Remark 5.1. A graph with no cycles is called an *acyclic* graph. Thus, we can say that a tree is a connected acyclic graph. Furthermore, a graph (not necessarily connected) with no cycles is called a *forest*. This implies that the components of a forest are trees.

Theorem 5.1. *Let $G = (V, E)$ be a graph and $p = |V|$ and $q = |E|$. The following statements are equivalent:*

1. *G is a tree.*
2. *For every pair of distinct vertices u and v in G, there is exactly one path from u to v;*
3. *G is connected and $p = q + 1$.*
4. *G is acyclic and $p = q + 1$.*
5. *G is acyclic and if any two nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle.*
6. *G is connected, is not K_p for $p \geq 3$, and if any two nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle.*
7. *G is not $K_3 \cup K_1$ or $K_3 \cup K_2$, $p = q + 1$, and if any nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle.*

Proof:

- $\{(1) \implies (2)\}$ Suppose G is a tree. Thus, G is connected. Let u and v be distinct vertices in G and P_1 and P_2 be two distinct $u - v$ paths in G. Starting with the initial vertex u, and since the paths are distinct, there is a vertex w (this maybe u itself) in P_1 and P_2 whose successor are two different vertices, say x_1 and x_2. Thus we have P_1, the path $v, \ldots, w, x_1, \ldots v$ and P_2, the path $u, \ldots, w, x_2, \ldots, v$. Clearly, this will form a cycle as these two paths will meet at another vertex, say y (this could be v). This contradicts the assumption that G is a tree and thus, do not contain any cycle.

- $\{(2) \implies (3)\}$ Suppose every pair of distinct vertices u and v in G is in exactly one $u - v$ path. This implies that G is connected. We show that $p = q + 1$ by induction. If $p = 1$, clearly $q = 0$ (that is one vertex and zero edge) and if $p = 2$, then $q = 1$ (that is two vertices and one edge). Let $p = q + 1$ be true when $p = k \in \mathbb{Z}$. Assume that $k = q + 1$ is true for all graphs of order k and size q with $k < p$. With a graph G of order p, we remove an edge. This together with the assumption will make G disconnected and having two components. Let these components be G_1 of order k_1 and size q_1 and G_2 of order k_2, and size q_2, with $k_1 > 0$ and $k_2 > 0$. Clearly $k_1 + k_2 = p$ and $k_1 < p$ and $k_2 < p$. Also, $q_1 + q_2 + 1 = q$. Thus, since G_1 and G_2 are of order less that p, the equations $k_1 = q_1 + 1$ and $k_2 = q_2 + 1$ are true. Therefore,

$$p = k_1 + k_2 = (q_1 + 1) + (q_2 + 1) = (q_1 + q_2 + 1) + 1 = q + 1.$$

- $\{(3) \implies (4)\}$ Suppose G is connected and $p = q + 1$. We need to show that G is acyclic. Suppose G is not acyclic. Thus it contains a cycle. Let this cycle contain n vertices and of course all n edges. Each of the remaining $p - n$ vertices is adjacent to another vertex on a geodesic to a vertex on the said cycle. Each of these edges are different. Thus, the number of edges of G is at least $n + p - n = p$, that is $q \geq p$. This contradicts the assumption that $p = q + 1$.

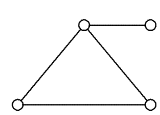
- $\{(4) \implies (5)\}$ Suppose G is acyclic and $p = q + 1$. Suppose G has k components, then each of these components is a tree. For $i = 1, 2, \dots k$, let p_i and q_i be the order and size, respectively of the k components. Since each component is a tree (and is thus connected), $p_i = q_i + 1$, is true for $i = 1, 2, \dots k$. We then have,

$$p = \sum_{i=1}^{k} p_i = \sum_{i=1}^{k} (q_i + 1) = q + k.$$

  But by assumption, $p = q + 1$, thus $k = 1$ and G is connected and is a tree. Thus, for every distinct pair of vertices u and v in G there is a unique $u - v$ path. If we add the edge $e = uv$ to G, (this is $G + e$) we form a cycle, and this cycle is unique because of the uniqueness of the $u - v$ path.

- $\{(5) \implies (6)\}$ Suppose G is acyclic and if any two nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle. We need to show that G can not be $K_p, p \geq 3$ and G is connected. Clearly G can not be $K_p, p \geq 3$ since K_p contains a cycle and G is assumed to be acyclic. Furthermore, G must be connected since if G contains two components say G_1 and G_2, then we can add an edge $e = x_1 x_2$ where $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$ then G remains acyclic. This contradicts the assumption that $G + e$ should have exactly one cycle.

