Axiom 1

Let $r \in \mathbb{R}$, and let $S \subseteq \mathbb{R}$ such that $S \neq \emptyset$.

1. *[The Trichotomy Law.] Exactly one of the statements $r > 0$, $r < 0$, $r = 0$ is true. [i.e., Any real number is either positive, negative or zero, where the 'or' is an exclusive disjunction.]*
2. *[The Completeness Axiom.] If there exists $u \in \mathbb{R}$ such that u is an upper bound of S, then there exists $s \in \mathbb{R}$ such that $s = \sup S$. [i.e., Any nonempty subset of $\mathbb{R}$ that has an upper bound has a least upper bound.]*
3. *[The Well-ordering Principle.] If $S \subseteq \mathbb{N}$, then there exists $m \in S$ such that $m = \min S$. [i.e., Any nonempty subset of $\mathbb{N}$ has a least element.]*

Corollary 2

Let $\varepsilon > 0$. Then for any $r \in \mathbb{R}$, the following statements hold.

1. *[Archimedean property of $\mathbb{R}$.] There exists $n \in \mathbb{N}$ such that $r < n\varepsilon$.*
2. *[Existence of the floor function.] There exists a unique $\lfloor r \rfloor \in \mathbb{Z}$ such that*
$$\lfloor r \rfloor \leq r < \lfloor r \rfloor + 1. \tag{1}$$
3. *[Denseness of $\mathbb{Q}$ in $\mathbb{R}$.]There exists $q \in \mathbb{Q}$ such that $-\varepsilon < r - q < \varepsilon$.*

Fundamental Properties of Absolute Value

Proposition 3

Let $a, b \in \mathbb{R}$.

1. $|ab| = |a| \cdot |b|$ *[i.e., the absolute value function is multiplicative].*
2. $\pm a \leq |a|$.
3. $\forall \varepsilon > 0\ [|x| < \varepsilon \iff -\varepsilon < x < \varepsilon]$.
4. $\forall \varepsilon > 0\ [|x| \leq \varepsilon \iff -\varepsilon \leq x \leq \varepsilon]$.
5. *If* $\pm a \leq b$ *and* $b \geq 0$, *then* $|a| \leq b$.
6. *If* $|a - b| < \varepsilon$ *for any* $\varepsilon > 0$, *then* $a = b$.
7. *If* $a \leq b + \varepsilon$ *for any* $\varepsilon > 0$, *then* $a \leq b$.

Further properties of inequalities and absolute value

Corollary 4

Let $a, b, c \in \mathbb{R}$.

1. $|a|^2 = |a^2| = a^2$.
2. $|a| = \sqrt{a^2}$.
3. $|a| \leq \sqrt{a^2 + b^2}$.
4. $|a - b| = 0 \iff a = b$.
5. $|a - b| = |b - a|$.
6. $|a + b| \leq |a| + |b|$.
7. $|a - b| \leq |a - c| + |c - b|$.
8. $||a| - |b|| \leq |a - b|$.

Properties of Lattice Operations

Proposition 6

For any $a, b \in \mathbb{R}$,

$$\begin{aligned} a \vee b &= \frac{1}{2}(a + b + |a - b|), \\ a \wedge b &= \frac{1}{2}(a + b - |a - b|). \end{aligned}$$

Proposition 7

For any $a, b \in \mathbb{R}$,

$$\begin{aligned} a \vee b &= b \vee a, \\ a \wedge b &= b \wedge a, \\ a \wedge b \leq r \leq a \vee b &\implies |r - a| \leq |a - b|, \\ & \qquad |r - b| \leq |a - b|. \end{aligned}$$

Problem Set 1: Prove Propositions 6 and 7.

Definition 2

A sequence $(a_n)_{n\in\mathbb{N}}$ *converges to* $a \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have $|a_n - a| < \varepsilon$. We write this in symbols as $a = \lim_{n\to\infty} a_n$. If there exists $a \in \mathbb{R}$ such that $a = \lim_{n\to\infty} a_n$, the sequence $(a_n)_{n\in\mathbb{N}}$ is said to be *convergent*.

