Contents

Chapter 1

Real Analysis

(1.5)
$OrderTrichotomy[<, S] := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$
$OrderTransitivity[<, S] := \forall_{x,y,z \in S}\big((x < y \wedge y < z) \implies x < z\big)$
$Order[<, S] := (OrderTrichotomy[<, S]) \wedge (OrderTransitivity[<, S])$

(1.7)
$BoundedAbove[E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge \Big(\exists_{\beta \in S}\forall_{x \in E}(x \leq \beta)\Big)$
$BoundedBelow[E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge \Big(\exists_{\beta \in S}\forall_{x \in E}(\beta \leq x)\Big)$
$UpperBound[\beta, E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge \big(\beta \in S \wedge \forall_{x \in E}(x \leq \beta)\big)$
$LowerBound[\beta, E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge \big(\beta \in S \wedge \forall_{x \in E}(\beta \leq x)\big)$

(1.8)
$LUB[\alpha, E, S, <] := (UpperBound[\alpha, E, S, <]) \wedge \Big(\forall_{\gamma}(\gamma < \alpha \implies \neg UpperBound[\gamma, E, S, <])\Big)$
$GLB[\alpha, E, S, <] := (LowerBound[\alpha, E, S, <]) \wedge \Big(\forall_{\beta}(\alpha < \beta \implies \neg LowerBound[\beta, E, S, <])\Big)$

(1.10)
$LUBProperty[S, <] := \forall_E\Big(\big((\emptyset \neq E \subset S) \wedge (BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <])\big)\Big)$
$GLBProperty[S, <] := \forall_E\Big(\big((\emptyset \neq E \subset S) \wedge (BoundedBelow[E, S, <]) \implies \exists_{\alpha \in S}(GLB[\alpha, E, S, <])\big)\Big)$

(1.11)
$LUBPropertyImpliesGLBProperty := LUBProperty[S, <] \implies GLBProperty[S, <]$

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(1) $LUBProperty[S, <] \implies \ldots$ — wts: 2

(1.1) $(\emptyset \neq B \subset S \wedge BoundedBelow[B, S, <]) \implies \ldots$ — wts: 1.2

(1.1.1) $Order[<, S] \wedge \exists_{\delta' \in S}(LowerBound[\delta', B, S, <])$ — from: $BoundedBelow$, 1.1

(1.1.2) $|B| = 1 \implies \ldots$ — wts: 1.1.3

(1.1.2.1) $\exists_{u'}(u' \in B)$ ▮ $u := choice(\{u' | u' \in B\})$ ▮ $B = \{u\}$ — from: 1.1.2

(1.1.2.2) $GLB[u, B, S, <]$ ▮ $\exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$

(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$

(1.1.4) $|B| \neq 1 \implies \ldots$ — wts: 1.1.5

(1.1.4.1) $\forall_E\big((\emptyset \neq E \subset S \wedge BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <])\big)$ — from: $LUBProperty$, 1

(1.1.4.2) $L := \{s \in S | LowerBound[s, B, S, <]\}$

(1.1.4.3) $|B| > 1 \wedge OrderTrichotomy[<, S]$ ▮ $\exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$ — from: $Order$, 1.1.1; wts: 1.1.4.7

(1.1.4.4) $b_1 := choice\Big(\{b_1' \in B | \exists_{b_0' \in B}(b_0' < b_1')\}\Big)$ ▮ $\neg LowerBound[b_1, B, S, <]$ — from: 1.1.4.2

(1.1.4.5) $b_1 \notin L$ ▮ $L \subset S$

(1.1.4.6) $\delta := choice(\{\delta' \in S | LowerBound[\delta', B, S, <]\})$ ▮ $\delta \in L$ ▮ $\emptyset \neq L$ — from: 1.1.1

(1.1.4.7) $\emptyset \neq L \subset S$ — from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y \in L}(LowerBound[y_0, B, S, <])$ ▮ $\forall_{y \in L}\forall_{x \in B}(y_0 \leq x)$ — from: $LowerBound$, 1.1.4.2; wts: 1.1.4.10

(1.1.4.9) $\forall_{x \in B}\Big(x \in S \land \forall_{y \in L}(y_0 \leq x)\Big) \;\blacksquare\; \forall_{x \in B}(UpperBound[x, L, S, <])$ from: UpperBound

(1.1.4.10) $\exists_{x \in S}(UpperBound[x, L, S, <]) \;\blacksquare\; BoundedAbove[L, S, <]$

(1.1.4.11) $\emptyset \neq L \subset S \land BoundedAbove[L, S, <]$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha' \in S}(LUB[\alpha', L, S, <]) \;\blacksquare\; \alpha := choice\Big(\{\alpha' \in S | (LUB[\alpha', L, S, <])\}\Big)$ from: 1.1.4.1 wts: 1.1.4.21

(1.1.4.13) $\forall_x(x \in B \implies UpperBound[x, L, S, <])$ from: 1.1.4.9 wts: 1.1.4.17

(1.1.4.14) $\forall_x(\neg UpperBound[x, L, S, <] \implies x \notin B)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$ wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound[\gamma, L, S, <] \;\blacksquare\; \gamma \notin B$ from: LUB, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B \;\blacksquare\; \gamma \in B \implies \gamma \geq \alpha$

(1.1.4.17) $\forall_{\gamma \in B}(\alpha \leq \gamma) \;\blacksquare\; LowerBound[\alpha, B, S, <]$ from: LowerBound

(1.1.4.18) $\alpha < \beta \implies \ldots$ wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y \in L}(y_0 \leq \alpha < \beta) \;\blacksquare\; \forall_{y \in L}(y_0 \neq \beta)$ from: LUB, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta \notin L \;\blacksquare\; \neg LowerBound[\beta, B, S, <]$ from: 1.1.4.2

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound[\beta, B, S, <] \;\blacksquare\; \forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.20) $LowerBound[\alpha, B, S, <] \land \forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB[\alpha, B, S, <] \;\blacksquare\; \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.5) $|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.6) $\Big(|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])\Big) \land \Big(|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])\Big)$ from: 1.1.3, 1.1.5

(1.1.7) $(|B| = 1 \lor |B| \neq 1) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <]) \;\blacksquare\; \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.2) $(\emptyset \neq B \subset S \land BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.3) $\forall_B\Big((\emptyset \neq B \subset S \land BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])\Big)$

(1.4) $GLBProperty[S, <]$

(2) $LUBProperty[S, <] \implies GLBProperty[S, <]$

(1.12)

$$Field[F, +, *] := \exists_{0,1 \in F}\forall_{x,y,z \in F}\left(\begin{array}{l} x + y \in F \quad \land \quad x * y \in F \quad \land \\ x + y = y + x \quad \land \quad x * y = y * x \quad \land \\ (x + y) + z = x + (y_0 + z) \quad \land \quad (x * y) * z = x * (y_0 * z) \quad \land \\ \mathbb{1} \neq \mathbb{0} \quad \land \quad x * (y_0 + z) = (x * y) + (x * z) \quad \land \\ \mathbb{0} + x = x \quad \land \quad \mathbb{1} * x = x \quad \land \\ \exists_{-x \in F}\big(x + (-x) = \mathbb{0}\big) \land \Big(x \neq \mathbb{0} \implies \exists_{1/x \in F}\big(x * (1/x) = \mathbb{1}\big)\Big) \end{array}\right)$$

************************ $(Field[F, +, *] \land x, y, z \in F) \implies \ldots$ ************************

(1.14)

$AdditiveCancellation := (x + y = x + z) \implies y = z$

(1) $y = \mathbb{0} + y = \big(x + (-x)\big) + y = \big((-x) + x\big) + y = (-x) + (x + y) = \ldots$ from: Field

(2) $(-x) + (x + z) = \big((-x) + x\big) + z = \big(x + (-x)\big) + z = \mathbb{0} + z = z$ from: Field

$AdditiveIdentityUniqueness := (x + y = x) \implies y = \mathbb{0}$

(1) $x + y = x = \mathbb{0} + x = x + \mathbb{0}$ from: Field

(2) $y = 0$ from: AdditiveCancellation

$AdditiveInverseUniqueness := (x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$ from: Field

(2) $y = -x$ from: AdditiveCancellation

$DoubleNegative := x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x \;\blacksquare\; \mathbb{0} = (-x) + x$ from: Field

(2) $x = -(-x)$	from: *AdditiveInverseUniqueness*

(1.15)

