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# Chapter 1

## Real Analysis

(1.5)

$$\text{OrderTrichotomy}(<, S) := \forall_{x,y \in S} (x < y \vee x = y \vee y < x)$$

$$\text{OrderTransitivity}(<, S) := \forall_{x,y,z \in S} ((x < y \wedge y < z) \implies x < z)$$

$$\text{Order}(<, S) := \text{OrderTrichotomy}(<, S) \wedge \text{OrderTransitivity}(<, S)$$

(1.7)

$$\text{BoundedAbove}(E, S, <) := \text{Order}(<, S) \wedge E \subset S \wedge \exists_{\beta \in S} \forall_{x \in E} (x \leq \beta)$$

$$\text{BoundedBelow}(E, S, <) := \text{Order}(<, S) \wedge E \subset S \wedge \exists_{\beta \in S} \forall_{x \in E} (\beta \leq x)$$

$$\text{UpperBound}(\beta, E, S, <) := \text{Order}(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E} (x \leq \beta)$$

$$\text{LowerBound}(\beta, E, S, <) := \text{Order}(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E} (\beta \leq x)$$

(1.8)

$$\text{LUB}(\alpha, E, S, <) := \text{UpperBound}(\alpha, E, S, <) \wedge \forall_{\gamma} (\gamma < \alpha \implies \neg \text{UpperBound}(\gamma, E, S, <))$$

$$\text{GLB}(\alpha, E, S, <) := \text{LowerBound}(\alpha, E, S, <) \wedge \forall_{\beta} (\alpha < \beta \implies \neg \text{LowerBound}(\beta, E, S, <))$$

(1.10)

$$\text{LUBProperty}(S, <) := \forall_E \left( (\emptyset \neq E \subset S \wedge \text{BoundedAbove}(E, S, <)) \implies \exists_{\alpha \in S} (\text{LUB}(\alpha, E, S, <)) \right)$$

$$\text{GLBProperty}(S, <) := \forall_E \left( (\emptyset \neq E \subset S \wedge \text{BoundedBelow}(E, S, <)) \implies \exists_{\alpha \in S} (\text{GLB}(\alpha, E, S, <)) \right)$$

(1.11)

$$\boxed{\text{LUBPropertyImpliesGLBProperty}} \quad \text{LUBProperty}(S, <) \implies \text{GLBProperty}(S, <)$$

$$(1) \quad \text{LUBProperty}(S, <) \implies \dots$$

wts: 2

$$(1.1) \quad (\emptyset \neq B \subset S \wedge \text{BoundedBelow}(B, S, <)) \implies \dots$$

wts: 1.2

$$(1.1.1) \quad \text{Order}(<, S) \wedge \exists_{\delta' \in S} (\text{LowerBound}(\delta', B, S, <))$$

from: [BoundedBelow](#), 1.1

$$(1.1.2) \quad |B| = 1 \implies \dots$$

wts: 1.1.3

$$(1.1.2.1) \quad \exists_{u'} (u' \in B) \blacksquare u := \text{choice}(\{u' \mid u' \in B\}) \blacksquare B = \{u\}$$

from: 1.1.2

$$(1.1.2.2) \quad \text{GLB}(u, B, S, <) \blacksquare \exists_{\epsilon_0 \in S} (\text{GLB}(\epsilon_0, B, S, <))$$

$$(1.1.3) \quad |B| = 1 \implies \exists_{\epsilon_0 \in S} (\text{GLB}(\epsilon_0, B, S, <))$$

$$(1.1.4) \quad |B| \neq 1 \implies \dots$$

wts: 1.1.5

$$(1.1.4.1) \quad \forall_E \left( (\emptyset \neq E \subset S \wedge \text{BoundedAbove}(E, S, <)) \implies \exists_{\alpha \in S} (\text{LUB}(\alpha, E, S, <)) \right)$$

from: [LUBProperty](#), 1

$$(1.1.4.2) \quad L := \{s \in S \mid \text{LowerBound}(s, B, S, <)\}$$

$$(1.1.4.3) \quad |B| > 1 \wedge \text{OrderTrichotomy}(<, S) \blacksquare \exists_{b_1' \in B} \exists_{b_0' \in B} (b_0' < b_1')$$

from: [Order](#), 1.1.1  
wts: 1.1.4.7

$$(1.1.4.4) \quad b_1 := \text{choice}(\{b_1' \in B \mid \exists_{b_0' \in B} (b_0' < b_1')\}) \blacksquare \neg \text{LowerBound}(b_1, B, S, <)$$

from: 1.1.4.2

$$(1.1.4.5) \quad b_1 \notin L \blacksquare L \subset S$$

$$(1.1.4.6) \quad \delta := \text{choice}(\{\delta' \in S \mid \text{LowerBound}(\delta', B, S, <)\}) \blacksquare \delta \in L \blacksquare \emptyset \neq L$$

