Preliminaries from Set Theory and Logic

Part 1

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Recall: Logical Connectives

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$	$p \oplus q$
T	T	T	T	T	T	F
T	F	F	T	F	F	T
F	T	F	T	T	F	T
F	F	F	F	T	T	F

The most important logical connective in Pure Math: The Mathematicians Arrow "$\Rightarrow$"

1. The Conditional
   1. Usage: $p \Rightarrow q$ means any of the following:
      1. If p, then q.
      2. p implies q.
      3. q if p.
      4. p only if q.
      5. q follows from p.
      6. p entails q.
      7. A sufficient condition for q is p.
      8. A necessary condition for p is q.
   2. Defining property: $p \Rightarrow q$ is false in one and only one combination of values: $p \equiv T$ and $q \equiv F$.
   3. Terminology: Given $p \Rightarrow q$, we have the following names:
      1. for p: *hypothesis, antecedent, premise*
      2. for q: *conclusion, consequent, consequence*

The Disjunctive Form of the Conditional

Theorem 1

For any propositions p, q, we have

$$p \Rightarrow q \equiv \neg p \vee q.$$

[For the proof, truth tables may be used.]

Negation of a Conditional: Use De Morgan's Laws on the Disjunctive Form

Theorem 2

For any propositions p, q, we have

$$\neg(p \Rightarrow q) \equiv p \wedge \neg q.$$

The Converse and the Contrapositive

Definition 3

1. The *converse* of the conditional statement $p \Rightarrow q$ is the statement
$$q \Rightarrow p.$$
2. The *contrapositive* of the conditional statement $p \Rightarrow q$ is the statement
$$\neg q \Rightarrow \neg p.$$

1. A conditional statement is NOT logically equivalent to its converse.
2. A conditional statement is logically equivalent to its contrapositive.

Tautologies and Contradictions

Definition 4

A statement that is always true is called *tautology*.

Corollary: Two propositions are logically equivalent iff their biconditional is a tautology.

Definition 5

A statement that is always false is called a *contradiction*.

Definition 6

A statement that is neither a tautology nor a contradiction is a *contingency*.

Arguments

Definition 7

An *argument* is a compound proposition of the form

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \quad \Rightarrow \quad q$$

where the propositions $p_1, p_2, \ldots, p_n$ are called the *premises* of the argument and the proposition q is called the *conclusion* of the argument.

Notation

We write an argument $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \Rightarrow q$ as

$$\begin{array}{l} p_1 \\ p_2 \\ \quad\vdots \\ p_n \\ \hline \therefore q \end{array}$$

Arguments

Notation

We write an argument $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \Rightarrow q$ as

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

The original expression $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \Rightarrow q$ is called the *propositional form* of the argument.

Arguments

Given the argument:

$$\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ p_4 \\ \hline \therefore q \end{array}$$

Are the following arguments different?

$$\begin{array}{c} p_2 \\ p_1 \\ p_3 \\ p_4 \\ \hline \therefore q \end{array} \qquad \begin{array}{r} p_1 \\ p_2 \wedge p_3 \\ p_4 \\ \hline \therefore q \end{array} \qquad \begin{array}{c} p_1 \\ p_2 \\ p_2 \\ p_2 \\ p_3 \\ p_4 \\ \hline \therefore q \end{array}$$

Valid and Invalid Arguments

Definition 8

An argument is *valid* if its propositional form is a tautology. Otherwise, [if the propositional form is a contigency *or* a contradiction], the argument is said to be *invalid*, or to be a *fallacy*.

Example 9

Fallacy of the Converse [Fallacy of Affirming the Conclusion]

$$\begin{array}{c} q \\ p \Rightarrow q \\ \hline \therefore p \end{array}$$

Some Standard Valid Arguments: The Rules of Inference

Modus ponens: $\dfrac{\begin{array}{c} p \\ p \Rightarrow q \end{array}}{\therefore q}$ Modus tollens: $\dfrac{\begin{array}{c} \neg q \\ p \Rightarrow q \end{array}}{\therefore \neg p}$

Hypothetical syllogism: $\dfrac{\begin{array}{c} p \Rightarrow q \\ q \Rightarrow r \end{array}}{\therefore p \Rightarrow r}$ Disjunctive syllogism: $\dfrac{\begin{array}{c} \neg p \\ p \vee q \end{array}}{\therefore q}$

Addition: $\dfrac{p}{\therefore p \vee q}$ Simplification: $\dfrac{p \wedge q}{\therefore p}$ Conjunction: $\dfrac{\begin{array}{c} p \\ q \end{array}}{\therefore p \wedge q}$

Rule of Conditional Proof (RCP): The arguments $\dfrac{p_1}{\therefore p \Rightarrow q}$ and $\dfrac{\begin{array}{c} p_1 \\ p \end{array}}{\therefore q}$ are the same.