$MultiplicativeCancellation := (x \neq 0 \wedge x * y = x * z) \implies y = z$ —

$MultiplicativeIdentityUniqueness := (x \neq 0 \wedge x * y = x) \implies y = \mathbb{1}$ —

$MultiplicativeInverseUniqueness := (x \neq 0 \wedge x * y = \mathbb{1}) \implies y = 1/x$ —

$DoubleReciprocal := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)

$Domination := \mathbb{0} * x = \mathbb{0}$

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x$ ▮ $\mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$	from: *Field*
(2) $\mathbb{0} * x = \mathbb{0}$	from: *AdditiveIdentityUniqueness*

$NonDomination := (x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies \dots$	
(1.1) $(x * y = 0) \implies \dots$	
(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = \big(x * (1/x)\big) * \big(y * (1/y)\big) = (x * y) * \big((1/x) * (1/y)\big) = \mathbb{0} * \big((1/x) * (1/y)\big) = \mathbb{0}$	from: *Field*, *Domination*, 1, 1.1
(1.1.2) $\mathbb{1} = \mathbb{0} \wedge \mathbb{1} \neq \mathbb{0}$ ▮ $\perp$	from: *Field*
(1.2) $(x * y = 0) \implies \perp$ ▮ $x * y \neq 0$	
(2) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq 0$	

$NegationCommutativity := (-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0}$ ▮ $x * y + (-x) * y = \mathbb{0}$	from: *Field*, *Domination* wts: 2
(2) $(-x) * y = -(x * y)$	from: *AdditiveInverseUniqueness*
(3) $x * y + x * (-y) = x * (y_0 + -y) = x * \mathbb{0} = \mathbb{0}$ ▮ $x * y + x * (-y) = \mathbb{0}$	from: *Field*, *Domination* wts: 4
(4) $x * (-y) = -(x * y)$	from: *AdditiveInverseUniqueness*
(5) $(-x) * y = -(x * y) = x * (-y)$	from: 2, 4

$NegativeMultiplication := (-x) * (-y) = x * y$

(1) $(-x) * (-y) = -\big(x * (-y)\big) = -\big(-(x * y)\big) = x * y$	from: *NegationCommutativity*, *DoubleNegative*

(1.17)

$$OrderedField[F, +, *, <] := \left(\begin{array}{l} Field[F, +, *] \quad \wedge \quad Order[<, F] \quad \wedge \\ \forall_{x,y,z \in F}(y_0 < z \implies x + y < x + z) \;\wedge \\ \forall_{x,y \in F}\big((x > \mathbb{0} \wedge y > \mathbb{0}) \implies x * y > \mathbb{0}\big) \end{array} \right)$$

************************ $(OrderedField[F, +, *, <] \wedge x, y, z \in F) \implies \dots$ ************************

(1.18)

$NegationOnOrder := x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \dots$	
(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x$ ▮ $\mathbb{0} > -x$ ▮ $-x < \mathbb{0}$	from: *OrderedField*
(2) $x > \mathbb{0} \implies -x < \mathbb{0}$	
(3) $-x < \mathbb{0} \implies \dots$	
(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x$ ▮ $\mathbb{0} < x$ ▮ $x > \mathbb{0}$	from: *OrderedField*
(4) $-x < \mathbb{0} \implies x > \mathbb{0}$	
(5) $x > \mathbb{0} \implies -x < \mathbb{0} \wedge -x < \mathbb{0} \implies x > \mathbb{0}$ ▮ $x > \mathbb{0} \iff -x < \mathbb{0}$	from: 2, 4

$PositiveFactorPreservesOrder := (x > \mathbb{0} \wedge y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \wedge y < z) \implies \dots$	
(1.1) $(-y) + z > (-y) + y = \mathbb{0}$ ▮ $z + (-y) = \mathbb{0}$	from: *OrderedField*
(1.2) $x * \big(z + (-y)\big) > \mathbb{0}$ ▮ $x * z + x * (-y) > \mathbb{0}$	from: *OrderedField*

(1.3) $x*z=\mathbb{0}+x*z=\big(x*y+-(x*y)\big)+x*z=\big(x*y+x*(-y)\big)+x*z=\ldots$ from: *Field*, *NegationCommutativity*

(1.4) $x*y+\big(x*z+x*(-y)\big)>x*y+\mathbb{0}=x*y$ from: *Field*, 1.2

(1.5) $x*z>x*y$ from: 1.3, 1.4

(2) $(x>\mathbb{0}\wedge y<z)\implies x*z>x*y$

NegativeFactorFlipsOrder $:=(x<\mathbb{0}\wedge y<z)\implies x*y>x*z$

(1) $(x<\mathbb{0}\wedge y<z)\implies\ldots$

(1.1) $-x>\mathbb{0}$ from: *NegationOnOrder*

(1.2) $(-x)*y<(-x)*z \blacksquare \mathbb{0}=x*y+(-x)*y<x*y+(-x)*z \blacksquare \mathbb{0}<x*y+(-x)*z$ from: *PositiveFactorPreservesOrder*

(1.3) $\mathbb{0}<(-x)*(-y+z) \blacksquare \mathbb{0}>x*(-y+z) \blacksquare \mathbb{0}>-(x*y)+x*z$ from: *NegationOnOrder*

(1.4) $x*y>x*z$

(2) $(x<\mathbb{0}\wedge y<z)\implies x*y>x*z$

SquareIsPositive $:=(x\neq\mathbb{0})\implies x*x>\mathbb{0}$

(1) $(x>\mathbb{0})\implies x*x>0$ from: *OrderedField*

(2) $(x<\mathbb{0})\implies\ldots$

(2.1) $-x>\mathbb{0} \blacksquare x*x=(-x)*(-x)>\mathbb{0} \blacksquare x*x>\mathbb{0}$ from: *NegationOnOrder*, *OrderedField*, *NegativeMultiplication*

(3) $(x<\mathbb{0})\implies x*x>\mathbb{0}$

(4) $x\neq\mathbb{0}\implies(x>\mathbb{0}\vee x<\mathbb{0})\implies x*x>\mathbb{0} \blacksquare x\neq\mathbb{0}\implies x*x>\mathbb{0}$ from: *OrderTrichotomy*, 1, 3

OneIsPositive $:=\mathbb{1}>\mathbb{0}$

(1) $\mathbb{1}\neq\mathbb{0} \blacksquare \mathbb{1}=\mathbb{1}*\mathbb{1}>\mathbb{0}$ from: *Field*, *SquareIsPositive*

ReciprocationOnOrder $:=(0<x<y)\implies 0<1/y<1/x$

(1) $(0<x<y)\implies\ldots$

(1.1) $x*(1/x)=\mathbb{1}>\mathbb{0} \blacksquare x*(1/x)>0$ from: *Field*, *OneIsPositive*

(1.2) $1/x<\mathbb{0}\implies x*(1/x)<0\wedge x*(1/x)>0\implies\bot \blacksquare 1/x>\mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

(1.3) $y*(1/y)=\mathbb{1}>\mathbb{0} \blacksquare y*(1/y)>0$ from: *Field*, *OneIsPositive*

(1.4) $1/y<\mathbb{0}\implies y*(1/y)<0\wedge y*(1/y)>0\implies\bot \blacksquare 1/y>\mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

(1.5) $(1/x)*(1/y)>\mathbb{0}$ from: *OrderedField*

(1.6) $\mathbb{0}<1/y=\big((1/x)*(1/y)\big)*x<\big((1/x)*(1/y)\big)*y=1/x$ from: *OrderedField*, 1, 1.4, 1.5

(1.19)
OrderedFieldQ $:=$ *OrderedField*$[\mathbb{Q},+,*,<]$ —

Subfield$[K,F,+,*]:=$ *Field*$[F,+,*]\wedge K\subset F\wedge$ *Field*$[K,+,*]$
OrderedSubfield$[K,F,+,*,<]:=$ *OrderedField*$[F,+,*,<]\wedge K\subset F\wedge$ *OrderedField*$[K,+,*,<]$
CutI$[\alpha]:=\emptyset\neq\alpha\subset\mathbb{Q}$
CutII$[\alpha]:=\forall_{p\in\alpha}\forall_{q\in\mathbb{Q}}(q<p\implies q\in\alpha)$
CutIII$[\alpha]:=\forall_{p\in\alpha}\exists_{r\in\alpha}(p<r)$
$\mathbb{R}:=\{\alpha\in\mathbb{Q}|$*CutI*$[\alpha]\wedge$ *CutII*$[\alpha]\wedge$ *CutIII*$[\alpha]\}$