from: 1.1.1

$$(1.1.4.7) \quad \emptyset \neq L \subset S$$

from: 1.1.4.5, 1.1.4.6

(1.1.4.8)	$\forall_{y \in L} (\text{LowerBound}(y, B, S, <)) \blacksquare \forall_{y \in L} \forall_{x \in B} (y \leq x)$	from: <i>LowerBound</i> , 1.1.4.2 wts: 1.1.4.10
(1.1.4.9)	$\forall_{x \in B} (x \in S \wedge \forall_{y \in L} (y \leq x)) \blacksquare \forall_{x \in B} (\text{UpperBound}(x, L, S, <))$	from: <i>UpperBound</i>
(1.1.4.10)	$\exists_{x \in S} (\text{UpperBound}(x, L, S, <)) \blacksquare \text{BoundedAbove}(L, S, <)$	
(1.1.4.11)	$\emptyset \neq L \subset S \wedge \text{BoundedAbove}(L, S, <)$	from: 1.1.4.7, 1.1.4.10
(1.1.4.12)	$\exists_{\alpha' \in S} (\text{LUB}(\alpha', L, S, <)) \blacksquare \alpha := \text{choice}(\{\alpha' \in S \mid (\text{LUB}(\alpha', L, S, <))\})$	from: 1.1.4.1 wts: 1.1.4.21
(1.1.4.13)	$\forall_x (x \in B \implies \text{UpperBound}(x, L, S, <))$	from: 1.1.4.9 wts: 1.1.4.17
(1.1.4.14)	$\forall_x (\neg \text{UpperBound}(x, L, S, <) \implies x \notin B)$	
(1.1.4.15)	$\gamma < \alpha \implies \dots$	wts: 1.1.4.16
(1.1.4.15.1)	$\neg \text{UpperBound}(\gamma, L, S, <) \blacksquare \gamma \notin B$	from: <i>LUB</i> , 1.1.4.12, 1.1.4.14
(1.1.4.16)	$\gamma < \alpha \implies \gamma \notin B \blacksquare \gamma \in B \implies \gamma \geq \alpha$	
(1.1.4.17)	$\forall_{\gamma \in B} (\alpha \leq \gamma) \blacksquare \text{LowerBound}(\alpha, B, S, <)$	from: <i>LowerBound</i>
(1.1.4.18)	$\alpha < \beta \implies \dots$	wts: 1.1.4.19
(1.1.4.18.1)	$\forall_{y \in L} (y \leq \alpha < \beta) \blacksquare \forall_{y \in L} (y \neq \beta)$	from: <i>LUB</i> , 1.1.4.12, 1.1.4.18
(1.1.4.18.2)	$\beta \notin L \blacksquare \neg \text{LowerBound}(\beta, B, S, <)$	from: 1.1.4.2
(1.1.4.19)	$\alpha < \beta \implies \neg \text{LowerBound}(\beta, B, S, <) \blacksquare \forall_{\beta \in S} (\alpha < \beta \implies \neg \text{LowerBound}(\beta, B, S, <))$	
(1.1.4.20)	$\text{LowerBound}(\alpha, B, S, <) \wedge \forall_{\beta \in S} (\alpha < \beta \implies \neg \text{LowerBound}(\beta, B, S, <))$	from: 1.1.4.17, 1.1.4.19
(1.1.4.21)	$\text{GLB}(\alpha, B, S, <) \blacksquare \exists_{\epsilon_1 \in S} (\text{GLB}(\epsilon_1, B, S, <))$	
(1.1.5)	$ B  \neq 1 \implies \exists_{\epsilon_1 \in S} (\text{GLB}(\epsilon_1, B, S, <))$	
(1.1.6)	$( B  = 1 \implies \exists_{\epsilon_0 \in S} (\text{GLB}(\epsilon_0, B, S, <))) \wedge ( B  \neq 1 \implies \exists_{\epsilon_1 \in S} (\text{GLB}(\epsilon_1, B, S, <)))$	from: 1.1.3, 1.1.5
(1.1.7)	$( B  = 1 \vee  B  \neq 1) \implies \exists_{\epsilon \in S} (\text{GLB}(\epsilon, B, S, <)) \blacksquare \exists_{\epsilon \in S} (\text{GLB}(\epsilon, B, S, <))$	
(1.2)	$(\emptyset \neq B \subset S \wedge \text{BoundedBelow}(B, S, <)) \implies \exists_{\epsilon \in S} (\text{GLB}(\epsilon, B, S, <))$	
(1.3)	$\forall_B ((\emptyset \neq B \subset S \wedge \text{BoundedBelow}(B, S, <)) \implies \exists_{\epsilon \in S} (\text{GLB}(\epsilon, B, S, <)))$	
(1.4)	$\text{GLBProperty}(S, <)$	
(2)	$\text{LUBProperty}(S, <) \implies \text{GLBProperty}(S, <)$	

(1.12)

$$\text{Field}(F, +, *) := \exists_{0, 1 \in F} \forall_{x, y, z \in F} \left( \begin{array}{llll} x + y \in F & \wedge & x * y \in F & \wedge \\ x + y = y + x & \wedge & x * y = y * x & \wedge \\ (x + y) + z = x + (y + z) & \wedge & (x * y) * z = x * (y * z) & \wedge \\ 1 \neq 0 & \wedge & x * (y + z) = (x * y) + (x * z) & \wedge \\ 0 + x = x & \wedge & 1 * x = x & \wedge \\ \exists_{-x \in F} (x + (-x) = 0) \wedge (x \neq 0 \implies \exists_{1/x \in F} (x * (1/x) = 1)) \end{array} \right)$$

$$\text{*****} (\text{Field}(F, +, *) \wedge x, y, z \in F) \implies \dots \text{*****}$$

(1.14)

$$\boxed{\text{AdditiveCancellation}} \quad (x + y = x + z) \implies y = z$$

$$(1) \quad y = 0 + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \dots$$

$$(2) \quad (-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = 0 + z = z$$

$$\boxed{\text{AdditiveIdentityUniqueness}} \quad (x + y = x) \implies y = 0$$

$$(1) \quad x + y = x = 0 + x = x + 0$$

$$(2) \quad y = 0$$

**AdditiveInverseUniqueness**  $(x + y = 0) \implies y = -x$

$$(1) \quad x + y = 0 = x + (-x)$$

from: *Field*

$$(2) \quad y = -x$$

from: *AdditiveCancellation*

**DoubleNegative**  $x = -(-x)$

$$(1) \quad 0 = x + (-x) = (-x) + x \quad \blacksquare \quad 0 = (-x) + x$$

from: *Field*

$$(2) \quad x = -(-x)$$

from: *AdditiveInverseUniqueness*

(1.15)

**MultiplicativeCancellation**  $(x \neq 0 \wedge x * y = x * z) \implies y = z$  —

**MultiplicativeIdentityUniqueness**  $(x \neq 0 \wedge x * y = x) \implies y = 1$  —

**MultiplicativeInverseUniqueness**  $(x \neq 0 \wedge x * y = 1) \implies y = 1/x$  —

**DoubleReciprocal**  $(x \neq 0) \implies x = 1/(1/x)$  —

(1.16)

**Domination**  $0 * x = 0$

$$(1) \quad 0 * x = (0 + 0) * x = 0 * x + 0 * x \quad \blacksquare \quad 0 * x = 0 * x + 0 * x$$

from: *Field*

$$(2) \quad 0 * x = 0$$

from: *AdditiveIdentityUniqueness*

**NonDomination**  $(x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$

$$(1) \quad (x \neq 0 \wedge y \neq 0) \implies \dots$$

$$(1.1) \quad (x * y = 0) \implies \dots$$

$$(1.1.1) \quad 1 = 1 * 1 = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = 0 * ((1/x) * (1/y)) = 0$$

from: *Field, Domination, 1, 1.1*

$$(1.1.2) \quad 1 = 0 \wedge 1 \neq 0 \quad \blacksquare \quad \perp$$

from: *Field*

$$(1.2) \quad (x * y = 0) \implies \perp \quad \blacksquare \quad x * y \neq 0$$

$$(2) \quad (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$$

**NegationCommutativity**  $(-x) * y = -(x * y) = x * (-y)$

$$(1) \quad x * y + (-x) * y = (x + -x) * y = 0 * y = 0 \quad \blacksquare \quad x * y + (-x) * y = 0$$

from: *Field, Domination*  
wts: 2

$$(2) \quad (-x) * y = -(x * y)$$

from: *AdditiveInverseUniqueness*

$$(3) \quad x * y + x * (-y) = x * (y + -y) = x * 0 = 0 \quad \blacksquare \quad x * y + x * (-y) = 0$$

from: *Field, Domination*  
wts: 4

$$(4) \quad x * (-y) = -(x * y)$$

from: *AdditiveInverseUniqueness*

$$(5) \quad (-x) * y = -(x * y) = x * (-y)$$

from: 2, 4

**NegativeMultiplication**  $(-x) * (-y) = x * y$

$$(1) \quad (-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$$

from: *NegationCommutativity, DoubleNegative*

(1.17)

$$\text{OrderedField}(F, +, *, <) := \left( \begin{array}{l} \text{Field}(F, +, *) \quad \wedge \quad \text{Order}(<, F) \quad \wedge \\ \forall_{x,y,z \in F} (y < z \implies x + y < x + z) \quad \wedge \\ \forall_{x,y \in F} ((x > 0 \wedge y > 0) \implies x * y > 0) \end{array} \right)$$

$$\text{*****} \quad (\text{OrderedField}(F, +, *, <) \wedge x, y, z \in F) \implies \dots \text{*****}$$