The statement $a = \lim_{n\to\infty} a_n$ can be written symbolically as

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \in [N, \infty[\cap \mathbb{N} \qquad [|a_n - a| < \varepsilon], \tag{1}$$

where $[N, \infty[:= \{x \in \mathbb{R} \ : \ x \geq N\}$. The set in the third quantifier in (1) seems to be an unnecessary complication. Since it can be seen anyway from the rest of the statement that n appears as an index of a sequence term, n must be an element of $\mathbb{N}$. Henceforth, we adopt the less precise but simpler notation

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \ [|a_n - a| < \varepsilon],$$

as the symbolic equivalent of $a = \lim_{n\to\infty} a_n$.

Uniqueness of Sequence Limits

Theorem 3

If $(a_n)_{n\in\mathbb{N}}$ *is convergent, then* $\lim\limits_{n\to\infty} a_n$ *is unique.*

The algebra of convergent sequences

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$, and let $c \in \mathbb{R}$. We define new sequences

$$(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} := (a_n + b_n)_{n\in\mathbb{N}}\,, \tag{6}$$

$$c\,(a_n)_{n\in\mathbb{N}} := (ca_n)_{n\in\mathbb{N}}\,. \tag{7}$$

That is, $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}$ is the sequence with $a_n + b_n$ as the nth term, while $c\,(a_n)_{n\in\mathbb{N}}$ is the sequence the nth term of which is ca_n.

Theorem 6

For any $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$, *the sequence*

$$(a_n)_{n\in\mathbb{N}} (b_n)_{n\in\mathbb{N}} := (a_n b_n)_{n\in\mathbb{N}},$$

is convergent [i.e., also in $\mathfrak{c}(\mathbb{R})$*], and furthermore,*

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n. \tag{16}$$

The presence of $(1)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$ such that (29) holds makes the associative algebra $\mathfrak{c}(\mathbb{R})$ *unital*, with $(1)_{n\in\mathbb{N}}$ as the *[multiplicative] identity*, while property (30) means that the associative algebra $\mathfrak{c}(\mathbb{R})$ is *commutative*. An element $(b_n)_{n\in\mathbb{N}}$ of $\mathfrak{c}(\mathbb{R})$ is a *unit* if there exists $(c_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$ such that $(b_n)_{n\in\mathbb{N}}(c_n)_{n\in\mathbb{N}} = (1)_{n\in\mathbb{N}} = (c_n)_{n\in\mathbb{N}}(b_n)_{n\in\mathbb{N}}$. The following gives us an insight about the units in $\mathfrak{c}(\mathbb{R})$.

Theorem 8

Let $(b_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$. *If* $\lim\limits_{n\to\infty} b_n \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\left(\frac{1}{b_n}\right)_{n\in\mathbb{N}}$ *is convergent, and*

$$\lim_{n\to\infty}\left(\frac{1}{b_n}\right) = \frac{1}{\lim\limits_{n\to\infty} b_n}. \tag{31}$$

Corollary 9

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$*, and let* $c \in \mathbb{R}$*. Assume that* $\lim_{n\to\infty} c_n \neq 0$*, and that* $c_n \neq 0$ *for any* $n \in \mathbb{N}$*. Then the sequences*

$$(c)_{n\in\mathbb{N}}, \quad (|a_n|)_{n\in\mathbb{N}}, \quad (a_n - b_n)_{n\in\mathbb{N}}, \quad \left(\frac{a_n}{c_n}\right)_{n\in\mathbb{N}}, \\ (a_n \vee b_n)_{n\in\mathbb{N}}, \quad (a_n \wedge b_n)_{n\in\mathbb{N}}, \tag{39}$$

are convergent, and furthermore,

1. $\lim_{n\to\infty} c = c$,
2. $\lim_{n\to\infty} |a_n| = \left|\lim_{n\to\infty} a_n\right|$,
3. $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$, *and*
4. $\lim_{n\to\infty} \frac{a_n}{c_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} c_n}$.
5. $\lim_{n\to\infty} (a_n \vee b_n) = a \vee b$.
6. $\lim_{n\to\infty} (a_n \wedge b_n) = a \wedge b$.

