AN OUTLINE SUMMARY OF BASIC POINT SET TOPOLOGY

J.P. MAY

We give a quick outline of a bare bones introduction to point set topology. The focus is on basic concepts and definitions rather than on the examples that give substance to the subject.

1. Topological spaces

Definition 1.1. A *topology* on a set X is a set of subsets, called the *open sets*, which satisfies the following conditions.

(i) The empty set $\emptyset$ and the set X are open.
(ii) Any finite intersection of open sets is open.
(iii) Any union of open sets is open.

A *neighborhood* of a point $x \in X$ is an open set U such that $x \in U$.

We sometimes write $\mathscr{T}$ for the set of open sets defining a topology, and write $(X, \mathscr{T})$ for the set X with the topology $\mathscr{T}$. More usually, when the topology $\mathscr{T}$ is understood, we just say that X is a topological space. We say that a topology $\mathscr{T}$ is *finer* than a topology $\mathscr{T}'$ if every set in $\mathscr{T}'$ is also in $\mathscr{T}$ ($\mathscr{T}$ has more open sets). We then say that $\mathscr{T}'$ is *coarser* than $\mathscr{T}$. The finest topology is the *discrete* topology, in which all subsets are open. The coarsest topology is the *trivial* topology, in which $\emptyset$ and X are the only open sets.

Definition 1.2. Let X be a topological space. A subset of X is *closed* if its complement is open. The closed sets satisfy the following conditions.

(i) The empty set $\emptyset$ and the set X are closed.
(ii) Any intersection of closed sets is closed.
(iii) Any finite union of closed sets is closed.

In practice, one often has a preferred collection of "small" or canonical open sets, a"basis" from which all other open sets are generated.

Definition 1.3. A *basis* for a topology on X is a set $\mathscr{B}$ of subsets of X such that

(i) For each $x \in X$, there is at least one $B \in \mathscr{B}$ such that $x \in B$.
(ii) If $x \in B' \cap B''$ where $B', B'' \in \mathscr{B}$, then there is at least one $B \in \mathscr{B}$ such that $x \in B \subset B' \cap B''$.

The topology $\mathscr{T}$ *generated* by the basis $\mathscr{B}$ is the set of subsets U such that, for every point $x \in U$, there is a $B \in \mathscr{B}$ such that $x \in B \subset U$. Equivalently, a set U is in $\mathscr{T}$ if and only if it is a union of sets in $\mathscr{B}$.

In the definition, we did not assume that we started with a topology on X. A given topology usually admits many different bases.

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Lemma 1.4. *Let $(X, \mathscr{T})$ be a topological space. A subset $\mathscr{B}$ of $\mathscr{T}$ is a basis that generates $\mathscr{T}$ if and only if for every $U \in \mathscr{T}$ and every $x \in U$, there is a $B \in \mathscr{B}$ such that $x \in B \subset U$.*

We can generate bases for topologies from subbases.

Definition 1.5. A *subbasis* for a topology on X is a set $\mathscr{S}$ of subsets of X whose union is X; that is, $\mathscr{S}$ is a *cover* of X. The set of finite intersections of sets in $\mathscr{S}$ is the basis generated by $\mathscr{S}$.

Definition 1.6. Let A be a subset of a topological space X. The *interior* $\mathring{A}$ of A is the union of the open subsets of X contained in A. The *closure* $\bar{A}$ of A is the intersection of the closed sets containing A. A point $x \in X$ is a *limit point* of A if every neighborhood of x contains a point $a \neq x$ of A. A is *dense* in X if $\bar{A} = X$.

Proposition 1.7. *A point $x \in X$ is in $\bar{A}$ if and only if every neighborhood of x contains a point of A, and $\bar{A}$ is the union of A and the set of limit points of A. The set A is closed if and only if it contains all of its limit points.*

There is a hierarchy of "separation properties" of spaces, beginning with the T_1 property. Perhaps the most fundamental is the Hausdorff property.

Definition 1.8. A space X is T_1 is each singleton set $\{x\}$ is closed. A space is Hausdorff if for each pair of points $x \neq y$, there are neighborhoods U of x and V of y such that $U \cap V = \emptyset$; Hausdorff spaces are also called T_2 spaces.

