Contents

Chapter 1

Real Analysis

(1.5)

$OrderTrichotomy(<, S) := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$

$OrderTransitivity(<, S) := \forall_{x,y,z \in S}\big((x < y \land y < z) \implies x < z\big)$

$Order(<, S) := OrderTrichotomy(<, S) \land OrderTransitivity(<, S)$

(1.7)

$BoundedAbove(E, S, <) := Order(<, S) \land E \subset S \land \exists_{\beta \in S}\forall_{x \in E}(x \le \beta)$

$BoundedBelow(E, S, <) := Order(<, S) \land E \subset S \land \exists_{\beta \in S}\forall_{x \in E}(\beta \le x)$

$UpperBound(\beta, E, S, <) := Order(<, S) \land E \subset S \land \beta \in S \land \forall_{x \in E}(x \le \beta)$

$LowerBound(\beta, E, S, <) := Order(<, S) \land E \subset S \land \beta \in S \land \forall_{x \in E}(\beta \le x)$

(1.8)

$LUB(\alpha, E, S, <) := UpperBound(\alpha, E, S, <) \land \forall_{\gamma}\big(\gamma < \alpha \implies \neg UpperBound(\gamma, E, S, <)\big)$

$GLB(\alpha, E, S, <) := LowerBound(\alpha, E, S, <) \land \forall_{\beta}\big(\alpha < \beta \implies \neg LowerBound(\beta, E, S, <)\big)$

(1.10)

$LUBProperty(S, <) := \forall_E\Big(\big(\emptyset \ne E \subset S \land BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$

$GLBProperty(S, <) := \forall_E\Big(\big(\emptyset \ne E \subset S \land BoundedBelow(E, S, <)\big) \implies \exists_{\alpha \in S}\big(GLB(\alpha, E, S, <)\big)\Big)$

(1.11)

$\boxed{LUBPropertyImpliesGLBProperty}$ $LUBProperty(S, <) \implies GLBProperty(S, <)$

(1) $LUBProperty(S, <) \implies \ldots$	wts: 2
(1.1) $\big(\emptyset \ne B \subset S \land BoundedBelow(B, S, <)\big) \implies \ldots$	wts: 1.2
(1.1.1) $Order(<, S) \land \exists_{\delta' \in S}\big(LowerBound(\delta', B, S, <)\big)$	from: BoundedBelow, 1.1
(1.1.2) $\lvert B \rvert = 1 \implies \ldots$	wts: 1.1.3
(1.1.2.1) $\exists_{u'}(u' \in B)$ ▮ $u := choice(\{u' \mid u' \in B\})$ ▮ $B = \{u\}$	from: 1.1.2
(1.1.2.2) $GLB(u, B, S, <)$ ▮ $\exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$	
(1.1.3) $\lvert B \rvert = 1 \implies \exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$	
(1.1.4) $\lvert B \rvert \ne 1 \implies \ldots$	wts: 1.1.5
(1.1.4.1) $\forall_E\Big(\big(\emptyset \ne E \subset S \land BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$	from: LUBProperty, 1
(1.1.4.2) $L := \{s \in S \mid LowerBound(s, B, S, <)\}$	
(1.1.4.3) $\lvert B \rvert > 1 \land OrderTrichotomy(<, S)$ ▮ $\exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$	from: Order, 1.1.1 wts: 1.1.4.7
(1.1.4.4) $b_1 := choice\Big(\{b_1' \in B \mid \exists_{b_0' \in B}(b_0' < b_1')\}\Big)$ ▮ $\neg LowerBound(b_1, B, S, <)$	from: 1.1.4.2
(1.1.4.5) $b_1 \notin L$ ▮ $L \subset S$	
(1.1.4.6) $\delta := choice\big(\{\delta' \in S \mid LowerBound(\delta', B, S, <)\}\big)$ ▮ $\delta \in L$ ▮ $\emptyset \ne L$	from: 1.1.1
(1.1.4.7) $\emptyset \ne L \subset S$	from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y\in L}\big(LowerBound(y_0, B, S, <)\big) \blacksquare \forall_{y\in L}\forall_{x\in B}(y_0 \le x)$ from: LowerBound, 1.1.4.2 wts: 1.1.4.10

(1.1.4.9) $\forall_{x\in B}\Big(x \in S \wedge \forall_{y\in L}(y_0 \le x)\Big) \blacksquare \forall_{x\in B}\big(UpperBound(x, L, S, <)\big)$ from: UpperBound

(1.1.4.10) $\exists_{x\in S}\big(UpperBound(x, L, S, <)\big) \blacksquare BoundedAbove(L, S, <)$

(1.1.4.11) $\emptyset \ne L \subset S \wedge BoundedAbove(L, S, <)$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha'\in S}\big(LUB(\alpha', L, S, <)\big) \blacksquare \alpha := choice\Big(\{\alpha' \in S | \big(LUB(\alpha', L, S, <)\big)\}\Big)$ from: 1.1.4.1 wts: 1.1.4.21

(1.1.4.13) $\forall_x\big(x \in B \implies UpperBound(x, L, S, <)\big)$ from: 1.1.4.9 wts: 1.1.4.17

(1.1.4.14) $\forall_x\big(\neg UpperBound(x, L, S, <) \implies x \notin B\big)$

(1.1.4.15) $\gamma < \alpha \implies \dots$ wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound(\gamma, L, S, <) \blacksquare \gamma \notin B$ from: LUB, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B \blacksquare \gamma \in B \implies \gamma \ge \alpha$

(1.1.4.17) $\forall_{\gamma\in B}(\alpha \le \gamma) \blacksquare LowerBound(\alpha, B, S, <)$ from: LowerBound

(1.1.4.18) $\alpha < \beta \implies \dots$ wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y\in L}(y_0 \le \alpha < \beta) \blacksquare \forall_{y\in L}(y_0 \ne \beta)$ from: LUB, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta \notin L \blacksquare \neg LowerBound(\beta, B, S, <)$ from: 1.1.4.2

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound(\beta, B, S, <) \blacksquare \forall_{\beta\in S}\big(\alpha < \beta \implies \neg LowerBound(\beta, B, S, <)\big)$

(1.1.4.20) $LowerBound(\alpha, B, S, <) \wedge \forall_{\beta\in S}\big(\alpha < \beta \implies \neg LowerBound(\beta, B, S, <)\big)$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB(\alpha, B, S, <) \blacksquare \exists_{\epsilon_1\in S}\big(GLB(\epsilon_1, B, S, <)\big)$

(1.1.5) $|B| \ne 1 \implies \exists_{\epsilon_1\in S}\big(GLB(\epsilon_1, B, S, <)\big)$

(1.1.6) $\Big(|B| = 1 \implies \exists_{\epsilon_0\in S}\big(GLB(\epsilon_0, B, S, <)\big)\Big) \wedge \Big(|B| \ne 1 \implies \exists_{\epsilon_1\in S}\big(GLB(\epsilon_1, B, S, <)\big)\Big)$ from: 1.1.3, 1.1.5

(1.1.7) $(|B| = 1 \vee |B| \ne 1) \implies \exists_{\epsilon\in S}\big(GLB(\epsilon, B, S, <)\big) \blacksquare \exists_{\epsilon\in S}\big(GLB(\epsilon, B, S, <)\big)$

(1.2) $(\emptyset \ne B \subset S \wedge BoundedBelow(B, S, <)) \implies \exists_{\epsilon\in S}\big(GLB(\epsilon, B, S, <)\big)$

