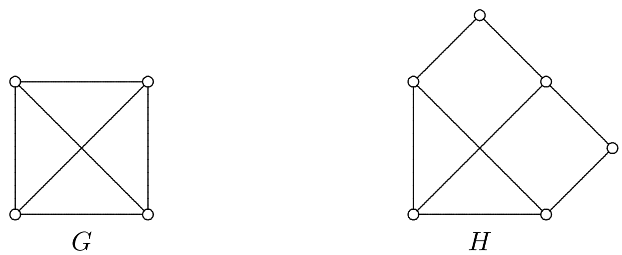

Figure 24: Subdivision H of G

6 Connectivity

6.1 Cutpoints, bridges and blocks

Definition 6.1. Let G be a graph. The vertex v of G is a *cutpoint* of G, when its removal increases the number of components of G. The edge e of G is a *bridge*, when its removal increases the number of components of G.

Example 6.1. Consider the graph in Figure 25

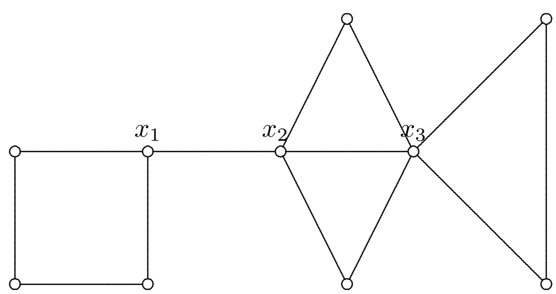

Figure 25: Example of a cutpoint and a bridge

The vertices x_1, x_2 and x_3 are cutpoints and the edge x_1x_2 is a bridge.

Theorem 6.1. *Let v be a vertex of a connected graph $G = (V, E)$. The following statements are equivalent:*

1. *v is a cutpoint of G;*
2. *There exists a partition of the set $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the point v is on every $u - w$ path.*
3. *There exists vertices u and w different from v, such that v is in every $u - w$ path;*

Proof:

- $\{(1) \implies (2)\}$ Suppose v is a cutpoint of G. Then the graph $G - v$ is a disconnected graph with two components. Let these components have as set of vertices U and W. Clearly, $\{U, W\}$ is a partition of $V = V(G)$. Then, if we consider a vertex $u \in U$ and a vertex $w \in W$, any u-w vertex in G will contain the vertex v.

- $\{(2) \implies (3)\}$ (3) immediately follows from (2), as a special case of (2).
- $\{(3) \implies (1)\}$ Suppose v is in every $u - w$ path in G, then there can not be a $u - w$ path in $G - v$. Thus, v is a cutpoint. $\square$

Theorem 6.2. *The e be an edge of a connected graph $G = (V, E)$. The following statements are equivalent:*

1. *e is a bridge of G;*
2. *e is not on any cycle of G;*
3. *There exists a partition of V into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the edge e is on every $u - w$ path.*
4. *There exists vertices u and v in G such that the edge e is on every $u - v$ path;*

Proof: *Exercise*

6.2 Point-connectivity and line-connectivity

Definition 6.2. Let G be a graph. The *point-connectivity* of G denoted by $\kappa(G)$ is the minimum number of vertices whose removal results in a disconnected or trivial graph.

Remark 6.1. We take note of the following:

1. If G is disconnected, then $\kappa(G) = 0$;
2. If G is connected with a cutpoint, then $\kappa(G) = 1$;
3. $\kappa(K_p) = p - 1$

Example 6.2. The graph is Figure 25 has $\kappa(G) = 1$, since the removal of any of the vertices x_1, x_2 or x_3 will result to a disconnected graph. Consider the graph in Figure 26.

Definition 6.3. Let G be a graph. The *line-connectivity* of G denoted by $\lambda(G)$ is the minimum number of edges whose removal results in a disconnected or trivial graph.

Remark 6.2. The following are true:

1. $\lambda(K_1) = 0$;
2. If G is a connected graph with a bridge, then $\lambda(G) = 1$.