- $\{(6) \implies (7)\}$ Suppose G is connected, is not K_p for $p \geq 3$, and if any two nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle. Since G is connected, then there is path connecting every pair of vertices. Suppose there are two paths connecting the same pair of vertices. Then, from the proof of $\{(1) \implies (2)\}$, the is a cycle in G. However, note that this cycle in G can not have more than three vertices, since if this were true, adding an edge e incident to two nonadjacent vertices in the cycle produces $G + e$ containing two cycles. Thus, the cycle must be K_3 and this is a proper subgraph of G, since it is assumed that G is not $K_p, p \geq 3$. This implies that there is at least one vertex adjacent to one of the vertices of K_3, since G is connected. Note that G is now



Clearly, that if any edge e is added to G , then one may be added so as to form two cycles in $G+e$. If no more edges maybe added we have formed $K_p, p \geq 3$. This contradicts the hypothesis, thus every pair of vertices in G are joined by a unique path and by $\{(2) \implies (3)\}$, $p = q + 1$. We note that G should contain K_3 as a proper subgraph, satisfy $p = q + 1$ and is connected. Thus it can not be $K_3 \cup K_1$ or $K_3 \cup K_2$

- $\{(7) \implies (1)\}$ Suppose G is not $K_3 \cup K_1$ or $K_3 \cup K_2$, $p = q + 1$, and if any nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle. Suppose G contains a cycle. Then from the argument above, this cycle must be K_3, with three vertices and three edges. Since $p = q + 1$, G contains another component and this component must be a tree. If the other component is a path of on three vertices with two edges, and adding an edge to G to form $G + e$ with result in a graph with two cycles. A contradiction to the hypothesis and thus G can only be either $K_3 \cup K_1$ or $K_3 \cup K_2$. These are the graphs excluded. Thus, G is acyclic. But then $p = q + 1$ as well, so since $\{(3) \implies (4)\}$ and $\{(4) \implies (5)\}$, then G is connected as well. Therefore, G is a tree. □

Remark 5.2. A graph $G = (V, E)$ with $|V| = 1$ and $|E| = 0$ is called a *trivial graph.*

Definition 5.2. Let G be a graph and v be a vertex of G. If deg $v = 0$, then we say that v is an *isolated vertex* and if deg $v = 1$, then we say that v is an *endpoint* of G.

Corollary 5.1.1. *Every nontrivial tree has at least two endpoints.*

Proof: We note that if $G = (V, E)$ where $V = \{x_1, x_2, \ldots, x_p\}$ and $|E| = q$. Suppose G is a tree, then G is connected and $p = q + 1$. Thus,

$$\sum_{i=1}^{p} deg(x_i) = 2q = 2(p - 1) = 2p - 2.$$

This implies that there are at least two vertices with degree less than 2. Since G is connected, then these vertices are of degree 1. So these vertices are endpoints.

5.2 Eulerian graphs and hamiltonian graphs

Definition 5.3. A *trail* in the graph G is a walk in which all lines are distinct. A closed trail in G is an *eulerian trail* that contains all vertices and edges of G. A graph G that contains an eulerial trail is called an *eulerian graph.*

Example 5.2. Consider the graph in Figure 22 below. The trail

$$x_1e_1x_2e_2x_3e_3x_4e_4x_5e_5x_3e_6x_6e_7x_1$$

is an eulerian trail. Thus, the graph is Figure 22 is an eulerian graph.

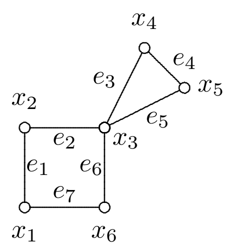


Figure 22: Example of Eulerian graph

Theorem 5.2. *The following statements are equivalent to any connected graph G*

1. *G is eulerian;*
2. *Every vertex in G has even degree;*
3. *The set of edges in G can be partitioned into cycles.*

Proof:

- $\{(1) \implies (2)\}$ Suppose G is eulerian thus, G contains an eulerian trail. Let T be this closed trail, then every occurrence of a vertex in T, contributes two units to the degree of that vertex. Also, since each edge in G occurs only once in T, then every vertex in G has even degree.
- $\{(2) \implies (3)\}$ Suppose every vertex in G is of even degree. Since G is connected and each vertex of degree at least 2, G contains a cycle. Let this cycle be C_1. The removal of the edges in C_1 from G gives a spanning subgraph of G, say G_1. We note that the degree of the vertices in G_1 are still even (Some of the vertices may be of degree zero!). We continue this process to get cycles C_2, C_3,...C_n, of spanning subgraphs G_2, G_3,..., G_n respectively, until a totally disconnected graph G_n is obtained. The set of cycles $\{C_1, C_2, \ldots, C_n\}$ is a partition of the edges of G.
- $\{(3) \implies (1)\}$ Let the set of cycles $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ be a partition of the edges of G. If there is only one cycle in the partition, then G is eulerian. Otherwise, if C_i is a cycle in $\mathcal{C}$ there is another cycle $C_j \in \mathcal{C}$, $i \neq j$ which has a common vertex, x with C_i. Then, the walk starting at x containing the edges of the cycles C_i and C_j in succession is a closed trail containing all the edges of these two cycles. We continue this process, until we obtain a closed trail in G containing all edges of G and each edge appearing only once in the trail. Thus, G is eulerian.

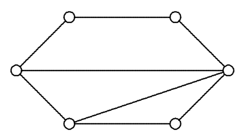
Remark 5.3. Theorem 5.2 suggests that if a connected graph G has no vertex of odd degree, then G contains a trail consisting of all vertices and edges of G.

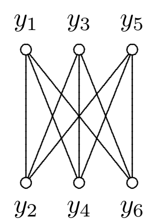
Corollary 5.2.1. *Let G be a connected graph with exactly $2n$ vertices of odd degree, $n \geq 1$. Then, the set of edges of G can be partitioned into n open trails.*

Corollary 5.2.2. *Let G be a connected graph with exactly two vertices of odd degree. Then G has an open trail containing all points and edges of G (which begins at one of the vertices of odd degree and ends at the other).*

Definition 5.4. Let G be a graph. If G has a spanning cycle, then G is called a *hamiltonian graph*. Suppose Z is a spanning cycle of G, then Z is called a *hamiltonian cycle*.

Example 5.3. The following graphs are hamiltonian





5.3 Planar graphs

Definition 5.5. A graph is said to be *embedded* in a surface S when it is drawn on S so that no two edges intersect. A graph is *planar* if it can be embedded in the plane.

Example 5.4. The complete graph K_4 is planar because it can be drawn in the plane where no two edges intersect.

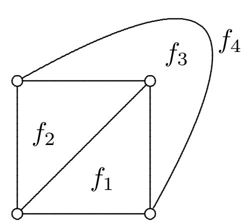


Figure 23: An embedding of K_4 in the plane and its faces

Definition 5.6. The regions defined by the plane graph as called its *interior faces* and the unbounded region is its *exterior face*.

Remark 5.4. If G is a tree, then G is planar and the number of faces of G is 1. The boundary of an interior face is the set of edges surrounding it. Every edge is a boundary of two faces.

Theorem 5.3. *If a connected planar graph G has p vertices, q edges and f faces, then*

$$p - q + f = 2.$$

Proof: *Exercise*

Corollary 5.3.1. *Let G be a planar graph of order p and size q. If each face of G is an n-cycle, then*

$$q = \frac{n(p-2)}{n-2}.$$

Proof: Since each face is an n-cycle and each edge is in two faces, then $nf = 2q$. So, $f = \frac{2q}{n}$. Thus,

$$\begin{aligned} p - q + \frac{2q}{n} &= 2 \\ q\left(1 - \frac{2}{n}\right) &= p - 2 \\ q &= \frac{n(p-2)}{n-2} \end{aligned}$$

Definition 5.7. A *maximal planar graph* is a graph in which no edge can be added without losing planarity.

Corollary 5.3.2. *Let G be a graph of order p and size q.*

1. *If G is a maximal planar graph, every face is a triangle and $q = 3p - 6$;*
2. *If G is planar in which every face is a 4-cycle, then $q = 2p - 4$.*

Remark 5.5. From Corollary 5.3.2, the maximum number of edges in a plane graph occurs when each face is a triangle, we have a necessary condition for planarity of a graph in terms of the number of edges as given in the next corollary.

Corollary 5.3.3. *If G is any planar graph of order p and size q with $p \geq 3$, then $q \leq 3p - 6$. Furthermore, if G has no triangles, then $q \leq 2p - 4$.*

Corollary 5.3.4. *The graph K_5 and $K_{3,3}$ are nonplanar.*

Definition 5.8. Let G be a graph. A graph H is said to be a *subdivision* of a graph G if H can be obtained from G by successively inserting a vertex in an edge of G.

Example 5.5. The graph H is a subdivision of G given in Figure 24.

Theorem 5.4. **(Kuratowski's Theorem)** *Let G be a graph. Then, G is planar if and only if G contains a subgraph that is a subdivision of either $K_{3,3}$ or K_5.*