(1.3) $\forall_B\Big((\emptyset \ne B \subset S \wedge BoundedBelow(B, S, <)) \implies \exists_{\epsilon\in S}\big(GLB(\epsilon, B, S, <)\big)\Big)$

(1.4) $GLBProperty(S, <)$

(2) $LUBProperty(S, <) \implies GLBProperty(S, <)$

(1.12)

$$Field(F, +, *) := \exists_{0,1\in F}\forall_{x,y,z\in F}\left(\begin{array}{l} x + y \in F \quad \wedge \quad x * y \in F \quad \wedge \\ x + y = y + x \quad \wedge \quad x * y = y * x \quad \wedge \\ (x + y) + z = x + (y_0 + z) \quad \wedge \quad (x * y) * z = x * (y_0 * z) \quad \wedge \\ \mathbb{1} \ne \mathbb{0} \quad \wedge \quad x * (y_0 + z) = (x * y) + (x * z) \quad \wedge \\ \mathbb{0} + x = x \quad \wedge \quad \mathbb{1} * x = x \quad \wedge \\ \exists_{-x\in F}\big(x + (-x) = \mathbb{0}\big) \wedge \Big(x \ne \mathbb{0} \implies \exists_{1/x\in F}\big(x * (1/x) = \mathbb{1}\big)\Big) \end{array}\right)$$

$$*********************** \big(Field(F, +, *) \wedge x, y, z \in F\big) \implies \dots ***********************$$

(1.14)

$\boxed{AdditiveCancellation}$ $(x + y = x + z) \implies y = z$

(1) $y = \mathbb{0} + y = \big(x + (-x)\big) + y = \big((-x) + x\big) + y = (-x) + (x + y) = \dots$ from: Field

(2) $(-x) + (x + z) = \big((-x) + x\big) + z = \big(x + (-x)\big) + z = \mathbb{0} + z = z$ from: Field

$\boxed{AdditiveIdentityUniqueness}$ $(x + y = x) \implies y = \mathbb{0}$

(1) $x + y = x = \mathbb{0} + x = x + \mathbb{0}$ from: Field

(2) $y = 0$ from: AdditiveCancellation

$\boxed{AdditiveInverseUniqueness}$ $(x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$ from: *Field*

(2) $y = -x$ from: *AdditiveCancellation*

$\boxed{DoubleNegative}$ $x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x$ ▮ $\mathbb{0} = (-x) + x$ from: *Field*

(2) $x = -(-x)$ from: *AdditiveInverseUniqueness*

(1.15)
$\boxed{MultiplicativeCancellation}$ $(x \neq 0 \wedge x * y = x * z) \implies y = z$ —
$\boxed{MultiplicativeIdentityUniqueness}$ $(x \neq 0 \wedge x * y = x) \implies y = \mathbb{1}$ —
$\boxed{MultiplicativeInverseUniqueness}$ $(x \neq 0 \wedge x * y = \mathbb{1}) \implies y = 1/x$ —
$\boxed{DoubleReciprocal}$ $(x \neq 0) \implies x = 1/(1/x)$ —

(1.16)
$\boxed{Domination}$ $\mathbb{0} * x = \mathbb{0}$

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x$ ▮ $\mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$ from: *Field*

(2) $\mathbb{0} * x = \mathbb{0}$ from: *AdditiveIdentityUniqueness*

$\boxed{NonDomination}$ $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies \ldots$

(1.1) $(x * y = 0) \implies \ldots$

(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = \big(x * (1/x)\big) * \big(y * (1/y)\big) = (x * y) * \big((1/x) * (1/y)\big) = \mathbb{0} * \big((1/x) * (1/y)\big) = \mathbb{0}$ from: *Field*, *Domination*, 1, 1.1

(1.1.2) $\mathbb{1} = \mathbb{0} \wedge \mathbb{1} \neq \mathbb{0}$ ▮ $\perp$ from: *Field*

(1.2) $(x * y = 0) \implies \perp$ ▮ $x * y \neq 0$

(2) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq 0$

$\boxed{NegationCommutativity}$ $(-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0}$ ▮ $x * y + (-x) * y = \mathbb{0}$ from: *Field*, *Domination* wts: 2

(2) $(-x) * y = -(x * y)$ from: *AdditiveInverseUniqueness*

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * \mathbb{0} = \mathbb{0}$ ▮ $x * y + x * (-y) = \mathbb{0}$ from: *Field*, *Domination* wts: 4

(4) $x * (-y) = -(x * y)$ from: *AdditiveInverseUniqueness*

(5) $(-x) * y = -(x * y) = x * (-y)$ from: 2, 4

$\boxed{NegativeMultiplication}$ $(-x) * (-y) = x * y$

(1) $(-x) * (-y) = -\big(x * (-y)\big) = -\big(-(x * y)\big) = x * y$ from: *NegationCommutativity*, *DoubleNegative*

(1.17)

$$OrderedField(F, +, *, <) := \left(\begin{array}{l} Field(F, +, *) \quad \wedge \quad Order(<, F) \quad \wedge \\ \forall_{x,y,z \in F}(y_0 < z \implies x + y < x + z) \ \wedge \\ \forall_{x,y \in F}\big((x > \mathbb{0} \wedge y > \mathbb{0}) \implies x * y > \mathbb{0}\big) \end{array}\right)$$

************************* $\big(OrderedField(F, +, *, <) \wedge x, y, z \in F\big) \implies \ldots$ *************************

(1.18)
$\boxed{NegationOnOrder}$ $x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \ldots$

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x \blacksquare \mathbb{0} > -x \blacksquare -x < \mathbb{0}$ from: *OrderedField*

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \ldots$

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x \blacksquare \mathbb{0} < x \blacksquare x > \mathbb{0}$ from: *OrderedField*

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \wedge -x < \mathbb{0} \implies x > \mathbb{0} \blacksquare x > \mathbb{0} \iff -x < \mathbb{0}$ from: 2, 4

$\boxed{\mathit{PositiveFactorPreservesOrder}}$ $(x > \mathbb{0} \wedge y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \wedge y < z) \implies \ldots$

(1.1) $(-y) + z > (-y) + y = \mathbb{0} \blacksquare z + (-y) = \mathbb{0}$ from: *OrderedField*

(1.2) $x * (z + (-y)) > \mathbb{0} \blacksquare x * z + x * (-y) > \mathbb{0}$ from: *OrderedField*

(1.3) $x * z = \mathbb{0} + x * z = \big(x * y + -(x * y)\big) + x * z = \big(x * y + x * (-y)\big) + x * z = \ldots$ from: *Field*, *NegationCommutativity*

(1.4) $x * y + \big(x * z + x * (-y)\big) > x * y + \mathbb{0} = x * y$ from: *Field*, 1.2

(1.5) $x * z > x * y$ from: 1.3, 1.4

(2) $(x > \mathbb{0} \wedge y < z) \implies x * z > x * y$

$\boxed{\mathit{NegativeFactorFlipsOrder}}$ $(x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \wedge y < z) \implies \ldots$

(1.1) $-x > \mathbb{0}$ from: *NegationOnOrder*

(1.2) $(-x) * y < (-x) * z \blacksquare \mathbb{0} = x * y + (-x) * y < x * y + (-x) * z \blacksquare \mathbb{0} < x * y + (-x) * z$ from: *PositiveFactorPreservesOrder*

(1.3) $\mathbb{0} < (-x) * (-y + z) \blacksquare \mathbb{0} > x * (-y + z) \blacksquare \mathbb{0} > -(x * y) + x * z$ from: *NegationOnOrder*

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

$\boxed{\mathit{SquareIsPositive}}$ $(x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ from: *OrderedField*

