Solutions Manual to Walter Rudin's *Principles of Mathematical Analysis*

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Chapter 6

The Riemann–Stieltjes Integral

Exercise 6.1 Suppose α increases on $[a, b]$, $a \le x_0 \le b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f\, d\alpha = 0$.

Solution. Let $\varepsilon > 0$, and let δ be such that $|\alpha(x) - \alpha(x_0)| < \varepsilon$ if $|x - x_0| < \delta$. Now consider any partiion $a = t_0 < t_1 < \cdots < t_n = b$ with $n \ge 2$ such that $|t_i - t_{i-1}| < \frac{\delta}{2}$. There exists an index i such that $t_{i-1} < x_0 < t_{i+1}$ (there may possibly be 2 such indices). We then have, for any choice of $t_0^*, t_1^*, \ldots, t_n^*$,

$$\begin{aligned}\left|\sum_{j=1}^{n} f(t_j^*)\big(\alpha(t_j) - \alpha(t_{j-1})\big)\right| &\le |f(t_i^*)||\alpha(t_i) - \alpha(t_{i-1})| + \\ &\quad + |f(t_{i+1}^*)||\alpha(t_{i+1}) - \alpha(t_i)] \\ &\le \alpha(t_{i+1}) - \alpha(t_{i-1}) < \varepsilon.\end{aligned}$$

By definition of the Riemann–Stieltjes integral, this means that $f \in \mathcal{R}(\alpha)$ and $\int f\, d\alpha = 0$.

Exercise 6.2 Suppose $f \ge 0$, f is continuous on $[a, b]$, and $\displaystyle\int_a^b f(x)\, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1.)

Solution. Suppose $f(x_0) \neq 0$ for some $x_0 \in [a, b]$. Since $f(x)$ is continuous on $[a, b]$ and $\dfrac{f(x_0)}{2} > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \dfrac{f(x_0)}{2}$ for all $x \in [a, b]$ such that $|x - x_0| < \delta$. Let $\eta = \min(\delta, \max(x_0 - a, b - x_0))$, so that $\eta > 0$. Let I be the interval $[x_0 - \eta, x_0]$ if it is contained in $[a, b]$; otherwise let $I = [x_0, x_0 + \eta]$. Whichever is the case, $I \subseteq [a, b]$ and $f(x) =$

$f(x_0) + (f(x) - f(x_0)) \geq f(x_0) - |f(x) - f(x_0)| > \dfrac{f(x_0)}{2}$ for all $x \in I$. The functions $f_1(x)$ and $f_2(x)$ defined as

$$f_1(x) = \begin{cases} f(x), & x \in I, \\ 0, & x \notin I, \end{cases} \qquad f_2(x) = \begin{cases} f(x), & x \notin I, \\ 0, & x \in I, \end{cases}$$

are both nonnegative, bounded, and continuous except possibly at the two endpoints of the interval I. They are therefore both Riemann-integrable. Consideration of Riemann sums shows that

$$\int_a^b f_1(x)\,dx \geq \eta\frac{\varepsilon}{2},$$

and

$$\int_a^b f_2(x)\,dx \geq 0,$$

It therefore follows that

$$\int_a^b f(x)\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx \geq \eta\frac{\varepsilon}{2} > 0,$$

contradicting the hypothesis that $\displaystyle\int_a^b f(x)\,dx = 0$.

Exercise 6.3 Define three functions $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$ and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if $f(0-) = f(0)$ and that then

$$\int f\,d\beta_i = f(0).$$

(b) State and prove a similar result for β_2.

(c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.

(d) If f is continuous at 0, prove that

$$\int f\,d\beta_1 = \int f\,d\beta_2 = \int f\,d\beta_3 = f(0).$$

Solution. Let $t_0 < t_1 < \cdots < t_{n-1} < t_n$ be any partition of any interval containing 0. Since the upper Riemann-Stieltjes sums become smaller and the lower ones larger when a point is added to any partition, in deciding whether a function is integrable or not, we may assume that 0 is one of the points of the partition. Let k be the index such that $t_k = 0$, so that the upper and lower Riemann-Stieltjes sums

$$\sum_{i=1}^{n} M_i(\beta_j(t_i) - \beta_j(t_{i-1})), \quad j = 1, 2, 3,$$

and

$$\sum_{i=1}^{n} m_i(\beta_j(t_i) - \beta_j(t_{i-1})), \quad j = 1, 2, 3,$$

are respectively M_k and m_k, M_{k-1} and m_{k-1}, $\dfrac{M_{k-1} + M_k}{2}$ and $\dfrac{m_{k-1} + m_k}{2}$.

