

# Contents



# Chapter 1

## Real Analysis

(1.5)

$$\textit{OrderTrichotomy}[\prec, S] := \forall_{x,y \in S} (x < y \vee x = y \vee y < x)$$

$$\textit{OrderTransitivity}[\prec, S] := \forall_{x,y,z \in S} ((x < y \wedge y < z) \implies x < z)$$

$$\textit{Order}[\prec, S] := (\textit{OrderTrichotomy}[\prec, S]) \wedge (\textit{OrderTransitivity}[\prec, S])$$

(1.7)

$$\textit{BoundedAbove}[E, S, \prec] := (\textit{Order}[\prec, S]) \wedge (E \subset S) \wedge \left( \exists_{\beta \in S} \forall_{x \in E} (x \leq \beta) \right)$$

$$\textit{BoundedBelow}[E, S, \prec] := (\textit{Order}[\prec, S]) \wedge (E \subset S) \wedge \left( \exists_{\beta \in S} \forall_{x \in E} (\beta \leq x) \right)$$

$$\textit{UpperBound}[\beta, E, S, \prec] := (\textit{Order}[\prec, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E} (x \leq \beta))$$

$$\textit{LowerBound}[\beta, E, S, \prec] := (\textit{Order}[\prec, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E} (\beta \leq x))$$

(1.8)

$$\textit{LUB}[\alpha, E, S, \prec] := (\textit{UpperBound}[\alpha, E, S, \prec]) \wedge \left( \forall_{\gamma} (\gamma < \alpha \implies \neg \textit{UpperBound}[\gamma, E, S, \prec]) \right)$$

$$\textit{GLB}[\alpha, E, S, \prec] := (\textit{LowerBound}[\alpha, E, S, \prec]) \wedge \left( \forall_{\beta} (\alpha < \beta \implies \neg \textit{LowerBound}[\beta, E, S, \prec]) \right)$$

(1.10)

$$\textit{LUBProperty}[S, \prec] := \forall_E \left( ((\emptyset \neq E \subset S) \wedge (\textit{BoundedAbove}[E, S, \prec]) \implies \exists_{\alpha \in S} (\textit{LUB}[\alpha, E, S, \prec])) \right)$$

$$\textit{GLBProperty}[S, \prec] := \forall_E \left( ((\emptyset \neq E \subset S) \wedge (\textit{BoundedBelow}[E, S, \prec]) \implies \exists_{\alpha \in S} (\textit{GLB}[\alpha, E, S, \prec])) \right)$$

(1.11)

$$\textit{LUBPropertyImpliesGLBProperty} := \textit{LUBProperty}[S, \prec] \implies \textit{GLBProperty}[S, \prec]$$

(1)  $\textit{LUBProperty}[S, \prec] \implies \dots$

wts: 2

(1.1)  $(\emptyset \neq B \subset S \wedge \textit{BoundedBelow}[B, S, \prec]) \implies \dots$

wts: 1.2

(1.1.1)  $\textit{Order}[\prec, S] \wedge \exists_{\delta' \in S} (\textit{LowerBound}[\delta', B, S, \prec])$

from: [BoundedBelow](#), 1.1

(1.1.2)  $|B| = 1 \implies \dots$

wts: 1.1.3

(1.1.2.1)  $\exists_{u'} (u' \in B) \blacksquare u := \textit{choice}(\{u' : u' \in B\}) \blacksquare B = \{u\}$

from: 1.1.2

(1.1.2.2)  $\textit{GLB}[u, B, S, \prec] \blacksquare \exists_{\epsilon_0 \in S} (\textit{GLB}[\epsilon_0, B, S, \prec])$

(1.1.3)  $|B| = 1 \implies \exists_{\epsilon_0 \in S} (\textit{GLB}[\epsilon_0, B, S, \prec])$

(1.1.4)  $|B| \neq 1 \implies \dots$

wts: 1.1.5

(1.1.4.1)  $\forall_E ((\emptyset \neq E \subset S \wedge \textit{BoundedAbove}[E, S, \prec]) \implies \exists_{\alpha \in S} (\textit{LUB}[\alpha, E, S, \prec]))$

from: [LUBProperty](#), 1

(1.1.4.2)  $L := \{s \in S : \textit{LowerBound}[s, B, S, \prec]\}$

(1.1.4.3)  $|B| > 1 \wedge \textit{OrderTrichotomy}[\prec, S] \blacksquare \exists_{b_1' \in B} \exists_{b_0' \in B} (b_0' < b_1')$

from: [Order](#), 1.1.1  
wts: 1.1.4.7

(1.1.4.4)  $b_1 := \textit{choice}(\{b_1' \in B : \exists_{b_0' \in B} (b_0' < b_1')\}) \blacksquare \neg \textit{LowerBound}[b_1, B, S, \prec]$

from: 1.1.4.2

(1.1.4.5)  $b_1 \notin L \blacksquare L \subset S$

(1.1.4.6)  $\delta := \textit{choice}(\{\delta' \in S : \textit{LowerBound}[\delta', B, S, \prec]\}) \blacksquare \delta \in L \blacksquare \emptyset \neq L$

from: 1.1.1

(1.1.4.7)  $\emptyset \neq L \subset S$

from: 1.1.4.5, 1.1.4.6

(1.1.4.8)  $\forall_{y \in L} (\textit{LowerBound}[y_0, B, S, \prec]) \blacksquare \forall_{y \in L} \forall_{x \in B} (y_0 \leq x)$

from: [LowerBound](#), 1.1.4.2  
wts: 1.1.4.10

$$(1.1.4.9) \quad \forall_{x \in B} \left( x \in S \wedge \forall_{y \in L} (y_0 \leq x) \right) \quad \blacksquare \quad \forall_{x \in B} (\text{UpperBound}[x, L, S, <])$$

from: *UpperBound*

$$(1.1.4.10) \quad \exists_{x \in S} (\text{UpperBound}[x, L, S, <]) \quad \blacksquare \quad \text{BoundedAbove}[L, S, <]$$

$$(1.1.4.11) \quad \emptyset \neq L \subset S \wedge \text{BoundedAbove}[L, S, <]$$

from: 1.1.4.7, 1.1.4.10

$$(1.1.4.12) \quad \exists_{\alpha' \in S} (\text{LUB}[\alpha', L, S, <]) \quad \blacksquare \quad \alpha := \text{choice}(\{\alpha' \in S : (\text{LUB}[\alpha', L, S, <])\})$$

from: 1.1.4.1  
wts: 1.1.4.21

$$(1.1.4.13) \quad \forall_x (x \in B \implies \text{UpperBound}[x, L, S, <])$$

from: 1.1.4.9  
wts: 1.1.4.17

$$(1.1.4.14) \quad \forall_x (\neg \text{UpperBound}[x, L, S, <] \implies x \notin B)$$

$$(1.1.4.15) \quad \gamma < \alpha \implies \dots$$

wts: 1.1.4.16

$$(1.1.4.15.1) \quad \neg \text{UpperBound}[\gamma, L, S, <] \quad \blacksquare \quad \gamma \notin B$$

from: *LUB*, 1.1.4.12, 1.1.4.14

$$(1.1.4.16) \quad \gamma < \alpha \implies \gamma \notin B \quad \blacksquare \quad \gamma \in B \implies \gamma \geq \alpha$$

$$(1.1.4.17) \quad \forall_{\gamma \in B} (\alpha \leq \gamma) \quad \blacksquare \quad \text{LowerBound}[\alpha, B, S, <]$$

from: *LowerBound*

$$(1.1.4.18) \quad \alpha < \beta \implies \dots$$

wts: 1.1.4.19

$$(1.1.4.18.1) \quad \forall_{y \in L} (y_0 \leq \alpha < \beta) \quad \blacksquare \quad \forall_{y \in L} (y_0 \neq \beta)$$

from: *LUB*, 1.1.4.12, 1.1.4.18

$$(1.1.4.18.2) \quad \beta \notin L \quad \blacksquare \quad \neg \text{LowerBound}[\beta, B, S, <]$$

from: 1.1.4.2

$$(1.1.4.19) \quad \alpha < \beta \implies \neg \text{LowerBound}[\beta, B, S, <] \quad \blacksquare \quad \forall_{\beta \in S} (\alpha < \beta \implies \neg \text{LowerBound}[\beta, B, S, <])$$

$$(1.1.4.20) \quad \text{LowerBound}[\alpha, B, S, <] \wedge \forall_{\beta \in S} (\alpha < \beta \implies \neg \text{LowerBound}[\beta, B, S, <])$$

from: 1.1.4.17, 1.1.4.19

$$(1.1.4.21) \quad \text{GLB}[\alpha, B, S, <] \quad \blacksquare \quad \exists_{\epsilon_1 \in S} (\text{GLB}[\epsilon_1, B, S, <])$$

$$(1.1.5) \quad |B| \neq 1 \implies \exists_{\epsilon_1 \in S} (\text{GLB}[\epsilon_1, B, S, <])$$

$$(1.1.6) \quad \left( |B| = 1 \implies \exists_{\epsilon_0 \in S} (\text{GLB}[\epsilon_0, B, S, <]) \right) \wedge \left( |B| \neq 1 \implies \exists_{\epsilon_1 \in S} (\text{GLB}[\epsilon_1, B, S, <]) \right)$$

from: 1.1.3, 1.1.5

$$(1.1.7) \quad (|B| = 1 \vee |B| \neq 1) \implies \exists_{\epsilon \in S} (\text{GLB}[\epsilon, B, S, <]) \quad \blacksquare \quad \exists_{\epsilon \in S} (\text{GLB}[\epsilon, B, S, <])$$

$$(1.2) \quad (\emptyset \neq B \subset S \wedge \text{BoundedBelow}[B, S, <]) \implies \exists_{\epsilon \in S} (\text{GLB}[\epsilon, B, S, <])$$

$$(1.3) \quad \forall_B ((\emptyset \neq B \subset S \wedge \text{BoundedBelow}[B, S, <]) \implies \exists_{\epsilon \in S} (\text{GLB}[\epsilon, B, S, <]))$$

$$(1.4) \quad \text{GLBProperty}[S, <]$$

$$(2) \quad \text{LUBProperty}[S, <] \implies \text{GLBProperty}[S, <]$$

$$(1.12)$$

$$\text{Field}[F, +, *] := \exists_{0, 1 \in F} \forall_{x, y, z \in F} \left( \begin{array}{l} x + y \in F \quad \wedge \quad x * y \in F \quad \wedge \\ x + y = y + x \quad \wedge \quad x * y = y * x \quad \wedge \\ (x + y) + z = x + (y + z) \quad \wedge \quad (x * y) * z = x * (y * z) \quad \wedge \\ 1 \neq 0 \quad \wedge \quad x * (y + z) = (x * y) + (x * z) \quad \wedge \\ 0 + x = x \quad \wedge \quad 1 * x = x \quad \wedge \\ \exists_{-x \in F} (x + (-x) = 0) \wedge \left( x \neq 0 \implies \exists_{1/x \in F} (x * (1/x) = 1) \right) \end{array} \right)$$

$$\text{***** (Field}[F, +, *] \wedge x, y, z \in F) \implies \dots \text{*****}$$

$$(1.14)$$

$$\text{AdditiveCancellation} := (x + y = x + z) \implies y = z$$

$$(1) \quad y = 0 + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \dots$$

from: *Field*

$$(2) \quad (-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = 0 + z = z$$

from: *Field*

$$\text{AdditiveIdentityUniqueness} := (x + y = x) \implies y = 0$$

$$(1) \quad x + y = x = 0 + x = x + 0$$

from: *Field*

$$(2) \quad y = 0$$

from: *AdditiveCancellation*

$$\text{AdditiveInverseUniqueness} := (x + y = 0) \implies y = -x$$

$$(1) \quad x + y = 0 = x + (-x)$$

from: *Field*

$$(2) \quad y = -x$$

from: *AdditiveCancellation*

$$\text{DoubleNegative} := x = -(-x)$$

$$(1) \quad 0 = x + (-x) = (-x) + x \quad \blacksquare \quad 0 = (-x) + x$$

from: *Field*

$$(2) \quad x = -(-x)$$

from: [AdditiveInverseUniqueness](#)

