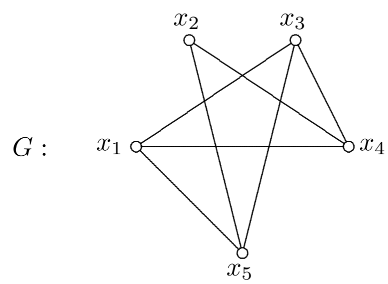
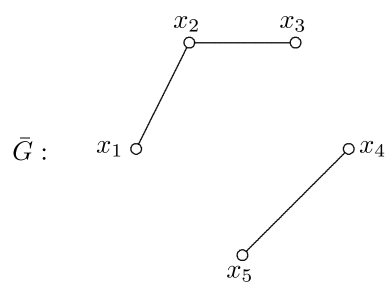
product of two graphs, the composition of two graphs and the conjunction of two graphs. These operations are formally defined as follows.

**Definition 4.1.** Let $G = (V(G), E(G)$ be a graph. The *complement* of $G$, denoted by $\bar{G} = (V(\bar{G}), E(\bar{G}))$, where $V(G) = V(\bar{G})$ and the edge $xy \in E(\bar{G})$ if and only if $xy \notin E(G)$.




The next operation is the rth power of a graph. This operation uses the notion of distance between vertices in a graph $G$. We define the distance between the vertices $x, y$ in $G$, denoted by $d(x, y)$, to be the length of the shortest path between $x$ and $y$, if any; otherwise $d(x, y) = \infty$. We note that in a connected graph, distance is a metric. This means that for any vertices $x, y, z$ of $G$,

1. $d(x, y) \geq 0$, with $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) + d(y, z) \geq d(x, z)$.

**Definition 4.2.** Let $G$ be a graph and $r$ be a positive integer. The *rth power of* $G$, denoted by $G^r$, is obtained by adding the edge $xy$ to the graph $G$ whenever the distance between the vertices $x$ and $y$ of $V(G)$ is less than or equal to $r$.

**Example 4.1.** The $2nd$ power of $C_6$, $C_6^2$ is illustrated in Figure 15.

**Remark 4.1.** Observe that $C_n^r$, reduces to the complete graph $K_n$ whenever $r \geq \frac{n-1}{2}$.

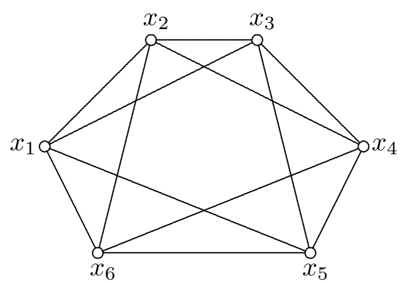



Figure 15: The graph $C_6^2$

**Definition 4.3.** Let $G_1$ and $G_2$ be graphs. The *sum* of $G_1$ and $G_2$, denoted by $G_1 + G_2$ is a graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{xy | x \in V(G_1), y \in V(G_2)\}.$$

**Example 4.2.** Consider the path, $P_2$ and the cycle, $C_3$. Suppose

$$V(P_2) = \{x_1, x_2\},\ E(P_2) = \{x_1x_2\}$$

and

$$V(C_3) = \{y_1, y_2, y_3\},\ E(C_3) = \{y_1y_2, y_2y_3, y_3y_1\}.$$

Then,

$$V(P_2 + C_3) = \{x_1, x_2, y_1, y_2, y_3\}$$

and

$$E(P_2 + C_3) = \{x_1x_2, y_1y_2, y_2y_3, y_3y_1, x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3\}.$$

A pictorial representation of $P_2 + C_3$ is given in Figure 16.

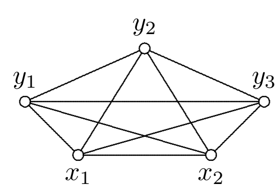



Figure 16: $P_2 + C_3$

**Remark 4.2.** Observe that $G_1 + G_2 \cong G_2 + G_1$.

For the next three operations, we will use the notation $[x, y]$ for the edge $xy$ in the graph $G$.