Proving Techniques [based on the properties of the conditional connective]

Standard theorem format: $p \Rightarrow q$.

1. **Direct proof:** Assume p is TRUE. Show that q is TRUE.
2. **Proof by contraposition:** Assume the falsehood of q. Prove the falsehood of p.
    1. $p \Rightarrow q \quad \equiv \quad \neg q \Rightarrow \neg p.$
3. **Proof by contradiction:** Assume the truth of p, and the falsehood of q. Produce a contradiction.
    1. $p \Rightarrow q \quad \equiv \quad (p \wedge \neg q) \Rightarrow F.$
4. **Proof by cases:** If $p \equiv p_1 \vee p_2$ for some propositions p_1, p_2, then prove both $p_1 \Rightarrow q$ and $p_2 \Rightarrow q$.
    1. $(p_1 \vee p_2) \Rightarrow q \quad \equiv \quad (p_1 \Rightarrow q) \wedge (p_2 \Rightarrow q).$
5. **Proof of a biconditional:** There are actually two theorems in $p \Leftrightarrow q$, which are *necessity* $p \Rightarrow q$, and *sufficiency* $q \Rightarrow p$. Both are to be proven.
    1. $p \Leftrightarrow q \quad \equiv \quad (p \Rightarrow q) \wedge (q \Rightarrow p).$
6. **Technique for disjunction in a conclusion:** The negation of one of the statements may be used as a hypothesis. Prove the other.
    1. $p \Rightarrow (q_1 \vee q_2) \quad \equiv \quad (p \wedge \neg q_1) \Rightarrow q_2 \quad \equiv \quad (p \wedge \neg q_2) \Rightarrow q_1.$

Quantifiers

1. *Universal Quantifier*: All, each, any, arbitrary, etc.
2. *Existential Quantifier*: Some, "There exists..."

Example 10

Recall that a sentence of the following form is NOT a proposition:

$$(x+y)^2 = x^2 + y^2,$$

but becomes a proposition when a quantifier is attached:

1. For all real numbers x and y, we find that $(x+y)^2 = x^2 + y^2$.
2. There exist real numbers x and y such that $(x+y)^2 = x^2 + y^2$.

Quantifiers

Rule

To negate a quantified statement, "reverse" the quantifier first before negating the actual statement.

Example 11 (The *Square of Opposition* in Classical Logic)

1. All women are mortal.
2. There exists a woman who is mortal.
3. All women are not mortal.
4. There exists a woman who is not mortal.

In Mathematics, logic statements are about collections of objects

Some notions from Set Theory: Class vs. Set

Undefined terms in [one formulation of] Set Theory: *class*, *element*, and the membership relation $\in$

Definition 12

A *set* is a class that is also an element.

Proposition 13 (Russell's Paradox)

The Russell Class $\{S \; : \; S \notin S\}$ *is not a set.*

Proof.

Suppose otherwise. Then the Russell Class $\mathcal{R}$ is either an element of itself or not.
If $\mathcal{R} \in \mathcal{R}$, then $\mathcal{R}$ satisfies the defining condition $S \notin S$ for the Russell class, i.e., $\mathcal{R} \notin \mathcal{R}$. $\lightning$
If $\mathcal{R} \notin \mathcal{R}$, then $\mathcal{R}$ satisfies the negation $S \in S$ of the defining condition for the Russell class, i.e., $\mathcal{R} \in \mathcal{R}$. $\lightning$ $\square$

Some notions from Set Theory: Proposiitonal Functions and Classes

Definition 14

A *propositional function* in n variables $x_1, x_2, \ldots, x_n$ is a proposition $P(x_1, x_2, \ldots, x_n)$ whose truth value depends on the values of the variables $x_1, x_2, \ldots, x_n$. The collection of all objects from which the values of $x_1, x_2, \ldots, x_n$ will be taken is called the *domain of discourse*, *universe of discourse*, or simply the *universe*.