(1.18)

**NegationOnOrder**  $x > 0 \iff -x < 0$

$$(1) \quad x > 0 \implies \dots$$

(1.1) $0 = (-x) + x > (-x) + 0 = -x \quad \blacksquare \quad 0 > -x \quad \blacksquare \quad -x < 0$	from: <i>OrderedField</i>
(2) $x > 0 \implies -x < 0$	
(3) $-x < 0 \implies \dots$	
(3.1) $0 = x + (-x) < x + 0 = x \quad \blacksquare \quad 0 < x \quad \blacksquare \quad x > 0$	from: <i>OrderedField</i>
(4) $-x < 0 \implies x > 0$	
(5) $x > 0 \implies -x < 0 \wedge -x < 0 \implies x > 0 \quad \blacksquare \quad x > 0 \iff -x < 0$	from: 2, 4

<b>PositiveFactorPreservesOrder</b> $(x > 0 \wedge y < z) \implies x * y < x * z$	
(1) $(x > 0 \wedge y < z) \implies \dots$	
(1.1) $(-y) + z > (-y) + y = 0 \quad \blacksquare \quad z + (-y) = 0$	from: <i>OrderedField</i>
(1.2) $x * (z + (-y)) > 0 \quad \blacksquare \quad x * z + x * (-y) > 0$	from: <i>OrderedField</i>
(1.3) $x * z = 0 + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \dots$	from: <i>Field, NegationCommutativity</i>
(1.4) $x * y + (x * z + x * (-y)) > x * y + 0 = x * y$	from: <i>Field, 1.2</i>
(1.5) $x * z > x * y$	from: 1.3, 1.4
(2) $(x > 0 \wedge y < z) \implies x * z > x * y$	

<b>NegativeFactorFlipsOrder</b> $(x < 0 \wedge y < z) \implies x * y > x * z$	
(1) $(x < 0 \wedge y < z) \implies \dots$	
(1.1) $-x > 0$	from: <i>NegationOnOrder</i>
(1.2) $(-x) * y < (-x) * z \quad \blacksquare \quad 0 = x * y + (-x) * y < x * y + (-x) * z \quad \blacksquare \quad 0 < x * y + (-x) * z$	from: <i>PositiveFactorPreservesOrder</i>
(1.3) $0 < (-x) * (-y + z) \quad \blacksquare \quad 0 > x * (-y + z) \quad \blacksquare \quad 0 > -(x * y) + x * z$	from: <i>NegationOnOrder</i>
(1.4) $x * y > x * z$	
(2) $(x < 0 \wedge y < z) \implies x * y > x * z$	

<b>SquareIsPositive</b> $(x \neq 0) \implies x * x > 0$	
(1) $(x > 0) \implies x * x > 0$	from: <i>OrderedField</i>
(2) $(x < 0) \implies \dots$	
(2.1) $-x > 0 \quad \blacksquare \quad x * x = (-x) * (-x) > 0 \quad \blacksquare \quad x * x > 0$	from: <i>NegationOnOrder, OrderedField, NegativeMultiplication</i>
(3) $(x < 0) \implies x * x > 0$	
(4) $x \neq 0 \implies (x > 0 \vee x < 0) \implies x * x > 0 \quad \blacksquare \quad x \neq 0 \implies x * x > 0$	from: <i>OrderTrichotomy, 1, 3</i>

<b>OneIsPositive</b> $1 > 0$	
(1) $1 \neq 0 \quad \blacksquare \quad 1 = 1 * 1 > 0$	from: <i>Field, SquareIsPositive</i>

<b>ReciprocationOnOrder</b> $(0 < x < y) \implies 0 < 1/y < 1/x$	
(1) $(0 < x < y) \implies \dots$	
(1.1) $x * (1/x) = 1 > 0 \quad \blacksquare \quad x * (1/x) > 0$	from: <i>Field, OneIsPositive</i>
(1.2) $1/x < 0 \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp \quad \blacksquare \quad 1/x > 0$	from: <i>NegativeFactorFlipsOrder, 1</i>
(1.3) $y * (1/y) = 1 > 0 \quad \blacksquare \quad y * (1/y) > 0$	from: <i>Field, OneIsPositive</i>
(1.4) $1/y < 0 \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp \quad \blacksquare \quad 1/y > 0$	from: <i>NegativeFactorFlipsOrder, 1</i>
(1.5) $(1/x) * (1/y) > 0$	from: <i>OrderedField</i>
(1.6) $0 < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$	from: <i>OrderedField, 1, 1.4, 1.5</i>

(1.19)

**OrderedFieldQ**   *OrderedField*( $\mathbb{Q}, +, *, <$ )   —

*Subfield*( $K, F, +, *$ ) := *Field*( $F, +, *$ )  $\wedge K \subset F \wedge$  *Field*( $K, +, *$ )

*OrderedSubfield*( $K, F, +, *, <$ ) := *OrderedField*( $F, +, *, <$ )  $\wedge K \subset F \wedge$  *OrderedField*( $K, +, *, <$ )

*CutI*( $\alpha$ ) :=  $\emptyset \neq \alpha \subset \mathbb{Q}$

*CutII*( $\alpha$ ) :=  $\forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$

*CutIII*( $\alpha$ ) :=  $\forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$

$\mathbb{R} := \{\alpha \in \mathbb{Q} \mid \text{CutI}(\alpha) \wedge \text{CutII}(\alpha) \wedge \text{CutIII}(\alpha)\}$

**CutCorollaryI**    $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1)    $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \dots$

(1.1)    $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$  from: *CutII*, 1

(1.2)    $q < p \implies q \in \alpha \blacksquare q \notin \alpha \implies q \geq p$  from: 1

(1.3)    $(q \notin \alpha) \implies \dots$

(1.3.1)    $q \geq p$  from: 1.2

(1.3.2)    $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp \blacksquare q \neq p$  from: 1, 1.3

(1.3.3)    $q \geq p \wedge q \neq p \blacksquare p < q$

(1.4)    $q \notin \alpha \implies p < q \blacksquare p < q$  from: 1

(2)    $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

**CutCorollaryII**    $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1)    $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \dots$

(1.1)    $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$  from: *CutII*, 1

(1.2)    $s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)) \blacksquare s \in \alpha \implies r \in \alpha$  from: 1, 1.1

