Solutions Manual to Walter Rudin's *Principles of Mathematical Analysis*

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Chapter 8

Some Special Functions

Exercise 8.1 Define

$$f(x) = \begin{cases} e^{-1/x^2} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$.

Solution. We have $\lim\limits_{x \to 0} x^k e^{-1/x^2} = 0$ for all $k = 0, \pm 1, \pm 2, \ldots$ by L'Hospital's rule. It is easily shown by induction that there is a polynomial p_n such that $f^{(n)}(x) = p_n(\frac{1}{x})e^{-1/x^2}$ for $x \neq 0$. Assuming (by induction) that $f^{(n)}(0) = 0$, we then have $f^{(n+1)}(0) = \lim\limits_{x \to 0} q_n(\dfrac{1}{x})e^{-1/x^2} = 0$, where $q_n(x) = xp_n(x)$.

Exercise 8.2 Let a_{ij} be the number in the ith row and jth column of the array

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & -1 & 0 & 0 & \ldots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \ldots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \quad \sum_j \sum_i = 0.$$

Solution. This is a routine computation:

$$\begin{aligned}\sum_i\sum_j a_{ij} &= \sum_{i=1}^{\infty}\left[-1+\sum_{j=1}^{i-1}2^{j-i}\right]\\ &= \sum_{i=1}^{\infty}[-1+(1-2^{1-i})]\\ &= \sum_{i=1}^{\infty}-2^{1-i}=-2,\end{aligned}$$

while

$$\begin{aligned}\sum_j\sum_i a_{ij} &= \sum_{j=1}^{\infty}\left[-1+\sum_{i=j+1}^{\infty}2^{j-i}\right]\\ &= \sum_{j=1}^{\infty}[-1+1]\\ &= 0.\end{aligned}$$

Exercise 8.3 Prove that

$$\sum_i\sum_j a_{ij}=\sum_j\sum_i a_{ij}$$

if $a_{ij}\ge 0$ for all i and j (the case $+\infty=+\infty$ may occur).

Solution. In fact the only case that we need to consider is the case when one of the two sums is infinite. If either sum is finite, we merely invoke Theorem 8.3, which explicitly states that the two sums are equal. Hence if either sum is infinite, then both are.

Exercise 8.4 Prove the following limit relations:

(a) $\displaystyle\lim_{x\to 0}\frac{b^x-1}{x}=\log b\quad(b>0).$

(b) $\displaystyle\lim_{x\to 0}\frac{\log(1+x)}{x}=1.$

(c) $\displaystyle\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e.$

(d) $\displaystyle\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x.$

Solution. (a) Consider the function $f(x)=b^x=e^{x\log b}$. The limit we are considering is $f'(0)$. By the chain rule

$$f'(x)=e^{x\log b}\log b.$$

Now take $x = 0$.

(*b*) Let $y = \log(1+x)$, so that $x = e^y - 1$. It is easy to justify the relation

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{y\to 0} \frac{y}{e^y - 1} = \frac{1}{\lim_{y\to 0} \frac{1}{\frac{e^y-1}{y}}} = 1,$$

since $\lim\limits_{y\to 0} \dfrac{e^y-1}{y} = E'(0)$.

(*c*) Consider the function $(1+x)^{1/x} = e^{\frac{\log(1+x)}{x}}$. By part (*b*) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e^1 = e$.

(*d*) As above, we have $\left(1+\dfrac{x}{n}\right)^n = \left[\left(1+\dfrac{x}{n}\right)^{1/(x/n)}\right]^x$, and by part (*c*) the limit of the expression inside the brackets is e.

Exercise 8.5 Find the following limits

(*a*) $\lim_{x\to 0} \frac{e-(1+x)^{1/x}}{x}$.

(*b*) $\lim_{n\to\infty} \frac{n}{\log n}[n^{1/n} - 1]$.

(*c*) $\lim_{x\to 0} \frac{\tan x - x}{x(1-\cos x)}$.

(*d*) $\lim_{x\to 0} \frac{x-\sin x}{\tan x - x}$.

Solution. (*a*) This limit is $f'(0)$, where $f(x) = (1+x)^{1/x}$ (by part (*c*) of the previous problem). Now for $x \neq 0$, we have

$$f'(x) = (1+x)^{1/x}\left[\frac{(1+x)\log(1+x) - x}{x^2(x+1)}\right].$$

Since we know that the limit of the first factor is e, we need only consider the limit inside the brackets. Since

$$(1+x)\log(1+x) = \left(x - \frac{x^2}{2} + \cdots\right) + x\left(x - \frac{x^2}{2} + \cdots\right),$$

we can cancel x^2 from the numerator and denominator of the expression in brackets, and we see that the limit of this expression is $\frac{1}{2}$. Hence the limit of $f'(x)$ as $x \to 0$ exists and equals $\frac{e}{2}$. It then follows from the mean-value theorem that $f'(0)$ equals this limit (see the corollary to Theorem 5.12).

(*b*) Write this expression as

$$\frac{e^{\frac{\log n}{n}} - 1}{\frac{\log n}{n}}.$$

Since $\frac{\log n}{n}$ tends to 0 as $n \to \infty$, this fraction tends to the derivative of e^x at 0, i.e., it tends to 1.

(*c*) Write this expression as

$$\frac{\sin x - x\cos x}{x\cos x(1-\cos x)}.$$

We can then use either Maclaurin series or L'Hospital's rule to prove that the limit is $\frac{2}{3}$.

(*d*) Write this expression as

$$\frac{(x-\sin x)\cos x}{\sin x - x\cos x}$$

and again either by Maclaurin series or L'Hospital's rule the limit is $\frac{1}{2}$.

Exercise 8.6 Suppose $f(x)f(y) = f(x+y)$ for all real x and y.

(*a*) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant.

(*b*) Prove the same thing, assuming only that f is continuous.

Solution. (*a*) Since f is not 0, it follows that $f(0) = 1$ (take $x = y = 0$ in the basic relation that defines f). It then follows that $f'(x) = f(x)f'(0)$, and hence that the function $g(x) = e^{-xf'(0)}f(x)$ satisfies $g'(0) = 0$ for all x. Therefore $g(x) = g(0) = f(0) = 1$ for all x, i.e., $f(x) = e^{cx}$, where $c = f'(0)$.