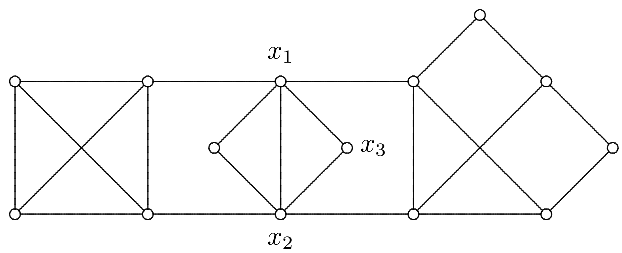

Figure 26: A graph with $\kappa(G) = 2$

Example 6.3. The graph in Figure 25 has $\lambda(G) = 1$, because it is connected with a bridge x_1x_2. The graph is Figure 26 is with $\lambda(G) = 2$. The edges x_1x_3 and x_2x_3 will disconnect the graph and two edges is the minimum number of edges that will make G disconnected. Also for the graph in Figure 26, $\kappa(G) = 2$. The removal of the vertices x_1 and x_2 will result to a disconnected graph.

Theorem 6.3. *For any graph G,*

$$\kappa(G) \leq \lambda(G).$$

Proof: Suppose if G is disconnected or trivial, then $\lambda(G) = \kappa(G) = 0$. Suppose G is connected with a bridge, the edge e. Thus, $\lambda(G) = 1$. Clearly, if either vertex which is incident to e is a cutpoint and thus if removed, we obtain a disconnected graph, thus $\kappa(G) = 1$. In both cases, $\kappa(G) \leq \lambda(G)$. Suppose $\lambda(G) = \lambda \geq 2$. We have λ edges whose removal from G produces a disconnected graph. However, we first remove $\lambda - 1$ edges. This produces a graph still connected but with a bridge, say $e = vw$. For each of the first $\lambda - 1$ edges, we remove a vertex incident to it, but different from either v or w. We note that the removal of these $\lambda - 1$ vertices will also result in the removal of at least $\lambda - 1$ edges. Two situations may happen. Either the resulting graph is disconnected or it has a bridge $e = uv$. If the former occurs, then $\kappa(G) < \lambda(G)$; if the latter occurs, then we remove one of the vertices u and v to obtain a disconnected graph and so $\kappa(G) = \lambda(G)$. Therefore, $\kappa(G) \leq \lambda(G)$. □

Definition 6.4. Let G be a graph. G is said to be *n-point-connected* if $\kappa(G) \geq n$, and G is said to be *n-line-connected* if $\lambda(G) \geq n$.

7 Some Graph Invariants

7.1 Introduction

We say that property P of a graph G is graph invariant if the property P is preserve by a graph isomorphism. In other words, if G has property P and $\phi : V(G) \longrightarrow V(G')$ is a graph isomorphism, then the graph G' has property P also. Graph invariants are non-negative numbers associated with a graph describing a property of the graph. The point-connectivity $\kappa(G)$ and the line-connectivity $\lambda(G)$ of a graph G are graph invariants.

7.2 Independence Number and Dominance Number

Definition 7.1. An *independent (stable) set* S of a graph G is a subset of $V(G)$ where no two elements of S are adjacent. The *independence (stability) number* of G, denoted by $\alpha(G)$, is defined by:

$$\alpha(G) = \max\{|S| : S \text{ is an independent set in } G\}$$

Example 7.1. The complete bipartite graph $K_{m,n}$ has a set of vertices which can be partitioned into two subsets, say V_1 and V_2, with cardinality m and n respectively. Note that none of the vertices in V_1 are adjacent, nor are the vertices in V_2. Thus, V_1 and V_2 are independent sets and that $\alpha(K_{m,n}) = \max\{m, n\}$. Moreover, for any bipartite graph G whose vertex set is partitioned by $\{V_1, V_2\}$, then $\alpha(G) = \max\{|V_1|, |V_2|\}$.

Definition 7.2. A *dominating set* S of a graph G is a subset of $V(G)$ where every element of $V(G) - S$ is adjacent to at least one element of S. The *dominance number* of G, denoted by $\beta(G)$, is defined by:

$$\beta(G) = \min\{|S| : S \text{ is a dominating set in } G\}$$

Remark 7.1. A dominating set S of G with $|S| = \beta(G)$ is called a *minimal dominating set* of G.

