Contents

Chapter 1

Real Analysis

(1.5)
$OrderTrichotomy[<, S] := \forall_{x,y \in S}(x < y \lor x = y \lor y < x)$
$OrderTransitivity[<, S] := \forall_{x,y,z \in S}((x < y \land y < z) \implies x < z)$
$Order[<, S] := (OrderTrichotomy[<, S]) \land (OrderTransitivity[<, S])$

(1.7)
$BoundedAbove[E, S, <] := (Order[<, S]) \land (E \subset S) \land (\exists_{\beta \in S} \forall_{x \in E}(x \le \beta))$
$BoundedBelow[E, S, <] := (Order[<, S]) \land (E \subset S) \land (\exists_{\beta \in S} \forall_{x \in E}(\beta \le x))$
$UpperBound[\beta, E, S, <] := (Order[<, S]) \land (E \subset S) \land (\beta \in S \land \forall_{x \in E}(x \le \beta))$
$LowerBound[\beta, E, S, <] := (Order[<, S]) \land (E \subset S) \land (\beta \in S \land \forall_{x \in E}(\beta \le x))$

(1.8)
$LUB[\alpha, E, S, <] := (UpperBound[\alpha, E, S, <]) \land (\forall_{\gamma}(\gamma < \alpha \implies \neg UpperBound[\gamma, E, S, <]))$
$GLB[\alpha, E, S, <] := (LowerBound[\alpha, E, S, <]) \land (\forall_{\beta}(\alpha < \beta \implies \neg LowerBound[\beta, E, S, <]))$

(1.10)
$LUBProperty[S, <] := \forall_E(((\emptyset \ne E \subset S) \land (BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <])))$
$GLBProperty[S, <] := \forall_E(((\emptyset \ne E \subset S) \land (BoundedBelow[E, S, <]) \implies \exists_{\alpha \in S}(GLB[\alpha, E, S, <])))$

(1.11)
$LUBPropertyImpliesGLBProperty := LUBProperty[S, <] \implies GLBProperty[S, <]$

(1) $LUBProperty[S, <] \implies \dots$		wts: 2
(1.1) $(\emptyset \ne B \subset S \land BoundedBelow[B, S, <]) \implies \dots$		wts: 1.2
(1.1.1) $Order[<, S] \land \exists_{\delta' \in S}(LowerBound[\delta', B, S, <])$		from: *BoundedBelow*, 1.1
(1.1.2) $|B| = 1 \implies \dots$		wts: 1.1.3
(1.1.2.1) $\exists_{u'}(u' \in B) \; \blacksquare \; u := choice(\{u' : u' \in B\}) \; \blacksquare \; B = \{u\}$		from: 1.1.2
(1.1.2.2) $GLB[u, B, S, <] \; \blacksquare \; \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$		
(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$		
(1.1.4) $|B| \ne 1 \implies \dots$		wts: 1.1.5
(1.1.4.1) $\forall_E((\emptyset \ne E \subset S \land BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <]))$		from: *LUBProperty*, 1
(1.1.4.2) $L := \{s \in S : LowerBound[s, B, S, <]\}$		
(1.1.4.3) $|B| > 1 \land OrderTrichotomy[<, S] \; \blacksquare \; \exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$		from: *Order*, 1.1.1 wts: 1.1.4.7
(1.1.4.4) $b_1 := choice(\{b_1' \in B : \exists_{b_0' \in B}(b_0' < b_1')\}) \; \blacksquare \; \neg LowerBound[b_1, B, S, <]$		from: 1.1.4.2
(1.1.4.5) $b_1 \notin L \; \blacksquare \; L \subset S$		
(1.1.4.6) $\delta := choice(\{\delta' \in S : LowerBound[\delta', B, S, <]\}) \; \blacksquare \; \delta \in L \; \blacksquare \; \emptyset \ne L$		from: 1.1.1
(1.1.4.7) $\emptyset \ne L \subset S$		from: 1.1.4.5, 1.1.4.6
(1.1.4.8) $\forall_{y \in L}(LowerBound[y_0, B, S, <]) \; \blacksquare \; \forall_{y \in L}\forall_{x \in B}(y_0 \le x)$		from: *LowerBound*, 1.1.4.2 wts: 1.1.4.10
(1.1.4.9) $\forall_{x \in B}(x \in S \land \forall_{y \in L}(y_0 \le x)) \; \blacksquare \; \forall_{x \in B}(UpperBound[x, L, S, <])$		from: *UpperBound*
(1.1.4.10) $\exists_{x \in S}(UpperBound[x, L, S, <]) \; \blacksquare \; BoundedAbove[L, S, <]$		

(1.1.4.11) $\emptyset \neq L \subset S \land BoundedAbove[L, S, <]$

(1.1.4.12) $\exists_{\alpha' \in S}(LUB[\alpha', L, S, <])$ ▌ $\alpha := choice(\{\alpha' \in S : (LUB[\alpha', L, S, <])\})$

(1.1.4.13) $\forall_x(x \in B \implies UpperBound[x, L, S, <])$

(1.1.4.14) $\forall_x(\neg UpperBound[x, L, S, <] \implies x \notin B)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$

   (1.1.4.15.1) $\neg UpperBound[\gamma, L, S, <]$ ▌ $\gamma \notin B$

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B$ ▌ $\gamma \in B \implies \gamma \geq \alpha$

(1.1.4.17) $\forall_{\gamma \in B}(\alpha \leq \gamma)$ ▌ $LowerBound[\alpha, B, S, <]$

(1.1.4.18) $\alpha < \beta \implies \ldots$

   (1.1.4.18.1) $\forall_{y \in L}(y_0 \leq \alpha < \beta)$ ▌ $\forall_{y \in L}(y_0 \neq \beta)$

   (1.1.4.18.2) $\beta \notin L$ ▌ $\neg LowerBound[\beta, B, S, <]$

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound[\beta, B, S, <]$ ▌ $\forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.20) $LowerBound[\alpha, B, S, <] \land \forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.21) $GLB[\alpha, B, S, <]$ ▌ $\exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.5) $|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.6) $(|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])) \land (|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <]))$

(1.1.7) $(|B| = 1 \lor |B| \neq 1) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$ ▌ $\exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.2) $(\emptyset \neq B \subset S \land BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.3) $\forall_B((\emptyset \neq B \subset S \land BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <]))$

(1.4) $GLBProperty[S, <]$

(2) $LUBProperty[S, <] \implies GLBProperty[S, <]$

(1.12)

$$Field[F, +, *] := \exists_{0,1 \in F} \forall_{x,y,z \in F} \begin{pmatrix} x + y \in F & \land & x * y \in F & \land \\ x + y = y + x & \land & x * y = y * x & \land \\ (x + y) + z = x + (y_0 + z) & \land & (x * y) * z = x * (y_0 * z) & \land \\ 1 \neq 0 & \land & x * (y_0 + z) = (x * y) + (x * z) & \land \\ 0 + x = x & \land & 1 * x = x & \land \\ \exists_{-x \in F}(x + (-x) = 0) \land (x \neq 0 \implies \exists_{1/x \in F}(x * (1/x) = 1)) \end{pmatrix}$$

*********************** $(Field[F, +, *] \land x, y, z \in F) \implies \ldots$ ***********************

(1.14)

$AdditiveCancellation := (x + y = x + z) \implies y = z$

(1) $y = 0 + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \ldots$

(2) $(-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = 0 + z = z$

$AdditiveIdentityUniqueness := (x + y = x) \implies y = 0$

(1) $x + y = x = 0 + x = x + 0$

(2) $y = 0$

$AdditiveInverseUniqueness := (x + y = 0) \implies y = -x$

(1) $x + y = 0 = x + (-x)$

(2) $y = -x$

$DoubleNegative := x = -(-x)$

(1) $0 = x + (-x) = (-x) + x$ ▌ $0 = (-x) + x$

(2) $x = -(-x)$

(1.15)

$MultiplicativeCancellation := (x \neq 0 \land x * y = x * z) \implies y = z$ —

$MultiplicativeIdentityUniqueness := (x \neq 0 \land x * y = x) \implies y = 1$ —

$MultiplicativeInverseUniqueness := (x \neq 0 \land x * y = 1) \implies y = 1/x$ —

$DoubleReciprocal := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)

$Domination := 0 * x = 0$

(1) $0 * x = (0 + 0) * x = 0 * x + 0 * x$ ▎ $0 * x = 0 * x + 0 * x$

(2) $0 * x = 0$

$NonDomination := (x \neq 0 \land y \neq 0) \implies x * y \neq 0$

(1) $(x \neq 0 \land y \neq 0) \implies \ldots$

  (1.1) $(x * y = 0) \implies \ldots$

    (1.1.1) $1 = 1 * 1 = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = 0 * ((1/x) * (1/y)) = 0$

    (1.1.2) $1 = 0 \land 1 \neq 0$ ▎ $\perp$

  (1.2) $(x * y = 0) \implies \perp$ ▎ $x * y \neq 0$

(2) $(x \neq 0 \land y \neq 0) \implies x * y \neq 0$

$NegationCommutativity := (-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = 0 * y = 0$ ▎ $x * y + (-x) * y = 0$

(2) $(-x) * y = -(x * y)$

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * 0 = 0$ ▎ $x * y + x * (-y) = 0$

(4) $x * (-y) = -(x * y)$

(5) $(-x) * y = -(x * y) = x * (-y)$

$NegativeMultiplication := (-x) * (-y) = x * y$

(1) $(-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$

(1.17)

$OrderedField[F, +, *, <] := \begin{pmatrix} Field[F, +, *] \quad \land \quad Order[<, F] \quad \land \\ \forall_{x,y,z \in F}(y_0 < z \implies x + y < x + z) \land \\ \forall_{x,y \in F}((x > 0 \land y > 0) \implies x * y > 0) \end{pmatrix}$

$******************** \; (OrderedField[F, +, *, <] \land x, y, z \in F) \implies \; \ldots \; ********************$

(1.18)

$NegationOnOrder := x > 0 \iff -x < 0$

(1) $x > 0 \implies \ldots$

  (1.1) $0 = (-x) + x > (-x) + 0 = -x$ ▎ $0 > -x$ ▎ $-x < 0$

(2) $x > 0 \implies -x < 0$

(3) $-x < 0 \implies \ldots$

  (3.1) $0 = x + (-x) < x + 0 = x$ ▎ $0 < x$ ▎ $x > 0$

(4) $-x < 0 \implies x > 0$

(5) $x > 0 \implies -x < 0 \land -x < 0 \implies x > 0$ ▎ $x > 0 \iff -x < 0$

$PositiveFactorPreservesOrder := (x > 0 \land y < z) \implies x * y < x * z$

(1) $(x > 0 \land y < z) \implies \ldots$

  (1.1) $(-y) + z > (-y) + y = 0$ ▎ $z + (-y) = 0$

  (1.2) $x * (z + (-y)) > 0$ ▎ $x * z + x * (-y) > 0$

  (1.3) $x * z = 0 + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \ldots$

  (1.4) $x * y + (x * z + x * (-y)) > x * y + 0 = x * y$

  (1.5) $x * z > x * y$

(2) $(x > \mathbb{0} \wedge y < z) \implies x * z > x * y$

$NegativeFactorFlipsOrder := (x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \wedge y < z) \implies \ldots$

  (1.1) $-x > \mathbb{0}$ from: *NegationOnOrder*

  (1.2) $(-x) * y < (-x) * z \ \blacksquare \ \mathbb{0} = x * y + (-x) * y < x * y + (-x) * z \ \blacksquare \ \mathbb{0} < x * y + (-x) * z$ from: *PositiveFactorPreservesOrder*

  (1.3) $\mathbb{0} < (-x) * (-y + z) \ \blacksquare \ \mathbb{0} > x * (-y + z) \ \blacksquare \ \mathbb{0} > -(x * y) + x * z$ from: *NegationOnOrder*

  (1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

$SquareIsPositive := (x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > \mathbb{0}$ from: *Ordered Field*

(2) $(x < \mathbb{0}) \implies \ldots$

  (2.1) $-x > \mathbb{0} \ \blacksquare \ x * x = (-x) * (-x) > \mathbb{0} \ \blacksquare \ x * x > \mathbb{0}$ from: *NegationOnOrder, Ordered Field, NegativeMultiplication*

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \vee x < \mathbb{0}) \implies x * x > \mathbb{0} \ \blacksquare \ x \neq \mathbb{0} \implies x * x > \mathbb{0}$ from: *OrderTrichotomy*, 1, 3

$OneIsPositive := \mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0} \ \blacksquare \ \mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ from: *Field, SquareIsPositive*

$ReciprocationOnOrder := (0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \ldots$

  (1.1) $x * (1/x) = \mathbb{1} > \mathbb{0} \ \blacksquare \ x * (1/x) > \mathbb{0}$ from: *Field, OneIsPositive*

  (1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp \ \blacksquare \ 1/x > \mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

  (1.3) $y * (1/y) = \mathbb{1} > \mathbb{0} \ \blacksquare \ y * (1/y) > \mathbb{0}$ from: *Field, OneIsPositive*

  (1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp \ \blacksquare \ 1/y > \mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

  (1.5) $(1/x) * (1/y) > \mathbb{0}$ from: *Ordered Field*

  (1.6) $\mathbb{0} < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$ from: *Ordered Field*, 1, 1.4, 1.5

(1.19)
$OrderedFieldQ := OrderedField[\mathbb{Q}, +, *, <]$ —

$Subfield[K, F, +, *] := Field[F, +, *] \wedge K \subset F \wedge Field[K, +, *]$
$OrderedSubfield[K, F, +, *, <] := OrderedField[F, +, *, <] \wedge K \subset F \wedge OrderedField[K, +, *, <]$
$CutI[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$
$CutII[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
$CutIII[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} : CutI[\alpha] \wedge CutII[\alpha] \wedge CutIII[\alpha]\}$

$CutCorollary1 := (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \ldots$

  (1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ from: *CutII*, 1

  (1.2) $q < p \implies q \in \alpha \ \blacksquare \ q \notin \alpha \implies q \geq p$ from: 1

  (1.3) $(q \notin \alpha) \implies \ldots$

    (1.3.1) $q \geq p$ from: 1.2

    (1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp \ \blacksquare \ q \neq p$ from: 1, 1.3

    (1.3.3) $q \geq p \wedge q \neq p \ \blacksquare \ p < q$

  (1.4) $q \notin \alpha \implies p < q \ \blacksquare \ p < q$ from: 1

(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

$CutCorollaryI1 := (\alpha \in \mathbb{R} \land r, s \in \mathbb{Q} \land r < s \land r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \land r, s \in \mathbb{Q} \land r < s \land r \notin \alpha) \implies \ldots$

   (1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ from: $Cut11, 1$

   (1.2) $s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha))$ ▮ $s \in \alpha \implies r \in \alpha$ from: $1, 1.1$

   (1.3) $r \notin \alpha \implies s \notin \alpha$ ▮ $s \notin \alpha$ from: $1, 1.2$

(2) $(\alpha \in \mathbb{R} \land r, s \in \mathbb{Q} \land r < s \land r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}} [\alpha, \beta] := \alpha, \beta \in \mathbb{R} \land \alpha \subset \beta$