(1.15)

$$\textcolor{red}{\textit{MultiplicativeCancellation}} := (x \neq 0 \wedge x * y = x * z) \implies y = z \quad \text{---}$$

$$\textcolor{red}{\textit{MultiplicativeIdentityUniqueness}} := (x \neq 0 \wedge x * y = x) \implies y = 1 \quad \text{---}$$

$$\textcolor{red}{\textit{MultiplicativeInverseUniqueness}} := (x \neq 0 \wedge x * y = 1) \implies y = 1/x \quad \text{---}$$

$$\textcolor{red}{\textit{DoubleReciprocal}} := (x \neq 0) \implies x = 1/(1/x) \quad \text{---}$$

(1.16)

$$\textcolor{red}{\textit{Domination}} := 0 * x = 0$$

from: [Field](#)

$$(1) \quad 0 * x = (0 + 0) * x = 0 * x + 0 * x \quad \blacksquare \quad 0 * x = 0 * x + 0 * x$$

$$(2) \quad 0 * x = 0$$

from: [AdditiveIdentityUniqueness](#)

$$\textcolor{red}{\textit{NonDomination}} := (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$$

$$(1) \quad (x \neq 0 \wedge y \neq 0) \implies \dots$$

$$(1.1) \quad (x * y = 0) \implies \dots$$

$$(1.1.1) \quad 1 = 1 * 1 = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = 0 * ((1/x) * (1/y)) = 0$$

from: [Field](#), [Domination](#), 1, 1.1

$$(1.1.2) \quad 1 = 0 \wedge 1 \neq 0 \quad \blacksquare \quad \perp$$

from: [Field](#)

$$(1.2) \quad (x * y = 0) \implies \perp \quad \blacksquare \quad x * y \neq 0$$

$$(2) \quad (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$$

$$\textcolor{red}{\textit{NegationCommutativity}} := (-x) * y = -(x * y) = x * (-y)$$

$$(1) \quad x * y + (-x) * y = (x + -x) * y = 0 * y = 0 \quad \blacksquare \quad x * y + (-x) * y = 0$$

from: [Field](#), [Domination](#)  
wts: 2

$$(2) \quad (-x) * y = -(x * y)$$

from: [AdditiveInverseUniqueness](#)

$$(3) \quad x * y + x * (-y) = x * (y_0 + -y) = x * 0 = 0 \quad \blacksquare \quad x * y + x * (-y) = 0$$

from: [Field](#), [Domination](#)  
wts: 4

$$(4) \quad x * (-y) = -(x * y)$$

from: [AdditiveInverseUniqueness](#)

$$(5) \quad (-x) * y = -(x * y) = x * (-y)$$

from: 2, 4

$$\textcolor{red}{\textit{NegativeMultiplication}} := (-x) * (-y) = x * y$$

$$(1) \quad (-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$$

from: [NegationCommutativity](#), [DoubleNegative](#)

(1.17)

$$\textcolor{red}{\textit{OrderedField}}[F, +, *, <] := \left( \begin{array}{l} \textcolor{blue}{\textit{Field}}[F, +, *] \quad \wedge \quad \textcolor{blue}{\textit{Order}}[<, F] \quad \wedge \\ \forall_{x,y,z \in F} (y_0 < z \implies x + y < x + z) \quad \wedge \\ \forall_{x,y \in F} ((x > 0 \wedge y > 0) \implies x * y > 0) \end{array} \right)$$

$$\textcolor{blue}{\textit{OrderedField}}[F, +, *, <] \wedge x, y, z \in F \implies \dots$$

(1.18)

$$\textcolor{red}{\textit{NegationOnOrder}} := x > 0 \iff -x < 0$$

$$(1) \quad x > 0 \implies \dots$$

$$(1.1) \quad 0 = (-x) + x > (-x) + 0 = -x \quad \blacksquare \quad 0 > -x \quad \blacksquare \quad -x < 0$$

from: [OrderedField](#)

$$(2) \quad x > 0 \implies -x < 0$$

$$(3) \quad -x < 0 \implies \dots$$

$$(3.1) \quad 0 = x + (-x) < x + 0 = x \quad \blacksquare \quad 0 < x \quad \blacksquare \quad x > 0$$

from: [OrderedField](#)

$$(4) \quad -x < 0 \implies x > 0$$

$$(5) \quad x > 0 \implies -x < 0 \wedge -x < 0 \implies x > 0 \quad \blacksquare \quad x > 0 \iff -x < 0$$

from: 2, 4

$$\textcolor{red}{\textit{PositiveFactorPreservesOrder}} := (x > 0 \wedge y < z) \implies x * y < x * z$$

$$(1) \quad (x > 0 \wedge y < z) \implies \dots$$

$$(1.1) \quad (-y) + z > (-y) + y = 0 \quad \blacksquare \quad z + (-y) = 0$$

from: [OrderedField](#)

$$(1.2) \quad x * (z + (-y)) > 0 \quad \blacksquare \quad x * z + x * (-y) > 0$$

from: [OrderedField](#)

$$(1.3) \quad x * z = 0 + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \dots$$

from: *Field, NegationCommutativity*

$$(1.4) \quad x * y + (x * z + x * (-y)) > x * y + 0 = x * y$$

from: *Field, 1.2*

$$(1.5) \quad x * z > x * y$$

from: 1.3, 1.4

$$(2) \quad (x > 0 \wedge y < z) \implies x * z > x * y$$

$$\textcolor{red}{NegativeFactorFlipsOrder} := (x < 0 \wedge y < z) \implies x * y > x * z$$

$$(1) \quad (x < 0 \wedge y < z) \implies \dots$$

$$(1.1) \quad -x > 0$$

from: *NegationOnOrder*

$$(1.2) \quad (-x) * y < (-x) * z \quad \blacksquare \quad 0 = x * y + (-x) * y < x * y + (-x) * z \quad \blacksquare \quad 0 < x * y + (-x) * z$$

from: *PositiveFactorPreservesOrder*

$$(1.3) \quad 0 < (-x) * (-y + z) \quad \blacksquare \quad 0 > x * (-y + z) \quad \blacksquare \quad 0 > -(x * y) + x * z$$

from: *NegationOnOrder*

$$(1.4) \quad x * y > x * z$$

$$(2) \quad (x < 0 \wedge y < z) \implies x * y > x * z$$

$$\textcolor{red}{SquareIsPositive} := (x \neq 0) \implies x * x > 0$$

$$(1) \quad (x > 0) \implies x * x > 0$$

from: *OrderedField*

$$(2) \quad (x < 0) \implies \dots$$

$$(2.1) \quad -x > 0 \quad \blacksquare \quad x * x = (-x) * (-x) > 0 \quad \blacksquare \quad x * x > 0$$

from: *NegationOnOrder, OrderedField, NegativeMultiplication*

$$(3) \quad (x < 0) \implies x * x > 0$$

$$(4) \quad x \neq 0 \implies (x > 0 \vee x < 0) \implies x * x > 0 \quad \blacksquare \quad x \neq 0 \implies x * x > 0$$

from: *OrderTrichotomy, 1.3*

$$\textcolor{red}{OneIsPositive} := 1 > 0$$

$$(1) \quad 1 \neq 0 \quad \blacksquare \quad 1 = 1 * 1 > 0$$

from: *Field, SquareIsPositive*

$$\textcolor{red}{ReciprocationOnOrder} := (0 < x < y) \implies 0 < 1/y < 1/x$$

$$(1) \quad (0 < x < y) \implies \dots$$

$$(1.1) \quad x * (1/x) = 1 > 0 \quad \blacksquare \quad x * (1/x) > 0$$

from: *Field, OneIsPositive*

$$(1.2) \quad 1/x < 0 \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp \quad \blacksquare \quad 1/x > 0$$

from: *NegativeFactorFlipsOrder, 1*

$$(1.3) \quad y * (1/y) = 1 > 0 \quad \blacksquare \quad y * (1/y) > 0$$

from: *Field, OneIsPositive*

$$(1.4) \quad 1/y < 0 \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp \quad \blacksquare \quad 1/y > 0$$

from: *NegativeFactorFlipsOrder, 1*

$$(1.5) \quad (1/x) * (1/y) > 0$$

from: *OrderedField*

$$(1.6) \quad 0 < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$$

from: *OrderedField, 1, 1.4, 1.5*

(1.19)

$$\textcolor{red}{OrderedFieldQ} := \textcolor{teal}{OrderedField}[\mathbb{Q}, +, *, <] \quad \text{---}$$

$$\textcolor{teal}{Subfield}[K, F, +, *] := \textcolor{teal}{Field}[F, +, *] \wedge K \subset F \wedge \textcolor{teal}{Field}[K, +, *]$$

$$\textcolor{teal}{OrderedSubfield}[K, F, +, *, <] := \textcolor{teal}{OrderedField}[F, +, *, <] \wedge K \subset F \wedge \textcolor{teal}{OrderedField}[K, +, *, <]$$

$$\textcolor{teal}{CutI}[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$$

$$\textcolor{teal}{CutII}[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$$

$$\textcolor{teal}{CutIII}[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$$

$$\mathbb{R} := \{\alpha \in \mathbb{Q} : \textcolor{teal}{CutI}[\alpha] \wedge \textcolor{teal}{CutII}[\alpha] \wedge \textcolor{teal}{CutIII}[\alpha]\}$$

$$\textcolor{red}{CutCorollaryI} := (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$$

$$(1) \quad (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \dots$$

$$(1.1) \quad \forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$$

from: *CutII, 1*

$$(1.2) \quad q < p \implies q \in \alpha \quad \blacksquare \quad q \notin \alpha \implies q \geq p$$

from: 1

$$(1.3) \quad (q \notin \alpha) \implies \dots$$

$$(1.3.1) \quad q \geq p$$

from: 1.2

$$(1.3.2) \quad (q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp \quad \blacksquare \quad q \neq p$$

from: 1, 1.3

(1.3.3)	$q \geq p \wedge q \neq p \blacksquare p < q$	
(1.4)	$q \notin \alpha \implies p < q \blacksquare p < q$	from: 1
(2)	$(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$	

**CutCorollaryI**  $:= (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1)	$(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \dots$	
(1.1)	$\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$	from: <a href="#">CutII, 1</a>
(1.2)	$s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)) \blacksquare s \in \alpha \implies r \in \alpha$	from: 1, 1.1
(1.3)	$r \notin \alpha \implies s \notin \alpha \blacksquare s \notin \alpha$	from: 1, 1.2
(2)	$(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$	