(*b*) The relation $f(x)f(y) = f(x+y)$ shows that either $f(x)$ is always zero, or it is never zero. In the latter case, since f is continuous, it cannot change sign, and therefore (since $f(0) = 1$) it is always positive. Let $g(x) = \log[f(x)]$. Then $g(x+y) = g(x) + g(y)$, and g is continuous. It suffices then to show that $g(x) = cx$ for some constant $c = g(1)$. To this end, we note that the additive property of g implies that $g(0) = 0$, $g(-x) = -g(x)$, and (by an easy induction) $g(nx) = ng(x)$ for all integers $n = 0, \pm 1, \pm 2, \ldots$. Consider the set of x such that $g(x) = g(1)x$. Obviously 0 and 1 belong to this set. If a belongs to this set, so does na for any n, since $g(na) = ng(a) = ng(1)a = g(1)(na)$. Finally, if a belongs to this set, so does $\frac{a}{n}$, $n = 1, 2, \ldots$, since $g(a) = g(n\frac{a}{n}) = ng(\frac{a}{n})$. That is, $g(\frac{a}{n}) = \frac{1}{n}g(a) = \frac{1}{n}g(1)a = g(1)\frac{a}{n}$. It now follows that r belongs to this set for all rational numbers r, that is, the two continuous functions $g(x)$ and $g(1)x$ have the same values at all rational numbers r. Since the rational numbers are dense, and the set of points at which two continuous functions are equal is a closed set, it follows that $g(x) = g(1)x$ for all x.

Exercise 8.7 If $0 < x < \frac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

Solution. To show the left-hand inequality, consider the function $f(x) = \sin x - \frac{2x}{\pi}$ on the interval $0 \le x \le \frac{\pi}{2}$. We have $f(0) = f(\frac{\pi}{2}) = 0$. Since $f''(x) = -\sin x \le 0$, the function $f'(x)$ is strictly decreasing on this interval. Therefore it has at most one zero on this interval; by Rolle's theorem, it has exactly one zero. Since $f''(x) < 0$ at that point, the function $f(x)$ has a maximum at that point. Therefore $f(x) > 0$ for $0 < x < \frac{\pi}{2}$.

The proof of the right-hand inequality is similar, but easier. The function $g(x) = x - \sin x$ has derivative $1 - \cos x$, which is nonnegative. Therefore $g(x)$ is strictly increasing, and so $g(x) > g(0) = 0$ for all $x > 0$ (the restriction $x < \frac{\pi}{2}$ is superfluous in this case).

Exercise 8.8 For $n = 0, 1, 2, \ldots$, and x real, prove that

$$|\sin nx| \le n|\sin x|.$$

Note that this inequality may be false for other values of n. For instance,

$$|\sin \frac{1}{2}\pi| > \frac{1}{2}|\sin \pi|.$$

Solution. The inequality is obvious if $n = 0$ or $n = 1$. Then by induction we have

$$\begin{aligned}
|\sin nx| &= |\sin((n-1)x + x)| \\
&= |\sin((n-1)x)\cos x + \cos((n-1)x)\sin x| \\
&\le |\sin((n-1)x)| + |\sin x| \\
&\le (n-1)|\sin x| + |\sin x| = |n|\,|\sin x|.
\end{aligned}$$

A stronger remark can be made: If c is *not* an integer, then $|\sin c\pi| > |c|\,|\sin \pi|$. Hence this inequality fails for $x = \pi$ unless c is an integer.

Exercise 8.9 (a) Put $s_N = 1 + (\frac{1}{2}) + \cdots + (1/N)$. Prove that

$$\lim_{N\to\infty} (s_N - \log N)$$

exists. (The limit, often denoted by γ, is called Euler's constant. Its numerical value is $0.5772\ldots$. It is not known whether γ is rational or not.)

(b) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$?

Solution. (a) We observe that $\log(N+1) - \log N = \displaystyle\int_N^{N+1} \frac{1}{t}\,dt$, so that $(s_{N+1} - \log(N+1)) - (s_N - \log N) = \frac{1}{N+1} - \int_N^{N+1} \frac{1}{t}\,dt < 0$. Thus the sequence is a decreasing sequence. On the other hand, it consists of positive numbers, since

$\log N = \int_1^N \frac{1}{t}\,dt < 1 + \frac{1}{2} + \cdots + \frac{1}{N-1} < s_N$. It follows that the sequence must converge to a nonnegative number γ.

(b) The answer here depends on how "rough" an estimate is desired. We observe that $s_{10^{N+1}} - s_{10^N}$ lies between $9 \cdot 10^N(\frac{1}{10^{N+1}})$ and $9 \cdot 10^N(\frac{1}{10^N})$, i.e., between 0.9 and 9. Hence by an easy induction $0.9N < s_{10^N} < 9N$. Thus $m = 112$ will certainly work, and m must be at least 12.

Exercise 8.10 Prove that $\sum 1/p$ diverges; the sum extends over all primes.

(This shows that the primes form a fairly substantial subset of the positive integers.)

Hint: Given N, let $p_1, \ldots, p_k$ be those primes that divide at least one integer $\leq N$. Then

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n} &\leq \prod_{j=1}^{k} \Big(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\Big) \\
&= \prod_{j=1}^{k} \Big(1 - \frac{1}{p_j}\Big)^{-1} \\
&\leq \exp \sum_{j=1}^{k} \frac{2}{p_j}.
\end{aligned}$$

The last inequality holds because

$$(1-x)^{-1} \leq e^{2x}$$

if $0 \leq x \leq \frac{1}{2}$.

(There are many proofs of this result. See, for instance, the article by I. Niven in *Amer. Math. Monthly*, vol. 78, 1971, pp. 272–273, and the one by R. Bellman in *Amer. Math. Monthly*, vol. 50, 1943, pp. 318–319.)

Solution. We observe that the primes $p_1, \ldots, p_k$ form the set of all primes not greater than N. Each of them is at least 2, and therefore each integer from 1 to N is a unique product of the form $p_1^{e_1} \cdots p_k^{e_k}$ for nonnegative integers e_j, $0 \leq e_j \leq \log_2 N$. For simplicity let m be the greatest integer in $\log_2 N$. Then certainly

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n} &\leq \sum_{e_1, \ldots, e_k = 0}^{m} \frac{1}{p_1^{e_1} \cdots p_k^{e_k}} \\
&= \prod_{j=1}^{k} \Big(1 + \frac{1}{p_j} + \cdots + \frac{1}{p_j^m}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{j=1}^{k}\Big(\frac{1-p_j^{-m-1}}{1-\frac{1}{p_j}}\Big)\\
&\le \prod_{j=1}^{k}\Big(\frac{1}{1-\frac{1}{p_j}}\Big)\\
&= \prod_{j=1}^{k}\Big(1-\frac{1}{p_j}\Big)^{-1}\\
&\le \exp\Big(\sum_{j=1}^{k}\frac{2}{p_j}\Big).
\end{aligned}
$$

To establish the inequality $(1-x)^{-1} \le e^{2x}$ on $[0, \frac{1}{2}]$, we simply observe that the function $f(x) = (1-x)e^{2x}$ has derivative $(1-2x)e^{2x}$, which is positive on this interval. Hence the smallest value this function has on the interval is its value at $x = 0$, which is 1.

We have now established the inequality

$$\sum_{j=1}^{k}\frac{1}{p_j} \ge \frac{1}{2}\log\Big(\sum_{n=1}^{N}\frac{1}{n}\Big)$$

for any integer N less than p_{k+1}. Since the right-hand side of this inequality tends to ∞, so does the left.