$OrderTrichotomyOfR := OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

   (1.1) $\neg(\alpha <_{\mathbb{R}} \beta \lor \alpha = \beta) \implies \ldots$

     (1.1.1) $\alpha \not\subset \beta \land \alpha \neq \beta$ from: $<_{\mathbb{R}}, 1.1$

     (1.1.2) $\exists_{p'} (p' \in \alpha \land p' \notin \beta)$ ▮ $p := choice(\{p' : p' \in \alpha \land p' \notin \beta\})$

     (1.1.3) $q \in \beta \implies \ldots$

       (1.1.3.1) $p, q \in \mathbb{Q}$

       (1.1.3.2) $q < p$ from: $CutCorollaryI$

       (1.1.3.3) $q \in \alpha$ from: $Cut11$

     (1.1.4) $q \in \beta \implies q \in \alpha$

     (1.1.5) $\forall_{q \in \beta} (q \in \alpha)$ ▮ $\beta \subseteq \alpha$

     (1.1.6) $\beta \subset \alpha$ ▮ $\beta <_{\mathbb{R}} \alpha$

   (1.2) $\neg(\alpha <_{\mathbb{R}} \beta \lor \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

   (1.3) $\neg(\alpha <_{\mathbb{R}} \beta \lor \alpha = \beta) \lor (\alpha <_{\mathbb{R}} \beta \lor \alpha = \beta)$ ▮ $(\beta <_{\mathbb{R}} \alpha) \lor (\alpha <_{\mathbb{R}} \beta \lor \alpha = \beta)$

   (1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \lor \beta <_{\mathbb{R}} \alpha)$

   (1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \lor \beta <_{\mathbb{R}} \alpha)$

   (1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \lor \alpha <_{\mathbb{R}} \beta)$

   (1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}} (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

$OrderTransitivityOfR := OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

   (1.1) $(\alpha <_{\mathbb{R}} \beta \land \beta <_{\mathbb{R}} \gamma) \implies \ldots$

     (1.1.1) $\alpha \subset \beta \land \beta \subset \gamma$

     (1.1.2) $\forall_{a \in \alpha} (a \in \beta) \land \forall_{b \in \beta} (b \in \gamma)$

     (1.1.3) $\forall_{a \in \alpha} (\alpha \in \gamma)$ ▮ $\alpha \subset \gamma$ ▮ $\alpha <_{\mathbb{R}} \gamma$

   (1.2) $(\alpha <_{\mathbb{R}} \beta \land \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha <_{\mathbb{R}} \beta \land \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}} ((\alpha <_{\mathbb{R}} \beta \land \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(4) $OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

$OrderOfR := Order[<_{\mathbb{R}}, \mathbb{R}]$ from: $OrderTrichotomyR, OrderTransitivityR$
wts:

$LUBPropertyOfR := LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\emptyset \neq A \subset \mathbb{R} \land BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \ldots$

   (1.1) $\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A} (p \in \alpha)\}$

   (1.2) $A \neq \emptyset$ ▮ $\exists_{\alpha} (\alpha \in A)$ ▮ $\alpha_0 := choice(\{\alpha : \alpha \in A\})$

   (1.3) $\alpha_0 \neq \emptyset$ ▮ $\exists_a (a \in \alpha_0)$ ▮ $a_0 := choice(\{a : a \in \alpha_0\})$ ▮ $a_0 \in \gamma$ ▮ $\gamma \neq \emptyset$

   (1.4) $BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]$ ▮ $\exists_{\beta} (UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.5) $\beta_0 := choice(\{\beta : UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}]\})$

(1.6) $UpperBound[\beta_0, A, \mathbb{R}, <_{\mathbb{R}}]$ ▌ $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \beta_0)$ ▌ $\forall_{\alpha \in A}(\alpha \subseteq \beta_0)$ ▌ $\forall_{\alpha \in A}\forall_{a \in \alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma$ ▌ $\forall_{a \in \gamma}(a \in \beta_0)$ ▌ $\gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q}$ ▌ $\gamma \subseteq \beta_0 \subset \mathbb{Q}$ ▌ $\gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q}$ ▌ $CutI[\gamma]$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$

 (1.10.1) $p \in \gamma$ ▌ $\exists_{\alpha \in A}(p \in \alpha)$ ▌ $\alpha_1 := choice(\{\alpha \in A : p \in \alpha\})$

 (1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p$ ▌ $q \in \alpha_1$ ▌ $q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma$ ▌ $\forall_{p \in \gamma}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma)$ ▌ $CutII[\gamma]$

(1.12) $p \in \gamma \implies \dots$

 (1.12.1) $\exists_{\alpha \in A}(p \in \alpha)$ ▌ $\alpha_2 := choice(\{\alpha \in A : p \in \alpha\})$

 (1.12.2) $\alpha_2 \in \mathbb{R}$ ▌ $CutII[\alpha_2]$ ▌ $\exists_{r \in \alpha_2}(p < r)$ ▌ $r_0 := choice(\{r \in \alpha_2 : p < r\})$

 (1.12.3) $r_0 \in \alpha_2$ ▌ $r_0 \in \gamma$

 (1.12.4) $p < r_0$ ▌ $p < r_0 \wedge r_0 \in \gamma$ ▌ $\exists_{r \in \gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r \in \gamma}(p < r)$ ▌ $\forall_{p \in \gamma}\exists_{r \in \gamma}(p < r)$ ▌ $CutIII[\gamma]$

(1.14) $CutI[\gamma] \wedge CutII[\gamma] \wedge CutIII[\gamma]$ ▌ $\gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha \in A}(\alpha \subseteq \gamma)$ ▌ $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma)$

(1.16) $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R}$ ▌ $UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.17) $\delta <_{\mathbb{R}} \gamma \implies \dots$

 (1.17.1) $\delta \subset \gamma$ ▌ $\exists_s(s \in \gamma \wedge s \notin \delta)$ ▌ $s_0 := choice(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$

 (1.17.2) $s_0 \in \gamma$ ▌ $\exists_{\alpha \in A}(s_0 \in \alpha)$ ▌ $\alpha_3 := choice(\{\alpha \in A : s_0 \in \alpha\})$

 (1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta$ ▌ $\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

 (1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \dots$

  (1.17.4.1) $\alpha_3 \subseteq \delta$ ▌ $\forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta)$ ▌ $\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

  (1.17.4.2) $\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$ ▌ $\bot$

 (1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \bot$ ▌ $\delta <_{\mathbb{R}} \alpha_3$ ▌ $\exists_{\alpha \in A}(\delta <_{\mathbb{R}} \alpha)$ ▌ $\exists_{\alpha \in A}(\neg(\alpha \leq_{\mathbb{R}} \delta))$

 (1.17.6) $\neg\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \delta)$ ▌ $\neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$ ▌ $\forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.19) $UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \wedge \forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.20) $LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$ ▌ $\exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(3) $\forall_A((\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]))$ ▌ $LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

$+_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$
$0_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 0\}$

$ZeroInR := 0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}}$ ▌ $\emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q}$ ▌ $CutI[0_{\mathbb{R}}]$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}}$ ▌ $\forall_{x \in 0_{\mathbb{R}}}\forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}})$ ▌ $CutII[0_{\mathbb{R}}]$

(3) $y := x/2$ ▌ $(x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $\forall_{x \in 0_{\mathbb{R}}}\exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $CutIII[0_{\mathbb{R}}]$

(4) $CutI[0_{\mathbb{R}}] \wedge CutII[0_{\mathbb{R}}] \wedge CutIII[0_{\mathbb{R}}]$ ▌ $0_{\mathbb{R}} \in \mathbb{R}$

$FieldAdditionClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \dots$

 (1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

 (1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

 (1.3) $\exists_a(a \in \alpha) \,;\, \exists_b(b \in \beta)$ ▌ $a_0 := choice(\{a : a \in \alpha\}) \,;\, b_0 := choice(\{b : b \in \beta\})$ ▌ $a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

 (1.4) $\exists_x(x \notin \alpha) \,;\, \exists_y(y_0 \notin \beta)$ ▌ $x_0 := choice(\{x : x \notin \alpha\}) \,;\, y_0 := choice(\{y : y \notin \beta\})$

 (1.5) $\forall_{r \in \alpha}(r < x_0) \,;\, \forall_{s \in \beta}(s < y_0)$ ▌ $\forall_{r \in \alpha}\forall_{s \in \beta}(r + s < x_0 + y_0)$ ▌ $x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

 (1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q}$ ▌ $CutI[\alpha +_{\mathbb{R}} \beta]$

(1.7) $(p \in \alpha +_\mathbb{R} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.7.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▮ $(r_0, s_0) := choice((r, s) \in \alpha \times \beta : p = r + s)$

(1.7.2) $q < p = r_0 + s_0$ ▮ $(q - s_0) < r_0$ ▮ $(q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta$ ▮ $q = (q - s_0) + s_0 \in \alpha +_\mathbb{R} \beta$ ▮ $q \in \alpha +_\mathbb{R} \beta$

(1.8) $(p \in \alpha +_\mathbb{R} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_\mathbb{R} \beta$ ▮ $\forall_{p \in \alpha +_\mathbb{R} \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_\mathbb{R} \beta)$ ▮ $CutII[\alpha +_\mathbb{R} \beta]$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▮ $(r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta : p = r + s\})$

(1.9.2) $r_1 \in \alpha$ ▮ $\exists_{t \in \alpha}(r_1 < t)$ ▮ $t_0 := choice(\{t \in \alpha : r_1 < t\})$

(1.9.3) $s_1 \in \beta$ ▮ $t + s_1 \in \alpha +_\mathbb{R} \beta \wedge p = r_1 + s_1 < t + s_1$ ▮ $\exists_{r \in \alpha +_\mathbb{R} \beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r \in \alpha +_\mathbb{R} \beta}(p < r)$ ▮ $\forall_{p \in \alpha +_\mathbb{R} \beta} \exists_{r \in \alpha +_\mathbb{R} \beta}(p < r)$ ▮ $CutIII[\alpha +_\mathbb{R} \beta]$

(1.11) $CutI[\alpha +_\mathbb{R} \beta] \wedge CutII[\alpha +_\mathbb{R} \beta] \wedge CutIII[\alpha +_\mathbb{R} \beta]$ ▮ $\alpha +_\mathbb{R} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_\mathbb{R} \beta) \in \mathbb{R})$

FieldAdditionCommutativityOfR := $(\alpha, \beta \in \mathbb{R}) \implies (\alpha +_\mathbb{R} \beta = \beta +_\mathbb{R} \alpha)$

(1) $\alpha +_\mathbb{R} \beta = \{r + s : r \in \alpha \wedge s \in \beta\} = \{s + r : s \in \beta \wedge r \in \alpha\} = \beta +_\mathbb{R} \alpha$

FieldAdditionAssociativityOfR := $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma))$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \{(a + b) + c : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma)$

FieldAdditionIdentityOfR := $(\alpha \in \mathbb{R}) \implies 0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \wedge s \in 0_\mathbb{R}) \implies \ldots$

(1.1.1) $s < 0$ ▮ $r + s < r + 0 = r$ ▮ $r + s < r$ ▮ $r + s \in \alpha$

(1.2) $(r \in \alpha \wedge s \in 0_\mathbb{R}) \implies r + s \in \alpha$ ▮ $\forall_{r \in \alpha} \forall_{s \in 0_\mathbb{R}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \wedge s \in 0_\mathbb{R}) \iff (r + s \in \alpha +_\mathbb{R} 0_\mathbb{R})$ ▮ $\forall_{p \in \alpha +_\mathbb{R} 0_\mathbb{R}}(p \in \alpha)$ ▮ $\alpha +_\mathbb{R} 0_\mathbb{R} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r \in \alpha}(p < r)$ ▮ $r_2 := choice(\{r \in \alpha : p < r\})$

(1.4.2) $p < r_2$ ▮ $p - r_2 < r_2 - r_2 = 0$ ▮ $(p - r_2) < 0$ ▮ $(p - r_2) \in 0_\mathbb{R}$

(1.4.3) $r_2 \in \alpha$ ▮ $p = r_2 + (p - r_2) \in \alpha +_\mathbb{R} 0_\mathbb{R}$ ▮ $p \in \alpha +_\mathbb{R} 0_\mathbb{R}$

(1.5) $p \in \alpha \implies p \in \alpha +_\mathbb{R} 0_\mathbb{R}$ ▮ $\forall_{p \in \alpha}(p \in \alpha +_\mathbb{R} 0_\mathbb{R})$ ▮ $\alpha \subseteq \alpha +_\mathbb{R} 0_\mathbb{R}$

(1.6) $\alpha +_\mathbb{R} 0_\mathbb{R} \subseteq \alpha \wedge \alpha \subseteq \alpha +_\mathbb{R} 0_\mathbb{R}$ ▮ $0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

FieldAdditionInverseOfR := $(\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}}(\alpha +_\mathbb{R} (-\alpha) = 0_\mathbb{R})$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} : \exists_{r > 0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q}$ ▮ $\exists_{s \in \mathbb{Q}}(s \notin \alpha)$ ▮ $s_0 := choice(\{s : s \notin \alpha\})$ ▮ $p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha$ ▮ $-p_0 - 1 \notin \alpha$ ▮ $\exists_{r > 0}(-p_0 - r \notin \alpha)$ ▮ $p_0 \in \beta$

(1.4) $\emptyset \neq \alpha$ ▮ $\exists_{q \in \alpha}$ ▮ $q_0 := choice(\{q \in \mathbb{Q} : q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha$ ▮ $-(-q_0) - r = q_0 - r < q_0$ ▮ $-(-q_0) - r < q_0$ ▮ $-(-q_0) - r \in \alpha$

(1.6) $\forall_{r > 0}(-(-q_0) - r \in \alpha)$ ▮ $\neg\exists_{r > 0}(-(-q_0) - r \notin \alpha)$ ▮ $-q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q}$ ▮ $CutI[\beta]$

(1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.8.1) $p \in \beta$ ▮ $\exists_{r > 0}(-p - r \notin \alpha)$ ▮ $r_0 := choice(\{r > 0 : -p - r \notin \alpha\})$

(1.8.2) $q < p$ ▮ $-p - r < -q - r$

(1.8.3) $-q - r \notin \alpha$ ▮ $q \in \beta$

(1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta$ ▮ $\forall_{p \in \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \beta)$ ▮ $CutII[\beta]$

(1.10) $p \in \beta \implies \ldots$

(1.10.1) $p \in \beta$ ▌ $\exists_{r>0}(-p - r \notin \alpha)$ ▌ $r_1 := choice(\{r > 0 : -p - r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0$ ▌ $r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p$ ▌ $t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha$ ▌ $-t_0 - (r_1/2) \notin \alpha$ ▌ $\exists_{r>0}(-t_0 - r \notin \alpha)$ ▌ $t_0 \in \beta$

(1.10.7) $t_0 > p \wedge t_0 \in \beta$ ▌ $\exists_{t \in \beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t \in \beta}(p < t)$ ▌ $\forall_{p \in \beta}\exists_{t \in \beta}(p < t)$ ▌ $CutIII[\beta]$