CutCorollaryI $:=(\alpha\in\mathbb{R}\wedge p\in\alpha\wedge q\in\mathbb{Q}\wedge q\notin\alpha)\implies p<q$

(1) $(\alpha\in\mathbb{R}\wedge p\in\alpha\wedge q\in\mathbb{Q}\wedge q\notin\alpha)\implies\ldots$

(1.1) $\forall_{p'\in\alpha}\forall_{q'\in\mathbb{Q}}(q'<p'\implies q'\in\alpha)$ from: *CutII*, 1

(1.2) $q<p\implies q\in\alpha \blacksquare q\notin\alpha\implies q\geq p$ from: 1

(1.3) $(q\notin\alpha)\implies\ldots$

(1.3.1) $q\geq p$ from: 1.2

(1.3.2) $(q=p)\implies(p\in\alpha\wedge p\notin\alpha)\implies\bot \blacksquare q\neq p$ from: 1, 1.3

(1.3.3) $q \geq p \wedge q \neq p$ ▮ $p < q$

(1.4) $q \notin \alpha \implies p < q$ ▮ $p < q$ from: 1

(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

$CutCorollaryII := (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$

(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ from: $CutII$, 1

(1.2) $s \in \alpha \implies \big(r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)\big)$ ▮ $s \in \alpha \implies r \in \alpha$ from: 1, 1.1

(1.3) $r \notin \alpha \implies s \notin \alpha$ ▮ $s \notin \alpha$ from: 1, 1.2

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$OrderTrichotomyOfR := OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$

(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$ from: $<_{\mathbb{R}}$, 1.1

(1.1.2) $\exists_{p'}(p' \in \alpha \wedge p' \notin \beta)$ ▮ $p := choice(\{p' | p' \in \alpha \wedge p' \notin \beta\})$

(1.1.3) $q \in \beta \implies \ldots$

(1.1.3.1) $p, q \in \mathbb{Q}$

(1.1.3.2) $q < p$ from: $CutCorollaryI$

(1.1.3.3) $q \in \alpha$ from: $CutII$

(1.1.4) $q \in \beta \implies q \in \alpha$

(1.1.5) $\forall_{q \in \beta}(q \in \alpha)$ ▮ $\beta \subseteq \alpha$

(1.1.6) $\beta \subset \alpha$ ▮ $\beta <_{\mathbb{R}} \alpha$

(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$ ▮ $(\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

$OrderTransitivityOfR := OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a \in \alpha}(a \in \beta) \wedge \forall_{b \in \beta}(b \in \gamma)$

(1.1.3) $\forall_{a \in \alpha}(\alpha \in \gamma)$ ▮ $\alpha \subset \gamma$ ▮ $\alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}}\big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(4) $OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

$OrderOfR := Order[<_{\mathbb{R}}, \mathbb{R}]$ from: $OrderTrichotomyR$, $OrderTransitivityR$

$LUBPropertyOfR := LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$ wts:

(1) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} | \exists_{\alpha \in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset \;\blacksquare\; \exists_\alpha(\alpha \in A) \;\blacksquare\; \alpha_0 := choice(\{\alpha|\alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset \;\blacksquare\; \exists_a(a \in \alpha_0) \;\blacksquare\; a_0 := choice(\{a|a \in \alpha_0\}) \;\blacksquare\; a_0 \in \gamma \;\blacksquare\; \gamma \neq \emptyset$

(1.4) $BoundedAbove[A, \mathbb{R}, <_\mathbb{R}] \;\blacksquare\; \exists_\beta(UpperBound[\beta, A, \mathbb{R}, <_\mathbb{R}])$

(1.5) $\beta_0 := choice(\{\beta|UpperBound[\beta, A, \mathbb{R}, <_\mathbb{R}]\})$

(1.6) $UpperBound[\beta_0, A, \mathbb{R}, <_\mathbb{R}] \;\blacksquare\; \forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \beta_0) \;\blacksquare\; \forall_{\alpha\in A}(\alpha \subseteq \beta_0) \;\blacksquare\; \forall_{\alpha\in A}\forall_{a\in\alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \;\blacksquare\; \forall_{a\in\gamma}(a \in \beta_0) \;\blacksquare\; \gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q} \;\blacksquare\; \gamma \subseteq \beta_0 \subset \mathbb{Q} \;\blacksquare\; \gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q} \;\blacksquare\; CutI[\gamma]$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.10.1) $p \in \gamma \;\blacksquare\; \exists_{\alpha\in A}(p \in \alpha) \;\blacksquare\; \alpha_1 := choice(\{\alpha \in A|p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \;\blacksquare\; q \in \alpha_1 \;\blacksquare\; q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \;\blacksquare\; \forall_{p\in\gamma}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \gamma) \;\blacksquare\; CutII[\gamma]$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha\in A}(p \in \alpha) \;\blacksquare\; \alpha_2 := choice(\{\alpha \in A|p \in \alpha\})$

(1.12.2) $\alpha_2 \in \mathbb{R} \;\blacksquare\; CutII[\alpha_2] \;\blacksquare\; \exists_{r\in\alpha_2}(p < r) \;\blacksquare\; r_0 := choice(\{r \in \alpha_2|p < r\})$

(1.12.3) $r_0 \in \alpha_2 \;\blacksquare\; r_0 \in \gamma$

(1.12.4) $p < r_0 \;\blacksquare\; p < r_0 \wedge r_0 \in \gamma \;\blacksquare\; \exists_{r\in\gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r\in\gamma}(p < r) \;\blacksquare\; \forall_{p\in\gamma}\exists_{r\in\gamma}(p < r) \;\blacksquare\; CutIII[\gamma]$

(1.14) $CutI[\gamma] \wedge CutII[\gamma] \wedge CutIII[\gamma] \;\blacksquare\; \gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha\in A}(\alpha \subseteq \gamma) \;\blacksquare\; \forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \gamma)$

(1.16) $\forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \gamma) \wedge \gamma \in \mathbb{R} \;\blacksquare\; UpperBound[\gamma, A, \mathbb{R}, <_\mathbb{R}]$

(1.17) $\delta <_\mathbb{R} \gamma \implies \ldots$

(1.17.1) $\delta \subset \gamma \;\blacksquare\; \exists_s(s \in \gamma \wedge s \notin \delta) \;\blacksquare\; s_0 := choice(\{s \in \mathbb{Q}|s \in \gamma \wedge s \notin \delta\})$

(1.17.2) $s_0 \in \gamma \;\blacksquare\; \exists_{\alpha\in A}(s_0 \in \alpha) \;\blacksquare\; \alpha_3 := choice(\{\alpha \in A|s_0 \in \alpha\})$

(1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta \;\blacksquare\; \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4) $\delta \geq_\mathbb{R} \alpha_3 \implies \ldots$

(1.17.4.1) $\alpha_3 \subseteq \delta \;\blacksquare\; \forall_{s\in\mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \;\blacksquare\; \neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4.2) $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \;\blacksquare\; \bot$

(1.17.5) $\delta \geq_\mathbb{R} \alpha_3 \implies \bot \;\blacksquare\; \delta <_\mathbb{R} \alpha_3 \;\blacksquare\; \exists_{\alpha\in A}(\delta <_\mathbb{R} \alpha) \;\blacksquare\; \exists_{\alpha\in A}\big(\neg(\alpha \leq_\mathbb{R} \delta)\big)$

(1.17.6) $\neg\forall_{\alpha\in A}(\alpha \leq_\mathbb{R} \delta) \;\blacksquare\; \neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}]$

(1.18) $\delta <_\mathbb{R} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}]) \;\blacksquare\; \forall_\delta(\delta <_\mathbb{R} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}])$

(1.19) $UpperBound[\gamma, A, \mathbb{R}, <_\mathbb{R}] \wedge \forall_\delta(\delta <_\mathbb{R} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}])$

(1.20) $LUB[\gamma, A, \mathbb{R}, <_\mathbb{R}] \;\blacksquare\; \exists_{\gamma\in S}(LUB[\gamma, A, \mathbb{R}, <_\mathbb{R}])$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_\mathbb{R}]) \implies \exists_{\gamma\in S}(LUB[\gamma, A, \mathbb{R}, <_\mathbb{R}])$

(3) $\forall_A\Big((\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_\mathbb{R}]) \implies \exists_{\gamma\in S}(LUB[\gamma, A, \mathbb{R}, <_\mathbb{R}])\Big) \;\blacksquare\; LUBProperty[\mathbb{R}, <_\mathbb{R}]$