$<_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

**OrderTrichotomyOfR**  $:= \text{OrderTrichotomy}[\mathbb{R}, <_{\mathbb{R}}]$

(1)	$(\alpha, \beta \in \mathbb{R}) \implies \dots$	
(1.1)	$\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \dots$	
(1.1.1)	$\alpha \not\subset \beta \wedge \alpha \neq \beta$	from: $<_{\mathbb{R}}$ , 1.1
(1.1.2)	$\exists_{p'} (p' \in \alpha \wedge p' \notin \beta) \blacksquare p := \text{choice}(\{p' : p' \in \alpha \wedge p' \notin \beta\})$	
(1.1.3)	$q \in \beta \implies \dots$	
(1.1.3.1)	$p, q \in \mathbb{Q}$	
(1.1.3.2)	$q < p$	from: <a href="#">CutCorollaryI</a>
(1.1.3.3)	$q \in \alpha$	from: <a href="#">CutII</a>
(1.1.4)	$q \in \beta \implies q \in \alpha$	
(1.1.5)	$\forall_{q \in \beta} (q \in \alpha) \blacksquare \beta \subseteq \alpha$	
(1.1.6)	$\beta \subset \alpha \blacksquare \beta <_{\mathbb{R}} \alpha$	
(1.2)	$\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$	
(1.3)	$\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \blacksquare (\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$	
(1.4)	$\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$	
(1.5)	$\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$	
(1.6)	$\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$	
(1.7)	$\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta$	
(2)	$(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$	
(3)	$\forall_{\alpha, \beta \in \mathbb{R}} (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$	
(4)	<a href="#">OrderTrichotomy</a> $[\mathbb{R}, <_{\mathbb{R}}]$	

**OrderTransitivityOfR**  $:= \text{OrderTransitivity}[\mathbb{R}, <_{\mathbb{R}}]$

(1)	$(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$	
(1.1)	$(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \dots$	
(1.1.1)	$\alpha \subset \beta \wedge \beta \subset \gamma$	
(1.1.2)	$\forall_{a \in \alpha} (a \in \beta) \wedge \forall_{b \in \beta} (b \in \gamma)$	
(1.1.3)	$\forall_{a \in \alpha} (\alpha \in \gamma) \blacksquare \alpha \subset \gamma \blacksquare \alpha <_{\mathbb{R}} \gamma$	
(1.2)	$(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$	
(2)	$(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$	
(3)	$\forall_{\alpha, \beta, \gamma \in \mathbb{R}} ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$	
(4)	<a href="#">OrderTransitivity</a> $[\mathbb{R}, <_{\mathbb{R}}]$	

**OrderOfR**  $:= \text{Order}[<_{\mathbb{R}}, \mathbb{R}]$

**LUBPropertyOfR**  $:= \text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}]$

(1)	$(\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \dots$	from: <a href="#">OrderTrichotomyR</a> , <a href="#">OrderTransitivityR</a> wts:
(1.1)	$\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A} (p \in \alpha)\}$	

(1.2)	$A \neq \emptyset \quad \blacksquare \quad \exists_{\alpha}(\alpha \in A) \quad \blacksquare \quad \alpha_0 := \text{choice}(\{\alpha : \alpha \in A\})$
(1.3)	$\alpha_0 \neq \emptyset \quad \blacksquare \quad \exists_a(a \in \alpha_0) \quad \blacksquare \quad a_0 := \text{choice}(\{a : a \in \alpha_0\}) \quad \blacksquare \quad a_0 \in \gamma \quad \blacksquare \quad \gamma \neq \emptyset$
(1.4)	$\text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}] \quad \blacksquare \quad \exists_{\beta}(\text{UpperBound}[\beta, A, \mathbb{R}, <_{\mathbb{R}}])$
(1.5)	$\beta_0 := \text{choice}(\{\beta : \text{UpperBound}[\beta, A, \mathbb{R}, <_{\mathbb{R}}]\})$
(1.6)	$\text{UpperBound}[\beta_0, A, \mathbb{R}, <_{\mathbb{R}}] \quad \blacksquare \quad \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \beta_0) \quad \blacksquare \quad \forall_{\alpha \in A}(\alpha \subseteq \beta_0) \quad \blacksquare \quad \forall_{\alpha \in A} \forall_{a \in \alpha}(a \in \beta_0)$
(1.7)	$(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \quad \blacksquare \quad \forall_{a \in \gamma}(a \in \beta_0) \quad \blacksquare \quad \gamma \subseteq \beta_0$
(1.8)	$\beta_0 \subset \mathbb{Q} \quad \blacksquare \quad \gamma \subseteq \beta_0 \subset \mathbb{Q} \quad \blacksquare \quad \gamma \subset \mathbb{Q}$
(1.9)	$\emptyset \neq \gamma \subset \mathbb{Q} \quad \blacksquare \quad \text{CutI}[\gamma]$
(1.10)	$(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
(1.10.1)	$p \in \gamma \quad \blacksquare \quad \exists_{\alpha \in A}(p \in \alpha) \quad \blacksquare \quad \alpha_1 := \text{choice}(\{\alpha \in A : p \in \alpha\})$
(1.10.2)	$p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \quad \blacksquare \quad q \in \alpha_1 \quad \blacksquare \quad q \in \gamma$
(1.11)	$(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \quad \blacksquare \quad \forall_{p \in \gamma} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma) \quad \blacksquare \quad \text{CutII}[\gamma]$
(1.12)	$p \in \gamma \implies \dots$
(1.12.1)	$\exists_{\alpha \in A}(p \in \alpha) \quad \blacksquare \quad \alpha_2 := \text{choice}(\{\alpha \in A : p \in \alpha\})$
(1.12.2)	$\alpha_2 \in \mathbb{R} \quad \blacksquare \quad \text{CutII}[\alpha_2] \quad \blacksquare \quad \exists_{r \in \alpha_2}(p < r) \quad \blacksquare \quad r_0 := \text{choice}(\{r \in \alpha_2 : p < r\})$
(1.12.3)	$r_0 \in \alpha_2 \quad \blacksquare \quad r_0 \in \gamma$
(1.12.4)	$p < r_0 \quad \blacksquare \quad p < r_0 \wedge r_0 \in \gamma \quad \blacksquare \quad \exists_{r \in \gamma}(p < r)$
(1.13)	$p \in \gamma \implies \exists_{r \in \gamma}(p < r) \quad \blacksquare \quad \forall_{p \in \gamma} \exists_{r \in \gamma}(p < r) \quad \blacksquare \quad \text{CutIII}[\gamma]$
(1.14)	$\text{CutI}[\gamma] \wedge \text{CutII}[\gamma] \wedge \text{CutIII}[\gamma] \quad \blacksquare \quad \gamma \in \mathbb{R}$
(1.15)	$\forall_{\alpha \in A}(\alpha \subseteq \gamma) \quad \blacksquare \quad \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma)$
(1.16)	$\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R} \quad \blacksquare \quad \text{UpperBound}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$
(1.17)	$\delta <_{\mathbb{R}} \gamma \implies \dots$
(1.17.1)	$\delta \subset \gamma \quad \blacksquare \quad \exists_s(s \in \gamma \wedge s \notin \delta) \quad \blacksquare \quad s_0 := \text{choice}(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$
(1.17.2)	$s_0 \in \gamma \quad \blacksquare \quad \exists_{\alpha \in A}(s_0 \in \alpha) \quad \blacksquare \quad \alpha_3 := \text{choice}(\{\alpha \in A : s_0 \in \alpha\})$
(1.17.3)	$s_0 \in \alpha_3 \wedge s_0 \notin \delta \quad \blacksquare \quad \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$
(1.17.4)	$\delta \geq_{\mathbb{R}} \alpha_3 \implies \dots$
(1.17.4.1)	$\alpha_3 \subseteq \delta \quad \blacksquare \quad \forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \quad \blacksquare \quad \neg \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$
(1.17.4.2)	$\neg \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \quad \blacksquare \quad \perp$
(1.17.5)	$\delta \geq_{\mathbb{R}} \alpha_3 \implies \perp \quad \blacksquare \quad \delta <_{\mathbb{R}} \alpha_3 \quad \blacksquare \quad \exists_{\alpha \in A}(\delta <_{\mathbb{R}} \alpha) \quad \blacksquare \quad \exists_{\alpha \in A}(\neg(\alpha \leq_{\mathbb{R}} \delta))$
(1.17.6)	$\neg \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \delta) \quad \blacksquare \quad \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$
(1.18)	$\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}] \quad \blacksquare \quad \forall_{\delta}(\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$
(1.19)	$\text{UpperBound}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \wedge \forall_{\delta}(\delta <_{\mathbb{R}} \gamma \implies \neg \text{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$
(1.20)	$\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \quad \blacksquare \quad \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$
(2)	$(\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$
(3)	$\forall_A \left( (\emptyset \neq A \subset \mathbb{R} \wedge \text{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(\text{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]) \right) \quad \blacksquare \quad \text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}]$

$$+_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$$

$$0_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 0\}$$

$$0InR := 0_{\mathbb{R}} \in \mathbb{R}$$

(1)	$-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}} \quad \blacksquare \quad \emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q} \quad \blacksquare \quad \text{CutI}[0_{\mathbb{R}}]$
(2)	$(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}} \quad \blacksquare \quad \forall_{x \in 0_{\mathbb{R}}} \forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}}) \quad \blacksquare \quad \text{CutII}[0_{\mathbb{R}}]$
(3)	$y := x/2 \quad \blacksquare \quad (x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y) \quad \blacksquare \quad \forall_{x \in 0_{\mathbb{R}}} \exists_{y \in 0_{\mathbb{R}}}(x < y) \quad \blacksquare \quad \text{CutIII}[0_{\mathbb{R}}]$
(4)	$\text{CutI}[0_{\mathbb{R}}] \wedge \text{CutII}[0_{\mathbb{R}}] \wedge \text{CutIII}[0_{\mathbb{R}}] \quad \blacksquare \quad 0_{\mathbb{R}} \in \mathbb{R}$

$$\text{FieldAdditionClosureOf } R := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$$

(1)	$(\alpha, \beta \in \mathbb{R}) \implies \dots$
(1.1)	$(\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$
(1.2)	$\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$



(1.3)	$\exists_a(a \in \alpha) ; \exists_b(b \in \beta) \blacksquare a_0 := choice(\{a : a \in \alpha\}) ; b_0 := choice(\{b : b \in \beta\}) \blacksquare a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$
(1.4)	$\exists_x(x \notin \alpha) ; \exists_y(y_0 \notin \beta) \blacksquare x_0 := choice(\{x : x \notin \alpha\}) ; y_0 := choice(\{y : y \notin \beta\})$
(1.5)	$\forall_{r \in \alpha}(r < x_0) ; \forall_{s \in \beta}(s < y_0) \blacksquare \forall_{r \in \alpha} \forall_{s \in \beta}(r + s < x_0 + y_0) \blacksquare x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$
(1.6)	$\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q} \blacksquare \textcolor{blue}{CutI}[\alpha +_{\mathbb{R}} \beta]$
(1.7)	$(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
(1.7.1)	$\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s) \blacksquare (r_0, s_0) := choice((r, s) \in \alpha \times \beta : p = r + s)$
(1.7.2)	$q < p = r_0 + s_0 \blacksquare (q - s_0) < r_0 \blacksquare (q - s_0) \in \alpha$
(1.7.3)	$s_0 \in \beta \blacksquare q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta \blacksquare q \in \alpha +_{\mathbb{R}} \beta$
(1.8)	$(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta) \blacksquare \textcolor{blue}{CutII}[\alpha +_{\mathbb{R}} \beta]$
(1.9)	$p \in \alpha \implies \dots$
(1.9.1)	$\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s) \blacksquare (r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta : p = r + s\})$
(1.9.2)	$r_1 \in \alpha \blacksquare \exists_{t \in \alpha}(r_1 < t) \blacksquare t_0 := choice(\{t \in \alpha : r_1 < t\})$
(1.9.3)	$s_1 \in \beta \blacksquare t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1 \blacksquare \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$
(1.10)	$p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r) \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} \beta} \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r) \blacksquare \textcolor{blue}{CutIII}[\alpha +_{\mathbb{R}} \beta]$
(1.11)	$\textcolor{blue}{CutI}[\alpha +_{\mathbb{R}} \beta] \wedge \textcolor{blue}{CutII}[\alpha +_{\mathbb{R}} \beta] \wedge \textcolor{blue}{CutIII}[\alpha +_{\mathbb{R}} \beta] \blacksquare \alpha +_{\mathbb{R}} \beta \in \mathbb{R}$
(2)	$(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