Some notes on subsequences

Given a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, and a function $\mathbb{N} \to \mathbb{N}$ denoted by $i \mapsto N_i$ such that

$$i < j \implies N_i < N_j, \tag{1}$$

we call $(a_{N_i})_{i\in\mathbb{N}}$ a *subsequence* of $(a_n)_{n\in\mathbb{N}}$.
Given a subsequence $(a_{N_i})_{i\in\mathbb{N}}$ of a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, by the Trichotomy Law, the condition $i \neq j$ means that either $i < j$ or $i > j$. Then by (1), we have either $N_i < N_j$ or $N_i > N_j$, which implies $N_i \neq N_j$. We have thus shown that $i \neq j$ implies $N_i \neq N_j$, and by contraposition,

$$N_i = N_j \implies i = j. \tag{2}$$

Therefore, $i \mapsto N_i$ is injective. The converse

$$i = j \implies N_i = N_j, \tag{3}$$

of (2) is true because $i \mapsto N_i$ is a function. Also, if $N_i < N_j$, then $N_i \neq N_j$, and by the contrapositive of (3), we have $i \neq j$.

Some notes on subsequences

If $i > j$, then we get, from (1), the contradiction $N_i > N_j$, and so the only possiblity is $i < j$. That is,

$$N_i < N_j \implies i < j. \tag{4}$$

From (1)–(4), we obtain

$$i \leq j \iff N_i \leq N_j. \tag{5}$$

Using an elementary proof, the equivalence (5) can be used to prove that the conditions

$$\forall \varepsilon > 0 \quad \exists N_I \in \mathbb{N} \quad \forall N_i \geq N_I \quad |a_{N_i} - a| < \varepsilon, \tag{6}$$

$$\forall \varepsilon > 0 \quad \exists I \in \mathbb{N} \quad \forall i \geq I \quad |a_{N_i} - a| < \varepsilon, \tag{7}$$

are equivalent. Hence, if the subsequence $(a_{N_i})_{i\in\mathbb{N}}$ converges to some $a \in \mathbb{R}$, both notations $\lim\limits_{N_i\to\infty} a_{N_i}$ and $\lim\limits_{i\to\infty} a_{N_i}$ are valid, and furthermore,

$$\lim_{N_i\to\infty} a_{N_i} = a \iff \lim_{i\to\infty} a_{N_i} = a.$$

Some notes on subsequences

i.e., The limiting process for the convergent subsequence $(a_{N_i})_{i\in\mathbb{N}}$ is the same regardless of whether we view this limiting process in terms of the original indices, as in $N_i \to \infty$, or in terms of the 'secondary' indices, as in $i \to \infty$.

Another important property of a subsequence $(a_{N_i})_{i\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ is that

$$\forall i \in \mathbb{N} \ \ [i \le N_i]\,. \tag{8}$$

If $i = 1$, then by the fact that $N_i \in \mathbb{N}$, we have $N_i \ge 1 = i$.
Suppose $i \le N_i$ for some $i \in \mathbb{N}$. Tending towards a contradiction, suppose $i + 1 > N_{i+1}$. Since both $i + 1$ and N_i are integers, we further have $i \ge N_{i+1}$. By the inductive hypothesis, $N_i \ge i \ge N_{i+1}$. But this contradicts $N_i < N_{i+1}$ because of (1) and $i < i + 1$. Therefore, $i + 1 \le N_{i+1}$, and we have proven (8) by induction.

Proposition 1

If $(a_n)_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$, then any convergent subsequence of $(a_n)_{n\in\mathbb{N}}$ also converges to a.