(1.3)    $r \notin \alpha \implies s \notin \alpha \blacksquare s \notin \alpha$  from: 1, 1.2

(2)    $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

**OrderTrichotomyOfR**   *OrderTrichotomy*( $\mathbb{R}, <_{\mathbb{R}}$ )

(1)    $(\alpha, \beta \in \mathbb{R}) \implies \dots$

(1.1)    $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \dots$

(1.1.1)    $\alpha \not\subset \beta \wedge \alpha \neq \beta$  from:  $<_{\mathbb{R}}$ , 1.1

(1.1.2)    $\exists_{p'} (p' \in \alpha \wedge p' \notin \beta) \blacksquare p := \text{choice}(\{p' \mid p' \in \alpha \wedge p' \notin \beta\})$

(1.1.3)    $q \in \beta \implies \dots$

(1.1.3.1)    $p, q \in \mathbb{Q}$

(1.1.3.2)    $q < p$  from: *CutCorollaryI*

(1.1.3.3)    $q \in \alpha$  from: *CutII*

(1.1.4)    $q \in \beta \implies q \in \alpha$

(1.1.5)    $\forall_{q \in \beta} (q \in \alpha) \blacksquare \beta \subseteq \alpha$

(1.1.6)    $\beta \subset \alpha \blacksquare \beta <_{\mathbb{R}} \alpha$

(1.2)    $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

(1.3)    $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \blacksquare (\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

(1.4)    $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.5)    $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.6)    $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

(1.7)    $\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta$

(2)    $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

$$(3) \quad \forall_{\alpha, \beta \in \mathbb{R}} (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$$

$$(4) \quad \text{OrderTrichotomy}(\mathbb{R}, <_{\mathbb{R}})$$

$$\boxed{\text{OrderTransitivityOf } R} \quad \text{OrderTransitivity}(\mathbb{R}, <_{\mathbb{R}})$$

$$(1) \quad (\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$$

$$(1.1) \quad (\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \dots$$

$$(1.1.1) \quad \alpha < \beta \wedge \beta < \gamma$$

$$(1.1.2) \quad \forall_{a \in \alpha} (a \in \beta) \wedge \forall_{b \in \beta} (b \in \gamma)$$

$$(1.1.3) \quad \forall_{a \in \alpha} (\alpha \in \gamma) \quad \blacksquare \quad \alpha < \gamma \quad \blacksquare \quad \alpha <_{\mathbb{R}} \gamma$$

$$(1.2) \quad (\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$$

$$(2) \quad (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$$

$$(3) \quad \forall_{\alpha, \beta, \gamma \in \mathbb{R}} ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$$

$$(4) \quad \text{OrderTransitivity}(\mathbb{R}, <_{\mathbb{R}})$$

$$\boxed{\text{OrderOf } R} \quad \text{Order}(<_{\mathbb{R}}, \mathbb{R})$$

from: *OrderTrichotomyR, OrderTransitivityR*  
wts:

$$\boxed{\text{LUBPropertyOf } R} \quad \text{LUBProperty}(\mathbb{R}, <_{\mathbb{R}})$$

$$(1) \quad (\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}(A, \mathbb{R}, <_{\mathbb{R}})) \implies \dots$$

$$(1.1) \quad \gamma := \{p \in \mathbb{Q} \mid \exists_{\alpha \in A} (p \in \alpha)\}$$

$$(1.2) \quad A \neq \emptyset \quad \blacksquare \quad \exists_{\alpha} (\alpha \in A) \quad \blacksquare \quad \alpha_0 := \text{choice}(\{\alpha \mid \alpha \in A\})$$

$$(1.3) \quad \alpha_0 \neq \emptyset \quad \blacksquare \quad \exists_{a} (a \in \alpha_0) \quad \blacksquare \quad a_0 := \text{choice}(\{a \mid a \in \alpha_0\}) \quad \blacksquare \quad a_0 \in \gamma \quad \blacksquare \quad \gamma \neq \emptyset$$

$$(1.4) \quad \text{BoundedAbove}(A, \mathbb{R}, <_{\mathbb{R}}) \quad \blacksquare \quad \exists_{\beta} (\text{UpperBound}(\beta, A, \mathbb{R}, <_{\mathbb{R}}))$$

$$(1.5) \quad \beta_0 := \text{choice}(\{\beta \mid \text{UpperBound}(\beta, A, \mathbb{R}, <_{\mathbb{R}})\})$$

$$(1.6) \quad \text{UpperBound}(\beta_0, A, \mathbb{R}, <_{\mathbb{R}}) \quad \blacksquare \quad \forall_{\alpha \in A} (\alpha \leq_{\mathbb{R}} \beta_0) \quad \blacksquare \quad \forall_{\alpha \in A} (\alpha \subseteq \beta_0) \quad \blacksquare \quad \forall_{\alpha \in A} \forall_{a \in \alpha} (a \in \beta_0)$$

$$(1.7) \quad (\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \quad \blacksquare \quad \forall_{a \in \gamma} (a \in \beta_0) \quad \blacksquare \quad \gamma \subseteq \beta_0$$

$$(1.8) \quad \beta_0 \subset \mathbb{Q} \quad \blacksquare \quad \gamma \subseteq \beta_0 \subset \mathbb{Q} \quad \blacksquare \quad \gamma \subset \mathbb{Q}$$

$$(1.9) \quad \emptyset \neq \gamma \subset \mathbb{Q} \quad \blacksquare \quad \text{CutI}(\gamma)$$

$$(1.10) \quad (p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$$

$$(1.10.1) \quad p \in \gamma \quad \blacksquare \quad \exists_{\alpha \in A} (p \in \alpha) \quad \blacksquare \quad \alpha_1 := \text{choice}(\{\alpha \in A \mid p \in \alpha\})$$

$$(1.10.2) \quad p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \quad \blacksquare \quad q \in \alpha_1 \quad \blacksquare \quad q \in \gamma$$

$$(1.11) \quad (p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \quad \blacksquare \quad \forall_{p \in \gamma} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \gamma) \quad \blacksquare \quad \text{CutII}(\gamma)$$

$$(1.12) \quad p \in \gamma \implies \dots$$

$$(1.12.1) \quad \exists_{\alpha \in A} (p \in \alpha) \quad \blacksquare \quad \alpha_2 := \text{choice}(\{\alpha \in A \mid p \in \alpha\})$$

$$(1.12.2) \quad \alpha_2 \in \mathbb{R} \quad \blacksquare \quad \text{CutII}(\alpha_2) \quad \blacksquare \quad \exists_{r \in \alpha_2} (p < r) \quad \blacksquare \quad r_0 := \text{choice}(\{r \in \alpha_2 \mid p < r\})$$

$$(1.12.3) \quad r_0 \in \alpha_2 \quad \blacksquare \quad r_0 \in \gamma$$

$$(1.12.4) \quad p < r_0 \quad \blacksquare \quad p < r_0 \wedge r_0 \in \gamma \quad \blacksquare \quad \exists_{r \in \gamma} (p < r)$$