$+_\mathbb{R}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_\mathbb{R} \beta) = \{r + s|r \in \alpha \wedge s \in \beta\}$
$0_\mathbb{R} := \{x \in \mathbb{Q}|x < 0\}$

$0InR := 0_\mathbb{R} \in \mathbb{R}$

(1) $-1 \in 0_\mathbb{R} \wedge 1 \notin 0_\mathbb{R} \;\blacksquare\; \emptyset \neq 0_\mathbb{R} \subseteq \mathbb{Q} \;\blacksquare\; CutI[0_\mathbb{R}]$

(2) $(x \in 0_\mathbb{R} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_\mathbb{R} \;\blacksquare\; \forall_{x\in 0_\mathbb{R}}\forall_{y\in\mathbb{Q}}(y_0 < x \implies y \in 0_\mathbb{R}) \;\blacksquare\; CutII[0_\mathbb{R}]$

(3) $y := x/2 \;\blacksquare\; (x \in 0_\mathbb{R}) \implies (x < y < 0) \implies \exists_{y\in 0_\mathbb{R}}(x < y) \;\blacksquare\; \forall_{x\in 0_\mathbb{R}}\exists_{y\in 0_\mathbb{R}}(x < y) \;\blacksquare\; CutIII[0_\mathbb{R}]$

(4) $CutI[0_\mathbb{R}] \wedge CutII[0_\mathbb{R}] \wedge CutIII[0_\mathbb{R}] \;\blacksquare\; 0_\mathbb{R} \in \mathbb{R}$

$FieldAdditionClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_\mathbb{R} \beta) \in \mathbb{R}\big)$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_\mathbb{R} \beta) = \{r + s|r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha) ; \exists_b(b \in \beta) \blacksquare a_0 := choice(\{a|a \in \alpha\}) ; b_0 := choice(\{b|b \in \beta\}) \blacksquare a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha) ; \exists_y(y_0 \notin \beta) \blacksquare x_0 := choice(\{x|x \notin \alpha\}) ; y_0 := choice(\{y|y \notin \beta\})$

(1.5) $\forall_{r\in\alpha}(r < x_0) ; \forall_{s\in\beta}(s < y_0) \blacksquare \forall_{r\in\alpha}\forall_{s\in\beta}(r + s < x_0 + y_0) \blacksquare x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q} \blacksquare CutI[\alpha +_{\mathbb{R}} \beta]$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.7.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s) \blacksquare (r_0, s_0) := choice\big((r, s) \in \alpha \times \beta | p = r + s\big)$

(1.7.2) $q < p = r_0 + s_0 \blacksquare (q - s_0) < r_0 \blacksquare (q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta \blacksquare q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta \blacksquare q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta \blacksquare \forall_{p\in\alpha+_{\mathbb{R}}\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta) \blacksquare CutII[\alpha +_{\mathbb{R}} \beta]$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s) \blacksquare (r_1, s_1) := choice\big(\{(r, s) \in \alpha \times \beta | p = r + s\}\big)$

(1.9.2) $r_1 \in \alpha \blacksquare \exists_{t\in\alpha}(r_1 < t) \blacksquare t_0 := choice(\{t \in \alpha | r_1 < t\})$

(1.9.3) $s_1 \in \beta \blacksquare t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1 \blacksquare \exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r) \blacksquare \forall_{p\in\alpha+_{\mathbb{R}}\beta}\exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r) \blacksquare CutIII[\alpha +_{\mathbb{R}} \beta]$

(1.11) $CutI[\alpha +_{\mathbb{R}} \beta] \wedge CutII[\alpha +_{\mathbb{R}} \beta] \wedge CutIII[\alpha +_{\mathbb{R}} \beta] \blacksquare \alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

$FieldAdditionCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s | r \in \alpha \wedge s \in \beta\} = \{s + r | s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

$FieldAdditionAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)\big)$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

$FieldAdditionIdentityOfR := (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \ldots$

(1.1.1) $s < 0 \blacksquare r + s < r + 0 = r \blacksquare r + s < r \blacksquare r + s \in \alpha$

(1.2) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha \blacksquare \forall_{r\in\alpha}\forall_{s\in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \forall_{p\in\alpha+_{\mathbb{R}}0_{\mathbb{R}}}(p \in \alpha) \blacksquare \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r\in\alpha}(p < r) \blacksquare r_2 := choice(\{r \in \alpha | p < r\})$

(1.4.2) $p < r_2 \blacksquare p - r_2 < r_2 - r_2 = 0 \blacksquare (p - r_2) < 0 \blacksquare (p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha \blacksquare p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare \forall_{p\in\alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

$FieldAdditionInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} | \exists_{r>0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q} \blacksquare \exists_{s\in\mathbb{Q}}(s \notin \alpha) \blacksquare s_0 := choice(\{s | s \notin \alpha\}) \blacksquare p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha \blacksquare -p_0 - 1 \notin \alpha \blacksquare \exists_{r>0}(-p_0 - r \notin \alpha) \blacksquare p_0 \in \beta$

(1.4) $\emptyset \neq \alpha \blacksquare \exists_{q\in\alpha} \blacksquare q_0 := choice(\{q \in \mathbb{Q} | q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha \blacksquare -(-q_0) - r = q_0 - r < q_0 \blacksquare -(-q_0) - r < q_0 \blacksquare -(-q_0) - r \in \alpha$

(1.6) $\forall_{r>0}\big(-(-q_0) - r \in \alpha\big) \blacksquare \neg\exists_{r>0}\big(-(-q_0) - r \notin \alpha\big) \blacksquare -q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q} \blacksquare CutI[\beta]$

(1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$

(1.8.1) $p \in \beta$ ▌ $\exists_{r>0}(-p-r \notin \alpha)$ ▌ $r_0 := choice(\{r > 0| -p-r \notin \alpha\})$

(1.8.2) $q < p$ ▌ $-p-r < -q-r$

(1.8.3) $-q-r \notin \alpha$ ▌ $q \in \beta$

(1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta$ ▌ $\forall_{p\in\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \beta)$ ▌ $CutII[\beta]$

(1.10) $p \in \beta \implies \dots$

(1.10.1) $p \in \beta$ ▌ $\exists_{r>0}(-p-r \notin \alpha)$ ▌ $r_1 := choice(\{r > 0| -p-r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0$ ▌ $r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p$ ▌ $t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -\big(p + (r_1/2)\big) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha$ ▌ $-t_0 - (r_1/2) \notin \alpha$ ▌ $\exists_{r>0}(-t_0 - r \notin \alpha)$ ▌ $t_0 \in \beta$

(1.10.7) $t_0 > p \wedge t_0 \in \beta$ ▌ $\exists_{t\in\beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t\in\beta}(p < t)$ ▌ $\forall_{p\in\beta}\exists_{t\in\beta}(p < t)$ ▌ $CutIII[\beta]$

(1.12) $CutI[\beta] \wedge CutII[\beta] \wedge CutIII[\beta]$ ▌ $\beta \in \mathbb{R}$

(1.13) $(r \in \alpha \wedge s \in \beta) \implies \dots$

(1.13.1) $s \in \beta$ ▌ $\exists_{t>0}(-s-t \notin \alpha)$ ▌ $t_1 := choice(\{t > 0| -s-t \notin \alpha\})$ ▌ $-s - t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha$ ▌ $-s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha$ ▌ $r < -s$ ▌ $r + s < 0$ ▌ $r + s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}}$ ▌ $\forall_{(r,s)\in\alpha\times\beta}(r + s \in 0_{\mathbb{R}})$ ▌ $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \dots$

(1.15.1) $v < 0$ ▌ $w_0 := -v/2$ ▌ $w > 0$

(1.15.2) $\exists_{n\in\mathbb{Z}}\big(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\big)$ ▌ $n_0 := choice\big(\{n \in \mathbb{Z}|nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\}\big)$ from: $ARCHIMEDEANPROPERTYOFQ + LUB$???