(a) Since $m_k \le f(x) \le M_k$ for $0 \le x \le t_{k+1}$ in the first case, the sets of upper and lower sums contain elements arbitrarily near to each other if and only if for each ε there is a partition with $M_k - m_k < \varepsilon$. If such a partition exists, let $\delta = t_{k+1}$. Then we have $|f(x) - f(0)| \le M_k - m_k < \varepsilon$ for $0 \le x \le \delta$, and hence $\lim\limits_{x \to 0+} = f(0)$. Conversely, if $\lim\limits_{x \to 0+} = f(0)$, then for any ε, let $\delta > 0$ be such that $|f(x) - f(0)| < \delta$ if $0 < x < \delta$, and let P be a partition with $t_k = 0$, $t_{k+1} < \delta$. It is then clear that both upper and lower Riemann sums differ from $f(0)$ by less than ε, i.e., $\displaystyle\int f\, d\beta_1 = f(0)$.

(b) $f \in \mathcal{R}(\beta_2)$ if and only if $\lim\limits_{x \to 0-} f(x) = f(0)$ and if this condition holds, then $\displaystyle\int f\, d\beta_2 = f(0)$. The proof is identical to the proof just given, except that "+" is replaced by "−."

(c) In the third case, the upper and lower Riemann-Stieltjes sums differ by $\dfrac{(M_k - m_k) + (M_{k-1} + m_{k-1}}{2}$. If, given ε, there exists a partition containing 0 for which this difference is less than $\dfrac{\varepsilon}{2}$, let $\delta = \min(t_{k+1}, -t_{k-1})$. Then for $-\delta \le x \le \delta$ we certainly have

$$|f(x) - f(0)| \le \max\Big(\frac{M_k - m_k}{2}, \frac{M_{k-1} - m_{k-1}}{2}\Big) \le M_k - m_k + M_{k-1} - m_{k-1} < \varepsilon,$$

so that f is continuous at 0. The same argument shows that in this case

$$\int f\, d\beta_3 = f(0).$$

(d) This result is contained in (a)–(c).

Exercise 6.4 If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Solution. Every upper Riemann sum equals $b - a$, and every lower Riemann sum equals 0. Hence the set of upper sums and the set of lower sums do not have a common bound.

Exercise 6.5 Suppose f is a bounded real function on $[a,b]$ and $f^2 \in \mathcal{R}$ on $[a,b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?

Solution. The integrability of f^2 does not imply the integrability of f. For example, one could let $f(x) = -1$ if x is irrational and $f(x) = 1$ if x is rational. Then every upper Riemann sum of f is $b-a$ and every lower sum is $a-b$. However, f^2, being the constant function 1, is integrable.

The integrability of f^3 does imply the integrability of f, by Theorem 6.11 with $\varphi(u) = \sqrt[3]{u}$.

Exercise 6.6 Le P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on $[0,1]$ which is continuous at every point outside P. Prove that $f \in \mathcal{R}$ on $[0,1]$. [*Hint:* P can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.]

Solution. Let $M = \sup\{|f(x)| : a \le x \le b\}$, and let $\varepsilon > 0$ be given. Cover P by a finite collection of open intervals $O = \bigcup\limits_{i=1}^{k} (a_i, b_i)$ such that $\sum(b_i - a_i) < \dfrac{\varepsilon}{4M}$. Let $\theta = \inf\{|x-y| : x \in P, y \in [a,b] \setminus O\}$. Since x and y range over disjoint compact sets, θ is a positive number. On the compact set $E = \{x : d(x,P) \ge \frac{1}{2}\theta\}$ the function f is uniformly continuous. Let $\delta > 0$ be such that $|f(x) - f(y)| < \dfrac{\varepsilon}{2(b-a)}$ if $x, y \in E$ and $|x-y| < \delta$. Then consider any partition $\{t_j\}$ of $[a,b]$ with $\max(t_j - t_{j-1}) < \min(\delta, \frac{1}{2}\theta)$. The difference between the upper and lower Riemann sums for this partition can be expressed as two sums:

$$\sum (M_j - m_j)(t_j - t_{j-1}) = \Sigma_1 + \Sigma_2,$$

where Σ_1 contains all the terms for which $[t_{j-1}, t_j]$ is contained in E and Σ_2 all the other terms. It is then obvious that

$$\Sigma_1 < \frac{\varepsilon}{2(b-a)} \sum (t_j - t_{j-1}) \le \frac{\varepsilon}{2},$$

and, since each interval $[t_{j-1}, t_j]$ that occurs in Σ_2 is contained in O,

$$\Sigma_2 < 2M \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}.$$

Therefore the upper and lower Riemann sums for any such partition differ by less than ε, and so f is Riemann integrable.

Exercise 6.7 Suppose f is a real function on $[0,1]$ and $f \in \mathcal{R}$ on $[c,1]$ for every $c > 0$. Define

$$\int_0^1 f(x)\,dx = \lim_{c \to 0+} \int_c^1 f(x)\,dx$$

if this limit exists (and is finite).

(a) If $f \in \mathcal{R}$ on $[0,1]$ show that this definition of the integral agrees with the old one.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

Solution. (a) Suppose $f \in \mathcal{R}$ on $[0,1]$. Let $\varepsilon > 0$ be given, and let $M = \sup\{|f(x)| : 0 \le x \le 1\}$. Let $c \in \left(0, \dfrac{\varepsilon}{4M}\right]$ be fixed, and consider any partition of $[0,1]$ containing c for which the upper and lower Riemann sums $\sum M_j(t_j - t_{j-1})$ and $\sum m_j(t_j - t_{j-1})$ of f differ by less than $\dfrac{\varepsilon}{4}$. Then the partition of $[c,1]$ formed by the points of this partition that lie in this interval certainly has the property that its upper and lower Riemann sums $\sum' M_j(t_j - t_{j-1})$ and $\sum' m_j(t_j - t_{j-1})$ differ by less than $\dfrac{\varepsilon}{4}$. Moreover, the terms of the original upper and lower Riemann sums *not* found in the sums for the smaller interval amount to less than $\dfrac{\varepsilon}{4}$. In short, we have shown that for $c < \dfrac{\varepsilon}{4M}$ and a suitable partition containing c,

$$\sum M_j(t - j - t_{j-1}) - \frac{\varepsilon}{4} < \int_0^1 f(x)\,dx \le< \sum m_j(t_j - t_{j-1}) + \frac{\varepsilon}{4}$$

and

$$\sum{}' M_j(t_j - t_{j-1}) - \frac{\varepsilon}{4} < \int_c^1 f(x)\,dx < \sum{}' m_j(t_j - + \frac{\varepsilon}{4}.$$

Moreover, we have also shown that

$$\left|\sum M_j(t_j - t_{j-1}) - \sum{}' M_j(t_j - t_{j-1})\right| < \frac{\varepsilon}{4}$$

and

$$\left|\sum m_j(t_j - t_{j-1}) - \sum{}' m_j(t_j - t_{j-1})\right| < \frac{\varepsilon}{4}.$$

combining these inequalities, we find that

$$\left|\int_0^1 f(x)\,dx - \int_c^1 f(x)\,dx\right| < \varepsilon$$

if $0 < c < \dfrac{\varepsilon}{4M}$.u

(b) Let

$$f(x) = (-1)^n(n+1)$$

for $\dfrac{1}{n+1} < x \le \dfrac{1}{n}$, $n = 1, 2, \ldots$. Then if $\dfrac{1}{N+1} \le c \le \dfrac{1}{N}$ we have

$$\int_c^1 f(x)\,dx = (-1)^N(N+1)\left(\frac{1}{N} - c\right) + \sum_{k=1}^{N-1} \frac{(-1)^k}{k}.$$

Since $0 \leq \frac{1}{N} - c \leq \frac{1}{N} - \frac{1}{N+1} = \frac{1}{N(N+1)}$, the first term on the right-hand side tends to zero as $c \downarrow 0$, while the sum approaches $\ln 2$. Hence this integral approaches a limit. However,

$$\int_c^1 |f(x)|\,dx = (N+1)\Big(\frac{1}{N} - c\Big) + \sum_{k=1}^{N-1} \frac{1}{k},$$

and in this case the first term on the right-hand side tends to zero as $c \downarrow 0$, while the sum tends to infinity.

