Answer the following as indicated and in the given order. **SHOW YOUR SOLUTIONS.**

1. Answer the following questions. (20 `Points`)

   (a) Given $a*b = 3a(b-a), a, b \in \mathbb{Z}$, does 2 divide $a*b$? Justify your answer.

   (b) Given a group G under the binary operation $*$, does it follow that $a * b = b * a$? Justify your answer.

   (c) If $3|c$ and $c| - 3$, then $c =$_____.

   (d) How many possible remainders are there if a positive integer is divided by a natural number $n + 1$?

   (e) If $\sim$ is an equivalence relation on a set S, it follows that _____.

   (f) Given the relation congruence modulo 8, in which congruence class does the integer 341 belong?

   (g) If $f : X \to Y$ and $g : Y \to Z$ are functions, find the domain of $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

   (h) Given $f(x) = \tan x$, is f surjective? Justify your answer.

   (i) If the order of the group G under $*$ is finite, how many times will you find the identity element e of G in a Cayley table?

   (j) If $G = \mathbb{Z}^+$ is a monoid under the binary operation $*$ and it was determined that $a^{-1} = \sqrt{a}$, is G a group?

2. Find $(1239, 735)$ using the Euclidean algorithm, and find x, y such that $(1239, 735) = 1239x + 735y$. (10 `Points`)

3. On the set of integers $\mathbb{Z}$, define $x \sim y$ if and only if $3x - 7y$ is even. Prove that $\sim$ is an equivalence relation. Determine the distinct equivalence classes. (12 `Points`)

4. Use the properties of groups to complete the following group table . (8 `Points`)

$*$	a	b	c	d
a			d	
b	a			
c			b	
d				

5. Write the Cayley table for $\mathbb{Z}_5 \backslash \{0\}$ under multiplication modulo 5 and determine the inverse of each element. (7 `Points`)

6. Determine if the given set is a group with respect to the operation *:

   (a) G = set of all positive rational numbers, $a * b = \dfrac{1}{a} + \dfrac{1}{b}$.

   (b) G = the set of all real numbers greater than 1, $a * b = ab - a - b + 2$.

   (15 `Points`)

7. Show that every group G with identity e and such that $x * x = e$ for all $x \in G$ is abelian. [Hint: Consider (a*b)*(a*b).] (5 `Points`)

BONUS. Show that if G is a group, then each element of G has a unique inverse. (3 `Points`)