(1.12) $CutI[\beta] \wedge CutII[\beta] \wedge CutIII[\beta]$ ▌ $\beta \in \mathbb{R}$

(1.13) $(r \in \alpha \wedge s \in \beta) \implies \ldots$

(1.13.1) $s \in \beta$ ▌ $\exists_{t>0}(-s - t \notin \alpha)$ ▌ $t_1 := choice(\{t > 0 : -s - t \notin \alpha\})$ ▌ $-s - t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha$ ▌ $-s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha$ ▌ $r < -s$ ▌ $r + s < 0$ ▌ $r + s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}}$ ▌ $\forall_{(r,s) \in \alpha \times \beta}(r + s \in 0_{\mathbb{R}})$ ▌ $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \ldots$

(1.15.1) $v < 0$ ▌ $w_0 := -v/2$ ▌ $w > 0$

(1.15.2) $\exists_{n \in \mathbb{Z}}(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha)$ ▌ $n_0 := choice(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$ from: *ARCHIMEDEANPROPERTYOFQ + LUB???*

(1.15.3) $p_0 := -(n_0 + 2)w_0$ ▌ $-p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha$ ▌ $-p_0 - w_0 \notin \alpha$ ▌ $p_0 \in \beta$

(1.15.4) $n_0 w_0 \in \alpha \wedge p_0 \in \beta$ ▌ $n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{v \in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta)$ ▌ $0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$ ▌ $\alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$ ▌ $\exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

$*_{\mathbb{R}}[\alpha, \beta] := \qquad —$

$1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$1IsNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}} \qquad —$

$1InR := 1_{\mathbb{R}} \in \mathbb{R} \qquad —$

$FieldMultiplicationClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R}) \qquad —$

$FieldMultiplicationCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha) \qquad —$

$FieldMultiplicationAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)) \qquad —$

$FieldMultiplicationIdentityOfR := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha \qquad —$

$FieldMultiplicationInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha \in \mathbb{R}}(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}) \qquad —$

$FieldDistributativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta \qquad —$

$FieldWithR := Field[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}] \qquad —$

$OrderedFieldWithR := OrderedField[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}] \qquad —$

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$

$QROrderedSubfieldOfR := OrderedSubfield[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}] \qquad —$

$QIsomorphicToQR := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q} \qquad —$

$CompletenessOfR := \exists_{\mathbb{R}}(LUBProperty[\mathbb{R}, <_{\mathbb{R}}] \wedge OrderedSubfield[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]) \qquad —$

(1.20)

$ArchimedeanPropertyOfR := \forall_{x,y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx : n \in \mathbb{N}^+\}$ ▌ $(\emptyset \neq A \subset \mathbb{R}) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$

(1.2) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n \in \mathbb{N}^+}(nx > y)$ ▌ $\forall_{n \in \mathbb{N}^+}(nx \leq y)$ ▌ $UpperBound[y_0, A, \mathbb{R}, <]$ ▌ $BoundedAbove[A, \mathbb{R}, <]$

(1.2.2) $CompletenessOfR$ ▌ $LUBProperty[\mathbb{R}, <]$

(1.2.3) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (BoundedAbove[A, \mathbb{R}, <])$ ▌ $\exists_{\alpha \in \mathbb{R}}(LUB[\alpha, A, \mathbb{R}, <])$ $\ldots$

(1.2.4) $\ldots \alpha_0 := choice(\{\alpha \in \mathbb{R} : LUB[\alpha, A, \mathbb{R}, <]\})$ ▌ $LUB[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0$ ▌ $\alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge (LUB[\alpha_0, A, \mathbb{R}, <])$ ▌ $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$ ▌ $\exists_{c \in A}(\alpha_0 - x < c)$ $\ldots$

(1.2.8) $\ldots c_0 := choice(\{c \in A : \alpha_0 - x < c\})$ ▌ $(c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$ ▌ $\exists_{m \in \mathbb{N}^+}(mx = c_0)$ $\ldots$

(1.2.10) $\ldots m_0 := choice(\{m \in \mathbb{N}^+ : mx = c_0\})$ ▌ $(m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0)$ ▌ $\alpha_0 - x < c_0 = m_0 x$ ▌ $\alpha_0 < m_0 x + x$ ▌ $\alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+$ ▌ $m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$ ▌ $(m_0 + 1)x \in A$

(1.2.14) $(\alpha_0 < (m_0 + 1)x) \wedge ((m_0 + 1)x \in A)$ ▌ $\exists_{c \in A}(\alpha_0 < c)$

(1.2.15) $LUB[\alpha_0, A, \mathbb{R}, <]$ ▌ $UpperBound[\alpha_0, A, \mathbb{R}, <]$ ▌ $\forall_{c \in A}(c \leq \alpha_0)$ ▌ $\neg\exists_{c \in A}(c > \alpha_0)$ ▌ $\neg\exists_{c \in A}(\alpha_0 < c)$

(1.2.16) $(\exists_{c \in A}(\alpha_0 < c)) \wedge (\neg\exists_{c \in A}(\alpha_0 < c))$ ▌ $\bot$

(1.3) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \bot$ ▌ $\exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y)$ ▌ $\forall_{x, y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

$QDenseInR := \forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y$ ▌ $(0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $ArchimedeanPropertyOfR \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R})$ ▌ $\exists_{n \in \mathbb{N}^+}(n(y - x) > 1)$ $\ldots$

(1.3) $\ldots n_0 := choice(\{n \in \mathbb{N}^+ : n(y - x) > 1\})$ ▌ $(n_0 \in \mathbb{N}^+) \wedge (n_0(y - x) > 1)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R})$ ▌ $n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m \in \mathbb{N}^+}(m(1) > n_0 x)$ $\ldots$

(1.6) $\ldots m_1 := choice(\{m \in \mathbb{N}^+ : m(1) > n_0 x\})$ ▌ $(m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m \in \mathbb{N}^+}(m(1) > -n_0 x)$ $\ldots$

(1.8) $\ldots m_2 := choice(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\})$ ▌ $(m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x)$ ▌ $-m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+$ ▌ $|m_1 - (-m_2)| \geq 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge (|m_1 - (-m_2)| \geq 2)$ ▌ $\exists_{m \in \mathbb{Z}}((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m))$ $\ldots$

(1.12) $\ldots m_0 := choice(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\})$ ▌ $(-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$

(1.13) $(n_0(y - x) > 1) \wedge (m_0 - 1 \leq n_0 x < m_0)$ ▌ $n_0 x < m_0 \leq 1 + n_0 x < n_0 y$ ▌ $n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y)$ ▌ $x < m_0 / n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z}$ ▌ $m_0 / n_0 \in \mathbb{Q}$

(1.16) $(m_0 / n_0 \in \mathbb{Q}) \wedge (x < m_0 / n_0 < y)$ ▌ $\exists_{p \in \mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y)$ ▌ $\forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1.21)

$RootLemma := (0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a) \sum_{i=1}^{n}(b^{n-i} a^{i-1})$

(1.2) $0 < a < b$ ▌ $b/a > 1$

(1.3) $b/a > 1$ ▌ $\sum_{i=1}^{n}(b^{n-i} a^{i-1}) \leq \sum_{i=1}^{n}(b^{n-i} a^{i-1}(b/a)^{i-1}) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$ ▌ $\sum_{i=1}^{n}(b^{n-i} a^{i-1}) \leq \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b - a) \sum_{i=1}^{n}(b^{n-i} a^{i-1}) \leq (b - a)nb^{n-1}$ ▌ $b^n - a^n \leq (b - a)nb^{n-1}$

(2) $(0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

$RootExistenceInR := \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}}(y_0{}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\}$ ▌ $t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$

(1.2) $t_0 := x/(1 + x)$ ▌ $(t_0 = x/(1 + x)) \wedge (t_0 \in \mathbb{R})$

(1.3) $0 < x$ ▌ $0 < x < 1 + x$ ▌ $t_0 = x/(1 + x) > 0$ ▌ $t_0 > 0$

(1.4) $1 = (1+x)/(1+x) > x/(1+x) = t_0$ ▌ $1 > t_0$

(1.5) $(t_0 > 0) \wedge (1 > t_0)$ ▌ $0 < t_0 < 1$

(1.6) $(0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1)$ ▌ $t_0{}^n \leq t_0$

(1.7) $0 < x$ ▌ $x > x/(1+x) = t_0$ ▌ $x > t_0$

(1.8) $(t_0{}^n \leq t_0) \wedge (x > t_0)$ ▌ $t_0{}^n < x$

(1.9) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge (t_0{}^n < x)$ ▌ $t_0 \in E$ ▌ $\emptyset \neq E$

(1.10) $t_1 := choice(\{t \in \mathbb{R} : t > 1 + x\})$ ▌ $(t_1 \in \mathbb{R}) \wedge (t_1 > 1 + x)$

(1.11) $x > 0$ ▌ $t_1 > 1 + x > 1$ ▌ $t_1 > 1$ ▌ $t_1{}^n \geq t_1$

(1.12) $(t_1{}^n \geq t_1) \wedge (t_1 > 1 + x) \wedge (1 > 0)$ ▌ $t_1{}^n \geq t_1 > 1 + x > x$ ▌ $t_1{}^n > x$

(1.13) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_1{}^n > x)$ ▌ $t_1 \notin E$ ▌ $E \subset \mathbb{R}$

(1.14) $(\emptyset \neq E) \wedge (E \subset \mathbb{R})$ ▌ $\emptyset \neq E \subset \mathbb{R}$

(1.15) $t \in E \implies \dots$

(1.15.1) $(t \in E) \wedge (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x))$ ▌ $t^n < x$

(1.15.2) $(t_1{}^n > x) \wedge (t^n < x)$ ▌ $t^n < x < t_1{}^n$ ▌ $t < t_1$

(1.16) $t \in E \implies t < t_1$ ▌ $\forall_{t \in E}(t \leq t_1)$ ▌ $UpperBound[t_1, E, \mathbb{R}, <]$ ▌ $BoundedAbove[E, \mathbb{R}, <]$

(1.17) $CompletenessOfR$ ▌ $LUBProperty[\mathbb{R}, <]$

(1.18) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (BoundedAbove[E, \mathbb{R}, <])$ ▌ $\exists_{y \in \mathbb{R}}(LUB[y, E, \mathbb{R}, <]) \dots$

(1.19) $\dots y_0 := choice(\{y \in \mathbb{R} : LUB[y, E, \mathbb{R}, <]\})$ ▌ $LUB[y_0, E, \mathbb{R}, <]$

(1.20) $(LUB[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0)$ ▌ $0 < t_0 \leq y_0 \in \mathbb{R}$ ▌ $0 < y_0 \in \mathbb{R}$

(1.21) $y_0{}^n < x \implies \dots$

(1.21.1) $k_0 := \frac{x - y_0{}^n}{n(y_0+1)^{n-1}}$ ▌ $k_0 \in \mathbb{R}$

(1.21.2) $y_0{}^n < x$ ▌ $0 < x - y_0{}^n$

(1.21.3) $(n > 0) \wedge (y_0 > 0)$ ▌ $0 < n(y_0 + 1)^{n-1}$

(1.21.4) $(0 < x - y_0{}^n) \wedge (0 < n(y_0 + 1)^{n-1})$ ▌ $0 < \frac{x - y_0{}^n}{n(y_0+1)^{n-1}} = k_0$ ▌ $0 < k_0$

(1.21.5) $(0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R})$ ▌ $0 < min(1, k_0) \in \mathbb{R}$

(1.21.6) $QDenseInR \wedge (0, min(1, k_0) \in \mathbb{R}) \wedge (0 < min(1, k_0))$ ▌ $\exists_{h \in \mathbb{Q}}(0 < h < min(1, k_0)) \dots$

(1.21.7) $\dots h_0 := choice(\{h \in \mathbb{Q} : 0 < h < min(1, k_0)\})$ ▌ $(0 < h_0 < 1) \wedge (h_0 < k_0 = \frac{x - y_0{}^n}{n(y_0+1)^{n-1}})$

(1.21.8) $(y_0 > 0) \wedge (h_0 > 0)$ ▌ $0 < y_0 < y_0 + h_0$

(1.21.9) $RootLemma \wedge (0 < y_0 < y_0 + h_0)$ ▌ $(y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + h_0)^{n-1}$

(1.21.10) $h_0 < 1$ ▌ $h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$

(1.21.11) $((y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + h_0)^{n-1}) \wedge (h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1})$ ▌ $(y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + 1)^{n-1}$

(1.21.12) $(0 < n(y_0 + 1)^{n-1}) \wedge (h_0 < k_0 = \frac{x - y_0{}^n}{n(y_0+1)^{n-1}})$ ▌ $h_0 n(y_0 + 1)^{n-1} < x - y_0{}^n$

(1.21.13) $((y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + 1)^{n-1}) \wedge (h_0 n(y_0 + 1)^{n-1} < x - y_0{}^n)$ ▌ $(y_0 + h_0)^n - y_0{}^n < x - y_0{}^n$ ▌ $(y_0 + h_0)^n < x$

(1.21.14) $(y_0 + h_0)^n - y_0{}^n < x - y_0{}^n$ ▌ $(y_0 + h_0)^n < x$

(1.21.15) $(0 < y_0 \mathbb{R}) \wedge (0 < h_0 < \mathbb{R})$ ▌ $0 < y_0 < y_0 + h_0 \in \mathbb{R}$

(1.21.16) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ((y_0 + h_0)^n < x) \wedge (0 < y_0 + h_0 \in \mathbb{R})$ ▌ $(y_0 + h_0)^n \in E$

(1.21.17) $((y_0 + h_0)^n \in E) \wedge (y_0 < y_0 + h_0)$ ▌ $\exists_{e \in E}(y_0 < e)$

(1.21.18) $LUB[y_0, E, \mathbb{R}, <]$ ▌ $UpperBound[y_0, E, \mathbb{R}, <]$ ▌ $\forall_{e \in E}(e \leq y_0)$ ▌ $\neg\exists_{e \in E}(e > y_0)$

(1.21.19) $(\exists_{e \in E}(e > y_0)) \wedge (\neg\exists_{e \in E}(e > y_0))$ ▌ $\bot$

(1.22) $y_0{}^n < x \implies \bot$ ▌ $y_0{}^n \geq x$

(1.23) $y_0{}^n > x \implies \dots$

(1.23.1) $k_1 := \frac{y_0{}^n - x}{n y_0{}^{n-1}}$ ▌ $(k_1 \in \mathbb{R}) \wedge (k_1 n y_0{}^{n-1} = y_0{}^n - x)$

(1.23.2) $(0 < x) \wedge (0 < n \in \mathbb{Z})$ ▌ $y_0{}^n - x < y_0{}^n \leq n y_0{}^n$ ▌ $y_0{}^n - x < n y_0{}^n$

(1.23.3) $y_0{}^n - x < n y_0{}^n$ ▌ $k_1 = \frac{y_0{}^n - x}{n y_0{}^{n-1}} < \frac{n y_0{}^n}{n y_0{}^{n-1}} = y_0$ ▌ $k_1 < y_0$