The limit superior of a sequence

Let us return our attention to an arbitrary sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Given $n \in \mathbb{N}$, let us collect the terms of the sequence "at index n and beyond" in the following set:

$$\{a_k \ : \ k \geq n\} = \{a_n, a_{n+1}, a_{n+2}, \ldots\}. \tag{11}$$

If the set (11) has an upper bound $M \in \mathbb{R}$, then its supremum

$$\sup_{k\geq n} a_k := \sup\{a_k \ : \ k \geq n\} = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}, \tag{12}$$

exists as an element of $\mathbb{R}$. Otherwise, we define $\sup\limits_{k\geq n} a_k$ as ∞. Note that the number $\sup\limits_{k\geq n} a_k$ depends on n, and so we now have a new sequence

$$\sup_{k\geq 1} a_k, \quad \sup_{k\geq 2} a_k, \quad \sup_{k\geq 3} a_k, \quad \ldots, \quad \sup_{k\geq n} a_k, \quad \ldots \tag{13}$$

of extended real numbers, where in the subscripts after the "$k \geq$" we find the indices of the terms of the sequence (13).

The limit superior of a sequence

Observe that the supremum (12) of (11) need not be one of the terms in (11), and so it is important to note here that (13) is not necessarily a subsequence of $(a_n)_{n\in\mathbb{N}}$. If the set of all terms in the sequence (13) has a lower bound $M' \in \mathbb{R}$, then the infimum

$$\limsup_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k := \inf\left\{\sup_{k\geq n} a_k : n \in \mathbb{N}\right\},$$
$$= \inf\left\{\sup_{k\geq 1} a_k, \quad \sup_{k\geq 2} a_k, \quad \ldots\right\},$$

of the set of all terms of (13) exists as an element of $\mathbb{R}$. Otherwise, we define $\limsup_{n\to\infty} a_n$ as $-\infty$. We call the number $\limsup_{n\to\infty} a_n$ the *limit superior* or *upper limit* of the sequence $(a_n)_{n\in\mathbb{N}}$.

Lemma 2

Let $M \in \mathbb{R}$*, and consider a sequence* $(a_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$*. If* $a_n \leq M$ *for any* $n \in \mathbb{N}$*, then* $\limsup_{n\to\infty} a_n \leq M$*.*

Lemma 3

If $-\infty < \limsup_{n\to\infty} a_n < \infty$, *then there exists a subsequence* $(a_{N_i})_{i\in\mathbb{N}}$ *of* $(a_n)_{n\in\mathbb{N}}$ *such that for any* $i \in \mathbb{N}$,

$$\left| a_{N_i} - \limsup_{n\to\infty} a_n \right| < \frac{1}{i}. \tag{14}$$

Bounded sequences and the Bolzano-Weierstrass Theorem

Given a real number $M > 0$, we say that a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is ***bounded by*** M if $|x_n| \leq M$ for all $n \in \mathbb{N}$. Any sequence bounded by some positive real number is a ***bounded* sequence**.

Lemma 4

If $c \in \mathbb{R}$ and if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded sequences in $\mathbb{R}$, then the sequences

$$(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}\,, \qquad c\,(a_n)_{n\in\mathbb{N}}\,, \qquad (a_n)_{n\in\mathbb{N}}\,(b_n)_{n\in\mathbb{N}}\,,$$

are also bounded.

Proof of Lemma 4

Suppose $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded by M and N, respectively. By routine computations using the properties of inequalities in $\mathbb{R}$, we find that the sequences $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}$, $c\,(a_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}\,(b_n)_{n\in\mathbb{N}}$ are bounded by $M+N$, $|c|\cdot M$ and MN, respectively. $\square$

Lemma 6

If $(a_n)_{n\in\mathbb{N}}$ *is bounded, then* $-\infty < \limsup_{n\to\infty} a_n < \infty$.

We summarize in the following the logical relationship between the notions of boundedness and convergence of a sequence in $\mathbb{R}$.

Theorem 7

1. *A convergent sequence in $\mathbb{R}$ is bounded.*
2. *A bounded sequence in $\mathbb{R}$ is not necessarily convergent.*
3. *[The Bolzano-Weierstrass Theorem.] A bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Cauchy Sequences

Definition 1

A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ *is Cauchy* or is a *Cauchy sequence* in $\mathbb{R}$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $m, n \geq N$, we have $|a_m - a_n| < \varepsilon$.