Exercise 8.11 Suppose $f \in \mathcal{R}$ on $[0, A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to +\infty$. Prove that

$$\lim_{t\to 0}\int_0^\infty e^{-tx}f(x)\,dx = 1 \quad (t > 0).$$

Solution. We first observe that the improper integral converges absolutely for all $t > 0$, since

$$\int_R^S e^{-tx}|f(x)|\,dx \le \frac{M}{t}(e^{-Rt} - e^{-St}) \to 0,$$

where $M = \sup\limits_{x\ge R}|f(x)|$, as $R, S \to \infty$.

We also note that

$$t\int_0^\infty e^{-tx}f(x)\,dx = \int_0^\infty e^{-u}f\Big(\frac{u}{t}\Big)\,du,$$

and this last improper integral also converges for all $t > 0$. Hence we have

$$
\begin{aligned}
\Big|t\int_0^\infty e^{-tx}f(x)\,dx - 1\Big| &= \Big|\int_0^\infty e^{-u}f\Big(\frac{u}{t}\Big)\,du - 1\Big|\\
&\le \int_0^\infty e^{-u}\Big|f\Big(\frac{u}{t}\Big) - 1\Big|\,dx.
\end{aligned}
$$

Since $f(x)$ has a limit at infinity and $f(x)$ is Riemann- integrable on $[0,1]$, it follows that $f(x) \leq K$ for some constant K and all x. Thus for any $\eta > 0$ we have

$$\begin{aligned}\left| t\int_0^\infty e^{-tx} f(x)\,dx - 1\right| &\leq (K+1)\int_0^\eta e^{-u}\,du + \int_\eta^\infty \left|f\Big(\frac{u}{t}\Big) - 1\right| du \\ &\leq \eta(K+1) + M_{\eta,t},\end{aligned}$$

where $M_{\eta,t} = \sup\limits_{z\geq\frac{\eta}{t}} |f(z) - 1|$. Hence, given $\varepsilon > 0$ we take $\eta = \frac{\varepsilon}{2(K+1)}$. We then choose $X > 0$ so large that $|f(z) - 1| < \frac{\varepsilon}{2}$ if $z > X$, and we let $\delta = \frac{\eta}{X}$. It then follows that $M_{\eta,t} < \frac{\varepsilon}{2}$ if $0 < t < \delta$.

Exercise 8.12 Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| < \pi$, and $f(x + 2\pi) = f(x)$ for all x.

(a) Compute the Fourier coefficients of f.

(b) Conclude that

$$\sum_{n=1}^\infty \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \qquad (0 < \delta < \pi).$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^\infty \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) Let $\delta \to 0$, and prove that

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

(e) Put $\delta = \pi/2$ in (c). What do you get?

Solution. (a) Since $f(x)$ is an even real-valued function, it makes sense to use the real form of the Fourier series, since symmetry shows that $b_n = 0$ for all n. Then $a_0 = \frac{2\delta}{\pi}$, and for $n \geq 1$ we have $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \dfrac{2}{\pi}\displaystyle\int_0^\delta \cos nx\,dx = \dfrac{2\sin n\delta}{\pi n}$.

(b) Since $f(x)$ satisfies the Lipschitz condition of Theorem 8.14 at $x = 0$, it follows that the series actually converges to $f(0)$ at that point, i.e.,

$$\frac{\delta}{\pi} + \frac{2}{\pi}\sum_{n=1}^\infty \frac{\sin(n\delta)}{n} = 1,$$

so that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}.$$

(*c*) Parseval's theorem now implies that

$$\frac{2\delta}{\pi} = \frac{1}{\pi}\int_{-\delta}^{\delta} |f(x)|^2\,dx = \frac{1}{2}\left(\frac{2\delta}{\pi}\right)^2 + \sum_{n=1}^{\infty} \frac{4\sin^2(n\delta)}{\pi^2 n^2}.$$

Now multiplying both sides by $\frac{\pi^2}{4\delta}$ gives the required result.

(*d*) Let R be any fixed number, N any positive integer, and let $\delta_N = \frac{R}{N}$. As $N \to \infty$ we have $\sum_{n=1}^{N} \frac{\sin^2(n\delta_N)}{n^2\delta_N} \to \int_0^R \left(\frac{\sin x}{x}\right)^2 dx$, since the left- hand side of this equality is a Riemann sum for this integral. Note that

$$\sum_{n=N+1}^{\infty} \frac{\sin^2(n\delta_N)}{n^2\delta_N} < \frac{1}{N\delta_N} = \frac{1}{R}.$$

(The inequality results from the fact that $\sum_{n=k}^{\infty} \frac{1}{n^2} < \int_{k-1}^{\infty} \frac{1}{t^2}\,dt = \frac{1}{k-1}$.) Given ε, choose $R > \frac{\varepsilon}{3}$ such that

$$\left|\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx - \int_0^{S} \left(\frac{\sin x}{x}\right)^2 dx\right| < \frac{\varepsilon}{3}$$

if $S > R$. Then choose $N_0 > \frac{3}{\varepsilon}$ so large that

$$\left|\sum_{n=1}^{N} \frac{\sin^2(n\delta_N)}{n^2\delta_N} - \int_0^{R} \left(\frac{\sin x}{x}\right)^2 dx\right| < \frac{\varepsilon}{3}$$

whenever $N > N_0$. Then for $N > N_0$, $\delta_N = \frac{R}{N}$ we have

$$\left|\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} - \int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx\right| < \varepsilon.$$

Consequently

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \lim_{N\to\infty} \frac{\pi - \delta_N}{2} = \frac{\pi}{2}.$$

(*e*) Taking $\delta = \pi/2$ yields

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

Exercise 8.13 Put $f(x) = x$ if $0 \le x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution. By computation we see that $a_n = 0$ for $n > 0$, and $a_0 = 2\pi$. Computation shows that $b_n = \frac{-2}{n}$. Hence Parseval's relation gives

$$\frac{8\pi^2}{3} = \frac{1}{2}(2\pi)^2 + 4\sum_{n=1}^{\infty} \frac{1}{n^2},$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

There is another way of deriving this result. Since

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2},$$

denoting this last sum by X, we find that

$$X - \frac{1}{4}X = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

and hence, by part (e) of the previous problem

$$X = \frac{4}{3}\frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

Exercise 8.14 If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(A recent article by E. L. Stark contains many references to series of the form $\sum n^{-s}$, where s is a positive integer. See *Math. Mag.*, vol. 47, 1974, pp. 197–202.)

Solution. Since $f(x)$ is an even function, $b_n = 0$ for all n. The a_n's are computed in a straightforward manner:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)^2\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3}\pi^2;$$

and

$$a_n = \frac{2}{\pi}\int_0^\pi (\pi - x)^2 \cos nx\, dx = (-1)^n \frac{2}{\pi}\int_0^\pi x^2 \cos nx\, dx,$$

so that, eventually, we find $a_n = \frac{4}{n^2}$.