(1.23.4) $y_0{}^n > x$ ▌ $0 < y_0{}^n - x$

(1.23.5) $(n > 0) \wedge (y_0 > 0)$ ▌ $0 < n y_0{}^{n-1}$

(1.23.6) $(0 < y_0{}^n - x) \wedge 0 < (n y_0{}^{n-1})$ ▌ $0 < \frac{y_0{}^n - x}{n y_0{}^{n-1}} = k_1$ ▌ $0 < k_1$

(1.23.7) $(k_1 < y_0) \wedge (0 < k_1)$ ▌ $(0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$

(1.23.8) $t \geq y_0 - k_1 \implies \ldots$

(1.23.8.1) $t \geq y_0 - k_1$ ▌ $t^n \geq (y_0 - k_1)^n$ ▌ $-t^n \leq -(y_0 - k_1)^n$ ▌ $y_0{}^n - t^n \leq y_0{}^n - (y_0 - k_1)^n$

(1.23.8.2) $Root\,Lemma \wedge (0 < y_0 - k_1 < y_0)$ ▌ $y_0{}^n - (y_0 - k_1)^n < k_1 n y_0{}^{n-1}$

(1.23.8.3) $(y_0{}^n - t^n \leq y_0{}^n - (y_0 - k_1)^n) \wedge (y_0{}^n - (y_0 - k_1)^n < k_1 n y_0{}^{n-1})$ ▌ $y_0{}^n - t^n < k_1 n y_0{}^{n-1}$

(1.23.8.4) $(k_1 n y_0{}^{n-1} = y_0{}^n - x) \wedge (y_0{}^n - t^n < k_1 n y_0{}^{n-1})$ ▌ $y_0{}^n - t^n < y_0{}^n - x$ ▌ $-t^n < -x$ ▌ $t^n > x$

(1.23.8.5) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t^n > x)$ ▌ $t \notin E$

(1.23.9) $t \geq y_0 - k_1 \implies t \notin E$ ▌ $t \in E \implies t < y_0 - k_1$ ▌ $\forall_{t \in E}(t \leq y_0 - k_1)$ ▌ $Upper\,Bound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.10) $(LUB[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0))$ ▌ $\neg Upper\,Bound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.11) $(Upper\,Bound[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg Upper\,Bound[y_0 - k_1, E, \mathbb{R}, <])$ ▌ $\bot$

(1.24) $y_0{}^n > x \implies \bot$ ▌ $y_0{}^n \leq x$

(1.25) $Order[\mathbb{R}, <]$ ▌ $OrderTrichotomy[\mathbb{R}, <]$

(1.26) $(OrderTrichotomy[\mathbb{R}, <]) \wedge (y_0{}^n \geq x) \wedge (y_0{}^n \leq x)$ ▌ $y_0{}^n = x$

(1.27) $(y_0{}^n = x) \wedge (y_0 \in \mathbb{R})$ ▌ $\exists_{y \in \mathbb{R}}(y^n = x)$

(1.28) $y_1, y_2 := choice(\{y \in \mathbb{R} : y^n = x\})$

(1.29) $y_1 \neq y_2 \implies \ldots$

(1.29.1) $(OrderTrichotomy[\mathbb{R}, <]) \wedge (y_1 \neq y_2)$ ▌ $(y_1 < y_2) \vee (y_2 < y_1) \ldots$

(1.29.2) $\ldots (x = y_1{}^n < y_2{}^n = x) \vee (x = y_2{}^n < y_1{}^n = x)$ ▌ $(x < x) \vee (x > x)$ ▌ $\bot \vee \bot$ ▌ $\bot$

(1.30) $y_1 \neq y_2 \implies \bot$ ▌ $y_1 = y_2$ ▌ $\forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b)$

(1.31) $(\exists_{y \in \mathbb{R}}(y^n = x)) \wedge (\forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b))$ ▌ $\exists!_{y \in \mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}}(y^n = x)$ ▌ $\forall_{0 < x \in \mathbb{R}}\forall_{0 < n \in \mathbb{Z}}\exists!_{0 < y \in \mathbb{R}}(y_0{}^n = x)$

$Root\,Existence\,In\,RCorollary := \forall_{0 < a \in \mathbb{R}}\forall_{0 < b \in \mathbb{R}}\forall_{0 < n \in \mathbb{Z}}((ab)^{1/n} = a^{1/n}b^{1/n})$ —

$Extended\,Real\,System[\bar{\mathbb{R}}, +, *, <] := \begin{pmatrix} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \wedge x - \infty = -\infty \wedge \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \wedge \\ (x > 0) \implies (x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty) \wedge \\ (x < 0) \implies (x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty) \end{pmatrix}$

$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$
$+_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d \rangle$
$*_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c \rangle$

$FieldC := Field[\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}}]$ —
$RSubfieldC := Subfield[\mathbb{R}, \mathbb{C}, +, *]$ —

$i := \langle 0, 1 \rangle \in \mathbb{C}$

$iProperty := i^2 = -1$ —
$CProperty := (a, b \in \mathbb{R}) \implies (\langle a, b \rangle = a + bi)$ —

$Conjugate[\overline{a + bi}] := a - bi$

$Conjugate\,Properties := (w, z \in \mathbb{C}) \implies \ldots$ —

(1) $\overline{z + w} = \overline{z} + \overline{w}$

(2) $\overline{z * w} = \overline{z} * \overline{w}$

(3) $Re(z) = (1/2)(z + \overline{z}) \wedge Im(z) = (1/2)(z - \overline{z})$

(4) $0 \leq z * \overline{z} \in \mathbb{R}$

$AbsoluteValueC[|z|] = (z * \overline{z})^{1/2}$
$AbsoluteValue\,Properties := (z, w \in \mathbb{C}) \implies \ldots$ —

(1) 123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$

$Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y}((x, y) \in f)$

$(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \ldots$

(1)	$Domain(f) := X; Codomain(f) := Y$
(2)	$Image(f, A) := \{f(a) : a \in A\}; Preimage(f, B) := \{a : f(a) \in B\}$
(3)	$Range(f) := Image(Domain(f))$

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X}(x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$

$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X}(y_0 = f(x))$

$Bijective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$

$SurjectiveEquivalent := (Range(f) = Codomain(f)) \implies Surjective(f)$

$(Function(f, X, Y) \wedge Function(g, Y, Z)) \implies (f \circ g)(x) := f(g(x)); Function(f \circ g, X, Z)$

$PropertiesofFunctions := (Function(f, A, B) \wedge Function(g, B, C) \wedge Function(h, C, D)) \implies \ldots$

(1)	$h \circ (g \circ f) = (h \circ g) \circ f$
(2)	$(Injective(f) \wedge Injective(g)) \implies Injective(g \circ f)$
(3)	$(Surjective(f) \wedge Surjective(g)) \implies Surjective(g \circ f)$
(4)	$(Bijective(f, A, B)) \implies \exists_{f^{-1}}(Function(f^{-1}, B, A) \wedge \forall_{a \in A}(f^{-1}(f(a)) = a) \wedge \forall_{b \in B}(f(f^{-1}(b)) = b))$

$|(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}}(b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \ldots$

(1)	$(a|b) \implies a|bc$
(2)	$(a|b \wedge b|c) \implies a|c$
(3)	$(a|b \wedge b|c) \implies a|(bx + cy)$
(4)	$(a|b \wedge b|a) \implies a = \pm b$
(5)	$(a|b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$
(6)	$(a|b) \iff (m \neq 0 \wedge ma|mb)$

$DivisionAlgorithm := (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}}(b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \wedge a : b \wedge a : c$

$GCD(a, b, c) := CD(a, b, c) \wedge \forall_d((d : b \wedge d : c) \implies d : a)$

$GCDEquivalent := 123123$

Chapter 3

Linear Algebra

3.1 Matrix Operations and Special Matrices

$Matrix[A, m, n] := [a_{i,j}]_{m \times n} :=$ m rows, n columns of real numbers
$\mathcal{M}_{m,n} := \{A : Matrix[A, m, n]\}$
$O_{m,n} := (Matrix[O, m, n]) \wedge (a_{i,j} = 0)$
$Square[A, n] := Matrix[A, n, n]$
$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$
$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$
$Diagonal[A, n] := (Square[A, n]) \wedge (i \neq j \implies a_{i,j} = 0)$
$Scalar[A, n, k] := (Diagonal[A, n]) \wedge (a_{i,i} = k)$
$I_n := Scalar[I, n, 1]$

$+(A, B) := ((Matrix[A, m, n]) \wedge (Matrix[B, m, n])) \implies (A + B = [a_{i,j} + b_{i,j}]_{m \times n})$
$*(r, A) := ((r \in \mathbb{R}) \wedge (Matrix[A, m, n])) \implies (r * A = [ra_{i,j}]_{m \times n})$
$*(A, B) := ((Matrix[A, m, p]) \wedge (Matrix[B, p, n])) \implies (A * B = \left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]_{m \times n})$
$^T[A] := (Matrix[A, m, n]) \implies (A^T = [a_{j,i}]_{n \times m})$

$AddCom := \forall_{A,B \in \mathcal{M}}(A + B = B + A)$

(1)	$A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C \in \mathcal{M}}((A + B) + C = A + (B + C))$

(1)	$(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A \in \mathcal{M}}\exists!_{O \in \mathcal{M}}(A + O = A = O + A)$

(1)	$A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$
(2)	$A + O_1 = A = A + O_2 \ \blacksquare\ O_1 = O_2$

$AddInv := \forall_{A \in \mathcal{M}}\exists!_{(-A) \in \mathcal{M}}(A + (-A) = O = (-A) + A)$

(1)	$A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$
(2)	$A + (-A_1) = O = A + (-A_2) \ \blacksquare\ -A_1 = -A_2 \ \blacksquare\ A_1 = A_2$

$MulAssoc := \forall_{A,B,C \in \mathcal{M}}((A * B) * C = A * (B * C))$

(1)	$(A * B) * C = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,j})\right] * C = \left[\sum_{k_2=1}^{p_2}(\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2})c_{k_2,j})\right] = \left[\sum_{k_2=1}^{p_2}\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right] = \ \ldots$
(2)	$\ldots \left[\sum_{k_1=1}^{p_1}\sum_{k_2=1}^{p_2}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right] = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1}\sum_{k_2=1}^{p_2}(b_{k_1,k_2}c_{k_2,j}))\right] = \ldots = A * (B * C)$

$MulId := \forall_{A : Square[A,n]}(A * I_n = A = I_n * A)$

(1) $A * I_n = \left[\sum_{k=1}^{n} \left(a_{i,k} \left(\begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \right) \right) \right] = [a_{i,j}] = A$

(2) $TODO = A$

$ScalAssoc := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} (r(sA) = (rs)A = s(rA))$

(1) $r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$

(2) $(rs)A = [rsa_{i,j}]$

(3) $s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$

$TransCancel := \forall_{A \in \mathcal{M}} (A = (A^T)^T)$

(1) $A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$

$ScalMulCom := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} ((rA) * B = r(A * B) = A * (rB))$

(1) $(rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[\sum_{k=1}^{p} (ra_{i,k} b_{k,j}) \right] = r(A * B)$

(2) $A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[\sum_{k=1}^{p} (a_{i,k} rb_{k,j}) \right] = \left[\sum_{k=1}^{p} (ra_{i,k} b_{k,j}) \right] = r(A * B)$

$ScalDistLeft := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} ((r + s)A = rA + sA)$

(1) TODO

$ScalDistRight := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} (r(A + B) = rA + rB)$

(1) TODO

$MulDistRight := \forall_{A,B,C \in \mathcal{M}} ((A + B) * C = A * C + B * C)$

(1) $(A + B) * C = [a_{i,j} + b_{i,j}] * C = \left[\sum_{k=1}^{p} ((a_{i,k} + b_{i,k}) c_{k,j}) \right] = \ldots$

(2) $\ldots \left[\sum_{k=1}^{p} (a_{i,k} c_{k,j} + b_{i,k} c_{k,j}) \right] = \left[\sum_{k=1}^{p} (a_{i,k} c_{k,j}) \right] + \left[\sum_{k=1}^{p} (b_{i,k} c_{k,j}) \right] = A * C + B * C$

$MulDistLeft := \forall_{A,B,C \in \mathcal{M}} (C * (A + B) = C * A + C * B)$

(1) TODO

$TransAddDist := \forall_{A,B \in \mathcal{M}} ((A + B)^T = A^T + B^T)$

(1) TODO

$TransMulDist := \forall_{A,B \in \mathcal{M}} ((A * B)^T = B^T * A^T)$

(1) $(A * B)^T = \left[\sum_{k=1}^{p} (a_{i,k} b_{k,j}) \right]^T = \left[\sum_{k=1}^{p} (a_{j,k} b_{k,i}) \right] = \left[\sum_{k=1}^{p} (b_{k,i} a_{j,k}) \right] = \left[\sum_{k=1}^{p} (b_{i,k}^T a_{k,j}^T) \right] = B^T * A^T$

$Sym[A] := A = A^T$
$SkewSym[A] := A = -A^T$
$Invertible[A] := (Square[A, n]) \wedge (\exists_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A))$

$SymGen := \forall_{A \in \mathcal{M}} (Sym[A + A^T])$

(1) $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$SkewSymGen := \forall_{A \in \mathcal{M}} (SkewSym[A - A^T])$

(1) $-(A - A^T)^T = -(A^T - (A^T)^T) = -(A^T - A) = (A - A^T)$

$SymDecomp := \forall_{A \in \mathcal{M}} \exists!_{B:Sym[B]} \exists!_{C:SkewSym[C]} (A = B + C)$

(1)	$B := (1/2) * (A + A^T)$; $C := (1/2) * (A - A^T)$
(2)	$SymGen[B] \wedge SkewSymGen[C]$
(3)	$A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$
(4)	$(1/2) * (A_1 + A_1{}^T) = (1/2) * (A_2 + A_2{}^T)$ ▌ $A_1 = A_2$
(5)	$(1/2) * (A_3 - A_3{}^T) = (1/2) * (A_4 - A_4{}^T)$ ▌ $A_3 = A_4$

$InvId := \forall_{A:Invertible[A]} (\exists!_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A))$

(1)	$A^{-1}{}_1 = A^{-1}{}_1 * I_n = A^{-1}{}_1 * (A * A^{-1}{}_2) = (A^{-1}{}_1 * A) * A^{-1}{}_2 = I_n * A^{-1}{}_2 = A^{-1}{}_2$

$InvCancel := \forall_{A:Invertible[A]} ((A^{-1})^{-1} = A)$

(1)	$(A * A^{-1})^{-1} = I_n{}^{-1} = I_n$
(2)	$(A^{-1})^{-1} * A^{-1} = I_n$ ▌ $A^{-1})^{-1} = I_n * A = A$

$InvDist := \forall_{A:Invertible[A]} \forall_{B:Invertible[B]} ((A * B)^{-1} = B^{-1} * A^{-1})$