Theorem 2

Every convergent sequence in $\mathbb{R}$ is Cauchy.

Proof.

We encounter in here another 'epsilon-over-two' technique. Suppose $(a_n)_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$, and let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - a| < \frac{\varepsilon}{2}. \tag{1}$$

In particular, for any two indices $m, n \geq N$ that satisfy the hypothesis of (1), we have $|a_m - a| < \frac{\varepsilon}{2}$ and $|a_n - a| = |a - a_n| < \frac{\varepsilon}{2}$. By the triangle inequality,

$$|a_m - a_n| = |(a_m - a) + (a - a_n)| \leq |a_m - a| + |a - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $(a_n)_{n\in\mathbb{N}}$ is Cauchy. $\square$

Cauchy Sequences

Proving the converse of Theorem 2 shall take us into some longer argumentation. First, we need the sort of dual of the notion of limit superior. The *limit inferior* or *lower limit* of a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is defined as $\liminf_{n\to\infty} a_n := \sup_{n\in\mathbb{N}} \inf_{k\geq n} a_k$, which is analogously defined as how we defined $\inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k$ in the previous lecture.

Lemma 3

For any sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$,

(i) $\liminf\limits_{n\to\infty} a_n = -\limsup\limits_{n\to\infty}(-a_n)$;

(ii) $\liminf\limits_{n\to\infty} a_n \leq \limsup\limits_{n\to\infty} a_n$;

(iii) *if M is a real number such that $M \leq a_n$ for any $n \in \mathbb{N}$, then $M \leq \liminf\limits_{n\to\infty} a_n$;*

(iv) *the condition $\liminf\limits_{n\to\infty} a_n \geq \limsup\limits_{n\to\infty} a_n$ holds if and only if $(a_n)_{n\in\mathbb{N}}$ is convergent;*

(v) *if $(a_n)_{n\in\mathbb{N}}$ is indeed convergent, then $(a_n)_{n\in\mathbb{N}}$ converges to the common value of $\liminf\limits_{n\to\infty} a_n$ and $\limsup\limits_{n\to\infty} a_n$.*

Theorem 4 (Cauchy convegence criterion)

Every Cauchy sequence in $\mathbb{R}$ is convergent.

Lemma 3.1. *If* $a, b \in \mathbb{R}$ *such that* $]a, b[\neq \emptyset$, *then there exist a sequence in* $]a, b[$ *that converges to* a *and a sequence that converges to* b.

Notation 3.2. Throughout Section 3, we fix $a, b \in \mathbb{R}$, and we use the symbol X to mean any of the intervals $]a, b[$, $[a, b]$, $]a, b]$, $[a, b[$. We call a, b the *endpoints* of X. We also assume henceforth that $|X| > 1$, i.e., $a < b$.

Lemma 3.3. *If c is an endpoint or an element of X, then there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in $X\setminus\{c\}$ such that $c = \lim_{n\to\infty} c_n$.*

Definition 3.4. Let c be an endpoint or an element of X, let $L \in \mathbb{R}$, and consider a function $f : X \backslash \{c\} \to \mathbb{R}$. We say that L is the *limit of* $f(x)$ *as* x *approaches* c, or that $f(x)$ *approaches* L *as* x *approaches* c, and denote this in symbols by $\lim\limits_{x \to c} f(x) = L$, if for any sequence $(c_n)_{n \in \mathbb{N}}$ in $X \backslash \{c\}$ such that $c = \lim\limits_{n \to \infty} c_n$, we have $\lim\limits_{n \to \infty} f(c_n) = L$. If there exists $L \in \mathbb{R}$ such that $\lim\limits_{x \to c} f(x) = L$, then we say that *the limit* $\lim\limits_{x \to c} f(x)$ *exists*.