(1.15.3) $p_0 := -(n_0 + 2)w_0$ ▌ $-p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha$ ▌ $-p_0 - w_0 \notin \alpha$ ▌ $p_0 \in \beta$

(1.15.4) $n_0 w_0 \in \alpha \wedge p_0 \in \beta$ ▌ $n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{v\in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta)$ ▌ $0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$ ▌ $\alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$ ▌ $\exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

$*_{\mathbb{R}}[\alpha, \beta] :=$ —
$1_{\mathbb{R}} := \{x \in \mathbb{Q}|x < 1\}$

$1IsNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ —
$1InR := 1_{\mathbb{R}} \in \mathbb{R}$ —
$FieldMultiplicationClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R}\big)$ —
$FieldMultiplicationCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ —
$FieldMultiplicationAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)\big)$ —
$FieldMultiplicationIdentityOfR := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ —
$FieldMultiplicationInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha\in\mathbb{R}}\big(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}\big)$ —

$FieldDistributativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ —

$FieldWithR := Field[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}]$ —
$OrderedFieldWithR := OrderedField[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q}|r < q\}|q \in \mathbb{Q}\}$
$QROrderedSubfieldOfR := OrderedSubfield[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —
$QIsomorphicToQR := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ —
$CompletenessOfR := \exists_{\mathbb{R}}(LUBProperty[\mathbb{R}, <_{\mathbb{R}}] \wedge OrderedSubfield[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}])$ —

(1.20)
$ArchimedeanPropertyOfR := \forall_{x,y\in\mathbb{R}}\big(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y)\big)$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx | n \in \mathbb{N}^+\} \blacksquare (\emptyset \neq A \subset \mathbb{R}) \wedge \big(a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)\big)$

(1.2) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \blacksquare \forall_{n \in \mathbb{N}^+}(nx \leq y) \blacksquare \mathit{UpperBound}[y_0, A, \mathbb{R}, <] \blacksquare \mathit{BoundedAbove}[A, \mathbb{R}, <]$

(1.2.2) $\mathit{CompletenessOfR} \blacksquare \mathit{LUBProperty}[\mathbb{R}, <]$

(1.2.3) $(\mathit{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (\mathit{BoundedAbove}[A, \mathbb{R}, <]) \blacksquare \exists_{\alpha \in \mathbb{R}}(\mathit{LUB}[\alpha, A, \mathbb{R}, <]) \ \ldots$

(1.2.4) $\ldots \alpha_0 := choice(\{\alpha \in \mathbb{R} | \mathit{LUB}[\alpha, A, \mathbb{R}, <]\}) \blacksquare \mathit{LUB}[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0 \blacksquare \alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge (\mathit{LUB}[\alpha_0, A, \mathbb{R}, <]) \blacksquare \neg\mathit{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg\mathit{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <] \blacksquare \exists_{c \in A}(\alpha_0 - x < c) \ \ldots$

(1.2.8) $\ldots c_0 := choice(\{c \in A | \alpha_0 - x < c\}) \blacksquare (c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge \big(a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)\big) \blacksquare \exists_{m \in \mathbb{N}^+}(mx = c_0) \ \ldots$

(1.2.10) $\ldots m_0 := choice(\{m \in \mathbb{N}^+ | mx = c_0\}) \blacksquare (m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0) \blacksquare \alpha_0 - x < c_0 = m_0 x \blacksquare \alpha_0 < m_0 x + x \blacksquare \alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+ \blacksquare m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge \big(a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)\big) \blacksquare (m_0 + 1)x \in A$

(1.2.14) $\big(\alpha_0 < (m_0 + 1)x\big) \wedge \big((m_0 + 1)x \in A\big) \blacksquare \exists_{c \in A}(\alpha_0 < c)$

(1.2.15) $\mathit{LUB}[\alpha_0, A, \mathbb{R}, <] \blacksquare \mathit{UpperBound}[\alpha_0, A, \mathbb{R}, <] \blacksquare \forall_{c \in A}(c \leq \alpha_0) \blacksquare \neg\exists_{c \in A}(c > \alpha_0) \blacksquare \neg\exists_{c \in A}(\alpha_0 < c)$

(1.2.16) $\big(\exists_{c \in A}(\alpha_0 < c)\big) \wedge \big(\neg\exists_{c \in A}(\alpha_0 < c)\big) \blacksquare \bot$

(1.3) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \bot \blacksquare \exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y) \blacksquare \forall_{x, y \in \mathbb{R}}\big(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y)\big)$

$\mathit{QDenseInR} := \forall_{x, y \in \mathbb{R}}\Big(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y)\Big)$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y \blacksquare (0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $\mathit{ArchimedeanPropertyOfR} \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R}) \blacksquare \exists_{n \in \mathbb{N}^+}\big(n(y - x) > 1\big) \ \ldots$

(1.3) $\ldots n_0 := choice\big(\{n \in \mathbb{N}^+ | n(y - x) > 1\}\big) \blacksquare (n_0 \in \mathbb{N}^+) \wedge \big(n_0(y - x) > 1\big)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R}) \blacksquare n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $\mathit{ArchimedeanPropertyOfR} \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R}) \blacksquare \exists_{m \in \mathbb{N}^+}\big(m(1) > n_0 x\big) \ \ldots$

(1.6) $\ldots m_1 := choice\big(\{m \in \mathbb{N}^+ | m(1) > n_0 x\}\big) \blacksquare (m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $\mathit{ArchimedeanPropertyOfR} \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R}) \blacksquare \exists_{m \in \mathbb{N}^+}\big(m(1) > -n_0 x\big) \ \ldots$

(1.8) $\ldots m_2 := choice\big(\{m \in \mathbb{N}^+ | m(1) > -n_0 x\}\big) \blacksquare (m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x) \blacksquare -m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+ \blacksquare |m_1 - (-m_2)| \geq 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge \big(|m_1 - (-m_2)| \geq 2\big) \blacksquare \exists_{m \in \mathbb{Z}}\big((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\big) \ \ldots$

(1.12) $\ldots m_0 := choice\big(\{m \in \mathbb{Z} | (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\}\big) \blacksquare (-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$

(1.13) $\big(n_0(y - x) > 1\big) \wedge (m_0 - 1 \leq n_0 x < m_0) \blacksquare n_0 x < m_0 \leq 1 + n_0 x < n_0 y \blacksquare n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y) \blacksquare x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z} \blacksquare m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y) \blacksquare \exists_{p \in \mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y) \blacksquare \forall_{x, y \in \mathbb{R}}\Big(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y)\Big)$

(1.21)
$\mathit{RootLemma} := (0 < a < b) \implies \Big(b^n - a^n \leq (b - a)nb^{n-1}\Big)$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b \blacksquare b/a > 1$

(1.3) $b/a > 1 \;\blacksquare\; \sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq \sum_{i=1}^{n}\left(b^{n-i}a^{i-1}(b/a)^{i-1}\right) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1} \;\blacksquare\; \sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b-a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq (b-a)nb^{n-1} \;\blacksquare\; b^n - a^n \leq (b-a)nb^{n-1}$

(2) $(0 < a < b) \implies \left(b^n - a^n \leq (b-a)nb^{n-1}\right)$

$\color{red}{RootExistenceInR} := \forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}({y_0}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} | t > 0 \wedge t^n < x\} \;\blacksquare\; t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$

(1.2) $t_0 := x/(1+x) \;\blacksquare\; \left(t_0 = x/(1+x)\right) \wedge (t_0 \in \mathbb{R})$

(1.3) $0 < x \;\blacksquare\; 0 < x < 1+x \;\blacksquare\; t_0 = x/(1+x) > 0 \;\blacksquare\; t_0 > 0$

(1.4) $1 = (1+x)/(1+x) > x/(1+x) = t_0 \;\blacksquare\; 1 > t_0$

(1.5) $(t_0 > 0) \wedge (1 > t_0) \;\blacksquare\; 0 < t_0 < 1$

(1.6) $(0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1) \;\blacksquare\; {t_0}^n \leq t_0$

(1.7) $0 < x \;\blacksquare\; x > x/(1+x) = t_0 \;\blacksquare\; x > t_0$

(1.8) $({t_0}^n \leq t_0) \wedge (x > t_0) \;\blacksquare\; {t_0}^n < x$

(1.9) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge ({t_0}^n < x) \;\blacksquare\; t_0 \in E \;\blacksquare\; \emptyset \neq E$

(1.10) $t_1 := choice(\{t \in \mathbb{R} | t > 1+x\}) \;\blacksquare\; (t_1 \in \mathbb{R}) \wedge (t_1 > 1+x)$

(1.11) $x > 0 \;\blacksquare\; t_1 > 1+x > 1 \;\blacksquare\; t_1 > 1 \;\blacksquare\; {t_1}^n \geq t_1$

(1.12) $({t_1}^n \geq t_1) \wedge (t_1 > 1+x) \wedge (1 > 0) \;\blacksquare\; {t_1}^n \geq t_1 > 1+x > x \;\blacksquare\; {t_1}^n > x$