(1)	$(A * B) * (A * B)^{-1} = I$ ▌ $B * (A * B)^{-1} = A^{-1}$ ▌ $(A * B)^{-1} = B^{-1} * A^{-1}$

$InvTrans := \forall_{A:Invertible[A]} ((A^T)^{-1} = (A^{-1})^T)$ ▌ $\Leftarrow$

(1)	$A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I$ ▌ $(A^{-1})^T = (A^T)^{-1}$

3.2 Elementary Matrices on Invertibility and Systems of Linear Equations

$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$
$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$
$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X (Sol[X, A, B])$
$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$
$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$

$HomoSysProps := (Sys[A, O]) \implies \dots$

(1)	$u_0 := O$; $u_1 := choice(\{X \in \mathcal{M} \mid X \neq O\})$; $k := choice(\mathbb{R})$
(2)	$TrivSol[u_0, A]$
(3)	$(NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$
(4)	$(TrivSol[\vec{X}, A]) \implies (TrivSol[LC(\vec{X}), A])$

$ElemMat[E] := (E = Swap[I_n, i, j]) \vee (Scale_*(I_n, i, c)) \vee (Combine_*(I_n, i, c, j))$
$ElemMatProd[E^*] := \exists_{\langle E \rangle} (\forall_{E_i \in E^*} (ElemMat[E_i]) \wedge (E^* = \Pi_{E_i \in E^*} (E_i)))$
$RowEquiv[A, B] := \exists_{E^*} ((ElemMatProd[E^*]) \wedge (B = E^* * A))$

$ElemMatInv := \forall_{E \in \mathcal{M}} ((ElemMat[E]) \implies (Invertible[E]))$

(1)	$E - RowSwap[E] \implies TODO$; $E - RowScale_*(E) \implies TODO$; $E - RowCombine_*(E) \implies TODO$

$ElemMatProdInv := \forall_{E^*} ((ElemMatProd[E^*]) \implies (Invertible[E^*]))$

(1)	TODO

$RowEquivSys := \forall_{A,B,C,D,X \in \mathcal{M}} (((Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]])) \implies (Sol[X, A, B] \iff Sol[X, C, D]))$

(1)	$\exists_{E^*:ElemMatProd[E^*]} ([CD] = E^* * [AB])$
(2)	$(E^* * A = C) \wedge (E^* * B = D)$
(3)	$Sol[Y, A, B] \implies \dots$

(3.1)	$A * Y = B$
(3.2)	$C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D$ ▮ $Sol[Y, C, D]$
(4)	$Sol[Y, A, B] \implies Sol[Y, C, D]$
(5)	$(A = (E^*)^{-1} * C) \wedge (B = (E^*)^{-1} * D)$
(6)	$Sol[Z, C, D] \implies \ldots$
(6.1)	$C * Z = D$
(6.2)	$A * Z = ((E^*)^{-1} * C) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$
(7)	$Sol[Z, C, D] \implies Sol[Z, A, B]$
(8)	$Sol[X, A, B] \iff Sol[X, C, D]$

$RowEquivHomoSysSol := \forall_{A,C,X \in \mathcal{M}}((RowEquiv[A,C]) \implies ((Sol[X,A,O]) \iff (Sol[X,C,O])))$

(1)	Set $B = D = O$

$RREF[A] := (A \in \mathcal{M}) \wedge \left\{ \begin{array}{ll} \text{All zero rows are at the bottom of the matrix.} & \wedge \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} \wedge \\ \text{The leading entry in any nonzero row is 1.} & \wedge \\ \text{All entries in the column above and below a leading 1 are zero.} & \wedge \end{array} \right\}$

$GaussJordanElim := \forall_{A \in \mathcal{M}} \exists!_{B \in \mathcal{M}}((RREF[B]) \wedge (RowEquiv[A, B]))$

(1)	Hit A with $ElemMat$'s until it becomes B
(2)	$(B = E^* * A) \wedge (RREF[B])$

$HasZero[A] := (Matrix(A, m, n)) \wedge (\exists_{i \le m}(A_{i,:} = O))$

$HasZeroNonInvertible := \forall_{A \in \mathcal{M}}((HasZero[A]) \implies (\neg Invertible[A]))$

(1)	$i := choice(\{i \le m | A_{i,:} = O\})$
(2)	$(B \in \mathcal{M}) \implies \ldots$
(2.1)	$(A * B)_{i,:} = O \neq I_{n_{i,:}}$ ▮ $A * B \neq I_n$
(3)	$(B \in \mathcal{M}) \implies (A * B \neq I_n)$ ▮ $\forall_{B \in \mathcal{M}}(A * B \neq I_n)$ ▮ $\neg Invertible[A]$

$InvIffRowEquivI := \forall_{A \in \mathcal{M}}((Invertible[A]) \iff (RowEquiv[A, I_n]))$

(1)	$(Invertible[A]) \implies \ldots$
(1.1)	$(RREF[B]) \wedge (RowEquiv[A, B])$
(1.2)	$B = E^* * A$
(1.3)	$(Invertible[E^*]) \wedge (Invertible[A])$ ▮ $Invertible[B]$
(1.4)	$Invertible[B]$ ▮ $\neg HasZero[B]$
(1.5)	$(RREF[B]) \wedge (\neg HasZero[B])$ ▮ $B = I_n$
(1.6)	$RowEquiv[A, I_n]$
(2)	$(Invertible[A]) \implies (RowEquiv[A, I_n])$
(3)	$(RowEquiv[A, I_n]) \implies \ldots$
(3.1)	$I_n = E^* * A$ ▮ $(E^*)^{-1} = A$
(3.2)	$A^{-1} = E^*_{DescSort}$ ▮ $Invertible[A]$
(4)	$(RowEquiv[A, I_n]) \implies (Invertible[A])$
(5)	$(Invertible[A]) \iff (RowEquiv[A, I_n])$

$RowEquivIIffTrivSol := \forall_{A \in \mathcal{M}}((RowEquiv[A, I_n]) \iff (\forall_X((X = O) \iff (Sol[X, A, O]))))$

(1)	$(RowEquiv[A, I_n]) \implies \ldots$
(1.1)	$RowEquiv[A, I_n]$ ▮ $Invertible[A]$
(1.2)	$(Sol[X, A, O]) \implies \ldots$
(1.2.1)	$A * X = O$ ▮ $X = A^{-1} * O = O$ ▮ $X = O$
(1.3)	$(Sol[X, A, O]) \implies (X = O)$
(1.4)	$(X = O) \implies (Sol[X, A, O])$

(1.5) $(X = O) \iff (Sol[X, A, O])$ ▌ $\forall_X((X = O) \iff (Sol[X, A, O]))$

(2) $(RowEquiv[A, I_n]) \implies (\forall_X((X = O) \iff (Sol[X, A, O])))$

(3) $(\forall_X((X = O) \iff (Sol[X, A, O]))) \implies \dots$

(3.1) $(RREF[B]) \wedge (RowEquiv[A, B])$

(3.2) $Sol[X, B, O]$

(3.3) $(B \neq I_n) \implies \dots$

$(3.3.1)$ $(\exists_{Y \neq X}(Sol[Y, B, O]))$

$(3.3.2)$ $Sol[Y, A, O]$ ▌ $Y = X$

$(3.3.3)$ $(Y \neq X) \wedge (Y = X)$ ▌ $\bot$

(3.4) $(B \neq I_n) \implies \bot$ ▌ $B = I_n$

(3.5) $(RowEquiv[A, B]) \wedge (B = I_n)$ ▌ $RowEquiv[A, I_n]$

(4) $(\forall_X((X = O) \iff (Sol[X, A, O]))) \implies (RowEquiv[A, I_n])$

(5) $(RowEquiv[A, I_n]) \iff (\forall_X((X = O) \iff (Sol[X, A, O])))$

$InvIffUniqSol := \forall_{A \in \mathcal{M}}((Invertible[A]) \iff (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])))$

(1) $(Invertible[A] \wedge B \in \mathcal{M}) \implies \dots$

(1.1) $(Invertible[A]) \wedge (Sys[A, B])$

(1.2) $(X = A^{-1} * B) \iff (Sol[X, A, B])$ ▌ $\exists!_{X \in \mathcal{M}}(Sol[X, A, B])$

(2) $(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies \dots$

(2.1) $X_i := choice(\{X_i | Sol[X_i, A, I_{n:,i}]\})$

(2.2) $A * [X_1 \dots X_n] = [(A * X_1) \dots (A * X_n)] = [I_{n:,1} \dots I_{n:,n}] = I_n$

(2.3) $A^{-1} = [X_1 \dots X_n]$

(3) $(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies (Invertible[A])$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}} \left\{ \begin{array}{l} (Invertible[A]) \qquad\qquad \iff \\ (RowEquiv[A, I_n]) \qquad\qquad \iff \\ (\forall_X((X = O) \iff (Sol[X, A, O]))) \iff \\ (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \end{array} \right\}$$

3.3 Vector Spaces

$$VectorSpace[V, +, *] := \exists_{O \in V} \forall_{\alpha, \beta \in \mathbb{R}} \forall_{u,v,w \in V} \left\{ \begin{array}{l} (u + v \in V) \ \wedge \ (u + v = v + u) \ \wedge \ ((u + v) + w = u + (v + w)) \ \wedge \\ (u + O = u) \qquad \wedge \qquad (\exists_{-u \in V}(u + (-u) = O)) \qquad \wedge \\ (\alpha * u \in V) \ \wedge \ (\alpha * (\beta * u) = (\alpha\beta) * u) \ \wedge \ (1 * u = u) \ \wedge \\ (\alpha * (u + v) = (\alpha * u) + (\alpha * v)) \wedge ((\alpha + \beta) * u = (\alpha * u) + (\beta * u)) \end{array} \right\}$$

$ZeroVectorUniq := \forall_{O', v \in V}((v + O' = v) \implies (O' = O))$

(1) $O' = O' + O = O + O' = O$ ▌ $O' = O$

$AddInvUnique := \forall_{-v', v \in V}((v + -v' = O) \implies (-v' = -v))$

(1) $-v' = -v' + O = -v' + (v + -v) = (-v' + v) + -v = (v + -v') + -v = O + -v = -v$ ▌ $-v' = -v$

$AddInvGen := \forall_{v \in V}((-1) * v = -v)$

(1) $v + (-1) * v = (1 - 1) * v = 0 * v = O$ ▌ $(-1) * v = -v$

$ZeroVectorGenLeft := \forall_{v \in V}(0 * v = O)$

(1) $0 * v = (0 + 0) * v = (0 * v) + (0 * v)$ ▌ $O = 0 * v$

$ZeroVectorGenRight := \forall_{r \in \mathbb{R}}(r * O = O)$

(1) $r * O = r * (O + O) = (r * O) + (r * O)$ ▌ $O = r * O$

$ZeroVectorEquiv := \forall_{r \in \mathbb{R}} \forall_{v \in V}((r * v = O) \iff ((v = O) \lor (r = 0)))$

(1)	$(ZeroVectorGenLeft) \land (ZeroVectorGenRight)$ ▋ $((v = O) \lor (r = 0)) \implies (r * v = O))$
(2)	$(r * v = O) \implies \dots$
(2.1)	$(r \neq 0) \implies \dots$
(2.1.1)	$r \neq 0$ ▋ $r^{-1} \in \mathbb{R}$
(2.1.2)	$ZeroVectorGenRight$ ▋ $O = r^{-1} * O = r^{-1} * (r * v) = (r^{-1}r) * v = 1 * v = v$ ▋ $O = v$
(2.2)	$(r \neq 0) \implies (v = O)$ ▋ $(r = 0) \lor (r \neq 0)$ ▋ $(r = 0) \lor (v = O)$
(3)	$(r * v = O) \implies ((r = 0) \lor (v = O))$
(4)	$(r * v = O) \iff ((r = 0) \lor (v = O))$

3.4   Subspaces and Special Subspaces

$Subspace[S, V, +, *] := (VectorSpace[V, +, *]) \land (\emptyset \neq S \subseteq V) \land (VectorSpace[S, +, *])$

$SubspaceEquiv := \forall_{V,S} \left(\begin{array}{l} (VectorSpace[V, +, *]) \qquad\qquad\qquad\qquad\qquad \implies \\ ((Subspace[S, V, +, *]) \iff ((\emptyset \neq S \subseteq V) \land (\forall_{r,s \in S}(r + s \in S)) \land (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)))) \end{array} \right)$

(1)	$(Subspace[S, V, +, *]) \implies \dots$
(1.1)	$Subspace[S, V, +, *]$ ▋ $\emptyset \neq S \subseteq V$
(1.2)	$VectorSpace[S, +, *]$ ▋ $(\forall_{r,s \in S}(r + s \in S)) \land (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S))$
(1.3)	$(\emptyset \neq S \subseteq V) \land (\forall_{r,s \in S}(r + s \in S)) \land (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S))$
(2)	$(Subspace[S, V, +, *]) \implies ((\emptyset \neq S \subseteq V) \land (\forall_{r,s \in S}(r + s \in S)) \land (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)))$
(3)	$((\emptyset \neq S \subseteq V) \land (\forall_{r,s \in S}(r + s \in S)) \land (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S))) \implies \dots$
(3.1)	$((\alpha, \beta \in \mathbb{R}) \land (\emptyset \neq S) \land (u, v, w \in S)) \implies \dots$
(3.1.1)	$u, v \in V$ ▋ $u + v = v + u$
(3.1.2)	$u, v, w \in V$ ▋ $(u + v) + w = u + (v + w)$
(3.1.3)	$(ZeroVectorGenLeft) \land (u \in S)$ ▋ $0 * u = O \in S$
(3.1.4)	$u \in V$ ▋ $u + O = u$
(3.1.5)	$(AddInvGen) \land (u \in S)$ ▋ $(-1) * u = -u \in S$
(3.1.6)	$u \in V$ ▋ $\alpha * (\beta * u) = (\alpha\beta) * u$
(3.1.7)	$u \in V$ ▋ $1 * u = u$
(3.1.8)	$u, v \in V$ ▋ $\alpha * (u + v) = (\alpha * u) + (\alpha * v)$
(3.1.9)	$u \in V$ ▋ $(\alpha + \beta) * u = (\alpha * u) + (\beta * u)$
(4)	$((\forall_{r,s \in S}(r + s \in S)) \land (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S))) \implies (Subspace[S, V, +, *])$
(5)	$(Subspace[S, V, +, *]) \iff ((\forall_{r,s \in S}(r + s \in S)) \land (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)))$

$SumSubContains := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A, V, +, *]) \land (Subspace[B, V, +, *]) \land (SetSum[A + B, A, B, V, +, *])) \implies \\ ((Subspace[A + B, V, +, *]) \land (A, B \subseteq A + B)) \end{array} \right)$