$$(1.13) \quad p \in \gamma \implies \exists_{r \in \gamma} (p < r) \quad \blacksquare \quad \forall_{p \in \gamma} \exists_{r \in \gamma} (p < r) \quad \blacksquare \quad \text{CutIII}(\gamma)$$

$$(1.14) \quad \text{CutI}(\gamma) \wedge \text{CutII}(\gamma) \wedge \text{CutIII}(\gamma) \quad \blacksquare \quad \gamma \in \mathbb{R}$$

$$(1.15) \quad \forall_{\alpha \in A} (\alpha \subseteq \gamma) \quad \blacksquare \quad \forall_{\alpha \in A} (\alpha \leq_{\mathbb{R}} \gamma)$$

$$(1.16) \quad \forall_{\alpha \in A} (\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R} \quad \blacksquare \quad \text{UpperBound}(\gamma, A, \mathbb{R}, <_{\mathbb{R}})$$

$$(1.17) \quad \delta <_{\mathbb{R}} \gamma \implies \dots$$

$$(1.17.1) \quad \delta \subset \gamma \quad \blacksquare \quad \exists_{s \in \gamma} (s \in \gamma \wedge s \notin \delta) \quad \blacksquare \quad s_0 := \text{choice}(\{s \in \mathbb{Q} \mid s \in \gamma \wedge s \notin \delta\})$$

$$(1.17.2) \quad s_0 \in \gamma \quad \blacksquare \quad \exists_{\alpha \in A} (s_0 \in \alpha) \quad \blacksquare \quad \alpha_3 := \text{choice}(\{\alpha \in A \mid s_0 \in \alpha\})$$

$$(1.17.3) \quad s_0 \in \alpha_3 \wedge s_0 \notin \delta \quad \blacksquare \quad \exists_{s \in \mathbb{Q}} (s \in \alpha_3 \wedge s \notin \delta)$$

$$(1.17.4) \quad \delta \geq_{\mathbb{R}} \alpha_3 \implies \dots$$

$$(1.17.4.1) \quad \alpha_3 \subseteq \delta \quad \blacksquare \quad \forall_{s \in \mathbb{Q}} (s \in \alpha_3 \implies s \in \delta) \quad \blacksquare \quad \neg \exists_{s \in \mathbb{Q}} (s \in \alpha_3 \wedge s \notin \delta)$$

$$(1.17.4.2) \quad \neg \exists_{s \in \mathbb{Q}} (s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}} (s \in \alpha_3 \wedge s \notin \delta) \quad \blacksquare \quad \perp$$

$$(1.17.5) \quad \delta \geq_{\mathbb{R}} \alpha_3 \implies \perp \quad \blacksquare \quad \delta <_{\mathbb{R}} \alpha_3 \quad \blacksquare \quad \exists_{\alpha \in A} (\delta <_{\mathbb{R}} \alpha) \quad \blacksquare \quad \exists_{\alpha \in A} (\neg(\alpha \leq_{\mathbb{R}} \delta))$$

$$(1.17.6) \quad \neg \forall_{\alpha \in A} (\alpha \leq_{\mathbb{R}} \delta) \quad \blacksquare \quad \neg \text{UpperBound}(\delta, A, \mathbb{R}, <_{\mathbb{R}})$$



$$(1.18) \quad \delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}(\delta, A, \mathbb{R}, <_{\mathbb{R}}) \quad \blacksquare \quad \forall_{\delta} (\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}(\delta, A, \mathbb{R}, <_{\mathbb{R}}))$$

$$(1.19) \quad \text{UpperBound}(\gamma, A, \mathbb{R}, <_{\mathbb{R}}) \wedge \forall_{\delta} (\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}(\delta, A, \mathbb{R}, <_{\mathbb{R}}))$$

$$(1.20) \quad \text{LUB}(\gamma, A, \mathbb{R}, <_{\mathbb{R}}) \quad \blacksquare \quad \exists_{\gamma \in S} (\text{LUB}(\gamma, A, \mathbb{R}, <_{\mathbb{R}}))$$

$$(2) \quad (\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}(A, \mathbb{R}, <_{\mathbb{R}})) \implies \exists_{\gamma \in S} (\text{LUB}(\gamma, A, \mathbb{R}, <_{\mathbb{R}}))$$

$$(3) \quad \forall_A \left( (\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}(A, \mathbb{R}, <_{\mathbb{R}})) \implies \exists_{\gamma \in S} (\text{LUB}(\gamma, A, \mathbb{R}, <_{\mathbb{R}})) \right) \quad \blacksquare \quad \text{LUBProperty}(\mathbb{R}, <_{\mathbb{R}})$$

$$+_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s \mid r \in \alpha \wedge s \in \beta\}$$

$$0_{\mathbb{R}} := \{x \in \mathbb{Q} \mid x < 0\}$$

$$\boxed{\text{0InR}} \quad 0_{\mathbb{R}} \in \mathbb{R}$$

$$(1) \quad -1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}} \quad \blacksquare \quad \emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q} \quad \blacksquare \quad \text{CutI}(0_{\mathbb{R}})$$

$$(2) \quad (x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}} \quad \blacksquare \quad \forall_{x \in 0_{\mathbb{R}}} \forall_{y \in \mathbb{Q}} (y < x \implies y \in 0_{\mathbb{R}}) \quad \blacksquare \quad \text{CutII}(0_{\mathbb{R}})$$

$$(3) \quad y := x/2 \quad \blacksquare \quad (x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}} (x < y) \quad \blacksquare \quad \forall_{x \in 0_{\mathbb{R}}} \exists_{y \in 0_{\mathbb{R}}} (x < y) \quad \blacksquare \quad \text{CutIII}(0_{\mathbb{R}})$$

$$(4) \quad \text{CutI}(0_{\mathbb{R}}) \wedge \text{CutII}(0_{\mathbb{R}}) \wedge \text{CutIII}(0_{\mathbb{R}}) \quad \blacksquare \quad 0_{\mathbb{R}} \in \mathbb{R}$$

$$\boxed{\text{FieldAdditionClosureOfR}} \quad (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$$

$$(1) \quad (\alpha, \beta \in \mathbb{R}) \implies \dots$$

$$(1.1) \quad (\alpha +_{\mathbb{R}} \beta) = \{r + s \mid r \in \alpha \wedge s \in \beta\}$$

$$(1.2) \quad \emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$$

$$(1.3) \quad \exists_a (a \in \alpha) ; \exists_b (b \in \beta) \quad \blacksquare \quad a_0 := \text{choice}(\{a \mid a \in \alpha\}) ; b_0 := \text{choice}(\{b \mid b \in \beta\}) \quad \blacksquare \quad a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$$