This gives the stated Fourier series, and since $f(x)$ satisfies the Lipschitz condition of Theorem 8.14, the series converges to $f(x)$ at every point. Taking $x = 0$ gives the first of the two desired equalities:

$$\pi^2 = f(0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Parseval's theorem yields

$$\frac{2\pi^4}{5} = \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4},$$

which easily transforms to the desired relation.

Exercise 8.15 With D_n as defined in (77), put

$$K_n(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1}\frac{1-\cos(N+1)x}{1-\cos x}$$

and that

(*a*) $K_n \geq 0$,

(*b*) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\, dx = 1$.

(*c*) $K_n(x) \leq \frac{1}{N+1}\frac{2}{1-\cos\delta}$ if $0 < \delta \leq |x| \leq \pi$.

If $s_N = s_N(f;x)$ is the Nth partial sum of the Fourier series of f, consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$\sigma_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\, dt$$

and hence prove Fejér's theorem:

If f is continuous, with period 2π, then $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi,\pi]$.

Hint: Use properties (a), (b), (c) to proceed as in Theorem 7.26.

Solution. Using the formula $1 - \cos\theta = 2\sin^2\frac{1}{2}\theta$, and the formula $D_n(x) = \frac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}$, we deduce that

$$(1-\cos x)K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} \sin\frac{1}{2}x\sin(n+\frac{1}{2})x.$$

Now, however, $\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \cos(\alpha+\beta)$, so that

$$\begin{aligned}(1-\cos x)K_N(x) &= \frac{1}{N+1}\sum_{n=0}^{N}\big(\cos(nx) - \cos((n+1)x)\big) = \\ &= \frac{1}{N+1}\big(1-\cos(N+1)x\big).\end{aligned}$$

The formula is now established. Notice that it could also be written

$$K_N(x) = \frac{1}{N+1}\left[\frac{\sin\left(\frac{N+1}{2}x\right)}{\sin\frac{1}{2}x}\right]^2.$$

(a) The nonnegativity of $K_N(x)$ is an immediate consequence of either of the formulas just written.

(b) It was established in the text that $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\,dx = 1$, and so the same result for $K_N(x)$, which is an average of the $D_n(x)$, must follow by routine computation.

(c) This inequality is an immediate consequence of the facts that $\cos(N+1)x \geq -1$ and that $\cos x$ is decreasing on $[0,\pi]$.

The formula for $\sigma_N(f;x)$ is an immediate consequence of the definition of $\sigma_N(f;x)$ and the corresponding formula for $s_n(f;x)$.

Now let $M = \sup|f(x)|$, the supremum being taken over all x. By (a) and (b) we have

$$\begin{aligned}|\sigma_N(x) - f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x-t) - f(x)]K_N(t)\,dt\right| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-t) - f(x)|K_N(t)\,dt \\ &\leq \frac{1}{2\pi}\int_{-\delta}^{\delta}|f(x-t) - f(x)|K_N(t)\,dt + \\ &\quad + \frac{1}{\pi}(\pi-\delta)\frac{1}{N+1}\frac{2}{1-\cos\delta}2M \\ &\leq \sup_{|t|\leq\delta}|f(x-t) - f(x)| + \frac{Q_\delta}{N+1}\end{aligned}$$

where $Q_\delta = \frac{4M(\pi-\delta)}{\pi(N+1)(1-\cos\delta)}$.

Given $\varepsilon > 0$, we first choose $\delta > 0$ so small that $\sup_{|t|\leq\delta} |f(x-t) - f(x)| < \frac{\varepsilon}{2}$ for all x. With this δ fixed, we then have $|\sigma_N(f;x) - f(x)| < \varepsilon$ for all $N > \frac{2Q_\delta}{\varepsilon}$ and all x.

Exercise 8.16 Prove a pointwise version of Fejér's Theorem:

If $f \in \mathcal{R}$ and $f(x+)$, $f(x-)$ exist for some x, then

$$\lim_{N\to\infty} \sigma_N(f;x) = \frac{1}{2}[f(x+) + f(x-)].$$

Solution. We need only a slight modification of the argument just given, namely the formula

$$\sigma_N(f;x) - \frac{1}{2}[f(x+) + f(x-)] =$$

$$= \frac{1}{2\pi}\int_0^\pi [f(x-t) - f(x-)]K_N(t)\,dt + \frac{1}{2\pi}\int_{-\pi}^0 [f(x-t) - f(x+)]K_N(t)\,dt.$$

Each of these two integrals can be broken up into an integral over a half-neighborhood of 0 and an integral outside that neighborhood. The first of the integrals can be made small if the neighborhood is taken small enough (independently of N). With that neighborhood fixed, the second integral in each case can be made small if N is large enough using the same inequalities just stated.

Exercise 8.17 Assume f is bounded and monotonic on $[-\pi, \pi)$ with Fourier coefficients c_n, as given by (62).

(a) Use Exercise 17 of Chap. 6 to prove that $\{nc_n\}$ is a bounded sequence.

(b) Combine (a) with Exercise 16 and with Exercise 14(e) of Chap. 3, to conclude that

$$\lim_{n\to\infty} s_N(f;x) = \frac{1}{2}[f(x+) + f(x-)]$$

for every x.

(c) Assume only that $f \in \mathcal{R}$ on $[-\pi, \pi]$ and that f is monotonic in some segment $(\alpha, \beta) \subset [-\pi, \pi]$. Prove that the conclusion of (b) holds for every $x \in (\alpha, \beta)$.

(This is an application of the localization theorem.)

Solution. (a) by Exercise 17 of Chap. 6 we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{-1}{2\pi n}\int_{-\pi}^{\pi} e^{-inx}\,df(x),$$

from which it follows that

$$|nc_n| \leq \frac{1}{2\pi}[f(\pi) - f(-\pi)].$$

(*b*) Since $f(x+)$ and $f(x-)$ exist at every point, it follows from the previous exercise that $\sigma_N(f;x) \to \frac{1}{2}[f(x+)+f(x-)]$. Then Exercise 14(*e*) of Chap. 3 assures us that $s_n(f;x)$ has the same limit.

(*c*) Let $g(x) = f(x)$ for $\alpha \le x \le \beta$, $g(x) = f(\alpha)$ for $0 \le x \le \alpha$ and $g(x) = f(\beta)$ for $\beta \le x \le 2\pi$. Then $s_N(g;x) \to \frac{1}{2}[g(x+)+g(x-)]$ for all x by part (*b*). Since $s_N(g;x) - s_N(f;x) \to 0$ for $\alpha < x < \beta$ by the Corollary to Theorem 8.14, it follows that $s_N(f;x) \to \frac{1}{2}[g(x+)+g(x-)] = \frac{1}{2}[f(x+)+f(x-)]$ for these values of x.

Exercise 8.18 Define

$$\begin{aligned} f(x) &= x^3 - \sin^2 x \tan x \\ g(x) &= 2x^2 - \sin^2 x - x\tan x. \end{aligned}$$

Find out, for each of these two functions, whether it is positive or negative for all $x \in (0, \pi/2)$, or whether it changes sign. Prove your answer.