Proposition 1.9. *If X is Hausdorff then X is T_1.*

Proposition 1.10. *If A is a subset of a Hausdorff space X, then x is a limit point of A if and only if every neighborhood of x contains infinitely many points in A.*

Lemma 1.11. *Let A be a subspace of a space X. A continuous function from A to a Hausdorff space Y admits at most one extension to a continuous map $\bar{A} \longrightarrow Y$.*

There are many ways to obtain new topological spaces from given spaces.

Definition 1.12. Let X and Y be topological spaces. The *product topology* on $X \times Y$ is the topology generated by the basis $\{U \times V\}$, where U is open in X and V is open in Y. More generally, for a set $\{X_i | i \in I\}$ of topological spaces, the product topology on the product set $\prod_{i \in I} X_i$ is the topology generated by the basis $\{\prod_{i \in I} U_i\}$ where U_i is open in X_i and $U_i = X_i$ for all but finitely many i.

Lemma 1.13. *A space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x)\}$ is a closed subset of $X \times X$.*

Definition 1.14. Let A be a subset of a topological space X. The *subspace topology* on A is the set of intersections $A \cap U$, where U is open in X.

We have a consistency observation relating the subspace and product topologies.

Proposition 1.15. *If $A \subset X$ and $B \subset Y$, then the subspace and product topologies on $A \times B$ coincide.*

Subspace topologies are defined for injective functions. There is a perhaps less intuitive analogue for surjective functions.

Definition 1.16. Let X be a topological space and $q : X \longrightarrow Y$ be a surjective function. The quotient topology on Y is the set of subsets U such that $q^{-1}(U)$ is open in X.

A subspace of a Hausdorff space is Hausdorff, a product of Hausdorff spaces is Hausdorff, but in general a quotient of a Hausdorff space need not be Hausdorff.

2. Continuous functions and homeomorphisms

Definition 2.1. Let X and Y be spaces. A function $f : X \longrightarrow Y$ is *continuous* if $f^{-1}(U)$ is open in X for all open subsets U of Y. A continuous function is often called a *map*.

It suffices for this that $f^{-1}(U)$ be open for each U in a basis for the topology for Y, or even for each U in a subbasis. By passage to complements, a function f is continuous if and only if $f^{-1}(C)$ is closed in X for all closed subsets C of Y. This can be reinterpreted in terms of closures (and thus of limit points).

Lemma 2.2. *A function $f : X \longrightarrow Y$ is continuous if and only if, for all $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.*

Clearly identity functions and composites of continuous functions are continuous.

Proposition 2.3. *Let X_i be spaces and let $\pi_i : \prod_i X_i \longrightarrow X_i$ be the projection. Then π_i is a continuous function. If Y is a space and $\rho_i : Y \longrightarrow X_i$ are continuous functions, then the map $Y \longrightarrow \prod X_i$ with ith coordinate ρ_i is continuous.*

Proposition 2.4. *Let X be a space, let $A \subset X$, and give A the subspace topology. Then the inclusion $i : A \longrightarrow X$ is a continuous function. If B is a space and $j : B \longrightarrow A$ is a function such that $i \circ j$ is continuous, then j is continuous.*

Proposition 2.5. *Let X be a space, let $q : X \longrightarrow Y$ be a surjective function, and give Y the quotient topology. Then q is a continuous function. If Z is a space and $r : Y \longrightarrow Z$ is a function such that $r \circ q$ is continuous, then r is continuous.*

The three previous propositions state that the product, subspace, and quotient topologies satisfy certain "universal properties". In each of these results, the specified topology is the only topology for which the last statement is true.

Continuity is a local condition on a function.

Lemma 2.6. *A function $f : X \longrightarrow Y$ is continuous if and only if for each $x \in X$ and neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

Lemma 2.7. *A function $f : X \longrightarrow Y$ is continuous if and only if its restriction to each set in an open cover of X is continuous.*

There is an analogue for finite closed covers.

Lemma 2.8. *A function $f : X \longrightarrow Y$ is continuous if and only if its restriction to each set in a finite closed cover of X is continuous.*

In particular, if $X = A \cup B$ where A and B are closed subsets of X, then continuous functions $A \longrightarrow Y$ and $B \longrightarrow Y$ that agree on $A \cap B$ induce a continuous function $X \longrightarrow Y$.

