \iff B = \Pi(E_i)) \wedge \left((Invertible(A)) \iff (A = \Pi(E_i)) \right)$$

$$\left(\forall_X(TrivSol(X, A, O)) \right) \implies (Invertible(A)) \quad \blacksquare \quad \Leftarrow$$

$$(1) \quad \left(\left(\forall_X(TrivSol(X, A, O)) \implies (RowEquiv(A, I)) \right) \wedge \left((Invertible(A)) \iff (RowEquiv(I, A)) \right) \right)$$

$$(Invertible(A)) \implies \left(\forall_X(TrivSol(X, A, O)) \right) \quad \blacksquare \quad \Leftarrow AX = O \quad \blacksquare \quad X = A^{-1}O = O \quad \blacksquare \quad \forall_X(TrivSol(X, A, O))$$