Exercise 6.8 Suppose $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$, where a is fixed. Define

$$\int_a^\infty f(x)\,dx = \lim_{x \to \infty} \int_a^b f(x)\,dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)\,dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called "integral test" for convergence of series.)

Solution. Since both the series and the integral are increasing functions of their upper limits, it suffices to show that they are bounded together. Define $f(x) = f(1)$ for $0 \leq x \leq 1$. Then consider a partition of $[0, n]$ consisting of the $n+1$ points $0, 1, 2, \ldots, n$. The upper Riemann sum for this partition is $\sum_{k=0}^{n-1} f(k)$ and the lower Riemann sum is $\sum_{k=1}^{n} f(k)$. Hence we have

$$\sum_{k=1}^{n} f(k) \leq \int_0^n f(x)\,dx = f(0) + \int_1^n f(x)\,dx \leq \sum_{k=0}^{n-1} f(k) = f(0) + \sum_{k=1}^{n-1} f(k).$$

This shows that

$$-f(0) + \sum_{k=1}^{n} f(k) \leq \int_1^n f(x)\,dx \leq \sum_{k=1}^{n-1} f(x),$$

and hence the sum and the integral converge or diverge together.

Exercise 6.9 Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx.$$

Solution. Without striving for ultimate generality we can get the main ideas in the following theorem:

Theorem. *Let $f(x)$ and $g(x)$ be continuously differentiable functions defined on $[a,\infty)$ such that $\lim_{b\to\infty} f(b)g(b)$ exists and the integral $\int_a^\infty f(x)g'(x)\,dx$ converges. Then $\int_a^\infty f'(x)g(x)\,dx$ converges and*

$$\int_a^\infty f'(x)g(x)\,dx = \lim_{b\to\infty}[f(b)g(b) - f(a)g(a)] - \int_a^\infty f(x)g'(x)\,dx.$$

Proof. For each finite value of b larger than a the standard rule for integration by parts gives

$$\int_a^b f'(x)g(x)\,dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f(x)g'(x)\,dx.$$

The hypotheses of the theorem guarantee that the limit on the right exists. Therefore, by definition, the integral on the left converges.

Applying this result with $f(x) = \sin x$, $g(x) = \frac{1}{1+x}$, we find, since $f(0)g(0) = 0$ and $\lim_{b\to\infty} f(b)g(b) = 0$, while $\int_0^\infty f(x)g'(x)\,dx$ converges absolutely, that

$$\int_0^\infty \frac{\cos x}{1+x}\,dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\,dx.$$

Exercise 6.10 Let p and q be positive real number ssuch that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(*a*) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) if $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha,$$

then

$$\int_a^b fg \, d\alpha \leq 1.$$

(c) If f and g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right|^{1/q}.$$

This is *Hölder's inequality*. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

(d) Show that Hölder's inequality is also true for the "improper" integrals described in Exercises 7 and 8.

Solution. (a) The inequality is obvious if either $u = 0$ or $v = 0$, and equality holds in that case if and only if $u = v = 0$. Hence assume $v > 0$. Keep v fixed. The inequality implies that $p > 1$ and $q > 1$, and hence the function $\varphi(u) = \dfrac{u^p}{p} + \dfrac{v^q}{q} - uv$ satisfies

$$\lim_{u \to +\infty} \varphi(u) = +\infty.$$

We also have $\varphi'(0) = -v < 0$. Hence the function $\varphi(u)$ has a minimum at some point u_0 on $(0, \infty)$ at which $0 = \varphi'(u_0) = u_0^{p-1} - v$, i.e., $u_0 = v^{\frac{1}{p-1}} = v^{q-1}$ and $u_0^p = v^q$. Note that $\varphi(u_0) = \dfrac{v^q}{p} + \dfrac{v^q}{q} - v^{q-1}v = v^q - v^q = 0$. Since this point is the only critical point for φ, we have $\varphi(u) > 0$ for all $u \neq u_0$, as required.