(1)	$(Subspace[A, V, +, *]) \land (Subspace[B, V, +, *])$ ▋ $(O \in A) \land (O \in B)$
(2)	$(SetSum[A + B, A, B, V, +, *]) \land (O \in A) \land (O \in B)$ ▋ $O \in A + B$ ▋ $\emptyset \neq A + B$
(3)	$(Subspace[A, V, +, *]) \land (Subspace[B, V, +, *])$ ▋ $A + B \subseteq V$ ▋ $\emptyset \neq A + B \subseteq V$
(4)	$(u, v \in A + B) \implies \dots$
(4.1)	$(\exists_{a_1 \in A} \exists_{b_1 \in B}(u = a_1 + b_1)) \land (\exists_{a_2 \in A} \exists_{b_2 \in B}(v = a_2 + b_2))$
(4.2)	$u + v = (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$
(4.3)	$(a_1 + a_2 \in A) \land (b_1 + b_2 \in B)$ ▋ $u + v \in A + B$
(5)	$(u, v \in A + B) \implies (u + v \in A + B)$ ▋ $\forall_{u,v \in A+B}(u + v \in A + B)$
(6)	$((r \in \mathbb{R}) \land (v \in A + B)) \implies \dots$
(6.1)	$\exists_{a \in A} \exists_{b \in B}(v = a + b)$
(6.2)	$r * v = r * (a + b) = r * a + r * b$
(6.3)	$(r * a \in A) \land (r * b) \in B$ ▋ $r * v \in A + B$
(7)	$((r \in \mathbb{R}) \land (v \in A + B)) \implies (r * v \in A + B)$ ▋ $\forall_{r \in \mathbb{R}} \forall_{v \in A+B}(r * v \in A + B)$

(8)	$(\emptyset \neq A + B \subseteq V) \wedge (\forall_{u,v \in A+B}(u + v \in A + B)) \wedge (\forall_{r \in \mathbb{R}}\forall_{v \in A+B}(r * v \in A + B))$ ▌ $Subspace[A + B, V, +, *]$
(9)	$(\forall_{a \in A}(a + O) = a) \wedge (O \in B)$ ▌ $A \subseteq A + B$
(10)	$(\forall_{b \in B}(b + O) = b) \wedge (O \in A)$ ▌ $B \subseteq A + B$
(11)	$(A, B \subseteq A + B) \wedge (Subspace[A + B, V, +, *])$

$$SumSubMinContains := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])) \implies \\ (\forall_C((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies (A + B \subseteq C)) \end{array} \right)$$

(1)	$SumSub$ ▌ $(A, B \subseteq A + B) \wedge (Subspace[A + B, V, +, *])$
(2)	$((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies \dots$
(2.1)	$(s \in A + B) \implies \dots$
(2.1.1)	$\exists_{a \in A}\exists_{b \in B}(s = a + b)$
(2.1.2)	$(A, B \subseteq C)$ ▌ $a, b \in C$
(2.1.3)	$Subspace[C, V, +, *]$ ▌ $s = a + b \in C$
(2.2)	$(s \in A + B) \implies (s \in C)$ ▌ $A + B \subseteq C$
(3)	$((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies (A + B \subseteq C)$

$NullSpace[N, A, m, n] := (Matrix[A, m, n]) \wedge (N = \{x \in \mathbb{R}^n | A * x = O\})$
$RowSpace[R, A, m, n] := (Matrix[A, m, n]) \wedge (R = \{x^T * A \in \mathbb{R}^n | x \in \mathbb{R}^m\})$
$ColSpace[C, A, m, n] := (Matrix[A, m, n]) \wedge (C = \{A * x \in \mathbb{R}^m | x \in \mathbb{R}^n\})$

$NullSubspace := (NullSpace[N, A, m, n]) \implies (Subspace[N, \mathbb{R}^n, +, *])$

(1)	TODO

$RowSubspace := (RowSpace[R, A, m, n]) \implies (Subspace[R, \mathbb{R}^n, +, *])$

(1)	TODO

$ColSubspace := (ColSpace[C, A, m, n]) \implies (Subspace[C, \mathbb{R}^m, +, *])$

(1)	TODO

3.5   Linear Combination, Linear Span, Linear Independence

$LinComb[c, U, K, V, +, *] := (VectorSpace[V, +, *]) \wedge (n \in \mathbb{N}) \wedge (U \in V^n) \wedge (K \in \mathbb{R}^n) \wedge (c = \sum_{i=1}^{n}(k_i * u_i))$

$$LinSpan[S', S, V, +, *] := \left(\begin{array}{l} (VectorSpace[V, +, *]) \wedge (S \in V^n) \wedge ((S = \emptyset) \implies (S' = \{O\})) \wedge \\ ((S \neq \emptyset) \implies (S' = \{c \in V | \exists_{K \in \mathbb{R}^n}(LinComb[c, S, K, V, +, *])\})) \end{array} \right)$$

$LinSpanSubContains := \forall_{S',S,V,+,*}((LinSpan[S', S, V, +, *]) \implies ((Subspace[S', V, +, *]) \wedge (S \subseteq S')))$

(1)	$(S = \emptyset) \implies (S' = \{O\}) \implies (\emptyset \neq S')$
(2)	$(S \neq \emptyset) \implies (LinComb[O, S, \{0\}^n, V, +, *]) \implies (O \in S') \implies (\emptyset \neq S')$
(3)	$((S = \emptyset) \vee (S \neq \emptyset)) \implies (\emptyset \neq S')$ ▌ $\emptyset \neq S'$
(4)	$LinSpan[S', S, V, +, *]$ ▌ $S' \subseteq V$ ▌ $\emptyset \neq S' \subseteq V$
(5)	$(a, b \in S') \implies \dots$
(5.1)	$(\exists_{K \in \mathbb{R}^n}(LinComb[a, S, K, V, +, *])) \wedge (\exists_{L \in \mathbb{R}^n}(LinComb[b, S, L, V, +, *]))$
(5.2)	$a + b = \sum_{i=1}^{n}(k_i * s_i) + \sum_{i=1}^{n}(l_i * s_i) = \sum_{i=1}^{n}((k_i + l_i) * s_i)$ ▌ $a + b = \sum_{i=1}^{n}((k_i + l_i) * s_i)$
(5.3)	$\langle k_i + l_i | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{N}^n$
(5.4)	$(a + b = \sum_{i=1}^{n}((k_i + l_i) * s_i)) \wedge (\langle k_i + l_i | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{N}^n)$ ▌ $\exists_{M \in \mathbb{N}^n}(a + b = \sum_{i=1}^{n}(m_i * s_i))$
(5.5)	$\exists_{M \in \mathbb{N}^n}(LinComb[a + b, S, M, V, +, *])$ ▌ $a + b \in S'$
(6)	$(a, b \in S') \implies (a + b \in S')$ ▌ $\forall_{a,b \in S'}(a + b \in S')$
(7)	$((r \in \mathbb{R}) \wedge (u \in S')) \implies \dots$
(7.1)	$\exists_{K \in \mathbb{R}^n}(LinComb[u, S, K, V, +, *])$ ▌ $\exists_{K \in \mathbb{R}^n}(u = \sum_{i=1}^{n}(k_i * s_i))$
(7.2)	$r * u = r * \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{n}(r * (k_i * s_i)) = \sum_{i=1}^{n}(rk_i) * s_i)$ ▌ $r * u = \sum_{i=1}^{n}(rk_i) * s_i$
(7.3)	$\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n$

(7.4)	$(\sum_{i=1}^{n}(rk_i * s_i)) \wedge (\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n)$
(7.5)	$\exists_{M \in \mathbb{R}^n}(r * u = \sum_{i=1}^{n}(m_i * s_i))$ ▮ $\exists_{M \in \mathbb{R}^n}(LinComb[r * u, S, M, V, +, *])$ ▮ $r * u \in S'$
(8)	$((r \in \mathbb{R}) \wedge (u \in S')) \implies (r * u \in S')$ ▮ $\forall_{r \in \mathbb{R}} \forall_{u \in S'}(r * u \in S')$
(9)	$(SubspaceEquiv) \wedge (\emptyset \neq S' \subseteq V) \wedge (\forall_{a,b \in S'}(a + b \in S')) \wedge (\forall_{r \in \mathbb{R}} \forall_{u \in S'}(r * u \in S'))$ ▮ $Subspace[S', V, +, *]$
(10)	$(s_j \in S) \implies \dots$

(10.1)	$K := \langle \left\{ \begin{matrix} 1 & i = j \\ 0 & i \neq j \end{matrix} \right\} | \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n$ ▮ $\sum_{i=1}^{n}(k_i * s_i) = s_j$
(10.2)	$\dots$ ▮ $\exists_{K \in \mathbb{R}^n}(LinComb[s_j, S, K, V, +, *])$ ▮ $s_j \in S'$
(11)	$(s_j \in S) \implies (s_j \in S')$ ▮ $\forall_{x \in S}(x \in S')$ ▮ $S \subseteq S'$
(12)	$(Subspace[S', V, +, *]) \wedge (S \subseteq S')$

$LinSpanSubMinContains := \forall_{S',S,V,+,*}((LinSpan[S', S, V, +, *]) \implies (\forall_W(((Subspace[W, V, +, *]) \wedge (S \subseteq W)) \implies (S' \subseteq W))))$

(1)	$(s' \in S') \implies \dots$
(1.1)	$\exists_{K \in \mathbb{R}^n}(LinComb[s', S, K, V, +, *])$ ▮ $s' = \sum_{i=1}^{n}(k_i * s_i)$
(1.2)	$(S \subseteq W) \wedge (Subspace[W, V, +, *])$ ▮ $s' = \sum_{i=1}^{n}(k_i * s_i) \in W$
(2)	$(s' \in S') \implies (s' \in W)$ ▮ $S' \subseteq W$

$LinInd[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (S \in V^n) \wedge ((S \neq \emptyset) \implies (\forall_{K \in \mathbb{R}^n}((LinComb[O, S, K, V, +, *]) \implies (K = \{0\}^n))))$

$ZeroDependent := (O \in S) \implies (\neg LinInd[S, V, +, *])$

(1)	$K := \langle \left\{ \begin{matrix} 1 & u_i = O \\ 0 & u_i \neq O \end{matrix} \right\} | (1 \leq i \leq n) \wedge (i \in \mathbb{N}) \rangle$ ▮ $K \in \mathbb{R}^n$
(2)	$(LinComb[O, S, K, V, +, *]) \wedge (K \neq \{O\}^n)$ ▮ $\neg LinInd[S, V, +, *]$

$SingletonNonZeroIndependent := (v \neq O) \implies (LinInd[\langle v \rangle, V, +, *])$

(1)	$(r * v = O) \iff ((r = 0) \vee (v \neq O))$
(2)	$v \neq O$ ▮ $r = 0$
(3)	$\forall_{r \in \mathbb{R}}((r * v = O) \implies (r = 0))$

$SubIndependent := \forall_{V,A,B}(((VectorSpace[V, +, *]) \wedge (A \subseteq B \in V^m)) \implies ((LinInd[B, V, +, *]) \implies (LinInd[A, V, +, *])))$

(1)	$(LinComb[O, A, K, V, +, *]) \implies \dots$
(1.1)	$L := \langle \left\{ \begin{matrix} 1 & j \leq n \\ 0 & j > n \end{matrix} \right\} | (1 \leq j \leq m \wedge (j \in \mathbb{N}) \rangle$ ▮ $L \in \mathbb{R}^m$
(1.2)	$A \subseteq B$ ▮ $\forall_{n \geq j \in \mathbb{N}}(a_j = b_j)$
(1.3)	$\forall_{n \geq j \in \mathbb{N}}(a_j = b_j)$ ▮ $\sum_{i=1}^{n}(k_i * a_i)) = \sum_{i=1}^{n}(k_i * a_i)) + O = \sum_{j=1}^{m}(l_j * b_j))$
(1.4)	$LinComb[O, A, K, V, +, *]$ ▮ $O = \sum_{i=1}^{n}(k_i * a_i)$
(1.5)	$O = \sum_{i=1}^{n}(k_i * a_i) = \sum_{j=1}^{m}(l_j * b_j)$ ▮ $LinComb[O, B, L, V, +, *]$
(1.6)	$(LinInd[B, V, +, *]) \wedge (LinComb[O, B, L, V, +, *])$ ▮ $L = \{0\}^m$
(1.7)	$(\forall_{n \geq j \in \mathbb{N}}(a_j = b_j)) \wedge (L = \{0\}^m)$ ▮ $\forall_{n \geq j \in \mathbb{N}}(k_j * a_j = l_j * b * j = l_j * a_j)$ ▮ $K = \{0\}^n$
(2)	$(LinComb[O, A, K, V, +, *]) \implies (K = \{0\}^n)$ ▮ $\forall_{K \in \subseteq \mathbb{R}^n}((LinComb[O, A, K, V, +, *]) \implies (K = \{0\}^n))$ ▮ $LinInd[A, V, +, *]$

$SuperDependent := \forall_{V,A,B}(((VectorSpace[V, +, *]) \wedge (A \subseteq B \subseteq V)) \implies ((\neg LinInd[A, V, +, *]) \implies (\neg LinInd[B, V, +, *])))$

(1)	TODO : A has a non trivial solution ▮ use the same non trivial solution in combination with B and L

$LinIndEquiv := \forall_{U,V}((LinInd[U, V, +, *]) \iff (\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])))$

(1)	$\Gamma' = \Gamma \setminus \{j\}$

(2)	$(\neg LinInd[U,V,+,*]) \implies \dots$
(2.1)	$(\exists_{\Gamma \in \mathbb{R}^{|U|}}((\sum(\gamma_i * u_i) = O) \wedge (\Gamma \neq \{0\}^{|U|})))$
(2.2)	$\exists_{\gamma_k \in \Gamma}(\gamma_k \neq 0)$
(2.3)	$\sum(\gamma_i' * u_i) = \sum(\gamma_i * u_i) - \gamma_k * u_k = -\gamma_k * u_j$
(2.4)	$u_k = (-1/\gamma_k)(\sum(\gamma_i' * u_i)) = \sum((-\gamma_i'/\gamma_k) * u_i)$ ▌ $\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])$
(3)	$(\neg LinInd[U,V,+,*]) \implies (\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *]))$
(4)	$(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])) \implies (LinInd[U,V,+,*])$
(5)	$(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])) \implies \dots$
(5.1)	$\exists_{j \in U}(j = \sum(\gamma_i' * u_i))$
(5.2)	$\Gamma := \Gamma' \cup \{-1\}$
(5.3)	$(\sum(\gamma_i * u_i) = \sum(\gamma_i' * u_i) + (-1) * \gamma_j = O) \wedge (\Gamma \neq \{0\}^n)$ ▌ $\neg LinInd[U,V,+,*]$
(6)	$(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])) \implies (\neg LinInd[U,V,+,*])$
(7)	$(LinInd[U,V,+,*]) \implies (\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *]))$
(8)	$(LinInd[U,V,+,*]) \iff (\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *]))$

$LinIndSuperspace := \forall_{U,V}((Subspace[U,V]) \implies (\forall_W((LinInd[W,U,+,*]) \implies (LinInd[W,V,+,*]))))$

(1)	$(\neg LinInd[W,V,+,*]) \implies \dots$
(1.1)	$\exists_{j \in W}(LinComb[j, W \setminus \{j\}, +, *])$ ▌ $\neg LinInd[W,U,+,*]$
(1.2)	$(LinInd[W,U,+,*]) \wedge (\neg LinInd[W,U,+,*])$ ▌ $\bot$
(2)	$(\neg LinInd[W,V,+,*]) \implies \bot$ ▌ $LinInd[W,V,+,*]$