Solution. Both functions tend to $-\infty$ as $x \to \frac{\pi}{2}$. Hence the only question is whether they ever become positive. We note that the derivative of the first function is $3x^2 - \sin^2 x - \tan^2 x$. By writing $\sin^2 x$ as $\frac{1}{2} - \frac{1}{2}\cos 2x$ and making repeated use of the formula $\frac{d}{dx}\tan^k x = k\tan^{k-1} x + k\tan^{k+1} x$, we find that the first six derivatives of this function vanish at 0, and that the sixth derivative is

$$-32\sin 2x - 272\tan x - 1232\tan^3 x - 1104\tan^5 x - 144\tan^7 x,$$

which is negative on $(0, \frac{\pi}{2})$. Hence all of the first six derivatives are negative on this interval, and therefore the function itself is negative.

The same technique applies to the second function. All of its first five derivatives vanish at $x = 0$ and the fifth is

$$\begin{aligned} &-[16\sin 2x + 16x + 80\tan x + 136x\tan^2 x + \\ &\qquad + 200\tan^3 x + 240x\tan^4 x + 120\tan^5 x + 120x\tan^6 x], \end{aligned}$$

which is negative on $(0, \frac{\pi}{2})$. Hence this function is always negative on that interval.

Exercise 8.19 Suppose f is a continuous function on R^1, $f(x+2\pi) = f(x)$, and α/π is irrational. Prove that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{\infty} f(x+n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

for every x. *Hint:* Do it first for $f(x) = e^{ikx}$.

Solution. Following the hint, we observe that both sides of the desired equality equal 1 trivially when $k = 0$. In any other case the right-hand side is zero, and the left-hand side is

$$\lim_{N\to\infty} e^{ikx} \frac{1 - e^{i(n+1)k\alpha}}{N(1 - e^{i\alpha})},$$

which tends to zero as $N \to \infty$.

Since both sides are linear functions of f, it now follows that the relation holds for all trigonometric polynomials. Finally, since both sides are bounded by the supremum of f, given ε, we can approximate f uniformly within ε by a trigonometric polynomial. It then follows that all the means on the left, for N sufficiently large, are within 2ε of the integral on the right. Since ε is arbitrary, it follows that the limit on the left equals the integral on the right.

Exercise 8.20 The following simple computation yields a good approximation to Stirling's formula.

For $m = 1, 2, 3, \ldots$, define

$$f(x) = (m + 1 - x)\log m - (x - m)\log(m + 1)$$

if $m \le x \le m + 1$, and define

$$g(x) = \frac{x}{m} - 1 + \log m$$

if $m - \frac{1}{2} \le x < m + \frac{1}{2}$. Draw the graphs of f and g. Note that $f(x) \le \log x \le g(x)$ if $x \ge 1$ and that

$$\int_1^n f(x)\,dx = \log(n!) - \frac{1}{2}\log n > -\frac{1}{8} + \int_1^n g(x)\,dx.$$

Integrate $\log x$ over $[1, n]$. Conclude that

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right)\log n + n < 1$$

for $n = 2, 3, 4, \ldots$. (*Note:* $\log\sqrt{2\pi} \sim 0.918\ldots$.) Thus

$$e^{7/8} < \frac{n!}{(n/e)^n\sqrt{n}} < e.$$

Solution. We first draw the graphs of f and g in the range $x = 1$ to $x = 10$. We note that f is merely the broken set of chords joining the points on the graph of $\log x$ at integer values of x, and g is made up of segments of the tangents at these points (g is not continuous). Because the downward side of the graph of $\log x$ is convex, $f(x) \le \log x \le g(x)$ for all x. The estimate for the integral of f is straightforward: The integral is the sum of the areas of one triangle and $n - 2$ trapezoids with base 1 and parallel sides $\log k$ and $\log(k + 1)$ ($k = 2, \ldots, n - 1$).

We find it equal to $\frac{1}{2}\log 1 + \log 2 + \log 3 + \cdots + \log(n-1) + \frac{1}{2}\log n = \log(n!) - \frac{1}{2}\log n$, as asserted. Meanwhile the integral of $g(x)$ is also a sum of trapezoids and two triangles and equals $\frac{1}{8} + \log(n!) - \frac{1}{2}\log n - \frac{1}{8n}$. Hence we have

$$\log(n!) - \frac{1}{2}\log n < \int_1^n \log x\, dx < \frac{1}{8} + \log(n!) - \frac{1}{2}\log n - \frac{1}{8n} < \frac{1}{8} + \log(n!) - \frac{1}{2}\log n.$$

Now straightforward computation reveals that

$$\int_1^n \log x\, dx = (n\log n - n) - (1\log 1 - 1) = n\log n - n + 1.$$

The desired inequalities are now deduced by taking exponentials of the three expressions.

Exercise 8.21 Let

$$L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|\, dt \quad (n = 1, 2, 3, \ldots).$$

Prove that there exists a constant $C > 0$ such that

$$L_n > C\log n \quad (n = 1, 2, 3, \ldots),$$

or, more precisely, that the sequence

$$\left\{L_n - \frac{4}{\pi^2}\log n\right\}$$

is bounded.

Solution. We observe that

$$L_n = \frac{1}{\pi}\int_0^{\frac{2\pi}{2n+1}} \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dt +$$

$$+ \sum_{k=1}^{n-1} \frac{1}{\pi}\int_{\frac{2\pi k}{2n+1}}^{\frac{2\pi(k+1)}{2n+1}} \frac{(-1)^k \sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dt + \frac{1}{\pi}\int_{\frac{2n\pi}{2n+1}}^{\pi} \frac{(-1)^n \sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\, dt$$

The substitution $u = (n+\frac{1}{2})t$ changes the first and last terms into the sum

$$\frac{1}{\pi}\int_0^{\pi} \frac{\sin u}{(n+\frac{1}{2})\sin(\frac{u}{2n+1})}\, du + \int_{n\pi}^{(n+\frac{1}{2})\pi} \frac{(-1)^n \sin u}{(n+\frac{1}{2})\sin(\frac{u}{2n+1})}\, du.$$

The first of these terms tends to $\frac{1}{2\pi}\int_0^{\pi} \sin u\, du = \frac{1}{\pi}$ as $n \to \infty$. The second tends to 0 (for $u \in [n\pi, (n+\frac{1}{2})\pi]$ we have $\sin(\frac{u}{2n+1}) \geq \sin\frac{n\pi}{2n+1}$, which tends to 1 as $n \to \infty$).

Thus we find that

$$\frac{1}{\pi} + \varepsilon_n + \sum_{k=1}^{n-1} \frac{1}{\pi \sin(\frac{\pi(k+1)}{2n+1})} \Big| \int_{\frac{2\pi k}{2n+1}}^{\frac{2\pi(k+1)}{2n+1}} \sin(n+\frac{1}{2})t\,dt \Big| < L_n,$$

where $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$.