(2) $(x < \mathbb{0}) \implies \ldots$

(2.1) $-x > \mathbb{0} \blacksquare x * x = (-x) * (-x) > \mathbb{0} \blacksquare x * x > \mathbb{0}$ from: *NegationOnOrder*, *OrderedField*, *NegativeMultiplication*

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \vee x < \mathbb{0}) \implies x * x > \mathbb{0} \blacksquare x \neq \mathbb{0} \implies x * x > \mathbb{0}$ from: *OrderTrichotomy*, 1, 3

$\boxed{\mathit{OneIsPositive}}$ $\mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0} \blacksquare \mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ from: *Field*, *SquareIsPositive*

$\boxed{\mathit{ReciprocationOnOrder}}$ $(0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \ldots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0} \blacksquare x * (1/x) > 0$ from: *Field*, *OneIsPositive*

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp \blacksquare 1/x > \mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0} \blacksquare y * (1/y) > 0$ from: *Field*, *OneIsPositive*

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp \blacksquare 1/y > \mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

(1.5) $(1/x) * (1/y) > \mathbb{0}$ from: *OrderedField*

(1.6) $\mathbb{0} < 1/y = \big((1/x) * (1/y)\big) * x < \big((1/x) * (1/y)\big) * y = 1/x$ from: *OrderedField*, 1, 1.4, 1.5

(1.19)
$\boxed{OrderedFieldQ}$ $OrderedField(\mathbb{Q},+,*,<)$ —

$Subfield(K,F,+,*) := Field(F,+,*) \wedge K \subset F \wedge Field(K,+,*)$
$OrderedSubfield(K,F,+,*,<) := OrderedField(F,+,*,<) \wedge K \subset F \wedge OrderedField(K,+,*,<)$
$CutI(\alpha) := \emptyset \neq \alpha \subset \mathbb{Q}$
$CutII(\alpha) := \forall_{p\in\alpha}\forall_{q\in\mathbb{Q}}(q<p \implies q\in\alpha)$
$CutIII(\alpha) := \forall_{p\in\alpha}\exists_{r\in\alpha}(p<r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} | CutI(\alpha) \wedge CutII(\alpha) \wedge CutIII(\alpha)\}$

$\boxed{CutCorollaryI}$ $(\alpha\in\mathbb{R}\wedge p\in\alpha\wedge q\in\mathbb{Q}\wedge q\notin\alpha) \implies p<q$

(1) $(\alpha\in\mathbb{R}\wedge p\in\alpha\wedge q\in\mathbb{Q}\wedge q\notin\alpha) \implies \dots$
(1.1) $\forall_{p'\in\alpha}\forall_{q'\in\mathbb{Q}}(q'<p' \implies q'\in\alpha)$ from: $CutII$, 1
(1.2) $q<p \implies q\in\alpha$ ▮ $q\notin\alpha \implies q\geq p$ from: 1
(1.3) $(q\notin\alpha) \implies \dots$
(1.3.1) $q\geq p$ from: 1.2
(1.3.2) $(q=p) \implies (p\in\alpha\wedge p\notin\alpha) \implies \bot$ ▮ $q\neq p$ from: 1, 1.3
(1.3.3) $q\geq p\wedge q\neq p$ ▮ $p<q$
(1.4) $q\notin\alpha \implies p<q$ ▮ $p<q$ from: 1
(2) $(\alpha\in\mathbb{R}\wedge p\in\alpha\wedge q\in\mathbb{Q}\wedge q\notin\alpha) \implies p<q$

$\boxed{CutCorollaryII}$ $(\alpha\in\mathbb{R}\wedge r,s\in\mathbb{Q}\wedge r<s\wedge r\notin\alpha) \implies s\notin\alpha$

(1) $(\alpha\in\mathbb{R}\wedge r,s\in\mathbb{Q}\wedge r<s\wedge r\notin\alpha) \implies \dots$
(1.1) $\forall_{s'\in\alpha}\forall_{r'\in\mathbb{Q}}(r'<s' \implies r'\in\alpha)$ from: $CutII$, 1
(1.2) $s\in\alpha \implies \big(r\in\mathbb{Q} \implies (r<s \implies r\in\alpha)\big)$ ▮ $s\in\alpha \implies r\in\alpha$ from: 1, 1.1
(1.3) $r\notin\alpha \implies s\notin\alpha$ ▮ $s\notin\alpha$ from: 1, 1.2
(2) $(\alpha\in\mathbb{R}\wedge r,s\in\mathbb{Q}\wedge r<s\wedge r\notin\alpha) \implies s\notin\alpha$

$<_{\mathbb{R}}(\alpha,\beta) := \alpha,\beta\in\mathbb{R}\wedge\alpha\subset\beta$

$\boxed{OrderTrichotomyOfR}$ $OrderTrichotomy(\mathbb{R},<_{\mathbb{R}})$

(1) $(\alpha,\beta\in\mathbb{R}) \implies \dots$
(1.1) $\neg(\alpha<_{\mathbb{R}}\beta\vee\alpha=\beta) \implies \dots$
(1.1.1) $\alpha\not\subset\beta\wedge\alpha\neq\beta$ from: $<_{\mathbb{R}}$, 1.1
(1.1.2) $\exists_{p'}(p'\in\alpha\wedge p'\notin\beta)$ ▮ $p := choice(\{p'|p'\in\alpha\wedge p'\notin\beta\})$
(1.1.3) $q\in\beta \implies \dots$
(1.1.3.1) $p,q\in\mathbb{Q}$
(1.1.3.2) $q<p$ from: $CutCorollaryI$
(1.1.3.3) $q\in\alpha$ from: $CutII$
(1.1.4) $q\in\beta \implies q\in\alpha$
(1.1.5) $\forall_{q\in\beta}(q\in\alpha)$ ▮ $\beta\subseteq\alpha$
(1.1.6) $\beta\subset\alpha$ ▮ $\beta<_{\mathbb{R}}\alpha$
(1.2) $\neg(\alpha<_{\mathbb{R}}\beta\vee\alpha=\beta) \implies \beta<_{\mathbb{R}}\alpha$
(1.3) $\neg(\alpha<_{\mathbb{R}}\beta\vee\alpha=\beta)\vee(\alpha<_{\mathbb{R}}\beta\vee\alpha=\beta)$ ▮ $(\beta<_{\mathbb{R}}\alpha)\vee(\alpha<_{\mathbb{R}}\beta\vee\alpha=\beta)$
(1.4) $\alpha=\beta \implies \neg(\alpha<_{\mathbb{R}}\beta\vee\beta<_{\mathbb{R}}\alpha)$
(1.5) $\alpha<_{\mathbb{R}}\beta \implies \neg(\alpha=\beta\vee\beta<_{\mathbb{R}}\alpha)$
(1.6) $\beta<_{\mathbb{R}}\alpha \implies \neg(\alpha=\beta\vee\alpha<_{\mathbb{R}}\beta)$
(1.7) $\alpha<_{\mathbb{R}}\beta \veebar \alpha=\beta \veebar \alpha<_{\mathbb{R}}\beta$
(2) $(\alpha,\beta\in\mathbb{R}) \implies (\alpha<_{\mathbb{R}}\beta \veebar \alpha=\beta \veebar \alpha<_{\mathbb{R}}\beta)$

(3) $\forall_{\alpha,\beta\in\mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $\mathit{OrderTrichotomy}(\mathbb{R}, <_{\mathbb{R}})$

$\boxed{\mathit{OrderTransitivityOfR}}$ $\mathit{OrderTransitivity}(\mathbb{R}, <_{\mathbb{R}})$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a\in\alpha}(a \in \beta) \wedge \forall_{b\in\beta}(b \in \gamma)$