**FieldAdditionCommutativityOf  $\mathbb{R}$**   $:= (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1)  $\alpha +_{\mathbb{R}} \beta = \{r + s : r \in \alpha \wedge s \in \beta\} = \{s + r : s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

**FieldAdditionAssociativityOf  $\mathbb{R}$**   $:= (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma))$

(1)  $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$

(1.1)  $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \dots$

(1.2)  $\{a + (b + c) : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2)  $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

**FieldAdditionIdentityOf  $\mathbb{R}$**   $:= (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1)  $\alpha \in \mathbb{R} \implies \dots$

(1.1)  $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \dots$

(1.1.1)  $s < 0 \blacksquare r + s < r + 0 = r \blacksquare r + s < r \blacksquare r + s \in \alpha$

(1.2)  $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha \blacksquare \forall_{r \in \alpha} \forall_{s \in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3)  $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \forall_{p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}}(p \in \alpha) \blacksquare \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4)  $p \in \alpha \implies \dots$

(1.4.1)  $\exists_{r \in \alpha}(p < r) \blacksquare r_2 := choice(\{r \in \alpha : p < r\})$

(1.4.2)  $p < r_2 \blacksquare p - r_2 < r_2 - r_2 = 0 \blacksquare (p - r_2) < 0 \blacksquare (p - r_2) \in 0_{\mathbb{R}}$

(1.4.3)  $r_2 \in \alpha \blacksquare p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5)  $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare \forall_{p \in \alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6)  $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2)  $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

**FieldAdditionInverseOf  $\mathbb{R}$**   $:= (\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(1)  $\alpha \in \mathbb{R} \implies \dots$

(1.1)  $\beta := \{p \in \mathbb{Q} : \exists_{r > 0}(-p - r \notin \alpha)\}$

(1.2)  $\alpha \subset \mathbb{Q} \blacksquare \exists_{s \in \mathbb{Q}}(s \notin \alpha) \blacksquare s_0 := choice(\{s : s \notin \alpha\}) \blacksquare p_0 := -s_0 - 1$

(1.3)  $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha \blacksquare -p_0 - 1 \notin \alpha \blacksquare \exists_{r > 0}(-p_0 - r \notin \alpha) \blacksquare p_0 \in \beta$

(1.4)  $\emptyset \neq \alpha \blacksquare \exists_{q \in \alpha} \blacksquare q_0 := choice(\{q \in \mathbb{Q} : q \in \alpha\})$

(1.5)  $r > 0 \implies \dots$

(1.5.1)  $q_0 \in \alpha \blacksquare -(-q_0) - r = q_0 - r < q_0 \blacksquare -(-q_0) - r < q_0 \blacksquare -(-q_0) - r \in \alpha$

(1.6)  $\forall_{r > 0}(-(-q_0) - r \in \alpha) \blacksquare \neg \exists_{r > 0}(-(-q_0) - r \notin \alpha) \blacksquare -q_0 \notin \beta$

(1.7)  $\emptyset \neq \beta \subset \mathbb{Q} \blacksquare \textcolor{blue}{CutI}[\beta]$

(1.8)	$(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$	
(1.8.1)	$p \in \beta \blacksquare \exists_{r>0}(-p-r \notin \alpha) \blacksquare r_0 := \text{choice}(\{r > 0 : -p-r \notin \alpha\})$	
(1.8.2)	$q < p \blacksquare -p-r < -q-r$	
(1.8.3)	$-q-r \notin \alpha \blacksquare q \in \beta$	
(1.9)	$(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta \blacksquare \forall_{p \in \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \beta) \blacksquare \text{CutII}[\beta]$	
(1.10)	$p \in \beta \implies \dots$	
(1.10.1)	$p \in \beta \blacksquare \exists_{r>0}(-p-r \notin \alpha) \blacksquare r_1 := \text{choice}(\{r > 0 : -p-r \notin \alpha\})$	
(1.10.2)	$t_0 := p + (r_1/2)$	
(1.10.3)	$r_1 > 0 \blacksquare r_1/2 > 0$	
(1.10.4)	$t_0 > t_0 - (r_1/2) = p \blacksquare t_0 > p$	
(1.10.5)	$-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$	
(1.10.6)	$-p - r_1 \notin \alpha \blacksquare -t_0 - (r_1/2) \notin \alpha \blacksquare \exists_{r>0}(-t_0 - r \notin \alpha) \blacksquare t_0 \in \beta$	
(1.10.7)	$t_0 > p \wedge t_0 \in \beta \blacksquare \exists_{t \in \beta}(p < t)$	
(1.11)	$p \in \beta \implies \exists_{t \in \beta}(p < t) \blacksquare \forall_{p \in \beta} \exists_{t \in \beta}(p < t) \blacksquare \text{CutIII}[\beta]$	
(1.12)	$\text{CutI}[\beta] \wedge \text{CutII}[\beta] \wedge \text{CutIII}[\beta] \blacksquare \beta \in \mathbb{R}$	
(1.13)	$(r \in \alpha \wedge s \in \beta) \implies \dots$	
(1.13.1)	$s \in \beta \blacksquare \exists_{t>0}(-s-t \notin \alpha) \blacksquare t_1 := \text{choice}(\{t > 0 : -s-t \notin \alpha\}) \blacksquare -s - t_1 < -s$	
(1.13.2)	$\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha \blacksquare -s \notin \alpha$	
(1.13.3)	$\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha \blacksquare r < -s \blacksquare r + s < 0 \blacksquare r + s \in 0_{\mathbb{R}}$	
(1.14)	$(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}} \blacksquare \forall_{(r,s) \in \alpha \times \beta}(r + s \in 0_{\mathbb{R}}) \blacksquare \alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$	
(1.15)	$v \in 0_{\mathbb{R}} \implies \dots$	
(1.15.1)	$v < 0 \blacksquare w_0 := -v/2 \blacksquare w > 0$	
(1.15.2)	$\exists_{n \in \mathbb{Z}}(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha) \blacksquare n_0 := \text{choice}(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$	from: ARCHIMEDEANPROPERTYOFQ + LUB???
(1.15.3)	$p_0 := -(n_0 + 2)w_0 \blacksquare -p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha \blacksquare -p_0 - w_0 \notin \alpha \blacksquare p_0 \in \beta$	
(1.15.4)	$n_0 w_0 \in \alpha \wedge p_0 \in \beta \blacksquare n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$	
(1.16)	$v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{v \in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta) \blacksquare 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$	
(1.17)	$\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta \blacksquare \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$	
(1.18)	$\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}} \blacksquare \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$	
(2)	$\alpha \in \mathbb{R} \implies \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$	

$*_{\mathbb{R}}[\alpha, \beta] := \quad \text{---}$   
 $1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$11s\text{Not}0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}} \quad \text{---}$   
 $11nR := 1_{\mathbb{R}} \in \mathbb{R} \quad \text{---}$

$\text{FieldMultiplicationClosureOf } \mathbb{R} := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta) \in \mathbb{R} \quad \text{---}$   
 $\text{FieldMultiplicationCommutativityOf } \mathbb{R} := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha) \quad \text{---}$   
 $\text{FieldMultiplicationAssociativityOf } \mathbb{R} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)) \quad \text{---}$   
 $\text{FieldMultiplicationIdentityOf } \mathbb{R} := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha \quad \text{---}$   
 $\text{FieldMultiplicationInverseOf } \mathbb{R} := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha \in \mathbb{R}}(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}) \quad \text{---}$

$\text{FieldDistributativityOf } \mathbb{R} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta \quad \text{---}$

$\text{FieldWith } \mathbb{R} := \text{Field}[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}] \quad \text{---}$   
 $\text{OrderedFieldWith } \mathbb{R} := \text{OrderedField}[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}] \quad \text{---}$

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$   
 $\text{QROrderedSubfieldOf } \mathbb{R} := \text{OrderedSubfield}[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}] \quad \text{---}$   
 $\text{QIsomorphicToQR} := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q} \quad \text{---}$   
 $\text{CompletenessOf } \mathbb{R} := \exists_{\mathbb{R}}(\text{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}] \wedge \text{OrderedSubfield}[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]) \quad \text{---}$

(1.20)  
 $\text{ArchimedeanPropertyOf } \mathbb{R} := \forall_{x,y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

(1)  $(x, y \in \mathbb{R} \wedge x > 0) \implies \dots$

(1.1)  $A := \{nx : n \in \mathbb{N}^+\} \blacksquare (\emptyset \neq A \subset \mathbb{R}) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$

(1.2)  $\neg \exists_{n \in \mathbb{N}^+}(nx > y) \implies \dots$

(1.2.1)  $\neg \exists_{n \in \mathbb{N}^+}(nx > y) \blacksquare \forall_{n \in \mathbb{N}^+}(nx \leq y) \blacksquare \text{UpperBound}[y_0, A, \mathbb{R}, <] \blacksquare \text{BoundedAbove}[A, \mathbb{R}, <]$

(1.2.2)  $\text{CompletenessOf } \mathbb{R} \blacksquare \text{LUBProperty}[\mathbb{R}, <]$

(1.2.3)  $(\text{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (\text{BoundedAbove}[A, \mathbb{R}, <]) \blacksquare \exists_{\alpha \in \mathbb{R}}(\text{LUB}[\alpha, A, \mathbb{R}, <]) \dots$

(1.2.4)  $\dots \alpha_0 := \text{choice}(\{\alpha \in \mathbb{R} : \text{LUB}[\alpha, A, \mathbb{R}, <]\}) \blacksquare \text{LUB}[\alpha_0, A, \mathbb{R}, <]$

(1.2.5)  $x > 0 \blacksquare \alpha_0 - x < \alpha_0$

(1.2.6)  $(\alpha_0 - x < \alpha_0) \wedge (\text{LUB}[\alpha_0, A, \mathbb{R}, <]) \blacksquare \neg \text{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7)  $\neg \text{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <] \blacksquare \exists_{c \in A}(\alpha_0 - x < c) \dots$

(1.2.8)  $\dots c_0 := \text{choice}(\{c \in A : \alpha_0 - x < c\}) \blacksquare (c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9)  $(c_0 \in A) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)) \blacksquare \exists_{m \in \mathbb{N}^+}(mx = c_0) \dots$