$$(1.4) \quad \exists_x (x \notin \alpha) ; \exists_y (y \notin \beta) \quad \blacksquare \quad x_0 := \text{choice}(\{x \mid x \notin \alpha\}) ; y_0 := \text{choice}(\{y \mid y \notin \beta\})$$

$$(1.5) \quad \forall_{r \in \alpha} (r < x_0) ; \forall_{s \in \beta} (s < y_0) \quad \blacksquare \quad \forall_{r \in \alpha} \forall_{s \in \beta} (r + s < x_0 + y_0) \quad \blacksquare \quad x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$$

$$(1.6) \quad \emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q} \quad \blacksquare \quad \text{CutI}(\alpha +_{\mathbb{R}} \beta)$$

$$(1.7) \quad (p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$$

$$(1.7.1) \quad \exists_{r \in \alpha} \exists_{s \in \beta} (p = r + s) \quad \blacksquare \quad (r_0, s_0) := \text{choice}(\{(r, s) \in \alpha \times \beta \mid p = r + s\})$$

$$(1.7.2) \quad q < p = r_0 + s_0 \quad \blacksquare \quad (q - s_0) < r_0 \quad \blacksquare \quad (q - s_0) \in \alpha$$

$$(1.7.3) \quad s_0 \in \beta \quad \blacksquare \quad q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta \quad \blacksquare \quad q \in \alpha +_{\mathbb{R}} \beta$$

$$(1.8) \quad (p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta \quad \blacksquare \quad \forall_{p \in \alpha +_{\mathbb{R}} \beta} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha +_{\mathbb{R}} \beta) \quad \blacksquare \quad \text{CutII}(\alpha +_{\mathbb{R}} \beta)$$

$$(1.9) \quad p \in \alpha \implies \dots$$

$$(1.9.1) \quad \exists_{r \in \alpha} \exists_{s \in \beta} (p = r + s) \quad \blacksquare \quad (r_1, s_1) := \text{choice}(\{(r, s) \in \alpha \times \beta \mid p = r + s\})$$

$$(1.9.2) \quad r_1 \in \alpha \quad \blacksquare \quad \exists_{t \in \alpha} (r_1 < t) \quad \blacksquare \quad t_0 := \text{choice}(\{t \in \alpha \mid r_1 < t\})$$

$$(1.9.3) \quad s_1 \in \beta \quad \blacksquare \quad t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1 \quad \blacksquare \quad \exists_{r \in \alpha +_{\mathbb{R}} \beta} (p < r)$$

$$(1.10) \quad p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta} (p < r) \quad \blacksquare \quad \forall_{p \in \alpha +_{\mathbb{R}} \beta} \exists_{r \in \alpha +_{\mathbb{R}} \beta} (p < r) \quad \blacksquare \quad \text{CutIII}(\alpha +_{\mathbb{R}} \beta)$$

$$(1.11) \quad \text{CutI}(\alpha +_{\mathbb{R}} \beta) \wedge \text{CutII}(\alpha +_{\mathbb{R}} \beta) \wedge \text{CutIII}(\alpha +_{\mathbb{R}} \beta) \quad \blacksquare \quad \alpha +_{\mathbb{R}} \beta \in \mathbb{R}$$

$$(2) \quad (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$$

$$\boxed{\text{FieldAdditionCommutativityOfR}} \quad (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$$

$$(1) \quad \alpha +_{\mathbb{R}} \beta = \{r + s \mid r \in \alpha \wedge s \in \beta\} = \{s + r \mid s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$$

$$\boxed{\text{FieldAdditionAssociativityOfR}} \quad (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma))$$

$$(1) \quad (\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$$

$$(1.1) \quad (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c \mid a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \dots$$

$$(1.2) \quad \{a + (b + c) \mid a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$$

$$(2) \quad (\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$$

$$\boxed{\text{FieldAdditionIdentityOfR}} \quad (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$$

- 
- (1)  $\alpha \in \mathbb{R} \implies \dots$
- (1.1)  $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \dots$
- (1.1.1)  $s < 0 \blacksquare r + s < r + 0 = r \blacksquare r + s < r \blacksquare r + s \in \alpha$
- (1.2)  $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha \blacksquare \forall_{r \in \alpha} \forall_{s \in 0_{\mathbb{R}}} (r + s \in \alpha)$
- (1.3)  $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}} (p \in \alpha) \blacksquare \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$
- (1.4)  $p \in \alpha \implies \dots$
- (1.4.1)  $\exists_{r \in \alpha} (p < r) \blacksquare r_2 := \text{choice}(\{r \in \alpha \mid p < r\})$
- (1.4.2)  $p < r_2 \blacksquare p - r_2 < r_2 - r_2 = 0 \blacksquare (p - r_2) < 0 \blacksquare (p - r_2) \in 0_{\mathbb{R}}$
- (1.4.3)  $r_2 \in \alpha \blacksquare p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$
- (1.5)  $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare \forall_{p \in \alpha} (p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$
- (1.6)  $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$
- 
- (2)  $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$
- 

**FieldAdditionInverseOfR**  $(\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}} (\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