Definition 2.9. A continuous bijection $f : X \longrightarrow Y$ is a *homeomorphism* if its inverse f^{-1} is also continuous. That is, a homeomorphism is a continuous bijection with a continuous inverse. An *inclusion* or *embedding* is a continuous injection that is a homeomorphism onto its image.

Intuitively, homeomorphism is the topological counterpart of the algebraic notion of isomorphism. Topologists are interested in properties of spaces that are invariant under homeomorphism. We shall later give conditions on X and Y that ensure that a continuous bijection is a homeomorphism.

3. Metric spaces

The intuition for and the most important examples of the general theory come from metric spaces, where the topology is defined in terms of a distance function.

Definition 3.1. A metric d on a set X is a function $d : X \times X \longrightarrow \mathbb{R}$ such that

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$.
(iii) $d(x, y) + d(y, z) \geq d(x, z)$.

The basis $\mathscr{B}$ determined by a metric d consists of the sets $B(x, r) = \{y|d(x, y) < r\}$. The topology generated by $\mathscr{B}$ is called the metric topology on X determined by d. A topological space X is *metrizable* if its topology is determined by a metric.

A subset A of a metric space X has an induced metric, and the metric and subspace topologies coincide. Any metric space is Hausdorff.

Of course, $\mathbb{R}^n$ has the standard metric

$$d(x, y) = (\sum(y_i - x_i)^2)^{1/2}.$$

The metric topology that it determines coincides with the product topology. The product of countably many copies of $\mathbb{R}$ is metrizable, but the product of uncountably many copies of $\mathbb{R}$ is not. There is a metric topology on any product of copies of $\mathbb{R}$, called the uniform topology, but it is finer than product topology when the product is infinite.

For metric spaces, Lemma 2.6 leads to the familiar ε, δ formulation of continuity.

Lemma 3.2. *A function $f : X \longrightarrow Y$ between metric spaces is continuous if and only if for each $x \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$f(B(x, \delta)) \subset B(f(x), \varepsilon);$$

that is, if the distance from x to y is less than δ, then the distance from $f(x)$ to $f(y)$ is less than ε.

Moreover, we can characterize continuity in terms of convergent sequences.

Definition 3.3. A sequence $\{x_n\}$ of points in a space X *converges* to a point x if every neighborhood of x contains all but finitely many of the x_n. We then write $\{x_n\} \to x$. If X is Hausdorff, then the limit of $\{x_n\}$ is unique if it exists.

Observe that if $\{x_n\} \subset A$ and $\{x_n\} \to x$, then $x \in \bar{A}$. The converse does not hold for general topological spaces, but it does for metric spaces. Actually, what is relevant is not the metric but something it implies.

Definition 3.4. A space X is *first countable* if for each $x \in X$, there is a countable set of neighborhoods U_n of x such that any neighborhood of x contains at least one of the U_n; X is *second countable* if its topology has a countable basis.

Using the neighborhoods $B(x, 1/n)$, we see that a metric space is first countable.

Lemma 3.5. *Let X be first countable. Then $x \in \bar{A}$ if and only if there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \to x$.*

Using Lemma 2.2 this leads to the promised characterization of continuity.

Proposition 3.6. *Let $f : X \longrightarrow Y$ be a function, where X is first countable and Y is any space. Then f is continuous if and only for every convergent sequence $\{x_n\} \to x$ in X, $\{f(x_n)\} \to f(x)$ in Y.*

4. Connected and locally connected spaces

As usual, let $I = [0, 1]$ denote the unit interval. A *path* in a space X is a map $f : I \longrightarrow X$. It is said to connect the points $f(0)$ and $f(1)$.

Definition 4.1. Let X be a space.

- (i) X is *connected* if the only subspaces of X that are both open and closed are $\emptyset$ and X.
- (ii) X is *path connected* if any two points of X can be connected by a path.

A path connected space is connected, but not conversely.

Lemma 4.2. *Let Y be a subspace of a space X and let $Y = A \cup B$. Then A and B are both open and closed in Y if and only if $\bar{A} \cap B$ and $A \cap \bar{B}$ are both empty or, equivalently, A contains no limit point of B and B contains no limit point of A. We then say that $Y = A \cup B$ is a* separation *of Y. Thus Y is connected if and only if it has no separation.*

The following consequence is used very frequently.