**Definition 4.4.** Let $G_1$ and $G_2$ be graphs. The *cartesian product of* $G_1$ *and* $G_2$, denoted by $G_1 \times G_2$ is graph with

$$V(G_1 \times G_2) = \{[x, y] | x \in V(G_1), y \in V(G_2)\}$$

and

$E(G_1 \times G_2) = \{[(a,b),(c,d)] |\, \text{either}\, \{a = b\, \text{and}\, [\text{b, d}] \in \text{E}(\text{G}_2)\}\, \text{or}\, \{\text{b} = \text{d}\, \text{and}\, [\text{a, c}] \in \text{E}(\text{G}_1)\}\}$

**Example 4.3.** Consider the path, $P_2$ and the cycle, $C_3$ in Example 4.2. Then the cartesian product of $P_2$ and $C_3$, $P_2 \times C_3$ has as its vertex set

$$V(P_2 \times C_3) = \{(x_1,y_1),(x_1,y_2),(x_1,y_3),(x_2,y_1),(x_2,y_2),(x_2,y_3)\}$$

and edge set

$$\begin{aligned} E(P_2 \times C_3) \quad = \quad & \{[(x_1,y_1),(x_1,y_2)],[(x_1,y_2),(x_1,y_3)],[(x_1,y_3),(x_1,y_1)], \\ & [(x_1,y_1),(x_2,y_1)],[(x_1,y_2),(x_2,y_2)],[(x_1,y_3),(x_2,y_3)], \\ & [(x_2,y_1),(x_2,y_2)],[(x_2,y_2),(x_2,y_3)],[(x_2,y_3),(x_2,y_1)]\} \end{aligned}$$

A graphical representation of $P_2 \times C_3$ is given in Figure 17

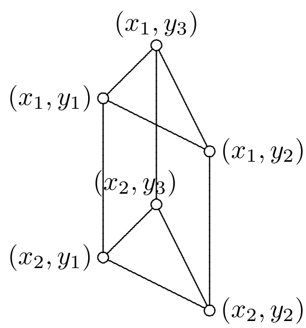



Figure 17: $P_2 \times C_3$

**Definition 4.5.** Let $G$ and $H$ be graphs. The *composition* of $G$ and $H$, denoted by $G \circ H$, is the graph with

$$V(G \circ H) = V(G) \times V(H)$$

and where two vertices $(a,b)$ and $(c,d)$ are adjacent if and only if (1.) $a = c$ and $[b,d] \in E(H)$ or (2.) $[a,c] \in E(G)$.

**Example 4.4.** Consider the cycle $C_4$ with $V(C_4) = \{x_1,x_2,x_3,x_4\}$ and the path $P_2$ with $V(P_2) = \{y_1,y_2\}$. Then the composition $C_4 \circ P_2$ has

$V(C_4 \circ P_2) = \{(x_1,y_1),(x_1,y_2),(x_2,y_1),(x_2,y_2),(x_3,y_1),(x_3,y_2),(x_4,y_1),(x_4,y_2)\}$

and

$$
\begin{aligned}
E(C_4 \circ P_2) \quad = \quad & \{[(x_1, y_1), (x_1, y_2)], [(x_2, y_1), (x_2, y_2)], [(x_3, y_1), (x_3, y_2)], \\
& [(x_4, y_1), (x_4, y_2)], [(x_1, y_1), (x_2, y_1)], [(x_1, y_1), (x_2, y_2)], \\
& [(x_1, y_2), (x_2, y_1)], [(x_1, y_2), (x_2, y_2)], [(x_2, y_1), (x_3, y_1)], \\
& [(x_2, y_1), (x_3, y_2)], [(x_2, y_2), (x_3, y_1)], [(x_2, y_2), (x_3, y_2)], \\
& [(x_3, y_1), (x_4, y_1)], [(x_3, y_1), (x_4, y_2)], [(x_3, y_2), (x_4, y_1)], \\
& [(x_3, y_2), (x_4, y_2)], [(x_4, y_1), (x_1, y_1)], [(x_4, y_1), (x_1, y_2)], \\
& [(x_4, y_2), (x_1, y_1)], [(x_4, y_2), (x_1, y_2)]\}.
\end{aligned}
$$