(b) Simply integrate the inequality

$$f(x)g(x) \leq \frac{f(x)^p}{p} + \frac{g(x)^q}{q}.$$

(c) The inequality is obviously equality if either of the two integrals on the right-hand side is zero. For the vanishing of, say $\int_a^b |f|^p \, d\alpha$ implies the vanishing of $\int_a^b M|f| \, d\alpha$ and hence the vanishing of $\int_a^b |g| \, |f| \, d\alpha$ if $|g(x)| \leq M$ for all x. Hence we now assume that $\int_a^b |f|^p \, d\alpha > 0$ and $\int_a^b |g|^q \, d\alpha > 0$. In part ($b$) we replace $f(x)$ by $\dfrac{|f(x)|}{\left(\int_a^b |f|^p \, d\alpha\right)^{1/p}}$ and $g(x)$ by $\dfrac{|g(x)|}{\left(\int_a^b |g|^q \, d\alpha\right)^{1/q}}$. We then need only invoke the inequality $\left| \int_a^b h \, d\alpha \right| \leq \int_a^b |h| \, d\alpha$.

(d) The inequality holds on each finite interval. If either of the factors on the right-hand side diverges as $b \to \infty$, the inequality is obvious. If they both converge, it follows that the left-hand side converges absolutely, and to a limit not larger than the limit of the right-hand side.

Exercise 6.11 Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$ define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 \, d\alpha \right\}^{1/2}.$$

Suppose f, g, and $h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Solution. We have

$$\begin{aligned}
\|f - h\|_2^{\frac{1}{2}} &= \int_a^b |f - h|^2 \, d\alpha \\
&= \int_a^b |(f - g) + (g - h)|^2 \, d\alpha \\
&= \int_a^b |f - g|^2 \, d\alpha + 2 \int_a^b |f - g| \, |g - h| \, d\alpha + \int_a^b |g - h|^2 \, d\alpha \\
&\leq \|f - g\|_2^2 + 2\|f - g\|_2 \|g - h\|_2 + \|g - h\|_2^2 \\
&= \left(\|f - g\|_2 + \|g - h\|_2\right)^2,
\end{aligned}$$

from which the desired inequality follow when square roots are taken.

Exercise 6.12 With the notations of Exercise 11, suppose $f \in \mathcal{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \varepsilon$.

Hint: Let $P = \{x_0, \ldots, x_n\}$ be a suitable partition of $[a, b]$, define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$.

Solution. Since $g(t)$ is defined on $[x_{i-1}, x_i]$ as the weighted average of the values of $f(x)$ at the endpoints, the weights being proportional to the distances from t to the endpoints, it is clear that $g(t)$ is piecewise linear, hence continuous. For the same reason the maximum value of the function $h = |g - f|$ on the interval $[x_{i-1}, x_i]$ will be at most $M_i - m_i$ where M_i and m_i are the maximum and minimum values of f on this interval. Let M be the maximum of $|f(x)|$ for $a \leq x \leq b$. If the partition is chosen so that

$$\sum (M_i - m_i)[\alpha(t_i) - \alpha(t_{i-1}] < \frac{\varepsilon^2}{2M},$$

then we will have

$$\sum (M_i - m_i)^2[\alpha(t_i) - \alpha(t_{i-1}] \leq 2M \sum (M_i - m_i)[\alpha(t_i) - \alpha(t_{i-1})] < \varepsilon^2,$$

and hence the upper Riemann integral for $|g - f|^2$ for this partition will also be less than ε^2. Therefore $\|g - f\|_2 < \varepsilon$, as required.

Exercise 6.13 Define

$$f(x) = \int_x^{x+1} \sin(t^2)\, dt.$$

(a) Prove that $|f(x)| < 1/x$ if $x > 0$.

Hint: Put $t^2 = u$ and integrate by parts to show that $f(x)$ is equal to

$$\frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}\, du.$$

Replace $\cos u$ by -1.

(b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x),$$

where $|r(x)| < c/x$, and c is constant.

(c) Find the upper and lower limits of $xf(x)$ as $x \to \infty$.

(d) Does $\int_0^\infty \sin(t^2)\, dt$ converge?