Example 7.2. Recall the graph in Figure 25, the set $S = \{x_1, x_3\}$ is a dominating set of G and there are no dominating set containing a single vertex of G. Thus, $\beta(G) = 2$.

Theorem 7.1. *A connected graph G of order at least two contains two disjoint dominating sets.*

Proof: Let S be a minimal dominating set. Consider the set $V - S$, where $V = V(G)$. Clearly S and $V - S$ are disjoint. We need to show that $V - S$ is a dominating set of G, that is every vertex $v \in S$ is adjacent to at least one element of $V - S$. However, this statement follows from the fact that S is a minimal dominating set of G. Thus, the disjoint sets of vertices S and $V - S$ of $V(G)$ are both dominating sets of G $\square$

Corollary 7.1.1. *Let G be a connected graph G of order n. If $n \geq 2$, then $\beta(G) \leq \dfrac{n}{2}$.*

Proof: Let S be a minimal dominating set of G. Then, $\beta(G) = |S|$. However, $V - S$ is also a dominating set of G and that $|V - S| \geq |S|$ and $|S| + |V - S| = n$.

$$n = |S| + |V - S| = \beta(G) + |V - S| \geq \beta(G) + |S| = \beta(G) + \beta(G) = 2\beta(G).$$

This gives, $\beta(G) \leq \frac{n}{2}$. $\square$

Theorem 7.2. *Let G be a graph and let S be an independent set in G. Then S is a maximal independent set if and only if S is a dominating set.*

Proof:$\{\Rightarrow\}$ Suppose S is a maximal independent set of G. Let $v \in V - S$, then the set $S \cup \{v\}$ is not an independent set. This implies that v is adjacent to at least one element in S. Thus, every element in $V - S$ is adjacent to at least one element in S. This makes S is a dominating set of G.

$\{\Leftarrow\}$ Let S be a dominating set of G. By hypothesis, S is also an independent set of G. We need to show that S is a maximal independent set of G. Let $v \in V - S$. Since S is a dominating set of G, then v is adjacent to at least one vertex in S. If we form the set $S \cup \{v\}$, this set is clearly not an independent set. This shows that adding a vertex v to S, $S \cup \{v\}$ fails to be an independent set. Thus, S is a maximal independent set. $\square$

Corollary 7.2.1. *For any graph G, $\beta(G) \leq \alpha(G)$.*

Proof: Let S be a maximal independent set and is also a dominating set. Thus,

$$\beta(G) \leq |S| = \alpha(G).$$

7.3 The chromatic number and colorings of graphs

Definition 7.3. Let G be a graph. A *k-coloring* of G is a mapping $\Lambda : V(G) \to \{1, 2, \ldots k\}$, $k > 0$ satisfying the following condition:

$$xy \in E(G) \Rightarrow \Lambda(x) \neq \Lambda(y).$$

In order words, no two adjacent vertices in G are assigned the same color. We wish to determine the minimum number of colors that can be used in a graph.

Definition 7.4. The *chromatic number* of a graph G, denoted by $\chi(G)$ is the least integer λ such that there exists a λ-coloring of G.

Example 7.3. Determine the chromatic number of the following special classes of graphs: P_n, C_n, K_n, S_n, W_n, $K_{m,n}$.

Theorem 7.3. *A graph G has chromatic number $\chi(G) = 2$ if and only if G is a nonempty bipartite graph.*

Proof: $\{\Rightarrow\}$ Suppose $\chi(G) = 2$. Let $C = \{1, 2\}$ be the set of colors of G. Define $V_1 = \{v \in V(G) | \Lambda(v) = 1\}$ and $V_2 = \{v \in V(G) | \Lambda(v) = 2\}$. Clearly the set $\{V_1, V_2\}$ is a partition of $V(G)$ and so G is a nonempty bipartite graph.