(1.1.3) $\forall_{a\in\alpha}(a \in \gamma) \blacksquare\ \alpha \subset \gamma \blacksquare\ \alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(3) $\forall_{\alpha,\beta,\gamma\in\mathbb{R}}\big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(4) $\mathit{OrderTransitivity}(\mathbb{R}, <_{\mathbb{R}})$

$\boxed{\mathit{OrderOfR}}$ $\mathit{Order}(<_{\mathbb{R}}, \mathbb{R})$

from: $\mathit{OrderTrichotomyR}$, $\mathit{OrderTransitivityR}$
wts:

$\boxed{\mathit{LUBPropertyOfR}}$ $\mathit{LUBProperty}(\mathbb{R}, <_{\mathbb{R}})$

(1) $\big(\emptyset \neq A \subset \mathbb{R} \wedge \mathit{BoundedAbove}(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} | \exists_{\alpha\in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset \blacksquare\ \exists_{\alpha}(\alpha \in A) \blacksquare\ \alpha_0 := \mathit{choice}(\{\alpha | \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset \blacksquare\ \exists_{a}(a \in \alpha_0) \blacksquare\ a_0 := \mathit{choice}(\{a | a \in \alpha_0\}) \blacksquare\ a_0 \in \gamma \blacksquare\ \gamma \neq \emptyset$

(1.4) $\mathit{BoundedAbove}(A, \mathbb{R}, <_{\mathbb{R}}) \blacksquare\ \exists_{\beta}\big(\mathit{UpperBound}(\beta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.5) $\beta_0 := \mathit{choice}(\{\beta | \mathit{UpperBound}(\beta, A, \mathbb{R}, <_{\mathbb{R}})\})$

(1.6) $\mathit{UpperBound}(\beta_0, A, \mathbb{R}, <_{\mathbb{R}}) \blacksquare\ \forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \beta_0) \blacksquare\ \forall_{\alpha\in A}(\alpha \subseteq \beta_0) \blacksquare\ \forall_{\alpha\in A}\forall_{a\in\alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \blacksquare\ \forall_{a\in\gamma}(a \in \beta_0) \blacksquare\ \gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q} \blacksquare\ \gamma \subseteq \beta_0 \subset \mathbb{Q} \blacksquare\ \gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q} \blacksquare\ \mathit{CutI}(\gamma)$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.10.1) $p \in \gamma \blacksquare\ \exists_{\alpha\in A}(p \in \alpha) \blacksquare\ \alpha_1 := \mathit{choice}(\{\alpha \in A | p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \blacksquare\ q \in \alpha_1 \blacksquare\ q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \blacksquare\ \forall_{p\in\gamma}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \gamma) \blacksquare\ \mathit{CutII}(\gamma)$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha\in A}(p \in \alpha) \blacksquare\ \alpha_2 := \mathit{choice}(\{\alpha \in A | p \in \alpha\})$

(1.12.2) $\alpha_2 \in \mathbb{R} \blacksquare\ \mathit{CutII}(\alpha_2) \blacksquare\ \exists_{r\in\alpha_2}(p < r) \blacksquare\ r_0 := \mathit{choice}(\{r \in \alpha_2 | p < r\})$

(1.12.3) $r_0 \in \alpha_2 \blacksquare\ r_0 \in \gamma$

(1.12.4) $p < r_0 \blacksquare\ p < r_0 \wedge r_0 \in \gamma \blacksquare\ \exists_{r\in\gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r\in\gamma}(p < r) \blacksquare\ \forall_{p\in\gamma}\exists_{r\in\gamma}(p < r) \blacksquare\ \mathit{CutIII}(\gamma)$

(1.14) $\mathit{CutI}(\gamma) \wedge \mathit{CutII}(\gamma) \wedge \mathit{CutIII}(\gamma) \blacksquare\ \gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha\in A}(\alpha \subseteq \gamma) \blacksquare\ \forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma)$

(1.16) $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R} \blacksquare\ \mathit{UpperBound}(\gamma, A, \mathbb{R}, <_{\mathbb{R}})$

(1.17) $\delta <_{\mathbb{R}} \gamma \implies \ldots$

(1.17.1) $\delta \subset \gamma \blacksquare\ \exists_{s}(s \in \gamma \wedge s \notin \delta) \blacksquare\ s_0 := \mathit{choice}(\{s \in \mathbb{Q} | s \in \gamma \wedge s \notin \delta\})$

(1.17.2) $s_0 \in \gamma \blacksquare\ \exists_{\alpha\in A}(s_0 \in \alpha) \blacksquare\ \alpha_3 := \mathit{choice}(\{\alpha \in A | s_0 \in \alpha\})$

(1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta \blacksquare\ \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \ldots$

(1.17.4.1) $\alpha_3 \subseteq \delta \blacksquare\ \forall_{s\in\mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \blacksquare\ \neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4.2) $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \blacksquare\ \bot$

(1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \bot \blacksquare\ \delta <_{\mathbb{R}} \alpha_3 \blacksquare\ \exists_{\alpha\in A}(\delta <_{\mathbb{R}} \alpha) \blacksquare\ \exists_{\alpha\in A}\big(\neg(\alpha \leq_{\mathbb{R}} \delta)\big)$

(1.17.6) $\neg\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \delta) \blacksquare\ \neg\mathit{UpperBound}(\delta, A, \mathbb{R}, <_{\mathbb{R}})$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}}))$ ▌ $\forall_{\delta}\big(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.19) $UpperBound(\gamma, A, \mathbb{R}, <_{\mathbb{R}}) \wedge \forall_{\delta}\big(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.20) $LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})$ ▌ $\exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(2) $\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(3) $\forall_{A}\Big(\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)\Big)$ ▌ $LUBProperty(\mathbb{R}, <_{\mathbb{R}})$

$+_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$
$0_{\mathbb{R}} := \{x \in \mathbb{Q} | x < 0\}$

$\boxed{0InR}$ $0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}}$ ▌ $\emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q}$ ▌ $CutI(0_{\mathbb{R}})$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}}$ ▌ $\forall_{x \in 0_{\mathbb{R}}} \forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}})$ ▌ $CutII(0_{\mathbb{R}})$

(3) $y := x/2$ ▌ $(x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $\forall_{x \in 0_{\mathbb{R}}} \exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $CutIII(0_{\mathbb{R}})$

(4) $CutI(0_{\mathbb{R}}) \wedge CutII(0_{\mathbb{R}}) \wedge CutIII(0_{\mathbb{R}})$ ▌ $0_{\mathbb{R}} \in \mathbb{R}$

$\boxed{FieldAdditionClosureOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_{a}(a \in \alpha)\, ;\, \exists_{b}(b \in \beta)$ ▌ $a_0 := choice(\{a | a \in \alpha\})\, ;\, b_0 := choice(\{b | b \in \beta\})$ ▌ $a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_{x}(x \notin \alpha)\, ;\, \exists_{y}(y_0 \notin \beta)$ ▌ $x_0 := choice(\{x | x \notin \alpha\})\, ;\, y_0 := choice(\{y | y \notin \beta\})$

(1.5) $\forall_{r \in \alpha}(r < x_0)\, ;\, \forall_{s \in \beta}(s < y_0)$ ▌ $\forall_{r \in \alpha} \forall_{s \in \beta}(r + s < x_0 + y_0)$ ▌ $x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q}$ ▌ $CutI(\alpha +_{\mathbb{R}} \beta)$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.7.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▌ $(r_0, s_0) := choice((r, s) \in \alpha \times \beta | p = r + s)$