(1.2.10)  $\dots m_0 := \text{choice}(\{m \in \mathbb{N}^+ : mx = c_0\}) \blacksquare (m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11)  $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0) \blacksquare \alpha_0 - x < c_0 = m_0 x \blacksquare \alpha_0 < m_0 x + x \blacksquare \alpha_0 < (m_0 + 1)x$

(1.2.12)  $m_0 \in \mathbb{N}^+ \blacksquare m_0 + 1 \in \mathbb{N}^+$

(1.2.13)  $(m_0 + 1 \in \mathbb{N}^+) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)) \blacksquare (m_0 + 1)x \in A$

(1.2.14)  $(\alpha_0 < (m_0 + 1)x) \wedge ((m_0 + 1)x \in A) \blacksquare \exists_{c \in A}(\alpha_0 < c)$

(1.2.15)  $\text{LUB}[\alpha_0, A, \mathbb{R}, <] \blacksquare \text{UpperBound}[\alpha_0, A, \mathbb{R}, <] \blacksquare \forall_{c \in A}(c \leq \alpha_0) \blacksquare \neg \exists_{c \in A}(c > \alpha_0) \blacksquare \neg \exists_{c \in A}(\alpha_0 < c)$

(1.2.16)  $(\exists_{c \in A}(\alpha_0 < c)) \wedge (\neg \exists_{c \in A}(\alpha_0 < c)) \blacksquare \perp$

(1.3)  $\neg \exists_{n \in \mathbb{N}^+}(nx > y) \implies \perp \blacksquare \exists_{n \in \mathbb{N}^+}(nx > y)$

(2)  $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y) \blacksquare \forall_{x, y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

**QDenseInR** :=  $\forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1)  $(x, y \in \mathbb{R} \wedge x < y) \implies \dots$

(1.1)  $x < y \blacksquare (0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2)  $\text{ArchimedeanPropertyOf } \mathbb{R} \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R}) \blacksquare \exists_{n \in \mathbb{N}^+}(n(y - x) > 1) \dots$

(1.3)  $\dots n_0 := \text{choice}(\{n \in \mathbb{N}^+ : n(y - x) > 1\}) \blacksquare (n_0 \in \mathbb{N}^+) \wedge (n_0(y - x) > 1)$

(1.4)  $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R}) \blacksquare n_0 x, -n_0 x \in \mathbb{R}$

(1.5)  $\text{ArchimedeanPropertyOf } \mathbb{R} \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R}) \blacksquare \exists_{m \in \mathbb{N}^+}(m(1) > n_0 x) \dots$

(1.6)  $\dots m_1 := \text{choice}(\{m \in \mathbb{N}^+ : m(1) > n_0 x\}) \blacksquare (m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7)  $\text{ArchimedeanPropertyOf } \mathbb{R} \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R}) \blacksquare \exists_{m \in \mathbb{N}^+}(m(1) > -n_0 x) \dots$

(1.8)  $\dots m_2 := \text{choice}(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\}) \blacksquare (m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9)  $(m_1 > n_0 x) \wedge (m_2 > -n_0 x) \blacksquare -m_2 < n_0 x < m_1$

(1.10)  $m_1, m_2 \in \mathbb{N}^+ \blacksquare |m_1 - (-m_2)| \geq 2$

(1.11)  $(-m_2 < n_0 x < m_1) \wedge (|m_1 - (-m_2)| \geq 2) \blacksquare \exists_{m \in \mathbb{Z}}((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)) \dots$

(1.12)  $\dots m_0 := \text{choice}(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\}) \blacksquare (-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$

(1.13)  $(n_0(y - x) > 1) \wedge (m_0 - 1 \leq n_0 x < m_0) \blacksquare n_0 x < m_0 \leq 1 + n_0 x < n_0 y \blacksquare n_0 x < m_0 < n_0 y$

(1.14)  $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y) \blacksquare x < m_0/n_0 < y$

(1.15)  $m_0, n_0 \in \mathbb{Z} \blacksquare m_0/n_0 \in \mathbb{Q}$

(1.16)  $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y) \blacksquare \exists_{p \in \mathbb{Q}}(x < p < y)$

(2)  $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y) \blacksquare \forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1.21)

**Root Lemma** :=  $(0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

(1)  $(0 < a < b) \implies \dots$

(1.1)  $b^n - a^n = (b - a) \sum_{i=1}^n (b^{n-i} a^{i-1})$

(1.2)  $0 < a < b \blacksquare b/a > 1$

$$(1.3) \quad b/a > 1 \quad \blacksquare \quad \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq \sum_{i=1}^n \left( b^{n-i} a^{i-1} (b/a)^{i-1} \right) = \sum_{i=1}^n (b^{n-1}) = nb^{n-1} \quad \blacksquare \quad \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq \sum_{i=1}^n (b^{n-1}) = nb^{n-1}$$

$$(1.4) \quad b^n - a^n = (b-a) \sum_{i=1}^n (b^{n-i} a^{i-1}) \leq (b-a)nb^{n-1} \quad \blacksquare \quad b^n - a^n \leq (b-a)nb^{n-1}$$

$$(2) \quad (0 < a < b) \implies \left( b^n - a^n \leq (b-a)nb^{n-1} \right)$$

$$\text{Root Existence In } \mathbb{R} := \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}} (y_0^n = x)$$

$$(1) \quad (0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \dots$$

$$(1.1) \quad E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\} \quad \blacksquare \quad t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$$

$$(1.2) \quad t_0 := x/(1+x) \quad \blacksquare \quad (t_0 = x/(1+x)) \wedge (t_0 \in \mathbb{R})$$

$$(1.3) \quad 0 < x \quad \blacksquare \quad 0 < x < 1+x \quad \blacksquare \quad t_0 = x/(1+x) > 0 \quad \blacksquare \quad t_0 > 0$$

$$(1.4) \quad 1 = (1+x)/(1+x) > x/(1+x) = t_0 \quad \blacksquare \quad 1 > t_0$$

$$(1.5) \quad (t_0 > 0) \wedge (1 > t_0) \quad \blacksquare \quad 0 < t_0 < 1$$

$$(1.6) \quad (0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1) \quad \blacksquare \quad t_0^n \leq t_0$$

$$(1.7) \quad 0 < x \quad \blacksquare \quad x > x/(1+x) = t_0 \quad \blacksquare \quad x > t_0$$

$$(1.8) \quad (t_0^n \leq t_0) \wedge (x > t_0) \quad \blacksquare \quad t_0^n < x$$

$$(1.9) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge (t_0^n < x) \quad \blacksquare \quad t_0 \in E \quad \blacksquare \quad \emptyset \neq E$$

$$(1.10) \quad t_1 := \text{choice}(\{t \in \mathbb{R} : t > 1+x\}) \quad \blacksquare \quad (t_1 \in \mathbb{R}) \wedge (t_1 > 1+x)$$

$$(1.11) \quad x > 0 \quad \blacksquare \quad t_1 > 1+x > 1 \quad \blacksquare \quad t_1 > 1 \quad \blacksquare \quad t_1^n \geq t_1$$

$$(1.12) \quad (t_1^n \geq t_1) \wedge (t_1 > 1+x) \wedge (1 > 0) \quad \blacksquare \quad t_1^n \geq t_1 > 1+x > x \quad \blacksquare \quad t_1^n > x$$

$$(1.13) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_1^n > x) \quad \blacksquare \quad t_1 \notin E \quad \blacksquare \quad E \subset \mathbb{R}$$

$$(1.14) \quad (\emptyset \neq E) \wedge (E \subset \mathbb{R}) \quad \blacksquare \quad \emptyset \neq E \subset \mathbb{R}$$

$$(1.15) \quad t \in E \implies \dots$$

$$(1.15.1) \quad (t \in E) \wedge (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \quad \blacksquare \quad t^n < x$$

$$(1.15.2) \quad (t_1^n > x) \wedge (t^n < x) \quad \blacksquare \quad t^n < x < t_1^n \quad \blacksquare \quad t < t_1$$

$$(1.16) \quad t \in E \implies t < t_1 \quad \blacksquare \quad \forall_{t \in E} (t \leq t_1) \quad \blacksquare \quad \text{Upper Bound}[t_1, E, \mathbb{R}, <] \quad \blacksquare \quad \text{Bounded Above}[E, \mathbb{R}, <]$$

$$(1.17) \quad \text{Completeness Of } \mathbb{R} \quad \blacksquare \quad \text{LUB Property}[\mathbb{R}, <]$$

$$(1.18) \quad (\text{LUB Property}[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (\text{Bounded Above}[E, \mathbb{R}, <]) \quad \blacksquare \quad \exists_{y \in \mathbb{R}} (\text{LUB}[y, E, \mathbb{R}, <]) \quad \dots$$

$$(1.19) \quad \dots y_0 := \text{choice}(\{y \in \mathbb{R} : \text{LUB}[y, E, \mathbb{R}, <]\}) \quad \blacksquare \quad \text{LUB}[y_0, E, \mathbb{R}, <]$$

$$(1.20) \quad (\text{LUB}[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0) \quad \blacksquare \quad 0 < t_0 \leq y_0 \in \mathbb{R} \quad \blacksquare \quad 0 < y_0 \in \mathbb{R}$$

$$(1.21) \quad y_0^n < x \implies \dots$$

$$(1.21.1) \quad k_0 := \frac{x - y_0^n}{n(y_0 + 1)^{n-1}} \quad \blacksquare \quad k_0 \in \mathbb{R}$$

$$(1.21.2) \quad y_0^n < x \quad \blacksquare \quad 0 < x - y_0^n$$

$$(1.21.3) \quad (n > 0) \wedge (y_0 > 0) \quad \blacksquare \quad 0 < n(y_0 + 1)^{n-1}$$

$$(1.21.4) \quad (0 < x - y_0^n) \wedge \left( 0 < n(y_0 + 1)^{n-1} \right) \quad \blacksquare \quad 0 < \frac{x - y_0^n}{n(y_0 + 1)^{n-1}} = k_0 \quad \blacksquare \quad 0 < k_0$$

$$(1.21.5) \quad (0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R}) \quad \blacksquare \quad 0 < \min(1, k_0) \in \mathbb{R}$$

$$(1.21.6) \quad \text{QDense In } \mathbb{R} \wedge (0, \min(1, k_0)) \in \mathbb{R} \wedge (0 < \min(1, k_0)) \quad \blacksquare \quad \exists_{h \in \mathbb{Q}} (0 < h < \min(1, k_0)) \quad \dots$$

$$(1.21.7) \quad \dots h_0 := \text{choice}(\{h \in \mathbb{Q} : 0 < h < \min(1, k_0)\}) \quad \blacksquare \quad (0 < h_0 < 1) \wedge \left( h_0 < k_0 = \frac{x - y_0^n}{n(y_0 + 1)^{n-1}} \right)$$

$$(1.21.8) \quad (y_0 > 0) \wedge (h_0 > 0) \quad \blacksquare \quad 0 < y_0 < y_0 + h_0$$

$$(1.21.9) \quad \text{Root Lemma} \wedge (0 < y_0 < y_0 + h_0) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < h_0 n(y_0 + h_0)^{n-1}$$

$$(1.21.10) \quad h_0 < 1 \quad \blacksquare \quad h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$$

$$(1.21.11) \quad \left( (y_0 + h_0)^n - y_0^n < h_0 n(y_0 + h_0)^{n-1} \right) \wedge \left( h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1} \right) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < h_0 n(y_0 + 1)^{n-1}$$

$$(1.21.12) \quad \left( 0 < n(y_0 + 1)^{n-1} \right) \wedge \left( h_0 < k_0 = \frac{x - y_0^n}{n(y_0 + 1)^{n-1}} \right) \quad \blacksquare \quad h_0 n(y_0 + 1)^{n-1} < x - y_0^n$$

$$(1.21.13) \quad \left( (y_0 + h_0)^n - y_0^n < h_0 n(y_0 + 1)^{n-1} \right) \wedge \left( h_0 n(y_0 + 1)^{n-1} < x - y_0^n \right) \quad \blacksquare \quad (y_0 + h_0)^n - y_0^n < x - y_0^n \quad \blacksquare \quad (y_0 + h_0)^n < x$$

$$(1.21.14) \quad (y_0 + h_0)^n - y_0^n < x - y_0^n \quad \blacksquare \quad (y_0 + h_0)^n < x$$

$$(1.21.15) \quad (0 < y_0 \in \mathbb{R}) \wedge (0 < h_0 \in \mathbb{R}) \quad \blacksquare \quad 0 < y_0 < y_0 + h_0 \in \mathbb{R}$$

$$(1.21.16) \quad (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ((y_0 + h_0)^n < x) \wedge (0 < y_0 + h_0 \in \mathbb{R}) \quad \blacksquare \quad (y_0 + h_0)^n \in E$$