$\{\Leftarrow\}$. Suppose G is a nonempty bipartite graph. Let $\{V_1, V_2\}$ be the partition of $V(G)$. We color the vertices in V_1 by color 1 and the vertices of V_2 by color 2. This satisfies the requirement that adjacent vertices are assigned different colors and two is the minimum number of colors we can use. Thus, $\chi(G) = 2$. $\square$

On the "Four Color Conjecture"

"Any map on a plane or a surface of a sphere can be colored with only four colors so that no two adjacent countries (countries sharing a common boundary) have the same color. Each country is assumed to consist of a single connected region."

Another version is:

" The chromatic number of every planar graph is at most 4".

A graph-theoretic version of this theorem is as follows: Each country or region is represented by a vertex and two vertices are joined by an edge whenever the corresponding countries are adjacent. (We can see that the graph obtained is planar.) We only need four colors to color any map so that adjacent countries are given different colors.

This conjecture has been proven using computer programs in 1976 by Appel and Haken after four years of work. The program used 1,200 hours of computer time. In 1994, another group of mathematicians led by Gonthiers established another proof, using a computer program too. A computer-program-free proof is still nonexistent.

8 Graph valued functions

In this chapter, we consider functions whose domain and co-domain is the set of all graphs, $\mathcal{G}$. Thus, we consider functions of the form $f : \mathcal{G} \to \mathcal{G}$. We will discuss the intersection graphs, line graphs and clique graphs.

Definition 8.1. Let $S = \{S_1, S_2, \ldots, S_n\}$ be a finite collection of nonempty sets. The *intersection graph of* S is the graph G, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ where $v_i = S_i$, $i = 1, 2, \ldots, n$; and $v_iv_j \in E(G)$, if and only if $S_i \cap S_j \neq \phi$. A graph G is said to be an intersection graph if there exists a collection S whose intersection graph is G.

Example 8.1. Let $S = \{S_1, S_2, S_3, S_4, S_5\}$, where $S_1 = \{a, e, i, o, u\}$, $S_2 = \{d, o, g\}$, $S_3 = \{c, a, t\}$, $S_4 = \{b, e, a, r\}$ and $S_5 = \{t, e, a\}$. The intersection graph on S is given in Figure 27.

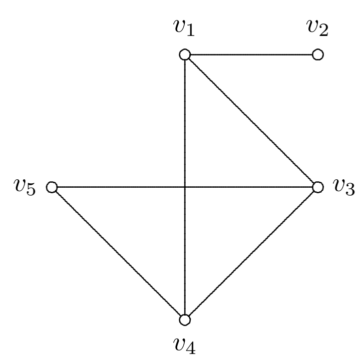

Figure 27: The intersection graph on S

Example 8.2. The null graph of order n, N_n is an intersection graph of a collection S of n disjoint sets.

Example 8.3. If S is a finite collection of closed intervals in $\mathbb{R}$, then the intersection graph of S is called an *interval graph*.

Definition 8.2. Let G be a nontrivial graph. The *line graph of* of G, denote by $L(G)$, is a graph whose vertex set is $V(L(G)) = E(G)$ and where two vertices are adjacent if the edges they represent are incident with a common vertex.

Example 8.4. Consider the star S_4. We note that $V(S_4) = \{x_0, x_1, x_2, x_3, x_4\}$ and edge set $E(S_4) = \{e_i = x_0x_i | i = 1, 2, 3, 4\}$. The line graph of S_4 is the graph $L(S_4)$, with

$$V(L(S_4)) = \{e_1, e_2, e_3, e_4\}$$

and

$$E(L(S_4)) = \{e_1e_2, e_1e_3, e_1e_3, e_1e_4, e_2e_3, e_2e_4, e_3e_4\}.$$

See Figure 28.