Proposition 4.3. *Let $X = A \cup B$ be a separation. If $Y \subset X$ is connected, then Y is contained in either A or B.*

Proposition 4.4. *A union of connected or path connected spaces that have a point in common is connected or path connected.*

Proposition 4.5. *If $f : X \longrightarrow Y$ is a continuous map and X is connected or path connected, then the image of f is connected or path connected.*

Proposition 4.6. *Any product of connected or path connected spaces is connected or path connected.*

Definition 4.7. Define two equivalence relations $\sim$ and $\approx$ on X.

- (i) $x \sim y$ if x and y are both in some connected subspace of X. A *component* of X is an equivalence class of points under $\sim$. Let $\pi_0'(X)$ denote the set of components of X.
- (ii) $x \approx y$ if there is a path connecting x and y. A *path component* of X is an equivalence class of points under $\approx$. Let $\pi_0(X)$ denote the set of path components of X.

If $x \approx y$, then $x \sim y$ since the image of a path connecting x and y is a connected subspace. Therefore each component of X is the union of some of its path components. For nice spaces, components and path components are the same.

Definition 4.8. Let X be a space.

- (i) X is *locally connected* if for each $x \in X$ and each neighborhood U of x, there is a connected neighborhood V of x contained in U.

(ii) X is *locally path connected* if for each $x \in X$ and each neighborhood U of x, there is a path connected neighborhood V of x contained in U.

Proposition 4.9. *Let X be a space.*

(i) *X is locally connected if and only if every component of an open subset U is open in X.*
(ii) *X is locally path connected if and only if every path component of an open subset U is open in X.*
(iii) *If X is locally path connected, then the components and path components of X coincide.*

5. Compact and locally compact spaces

Definition 5.1. A space X is *compact* if every open cover contains a finite subcover. That is, if X is the union of open sets U_i, then there are finitely many indices i_1, $\cdots$, i_n such that X is the union of the U_{i_n}.

Using standard facts about complements, one can reformulate the notion of compactness as follows. Say that a set of subsets of X has the finite intersection property if any finite subset has nonempty intersection.

Proposition 5.2. *A space X is compact if and only if any set of closed subsets of X with the finite intersection property has nonempty intersection. In particular, if X is compact and if $\{C_n\}$ is a nested sequence of closed subsets of X, $C_n \supset C_{n+1}$, then $\cap C_n$ is nonempty.*

A metric space X is *bounded* if $d(x, y) \leq D$ for some fixed D and all $x, y \in X$; the least such D is called the *diameter* of X. Boundedness is not a "topological" property, since it depends on the choice of metric: different metrics can define the same topology but have very different bounded subsets. With the standard Euclidean metric, we have the following result.

Theorem 5.3 (Heine-Borel)**.** *A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

In general, we have the following observations about the compactness of subspaces. For a subset A of a space X, a cover of A in X is a set of subsets of X whose union contains A.

Proposition 5.4. *Let A be a subspace of a space X. Then A is compact if and only if every cover of A in X has a finite subcover. If X is compact, then every closed subspace of X is compact.*

For compact Hausdorff spaces, the second statement has a converse.

Proposition 5.5. *Every compact subspace of a compact Hausdorff space is closed.*

Proposition 5.6. *If $f : X \longrightarrow Y$ is a continuous function and X is compact, then the image of f is a compact subspace of Y. In particular, any quotient space of a compact space is compact.*

Theorem 5.7. *Let X be compact and Y be Hausdorff. Then a continuous bijection $f : X \longrightarrow Y$ is a homeomorphism (hence X is Hausdorff and Y is compact).*

Proof. If C is closed in X, then C is compact, hence $f(C)$ is compact, hence $f(C)$ is closed in Y. This proves that f^{-1} is continuous. $\square$

The results above give the behavior of compactness with respect to subspaces and quotient spaces. The behavior with respect to products is deeper than anything that we have stated so far.

Theorem 5.8 (Tychonoff)**.** *Any product of compact spaces is compact.*

The case of finite products is not difficult, but the general case is.

For metric spaces, compactness can be characterized in terms of limit points and convergent sequences.

Theorem 5.9. *Consider the following conditions on a space X.*

(i) *X is compact.*
(ii) *Every infinite subset of X has a limit point.*
(iii) *Every sequence in X has a convergent subsequence.*

In general, $1 \Rightarrow 2 \Rightarrow 3$. If X is a metric space, the three conditions are equivalent.