- 
- (1)  $\alpha \in \mathbb{R} \implies \dots$
- (1.1)  $\beta := \{p \in \mathbb{Q} \mid \exists_{r > 0} (-p - r \notin \alpha)\}$
- (1.2)  $\alpha \subset \mathbb{Q} \blacksquare \exists_{s \in \mathbb{Q}} (s \notin \alpha) \blacksquare s_0 := \text{choice}(\{s \mid s \notin \alpha\}) \blacksquare p_0 := -s_0 - 1$
- (1.3)  $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha \blacksquare -p_0 - 1 \notin \alpha \blacksquare \exists_{r > 0} (-p_0 - r \notin \alpha) \blacksquare p_0 \in \beta$
- (1.4)  $\emptyset \neq \alpha \blacksquare \exists_{q \in \alpha} \blacksquare q_0 := \text{choice}(\{q \in \mathbb{Q} \mid q \in \alpha\})$
- (1.5)  $r > 0 \implies \dots$
- (1.5.1)  $q_0 \in \alpha \blacksquare -(-q_0) - r = q_0 - r < q_0 \blacksquare -(-q_0) - r < q_0 \blacksquare -(-q_0) - r \in \alpha$
- (1.6)  $\forall_{r > 0} (-(-q_0) - r \in \alpha) \blacksquare \neg \exists_{r > 0} (-(-q_0) - r \notin \alpha) \blacksquare -q_0 \notin \beta$
- (1.7)  $\emptyset \neq \beta \subset \mathbb{Q} \blacksquare \text{CutI}(\beta)$
- (1.8)  $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
- (1.8.1)  $p \in \beta \blacksquare \exists_{r > 0} (-p - r \notin \alpha) \blacksquare r_0 := \text{choice}(\{r > 0 \mid -p - r \notin \alpha\})$
- (1.8.2)  $q < p \blacksquare -p - r < -q - r$
- (1.8.3)  $-q - r \notin \alpha \blacksquare q \in \beta$
- (1.9)  $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta \blacksquare \forall_{p \in \beta} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \beta) \blacksquare \text{CutII}(\beta)$
- (1.10)  $p \in \beta \implies \dots$
- (1.10.1)  $p \in \beta \blacksquare \exists_{r > 0} (-p - r \notin \alpha) \blacksquare r_1 := \text{choice}(\{r > 0 \mid -p - r \notin \alpha\})$
- (1.10.2)  $t_0 := p + (r_1/2)$
- (1.10.3)  $r_1 > 0 \blacksquare r_1/2 > 0$
- (1.10.4)  $t_0 > t_0 - (r_1/2) = p \blacksquare t_0 > p$
- (1.10.5)  $-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$
- (1.10.6)  $-p - r_1 \notin \alpha \blacksquare -t_0 - (r_1/2) \notin \alpha \blacksquare \exists_{r > 0} (-t_0 - r \notin \alpha) \blacksquare t_0 \in \beta$
- (1.10.7)  $t_0 > p \wedge t_0 \in \beta \blacksquare \exists_{t \in \beta} (p < t)$
- (1.11)  $p \in \beta \implies \exists_{t \in \beta} (p < t) \blacksquare \forall_{p \in \beta} \exists_{t \in \beta} (p < t) \blacksquare \text{CutIII}(\beta)$
- (1.12)  $\text{CutI}(\beta) \wedge \text{CutII}(\beta) \wedge \text{CutIII}(\beta) \blacksquare \beta \in \mathbb{R}$
- (1.13)  $(r \in \alpha \wedge s \in \beta) \implies \dots$
- (1.13.1)  $s \in \beta \blacksquare \exists_{t > 0} (-s - t \notin \alpha) \blacksquare t_1 := \text{choice}(\{t > 0 \mid -s - t \notin \alpha\}) \blacksquare -s - t_1 < -s$
- (1.13.2)  $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha \blacksquare -s \notin \alpha$
- (1.13.3)  $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha \blacksquare r < -s \blacksquare r + s < 0 \blacksquare r + s \in 0_{\mathbb{R}}$
- (1.14)  $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}} \blacksquare \forall_{(r,s) \in \alpha \times \beta} (r + s \in 0_{\mathbb{R}}) \blacksquare \alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$
- (1.15)  $v \in 0_{\mathbb{R}} \implies \dots$
- (1.15.1)  $v < 0 \blacksquare w_0 := -v/2 \blacksquare w > 0$
- (1.15.2)  $\exists_{n \in \mathbb{Z}} (nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha) \blacksquare n_0 := \text{choice}(\{n \in \mathbb{Z} \mid nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$
- (1.15.3)  $p_0 := -(n_0 + 2)w_0 \blacksquare -p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha \blacksquare -p_0 - w_0 \notin \alpha \blacksquare p_0 \in \beta$
- (1.15.4)  $n_0 w_0 \in \alpha \wedge p_0 \in \beta \blacksquare n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2)w_0 = v \in \alpha +_{\mathbb{R}} \beta$
- 

from: ARCHIMEDEANPROPERTYOFQ + LUB???

$$(1.16) \quad v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta \quad \blacksquare \quad \forall v \in 0_{\mathbb{R}} (v \in \alpha +_{\mathbb{R}} \beta) \quad \blacksquare \quad 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$$

$$(1.17) \quad \alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta \quad \blacksquare \quad \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$$

$$(1.18) \quad \beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}} \quad \blacksquare \quad \exists_{-\alpha \in \mathbb{R}} (\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$$

$$(2) \quad \alpha \in \mathbb{R} \implies \exists_{-\alpha \in \mathbb{R}} (\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$$

$$*_{\mathbb{R}}(\alpha, \beta) := \quad \text{---}$$

$$1_{\mathbb{R}} := \{x \in \mathbb{Q} \mid x < 1\}$$

$$\boxed{\text{IsNot0}} \quad 0_{\mathbb{R}} \neq 1_{\mathbb{R}} \quad \text{---}$$

$$\boxed{\text{InR}} \quad 1_{\mathbb{R}} \in \mathbb{R} \quad \text{---}$$

$$\boxed{\text{FieldMultiplicationClosureOfR}} \quad (\alpha, \beta \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R}) \quad \text{---}$$

$$\boxed{\text{FieldMultiplicationCommutativityOfR}} \quad (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha) \quad \text{---}$$

$$\boxed{\text{FieldMultiplicationAssociativityOfR}} \quad (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)) \quad \text{---}$$

$$\boxed{\text{FieldMultiplicationIdentityOfR}} \quad (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha \quad \text{---}$$

$$\boxed{\text{FieldMultiplicationInverseOfR}} \quad (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha \in \mathbb{R}} (\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}) \quad \text{---}$$

$$\boxed{\text{FieldDistributativityOfR}} \quad (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta \quad \text{---}$$

$$\boxed{\text{FieldWithR}} \quad \text{Field}(\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}) \quad \text{---}$$

$$\boxed{\text{OrderedFieldWithR}} \quad \text{OrderedField}(\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}) \quad \text{---}$$

$$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} \mid r < q\} \mid q \in \mathbb{Q}\}$$

$$\boxed{\text{QROrderedSubfieldOfR}} \quad \text{OrderedSubfield}(\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}) \quad \text{---}$$

$$\boxed{\text{QIsomorphicToQR}} \quad \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q} \quad \text{---}$$

$$\boxed{\text{ExistenceOfR}} \quad \exists_{\mathbb{R}} (\text{LUBProperty}(\mathbb{R}, <_{\mathbb{R}}) \wedge \text{OrderedSubfield}(\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}})) \quad \text{---}$$

$$(1.20)$$

$$\boxed{\text{ArchimedeanPropertyOfR}} \quad \forall_{x,y \in \mathbb{R}} (x > 0 \implies \exists_{n \in \mathbb{N}^+} (nx > y))$$

$$(1) \quad (x, y \in \mathbb{R} \wedge x > 0) \implies \dots$$

$$(1.1) \quad A := \{nx \mid n \in \mathbb{N}^+\} \quad \blacksquare \quad \emptyset \neq A \subset \mathbb{R}$$

$$(1.2) \quad \neg \exists_{n \in \mathbb{N}^+} (nx > y) \implies \dots$$

$$(1.2.1) \quad \neg \exists_{n \in \mathbb{N}^+} (nx > y) \quad \blacksquare \quad \forall_{n \in \mathbb{N}^+} (nx \leq y)$$

$$(1.2.2) \quad \text{UpperBound}(y, A, \mathbb{R}, <) \quad \blacksquare \quad \text{BoundedAbove}(A, \mathbb{R}, <)$$

$$(1.2.3) \quad \text{LUBProperty}(\mathbb{R}, <) \wedge \emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}(A, \mathbb{R}, <) \quad \blacksquare \quad \exists_{\alpha \in \mathbb{R}} (\text{LUB}(\alpha, A, \mathbb{R}, <))$$