(1.21.17)	$((y_0 + h_0)^n \in E) \wedge (y_0 < y_0 + h_0) \quad \blacksquare \quad \exists_{e \in E} (y_0 < e)$
(1.21.18)	$\textcolor{teal}{LUB}[y_0, E, \mathbb{R}, <] \quad \blacksquare \quad \textcolor{teal}{UpperBound}[y_0, E, \mathbb{R}, <] \quad \blacksquare \quad \forall_{e \in E} (e \leq y_0) \quad \blacksquare \quad \neg \exists_{e \in E} (e > y_0)$
(1.21.19)	$(\exists_{e \in E} (e > y_0)) \wedge (\neg \exists_{e \in E} (e > y_0)) \quad \blacksquare \quad \perp$
(1.22)	$y_0^n < x \implies \perp \quad \blacksquare \quad y_0^n \geq x$
(1.23)	$y_0^n > x \implies \dots$
(1.23.1)	$k_1 := \frac{y_0^n - x}{ny_0^{n-1}} \quad \blacksquare \quad (k_1 \in \mathbb{R}) \wedge (k_1 ny_0^{n-1} = y_0^n - x)$
(1.23.2)	$(0 < x) \wedge (0 < n \in \mathbb{Z}) \quad \blacksquare \quad y_0^n - x < y_0^n \leq ny_0^n \quad \blacksquare \quad y_0^n - x < ny_0^n$
(1.23.3)	$y_0^n - x < ny_0^n \quad \blacksquare \quad k_1 = \frac{y_0^n - x}{ny_0^{n-1}} < \frac{ny_0^n}{ny_0^{n-1}} = y_0 \quad \blacksquare \quad k_1 < y_0$
(1.23.4)	$y_0^n > x \quad \blacksquare \quad 0 < y_0^n - x$
(1.23.5)	$(n > 0) \wedge (y_0 > 0) \quad \blacksquare \quad 0 < ny_0^{n-1}$
(1.23.6)	$(0 < y_0^n - x) \wedge 0 < (ny_0^{n-1}) \quad \blacksquare \quad 0 < \frac{y_0^n - x}{ny_0^{n-1}} = k_1 \quad \blacksquare \quad 0 < k_1$
(1.23.7)	$(k_1 < y_0) \wedge (0 < k_1) \quad \blacksquare \quad (0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$
(1.23.8)	$t \geq y_0 - k_1 \implies \dots$
(1.23.8.1)	$t \geq y_0 - k_1 \quad \blacksquare \quad t^n \geq (y_0 - k_1)^n \quad \blacksquare \quad -t^n \leq -(y_0 - k_1)^n \quad \blacksquare \quad y_0^n - t^n \leq y_0^n - (y_0 - k_1)^n$
(1.23.8.2)	$\textcolor{teal}{RootLemma} \wedge (0 < y_0 - k_1 < y_0) \quad \blacksquare \quad y_0^n - (y_0 - k_1)^n < k_1 ny_0^{n-1}$
(1.23.8.3)	$(y_0^n - t^n \leq y_0^n - (y_0 - k_1)^n) \wedge (y_0^n - (y_0 - k_1)^n < k_1 ny_0^{n-1}) \quad \blacksquare \quad y_0^n - t^n < k_1 ny_0^{n-1}$
(1.23.8.4)	$(k_1 ny_0^{n-1} = y_0^n - x) \wedge (y_0^n - t^n < k_1 ny_0^{n-1}) \quad \blacksquare \quad y_0^n - t^n < y_0^n - x \quad \blacksquare \quad -t^n < -x \quad \blacksquare \quad t^n > x$
(1.23.8.5)	$(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t^n > x) \quad \blacksquare \quad t \notin E$
(1.23.9)	$t \geq y_0 - k_1 \implies t \notin E \quad \blacksquare \quad t \in E \implies t < y_0 - k_1 \quad \blacksquare \quad \forall_{t \in E} (t \leq y_0 - k_1) \quad \blacksquare \quad \textcolor{teal}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$
(1.23.10)	$(\textcolor{teal}{LUB}[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)) \quad \blacksquare \quad \neg \textcolor{teal}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$
(1.23.11)	$(\textcolor{teal}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg \textcolor{teal}{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \quad \blacksquare \quad \perp$
(1.24)	$y_0^n > x \implies \perp \quad \blacksquare \quad y_0^n \leq x$
(1.25)	$\textcolor{teal}{Order}[\mathbb{R}, <] \quad \blacksquare \quad \textcolor{teal}{OrderTrichotomy}[\mathbb{R}, <]$
(1.26)	$(\textcolor{teal}{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_0^n \geq x) \wedge (y_0^n \leq x) \quad \blacksquare \quad y_0^n = x$
(1.27)	$(y_0^n = x) \wedge (y_0 \in \mathbb{R}) \quad \blacksquare \quad \exists_{y \in \mathbb{R}} (y^n = x)$
(1.28)	$y_1, y_2 := \textit{choice}(\{y \in \mathbb{R} : y^n = x\})$
(1.29)	$y_1 \neq y_2 \implies \dots$
(1.29.1)	$(\textcolor{teal}{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_1 \neq y_2) \quad \blacksquare \quad (y_1 < y_2) \vee (y_2 < y_1) \quad \dots$
(1.29.2)	$\dots (x = y_1^n < y_2^n = x) \vee (x = y_2^n < y_1^n = x) \quad \blacksquare \quad (x < x) \vee (x > x) \quad \blacksquare \quad \perp \vee \perp \quad \blacksquare \quad \perp$
(1.30)	$y_1 \neq y_2 \implies \perp \quad \blacksquare \quad y_1 = y_2 \quad \blacksquare \quad \forall_{a,b \in \mathbb{R}} ((a^n = x \wedge b^n = x) \implies a = b)$
(1.31)	$(\exists_{y \in \mathbb{R}} (y^n = x)) \wedge (\forall_{a,b \in \mathbb{R}} ((a^n = x \wedge b^n = x) \implies a = b)) \quad \blacksquare \quad \exists!_{y \in \mathbb{R}} (y^n = x)$
(2)	$(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}} (y^n = x) \quad \blacksquare \quad \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}} (y_0^n = x)$

$$\textcolor{red}{RootExistenceInRCorollary} := \forall_{0 < a \in \mathbb{R}} \forall_{0 < b \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \left( (ab)^{1/n} = a^{1/n} b^{1/n} \right) \quad \text{---}$$

$$\textcolor{red}{ExtendedRealSystem}[\bar{\mathbb{R}}, +, *, <] := \left( \begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \quad \wedge \quad x - \infty = -\infty \quad \wedge \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \quad \wedge \\ (x > 0) \implies (x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty) \quad \wedge \\ (x < 0) \implies (x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty) \end{array} \right)$$

$$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$$

$$+_C[\langle a, b \rangle, \langle c, d \rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d \rangle$$

$$*_C[\langle a, b \rangle, \langle c, d \rangle] := \langle a *_R c - b *_R d, a *_R d + b *_R c \rangle$$

$$\textcolor{red}{FieldC} := \textcolor{teal}{Field}[\mathbb{C}, +_C, *_C] \quad \text{---}$$

$$\textcolor{red}{RSubfieldC} := \textcolor{teal}{Subfield}[\mathbb{R}, \mathbb{C}, +, *] \quad \text{---}$$

$$i := \langle 0, 1 \rangle \in \mathbb{C}$$

$$\textcolor{red}{iProperty} := i^2 = -1 \quad \text{---}$$

$$\textcolor{red}{CProperty} := (a, b \in \mathbb{R}) \implies (\langle a, b \rangle = a + bi) \quad \text{---}$$



# Chapter 2

## Abstract Algebra

$$\begin{aligned} \text{Relation}(f, X) &:= f \subseteq X \\ \text{Function}(f, X, Y) &:= X \neq \emptyset \neq Y \wedge \text{Relation}(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y} ((x, y) \in f) \end{aligned}$$

$$(\text{Function}(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \dots$$

$$(1) \quad \text{Domain}(f) := X; \text{Codomain}(f) := Y$$

$$(2) \quad \text{Image}(f, A) := \{f(a) : a \in A\}; \text{Preimage}(f, B) := \{a : f(a) \in B\}$$

$$(3) \quad \text{Range}(f) := \text{Image}(f, \text{Domain}(f))$$

$$\text{Injective}(f, X, Y) := \text{Function}(f, X, Y) \wedge \forall_{x_1, x_2 \in X} (x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$$

$$\text{Surjective}(f, X, Y) := \text{Function}(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X} (y = f(x))$$

$$\text{Bijective}(f, X, Y) := \text{Injective}(f, X, Y) \wedge \text{Surjective}(f, X, Y)$$

$$\text{SurjectiveEquivalent} := (\text{Range}(f) = \text{Codomain}(f)) \implies \text{Surjective}(f)$$

$$(\text{Function}(f, X, Y) \wedge \text{Function}(g, Y, Z)) \implies (f \circ g)(x) := f(g(x)); \text{Function}(f \circ g, X, Z)$$

$$\text{Properties of Functions} := (\text{Function}(f, A, B) \wedge \text{Function}(g, B, C) \wedge \text{Function}(h, C, D)) \implies \dots$$

$$(1) \quad h \circ (g \circ f) = (h \circ g) \circ f$$

$$(2) \quad (\text{Injective}(f) \wedge \text{Injective}(g)) \implies \text{Injective}(g \circ f)$$

$$(3) \quad (\text{Surjective}(f) \wedge \text{Surjective}(g)) \implies \text{Surjective}(g \circ f)$$

$$(4) \quad (\text{Bijective}(f, A, B)) \implies \exists_{f^{-1}} \left( \text{Function}(f^{-1}, B, A) \wedge \forall_{a \in A} (f^{-1}(f(a)) = a) \wedge \forall_{b \in B} (f(f^{-1}(b)) = b) \right)$$

$$(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}} (b = ac)$$

$$\text{Divisibility Theorems} := (a, b, c, m, x, y \in \mathbb{Z}) \implies \dots$$

$$(1) \quad (a|b) \implies a|bc$$

$$(2) \quad (a|b \wedge b|c) \implies a|c$$

$$(3) \quad (a|b \wedge b|c) \implies a|(bx + cy)$$

$$(4) \quad (a|b \wedge b|a) \implies a = \pm b$$

$$(5) \quad (a|b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$$

$$(6) \quad (a|b) \iff (m \neq 0 \wedge ma|mb)$$

$$\text{Division Algorithm} := (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}} (b = aq + r)$$

$$\text{CD}(a, b, c) := a, b, c \in \mathbb{Z} \wedge a : b \wedge a : c$$

$$\text{GCD}(a, b, c) := \text{CD}(a, b, c) \wedge \forall_d ((d : b \wedge d : c) \implies d : a)$$

$$\text{GCDEquivalent} := 123123$$





## Chapter 3

# Linear Algebra

$Matrix[A, m, n] := [a_{i,j}]_{m \times n} := m \text{ rows, } n \text{ columns of real numbers}$