We say that X is *sequentially compact* if it satisfies (iii). The following important fact is used in proving that $(iii) \Rightarrow (i)$ when X is a metric space.

Lemma 5.10 (Lebesque Lemma)**.** *Let $\mathscr{U}$ be an open cover of a sequentially compact metric space X. Then there is a $\delta > 0$ such that if $A \subset X$ is bounded with diameter less than δ, then A is contained in some $U \in \mathscr{U}$.*

Proof. If not, then for each n we can choose a subset A_n of diameter less than $1/n$ which is not contained in any $U \in \mathscr{U}$. Choose a point $x_n \in A_n$ for each n. Suppose that $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ that converges to some x. Certainly $x \in U$ for some $U \in \mathscr{U}$. For small enough ε and large enough n_i, $B(x, 2\varepsilon) \subset U$, $d(x, x_{n_i}) < \varepsilon$ and $1/n_i < \varepsilon$. It follows easily that $A_{n_i} \subset U$, which is a contradiction. □

Definition 5.11. A space X is locally compact if each point of X has a neighborhood that is contained in a compact subspace of X.

Clearly $\mathbb{R}^n$ is locally compact but not compact.

Lemma 5.12. *Let X be a Hausdorff space. Then X is locally compact if and only if for any point x and any neighborhood U of x, there is a smaller neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.*

This criterion is needed to prove the second part of the following result.

Lemma 5.13. *Let A be a subspace of a locally compact subspace X. If A is closed or if A is open and X is Hausdorff, then A is locally compact.*

Locally compact Hausdorff spaces admit a canonical compactification, as we now make precise.

Definition 5.14. A *compactification* of a space X is an inclusion of X as a dense subspace in a compact Hausdorff space Y. Observe that a compactification of a compact Hausdorff space must be a homeomorphism. Two compactifications Y and Y' are equivalent if there is a homeomorphism $Y \longrightarrow Y'$ which restricts to the identity map on X.

Compactifications are of fundamental importance in topology and algebraic geometry. The most naive example is the one-point compactification. The construction applies to any Hausdorff space, but it only gives a Hausdorff space when X is locally compact.

Construction 5.15. Let X be a Hausdorff space and let Y be the union of X and a disjoint point denoted ∞. Then Y is a topological space whose open sets are the open sets in X together with the complements of the compact sets in X. The space Y is called the *one point compactification of* X.

If X is itself compact, then $\{\infty\}$ is open and closed in Y and Y is the union of its components X and $\{\infty\}$.

Proposition 5.16. *If X is a locally compact Hausdorff space that is not compact, then the one point compactification Y of X is in fact a compactification: Y is compact Hausdorff and X is a dense subspace.*

Since X is itself one of the open sets in Y, Lemma 5.13 gives the following implication.

Corollary 5.17. *A space X is locally compact and Hausdorff if and only if it is homeomorphic to an open subset of a compact Hausdorff space.*

6. Further separation properties

We have defined T_1 spaces and Hausdorff spaces. We give three analogous definitions, and we describe various implications relating these separation properties to each other and to local compactness.

Definition 6.1. Let X be a T_1 space (points are closed), let $x \in X$, and let A and B be closed subsets of X.

(i) X is *regular* if whenever $x \notin A$, there are open subsets U and V such that $x \in U$ and $A \subset V$.
(ii) X is *completely regular* if whenever $x \notin A$, there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ and $f(a) = 1$ for $a \in A$.
(iii) X is *normal* if whenever $A \cap B = \emptyset$, there are open subsets U and V such that $A \subset U$ and $B \subset V$.

Together with Lemma 5.12, the following result makes clear that these separation properties are closely related to local compactness.

Lemma 6.2. *Let X be a T_1 space.*

(i) *X is regular if and only if for any point x and any neighborhood U of x, there is a smaller neighborhood V of x such that $\bar{V} \subset U$.*
(ii) *X is normal if and only if for any closed set A contained in an open set U, there is an open set V such that $A \subset V$ and $\bar{V} \subset U$.*

Language varies. The terms regular, completely regular, and normal are often defined without assuming that X is T_1. Then what we call regular and normal spaces are called T_3 and T_4 spaces and what we call completely regular spaces are called Tychonoff spaces. (The T_i notation goes back to a 1935 paper of Alexandroff and Hopf, but some later references confuse things further by forgetting history and using T_i differently).