(1.7.2) $q < p = r_0 + s_0$ ▌ $(q - s_0) < r_0$ ▌ $(q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta$ ▌ $q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta$ ▌ $q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta)$ ▌ $CutII(\alpha +_{\mathbb{R}} \beta)$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▌ $(r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta | p = r + s\})$

(1.9.2) $r_1 \in \alpha$ ▌ $\exists_{t \in \alpha}(r_1 < t)$ ▌ $t_0 := choice(\{t \in \alpha | r_1 < t\})$

(1.9.3) $s_1 \in \beta$ ▌ $t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1$ ▌ $\exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} \beta} \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▌ $CutIII(\alpha +_{\mathbb{R}} \beta)$

(1.11) $CutI(\alpha +_{\mathbb{R}} \beta) \wedge CutII(\alpha +_{\mathbb{R}} \beta) \wedge CutIII(\alpha +_{\mathbb{R}} \beta)$ ▌ $\alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

$\boxed{FieldAdditionCommutativityOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s | r \in \alpha \wedge s \in \beta\} = \{s + r | s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

$\boxed{FieldAdditionAssociativityOfR}$ $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)\big)$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

$\boxed{FieldAdditionIdentityOfR}$ $(\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \ldots$

(1.1.1) $s < 0 \blacksquare r + s < r + 0 = r \blacksquare r + s < r \blacksquare r + s \in \alpha$

(1.2) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha \blacksquare \forall_{r\in\alpha}\forall_{s\in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \forall_{p\in\alpha+_{\mathbb{R}}0_{\mathbb{R}}}(p \in \alpha) \blacksquare \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r\in\alpha}(p < r) \blacksquare r_2 := choice(\{r \in \alpha | p < r\})$

(1.4.2) $p < r_2 \blacksquare p - r_2 < r_2 - r_2 = 0 \blacksquare (p - r_2) < 0 \blacksquare (p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha \blacksquare p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare \forall_{p\in\alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \blacksquare \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \blacksquare 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

$\boxed{FieldAdditionInverseOfR}$ $(\alpha \in \mathbb{R}) \implies \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} | \exists_{r>0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q} \blacksquare \exists_{s\in\mathbb{Q}}(s \notin \alpha) \blacksquare s_0 := choice(\{s | s \notin \alpha\}) \blacksquare p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha \blacksquare -p_0 - 1 \notin \alpha \blacksquare \exists_{r>0}(-p_0 - r \notin \alpha) \blacksquare p_0 \in \beta$

(1.4) $\emptyset \neq \alpha \blacksquare \exists_{q\in\alpha} \blacksquare q_0 := choice(\{q \in \mathbb{Q} | q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha \blacksquare -(-q_0) - r = q_0 - r < q_0 \blacksquare -(-q_0) - r < q_0 \blacksquare -(-q_0) - r \in \alpha$

(1.6) $\forall_{r>0}\big(-(-q_0) - r \in \alpha\big) \blacksquare \neg\exists_{r>0}(-(-q_0) - r \notin \alpha) \blacksquare -q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q} \blacksquare CutI(\beta)$

(1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.8.1) $p \in \beta \blacksquare \exists_{r>0}(-p - r \notin \alpha) \blacksquare r_0 := choice(\{r > 0 | -p - r \notin \alpha\})$

(1.8.2) $q < p \blacksquare -p - r < -q - r$

(1.8.3) $-q - r \notin \alpha \blacksquare q \in \beta$

(1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta \blacksquare \forall_{p\in\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \beta) \blacksquare CutII(\beta)$

(1.10) $p \in \beta \implies \ldots$

(1.10.1) $p \in \beta \blacksquare \exists_{r>0}(-p - r \notin \alpha) \blacksquare r_1 := choice(\{r > 0 | -p - r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0 \blacksquare r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p \blacksquare t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha \blacksquare -t_0 - (r_1/2) \notin \alpha \blacksquare \exists_{r>0}(-t_0 - r \notin \alpha) \blacksquare t_0 \in \beta$

(1.10.7) $t_0 > p \wedge t_0 \in \beta \blacksquare \exists_{t\in\beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t\in\beta}(p < t) \blacksquare \forall_{p\in\beta}\exists_{t\in\beta}(p < t) \blacksquare CutIII(\beta)$

(1.12) $CutI(\beta) \wedge CutII(\beta) \wedge CutIII(\beta) \blacksquare \beta \in \mathbb{R}$

(1.13) $(r \in \alpha \wedge s \in \beta) \implies \ldots$

(1.13.1) $s \in \beta \blacksquare \exists_{t>0}(-s - t \notin \alpha) \blacksquare t_1 := choice(\{t > 0 | -s - t \notin \alpha\}) \blacksquare -s - t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha \blacksquare -s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha \blacksquare r < -s \blacksquare r + s < 0 \blacksquare r + s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}} \blacksquare \forall_{(r,s)\in\alpha\times\beta}(r + s \in 0_{\mathbb{R}}) \blacksquare \alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \ldots$

(1.15.1) $v < 0 \blacksquare w_0 := -v/2 \blacksquare w > 0$

(1.15.2) $\exists_{n\in\mathbb{Z}}(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha) \blacksquare n_0 := choice(\{n \in \mathbb{Z} | nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$ from: $ARCHIMEDEANPROPERTYOFQ + LUB???$

(1.15.3) $p_0 := -(n_0 + 2)w_0 \blacksquare -p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha \blacksquare -p_0 - w_0 \notin \alpha \blacksquare p_0 \in \beta$

(1.15.4) $n_0 w_0 \in \alpha \wedge p_0 \in \beta \blacksquare n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{v \in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta)$ ▌ $0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$ ▌ $\alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$ ▌ $\exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha \in \mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

$*_{\mathbb{R}}(\alpha, \beta) :=$ —

$1_{\mathbb{R}} := \{x \in \mathbb{Q} | x < 1\}$

$\boxed{1IsNot0}$ $0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ —

$\boxed{1InR}$ $1_{\mathbb{R}} \in \mathbb{R}$ —

$\boxed{FieldMultiplicationClosureOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R}\big)$ —

$\boxed{FieldMultiplicationCommutativityOfR}$ $(\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ —

$\boxed{FieldMultiplicationAssociativityOfR}$ $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)\big)$ —

$\boxed{FieldMultiplicationIdentityOfR}$ $(\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ —

$\boxed{FieldMultiplicationInverseOfR}$ $(\alpha \in \mathbb{R}) \implies \exists_{1/\alpha \in \mathbb{R}}\big(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}\big)$ —

$\boxed{FieldDistributativityOfR}$ $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ —

$\boxed{FieldWithR}$ $Field(\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}})$ —

$\boxed{OrderedFieldWithR}$ $OrderedField(\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}})$ —

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} | r < q\} | q \in \mathbb{Q}\}$

$\boxed{QROrderedSubfieldOfR}$ $OrderedSubfield(\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}})$ —

$\boxed{QIsomorphicToQR}$ $\mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ —

$\boxed{ExistenceOfR}$ $\exists_{\mathbb{R}}\big(LUBProperty(\mathbb{R}, <_{\mathbb{R}}) \wedge OrderedSubfield(\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}})\big)$ —

$\boxed{OrderedFieldR}$ $OrderedField(\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}})$ —

(1.20)

$\boxed{ArchimedeanPropertyOfR}$ $\forall_{x,y \in \mathbb{R}}\big(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y)\big)$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx | n \in \mathbb{N}^+\}$ ▌ $\emptyset \neq A \subset \mathbb{R}$

