MT621M-Elementary Number Theory

Mathematical Preliminaries

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Introduction

In this chapter, we recall some mathematical tools needed to prove theorems and results in number theory. Most statements in mathematics are stated in a form of propositions, theorems or implications which are often made up of the HYPOTHESIS and the CONCLUSION.

These statements are often phrased in the following manner:

- **Conditional**: $P \Rightarrow Q$ which is read as "*P implies Q*" or IF P, then Q.
- **Equivalence**: $P \Leftrightarrow Q$ which is read as "*P if and only if Q*". This means that P is both necessary and sufficient condition for Q.

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Methods of Proof

There are four main methods of proving propositions.

1. Direct Proof
2. Indirect Proof
3. Proof by Contradiction
4. Proof by Mathematical Induction

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Direct Proof

Direct Proof: In this method, we begin with the hypothesis and progress logically using known results or facts and definitions to arrive at the conclusion.

Example

If x, y are odd integers, then $x + y$ is even.

Proof.

Let $x, y \in \mathbb{Z}$. Since, x, y are odd integers, there exists intgers a, b such that $x = 2a + 1, y = 2b + 1$. This gives us,

$$\begin{aligned} x + y &= 2a + 1 + 2b + 1 \\ &= 2(a + b + 1) \end{aligned}$$

Since $x + y = 2(a + b + 1)$, the conclusion follows. $\square$

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Indirect Proof

Indirect Proof: In this method, we use the fact that the conditional statement $P \Rightarrow Q$ is equivalent to its contrapositive that is $\neg Q \Rightarrow \neg P$

Example

If x, y are odd integers, then $x + y$ is even.

Show that if $x + y$ is not even, then exactly one of x or y must be even.

Indirect Proof Example

Proof by Contradiction

Proof by Contradiction: In this method, we use the negation of the conclusion together with our hypothesis to obtain a false statement or contradiction.

Example

Show that there is no largest even integer.

Proof.

Suppose now. That is, suppose there is a largest even integer, say k.Since k is even, there exist $n \in \mathbb{Z}$ such that $k = 2n$. Now, consider $k + 2$. $k + 2 = (2n) + 2 = 2(n + 1)$. So $k + 2$ is even. But $k + 2$ is larger than k. This contradicts our assumption that k was the largest even integer. So our original claim must have been true. $\square$

If x, y are odd integers, then $x + y$ is even: Proof By Contradiction

Proof by Mathematical Induction

Theorem (The Well-Ordering Property)

Every non-empty set of positive integers has a least element.

Mathematical Induction :This method of proof is used when the statement to be proven contains a function of an integer variable.

Definition (First Principle of Mathematical Induction)

Let $S_m, S_{m+1}, \ldots$ be a sequence of statements, where m is a positive integer. Suppose that the following conditions are true:

1. S_m is true
2. If an arbitrary statement S_k is true, then the next statement S_{k+1} is also true.

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Definition (Second Principle of Mathematical Induction)

A set of positive integers which contains the integer 1, and which has the property that if it contains all the positive integers $1, 2, ..., k$, then it also contains the integer $k + l$, must be the set of all positive integers.

An alternative definition for the Second Principle of Mathematical Induction is given below:

Definition (Second Principle of Mathematical Induction)

Suppose that a property $P(n)$ is true for $n = 1$. If $P(n)$ is also true for all $n < k$, where k is an arbitrary integer greater than or equal to 1 implies $P(n)$ is true for $n = k$ then $P(n)$ must be true for all $n \in \mathbb{N}$.

EXERCISES Part 1

1. Show that the sum of two consecutive integers is odd.
2. Prove that if an integer k is odd then k^2 is odd.
3. Use Mathematical induction to prove the following statements
   a) Show that $4^n - 1$ is divisible by 3 for all positive integer n.
   b) Show that for $n \in \mathbb{Z}^+$, the sum of the first n consecutive integers is $\dfrac{n(n+1)}{2}$.
   c) Show that the sum of the first n odd natural numbers is n^2.
   d) Show that for all positive integers n, $3^n > 3n - 1$.
   e) Show that for any natural number n, $n^3 - n$ is divisibe by 3.

EXERCISES Part 2

4. A triangular number is a number that represents the number of dots that can be arranges evenly in an equilateral triangle. Equivalently, the k^{th} triangular number is given by $t_k = \dfrac{k(k+1)}{2}$.
   (a) List the first 6 triangular numbers.
   (b) Prove that an integer n is a triangular number if and only if $8n+1$ is a perfect square.
   (c) Let n be a triangular number. Prove that $(2k+1)2n+t_k$ is also a triangular number.