Lemma 3.5. *If* $\lim_{x \to c} f(x)$ *exists, then* $\lim_{x \to c} f(x)$ *is unique.*

Theorem 3.6 (The Squeeze Theorem). *Let $c \in X$, and consider functions $f, g, h : X \setminus \{c\} \to \mathbb{R}$ such that*

(i) $g(x) \leq f(x) \leq h(x)$ *for any $x \in X \setminus \{c\}$; and*

(ii) $L := \lim_{x \to c} g(x) = \lim_{x \to c} h(x)$.

Then $\lim_{x \to c} f(x)$ exists and is equal to L.

770 **Lemma 3.7.** *Let $k \in \mathbb{R}$, $c \in X$, and consider functions $f_1, f_2, f_3 : X\backslash\{c\} \to \mathbb{R}$. Suppose further that $f_3(x) \neq 0$ for any $x \in X\backslash\{c\}$, and $\lim_{x\to c} f_3(x) \neq 0$. We have*

$$\lim_{x\to c} x = c, \tag{3.11}$$

$$\lim_{x\to c} k = k, \tag{3.12}$$

$$\lim_{x\to c} [f_1(x) + f_2(x)] = \lim_{x\to c} f_1(x) + \lim_{x\to c} f_2(x), \tag{3.13}$$

$$\lim_{x\to c} k \cdot f_1(x) = k \lim_{x\to c} f_1(x), \tag{3.14}$$

$$\lim_{x\to c} [f_1(x) f_2(x)] = \lim_{x\to c} f_1(x) \cdot \lim_{x\to c} f_2(x), \tag{3.15}$$

$$\lim_{x\to c} \frac{1}{f_3(x)} = \frac{1}{\lim_{x\to c} f_3(x)}, \tag{3.16}$$

$$\lim_{x\to c} |f_1(x)| = \left|\lim_{x\to c} f_1(x)\right|, \tag{3.17}$$

$$\lim_{x\to c} (f_1(x) \vee f_2(x)) = \left(\lim_{x\to c} f_1(x)\right) \vee \left(\lim_{x\to c} f_2(x)\right), \tag{3.18}$$

$$\lim_{x\to c} (f_1(x) \wedge f_2(x)) = \left(\lim_{x\to c} f_1(x)\right) \wedge \left(\lim_{x\to c} f_2(x)\right). \tag{3.19}$$

Definition 3.8. A function $f : X \to \mathbb{R}$ is *continuous at* $c \in X$ if for any sequence $(c_n)_{n\in\mathbb{N}}$ in X that converges to c, we have $\lim_{n\to\infty} f(c_n) = f(c)$. Continuity of a function on some element of its domain is often called the *local definition of continuity* or *continuity at a point*. If f is continuous on every $c \in X$, we say that f is *continuous on* X.

Definition 3.9. Let $C(X)$ be the set of all functions continuous on the interval X.

Theorem 3.10. *Let $k \in \mathbb{R}$, let $f_1, f_2, f_3 \in C(X)$, and let $p \in \mathbb{R}[x]$. Suppose further that $f_3(x) \neq 0$ for all $x \in X$, and that for any convergent sequence $(c_n)_{n\in\mathbb{N}}$ in X, we have $\lim\limits_{n\to\infty} f_3(c_n) \neq 0$. Then*

$$\mathrm{id},\ k,\ f_1 + f_2,\ kf_1,\ f_1 f_2,\ \frac{1}{f_3},\ |f_1|,\ f_1 \vee f_2,\ f_1 \wedge f_2,\ p|_X \in C(X).$$

(i) [The Intermediate Value Theorem.] *If $f(a) \neq f(b)$ and*

$$k \in [f(a) \wedge f(b), f(b) \vee f(a)], \tag{3.36}$$

then there exists $c \in X$ such that $f(c) = k$. [i.e., The equation $f(x)=k$ always has a solution in X for any k that satisfies (3.36).]