If we take out the first two terms of the sum instead of just the first, we find similarly that

$$L_n = \frac{1}{\pi} \int_0^{\frac{4\pi}{2n+1}} \frac{|\sin(n+\frac{1}{2})t|}{\sin\frac{1}{2}t}\,dt +$$

$$+ \sum_{k=2}^{n-1} \frac{1}{\pi} \int_{\frac{2\pi k}{2n+1}}^{\frac{2\pi(k+1)}{2n+1}} \frac{(-1)^k \sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt + \frac{1}{\pi} \int_{\frac{2n\pi}{2n+1}}^{\pi} \frac{(-1)^n \sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt.$$

Again the substitution $u = (n+\frac{1}{2})t$ changes the first and last terms into the sum

$$\frac{1}{\pi} \int_0^{2\pi} \frac{\sin u}{(n+\frac{1}{2})\sin(\frac{u}{2n+1})}\,du + \int_{n\pi}^{(n+\frac{1}{2})\pi} \frac{(-1)^n \sin u}{(n+\frac{1}{2})\sin(\frac{u}{2n+1})}\,du.$$

The first of these terms tends to $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} |\sin u|\,du = \frac{2}{\pi}$ as $n \longrightarrow \infty$, and once again the second tends to zero.

Thus we find that

$$L_n < \frac{2}{\pi} + \eta_n + \sum_{k=2}^{n-1} \frac{1}{\pi \sin(\frac{\pi k}{2n+1})} \Big| \int_{\frac{2\pi k}{2n+1}}^{\frac{2\pi(k+1)}{2n+1}} \sin(n+\frac{1}{2})t\,dt \Big|,$$

where $\eta_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Once again, in each of the integrals under the sigma in the last two inequalities we make the substitution $u = (n+\frac{1}{2})t$. When we do so, we have

$$\frac{1}{\pi} + \varepsilon + \sum_{k=1}^{n-1} \frac{2}{(n+\frac{1}{2})\pi \sin(\frac{\pi(k+1)}{2n+1})} < L_n < \frac{2}{\pi} + \eta_n + \sum_{k=2}^{n-1} \frac{2}{(n+\frac{1}{2})\pi \sin(\frac{\pi k}{2n+1})},$$

where $\varepsilon_n \longrightarrow 0$ and $\eta_n \longrightarrow 0$. It therefore follows that

$$\frac{1}{\pi} + \varepsilon_n < L_n - \sum_{k=1}^{n-1} \frac{2}{\pi} \frac{1}{(n+\frac{1}{2})\sin(\frac{\pi(k+1)}{2n+1})} < \frac{2}{\pi} + \eta_n - \frac{1}{(n+\frac{1}{2})\sin(\frac{\pi n}{2n+1})}.$$

The extremes in these inequalities are both bounded. Hence we will be done if we can show that

$$\frac{2}{\pi}\log n - \sum_{k=1}^{n-1} \frac{1}{(n+\frac{1}{2})\sin(\frac{\pi(k+1)}{2n+1})}$$

remains bounded. To do this, we use the fact that there is a constant K such that

$$\left|\frac{1}{\sin x} - \frac{1}{x}\right| \leq Kx$$

for $0 < x \leq \frac{\pi}{2}$. This fact in turn is a consequence of the fact that, by L'Hospital's rule,

$$\lim_{x \to 0} \frac{x - \sin x}{x^2 \sin x} = -\frac{1}{2}.$$

We thus have

$$\sum_{k=1}^{n-1} \frac{1}{(n+\frac{1}{2})\sin(\frac{\pi(k+1)}{2n+1})} = E_n + \sum_{k=1}^{n-1} \frac{1}{(n+\frac{1}{2})(\frac{\pi(k+1)}{2n+1})},$$

where

$$|E_n| \leq K \frac{1}{n+\frac{1}{2}} \sum_{k=1}^{n-1} \frac{\pi(k+1)}{2n+1} =$$

$$= \frac{2K}{\pi(2n+1)^2} \sum_{k=1}^{n} k+1 = \frac{2K}{\pi(2n+1)^2}\left[\frac{(n+1)(n+2)}{2} - 1\right].$$

Since the right-hand side tends to $\frac{K}{4\pi}$ as $n \to \infty$, we see that E_n remains bounded as $n \to \infty$. We will be finished if we can show that

$$\log n - \sum_{k=1}^{n-1} \frac{1}{k+1}$$

remains bounded. But this was done in Exercise 9 above.

Exercise 8.22 If α is real and $-1 < x < 1$, prove Newton's binomial theorem

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n.$$

Hint: Denote the right side by $f(x)$. Prove that the series converges. Prove that

$$(1+x)f'(x) = \alpha f(x).$$

and solve this differential equation.

Show also that

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^n$$

if $-1 < x < 1$ and $\alpha > 0$.

Solution. Following the hint, we use the ratio test to establish that the radius of convergence of the power series that defines $f(x)$ is 1. This amounts merely to the statement that

$$\lim_{n\to\infty}\left|\frac{\alpha-n}{n+1}\right|=1.$$

The differential equation then results from termwise operations on the series and the fact that

$$\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}=\alpha\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Then, given that $f(0)=1\neq 0$, it follows that for x near 0 we have

$$\frac{f'(x)}{f(x)}=\frac{\alpha}{1+x}.$$

so that $\log f(x)$ and $\log(1+x)^\alpha$ have the same derivative, and hence differ by a constant, which turns out to be zero, since both equal 1 at $x=0$. It thus follows that $f(x)=(1+x)^\alpha$.

To prove the other relation, we merely observe that

$$(-\alpha(-\alpha-1)\cdots(-\alpha-n+1)=(-1)^n\alpha(\alpha+1)\cdots(\alpha+n-1)=(-1)^n\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}.$$

Exercise 8.23 Let γ be a continuously differentiable *closed* curve in the complex plane with parameter interval $[a,b]$, and assume that $\gamma(t)\neq 0$ for every $t\in[a,b]$. Define the *index* of γ to be

$$\mathrm{Ind}\,(\gamma)=\frac{1}{2\pi i}\int_a^b\frac{\gamma'(t)}{\gamma(t)}\,dt.$$

Prove that $\mathrm{Ind}\,(\gamma)$ is always an integer.

Hint: There exists φ on $[a,b]$ with $\varphi'=\gamma'/\gamma$, $\varphi(a)=0$. Hence $\gamma\exp(-\varphi)$ is constant. Since $\gamma(a)=\gamma(b)$, it follows that $\exp(\varphi(a))=\exp(\varphi(b))=1$. Note that $\varphi(b)=2\pi i\mathrm{Ind}\,(\gamma)$.

Compute $\mathrm{Ind}\,(\gamma)$ when $\gamma(t)=e^{int}$, $a=0$, $b=2\pi$.

Explain why $\mathrm{Ind}\,(\gamma)$ is often called the *winding number* of γ around 0.