Lemma 6.3. *The following implications hold:*

$$\textit{normal} \Rightarrow \textit{completely regular} \Rightarrow \textit{regular} \Rightarrow \textit{Hausdorff} \Rightarrow T_1.$$

The implications normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff are obvious. The implication normal $\Rightarrow$ completely regular is a consequence of the following important result.

Theorem 6.4 (Urysshon's lemma)**.** *If X is normal and A and B are disjoint closed subsets of X, then there is a continuous function $f : X \longrightarrow I$ such that $f(a) = 0$ if $a \in A$ and $f(b) = 1$ if $b \in B$.*

The proof is non-trivial, and the closely analogous assertion that regular implies completely regular is false. Uryssohn's lemma can be used to prove the following equally important result.

Theorem 6.5 (Tietze extension theorem)**.** *If A is a closed subspace of a normal space X and $f : A \longrightarrow I$ is a continuous function, then f can be extended to a continuous function $X \longrightarrow I$.*

Normality is the most desirable separation property, but it is much less nicely behaved than our other separation properties.

Proposition 6.6. *A subspace of a Hausdorff, regular, or completely regular space is again Hausdorff, regular, or completely regular. A product of Hausdorff, regular, or completely regular spaces is again Hausdorff, regular, or completely regular. Neither of these assertions is true in general for normal spaces.*

For example, the product of uncountably many copies of $\mathbb{R}$ is not normal. Since $\mathbb{R}$ is homeomorphic to the open interval $(0, 1)$ and Tychonoff's theorem implies that the product of uncountably many copies of I is compact Hausdorff, this example also shows that a subspace of a normal space need not be normal. Nevertheless, most spaces of interest are normal.

Theorem 6.7. *If X is metrizable or compact Hausdorff, then X is normal.*

Some indication of the importance of complete regularity is given by the following sequence of results, the second of which should be compared with Corollary 5.17.

Theorem 6.8. *If X is completely regular, then it can be embedded as a subspace of a product of copies of the unit interval.*

Corollary 6.9. *The following conditions on a space X are equivalent.*

(i) *X is completely regular.*
(ii) *X is homeomorphic to a subspace of a compact Hausdorff space.*
(iii) *X is homeomorphic to a subspace of a normal space.*

Corollary 6.10. *A space X admits a compactification if and only if it is completely regular.*

Proof. If Y is a compactification of X, then X is a subspace of the compact Hausdorff space Y and is thus completely regular. Conversely, if X is completely regular and thus homeomorphic to a subspace of some compact Hausdorff space Z, then the closure of the image of X in Z is a compactification of X, called the compactification induced by the inclusion of X in Z. □

The very definition of complete regularity leads to a canonical compactification.

Construction 6.11. Let X be completely regular. Let $F = F(X)$ be the set of all continuous functions $f : X \longrightarrow I$, let $Z = Z(X)$ be the product of copies of I indexed on the set F, and let $i : X \longrightarrow Z$ be the map whose fth coordinate is the map f. Then i is an inclusion. The induced compactification is denoted βX and called the *Stone-Čech compactification* of X.

The Stone-Čech compactification is characterized as the unique compactification (up to equivalence) that satisfies the following "universal property".

Proposition 6.12. *Let X be a completely regular space. A map $f : X \longrightarrow Y$, where Y is a compact Hausdorff space, extends uniquely to a map $\tilde{f} : \beta X \longrightarrow Y$.*

Proof. Uniqueness holds by Lemma 1.11. When $Y = I$, the existence is immediate from the construction: f is one of the coordinate maps, and the projection from $Z(X)$ to this coordinate restricts to $\tilde{f} : \beta X \longrightarrow I$. In general, Y is homeomorphic to $\beta Y \subset Z(Y)$. The map $f_g : X \xrightarrow{f} Y \cong \beta Y \subset Z(Y) \xrightarrow{\pi_g} I$ obtained from the gth coordinate projection π_g, $g \in Z(Y)$, extends to a map $\tilde{f}_g : \beta X \longrightarrow I$, and $\tilde{f}_g$ is the gth coordinate of a map $\beta X \longrightarrow Z(Y)$. This map sends X into the closed set βY, hence it sends the closure βX into $\beta Y \cong Y$, giving $\tilde{f}$. □

7. Metrization theorems and paracompactness

Since we are much more comfortable with metric spaces than with general spaces, it is important to be able to recognize when the topology on a given space is that induced by some metric. The simplest criterion is the following. Metrization theorems are proven by embedding a given space as a subspace of a space that is known to be metrizable. Let I^ω denote the product of countably many copies of I. It is a metric space, which would be false for an uncountable product.