Axiom 15 (Axiom of Class Construction)

For any propositional function P, the class $\{x \ : \ P(x)\}$ exists.

Symbolic Logic with Quantifiers

Definition 16

The *universal quantification* of the statement $P(x)$ is the statement

$$\forall x \; P(x) \quad : \quad P(x) \text{ is true for all x in the universe of discourse.}$$

The *existential quantification* of the statement $P(x)$ is the statement

$$\exists x \; P(x) \quad : \quad \text{There exists } x \text{ in the universe of discourse such that } P(x) \text{ is true.}$$

Symbolic Logic with Quantifiers: Finite Universe of Discourse

Example 17

Let $S = \{1, 2, 3, 4\}$ denote the universe of discourse.

$$\begin{aligned} P(x) &: \quad x \text{ is even.} \\ \forall x\ P(x) &\Leftrightarrow P(1) \wedge P(2) \wedge P(3) \wedge P(4) \\ \exists x\ P(x) &\Leftrightarrow P(1) \vee P(2) \vee P(3) \vee P(4) \end{aligned}$$

In general, if the universe of discourse is a finite set $\{x_1, x_2, \ldots, x_n\}$,

$$\begin{aligned} \forall x\ P(x) &\Leftrightarrow P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n) \\ \exists x\ P(x) &\Leftrightarrow P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n) \end{aligned}$$

Theorem 18

$$\begin{aligned} \neg\forall x\ P(x) &\Leftrightarrow \exists x\ \neg P(x) \\ \neg\exists x\ P(x) &\Leftrightarrow \forall x\ \neg P(x) \end{aligned}$$

In general, if the universe of discourse is a finite set $\{x_1, x_2, \ldots, x_n\}$,

$$\begin{aligned} \forall x\ P(x) &\Leftrightarrow P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n) \\ \exists x\ P(x) &\Leftrightarrow P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n) \end{aligned}$$

Symbolic Logic with Quantifiers

Theorem 19

$$\begin{aligned} \neg\forall x\ P(x) &\Leftrightarrow \exists x\ \neg P(x) \\ \neg\exists x\ P(x) &\Leftrightarrow \forall x\ \neg P(x) \end{aligned}$$

In general, if the universe of discourse is a finite set $\{x_1, x_2, \ldots, x_n\}$,

$$\begin{aligned} \neg\forall x\ P(x) &\Leftrightarrow \neg P(x_1) \vee \neg P(x_2) \vee \cdots \vee \neg P(x_n) \\ \neg\exists x\ P(x) &\Leftrightarrow \neg P(x_1) \wedge \neg P(x_2) \wedge \cdots \wedge \neg P(x_n) \end{aligned}$$

Non-predicate Clauses

The statement "$x \in A$" (for some x in the universe and some subset A of the universe)

$$\begin{aligned} \forall x \in A \;\; P(x) \quad &\Leftrightarrow \quad \forall x \;\; (x \in A \Longrightarrow P(x)) \\ \exists x \in A \;\; P(x) \quad &\Leftrightarrow \quad \exists x \;\; (x \in A \wedge P(x)) \end{aligned}$$

Rules of Inference for Quantifiers

1. **Existential Instantiation** (EI):
$$\frac{\exists x \; P(x)}{\therefore P(a)}$$
where the symbol a has no previous appearance in the proof
2. **Universal Instantiation** (UI):
$$\frac{\forall x \; P(x)}{\therefore P(a)}$$
3. **Existential Generalization** (EG):
$$\frac{P(a)}{\therefore \exists x \; P(x)}$$
4. **Universal Generalization** (UG):
$$\frac{P(a)}{\therefore \forall x \; P(x)}$$
where the symbol a is not a result of **EI**

Class Containment and Class Equality

Let A and B be classes.

Suppose $x \in A \Longrightarrow x \in B$.
UG: $\forall x\ [x \in A \Longrightarrow x \in B]$.
Rule for non-predicate clauses: $\forall x \in A\ [x \in B]$.

Conversely:
Suppose $\forall x \in A\ [x \in B]$.
Rule for non-predicate clauses: $\forall x\ [x \in A \Longrightarrow x \in B]$.
UI: $x \in A \Longrightarrow x \in B$.