(1.2) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n \in \mathbb{N}^+}(nx > y)$ ▌ $\forall_{n \in \mathbb{N}^+}(nx \leq y)$

(1.2.2) $UpperBound(y_0, A, \mathbb{R}, <)$ ▌ $BoundedAbove(A, \mathbb{R}, <)$

(1.2.3) $LUBProperty(\mathbb{R}, <) \wedge \emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <)$ ▌ $\exists_{\alpha \in \mathbb{R}}(LUB(\alpha, A, \mathbb{R}, <))$

(1.2.4) $\alpha_0 := choice(\{\alpha \in \mathbb{R} | LUB(\alpha, A, \mathbb{R}, <)\})$

(1.2.5) $x > 0$ ▌ $\alpha_0 - x < \alpha_0$

(1.2.6) $LUB(\alpha_0, A, \mathbb{R}, <) \wedge \alpha_0 - x < \alpha_0$ ▌ $\neg UpperBound(\alpha_0 - x, A, \mathbb{R}, <)$

(1.2.7) $\exists_{c \in A}(\alpha_0 - x < c)$ ▌ $c_0 := choice(\{c \in A | \alpha_0 - x < c\})$

(1.2.8) $c_0 \in A$ ▌ $\exists_{m \in \mathbb{N}^+}(mx = c_0)$ ▌ $m_0 := choice(\{m \in \mathbb{N}^+ | mx = c_0\})$

(1.2.9) $\alpha_0 - x < c_0 = m_0 x$ ▌ $\alpha_0 - x < m_0 x$ ▌ $\alpha_0 < (m_0 + 1)x$

(1.2.10) $m_0 + 1 \in \mathbb{N}^+$ ▌ $(m_0 + 1)x \in A$

(1.2.11) $\alpha_0 < (m_0 + 1)x \wedge (m_0 + 1)x \in A$ ▌ $\exists_{c \in A}(\alpha_0 < c)$

(1.2.12) $LUB(\alpha_0, A, \mathbb{R}, <)$ ▌ $UpperBound(\alpha_0, A, \mathbb{R}, <)$ ▌ $\forall_{c \in A}(c \leq \alpha_0)$ ▌ $\neg\exists_{c \in A}(\alpha_0 < c)$

(1.2.13) $\exists_{c \in A}(\alpha_0 < c) \wedge \neg\exists_{c \in A}(\alpha_0 < c)$ ▌ $\perp$

(1.3) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \perp$ ▌ $\exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y)$ ▌ $\forall_{x,y \in \mathbb{R}}\big(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y)\big)$

$\boxed{QDenseInR}$ $\forall_{x,y \in \mathbb{R}}\Big(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y)\Big)$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y \;\blacksquare\; y - x > 0 \;\blacksquare\; \exists_{n\in\mathbb{N}^+}(n(y_0 - x) > 1) \;\blacksquare\; n_0 := choice(\{n \in \mathbb{N}^+ | n(y_0 - x) > 1\}) \;\blacksquare\; n_0(y_0 - x) > 1$

(1.2) $1 > 0 \;\blacksquare\; \exists_{m\in\mathbb{N}^+}(m(1) > n_0 x) \;\blacksquare\; m_1 := choice(\{m \in \mathbb{N}^+ | m(1) > n_0 x\}) \;\blacksquare\; m_1 > n_0 x$

(1.3) $1 > 0 \;\blacksquare\; \exists_{m\in\mathbb{N}^+}(m(1) > -n_0 x) \;\blacksquare\; m_2 := choice(\{m \in \mathbb{N}^+ | m(1) > -n_0 x\}) \;\blacksquare\; m_2 > -n_0 x \;\blacksquare\; -m_2 < n_0 x$

(1.4) $-m_2 < n_0 x < m_1$

(1.5) $m_1, m_2 \in \mathbb{N}^+ \;\blacksquare\; |m_1 - (-m_2)| \geq 2$

(1.6) $-m_2 < n_0 x < m_1 \wedge |m_1 - (-m_2)| \geq 2 \;\blacksquare\; \exists_{m\in\mathbb{Z}}(-m_2 < m < m_1 \wedge m - 1 \leq n_0 x < m)$

(1.7) $m_0 := choice(\{m \in \mathbb{Z} | -m_2 < m < m_1 \wedge m - 1 \leq n_0 x < m\}) \;\blacksquare\; -m_2 < m_0 < m_1 \wedge m_0 - 1 \leq n_0 x < m_0$

(1.8) $(n_0(y_0 - x) > 1) \wedge (m_0 - 1 \leq n_0 x < m_0) \;\blacksquare\; n_0 x < m_0 \leq 1 + n_0 x < n_0 y \;\blacksquare\; n_0 x < m_0 < n_0 y$

(1.9) $n_0 \in \mathbb{N}^+ \;\blacksquare\; n_0 > 0 \;\blacksquare\; x < m_0/n_0 < y$

(1.10) $m_0, n_0 \in \mathbb{Z} \;\blacksquare\; m_0/n_0 \in \mathbb{Q}$

(1.11) $m_0/n_0 \in \mathbb{Q} \wedge x < m_0/n_0 < y \;\blacksquare\; \exists_{p\in\mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p\in\mathbb{Q}}(x < p < y) \;\blacksquare\; \forall_{x,y\in\mathbb{R}}\exists_{p\in\mathbb{Q}}(x < p < y)$

(1.21)

$\boxed{RootExistenceInRLemma}$ $(0 < a < b) \implies b^n - a^n \leq (b - a)nb^{n-1}$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b \;\blacksquare\; b/a > 1$

(1.3) $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq \sum_{i=1}^{n}(b^{n-i}a^{i-1}(b/a)^{i-1}) \leq \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq (b - a)nb^{n-1} \;\blacksquare\; b^n - a^n \leq (b - a)nb^{n-1}$

(2) $(0 < a < b) \implies b^n - a^n \leq (b - a)nb^{n-1}$

$\boxed{RootExistenceInR}$ $\forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}({y_0}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} | t > 0 \wedge t^n < x\}$

(1.2) $t_0 := x/(1 + x)$

(1.3) $1 + x > x > 0 \;\blacksquare\; t_0 = x/(1 + x) > 0 \;\blacksquare\; t_0 > 0$

(1.4) $1 = (1 + x)/(1 + x) > x/(1 + x) = t_0 \;\blacksquare\; 1 > t_0$

(1.5) $0 < t_0 < 1 \;\blacksquare\; {t_0}^n \leq t_0$

(1.6) $x < 0 \;\blacksquare\; 1 + x > 1 \;\blacksquare\; x > x/(1 + x) = t_0 \;\blacksquare\; x > t_0$

(1.7) ${t_0}^n \leq t_0 < x \;\blacksquare\; {t_0}^n < x$

(1.8) $t_0 > 0 \wedge {t_0}^n < x \;\blacksquare\; t_0 \in E \;\blacksquare\; \emptyset \neq E$

(1.9) $t_1 := choice(\{t \in \mathbb{R} | t > 1 + x\}) \;\blacksquare\; t_1 > 1 + x$

(1.10) $t_1 > 1 + x \;\blacksquare\; {t_1}^n \geq t_1 > 1 + x > x \;\blacksquare\; {t_1}^n > x \;\blacksquare\; \neg({t_1}^n < x) \;\blacksquare\; t_1 \notin E \;\blacksquare\; E \subset \mathbb{R}$

(1.11) $\emptyset \neq E \wedge E \subset \mathbb{R} \;\blacksquare\; \emptyset \neq E \subset \mathbb{R}$