$\mathcal{M}_{m,n} := \{A : Matrix[A, m, n]\}$

$O_{m,n} := (Matrix[O, m, n]) \wedge (a_{i,j} = 0)$

$Square[A, n] := Matrix[A, n, n]$

$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$

$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$

$Diagonal[A, n] := (Square[A, n]) \wedge (i \neq j \implies a_{i,j} = 0)$

$Scalar[A, n, k] := (Diagonal[A, n]) \wedge (a_{i,i} = k)$

$I_n := Scalar[I, n, 1]$

$+(A, B) := ((Matrix[A, m, n]) \wedge (Matrix[B, m, n])) \implies (A + B = [a_{i,j} + b_{i,j}]_{m \times n})$

$*(r, A) := ((r \in \mathbb{R}) \wedge (Matrix[A, m, n])) \implies (r * A = [ra_{i,j}]_{m \times n})$

$*(A, B) := ((Matrix[A, m, p]) \wedge (Matrix[B, p, n])) \implies \left( A * B = \left[ \sum_{k=1}^p (a_{i,k} b_{k,j}) \right]_{m \times n} \right)$

$^T[A] := (Matrix[A, m, n]) \implies (A^T = [a_{j,i}]_{n \times m})$

$AddCom := \forall_{A,B \in \mathcal{M}} (A + B = B + A)$

(1)  $A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C \in \mathcal{M}} ((A + B) + C = A + (B + C))$

(1)  $(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A \in \mathcal{M}} \exists!_{O \in \mathcal{M}} (A + O = A = O + A)$

(1)  $A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$

(2)  $A + O_1 = A = A + O_2 \quad \blacksquare \quad O_1 = O_2$

$AddInv := \forall_{A \in \mathcal{M}} \exists!_{(-A) \in \mathcal{M}} (A + (-A) = O = (-A) + A)$

(1)  $A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$

(2)  $A + (-A_1) = O = A + (-A_2) \quad \blacksquare \quad -A_1 = -A_2 \quad \blacksquare \quad A_1 = A_2$

$MulAssoc := \forall_{A,B,C \in \mathcal{M}} ((A * B) * C = A * (B * C))$

(1)  $(A * B) * C = \left[ \sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,j}) \right] * C = \left[ \sum_{k_2=1}^{p_2} \left( \sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,k_2}) c_{k_2,j} \right) \right] = \left[ \sum_{k_2=1}^{p_2} \sum_{k_1=1}^{p_1} (a_{i,k_1} b_{k_1,k_2} c_{k_2,j}) \right] = \dots$

(2)  $\dots \left[ \sum_{k_1=1}^{p_1} \sum_{k_2=1}^{p_2} (a_{i,k_1} b_{k_1,k_2} c_{k_2,j}) \right] = \left[ \sum_{k_1=1}^{p_1} \left( a_{i,k_1} \sum_{k_2=1}^{p_2} (b_{k_1,k_2} c_{k_2,j}) \right) \right] = \dots = A * (B * C)$

$MulId := \forall_{A: Square[A,n]} (A * I_n = A = I_n * A)$

$$(1) \quad A * I_n = \left[ \sum_{k=1}^n \left( a_{i,k} \begin{pmatrix} 1 & k=j \\ 0 & k \neq j \end{pmatrix} \right) \right] = [a_{i,j}] = A$$

$$(2) \quad \text{TODO} = A$$

$$\text{ScalAssoc} := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} (r(sA) = (rs)A = s(rA))$$

$$(1) \quad r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$$

$$(2) \quad (rs)A = [rsa_{i,j}]$$

$$(3) \quad s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$$

$$\text{TransCancel} := \forall_{A \in \mathcal{M}} (A = (A^T)^T)$$

$$(1) \quad A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$$

$$\text{ScalMulCom} := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} ((rA) * B = r(A * B) = A * (rB))$$

$$(1) \quad (rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[ \sum_{k=1}^p (ra_{i,k} b_{k,j}) \right] = r(A * B)$$

$$(2) \quad A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[ \sum_{k=1}^p (a_{i,k} rb_{k,j}) \right] = \left[ \sum_{k=1}^p (ra_{i,k} b_{k,j}) \right] = r(A * B)$$

$$\text{ScalDistLeft} := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} ((r+s)A = rA + sA)$$

$$(1) \quad \text{TODO}$$

$$\text{ScalDistRight} := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} (r(A+B) = rA + rB)$$

$$(1) \quad \text{TODO}$$

$$\text{MulDistRight} := \forall_{A,B,C \in \mathcal{M}} ((A+B) * C = A * C + B * C)$$

$$(1) \quad (A+B) * C = [a_{i,j} + b_{i,j}] * C = \left[ \sum_{k=1}^p ((a_{i,k} + b_{i,k}) c_{k,j}) \right] = \dots$$

$$(2) \quad \dots \left[ \sum_{k=1}^p (a_{i,k} c_{k,j} + b_{i,k} c_{k,j}) \right] = \left[ \sum_{k=1}^p (a_{i,k} c_{k,j}) \right] + \left[ \sum_{k=1}^p (b_{i,k} c_{k,j}) \right] = A * C + B * C$$

$$\text{MulDistLeft} := \forall_{A,B,C \in \mathcal{M}} (C * (A+B) = C * A + C * B)$$

$$(1) \quad \text{TODO}$$

$$\text{TransAddDist} := \forall_{A,B \in \mathcal{M}} ((A+B)^T = A^T + B^T)$$

$$(1) \quad \text{TODO}$$

$$\text{TransMulDist} := \forall_{A,B \in \mathcal{M}} ((A * B)^T = B^T * A^T)$$

$$(1) \quad (A * B)^T = \left[ \sum_{k=1}^p (a_{i,k} b_{k,j}) \right]^T = \left[ \sum_{k=1}^p (a_{j,k} b_{k,i}) \right] = \left[ \sum_{k=1}^p (b_{k,i} a_{j,k}) \right] = \left[ \sum_{k=1}^p (b_{i,k}^T a_{k,j}^T) \right] = B^T * A^T$$

$$\text{Sym}[A] := A = A^T$$

$$\text{SkewSym}[A] := A = -A^T$$

$$\text{Invertible}[A] := (\text{Square}[A, n]) \wedge \left( \exists_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A) \right)$$

$$\text{SymGen} := \forall_{A \in \mathcal{M}} (\text{Sym}[A + A^T])$$

$$(1) \quad (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

$$\text{SkewSymGen} := \forall_{A \in \mathcal{M}} (\text{SkewSym}[A - A^T])$$

$$(1) \quad -(A - A^T)^T = -(A^T - (A^T)^T) = -(A^T - A) = (A - A^T)$$

$$SymDecomp := \forall_{A \in \mathcal{M}} \exists!_{B: Sym[B]} \exists!_{C: SkewSym[C]} (A = B + C)$$

$$(1) \quad B := (1/2) * (A + A^T) ; C := (1/2) * (A - A^T)$$

$$(2) \quad SymGen[B] \wedge SkewSymGen[C]$$

$$(3) \quad A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$$

$$(4) \quad (1/2) * (A_1 + A_1^T) = (1/2) * (A_2 + A_2^T) \quad \blacksquare \quad A_1 = A_2$$

$$(5) \quad (1/2) * (A_3 - A_3^T) = (1/2) * (A_4 - A_4^T) \quad \blacksquare \quad A_3 = A_4$$

$$InvId := \forall_{A: Invertible[A]} \left( \exists!_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A) \right)$$

$$(1) \quad A^{-1}_1 = A^{-1}_1 * I_n = A^{-1}_1 * (A * A^{-1}_2) = (A^{-1}_1 * A) * A^{-1}_2 = I_n * A^{-1}_2 = A^{-1}_2$$

$$InvCancel := \forall_{A: Invertible[A]} \left( (A^{-1})^{-1} = A \right)$$

$$(1) \quad (A * A^{-1})^{-1} = I_n^{-1} = I_n$$

$$(2) \quad (A^{-1})^{-1} * A^{-1} = I_n \quad \blacksquare \quad A^{-1})^{-1} = I_n * A = A$$

$$InvDist := \forall_{A: Invertible[A]} \forall_{B: Invertible[B]} \left( (A * B)^{-1} = B^{-1} * A^{-1} \right)$$

$$(1) \quad (A * B) * (A * B)^{-1} = I \quad \blacksquare \quad B * (A * B)^{-1} = A^{-1} \quad \blacksquare \quad (A * B)^{-1} = B^{-1} * A^{-1}$$

$$InvTrans := \forall_{A: Invertible[A]} \left( (A^T)^{-1} = (A^{-1})^T \right) \quad \blacksquare \quad \Leftarrow$$

$$(1) \quad A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I \quad \blacksquare \quad (A^{-1})^T = (A^T)^{-1}$$

$$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$$

$$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$$

$$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X (Sol[X, A, B])$$

$$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$$

$$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$$

$$HomoSysProps := (Sys[A, O]) \implies \dots$$

$$(1) \quad u_0 := O ; u_1 := choice(\{X \in \mathcal{M} | X \neq O\}) ; k := choice(\mathbb{R})$$

$$(2) \quad TrivSol[u_0, A]$$

$$(3) \quad (NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$$

$$(4) \quad (TrivSol[\vec{X}, A]) \implies (TrivSol[LC(\vec{X}), A])$$

$$ElemMat[E] := (E = Swap[I_n, i, j]) \vee (Scale_*(I_n, i, c)) \vee (Combine_*(I_n, i, c, j))$$

$$ElemMatProd[E^*] := \exists_{\langle E \rangle} \left( \forall_{E_i \in E^*} (ElemMat[E_i]) \wedge \left( E^* = \prod_{E_i \in E^*} (E_i) \right) \right)$$

$$RowEquiv[A, B] := \exists_{E^*} ((ElemMatProd[E^*]) \wedge (B = E^* * A))$$

$$ElemMatInv := \forall_{E \in \mathcal{M}} ((ElemMat[E]) \implies (Invertible[E]))$$

$$(1) \quad E - RowSwap[E] \implies TODO ; E - RowScale_*(E) \implies TODO ; E - RowCombine_*(E) \implies TODO$$

$$ElemMatProdInv := \forall_{E^*} ((ElemMatProd[E^*]) \implies (Invertible[E^*]))$$

$$(1) \quad TODO$$

$$RowEquivSys := \forall_{A, B, C, D, X \in \mathcal{M}} \left( ((Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]])) \implies (Sol[X, A, B] \iff Sol[X, C, D]) \right)$$