$$(1.2.4) \quad \alpha_0 := \text{choice}(\{\alpha \in \mathbb{R} \mid \text{LUB}(\alpha, A, \mathbb{R}, <)\})$$

$$(1.2.5) \quad x > 0 \quad \blacksquare \quad \alpha_0 - x < \alpha_0$$

$$(1.2.6) \quad \text{LUB}(\alpha_0, A, \mathbb{R}, <) \wedge \alpha_0 - x < \alpha_0 \quad \blacksquare \quad \neg \text{UpperBound}(\alpha_0 - x, A, \mathbb{R}, <)$$

$$(1.2.7) \quad \exists_{c \in A} (\alpha_0 - x < c) \quad \blacksquare \quad c_0 := \text{choice}(\{c \in A \mid \alpha_0 - x < c\})$$

$$(1.2.8) \quad c_0 \in A \quad \blacksquare \quad \exists_{m \in \mathbb{N}^+} (mx = c_0) \quad \blacksquare \quad m_0 := \text{choice}(\{m \in \mathbb{N}^+ \mid mx = c_0\})$$

$$(1.2.9) \quad \alpha_0 - x < c_0 = m_0 x \quad \blacksquare \quad \alpha_0 - x < m_0 x \quad \blacksquare \quad \alpha_0 < (m_0 + 1)x$$

$$(1.2.10) \quad m_0 + 1 \in \mathbb{N}^+ \quad \blacksquare \quad (m_0 + 1)x \in A$$

$$(1.2.11) \quad \alpha_0 < (m_0 + 1)x \wedge (m_0 + 1)x \in A \quad \blacksquare \quad \exists_{c \in A} (\alpha_0 < c)$$

$$(1.2.12) \quad \text{LUB}(\alpha_0, A, \mathbb{R}, <) \quad \blacksquare \quad \text{UpperBound}(\alpha_0, A, \mathbb{R}, <) \quad \blacksquare \quad \forall_{c \in A} (c \leq \alpha_0) \quad \blacksquare \quad \neg \exists_{c \in A} (\alpha_0 < c)$$

$$(1.2.13) \quad \exists_{c \in A} (\alpha_0 < c) \wedge \neg \exists_{c \in A} (\alpha_0 < c) \quad \blacksquare \quad \perp$$

$$(1.3) \quad \neg \exists_{n \in \mathbb{N}^+} (nx > y) \implies \perp \quad \blacksquare \quad \exists_{n \in \mathbb{N}^+} (nx > y)$$

$$(2) \quad (x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+} (nx > y) \quad \blacksquare \quad \forall_{x,y \in \mathbb{R}} (x > 0 \implies \exists_{n \in \mathbb{N}^+} (nx > y))$$

$$\boxed{\text{QDenseInR}} \quad \forall_{x,y \in \mathbb{R}} (x < y \implies \exists_{p \in \mathbb{Q}} (x < p < y))$$

$$(1) \quad (x, y \in \mathbb{R} \wedge x < y) \implies \dots$$

$$(1.1) \quad x < y \quad \blacksquare \quad y - x > 0 \quad \blacksquare \quad \exists_{n \in \mathbb{N}^+} (n(y - x) > 1) \quad \blacksquare \quad n_0 := \text{choice}(\{n \in \mathbb{N}^+ \mid n(y - x) > 1\}) \quad \blacksquare \quad n_0(y - x) > 1$$

(1.2)	$1 > 0 \quad \blacksquare \quad \exists_{m \in \mathbb{N}^+} (m(1) > n_0 x) \quad \blacksquare \quad m_1 := \text{choice}(\{m \in \mathbb{N}^+   m(1) > n_0 x\}) \quad \blacksquare \quad m_1 > n_0 x$
(1.3)	$\exists_{m \in \mathbb{N}^+} (m(1) > -n_0 x) \quad \blacksquare \quad m_2 := \text{choice}(\{m \in \mathbb{N}^+   m(1) > -n_0 x\}) \quad \blacksquare \quad m_2 > -n_0 x \quad \blacksquare \quad -m_2 < n_0 x$
(1.4)	$-m_2 < n_0 x < m_1$
(1.5)	$m_1, m_2 \in \mathbb{N}^+ \quad \blacksquare \quad  m_1 - (-m_2)  \geq 2$
(1.6)	$-m_2 < n_0 x < m_1 \wedge  m_1 - (-m_2)  \geq 2 \quad \blacksquare \quad \exists_{m \in \mathbb{Z}} (-m_2 < m < m_1 \wedge m - 1 \leq n_0 x < m)$
(1.7)	$m_0 := \text{choice}(\{m \in \mathbb{Z}   -m_2 < m < m_1 \wedge m - 1 \leq n_0 x < m\}) \quad \blacksquare \quad -m_2 < m_0 < m_1 \wedge m_0 - 1 \leq n_0 x < m_0$
(1.8)	$(n_0(y - x) > 1) \wedge (m_0 - 1 \leq n_0 x < m_0) \quad \blacksquare \quad n_0 x < m_0 \leq 1 + n_0 x < n_0 y \quad \blacksquare \quad n_0 x < m_0 < n_0 y$
(1.9)	$n_0 \in \mathbb{N}^+ \quad \blacksquare \quad n_0 > 0 \quad \blacksquare \quad x < m_0/n_0 < y$
(1.10)	$m_0, n_0 \in \mathbb{Z} \quad \blacksquare \quad m_0/n_0 \in \mathbb{Q}$
(1.11)	$m_0/n_0 \in \mathbb{Q} \wedge x < m_0/n_0 < y \quad \blacksquare \quad \exists_{p \in \mathbb{Q}} (x < p < y)$
(2)	$(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}} (x < p < y) \quad \blacksquare \quad \forall_{x, y \in \mathbb{R}} \exists_{p \in \mathbb{Q}} (x < p < y)$

(1.21)

Root 123123

(1) 123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

## Chapter 2

# Abstract Algebra

$Relation(f, X) := f \subseteq X$

$Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y} ((x, y) \in f)$

$(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \dots$

---

(1)  $Domain(f) := X; Codomain(f) := Y$

---

(2)  $Image(f, A) := \{f(a) | a \in A\}; Preimage(f, B) := \{a | f(a) \in B\}$

---

(3)  $Range(f) := Image(Domain(f))$

---

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X} (x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$

$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X} (y = f(x))$

$Bijective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$

TODO: Definition properties Surjective Equivalent:  $(Range(f) = Codomain(f)) \implies Surjective(f)$



## Chapter 3

# Linear Algebra

$A_{square} = B_{sym} + C_{skewsym}$   $A + A^T = B_{sym}$   $A - A^T = B_{skewsym}$   $A = (1/2)(A + A^T) + (1/2)(A - A^T)$   $(AB)^{-1} = B^{-1}A^{-1}$   
 $AB * \dots \dots \dots$   
 $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$   
 $A^123123$   
TODO PROOFS