(1.12) $t \in E \implies \ldots$

(1.12.1) $t^n < x \wedge {t_1}^n > x \;\blacksquare\; t^n < x < {t_1}^n \;\blacksquare\; t < t_1$

(1.13) $t \in E \implies t < t_1 \;\blacksquare\; \forall_{t\in E}(t \leq t_1) \;\blacksquare\; UpperBound(t_1, E, \mathbb{R}, <) \;\blacksquare\; BoundedAbove(E, \mathbb{R}, <)$

(1.14) $LUBProperty(\mathbb{R}, <) \wedge \emptyset \neq E \subset \mathbb{R} \wedge BoundedAbove(E, \mathbb{R}, <) \;\blacksquare\; \exists_{y\in\mathbb{R}}(LUB(y_0, E, \mathbb{R}, <))$

(1.15) $y_0 := choice(\{y \in \mathbb{R} | LUB(y, E, \mathbb{R}, <)\})$

(1.16) $LUB(y_0, E, \mathbb{R}, <) \wedge t_0 \in E \;\blacksquare\; 0 < t_0 < y_0 \;\blacksquare\; y_0 > 0$

(1.17) ${y_0}^n < x \implies \ldots$

(1.17.1) $k_0 := \frac{x - {y_0}^n}{n(y_0 + 1)^{n-1}} \;\blacksquare\; k_0 \in \mathbb{R}$

(1.17.2) ${y_0}^n < x \;\blacksquare\; 0 < x - {y_0}^n$

(1.17.3) $n > 0 \wedge y_0 > 0 \;\blacksquare\; 0 < n(y_0 + 1)^{n-1}$

(1.17.4) $0 < x - {y_0}^n \wedge 0 < n(y_0 + 1)^{n-1} \;\blacksquare\; 0 < \frac{x - {y_0}^n}{n(y_0 + 1)^{n-1}} = k_0 \;\blacksquare\; 0 < k_0$

(1.17.5) $\forall_{x',y'\in\mathbb{R}}\left(x'<y' \implies \exists_{p'\in\mathbb{Q}}(x'<p'<y')\right)\land 0, min(1,k_0)\in\mathbb{R}\land 0<min(1,k_0)$

(1.17.6) $\exists_{h\in\mathbb{Q}}(0<h<min(1,k_0)) \blacksquare h_0 := choice(\{h\in\mathbb{Q}|0<h<min(1,k_0)\})$

(1.17.7) $0<h_0<1\land h_0<k_0=\frac{x-{y_0}^n}{n(y_0+1)^{n-1}}$

(1.17.8) $0<n(y_0+1)^{n-1}\land h_0<k_0=\frac{x-{y_0}^n}{n(y_0+1)^{n-1}} \blacksquare h_0n(y_0+1)^{n-1}<x-{y_0}^n$

(1.17.9) $n>0\land y_0>0 \blacksquare 0<n(y_0+1)^{n-1} \blacksquare h_0n(y_0+1)^{n-1}<x-{y_0}^n$

(1.17.10) $(y_0+h_0)^n-{y_0}^n<(y_0+h_0-y_0)n(y_0+h_0)^{n-1}=h_0n(y_0+h_0)^{n-1}<h_0n(y_0+1)^{n-1}<x-{y_0}^n$

(1.17.11) $(y_0+h_0)^n-{y_0}^n<x-{y_0}^n \blacksquare (y_0+h_0)^n<x$

(1.17.12) $y_0>0\land h_0>0 \blacksquare (y_0+h_0)>h_0>0$

(1.17.13) $(y_0+h_0)>0\land(y_0+h_0)^n<x \blacksquare (y_0+h_0)^n\in E$

(1.17.14) $(y_0+h_0)^n\in E\land y_0+h_0>y_0 \blacksquare \exists_{e\in E}(e>y_0)$

(1.17.15) $\mathit{LUB}(y_0,E,\mathbb{R},<) \blacksquare \mathit{UpperBound}(y_0,E,\mathbb{R},<) \blacksquare \forall_{e\in E}(e\le y_0) \blacksquare \neg\exists_{e\in E}(e>y_0)$

(1.17.16) $\exists_{e\in E}(e>y_0)\land\neg\exists_{e\in E}(e>y_0) \blacksquare \bot$

(1.18) ${y_0}^n<x \implies \bot \blacksquare {y_0}^n\ge x$

(1.19) ${y_0}^n>x \implies \ldots$

(1.19.1) $k_1 := \frac{{y_0}^n-x}{n{y_0}^{n-1}} \blacksquare k_1\in\mathbb{R}\land k_1n{y_0}^{n-1}={y_0}^n-x$

(1.19.2) $0<x\in\mathbb{R}\land 0<n\in\mathbb{Z} \blacksquare {y_0}^n-x<{y_0}^n\le n{y_0}^n \blacksquare {y_0}^n-x<n{y_0}^n$

(1.19.3) $k_1=\frac{{y_0}^n-x}{n{y_0}^{n-1}}<\frac{n{y_0}^n}{n{y_0}^{n-1}}=y_0 \blacksquare k_1<y_0$

(1.19.4) ${y_0}^n>x \blacksquare {y_0}^n-x>0$

(1.19.5) $n>0\land y_0>0 \blacksquare 0<n{y_0}^{n-1}$

(1.19.6) $0<{y_0}^n-x\land 0<n{y_0}^{n-1} \blacksquare 0<\frac{{y_0}^n-x}{n{y_0}^{n-1}}=k_1 \blacksquare 0<k_1$

(1.19.7) $k_1<y_0\land 0<k_1 \blacksquare 0<k_1<y_0$

(1.19.8) $t\ge y_0-k_1 \implies \ldots$

(1.19.8.1) $t\ge y_0-k_1 \blacksquare t^n\ge(y_0-k_1)^n \blacksquare -t^n\le-(y_0-k_1)^n \blacksquare {y_0}^n-t^n\le{y_0}^n-(y_0-k_1)^n$

(1.19.8.2) ${y_0}^n-(y_0-k_1)^n<(y_0-y_0+k_1)n{y_0}^{n-1}=k_1n{y_0}^{n-1}={y_0}^n-x$

(1.19.8.3) ${y_0}^n-t^n<{y_0}^n-x \blacksquare -t^n<-x \blacksquare t_n>x \blacksquare \neg(t_n<x) \blacksquare t\notin E$

(1.19.9) $t\ge y_0-k_1 \implies t\notin E \blacksquare t\in E \implies t<y_0-k_1$

(1.19.10) $\forall_{t\in E}(t\le y_0-k_1) \blacksquare \mathit{UpperBound}(y_0-k_1,E,\mathbb{R},<)$

(1.19.11) $\mathit{LUB}(y_0,E,\mathbb{R},<) \blacksquare \forall_{z\in\mathbb{R}}(z<y_0 \implies \neg\mathit{UpperBound}(z,E,\mathbb{R},<))$

(1.19.12) $k_1>0 \blacksquare y-k_1<y_0 \blacksquare \neg\mathit{UpperBound}(y_0-k_1,E,\mathbb{R},<)$

(1.19.13) $\mathit{UpperBound}(y_0-k_1,E,\mathbb{R},<)\land\neg\mathit{UpperBound}(y_0-k_1,E,\mathbb{R},<) \blacksquare \bot$

(1.20) ${y_0}^n>x \implies \bot \blacksquare {y_0}^n\le x$

(1.21) $({y_0}^n<x\veebar{y_0}^n=x\veebar x<{y_0}^n)\land({y_0}^n\ge x)\land({y_0}^n\le x) \blacksquare {y_0}^n=x$