Solution. (a) This inequality is obvious if $0 < x \leq 1$. Hence we assume $x > 1$. Following the hint, we observe that

$$\begin{aligned} f(x) &< \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &= \frac{1+\cos(x^2)}{2x} - \frac{1+\cos[(x+1)^2]}{2(x+1)} \\ &\leq \frac{1+\cos(x^2)}{2x} \\ &\leq \frac{1}{x}. \end{aligned}$$

A similar argument shows that

$$\begin{aligned}
f(x) &> \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \frac{1}{2x} + \frac{1}{2(x+1)} \\
&= \frac{-1+\cos(x^2)}{2x} - \frac{-1+\cos[(x+1)^2]}{2(x+1)} \\
&= \frac{-1+\cos(x^2)}{2x} + \frac{1-\cos[(x+1)^2]}{2(x+1)} \\
&\geq \frac{-1+\cos(x^2)}{2x} \\
&\geq \frac{-1}{x}.
\end{aligned}$$

(*b*) The expression just written for $f(x)$ shows that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x),$$

where

$$r(x) = \Big(\frac{1}{x+1}\Big)\cos[(x+1)^2] - \frac{x}{2}\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}\,du.$$

If we integrate by parts again, we find that

$$\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}\,du = \frac{\sin[(x+1)^2]}{(x+1)^3} - \frac{\sin(x^2)}{x^3} + \frac{3}{2}\int_{x^2}^{(x+1)^2} \frac{\sin u}{x^{5/2}}\,du.$$

We now observe that the absolute value of this last integral is at most

$$\frac{3}{2}\int_{x^2}^{\infty} \frac{1}{u^{5/2}}\,du = -u^{-3/2}\Big|_{x^2}^{\infty} = x^{-3}.$$

It then follows by collecting the terms that

$$|r(x)| < \frac{3}{x}.$$

(*c*) Since $r(x) \longrightarrow 0$, the upper and lower limits of $xf(x)$ will be the corresponding limits of

$$\frac{\cos(x^2) - \cos[(x+1)^2]}{2} = \sin\Big(x^2 + x + \frac{1}{2}\Big)\sin\Big(x + \frac{1}{2}\Big).$$

We can write this last expression as $\sin s \sin\big(s^2 + \frac{1}{4}\big)$, where $s = x + \frac{1}{2}$. We claim that the upper limit of this expression is 1 and the lower limit is -1. Indeed, let $\varepsilon > 0$ be given. Choose n to be any positive integer larger than $\dfrac{2-\varepsilon}{8\varepsilon}$. Then the interval $\left(\frac{1}{4} + \left(\left(2n+\frac{1}{2}\right)\pi - \varepsilon\right)^2, \frac{1}{4} + \left(\left(2n+\frac{1}{2}\right)\pi + \varepsilon\right)^2\right)$ is longer than 2π, and hence there exists a point $t \in \left(\left(2n+\frac{1}{2}\right)\pi - \varepsilon, \left(2n+\frac{1}{2}\right)\pi + \varepsilon\right)$

at which $\sin\left(t^2+\frac{1}{4}\right)=1$ and also a point u in the same interval at which $\sin\left(u^2+\dfrac{1}{4}\right)=-1$. But then $tf(t)>1-\varepsilon$ and $uf(u)<-1+\varepsilon$. It follows that the upper limit is 1 and the lower limit is -1. (This argument actually shows that the limit points of $xf(x)$ fill up the entire interval $[-1,1]$.)

(d) The integral does converge. We observe that for integers N we have

$$\begin{aligned}\int_0^N \sin(t^2)\,dt &= \sum_{k=0}^{N} f(k)\\ &= f(0)+\sum_{k=1}^{N}\frac{r(k)}{k}+\sum_{k=1}^{N}\frac{\cos(k^2)-\cos[(k+1)^2]}{k}\\ &= f(0)+\sum_{k=1}^{N}\frac{r(k)}{k}+\left[\frac{\cos 1}{2}-\frac{\cos[(N+1)^2]}{N}\right]+\sum_{k=2}^{N}\frac{\cos(k^2)}{k(k-1)}.\end{aligned}$$

The first sum on the right converges since $|r(k)|<\dfrac{3}{k}$, and the rest obviously converges. Hence we will be finished if we show that

$$\lim_{x\to\infty}\int_{[x]}^{x}\sin(t^2)\,dt=0,$$

where $[x]$ is the integer such that $[x]\le x<[x]+1$. But this is easily done using integration by parts. The integral equals

$$\frac{\cos([x]^2)}{2[x]}-\frac{\cos(x^2)}{x^2}-\int_{[x]^2}^{x^2}\frac{\cos u}{4u^{3/2}}\,du,$$

and this expression obviously tends to zero as $x\to\infty$.