Theorem 7.1 (Urysohn metrization theorem)**.** *The following conditions on a T_1 space X are equivalent.*

(1) *X is regular and second countable.*
(2) *X is homeomorphic to a subspace of I^ω.*
(3) *X is metrizable and has a countable dense subset.*

Remember that second countable means that there is a countable basis for the topology. This ensures the following analogue of compactness.

Lemma 7.2. *If X is second countable, then any open cover of X has a countable subcover and X has a countable dense subset.*

Second countability is a strong condition, and a weaker countability condition, plus regularity, is necessary and sufficient for metrizability.

Definition 7.3. A set $\mathscr{V}$ of subsets of X is *locally finite* if each $x \in X$ has a neighborhood that intersects at most finitely many subsets of $\mathscr{V}$. A cover $\mathscr{U}$ of X is *σ-locally finite* if it is the union of countably many locally finite subsets.

Theorem 7.4 (Nagata-Smirnov metrization theorem)**.** *A space is metrizable if and only if it is regular and has a σ-locally finite basis.*

The "σ" here is essential: if a Hausdorff space has a locally finite cover, then it is discrete.

There is another characterization of metrizability that is perhaps more intuitive.

Definition 7.5. A space X is locally metrizable if every point $x \in X$ has a neighborhood U such that U (with its subspace topology) is metrizable.

Clearly any metric space is locally metrizable. There is a property, called paracompactness, that is very often used to patch local conditions to obtain a global condition, and Stone proved that any metric space is paracompact.

Theorem 7.6 (Smirnov metrization theorem)**.** *A space is metrizable if and only if it is paracompact and locally metrizable.*

We explain paracompactness. A *refinement* of a cover $\mathscr{U}$ of X is a collection of subspaces each of which is contained in at least one of the spaces in $\mathscr{U}$.

Definition 7.7. A space X is paracompact if every open cover of X has a locally finite refinement that is again an open cover of X.

Clearly a compact Hausdorff space is paracompact. The following sharpening of part of Theorem 6.7 holds.

Theorem 7.8. *A paracompact space X is normal.*

Like normality, paracompactness is not preserved by standard constructions. For this reason, Stone's theorem that metrizable $\Rightarrow$ paracompact is more useful than the converse implication of Smirnov's metrization theorem.

Proposition 7.9. *A closed subspace of a paracompact space is paracompact. In general, subspaces of paracompact spaces and products of paracompact spaces need not be paracompact.*

The point of paracompactness is that it ensures the existence of particularly convenient open covers. This is very important in the theory of fiber bundles in algebraic topology.

Definition 7.10. An open cover $\mathscr{U}$ of X is *numerable* if it is locally finite and for each $U \in \mathscr{U}$ there is a continuous function $\phi_U : X \longrightarrow I$ such that $\phi_U(x) > 0$ only if $x \in U$. A numerable cover $\mathscr{U}$ is a *partition of unity* if $\sum_U \phi_U(x) = 1$ for each $x \in X$.

Given a numerable cover $\mathscr{U}$, we can define $\phi(x) = \sum_U \phi_U(x)$ and $\psi_U(x) = \phi_U(x)/\phi(x)$, thereby obtaining a partition of unity.

Proposition 7.11. *If X is paracompact, then any open cover of X has a numerable refinement.*

Definition 7.12. An *n-manifold* M is a second countable Hausdorff space each point of which has a neighborhood homeomorphic to $\mathbb{R}^n$.

By the Urysohn metrization theorem, an n-manifold is metrizable. By Stone's theorem, it is therefore paracompact. The following theorem can be proven by use of a numerable cover of M.

Theorem 7.13. *Any n-manifold M can be embedded as a subspace of $\mathbb{R}^N$ for a sufficiently large N.*