(ii) [The Extreme Value Theorem.] *There exist $c_1, c_2 \in X$ such that for any $x \in X$, we have $f(x) \in [f(c_1), f(c_2)]$. [i.e., The function f attains its maximum and minimum values—extreme values—in X.]*

905 **Corollary 3.12.** *(i) If* $f \in C(X)$, $c_1, c_2 \in X$ *and*

$$k \in [f(c_1) \wedge f(c_2), f(c_1) \vee f(c_2)],$$

then there exists $c \in X$ *such that* $f(c) = k$.

(ii) If $X = [a, b]$, *then for any* $f \in C(X)$, *the set* $f(X)$ *is a closed interval.*

Theorem 3.13. *A function $f : X \to \mathbb{R}$ is continuous at $c \in X$ if and only if*

$$\lim_{x \to c} f|_{X \setminus \{c\}}(x) = f(c). \tag{3.52}$$

(i) for any sequence $(c_n)_{n\in\mathbb{N}}$ in X that converges to c, we have $\lim_{n\to\infty} f(c_n) = f(c)$,

(ii) for any sequence $(c_n)_{n\in\mathbb{N}}$ in $X\backslash\{c\}$ that converges to c, we have $\lim_{n\to\infty} f(c_n) = f(c)$,

are equivalent. The only difference in the phrasing of the said statements is whether the sequence $(c_n)_{n\in\mathbb{N}}$ is in X or in $X\backslash\{c\}$. If the condition $\lim_{n\to\infty} f(c_n) = f(c)$ holds for has an upper bound in $\mathbb{R}$, and so

$$\|f\|_\infty := \sup\{|f(x)| \,:\, x \in X\},$$

exists as a real number. By the uniqueness of the supremum, this gives us a function $C(X) \to \mathbb{R}$ given by $f \mapsto \|f\|_\infty$, which we call the *uniform norm*, the *sup norm*, the *supremum norm*, the *Chebyshev norm* or the *infinity norm on* $C(X)$.

Proposition 3.15. *For any $f, g \in C(X)$ and any $k \in \mathbb{R}$,*

(i) $\|f\|_\infty = 0 \quad \iff \quad f = 0.$

(ii) $\|kf\|_\infty \le |k| \cdot \|f\|_\infty.$

(iii) $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty.$

Corollary 3.16. *For any* $f, g, h \in C(X)$,

(i) $\|f - g\|_\infty = 0 \iff f = g.$

(ii) $\|f - g\|_\infty = \|g - f\|_\infty.$

(iii) $\|f - g\|_\infty \leq \|f - h\|_\infty + \|h - g\|_\infty.$

1050 The properties of the sup norm on $C(X)$ outlined in Corollary 3.16 are analogous to the properties of the distance function in $\mathbb{R}$ from Corollary 1.5*(iv)*,*(v)*,*(vii)*. Thus, we have, for $C(X)$, a function $C(X) \times C(X) \to \mathbb{R}$ given by $(f, g) \mapsto \|f - g\|_\infty$, which satisfies an identity of indiscernibles, a property of symmetry, and a triangle inequality, if X is a closed interval.

Theorem 3.17. *Let $L \in \mathbb{R}$, let c be an element or an endpoint of X, and consider a function $f : X \setminus \{c\} \to \mathbb{R}$. The condition $\lim_{x \to c} f(x) = L$ holds if and only if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X \ [0 < |x - c| < \delta \Longrightarrow |f(x) - L| < \varepsilon]. \tag{3.69}$$

Theorem 3.18. *A function $f : X \to \mathbb{R}$ is continuous at $c \in X$ if and only if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X \ \left[|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon \right]. \tag{3.75}$$

Definition 3.19. A function $f : X \to \mathbb{R}$ is *uniformly continuous on* X if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall c \in X \quad \forall x \in X \ \left[|x - c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon \right]. \quad (3.81)$$

If indeed f is uniformly continuous then the number $\delta = \delta(\varepsilon)$ in (3.81) is called a *modulus of continuity* of f.

Theorem 3.20. *(i) A uniformly continous function on X is continuous on X.*

(ii) A continuous function on X is not necessarily uniformly continuous on X.

(iii) If X is a closed interval, then any continuous function on X is uniformly continuous on X.