A pictorial representation of $C_4 \circ P_2$ is given in Figure 18

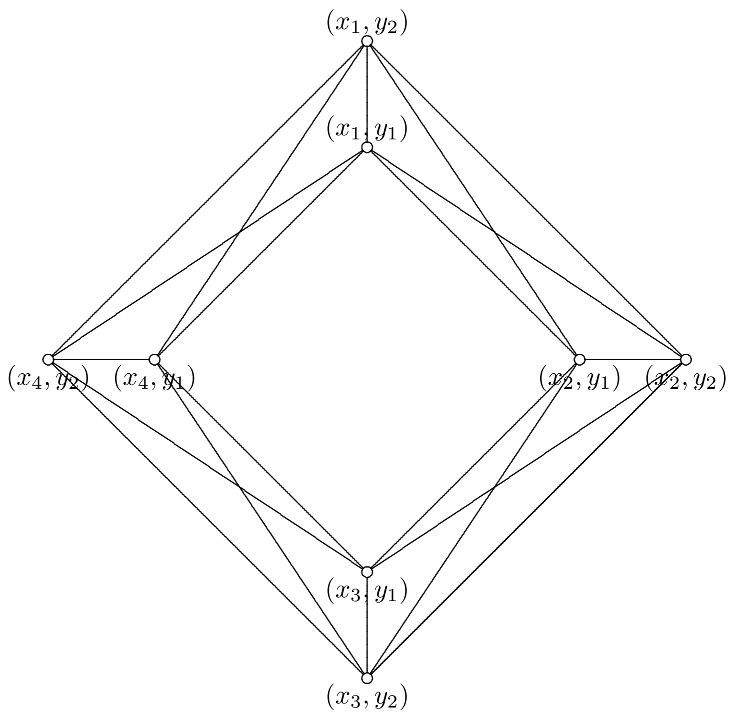



Figure 18: The composition $C_4 \circ P_2$

**Definition 4.6.** Let $G$ and $H$ be graphs. The *conjunction* of $G$ and $H$, denoted by $G \wedge H$ is a graph with

$$V(G \wedge H) = V(G) \times V(H)$$

and

$$E(G \wedge H) = \{[(x_1, x_2), (y_1, y_2)] \mid [x_1, y_1] \in E(G) \, \text{and} \, [x_2, y_2] \in E(H)\}.$$

**Example 4.5.** Consider the cycle $C_4$ with $V(C_4) = \{x_1, x_2, x_3, x_4\}$ and the path $P_2$ with $V(P_2) = \{y_1, y_2\}$. Then the conjunction $C_4 \wedge P_2$ has

$$V(C_4 \wedge P_2) = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2), (x_4, y_1), (x_4, y_2)\}$$

and

$$\begin{aligned} E(C_4 \wedge P_2) \quad = \quad & \{[(x_1, y_1), (x_2, y_2)], [(x_2, y_2), (x_3, y_1)], [(x_3, y_1), (x_4, y_2)], [(x_4, y_2), (x_1, y_1)], \\ & [(x_1, y_2), (x_2, y_1)], [(x_2, y_1), (x_3, y_2)], [(x_3, y_2), (x_4, y_1)], [(x_4, y_1), (x_1, y_2)]\} \end{aligned}$$