$Spans[S,V,+,*] := LinSpan[V,S,V,+,*]$
$FinDim[V,+,*] := \exists_{S \in V^n}(Spans[S,V,+,*])$

$LinDepLemma := \forall_{S,V}\left(\begin{array}{l} (\neg LinInd[S,V,+,*]) \\ \exists_{j \in \mathbb{N}_{1,n}}((s_j \in LinSpan[P_1, S_{1,j-1}, V, +, *]) \wedge (LinSpan[P_2, S, V, +, *] = LinSpan[P_3, S \setminus \{s_j\}, V, +, *])) \end{array} \right)$ $\implies$

(1)	$\neg LinInd[S,V,+,*]$ ▌ $\exists_{K \in \mathbb{R}^n}((LinComb[O, S, K, V, +, *]) \wedge (K \neq \{0\}^n))$
(2)	$\exists_{j \in \mathbb{N}_{1,n}}((k_j \neq 0) \wedge (\forall_{i \in \mathbb{N}_{1,n}}((i > j) \implies (k_i = 0))))$
(3)	$s_j = (-1/k_j)\sum_{i=1}^{j-1}(k_i * s_i) = \sum_{i=1}^{j-1}((-k_i/k_j) * s_i)$
(4)	$\langle -k_i/k_j | i \in \mathbb{N}_{1,j-1} \rangle \in \mathbb{R}^{j-1}$
(5)	$\exists_{M \in \mathbb{R}^{j-1}}(LinComb[s_j, S_{1,j-1}, M, V, +, *])$ ▌ $s_j \in LinSpan[P_1, S_{1,j-1}, V, +, *]$
(6)	$(v \in P_2) \iff (v \in LinSpan[P_2, S, V, +, *]) \iff \dots$
(7)	$\dots (v = \sum_{i=1}^n(k_i * s_i)) = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^n(k_i * s_i)) + k_j * s_j = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^n(k_i * s_i)) + k_j * \sum_{i=1}^{j-1}((-k_i/k_j) * s_i)) \iff \dots$
(8)	$(v \in LinSpan[P_3, S \setminus \{s_j\}, V, +, *]) \iff (v \in P_3)$ ▌ $(v \in P_2) \iff (v \in P_3)$ ▌ $P_2 = P_3$

$LinIndLengthLeqSpan := \forall_{L,S}(((LinInd[L,V,+,*]) \wedge (Spans[S,V,+,*])) \implies (|L| \leq |S|))$

(1)	TODO : form $B = L \cup S$, remove dependent elements in S such that $(Spans[B,V,+,*]) \wedge (|B| = |S|)$ by $LinDepLemma$, $|L| \leq |B| = |S|$
(2)	$\forall_{l_i \in L} \dots$
(2.1)	$l_i \in V$ ▌ $LinComb[l_i, S, K, V, +, *]$ ▌ $\neg LinInd[\langle l_i \rangle \cup S, V, +, *]$
(2.2)	$LinDepLemma$ ▌ $\exists_{j \in \mathbb{N}_{1,n}}(LinSpan[V, S, V, +, *] = LinSpan[V, S \setminus \{s_j\}, V, +, *])$
(2.3)	$B := \langle l_i \rangle \cup S \setminus \{s_j\}$ ▌ $|B| = 1 + |S| - 1 = |S|$
(3)	$|L| \leq |B| = |S|$ ▌ $|L| \leq |S|$

$FinSubSpace := \forall_{U,V}(((Subspace[U,V,+,*]) \wedge (FinDim[V,+,*])) \implies (FinDim[U,+,*]))$

(1)	TODO : take $Spans[S,V,+,*]$, remove all $s_j \in S$ such that $U \subseteq LinSpan[S \setminus \{s_j\}]$ ▌ $S' = S \setminus \{s_j\}$ $(LinSpan[U, S', V, +, *]) \wedge (|S'| \leq |S|)$ ▌ $FinDim[U,+,*]$

(2) $FinDim[V,+,*]$ ▋ $\exists_{S\in V^n}(Spans[S,V,+,*])$

(3) $\forall_{(u_j\in U)\wedge(\neg LinSpan[U,S,V,+,*])}\cdots$

  (3.1) $\neg LinSpan[U,S,V,+,*]$ ▋ $\exists_{u_j\in U}(\neg LinComb[u_j,S_{1,j-1},K_{1,j-1},V,+,*])$

  (3.2) $B:=S_{1,j-1}$ ▋ $|B|=|S|-1<|S|$

(4) $LinSpan[U,B,V,+,*]$ ▋ $\exists_{B\in V^M}(Spans[B,U,+,*])$ ▋ $FinDim[U,+,*$

3.6   Bases and Dimensions

$Basis[S,V,+,*]:=(Spans[S,V,+,*])\wedge(LinInd[S,V,+,*])$

$BasisEquiv:=\forall_{S,V}((Basis[S,V,+,*])\iff(\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*]))$

(1) $(Basis[S,V,+,*])\implies\ldots$

  (1.1) $(v\in V)\implies\ldots$

    (1.1.1) $Basis[S,V,+,*]$ ▋ $Spans[V,S,+,*]$ ▋ $\exists_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])$

    (1.1.2) $((K_1,K_2\in\mathbb{R}^n)\wedge(LinComb[v,S,K_1,V,+,*])\wedge(LinComb[v,S,K_2,V,+,*]))\implies\ldots$

      (1.1.2.1) $(v=\sum(k_{1_i}*s_i))\wedge(v=\sum(k_{2_i}*s_i))$

      (1.1.2.2) $O=v-v=\sum(k_{1_i}*s_i)-\sum(k_{2_i}*s_i)=\sum((k_{1_i}-k_{2_i})*s_i)$

      (1.1.2.3) $L:=\langle k_{1_i}-k_{2_i}|i\in\mathbb{N}_{i=1}^n\rangle\in\mathbb{R}^n$

      (1.1.2.4) $(LinInd[S,V,+,*])\wedge(LinComb[O,S,L,V,+,*])$ ▋ $L=\{0\}^n$ ▋ $K_2=K_1$

    (1.1.3) $((K_1,K_2\in\mathbb{R}^n)\wedge(LinComb[v,S,K_1,V,+,*])\wedge(LinComb[v,S,K_2,V,+,*]))\implies(K_1=K_2)$

    (1.1.4) $\forall_{K_1,K_2\in\mathbb{R}^n}((LinComb[v,S,K_1,V,+,*])\wedge(LinComb[v,S,K_2,V,+,*])\implies(K_1=K_2))$

    (1.1.5) $\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])$

  (1.2) $(v\in V)\implies(\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*]))$

(2) $(Basis[S,V,+,*])\implies(\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*]))$

(3) $(\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*]))\implies\ldots$

  (3.1) $\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])$ ▋ $\forall_{v\in V}\exists_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])$ ▋ $Spans[S,V,+,*]$

  (3.2) $O\in V$ ▋ $\exists!_{K\in\mathbb{R}^n}(LinComb[O,S,K,V,+,*])$

  (3.3) $(K\neq\{0\}^n)\implies(\neg\exists!_{K\in\mathbb{R}^n}(LinComb[O,S,K,V,+,*]))\implies\perp$ ▋ $K=\{0\}^n$

  (3.4) $(\exists!_{K\in\mathbb{R}^n}(LinComb[O,S,K,V,+,*]))\wedge(K=\{0\}^n)$ ▋ $LinInd[S,V,+,*]$

  (3.5) $(Spans[S,V,+,*])\wedge(LinInd[S,V,+,*])$ ▋ $Basis[S,V,+,*]$

(4) $(\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*]))\implies(Basis[S,V,+,*])$

$SpanReduceBasis:=\forall_{S,V}(Spans[S,V,+,*]\implies\exists_B((B\subseteq S)\wedge(Basis[B,V,+,*])))$

(1)   TODO : Remove all dependent $s_j\in S$ ▋ $(S'=S\setminus\{s_j\})\wedge(LinSpan[S']=LinSpan[S])$ until $LinInd[S']$ ▋ $Basis[S']$

$FinDimBasis:=\forall_V((FinDim[V,+,*])\implies(\exists_B(Basis[B,V,+,*])))$

(1) $FinDim[V,+,*]$ ▋ $\exists_{S\in V^n}(Spans[S,V,+,*])$

(2) $(SpanReduceBasis)\wedge(Spans[S,V,+,*])$ ▋ $\exists_B(Basis[B,V,+,*])$

$LinIndExpandBasis:=\forall_{L,V}(LinInd[L,V,+,*]\implies\exists_B((L\subseteq B)\wedge(Basis[B,V,+,*])))$

(1)   TODO : $FinDimBasis$ ▋ $\exists_A(Basis[A,V,+,*])$, form $B=L\cup A$ ▋ $Span[B]$,
  use $SpanReduceBasis$ call it B', $(L\subseteq B')\wedge(Basis[B'])$

$BasisLinearIndCard:=\forall_{S,T,V}(((Basis[S,V,+,*])\wedge(LinInd[T,V,+,*]))\implies(|T|\leq|S|))$

(1) $(Basis[S,V,+,*])\implies\ldots$

  (1.1) $(|T|>|S|)\implies\ldots$

    (1.1.1) $(Spans[S,V,+,*])\wedge(T\subseteq V)$ ▋ $t_{1\ldots t_j}=\sum(\gamma_i*s*i)\ldots$

    (1.1.2) $\ldots t_j=\sum(\gamma_i'*t_i)$ ▋ $\neg LinInd[T,V,+,*]$

  (1.2) $(|T|>|S|)\implies(\neg LinInd[T,V,+,*])$ ▋ $(LinInd[T,V,+,*])\implies(|T|\leq|S|)$

(2) $((Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*])) \implies (|T| \leq |S|)$

$BasisCard := \forall_{S,T,V}(((Basis[S,V,+,*]) \wedge (Basis[T,V,+,*])) \implies (|T| = |S|))$

(1)	$Basis[S,V,+,*]$ ▌ $LinInd[S,V,+,*]$
(2)	$(Basis[T,V,+,*]) \wedge (LinInd[S,V,+,*])$ ▌ $|S| \leq |T|$
(3)	$Basis[T,V,+,*]$ ▌ $LinInd[T,V,+,*]$
(4)	$(Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*])$ ▌ $|T| \leq |S|$
(5)	$(|S| \leq |T|) \wedge (|T| \leq |S|)$ ▌ $|T| = |S|$

$Dim[d,V,+,*] := (\exists_B(Basis[B,V,+,*])) \wedge ((V = \{O\}) \implies (d = 0)) \wedge ((V \neq \{O\}) \implies (d = |B|))$

$LinIndLengthDim := \forall_{U,V}(((LinInd[U,V,+,*]) \wedge (Dim[|U|,V,+,*])) \implies (Basis[U,V,+,*]))$

(1)	$(LinIndExpandBasis) \wedge (LinInd[U,V,+,*])$ ▌ $\exists_B((U \subseteq B) \wedge (Basis[B,V,+,*]))$
(2)	$(BasisCard) \wedge (Dim[|U|,V,+,*]) \wedge (Basis[B,V,+,*])$ ▌ $|B| = |U|$ ▌ $B = U$ ▌ $Basis[U,V,+,*]$

$SpanLengthDim := \forall_{U,V}(((Spans[U,V,+,*]) \wedge (Dim[|U|,V,+,*])) \implies (Basis[U,V,+,*]))$

(1)	$(SpanReduceBasis) \wedge (Spans[U,V,+,*])$ ▌ $\exists_B((B \subseteq U) \wedge (Basis[B,V,+,*]))$
(2)	$(BasisCard) \wedge (Dim[|U|,V,+,*]) \wedge (Basis[B,V,+,*])$ ▌ $|B| = |U|$ ▌ $B = U$ ▌ $Basis[U,V,+,*]$

$LinDepLengthDim := \forall_{U,V}(((U \subseteq V) \wedge (|U| > Dim[V])) \implies (\neg LinInd[U,V,+,*]))$

(1)	Contrapositive of $BasisLinearIndCard$

$LinDepLengthDim := \forall_{U,V}(((U \subseteq V) \wedge (|U| < Dim[V])) \implies (\neg Spans[U,V,+,*]))$

(1)	Suppose $Spans[U,V,+,*]$, $B = SpanReduceBasis[U]$ to form a basis, $(|B| \leq |U| < Dim[V]) \wedge |B| = Dim[V]$ ▌ $\perp$
(2)	$\neg Spans[U,V,+,*]$

3.7   Rank

$Nullity[n,A] := (NullSpace[N,A]) \wedge (Dim[n,N,+,*])$
$Rank[r,A,m,n] := (Matrix[A,m,n]) \wedge (RowSpace[R,A,m,n]) \wedge (Dim[r,R,A,+,*])$

$RowRankEqColRank := \forall_A(TODO)$

(1)	TODO

$RankNullity := \forall_A((Matrix[A,m,n]) \implies (Rank[A] + Nullity[A] = n))$

(1)	TODO

$RankInv := \forall_A((Matrix[A,m,n]) \implies ((Rank[A] = n) \iff (Inv[A])))$

(1)	TODO

$RankNonTrivialSol := (\exists_X((A * X = O) \wedge (X \neq O))) \iff (Rank[A] < n)$

(1)	TODO

$RankUniqueSol := (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X,A,B])) \iff (Rank[A] = n)$

(1)	TODO

$$SquareTheorems_{()} := \forall_{A \in \mathcal{M}} \left\{ \begin{array}{ll} (Invertible[A]) & \Longleftrightarrow \\ (RowEquiv[A, I_n]) & \Longleftrightarrow \\ (\forall_X((X = O) \Longleftrightarrow (Sol[X, A, O]))) & \Longleftrightarrow \\ (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) & \Longleftrightarrow \\ (Rank[A] = n) & \Longleftrightarrow \\ (Nullity[A] = 0) & \Longleftrightarrow \\ \text{(The rows form a linearly independent set of vectors (to get full rank))} & \Longleftrightarrow \\ \text{(The columns form a linearly independent set of vectors (to get full rank))} & \Longleftrightarrow \end{array} \right.$$

3.8 Linear Transformations

$$LinTrans[L, V, +_v, *_v, W, +_w, *_w] := \left(\begin{array}{c} (Function[f, V, W]) \wedge (VectorSpace[V, +_v, *_v]) \wedge (VectorSpace[W, +_w, *_w]) \wedge \\ (\forall_{\alpha, \beta \in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))) \quad \wedge \quad (\forall_{r \in \mathbb{R}} \forall_{\alpha \in V}(L(r *_v \alpha) = r *_w L(\alpha))) \end{array} \right)$$

$LinOp[L, V, +_v, *_v] := LinTrans[L, V, +_v, *_v, V, +_v, *_v]$

$\mathcal{L}[V, W] := \{L | LinTrans[L, V, +_v, *_v, W, +_w, *_w]\}$

$ZeroMapsToZero := \forall_{L,V,W}((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \Longrightarrow (L(O_v) = O_w))$