(1.22) ${y_0}^n=x\land y_0\in\mathbb{R} \blacksquare \exists_{y\in\mathbb{R}}(y^n=x)$

(1.23) $y_1,y_2 := choice(\{y\in\mathbb{R}|y^n=x\}) \blacksquare y_1\ne y_2 \implies \ldots$

(1.23.1) $(y_1<y_2)\veebar(y_2<y_1) \blacksquare (x={y_1}^n<{y_2}^n=x)\veebar(x={y_2}^n<{y_1}^n=x) \blacksquare (x<x)\veebar(x>x) \blacksquare \bot\veebar\bot \blacksquare \bot$

(1.24) $y_1\ne y_2 \implies \bot \blacksquare y_1=y_2 \blacksquare \forall_{a,b\in\mathbb{R}}((a^n=x\land b^n=x) \implies a=b)$

(1.25) $\exists_{y\in\mathbb{R}}(y^n=x)\land\forall_{a,b\in\mathbb{R}}((a^n=x\land b^n=x) \implies a=b) \blacksquare \exists!_{y\in\mathbb{R}}(y^n=x)$

(2) $(0<x\in\mathbb{R}\land 0<n\in\mathbb{Z}) \implies \exists!_{y\in\mathbb{R}}(y^n=x) \blacksquare \forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}({y_0}^n=x)$

$\boxed{\mathit{RootExistenceInRCorollary}}$ $\forall_{0<a\in\mathbb{R}}\forall_{0<b\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}((ab)^{1/n}=a^{1/n}b^{1/n})$ —

$$\mathit{ExtendedRealSystem}(\bar{\mathbb{R}},+,*,<) := \left(\begin{array}{l} \bar{\mathbb{R}}=\mathbb{R}\cup\{-\infty,+\infty\} \quad\land\quad -\infty<x<\infty \quad\land \\ x+\infty=+\infty \;\land\; x-\infty=-\infty \;\land\; \frac{x}{+\infty}=\frac{x}{-\infty}=0\;\land \\ (x>0) \implies (x*(+\infty)=+\infty\land x*(-\infty)=-\infty)\land \\ (x<0) \implies (x*(+\infty)=-\infty\land x*(-\infty)=+\infty) \end{array}\right)$$

$\mathbb{C} := \{\langle a,b\rangle\in\mathbb{R}\times\mathbb{R}\}$

$+_{\mathbb{C}}(\langle a,b\rangle,\langle c,d\rangle) := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d\rangle$
$*_{\mathbb{C}}(\langle a,b\rangle,\langle c,d\rangle) := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c\rangle$

$\boxed{FieldC}$ $Field(\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}})$ —
$\boxed{RSubfieldC}$ $Subfield(\mathbb{R}, \mathbb{C}, +, *)$ —

$i := \langle 0, 1\rangle \in \mathbb{C}$

$\boxed{iProperty}$ $i^2 = -1$ —
$\boxed{CProperty}$ $(a, b \in \mathbb{R}) \implies (\langle a,b\rangle = a + bi)$ —

$Conjugate(\overline{a+bi}) := a - bi$

$\boxed{ConjugateProperties}$ $(w, z \in \mathbb{C}) \implies \dots$ —

(1)	$\overline{z+w} = \overline{z} + \overline{w}$
(2)	$\overline{z*w} = \overline{z} * \overline{w}$
(3)	$Re(z) = (1/2)(z + \overline{z}) \wedge Im(z) = (1/2)(z - \overline{z})$
(4)	$0 \leq z * \overline{z} \in \mathbb{R}$

$AbsoluteValueC(|z|) = (z * \overline{z})^{1/2}$
$\boxed{AbsoluteValueProperties}$ $(z, w \in \mathbb{C}) \implies \dots$ —

(1)	123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y}((x, y) \in f)$

$(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \ldots$

(1) $Domain(f) := X; Codomain(f) := Y$

(2) $Image(f, A) := \{f(a) | a \in A\}; Preimage(f, B) := \{a | f(a) \in B\}$

(3) $Range(f) := Image(Domain(f))$

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X}(x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$
$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X}(y_0 = f(x))$
$Bijective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$
$\boxed{SurjectiveEquivalent}$ $(Range(f) = Codomain(f)) \implies Surjective(f)$

$(Function(f, X, Y) \wedge Function(g, Y, Z)) \implies (f \circ g)(x) := f(g(x)); Function(f \circ g, X, Z)$

$\boxed{PropertiesofFunctions}$ $(Function(f, A, B) \wedge Function(g, B, C) \wedge Function(h, C, D)) \implies \ldots$

(1) $h \circ (g \circ f) = (h \circ g) \circ f$

(2) $(Injective(f) \wedge Injective(g)) \implies Injective(g \circ f)$

(3) $(Surjective(f) \wedge Surjective(g)) \implies Surjective(g \circ f)$

(4) $(Bijective(f, A, B)) \implies \exists_{f^{-1}}(Function(f^-1, B, A) \wedge \forall_{a \in A}(f^{-1}(f(a)) = a) \wedge \forall_{b \in B}(f(f^{-1}(b)) = b))$

$|(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}}(b = ac)$

$\boxed{DivisibilityTheorems}$ $(a, b, c, m, x, y \in \mathbb{Z}) \implies \ldots$

(1) $(a|b) \implies a|bc$

(2) $(a|b \wedge b|c) \implies a|c$

(3) $(a|b \wedge b|c) \implies a|(bx + cy)$

(4) $(a|b \wedge b|a) \implies a = \pm b$

(5) $(a|b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$

(6) $(a|b) \iff (m \neq 0 \wedge ma|mb)$

$\boxed{DivisionAlgorithm}$ $(a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}}(b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \wedge a|b \wedge a|c$
$GCD(a, b, c) := CD(a, b, c) \wedge \forall_d((d|b \wedge d|c) \implies d|a)$
$\boxed{GCDEquivalent}$ 123123

Chapter 3

Linear Algebra

$EquivalentSystem()\ldots$
$(AB)^T = B^T A^T$
$Sym(A) := A^T = A$
$Skew(A) := A^T = -A$

$(B = A + A^T) \implies Sym(B)$
$(B = A - A^T) \implies Skew(B)$
$A = (1/2)(A + A^T) + (1/2)(A - A^T) = Sym(B_1) + Skew(B_2)$

$Invertible(A) := \exists_{A^{-1}}(AA^{-1} = I = A^{-1}A)$

$\big(Invertible(A) \wedge Invertible(B)\big) \implies \Big(Invertible(AB) \wedge (AB)^{-1} = B^{-1}A^{-1}\Big)$
$\big(Invertible(A)\big) \implies \Big(Invertible(A^{-1}) \wedge (A^{-1})^{-1} = A\Big)$
$\big(Invertible(A)\big) \implies \Big(Invertible(A^T) \wedge (A^T)^{-1} = (A^{-1})^T\Big)$

$RREF(A) := (Definition 1.18)$
$ElementaryRowOperation(\phi) := (Definition 1.19)$
$RowEquivalent(A, B) := \exists_{\Phi}(\forall_{\phi\in\Phi}(ElementaryRowOperation(\phi)) \wedge |\Phi| \in \mathbb{N} \wedge \Phi(A) = B)$

By Gauss-Jordan Elimination: $NonZero(A) \implies \exists_B(RREF(B) \wedge RowEquivalent(A, B))$

$\big(AX = B \wedge CX = D \wedge RowEquivalent([A|B], [C|D])\big) \implies ([AX = B] \equiv [CX = D])$
$\big(RowEquivalent(A, B)\big) \implies ([AX = \mathbb{O}] \equiv [BX = \mathbb{O}])$