Therefore, $x \in A \Longrightarrow x \in B$ is equivalent to $x \in A \Longrightarrow x \in B$.

Definition 20

$$A \subseteq B \iff [x \in A \Longrightarrow x \in B],$$
$$A = B \iff [A \subseteq B] \wedge [B \subseteq A].$$

Relations

Let A and B be sets, and let a and b be elements.

Definition 21

$$\begin{aligned} (a,b) &:= \{a,\{a,b\}\}, \\ A\times B &:= \{(a,b) \ : \ a\in A,\ b\in B\}. \end{aligned}$$

Definition 22

A *relation* on a set A is a subset of $A\times A$.

Notation: If $\propto$ is a relation on a set A, then we write $a\propto b$ if [and only if] $(a,b)\in\propto$.
If $(a,b)\notin\propto$, then we write $a\not\propto b$.
If $\propto$ is a relation on A, then given $a\in A$, by the Axiom of Class Construction, $\{x \ : \ x\propto a\}$ exists, and $b\propto a$ iff $b\in\{x \ : \ x\propto a\}$.

$$\begin{aligned} \forall x\propto a \ \ P(x) &\iff \forall x \ \ (x\propto a\implies P(x)) \\ \exists x\propto a \ \ P(x) &\iff \exists x \ \ (x\propto a\wedge P(x)) \end{aligned}$$

Functions

We say that $f \subseteq X \times Y$ is a *function of X into Y*, or a *function from X to Y* if [and only if] the following two conditions both hold:

$$\forall x \in X \quad \exists y \in Y \;\; [(x, y) \in f],$$
$$\forall (x_1, y_1), (x_2, y_2) \in f \;\; [x_1 = x_2 \quad \Longrightarrow \quad y_1 = y_2].$$

Functions: Traditional Notation

We say that f is a *function of* X *into* Y, or a *function from* X *to* Y, in symbols $f : X \to Y$, if [and only if] the following two conditions both hold:

$$\forall x \in X \quad \exists y \in Y \;\; [y = f(x)]\,,$$
$$\forall x_1, x_2 \in X \;\; [x_1 = x_2 \quad \Longrightarrow \quad f(x_1) = f(x_2)]\,.$$

Surjectivity:

$$\forall y \in Y \quad \exists x \in X \;\; [y = f(x)]\,.$$

Injectivity:

$$\forall x_1, x_2 \in X \;\; [f(x_1) = f(x_2) \quad \Longrightarrow \quad x_1 = x_2]\,.$$

Notational Conventions on Functions

1. In the symbolic notation $f : X \to Y$, if the name of a function or functions is not relevant, we often write $X \to Y$.
   e.g. Let $\mathbb{R}$ denote the set of all real numbers, and $\mathbb{N}$ the set of all positive integers. Sequences in $\mathbb{R}$ are functions $\mathbb{N} \to \mathbb{R}$.

2. An alternative to the traditional function notation $y = f(x)$ is $x \mapsto f(x)$.
   e.g., Complicated function compositions: Given functions $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$, we shall consider the function $\mathbb{R} \to \mathbb{R}$ defined by the rule

   $$t \mapsto M(\varphi(t, t), v(t)).$$

   Alternative: define $F : \mathbb{R} \to \mathbb{R}$ by $F(t) = M(\varphi(t, t), t)$ for all $t \in \mathbb{R}$.
   If the name of a function is still not relevant: e.g., the square root function is $x \mapsto \sqrt{x}$.

Image and Inverse Image

Consider a function $f : X \to Y$ and let $A \subseteq X$ and $B \subseteq Y$.
The *image* of A under f is $f(A) := \{y \ : \ \exists x \in A \ [y = f(x)]\}$.
The *inverse image* of B under f is $f^{-1}(B) := \{x \in X \ : \ f(x) \in B\}$.
WARNING: $f^{-1}(B)$ exists for any $B \subseteq Y$, even if the inverse function f^{-1} does not [i.e, f is not surjective or not injective].