Solution. Again, following the hint leaves very little to do. We define

$$\varphi(x)=\int_a^x\frac{\gamma'(t)}{\gamma(t)}\,dt,$$

so that we automatically have $\varphi'(t)=\frac{\gamma'(t)}{\gamma(t)}$. The fact that $\gamma\exp(-\varphi)$ is constant is now a consequence of the chain rule. It then follows immediately that $\exp(\varphi(b))=1$, so that $\varphi(b)=2\pi i n$ for some integer n.

Routine computation shows that $\mathrm{Ind}\,(\gamma)=n$ if $\gamma(t)=e^{int}$, $0\le t\le 2\pi$. Since this curve winds counterclockwise about 0 a total of n times, the name *winding number* is appropriate.

Exercise 8.24 Let γ be as in Exercise 23, and assume in addition that the range of γ does not intersect the negative real axis. Prove that $\operatorname{Ind}(\gamma) = 0$. *Hint:* For $0 \le c < \infty$, $\operatorname{Ind}(\gamma + c)$ is a continuous integer-valued function of c. Also, $\operatorname{Ind}(\gamma + c) \to 0$ as $c \to \infty$.

Solution. Following the hint, we observe that

$$f(c) = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)+c}\,dt,$$

is a continuous function of c on $[0, \infty)$, since

$$|f(c_1) - f(c_2)| = \frac{1}{2\pi}\left|\int_a^b \frac{\gamma'(t)(c_1 - c_2)}{(\gamma(t)+c_1)(\gamma(t)+c_2)}\,dt\right| \le K|c_1 - c_2|,$$

where $K = \dfrac{1}{2\pi r^2}\displaystyle\int_a^b |\gamma'(t)|\,dt$ and r is the supremum of the integrand for $c_1, c_2 \ge 0$ and $0 \le t \le 2\pi$. (This supremum is finite, since the integrand tends to zero as either c_1 or c_2 tends to infinity.) Furthermore

$$|f(c)| \le \frac{1}{2\pi c}\int_a^b \frac{|\gamma'(t)|}{|1 + \frac{\gamma(t)}{c}|}\,dt,$$

and this last expression tends to 0 as $c \to \infty$. It follows, since f assumes only integer values, that $f(c) \equiv 0$. In particular $f(0) = \operatorname{Ind}(\gamma) = 0$.

Exercise 8.25 Suppose γ_1 and γ_2 are curves as in Exercise 23, and

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)| \quad (a \le t \le b)$$

Prove that $\operatorname{Ind}(\gamma_1) = \operatorname{Ind}(\gamma_2)$.

Hint: Put $\gamma = \gamma_2/\gamma_1$. Then $|1 - \gamma| < 1$. Hence $\operatorname{Ind}(\gamma) = 0$ by Exercise 24. Also,

$$\frac{\gamma'}{\gamma} = \frac{\gamma_2'}{\gamma_2} - \frac{\gamma_1'}{\gamma_1}.$$

Solution. The hint leaves almost nothing to be done. The inequality established for γ shows that the real part of γ is always positive, so that the hypotheses of Exercise 24 are satisfied. The relation for $\frac{\gamma'}{\gamma}$ is a routine computation, and shows in general that $\operatorname{Ind}(\gamma\delta) = \operatorname{Ind}(\gamma) + \operatorname{Ind}(\delta)$.

Exercise 8.26 Let γ be a *closed* curve in the complex plane (not necessarily differentiable) with parameter interval $[0, 2\pi]$, such that $\gamma(t) \neq 0$ for every $t \in [0, 2\pi]$.

Choose $\delta > 0$ such that $|\gamma(t)| > \delta$ for all $t \in [0, 2\pi]$. If P_1 and P_2 are trigonometric polynomials such that $|P_i(t) - \gamma(t)| < \delta/4$ for all $t \in [0, 2\pi]$, (their existence is assured by Theorem 8.15), prove that

$$\operatorname{Ind}(P_1) = \operatorname{Ind}(P_2)$$

by applying Exercise 25.

Define this common value to be $\operatorname{Ind}(\gamma)$.

Prove that the statements of Exercises 24 and 25 hold without any differentiability assumptions.

Solution. Since $|P_1(t) - P_2(t)| < \frac{\delta}{2} < |P_1(t)|$, (because $|P_1(t)| \geq |f(t)| - |f(t) - P_1(t)| > \frac{3\delta}{4}$), the equality of the indices follows from Exercise 25, as stated.

Exercise 24 remains valid, since if $\gamma(t)$ does not intersect the negative real axis, there is a positive number $\delta > 0$ such that $|\gamma(t) - x| \geq \delta$ for all $x \leq 0$. Then if $|P_j(t) - \gamma(t)| < \delta$ for all $t \in [0, 2\pi]$, it follows that $P_j(t)$ also does not intersect the negative real axis, hence has winding number 0.

Exercise 25 remains valid, since if $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$ for all t, we can let $\delta = \min_t |\gamma_1(t)| - |\gamma_1(t) - \gamma_2(t)|$. Then if $|P_i(t) - \gamma_i(t)| < \delta/4$ for all t, it follows that $|P_1(t) - P_2(t)| \leq |\gamma_1(t) - \gamma_2(t)| + (\delta/2) < |\gamma_1(t)| - (\delta/4) \leq |P_1(t)|$, and so $\operatorname{Ind}(P_1) = \operatorname{Ind}(P_2)$, by Exercise 25.

Exercise 8.27 Let f be a continuous complex function defined in the complex plane. Suppose there is a positive integer n and a complex number $c \neq 0$ such that

$$\lim_{|z| \to \infty} z^{-n} \gamma(z) = c.$$

Prove that $f(z) = 0$ for at least one complex number z.

Note that this is a generalization of Theorem 8.8.

Hint: Assume $f(z) \neq 0$ for all z, define

$$\gamma_r(t) = f(re^{it\theta})$$

for $0 \leq r < \infty$, $0 \leq t \leq 2\pi$, and prove the following statements about the curve γ.

(a) $\operatorname{Ind}(\gamma_0) = 0$.

(b) $\operatorname{Ind}(\gamma_r) = n$ for all sufficiently large r.

(c) $\operatorname{Ind}(\gamma_r)$ is a continuous function of r on $[0, \infty)$.

[In (b) and (c), use the last part of Exercise 26.]

Show that (a), (b), and (c) are contradictory, since $n > 0$.

Solution. (a) Since $\gamma_0(t) = f(0)$ for all t, we have $\gamma_0'(t) = 0$ for all t, and hence by definition $\operatorname{Ind}(\gamma_0) = 0$.