$$(1) \quad \exists_{E^*: ElemMatProd[E^*]} ([CD] = E^* * [AB])$$

$$(2) \quad (E^* * A = C) \wedge (E^* * B = D)$$

$$(3) \quad Sol[Y, A, B] \implies \dots$$

$$(3.1) \quad A * Y = B$$

$$(3.2) \quad C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D \quad \blacksquare \text{ Sol}[Y, C, D]$$

$$(4) \quad \text{Sol}[Y, A, B] \implies \text{Sol}[Y, C, D]$$

$$(5) \quad \left( A = (E^*)^{-1} * C \right) \wedge \left( B = (E^*)^{-1} * D \right)$$

$$(6) \quad \text{Sol}[Z, C, D] \implies \dots$$

$$(6.1) \quad C * Z = D$$

$$(6.2) \quad A * Z = \left( (E^*)^{-1} * C \right) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$$

$$(7) \quad \text{Sol}[Z, C, D] \implies \text{Sol}[Z, A, B]$$

$$(8) \quad \text{Sol}[X, A, B] \iff \text{Sol}[X, C, D]$$

$$\text{RowEquivHomoSysSol} := \forall_{A, C, X \in \mathcal{M}} \left( (\text{RowEquiv}[A, C]) \implies ((\text{Sol}[X, A, O]) \iff (\text{Sol}[X, C, O])) \right)$$

$$(1) \quad \text{Set } B = D = O$$

$$\text{RREF}[A] := (A \in \mathcal{M}) \wedge \left( \begin{array}{l} \text{All zero rows are at the bottom of the matrix.} \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} \\ \text{The leading entry in any nonzero row is 1.} \\ \text{All entries in the column above and below a leading 1 are zero.} \end{array} \right)$$

$$\text{GaussJordanElim} := \forall_{A \in \mathcal{M}} \exists!_{B \in \mathcal{M}} ((\text{RREF}[B]) \wedge (\text{RowEquiv}[A, B]))$$

$$(1) \quad \text{Hit } A \text{ with } \text{ElemMat}'\text{s until it becomes } B$$

$$(2) \quad (B = E^* * A) \wedge (\text{RREF}[B])$$

$$\text{HasZero}[A] := (\text{Matrix}(A, m, n)) \wedge (\exists_{i \leq m} (A_{i,:} = O))$$

$$\text{HasZeroNonInvertible} := \forall_{A \in \mathcal{M}} ((\text{HasZero}[A]) \implies (\neg \text{Invertible}[A]))$$

$$(1) \quad i := \text{choice}(\{i \leq m \mid A_{i,:} = O\})$$

$$(2) \quad (B \in \mathcal{M}) \implies \dots$$

$$(2.1) \quad (A * B)_{i,:} = O \neq I_{ni,:} \quad \blacksquare \quad A * B \neq I_n$$

$$(3) \quad (B \in \mathcal{M}) \implies (A * B \neq I_n) \quad \blacksquare \quad \forall_{B \in \mathcal{M}} (A * B \neq I_n) \quad \blacksquare \quad \neg \text{Invertible}[A]$$

$$\text{InvIf f RowEquivI} := \forall_{A \in \mathcal{M}} ((\text{Invertible}[A]) \iff (\text{RowEquiv}[A, I_n]))$$

$$(1) \quad (\text{Invertible}[A]) \implies \dots$$

$$(1.1) \quad (\text{RREF}[B]) \wedge (\text{RowEquiv}[A, B])$$

$$(1.2) \quad B = E^* * A$$

$$(1.3) \quad (\text{Invertible}[E^*]) \wedge (\text{Invertible}[A]) \quad \blacksquare \quad \text{Invertible}[B]$$

$$(1.4) \quad \text{Invertible}[B] \quad \blacksquare \quad \neg \text{HasZero}[B]$$

$$(1.5) \quad (\text{RREF}[B]) \wedge (\neg \text{HasZero}[B]) \quad \blacksquare \quad B = I_n$$

$$(1.6) \quad \text{RowEquiv}[A, I_n]$$

$$(2) \quad (\text{Invertible}[A]) \implies (\text{RowEquiv}[A, I_n])$$

$$(3) \quad (\text{RowEquiv}[A, I_n]) \implies \dots$$

$$(3.1) \quad I_n = E^* * A \quad \blacksquare \quad (E^*)^{-1} = A$$

$$(3.2) \quad A^{-1} = E^*_{\text{DescSort}} \quad \blacksquare \quad \text{Invertible}[A]$$

$$(4) \quad (\text{RowEquiv}[A, I_n]) \implies (\text{Invertible}[A])$$

$$(5) \quad (\text{Invertible}[A]) \iff (\text{RowEquiv}[A, I_n])$$

$$\text{RowEquivIf f TrivSol} := \forall_{A \in \mathcal{M}} \left( (\text{RowEquiv}[A, I_n]) \iff \left( \forall_X ((X = O) \iff (\text{Sol}[X, A, O])) \right) \right)$$

$$(1) \quad (\text{RowEquiv}[A, I_n]) \implies \dots$$

$$(1.1) \quad \text{RowEquiv}[A, I_n] \quad \blacksquare \quad \text{Invertible}[A]$$

$$(1.2) \quad (\text{Sol}[X, A, O]) \implies \dots$$

$$(1.2.1) \quad A * X = O \quad \blacksquare \quad X = A^{-1} * O = O \quad \blacksquare \quad X = O$$

$$(1.3) \quad (\text{Sol}[X, A, O]) \implies (X = O)$$

(1.4)	$(X = O) \implies (Sol[X, A, O])$
(1.5)	$(X = O) \iff (Sol[X, A, O]) \blacksquare \forall_X((X = O) \iff (Sol[X, A, O]))$
(2)	$(RowEquiv[A, I_n]) \implies \left( \forall_X((X = O) \iff (Sol[X, A, O])) \right)$
(3)	$\left( \forall_X((X = O) \iff (Sol[X, A, O])) \right) \implies \dots$
(3.1)	$(RREF[B]) \wedge (RowEquiv[A, B])$
(3.2)	$Sol[X, B, O]$
(3.3)	$(B \neq I_n) \implies \dots$
(3.3.1)	$\left( \exists_{Y \neq X}(Sol[Y, B, O]) \right)$
(3.3.2)	$Sol[Y, A, O] \blacksquare Y = X$
(3.3.3)	$(Y \neq X) \wedge (Y = X) \blacksquare \perp$
(3.4)	$(B \neq I_n) \implies \perp \blacksquare B = I_n$
(3.5)	$(RowEquiv[A, B]) \wedge (B = I_n) \blacksquare RowEquiv[A, I_n]$
(4)	$\left( \forall_X((X = O) \iff (Sol[X, A, O])) \right) \implies (RowEquiv[A, I_n])$
(5)	$(RowEquiv[A, I_n]) \iff \left( \forall_X((X = O) \iff (Sol[X, A, O])) \right)$

$$InvIf fUniqSol := \forall_{A \in \mathcal{M}} \left( (Invertible[A]) \iff \left( \forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}} (Sol[X, A, B]) \right) \right)$$

(1)	$(Invertible[A] \wedge B \in \mathcal{M}) \implies \dots$
(1.1)	$(Invertible[A]) \wedge (Sys[A, B])$
(1.2)	$(X = A^{-1} * B) \iff (Sol[X, A, B]) \blacksquare \exists!_{X \in \mathcal{M}} (Sol[X, A, B])$
(2)	$\left( \forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}} (Sol[X, A, B]) \right) \implies \dots$
(2.1)	$X_i := choice(\{X_i   Sol[X_i, A, I_{n:,i}]\})$
(2.2)	$A * [X_1 \dots X_n] = [(A * X_1) \dots (A * X_n)] = [I_{n:,1} \dots I_{n:,n}] = I_n$
(2.3)	$A^{-1} = [X_1 \dots X_n]$
(3)	$\left( \forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}} (Sol[X, A, B]) \right) \implies (Invertible[A])$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}} \left( \begin{array}{c} (Invertible[A]) \iff \\ (RowEquiv[A, I_n]) \iff \\ \left( \forall_X((X = O) \iff (Sol[X, A, O])) \right) \iff \\ \left( \forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}} (Sol[X, A, B]) \right) \end{array} \right)$$

$$VectorSpace[V, +, *] := \forall_{\alpha, \beta \in \mathbb{R}} \forall_{u, v, w \in V} \exists_{O \in V} \left( \begin{array}{c} (u + v \in V) \wedge (u + v = v + u) \wedge ((u + v) + w = u + (v + w)) \wedge \\ (u + O = u) \wedge \left( \exists_{-u \in V} (u + (-u) = O) \right) \wedge \\ (r * u \in V) \wedge (\alpha * (\beta * u) = (\alpha\beta) * u) \wedge (1 * u = u) \wedge \\ (\alpha * (u + v) = (\alpha * u) + (\alpha * v)) \wedge ((\alpha + \beta) * u = (\alpha * u) + (\beta * u)) \end{array} \right)$$

$$ZeroVectorUniq := \forall_{u, v \in V} ((v + u = v) \implies (u = O))$$

(1)	$(v + u = v) \blacksquare -v + v + u = -v + v \blacksquare u = O$
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$$ZeroVectorGen := \forall_{\alpha \in \mathbb{R}} \forall_{u \in V} \left( (\alpha * u = O) \iff ((u = O) \vee (\alpha = 0)) \right)$$

(1)	$(u = O) \implies \dots$
(1.1)	$\alpha * u = \alpha * O = \alpha * (O + O) = (\alpha * O) + (\alpha * O)$
(1.2)	$\alpha * O = (\alpha * O) + (\alpha * O) \blacksquare \alpha * O = O \blacksquare \alpha * u = O$
(2)	$(u = O) \implies (\alpha * u = O)$
(3)	$(\alpha = 0) \implies \dots$
(3.1)	$\alpha * u = 0 * u = (0 + 0) * u = (0 * u) + (0 * u)$
(3.2)	$0 * u = (0 * u) + (0 * u) \blacksquare 0 * u = O \blacksquare \alpha * u = O$
(4)	$(\alpha = 0) \implies (\alpha * u = O)$
(5)	$((u = O) \implies (\alpha * u = O)) \wedge ((\alpha = 0) \implies (\alpha * u = O)) \blacksquare ((u = O) \vee (\alpha = 0)) \implies (\alpha * u = O)$

$$(6) \quad (\alpha * u = O) \implies \dots$$

$$(6.1) \quad (\alpha \neq 0) \implies \dots$$

$$(6.1.1) \quad \alpha^{-1} \in \mathbb{R}$$

$$(6.1.2) \quad O = \alpha^{-1} * O = \alpha^{-1} * (\alpha * u) = (\alpha^{-1} * \alpha) * u = 1 * u = u \quad \blacksquare \quad u = O$$

$$(6.2) \quad (\alpha \neq 0) \implies (u = O)$$

$$(6.3) \quad (\alpha = 0) \vee (\alpha \neq 0) \quad \blacksquare \quad (\alpha = 0) \vee (u = O)$$

$$(7) \quad (\alpha * u = O) \implies ((\alpha = 0) \vee (u = O))$$

$$(8) \quad (\alpha * u = O) \iff ((\alpha = 0) \vee (u = O))$$

$$NegVectorGen := \forall_{u \in V} ((-1) * u = -u)$$

$$(1) \quad O = 0 * u = (1 + (-1)) * u = (1 * u) + ((-1) * u) = u + ((-1) * u) \quad \blacksquare \quad O = u + ((-1) * u) \quad \blacksquare \quad -u = (-1) * u$$

TODO SUBSPACE p 64

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