A pictorial representation of $C_4 \wedge P_2$ is given in Figure 19

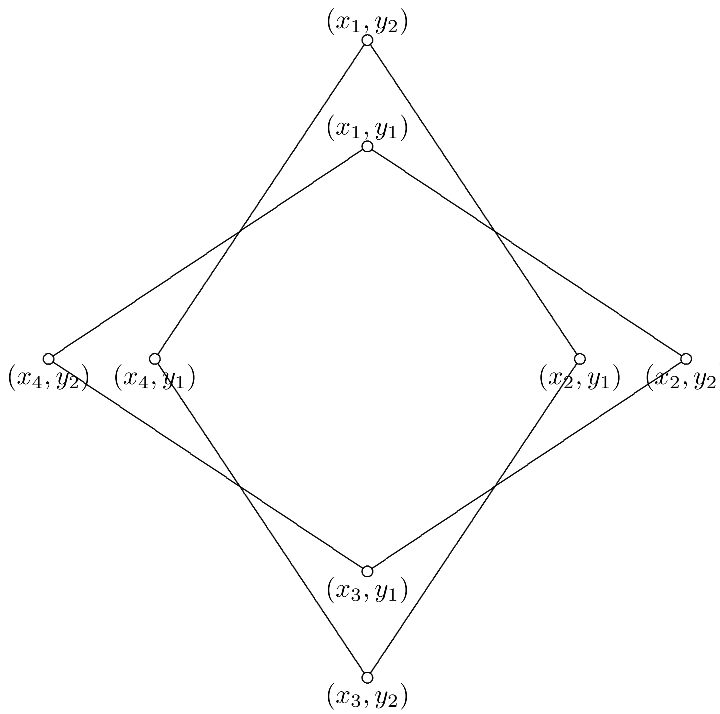



Figure 19: The conjunction $C_4 \wedge P_2$

The adjacency matrix of the conjunction of two graphs can be obtained by performing an operation on matrices called the Kronecker product. Below is its definition and an example.

**Definition 4.7.** Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix and $\mathbf{B} = [b_{ij}]$ be a $p \times q$ matrix. The *Kronecker product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is an $mp \times nq$ matrix defined by

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}.$$

**Example 4.6.** Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix}.$$

Then,

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} 1\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} & 2\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} & 3\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} \\ 4\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} & 5\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} & 6\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 7 & 8 & 14 & 16 & 21 & 24 \\ 9 & 0 & 18 & 0 & 27 & 0 \\ 28 & 32 & 35 & 40 & 42 & 48 \\ 36 & 0 & 45 & 0 & 56 & 0 \end{bmatrix}.$$

**Remark 4.3.** It can be shown that the adjacency matrix of the conjunction given in Example 4.5 can be expressed as

$$\mathcal{A}(C_4 \wedge P_2) = \mathcal{A}(P_2) \otimes \mathcal{A}(C_4).$$

In general if $G$ and $H$ are graphs, then

$$\mathcal{A}(G \wedge H) = \mathcal{A}(H) \otimes \mathcal{A}(G).$$

# 5 More on special classes of graphs

## 5.1 Trees

**Definition 5.1.** A *tree* is a connected graph with no cycles.

**Example 5.1.** Give all distinct trees with orders 1,2,3,4,5,6,7.

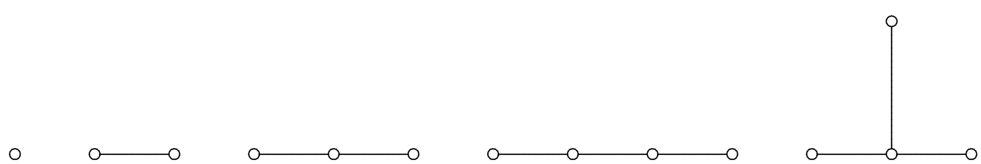


Figure 20: Trees of orders 1, 2, 3, 4

**Remark 5.1.** A graph with no cycles is called an *acyclic* graph. Thus, we can say that a tree is a connected acyclic graph. Furthermore, a graph (not necessarily connected) with no cycles is called a *forest*. This implies that the components of a forest are trees.

**Theorem 5.1.** *Let $G = (V, E)$ be a graph. The following statements are equivalent:*

1. *$G$ is a tree.*