(1.13) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge ({t_1}^n > x) \;\blacksquare\; t_1 \notin E \;\blacksquare\; E \subset \mathbb{R}$

(1.14) $(\emptyset \neq E) \wedge (E \subset \mathbb{R}) \;\blacksquare\; \emptyset \neq E \subset \mathbb{R}$

(1.15) $t \in E \implies \ldots$

(1.15.1) $(t \in E) \wedge \left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \;\blacksquare\; t^n < x$

(1.15.2) $({t_1}^n > x) \wedge (t^n < x) \;\blacksquare\; t^n < x < {t_1}^n \;\blacksquare\; t < t_1$

(1.16) $t \in E \implies t < t_1 \;\blacksquare\; \forall_{t\in E}(t \leq t_1) \;\blacksquare\; \color{cyan}{UpperBound}[t_1, E, \mathbb{R}, <] \;\blacksquare\; \color{cyan}{BoundedAbove}[E, \mathbb{R}, <]$

(1.17) $\color{cyan}{CompletenessOfR} \;\blacksquare\; \color{cyan}{LUBProperty}[\mathbb{R}, <]$

(1.18) $(\color{cyan}{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (\color{cyan}{BoundedAbove}[E, \mathbb{R}, <]) \;\blacksquare\; \exists_{y\in\mathbb{R}}(\color{cyan}{LUB}[y, E, \mathbb{R}, <]) \;\ldots$

(1.19) $\ldots y_0 := choice(\{y \in \mathbb{R} | \color{cyan}{LUB}[y, E, \mathbb{R}, <]\}) \;\blacksquare\; \color{cyan}{LUB}[y_0, E, \mathbb{R}, <]$

(1.20) $(\color{cyan}{LUB}[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0) \;\blacksquare\; 0 < t_0 \leq y_0 \in \mathbb{R} \;\blacksquare\; 0 < y_0 \in \mathbb{R}$

(1.21) ${y_0}^n < x \implies \ldots$

(1.21.1) $k_0 := \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} \;\blacksquare\; k_0 \in \mathbb{R}$

(1.21.2) ${y_0}^n < x \;\blacksquare\; 0 < x - {y_0}^n$

(1.21.3) $(n > 0) \wedge (y_0 > 0) \;\blacksquare\; 0 < n(y_0+1)^{n-1}$

(1.21.4) $(0 < x - {y_0}^n) \wedge \left(0 < n(y_0+1)^{n-1}\right) \;\blacksquare\; 0 < \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} = k_0 \;\blacksquare\; 0 < k_0$

(1.21.5) $(0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R}) \;\blacksquare\; 0 < min(1, k_0) \in \mathbb{R}$

(1.21.6) $\color{cyan}{QDenseInR} \wedge \left(0, min(1, k_0) \in \mathbb{R}\right) \wedge \left(0 < min(1, k_0)\right) \;\blacksquare\; \exists_{h\in\mathbb{Q}}\left(0 < h < min(1, k_0)\right) \;\ldots$

(1.21.7) $\ldots h_0 := choice\left(\{h \in \mathbb{Q} | 0 < h < min(1, k_0)\}\right) \;\blacksquare\; (0 < h_0 < 1) \wedge \left(h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}\right)$

(1.21.8) $(y_0 > 0) \wedge (h_0 > 0) \;\blacksquare\; 0 < y_0 < y_0 + h_0$

(1.21.9) $\color{cyan}{RootLemma} \wedge (0 < y_0 < y_0 + h_0) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}$

(1.21.10) $h_0 < 1 \;\blacksquare\; h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0+1)^{n-1}$

(1.21.11) $\left((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}\right) \wedge \left(h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0+1)^{n-1}\right) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0+1)^{n-1}$

(1.21.12) $\left(0 < n(y_0+1)^{n-1}\right) \wedge \left(h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}\right) \;\blacksquare\; h_0 n(y_0+1)^{n-1} < x - {y_0}^n$

(1.21.13) $\left((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0+1)^{n-1}\right) \wedge \left(h_0 n(y_0+1)^{n-1} < x - {y_0}^n\right) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \;\blacksquare\; (y_0 + h_0)^n < x$

(1.21.14) $(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \;\blacksquare\; (y_0 + h_0)^n < x$

(1.21.15) $(0 < y_0\mathbb{R}) \wedge (0 < h_0 < \mathbb{R}) \;\blacksquare\; 0 < y_0 < y_0 + h_0 \in \mathbb{R}$

(1.21.16) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge \left((y_0 + h_0)^n < x\right) \wedge (0 < y_0 + h_0 \in \mathbb{R}) \;\blacksquare\; (y_0 + h_0)^n \in E$

(1.21.17) $\big((y_0 + h_0)^n \in E\big) \wedge (y_0 < y_0 + h_0) \blacksquare \exists_{e \in E}(y_0 < e)$

(1.21.18) $LUB[y_0, E, \mathbb{R}, <] \blacksquare UpperBound[y_0, E, \mathbb{R}, <] \blacksquare \forall_{e \in E}(e \leq y_0) \blacksquare \neg\exists_{e \in E}(e > y_0)$

(1.21.19) $\big(\exists_{e \in E}(e > y_0)\big) \wedge \big(\neg\exists_{e \in E}(e > y_0)\big) \blacksquare \bot$

(1.22) ${y_0}^n < x \implies \bot \blacksquare {y_0}^n \geq x$

(1.23) ${y_0}^n > x \implies \ldots$

(1.23.1) $k_1 := \frac{{y_0}^n - x}{n{y_0}^{n-1}} \blacksquare (k_1 \in \mathbb{R}) \wedge (k_1 n {y_0}^{n-1} = {y_0}^n - x)$

(1.23.2) $(0 < x) \wedge (0 < n \in \mathbb{Z}) \blacksquare {y_0}^n - x < {y_0}^n \leq n{y_0}^n \blacksquare {y_0}^n - x < n{y_0}^n$

(1.23.3) ${y_0}^n - x < n{y_0}^n \blacksquare k_1 = \frac{{y_0}^n - x}{n{y_0}^{n-1}} < \frac{n{y_0}^n}{n{y_0}^{n-1}} = y_0 \blacksquare k_1 < y_0$

(1.23.4) ${y_0}^n > x \blacksquare 0 < {y_0}^n - x$

(1.23.5) $(n > 0) \wedge (y_0 > 0) \blacksquare 0 < n{y_0}^{n-1}$

(1.23.6) $(0 < {y_0}^n - x) \wedge 0 < (n{y_0}^{n-1}) \blacksquare 0 < \frac{{y_0}^n - x}{n{y_0}^{n-1}} = k_1 \blacksquare 0 < k_1$

(1.23.7) $(k_1 < y_0) \wedge (0 < k_1) \blacksquare (0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$

(1.23.8) $t \geq y_0 - k_1 \implies \ldots$

(1.23.8.1) $t \geq y_0 - k_1 \blacksquare t^n \geq (y_0 - k_1)^n \blacksquare -t^n \leq -(y_0 - k_1)^n \blacksquare {y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n$

(1.23.8.2) $RootLemma \wedge (0 < y_0 - k_1 < y_0) \blacksquare {y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}$

(1.23.8.3) $\big({y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n\big) \wedge \Big({y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}\Big) \blacksquare {y_0}^n - t^n < k_1 n {y_0}^{n-1}$

(1.23.8.4) $(k_1 n {y_0}^{n-1} = {y_0}^n - x) \wedge ({y_0}^n - t^n < k_1 n {y_0}^{n-1}) \blacksquare {y_0}^n - t^n < {y_0}^n - x \blacksquare -t^n < -x \blacksquare t^n > x$

(1.23.8.5) $\big(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\big) \wedge (t^n > x) \blacksquare t \notin E$

(1.23.9) $t \geq y_0 - k_1 \implies t \notin E \blacksquare t \in E \implies t < y_0 - k_1 \blacksquare \forall_{t \in E}(t \leq y_0 - k_1) \blacksquare UpperBound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.10) $\big(LUB[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)\big) \blacksquare \neg UpperBound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.11) $(UpperBound[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg UpperBound[y_0 - k_1, E, \mathbb{R}, <]) \blacksquare \bot$

(1.24) ${y_0}^n > x \implies \bot \blacksquare {y_0}^n \leq x$

(1.25) $Order[\mathbb{R}, <] \blacksquare OrderTrichotomy[\mathbb{R}, <]$

(1.26) $(OrderTrichotomy[\mathbb{R}, <]) \wedge ({y_0}^n \geq x) \wedge ({y_0}^n \leq x) \blacksquare {y_0}^n = x$