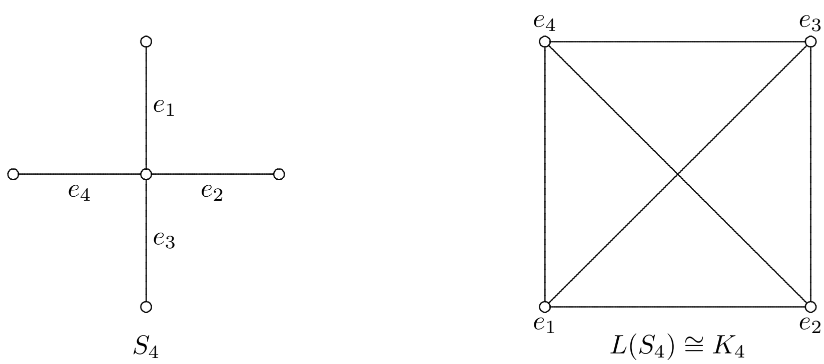

Figure 28: The graph S_4 and its line graph $L(S_4) \cong K_4$

Remark 8.1. The line graph $L(G)$ of the graph G may be viewed as the intersection graph of S, where the elements of the collection S are two-element sets whose elements are the vertices of each edge of G.

Definition 8.3. Let G be a graph. A *clique* of G is a complete subgraph of G. A *maximum clique* of G is a clique which is not properly contained in a larger clique of G. A *maximum clique* of G is a clique in G of maximum order. The order of a maximum clique of G, is called the *clique number* of G, and is denoted by $\omega(G)$.

Example 8.5. Consider the graph in Figure 26. The cliques are K_2, K_3 and K_4. Thus, $\omega(G) = 4$.

Definition 8.4. The G be a graph. Let S be a collection of sets, where each $S_i \in S$ is the set of vertices of a maximal clique in G. Then, the intersection graph of S is called the clique graph of G.

Example 8.6. Let G be the graph given in Figure 29. We want to find its clique graph.

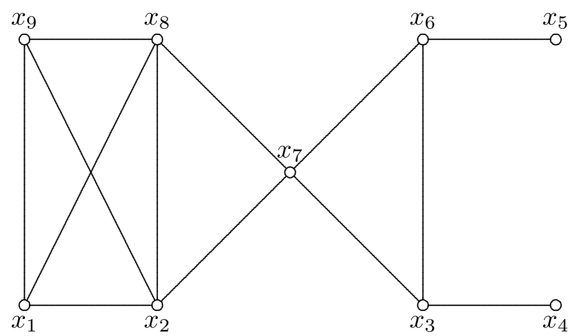

Figure 29: A graph G

The maximal cliques of G are given in Figure 30

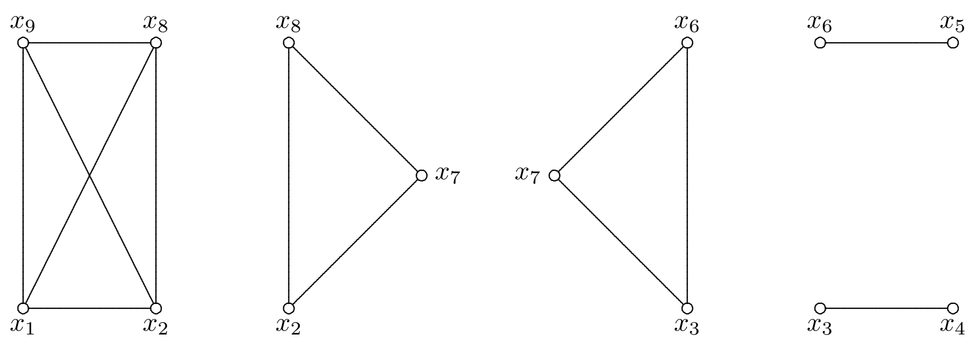

Figure 30: The maximal cliques of G

Thus, $S = \{S_1, S_2, S_3, S_4, S_5\}$, where $S_1 = \{x_1, x_2, x_8, x_9\}$, $S_2 = \{x_2, x_7, x_8\}$, $S_3 = \{x_7, x_3, x_6\}$, $S_4 = \{x_3, x_4\}$ and $S_5 = \{x_6, x_5\}$. Thus, letting $S_i = v_i$, the clique graph of G is given Figure 31.

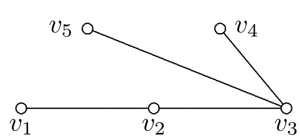

Figure 31: The clique graph of G