Exercise 6.14 Deal similarly with

$$f(x)=\int_x^{x+1} sin(e^t)\,dt.$$

Show that

$$e^x|f(x)|<2$$

and that

$$e^x f(x)=\cos(e^x)-e^{-1}\cos(e^{x+1})+r(x),$$

where $|r(x)|<Ce^{-x}$ for some constant C.

Solution. The arguments are completely analogous to the preceding problem. The substitution $u=e^t$ changes $f(x)$ into

$$f(x)=\int_{e^x}^{e^{x+1}}\frac{\sin u}{u}\,du,$$

and then integration by parts yields

$$f(x) = \frac{\cos(e^x)}{e^x} - \frac{\cos(e^{x+1})}{e^{x+1}} - \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\, du$$

from which it then follows that

$$-\frac{1-\cos(e^x)}{e^x} \le f(x) \le \frac{1+\cos(e^x)}{e^x}.$$

We have the equality

$$e^x f(x) = \cos(e^x) - e^{-1}\cos(e^{x+1}) - e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\, du,$$

and one more integration by parts shows that

$$\left| e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2}\, du \right| < \frac{3}{e^x}.$$

In this case $f(x)$ decreases so rapidly that there is no difficulty at all proving the convergence of the integral.

Exercise 6.15 Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x)\, dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x)\, dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2\, dx \cdot \int_a^b x^2 f^2(x)\, dx \ge \frac{1}{4}.$$

Solution. To prove the first assertion we merely integrate by parts, taking $u = x$, $dv = f(x)f'(x)\, dx$, so that $du = dx$ and $v = \frac{1}{2}f^2(x)$. Since v vanishes at both endpoints, the result is

$$\int_a^b x f(x) f'(x)\, dx = -\frac{1}{2}\int_a^b f^2(x)\, dx = -\frac{1}{2}.$$

The second inequality is an immediate consequence of the Schwarz inequality applied to the two functions $xf(x)$ and $f'(x)$.

Exercise 6.16 For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(This is Riemann's zeta function, of great importance in the study of the distribution of prime numbers.) Prove that

(*a*) $\qquad \zeta(s) = s\int_1^{\infty} \frac{[x]}{x^{s+1}}\,dx$

and that

(*b*) $\qquad \zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \frac{x-[x]}{x^{s+1}}\,dx,$

where $[x]$ denotes the greatest integer $\leq x$.

Prove that the integral in (*b*) converges for all $x > 0$.

Hint: To prove (*a*) compute the difference between the integral over $[1, N]$ and the Nth partial sum of the series that defines $\zeta(s)$.

Solution. (*a*) Ignoring the author's advice, we note that

$$\begin{aligned} s\int_1^{\infty} \frac{[x]}{x^{s+1}}\,dx &= s\sum_{n=1}^{\infty} n\int_n^{n+1} \frac{1}{x^{s+1}}\,dx \\ &= \sum_{n=1}^{\infty} n\left[\frac{1}{n^s} - \frac{1}{(n+1)^s}\right] \\ &= 1\left[\frac{1}{1^s} - \frac{1}{2^s}\right] + 2\left[\frac{1}{2^s} - \frac{1}{3^s}\right] + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \zeta(s). \end{aligned}$$

(*b*) This result is a trivial consequence of (*a*) and the identity

$$\frac{s}{s-1} = \int_1^{\infty} \frac{x}{x^{s+1}}\,dx.$$

Exercise 6.17 Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that

$$\int_a^b \alpha(x)g(x)\,dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_z^b G\,d\alpha.$$

Hint: Take g real, without loss of generality. Given $P = \{x_0, x_1, \ldots, x_n\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Show that

$$\sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i.$$

Solution. The identity just given is a trivial consequence of Abel's method of rearranging the sums:

$$\begin{aligned}\sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i &= \sum_{i=1}^{n} \alpha(x_i)\big(G(x_i) - G(x_{i-1})\big)\\ &= G(x_n)\alpha(x_n) - G(x_0)\alpha(x_0) - \sum_{i=1}^{G}(x_{i-1})\big(\alpha(x_i) - \alpha_{i-1})\big).\end{aligned}$$

Now the fact that $G(x)$ is continuous and α is nondecreasing means that the right-hand side can be made arbitrarily close to

$$G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G\,d\alpha,$$

whenever the partition is sufficiently fine. It does not follow immediately that the function $\alpha(x)g(x)$ is integrable on $[a, b]$. However, since α is nondecreasing, its only discontinuties are jumps, and for any given $\varepsilon > 0$ there can be only a finite number of jumps larger than ε. These can be enclosed in a finite number of open intervals of arbitrarily small length. We can then argue, as in Exercise 6 above, that *any* partition that is sufficiently fine will have upper and lower Riemann sums that differ by less than ε. Hence $\alpha(x)g(x)$ is integrable, and its integral is given by the stated relation.