(1.27) $({y_0}^n = x) \wedge (y_0 \in \mathbb{R}) \blacksquare \exists_{y \in \mathbb{R}}(y^n = x)$

(1.28) $y_1, y_2 := choice(\{y \in \mathbb{R} | y^n = x\})$

(1.29) $y_1 \neq y_2 \implies \ldots$

(1.29.1) $(OrderTrichotomy[\mathbb{R}, <]) \wedge (y_1 \neq y_2) \blacksquare (y_1 < y_2) \veebar (y_2 < y_1) \ \ldots$

(1.29.2) $\ldots (x = {y_1}^n < {y_2}^n = x) \veebar (x = {y_2}^n < {y_1}^n = x) \blacksquare (x < x) \veebar (x > x) \blacksquare \bot \veebar \bot \blacksquare \bot$

(1.30) $y_1 \neq y_2 \implies \bot \blacksquare y_1 = y_2 \blacksquare \forall_{a,b \in \mathbb{R}}\big((a^n = x \wedge b^n = x) \implies a = b\big)$

(1.31) $\Big(\exists_{y \in \mathbb{R}}(y^n = x)\Big) \wedge \Big(\forall_{a,b \in \mathbb{R}}\big((a^n = x \wedge b^n = x) \implies a = b\big)\Big) \blacksquare \exists!_{y \in \mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}}(y^n = x) \blacksquare \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}}({y_0}^n = x)$

$RootExistenceInRCorollary := \forall_{0 < a \in \mathbb{R}} \forall_{0 < b \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \Big((ab)^{1/n} = a^{1/n} b^{1/n}\Big)$ —

$$ExtendedRealSystem[\bar{\mathbb{R}}, +, *, <] := \begin{pmatrix} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \ \wedge \ x - \infty = -\infty \ \wedge \ \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \ \wedge \\ (x > 0) \implies \big(x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty\big) \wedge \\ (x < 0) \implies \big(x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty\big) \end{pmatrix}$$

$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$

$+_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d \rangle$

$*_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c \rangle$

$FieldC := Field[\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}}]$ —

$RSubfieldC := Subfield[\mathbb{R}, \mathbb{C}, +, *]$ —

$i := \langle 0, 1 \rangle \in \mathbb{C}$

$iProperty := i^2 = -1$ —

$CProperty := (a, b \in \mathbb{R}) \implies (\langle a, b \rangle = a + bi)$ —

$Conjugate[\overline{a+bi}] := a - bi$

$ConjugateProperties := (w, z \in \mathbb{C}) \implies \ldots$ —

(1)	$\overline{z+w} = \overline{z} + \overline{w}$
(2)	$\overline{z*w} = \overline{z} * \overline{w}$
(3)	$Re(z) = (1/2)(z + \overline{z}) \wedge Im(z) = (1/2)(z - \overline{z})$
(4)	$0 \leq z * \overline{z} \in \mathbb{R}$

$AbsoluteValueC[|z|] = (z * \overline{z})^{1/2}$
$AbsoluteValueProperties := (z, w \in \mathbb{C}) \implies \ldots$ —

(1)	123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \land Relation(f, X \times Y) \land \forall_{x \in X} \exists!_{y \in Y} \big((x, y) \in f\big)$

$\big(Function(f, X, Y) \land A \subseteq X \land B \subseteq Y\big) \implies \dots$

(1)	$Domain(f) := X; Codomain(f) := Y$
(2)	$Image(f, A) := \{f(a) \vert a \in A\}; Preimage(f, B) := \{a \vert f(a) \in B\}$
(3)	$Range(f) := Image\big(Domain(f)\big)$

$Injective(f, X, Y) := Function(f, X, Y) \land \forall_{x_1, x_2 \in X} \big(x_1 \neq x_2 \implies f(x_1) \neq f(x_2)\big)$
$Surjective(f, X, Y) := Function(f, X, Y) \land \forall_{y \in Y} \exists_{x \in X} \big(y_0 = f(x)\big)$
$Bijective(f, X, Y) := Injective(f, X, Y) \land Surjective(f, X, Y)$
$SurjectiveEquivalent := \big(Range(f) = Codomain(f)\big) \implies Surjective(f)$

$\big(Function(f, X, Y) \land Function(g, Y, Z)\big) \implies (f \circ g)(x) := f\big(g(x)\big); Function(f \circ g, X, Z)$

$Properties of Functions := \big(Function(f, A, B) \land Function(g, B, C) \land Function(h, C, D)\big) \implies \dots$

(1)	$h \circ (g \circ f) = (h \circ g) \circ f$
(2)	$\big(Injective(f) \land Injective(g)\big) \implies Injective(g \circ f)$
(3)	$\big(Surjective(f) \land Surjective(g)\big) \implies Surjective(g \circ f)$
(4)	$\big(Bijective(f, A, B)\big) \implies \exists_{f^{-1}} \Bigg(Function(f^{-1}1, B, A) \land \forall_{a \in A} \Big(f^{-1}\big(f(a)\big) = a\Big) \land \forall_{b \in B} \Big(f\Big(f^{-1}(b)\Big) = b\Big)\Bigg)$

$\vert(a, b) := a, b \in \mathbb{Z} \land a \neq 0 \land \exists_{c \in \mathbb{Z}} (b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \dots$

(1)	$(a \vert b) \implies a \vert bc$
(2)	$(a \vert b \land b \vert c) \implies a \vert c$
(3)	$(a \vert b \land b \vert c) \implies a \vert (bx + cy)$
(4)	$(a \vert b \land b \vert a) \implies a = \pm b$
(5)	$(a \vert b \land a > 0 \land b > 0) \implies (a \leq b)$
(6)	$(a \vert b) \iff (m \neq 0 \land ma \vert mb)$

$DivisionAlgorithm := (a, b \in \mathbb{Z} \land a > 0) \implies \exists!_{q, r \in \mathbb{Z}} (b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \land a \vert b \land a \vert c$
$GCD(a, b, c) := CD(a, b, c) \land \forall_d \big((d \vert b \land d \vert c) \implies d \vert a\big)$
$GCDEquivalent := 123123$

Chapter 3

Linear Algebra

$Matrix(A,m,n) := [a_{i,j}]_{m\times n}$:= m rows, n columns of real numbers
$+(A,B) := (A+B=[a_{i,j}+b_{i,j}]_{m\times n}) \wedge \big(Matrix(A,m,n)\big) \wedge \big(Matrix(B,m,n)\big)$
$*\,(r,A) := (r*A=[ra_{i,j}]_{m\times n}) \wedge \big(Matrix(A,m,n)\big)$
$*\,(A,B) := \left(A*B=\left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]_{m\times n}\right) \wedge \big(Matrix(A,m,p)\big) \wedge \big(Matrix(B,p,n)\big)$
${}^{T}(A) := (A^T=[a_{j,i}]_{n\times m}) \wedge \big(Matrix(A,m,n)\big)$

$AddCom := A+B=B+A \;\blacksquare \leftarrow A+B=[a_{i,j}+b_{i,j}]=[b_{i,j}+a_{i,j}]=B+A$
$AddAssoc := (A+B)+C=A+(B+C) \;\blacksquare \leftarrow (A+B)+C=[(a_{i,j}+b_{i,j})+c_{i,j}]=[a_{i,j}+(b_{i,j}+c_{i,j})]=A+(B+C)$
$AddId := A+O=A=O+A \;\blacksquare \leftarrow A+O=[a_{i,j}+0]=A=[0+a_{i,j}]=O+A$
$AddInv := A+(-A)=O=(-A)+A \;\blacksquare \leftarrow A+(-A)=[a_{i,j}-a_{i,j}]=O=[-a_{i,j}+a_{i,j}]=(-A)+A$

$MulAssoc := (A*B)*C=A*(B*C) \;\blacksquare \leftarrow$

(1)	$(A*B)*C=\left[\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,j})\right]*C=\left[\sum_{k_2=1}^{p_2}\left(\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2})c_{k_2,j}\right)\right]=\left[\sum_{k_2=1}^{p_2}\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right]=\;...$
(2)	$...\left[\sum_{k_1=1}^{p_1}\sum_{k_2=1}^{p_2}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right]=\left[\sum_{k_1=1}^{p_1}\left(a_{i,k_1}\sum_{k_2=1}^{p_2}(b_{k_1,k_2}c_{k_2,j})\right)\right]=\ldots=A*(B*C)$