(1)	$L(O_v) = L(O_v +_v O_v) = L(O_v) +_w L(O_v)$
(2)	$O_w = L(O_v) - L(O_v) = L(O_v)$

$SplitAddInv := \forall_{L,V,W}((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \Longrightarrow (\forall_{\alpha, \beta \in V}(L(\alpha -_v \beta) = L(\alpha) -_w L(\beta))))$

(1)	$L(\alpha - \beta) = L(\alpha + (-\beta)) = L(\alpha) + L(-\beta) = L(\alpha) + (-1) * L(\beta) = L(\alpha) - L(\beta)$

$$BasisDomainInduceLinTrans := \forall_{V,W} \left(\begin{array}{c} ((Basis[A, V, +_v, *_v]) \wedge (B \subseteq W) \wedge (n = |B| = |A|) \wedge (VectorSpace[W, +_w, *_w])) \Longrightarrow \\ (\exists!_T((LinTrans[T, V, +_v, *_v, W, +_w, *_w]) \wedge (\forall_{i \in \mathbb{N}_{1,n}}(T(a_i) = b_i)))) \end{array} \right)$$

(1) $T(\sum_{i=1}^{n}(k_i * a_i)) := \sum_{i=1}^{n}(k_i * b_i)$

(2) $(i \in \mathbb{N}_{1,n}) \Longrightarrow \ldots$

(2.1) $L := \left\langle \left\{ \begin{array}{ll} 1 & j = i \\ 0 & j \neq i \end{array} \right| j \in \mathbb{N}_{1,n} \right\rangle \blacksquare L \in \mathbb{R}^n$

(2.2) $T(a_i) = T(\sum_{i=1}^{n}(l_i * a_i)) = \sum_{i=1}^{n}(l_i * b_i) = b_i \blacksquare T(a_i) = b_i$

(3) $(i \in \mathbb{N}_{1,n}) \Longrightarrow (T(a_i) = b_i) \blacksquare \forall_{i \in \mathbb{N}_{1,n}}(T(a_i) = b_i)$

(4) $(BasisEquiv) \wedge (Basis[A, V, +_v, *_v]) \blacksquare \forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, A, K, V, +, *]) \blacksquare T : Span[A] \to Span[B] \blacksquare Function[T, V, W]$

(5) $(\alpha, \beta \in V) \Longrightarrow \ldots$

(5.1) $(\exists_{K_\alpha}(LinComb[\alpha, A, K_\alpha, V, +_v, *_v])) \wedge (\exists_{K_\beta}(LinComb[\beta, A, K_\beta, V, +_v, *_v])) \blacksquare (\alpha = \sum_{i=1}^{n}(k_{\alpha i} * a_i)) \wedge (\beta = \sum_{i=1}^{n}(k_{\beta_i} * a_i))$

(5.2) $T(\alpha + \beta) = T(\sum_{i=1}^{n}(k_{\alpha i} * a_i) + \sum_{i=1}^{n}(k_{\beta_i} * a_i)) = T(\sum_{i=1}^{n}((k_{\alpha i} + k_{\beta_i}) * a_i))) = \sum_{i=1}^{n}((k_{\alpha i} + k_{\beta_i}) * b_i) = \ldots$

(5.3) $\ldots \sum_{i=1}^{n}(k_{\alpha i} * b_i) + \sum_{i=1}^{n}(k_{\beta_i} * b_i) = T(\sum_{i=1}^{n}(k_{\alpha i} * a_i)) + T(\sum_{i=1}^{n}(k_{\beta_i} * a_i)) = T(\alpha) + T(\beta)$

(6) $(\alpha, \beta \in V) \Longrightarrow (L(\alpha +_v \beta) = L(\alpha) +_w L(\beta)) \blacksquare \forall_{\alpha, \beta \in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))$

(7) $((r \in \mathbb{R}) \wedge (\alpha \in V)) \Longrightarrow \ldots$

(7.1) $\exists_K(LinComb[\alpha, A, K, V, +_v, *_v]) \blacksquare \alpha = \sum_{i=1}^{n}(k_i * a_i)$

(7.2) $L(r *_v \alpha) = L(r *_v \sum_{i=1}^{n}(k_i *_v a_i)) = L(\sum_{i=1}^{n}((rk_i) *_v a_i)) = \ldots$

(7.3) $\ldots \sum_{i=1}^{n}((rk_i) *_w b_i) = r *_w \sum_{i=1}^{n}(k_i *_w b_i) = r *_w L(\sum_{i=1}^{n}(k_i *_v a_i)) = r *_w L(\alpha)$

(8) $((r \in \mathbb{R}) \wedge (\alpha \in V)) \Longrightarrow (L(r *_v \alpha) = r *_w L(\alpha)) \blacksquare \forall_{r \in \mathbb{R}} \forall_{\alpha \in V}(L(r *_v \alpha) = r *_w L(\alpha))$

(9) $(\forall_{i \in \mathbb{N}_{1,n}}(T(a_i) = b_i)) \wedge (Function[T, V, W]) \wedge (\forall_{\alpha, \beta \in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))) \wedge (\forall_{r \in \mathbb{R}} \forall_{\alpha \in V}(L(r *_v \alpha) = r *_w L(\alpha))) \wedge \ldots$

(10) $\ldots (VectorSpace[V, +_v, *_v]) \wedge (VectorSpace[W, +_w, *_w]) \blacksquare LinTrans[T, V, +_v, *_v, W, +_w, *_w]$

$Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (ker_L = \{\alpha \in V | L(\alpha) = O_w\})$

$KerSub := \forall_{L,V,W}((Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w]) \Longrightarrow (Subspace[ker_L, V, +_v, *_v]))$

(1) $ZeroMapsToZero \blacksquare L(O_v) = O_w \blacksquare O_v \in ker_L \blacksquare \emptyset \neq ker_L \blacksquare \emptyset \neq ker_L \subseteq V$

(2) $(\alpha, \beta \in ker_L) \Longrightarrow \ldots$

(2.1) $(L(\alpha) = O_w) \wedge (L(\beta) = O_w)$

(2.2)	$L(\alpha + \beta) = L(\alpha) + L(\beta) = O_w + O_w = O_w$ ▌ $L(\alpha + \beta) \in ker_L$	

(3) $(\alpha, \beta \in ker_L) \implies (\alpha + \beta \in ker_L)$ ▌ $\forall_{\alpha, \beta \in ker_L}(\alpha + \beta \in ker_L)$

(4) $((r \in \mathbb{R}) \wedge (\alpha \in ker_L)) \implies \dots$

  (4.1) $L(\alpha) = O_w$ ▌ $L(r * \alpha) = r * L(\alpha) = r * O_w = O_w$ ▌ $r * \alpha \in ker_L$

(5) $((r \in \mathbb{R}) \wedge (\alpha \in ker_L)) \implies (r * \alpha \in ker_L)$ ▌ $\forall_{r \in \mathbb{R}} \forall_{\alpha \in ker_L}(r * \alpha \in ker_L)$

(6) $(SubspaceEquiv) \wedge (\emptyset \neq ker_L \subseteq V) \wedge (\forall_{\alpha, \beta \in ker_L}(\alpha + \beta \in ker_L)) \wedge (\forall_{r \in \mathbb{R}} \forall_{\alpha \in ker_L}(r * \alpha \in ker_L))$ ▌ $Subspace[ker_L, V, +_v, *_v]$

$KerInjective := \forall_{L,V,W}((Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w]) \implies ((Injective[L, V, W]) \iff (ker_L = \{O_v\})))$

(1) $(Injective[L, V, W]) \implies \dots$

  (1.1) $ZeroMapsToZero$ ▌ $L(O_v) = O_w$

  (1.2) $O_v \in ker_L$ ▌ $\{O_v\} \subseteq ker_L$

  (1.3) $(v \in ker_L) \implies \dots$

    (1.3.1) $L(v) = O_w$

    (1.3.2) $(Injective[L, V, W]) \wedge (L(O_v) = O_w)$ ▌ $O_v = v$

  (1.4) $(v \in ker_L) \implies (v = O_v)$ ▌ $ker_L \subseteq \{O_v\}$

  (1.5) $(\{O_v\} \subseteq ker_L) \wedge (ker_L \subseteq \{O_v\})$ ▌ $ker_L = \{O_v\}$

(2) $(Injective[L, V, W]) \implies (ker_L = \{O_v\})$

(3) $(ker_L = \{O_v\}) \implies \dots$

  (3.1) $((u, v \in V) \wedge (L(u) = L(v))) \implies \dots$

    (3.1.1) $O_w = L(u) - L(v) = L(u - v)$ ▌ $u - v \in ker_L$

    (3.1.2) $ker_L = \{O_v\}$ ▌ $u - v = O_v$ ▌ $u = v$

  (3.2) $((u, v \in V) \wedge (L(u) = L(v))) \implies (u = v)$ ▌ $\forall_{u,v \in V}((L(u) = L(v)) \implies (u = v))$ ▌ $Injective[L, V, W]$

(4) $(ker_L = \{O_v\}) \implies (Injective[L, V, W])$

(5) $(Injective[L, V, W]) \iff (ker_L = \{O_v\})$

$Rng[rng_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (rng_L = \{\beta \in W \mid \exists_{\alpha \in V}(\beta = L(\alpha))\})$

$RangeSub := \forall_{L,V,W}((Ran[rng_L, L, V, +_v, *_v, W, +_w, *_w]) \implies (Subspace[rng_L, W, +_w, *_w]))$

(1) $ZeroMapsToZero$ ▌ $O_w = L(O_v)$ ▌ $\exists_{\alpha \in V}(O_w = L(\alpha))$ ▌ $O_w \in rng_L$ ▌ $\emptyset \neq rng_L$ ▌ $\emptyset \neq rng_L \subseteq W$

(2) $(\alpha, \beta \in rng_L) \implies \dots$

  (2.1) $(\exists_{u \in V}(\alpha = L(u))) \wedge (\exists_{v \in V}(\beta = L(v)))$

  (2.2) $\alpha + \beta = L(u) + L(v) = L(u + v)$ ▌ $\exists_{w \in V}(\alpha + \beta = L(w))$ ▌ $\alpha + \beta \in rng_L$

(3) $(\alpha, \beta \in rng_L) \implies (\alpha + \beta \in rng_L)$ ▌ $\forall_{\alpha, \beta \in rng_L}(\alpha + \beta \in rng_L)$

(4) $((r \in \mathbb{R}) \wedge (\alpha \in rng_L)) \implies \dots$

  (4.1) $\exists_{v \in V}(\alpha = L(v))$ ▌ $L(r * v) = r * L(v) = r * \alpha$ ▌ $\exists_{w \in V}(r * \alpha = L(w))$ ▌ $r * \alpha \in rng_L$

(5) $((r \in \mathbb{R}) \wedge (\alpha \in rng_L)) \implies (r * \alpha \in rng_L)$ ▌ $\forall_{r \in \mathbb{R}} \forall_{\alpha \in rng_L}(r * \alpha \in rng_L)$

(6) $(SubspaceEquiv) \wedge (\emptyset \neq rng_L \subseteq W) \wedge (\forall_{\alpha, \beta \in rng_L}(\alpha + \beta \in rng_L)) \wedge (\forall_{r \in \mathbb{R}} \forall_{\alpha \in rng_L}(r * \alpha \in rng_L))$ ▌ $Subspace[rng_L, W, +_w, *_w]$

$RankKer := \forall_{L,V,W}((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies (Dim[V] = Dim[ker_L] + Dim[rng_L]))$ TODO

(1) $\exists_U(Basis[U, ker_L, +_v, *_v])$ ▌ $Dim[ker_L] = |U|$

(2) $(LinIndSuperspace) \wedge (LinInd[U, ker_L, +_v, *_v])$ ▌ $LinInd[U, V, +_v, *_v]$

(3) $LinIndExpandBasis$ ▌ $\exists_B((U \subseteq B) \wedge (Basis[B, V, +_v, *_v]))$ ▌ $Dim[V] = |B|$

(4) $T := B \setminus U$ ▌ $B = U \cup T$

(5) $m := |U| \,; n := |T| \,; p := |B|$

(6) $L(T) := \langle L(t_i) \mid i \in \mathbb{N}_{1,n} \rangle \subseteq W^n$

(7) $(w \in W) \implies \dots$

  (7.1) $\exists_{v \in V}(w = L(v))$

  (7.2) $Basis[B, V, +_v, *_v]$ ▌ $\exists_{K \in \mathbb{R}^p}(v = \sum_{i=1}^p (k_i * b_i))$

  (7.3) $v = \sum_{i=1}^p (k_i * b_i) = \sum_{i=1}^m (k_i * u_i) + \sum_{i=1}^n (\kappa_i * t_i)$

  (7.4) $w = L(v) = L(\sum_{i=1}^m (k_i * u_i) + \sum_{i=1}^n (k_i * t_i)) = \sum_{i=1}^m (k_i * L(u_i)) + \sum_{i=1}^n (k_i * L(t_i)) = O_w + \sum_{i=1}^n (k_i * L(t_i)) = \sum_{i=1}^n (k_i * L(t_i))$

(7.5)	$\exists_K(LinComb[w, L(T), K, W, +_w, *_w])$
(8)	$(w \in W) \implies (\exists_K(LinComb[w, L(T), K, W, +_w, *_w])) \ \blacksquare \ \forall_{w \in W}(\exists_K(LinComb[w, L(T), K, W, +_w, *_w]))$
(9)	$Spans[L(T), W, +_w, *_w]$
(10)	$((K \in \mathbb{R}^n) \wedge (LinComb[O_w, L(T), K, W, +_w, *_w])) \implies \ldots$
(10.1)	$O_w = \sum_{i=1}^n (k_i * L(t_i)) = L(\sum_{i=1}^n (k_i * t_i)) \ \blacksquare \ \sum_{i=1}^n (k_i * t_i) \in ker_L$
(10.2)	$Basis[U, ker_L, +_v, *_v] \ \blacksquare \ \exists_{D \in \mathbb{R}^m}(\sum_{i=1}^n (k_i * t_i) = \sum_{i=1}^m (d_i * u_i))$
(10.3)	$(LinInd[T \cup U]) \wedge (\sum_{i=1}^n (k_i * t_i) = \sum_{i=1}^m (d_i * u_i)) \ \blacksquare \ K = D = \{O\} \ \blacksquare \ K = \{O\}$
(11)	$((K \in \mathbb{R}^n) \wedge (LinComb[O_w, L(T), K, W, +_w, *_w])) \implies (K = \{O\})$
(12)	$\forall_{K \in \mathbb{R}^n}((LinComb[O_w, L(T), K, W, +_w, *_w]) \implies (K = \{O\}))$
(13)	$\forall_{K \in \mathbb{R}^n}((LinComb[O_w, L(T), K, W, +_w, *_w]) \iff (K = \{O\})) \ \blacksquare \ LinInd[L(T), W, +_w, *_w]$
(14)	$Basis[L(T), W, +_w, *_w] \ \blacksquare \ Dim[V] = |B| = |U| + |L(T)| = Dim[ker_L] + Dim[rng_L]$

TODO INJ -> SUR, SUR -> INJ