(b) Choose R so large that $|z^{-n} f(z) - c| < \frac{|c|}{2}$ whenever $|z| > R$. Then for all r we have $\operatorname{Ind}(\gamma_r) = \operatorname{Ind}(\gamma_{r1}) + \operatorname{Ind}(\gamma_{r2})$, where $\gamma_{r1}(t) = r^n e^{int}$ and

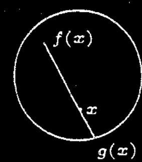

Figure 8.1: The Brouwer fixed-point theorem

$\gamma_{r2}(t) = r^{-n}e^{-int}f(re^{it})$. By Exercise 25 we have $\operatorname{Ind}(\gamma_{r2}) = 0$ for $r > R$, and by direct computation we have $\operatorname{Ind}(\gamma_{r1}) = n$ for all r.

(c) Fix $r_0 > 0$, and let $\varepsilon = \min_{0\le t\le 2\pi} |f(r_0e^{it})|$. Then choose $\delta \in (0, r_0)$ such that $|f(r_0e^{it}) - f(re^{it})| < \varepsilon$ if $|r - r_0| < \delta$. Then by Exercise 25 we again have $\operatorname{Ind}(\gamma_r) = \operatorname{Ind}(\gamma_{r_0})$ for $|r - r_0| < \delta$. Hence $\operatorname{Ind}(\gamma_r)$ is a locally constant function of r. By the connectivity of $[0, \infty)$, it follows that it is globally constant, which contradicts (a) and (b).

Exercise 8.28 Let $\overline{D}$ be the closed unit disc in the complex plane. (Thus $z \in \overline{D}$ if and only if $|z| \le 1$.) Let g be a continuous mapping of $\overline{D}$ into the unit circle T. (Thus $|g(z)| = 1$ for every $z \in \overline{D}$.)

Prove that $g(z) = -z$ for at least one $z \in T$.

Hint: For $0 \le r \le 1$, $0 \le t \le 2\pi$, put

$$\gamma_r(t) = g(re^{it}),$$

and put $\psi(t) = e^{-it}\gamma_1(t)$. If $g(z) \ne -z$ for every $z \in T$, then $\psi(t) \ne -1$ for every $t \in [0, 2\pi]$. Hence $\operatorname{Ind}(\psi) = 0$, by Exercises 24 and 25. It follows that $\operatorname{Ind}(\gamma_1) = 1$. But $\operatorname{Ind}(\gamma_0) = 0$. Derive a contradiction, as in Exercise 27.

Solution. The hint tells us that $\psi(t)$ does not meet the negative real axis, hence has index 0, by Exercise 24. Hence by Exercise 25, γ_1 has index 1. Again, since $\gamma_0 = g(0) \ne 0$ (since $g(0) \ne -0 = 0$), it follows that $\operatorname{Ind}(\gamma_0) = 0$. But, as before, since $|g(z)| = 1$ for all z, it follows that $\operatorname{Ind}(\gamma_r)$ is locally constant and hence by the connectivity of $[0, 1]$, globally constant. Thus, once again, we have a contradiction.

Exercise 8.29 Prove that every continuous mapping f of $\overline{D}$ into $\overline{D}$ has a fixed point in $\overline{D}$.

(This is the 2-dimensional case of Brouwer's fixed-point theorem.)

Hint: Assume $f(z) \ne z$ for every $z \in \overline{D}$. Associate to each $z \in \overline{D}$ the point $g(z) \in T$ which lies on the ray that starts at $f(z)$ and passes through z. Then g maps $\overline{D}$ into T, $g(z) = z$ if $z \in T$, and g is continuous, because

$$g(z) = z - s(z)[f(z) - z],$$

where $s(z)$ is the unique nonnegative root of a certain quadratic equation whose coefficients are continuous functions of f and z. Apply Exercise 28.

Solution. The number $s = s(z)$ is a nonnegative real number because of the geometry of the situation (see figure). The quadratic equation in question is given by the relation $|g(z)|^2 = 1$, i.e.,

$$|f(z) - z|^2 s^2 + 2\big(|z|^2 - \mathrm{Re}\,(\bar{z}f(z))\big)s + |z|^2 - 1 = 0.$$

It is well-known that a quadratic equation $az^2 + bz + c = 0$ has one and only one nonnegative root if a, b, and c are real and $ac < 0$. We can write explicitly

$$s(z) = \frac{|z|^2 - \mathrm{Re}\,(\bar{z}f(z)) + \sqrt{(|z|^2 - \mathrm{Re}\,(\bar{z}f(z))^2 + |f(z) - z|^2(1 - |z|^2)}}{|f(z) - z|^2}.$$

which makes it clear that $s(z)$ is a continuous function of z. Hence $g(z)$ is continuous.

We now know that there must be a value at which $g(z) = -z$. But this is impossible, since $|g(z)| = 1$ for all z and $g(z) = z$ if $|z| = 1$.

Exercise 8.30 Use Stirling's formula to prove that

$$\lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} = 1$$

for every real constant c.

Solution. We need Stirling's formula in the form

$$\lim_{z\to\infty} \frac{\Gamma(z)}{\left(\frac{z-1}{e}\right)^{z-1}\sqrt{2\pi(z-1)}} = 1.$$

Applying this result with $z = x + c$ and $z = x$, we get

$$\lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} =$$

$$= \lim_{x\to\infty} f(x) \cdot \frac{\Gamma(x+c)}{(\frac{x+c-1}{e})^{x+c-1}\sqrt{2\pi(x+c-1)}} \cdot \frac{(\frac{x-1}{e})^{x-1}\sqrt{2\pi(x-1)}}{\Gamma(x)},$$

where

$$f(x) = \frac{1}{x^c} \cdot \frac{(\frac{x+c-1}{e})^{x+c-1}}{(\frac{x-1}{e})^{x-1}} \cdot \sqrt{\frac{x+c-1}{x-1}} = \frac{(1+\frac{c-1}{x})^x(1+\frac{c-1}{x})^{c-1}}{e^c(1-\frac{1}{x})^x(1-\frac{1}{x})^{-1}} \cdot \sqrt{\frac{x+c-1}{x-1}}.$$

Since $x^x \to 1$ as $x \to \infty$, it now follows that $\lim_{x\to\infty} f(x) = 1$, which, combined with Stirling's formula, gives the desired result.

Exercise 8.31 In the proof of Theorem 7.26 it was shown that

$$\int_{-1}^{1}(1-x^2)^n\,dx \geq \frac{4}{3\sqrt{\pi}}.$$

for $n = 1, 2, 3, \ldots$. Use Theorem 8.20 and Exercise 30 to show the more precise result

$$\lim_{n\to\infty}\sqrt{n}\int_{-1}^{1}(1-x^2)^n\,dx = \sqrt{\pi}.$$

Solution. Let $u = x^2$ in the integral, so that $dx = \frac{1}{2}u^{-\frac{1}{2}}\,du$. We then have

$$\sqrt{n}\int_{-1}^{1}(1-x^2)^n\,dx = \sqrt{n}\int_{0}^{1}(1-u)^n u^{-\frac{1}{2}}\,du = \frac{\sqrt{n}\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(n+1)},$$

and taking $c = \frac{1}{2}$ in Exercise 30, we find that this last expression tends to $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.