$$\begin{array}{rcl} A_1 \subseteq A_2 & \Longrightarrow & f(A_1) \subseteq f(A_2), \\ f(A_1 \cup A_2) & = & f(A_1) \cup f(A_2), \\ f(A_1 \cap A_2) & \subseteq & f(A_1) \cap f(A_2), \ \text{injectiveness is required for equality,} \\ (f(A_1))^c & \subseteq & f(A_1^c), \ \text{surjectiveness is required for equality.} \end{array}$$

$$\begin{array}{rcl} B_1 \subseteq B_2 & \Longrightarrow & f^{-1}(B_1) \subseteq f^{-1}(B_2), \\ f^{-1}(B_1 \cup B_2) & = & f^{-1}(B_1) \cup f^{-1}(B_2), \\ f^{-1}(B_1 \cap B_2) & = & f^{-1}(B_1) \cap f^{-1}(B_2), \\ (f^{-1}(B_1))^c & = & f^{-1}(B_1^c). \end{array}$$

Definition 23

A *partial order* on a set A is a relation $\propto$ on A that is:

1. *reflexive*: $\forall a \in A \ [a \propto a]$;
2. *antisymmetric*: $\forall a, b \in A \ [\ [a \propto b \wedge b \propto a] \implies a = b]$; and
3. *transitive*: $\forall a, b, c \in A \ [\ [a \propto b \wedge b \propto c] \implies a \propto c]$.

A *partially ordered set* is a set on which a partial order exists. If $\propto$ is a partial order on A, two elements a and b of A are said to be *comparable* if $a \propto b$ or $b \propto a$. A partially ordered set in which any two elements are comparable is said to be *linearly ordered* or *totally ordered*, and the underlying partial order is called a *total order* or *linear order*.

Suppose $\lhd$ is a partial order on a set A.

1. An element a of A is a *largest element* of A if [and only if] $\forall x \in A \ [a \lhd x \implies x = a]$.
    1. If a is a largest element of A, then $\forall x \in A \ [x \lhd a]$ need not be true. e.g., $\exists c \in A \ [c \not\lhd a]$.
    2. It is possible for a set to have several or no largest element under a given partial order.
2. An element a of A is a *least element* of A if $\forall x \in A \ [x \lhd a \implies x = a]$.
3. If $\emptyset \neq B \subseteq A$, then $a \in A$ is an *upper bound* of B if $\forall b \in B \ [b \lhd a]$.
    1. An upper bound of B need not be an element of B.

Suprema and Infima

Basic Uniqueness Proof

To prove that the element x of the universe of discourse is the unique element with property P, prove that

$$[P(x) \wedge P(y)] \Longrightarrow x = y. \tag{1}$$

Proposition 24

The least upper bound of a subset of a partially ordered set is unique.

Proof.

Let B be a subset of a set A partially ordered by $\lhd$. Let $\mathscr{U}(B)$ be the set of all upper bounds of B. Suppose B has two least upper bounds x and y. Since $y \in \mathscr{U}(B)$ and x is a least element of $\mathscr{U}(B)$, we have $x \lhd y$. Since $x \in \mathscr{U}(B)$ and y is a least element of $\mathscr{U}(B)$, we have $y \lhd x$. By anti-symmetry, we have $x = y$. □

Suprema and Infima

1. *The* least upper bound of a set is called the *supremum* of the set.
2. We have analogous notions for *lower bound* and *greatest lower bound* or *infimum*.

Axiom 25

There exists a total order $<$ on the set $\mathbb{R}$ of all real numbers such that

1. $a < b \Longrightarrow a + c < b + c.$
2. $[a < b \wedge 0 < c] \Longrightarrow ac < bc.$

Other relations $>$, $\leq$ and $\geq$ are defined by

$$\begin{aligned} a > b &\iff b < a, \\ a \leq b &\iff [a < b \vee a = b], \\ a \geq b &\iff [b < a \vee a = b]. \end{aligned}$$

Order Axioms for the Real Number System

Proposition 26

$\forall a \in \mathbb{R}\ [a > 0 \iff -a < 0]$.

Proof.

By UG, a, b, c in the condition $a < b \implies a + c < b + c$ is arbitrary. If we make replacements: $a \mapsto 0$, $b \mapsto a$ and $c \mapsto -a$, we have $a > 0 \implies -a < 0$.

If we make the replacements, $a \mapsto -a$, $b \mapsto 0$ and $c \mapsto a$, we have $-a < 0 \implies a > 0$. □

Proposition 27

$\forall a, b, c \in \mathbb{R}\ [\ [a < b \wedge c < 0] \implies ac > bc]$.

Proof.

Exercise. □