Exercise 6.18 Let γ_1, γ_2, γ_3 be curves in the complex plane defined on $[0, 2\pi]$ by

$$\gamma_1(t) = e^{it}, \quad \gamma_2(t) = e^{2it}, \quad \gamma_3(t) = e^{2\pi it\sin(1/t)}.$$

Show that these curves have the same range, that γ_1 and γ_2 are rectifiable, that the length of γ_1 is 2π, that the length of γ_2 is 4π, and that γ_3 is not rectifiable.

Solution. Since e^{it} has period 2π it is obvious that γ_1 and γ_2 have the same range, namely the set of all complex numbers of absolute value 1. To show that this is also the range of γ_3, we need to show that the mapping $t \mapsto 2\pi t\sin(1/t)$, $0 \le t \le 2pi$, covers an interval of length 2π, i.e., that the mapping $t \mapsto t\sin(1/t)$, $0 \le t \le 2\pi$ covers an interval of length 1. (We naturally take the value to be zero when $t = 0$.) Since this range is connected, it suffices to find two points a and b in the range with $a - b > 1$. We choose those points to be $a = \dfrac{3}{\pi}$ (the image of $t = \frac{6}{\pi}$) and $b = \dfrac{-2}{3\pi}$, (the image of $t = \frac{2}{3\pi}$). We have $a - b = \frac{11}{3\pi} > 1$.

The rectification of γ_1 and γ_2 is straightforward:

$$l(\gamma_1) = \int_0^{2\pi} |\gamma_1'(t)|\,dt = 2\pi,$$

$$l(\gamma_2) = \int_0^{2\pi} |\gamma_2'(t)|\,dt = \int_0^{2\pi} 2\,dt = 4\pi.$$

To show that γ_3 is not rectifiable, we observe that its length would be

$$\int_0^{2\pi} \left| \sin(1/t) - \frac{1}{t}\cos(1/t) \right| dt \geq \int_0^{2\pi} \left| \frac{\cos(1/t)}{t} \right| dt - 2\pi.$$

By making the substitution $u = \frac{1}{t}$ in this last integral we get

$$\int_{\frac{1}{2\pi}}^{\infty} \left| \frac{\cos u}{u} \right| du.$$

But we already know that this integral diverges, since

$$\sum_{n=1}^{\infty} \int_{2n\pi}^{(2n+\frac{1}{2})\pi} \frac{\cos u}{u}\, du \geq \sum_{n=1}^{\infty} \frac{1}{(2n+\frac{1}{2})\pi} = \infty.$$

Exercise 6.19 Let γ_1 be a curve in R^k defined on $[a, b]$; let ϕ be a continuous 1-1 mapping of $[c, d]$ onto $[a, b]$ such that $\phi(c) = a$, and define $\gamma_2(x) = \gamma_1(\phi(x))$. Prove that γ_2 is an arc, a closed curve, or a rectifiable curve if and only if the same is true of γ_1. Prove that γ_1 and γ_2 have the same length.

Solution. We know that ϕ has a continuous 1-1 inverse φ, and that the composition of one-to-one functions is one-to-one. Hence, since $\gamma_1(x) = \gamma_2(\varphi(x))$, we see that γ_1 and γ_2 are both arcs (one-to-one) if either is. Since necessarily $\phi(d) = b$, we see that $\gamma_1(a) = \gamma_1(b)$ if and only if $\gamma_2(c) = \gamma_2(d)$. Hence both are closed curves if either is. Finally, since ϕ and φ establish a one-to-one correspondence between partitions $\{s_i\}$ of $[a, b]$ and $\{t_i\}$ of $[c, d]$ such that $\sum |\gamma_1(s_i) - \gamma_1(s_{i-1})| = \sum |\gamma_2(t_i) - \gamma_2(t_{i-1})|$, it follows that the two curves have the same length.