$LeftDist := (A+B)*C=A*C+B*C \;\blacksquare \leftarrow$

(1)	$(A+B)*C=[a_{i,j}+b_{i,j}]*C=\left[\sum_{k=1}^{p}\left((a_{i,k}+b_{i,k})c_{k,j}\right)\right]=\;...$
(2)	$...\left[\sum_{k=1}^{p}(a_{i,k}c_{k,j}+b_{i,k}c_{k,j})\right]=\left[\sum_{k=1}^{p}(a_{i,k}c_{k,j})\right]+\left[\sum_{k=1}^{p}(b_{i,k}c_{k,j})\right]=A*C+B*C$

$RightDist := C*(A+B)*C=C*A+C*B$ —
$Scalar1 := r(sA)=(rs)A=s(rA)$ —
$Scalar2 := A(rb)=r(AB)$ —
$Scalar3 := (r+s)A=rA+rS$ —
$Scalar4 := r(A+B)=rA+rB$ —

$Trans1 := A=(A^T)^T \;\blacksquare \leftarrow A=[a_{i,j}]=[a_{j,i}]^T=([a_{i,j}]^T)^T=(A^T)^T$
$Trans2 := (A+B)^T=A^T+B^T$ —
$Trans3 := (A*B)^T=B^T*A^T \;\blacksquare \leftarrow (A*B)^T=\left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]^T=\left[\sum_{k=1}^{p}(a_{j,k}b_{k,i})\right]=\left[\sum_{k=1}^{p}(b_{k,i}a_{j,k})\right]=\left[\sum_{k=1}^{p}(b^T_{i,k}a^T_{k,j})\right]=B^T*A^T$

$Sym(A) := A=A^T$; $SkewSym(A) := A=-A^T$
$SymGen := Sym(A+A^T) \;\blacksquare \leftarrow (A+A^T)^T=A^T+(A^T)^T=A^T+A=A+A^T$
$SkewSymGem := SkewSym(A-A^T) \;\blacksquare \leftarrow -(A-A^T)^T=-\big(A^T-(A^T)^T\big)=-(A^T-A)=(A-A^T)$
$SymDec := A=(1/2)*(A+A^T)+(1/2)*(A-A^T)$ —

$Square(A) := Matrix(A,n,n)$
$Diagonal(A) := \big(Square(A)\big) \wedge (i\neq j \implies a_{i,j}=0)$
$Scalar(A) := \big(Diagonal(A)\big) \wedge (a_{i,i}=k)$
$I := \big(Scalar(I)\big) \wedge (i_{d,d}=1)$

$MulId := A * I = A = I * A$ —

$UpperTriangular(A) := \left(Square(A)\right) \wedge (i > \implies a_{i,j} = 0)$
$LowerTriangular(A) := \left(Square(A)\right) \wedge (i < j \implies a_{i,j} = 0)$

$Invertible(A) := \exists_{A^{-1}}(A * A^{-1} = I = A^{-1} * A)$
$UniqueInverse := \left((A * B_1 = I) \wedge (A * B_2 = I)\right) \implies B_1 = B_2$ —
$Inv1 := \left(Invertible(A) \wedge Invertible(B)\right) \implies (A * B)^{-1} = B^{-1} * A^{-1}$ ▮ ←

(1) $(A * B) * (A * B)^{-1} = I$ ▮ $B * (A * B)^{-1} = A^{-1}$ ▮ $(A * B)^{-1} = B^{-1} * A^{-1}$

$Inv2 := \left(Invertible(A)\right) \implies \left(\left(Invertible(A^{-1}) \wedge (A = (A^{-1})^{-1}\right)\right)$ —

$InvTrans := \left(Invertible(A)\right) \implies \left(\left(Invertible(A^T) \wedge ((A^T)^{-1} = (A^{-1})^T\right)\right)$ ▮ ←

(1) $A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I$ ▮ $(A^{-1})^T = (A^T)^{-1}$

$Sys(A, X, B) := (A * X = B) \wedge \left(Matrix(A, m, n)\right) \wedge \left(Matrix(X, n, 1)\right) \wedge \left(Matrix(B, m, 1)\right)$
$Sol(X, A, B) := Sys(A, X, B)$
$TrivSol(X, A, B) := \left(Sol(X, A, B)\right) \wedge (a_{i,j} = 0)$

$RREF(A) := (Definition 1.18)$
$ElementaryRowOperation(\phi) := (Definition 1.19)$
$RowEquiv(A, B) := \exists_{\Phi}\left(\forall_{\phi \in \Phi}\left(ElementaryRowOperation(\phi)\right) \wedge |\Phi| \in \mathbb{N} \wedge \Phi(A) = B\right)$

$(A \neq O) \implies \left(\exists_B\left(RREF(B) \wedge RowEquiv(A, B)\right)\right)$ (By Gauss-Jordan Elimination) —
$\left(Sys(A, X_1, B) \wedge Sys(C, X_2, D) \wedge RowEquiv([A|B], [C|D])\right) \implies (X_1 = X_2)$ (By algebra on systems of equations) —
$\left(RowEquiv(A, B)\right) \implies \left(Sys(A, X_1, O) \wedge Sys(B, X_1, O)\right)$ (Corollary) —
$(m < n) \implies \left(\exists_X\left(\neg TrivSol(X, A, O)\right)\right)$ ▮ $\left(\forall_X(TrivSol(X, A, O)\right) \implies (m \geq n)$ ▮ ←

(1) Let B = GaussJordan(A) and let r be the number of non-zero rows of B ▮ $r \leq m$

(2) $(r \leq m) \wedge (m < n)$ ▮ $r < n$ ▮ $0 < n - r$

(3) The solution of B will have r fixed variables and n - r free variables which is also more then zero

(4) The solution of B will have at least one free variable that can be non-zero, i.e., non-trivial

$Elem(A) := RowEquiv(A, I)$
$\left(Elem_i(A)\right) \implies \left(Invertible(A) \wedge Elem_i(A^{-1})\right)$
$\left(ElementaryRowOperation(\phi) \wedge \left(B = \phi(A)\right)\right) \implies \left(Elem(E) \wedge B = E * A\right)$
$\left(RowEquiv(A, B)\right) \iff \left(B = \Pi(E_i) * A\right)$
$\left(RowEquiv(I, B)\right) \iff \left(B = \Pi(E_i)\right)$

$\left(\forall_X(TrivSol(X, A, O)\right) \implies \left(RowEquiv(A, I)\right)$ ▮ ←

(1) Let B = GaussJordan(A) ▮ $RowEquiv(A, B)$

(2) $\forall_X(TrivSol(X, B, O)$ ▮ $n \geq n$ ▮ $n = n$ ▮ $B = I$

(3) $RowEquiv(A, B)$ ▮ $RowEquiv(A, I)$

$\left(Invertible(A)\right) \iff \left(A = \Pi(E_i)\right)$ ▮ ←

(1) $Invertible(A) \implies \ldots$

(1.1) $AX = O$ ▮ $X = A^{-1}O = O$ ▮ $\forall_X(TrivSol(X, A, O)$

(1.2) $\forall_X(TrivSol(X, A, O)$ ▮ $RowEquiv(A, I)$ ▮ $A = \Pi(E_i)$

(2) $A = \Pi(E_i) \implies \ldots$

(2.1) $\Pi({E_{-i}}^{-1}) = A^{-1}$ ▮ $Invertible(A)$

(3) $\left(Invertible(A)\right) \iff \left(A = \Pi(E_i)\right)$

$\left(Invertible(A)\right) \iff \left(RowEquiv(I, A)\right)$ ▮ ← $\left(RowEquiv(I, B) \iff B = \Pi(E_i)\right) \wedge \left(\left(Invertible(A)\right) \iff \left(A = \Pi(E_i)\right)\right)$

$\left(\forall_X \left(TrivSol(X, A, O)\right)\right) \implies \left(Invertible(A)\right)$ ▮ $\leftarrow$

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(1) $\left(\left(\forall_X \left(TrivSol(X, A, O)\right) \implies \left(RowEquiv(A, I)\right)\right) \land \left(\left(Invertible(A)\right) \iff \left(RowEquiv(I, A)\right)\right)\right)$

$\left(Invertible(A)\right) \implies \left(\forall_X \left(TrivSol(X, A, O)\right)\right)$ ▮ $\leftarrow AX = O$ ▮ $X = A^{-1}O = O$ ▮ $\forall_X (TrivSol(X, A, O)$