Contents

Chapter 1

Real Analysis

(1.5)
$OrderTrichotomy(<, S) := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$
$OrderTransitivity(<, S) := \forall_{x,y,z \in S}\big((x < y \wedge y < z) \implies x < z\big)$
$Order(<, S) := OrderTrichotomy(<, S) \wedge OrderTransitivity(<, S)$

(1.7)
$BoundedAbove(E, S, <) := Order(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(x \leq \beta)$
$BoundedBelow(E, S, <) := Order(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(\beta \leq x)$
$UpperBound(\beta, E, S, <) := Order(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(x \leq \beta)$
$LowerBound(\beta, E, S, <) := Order(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(\beta \leq x)$

(1.8)
$LUB(\alpha, E, S, <) := UpperBound(\alpha, E, S, <) \wedge \forall_{\gamma}\big(\gamma < \alpha \implies \neg UpperBound(\gamma, E, S, <)\big)$
$GLB(\alpha, E, S, <) := LowerBound(\alpha, E, S, <) \wedge \forall_{\beta}\big(\alpha < \beta \implies \neg LowerBound(\beta, E, S, <)\big)$

(1.10)
$LUBProperty(S, <) := \forall_E\Big(\big(\emptyset \neq E \subset S \wedge BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$

$GLBProperty(S, <) := \forall_E\Big(\big(\emptyset \neq E \subset S \wedge BoundedBelow(E, S, <)\big) \implies \exists_{\alpha \in S}\big(GLB(\alpha, E, S, <)\big)\Big)$

(1.11)

$LUBPropertyImpliesGLBProperty$	$LUBProperty(S, <) \implies GLBProperty(S, <)$

(1)	$LUBProperty(S, <) \implies \ldots$	wts: 2
(1.1)	$\big(\emptyset \neq B \subset S \wedge BoundedBelow(B, S, <)\big) \implies \ldots$	wts: 1.2
(1.1.1)	$Order(<, S) \wedge \exists_{\delta' \in S}\big(LowerBound(\delta', B, S, <)\big)$	from: $BoundedBelow$, 1.1
(1.1.2)	$|B| = 1 \implies \ldots$	wts: 1.1.3
(1.1.2.1)	$\exists_{u'}(u' \in B) \ \blacksquare\ u := choice(\{u' | u' \in B\}) \ \blacksquare\ B = \{u\}$	from: 1.1.2
(1.1.2.2)	$GLB(u, B, S, <) \ \blacksquare\ \exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$	
(1.1.3)	$|B| = 1 \implies \exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$	
(1.1.4)	$|B| \neq 1 \implies \ldots$	wts: 1.1.5
(1.1.4.1)	$\forall_E\Big(\big(\emptyset \neq E \subset S \wedge BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$	from: $LUBProperty$, 1
(1.1.4.2)	$L := \{s \in S | LowerBound(s, B, S, <)\}$	
(1.1.4.3)	$|B| > 1 \wedge OrderTrichotomy(<, S) \ \blacksquare\ \exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$	from: $Order$, 1.1.1 wts: 1.1.4.7
(1.1.4.4)	$b_1 := choice\Big(\{b_1' \in B | \exists_{b_0' \in B}(b_0' < b_1')\}\Big) \ \blacksquare\ \neg LowerBound(b_1, B, S, <)$	from: 1.1.4.2
(1.1.4.5)	$b_1 \notin L \ \blacksquare\ L \subset S$	
(1.1.4.6)	$\delta := choice\big(\{\delta' \in S | LowerBound(\delta', B, S, <)\}\big) \ \blacksquare\ \delta \in L \ \blacksquare\ \emptyset \neq L$	from: 1.1.1
(1.1.4.7)	$\emptyset \neq L \subset S$	from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y \in L}\Big(LowerBound(y, B, S, <)\Big)$ ▮ $\forall_{y \in L}\forall_{x \in B}(y \leq x)$

(1.1.4.9) $\forall_{x \in B}\Big(x \in S \land \forall_{y \in L}(y \leq x)\Big)$ ▮ $\forall_{x \in B}\Big(UpperBound(x, L, S, <)\Big)$

(1.1.4.10) $\exists_{x \in S}\Big(UpperBound(x, L, S, <)\Big)$ ▮ $BoundedAbove(L, S, <)$

(1.1.4.11) $\emptyset \neq L \subset S \land BoundedAbove(L, S, <)$

(1.1.4.12) $\exists_{\alpha' \in S}\Big(LUB(\alpha', L, S, <)\Big)$ ▮ $\alpha := choice\Big(\{\alpha' \in S \,|\, (LUB(\alpha', L, S, <))\}\Big)$

(1.1.4.13) $\forall_x\Big(x \in B \implies UpperBound(x, L, S, <)\Big)$

(1.1.4.14) $\forall_x\Big(\neg UpperBound(x, L, S, <) \implies x \notin B\Big)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$

  (1.1.4.15.1) $\neg UpperBound(\gamma, L, S, <)$ ▮ $\gamma \notin B$

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B$ ▮ $\gamma \in B \implies \gamma \geq \alpha$

(1.1.4.17) $\forall_{\gamma \in B}(\alpha \leq \gamma)$ ▮ $LowerBound(\alpha, B, S, <)$

(1.1.4.18) $\alpha < \beta \implies \ldots$

  (1.1.4.18.1) $\forall_{y \in L}(y \leq \alpha < \beta)$ ▮ $\forall_{y \in L}(y \neq \beta)$

  (1.1.4.18.2) $\beta \notin L$ ▮ $\neg LowerBound(\beta, B, S, <)$

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound(\beta, B, S, <)$ ▮ $\forall_{\beta \in S}\Big(\alpha < \beta \implies \neg LowerBound(\beta, B, S, <)\Big)$

(1.1.4.20) $LowerBound(\alpha, B, S, <) \land \forall_{\beta \in S}\Big(\alpha < \beta \implies \neg LowerBound(\beta, B, S, <)\Big)$

(1.1.4.21) $GLB(\alpha, B, S, <)$ ▮ $\exists_{\epsilon_1 \in S}\Big(GLB(\epsilon_1, B, S, <)\Big)$

(1.1.5) $|B| \neq 1 \implies \exists_{\epsilon_1 \in S}\Big(GLB(\epsilon_1, B, S, <)\Big)$

(1.1.6) $\Big(|B| = 1 \implies \exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)\Big) \land \Big(|B| \neq 1 \implies \exists_{\epsilon_1 \in S}\big(GLB(\epsilon_1, B, S, <)\big)\Big)$

(1.1.7) $(|B| = 1 \lor |B| \neq 1) \implies \exists_{\epsilon \in S}\Big(GLB(\epsilon, B, S, <)\Big)$ ▮ $\exists_{\epsilon \in S}\Big(GLB(\epsilon, B, S, <)\Big)$

(1.2) $\Big(\emptyset \neq B \subset S \land BoundedBelow(B, S, <)\Big) \implies \exists_{\epsilon \in S}\Big(GLB(\epsilon, B, S, <)\Big)$

(1.3) $\forall_B\Big(\big(\emptyset \neq B \subset S \land BoundedBelow(B, S, <)\big) \implies \exists_{\epsilon \in S}\big(GLB(\epsilon, B, S, <)\big)\Big)$

(1.4) $GLBProperty(S, <)$

(2) $LUBProperty(S, <) \implies GLBProperty(S, <)$

(1.12)

$$Field(F, +, *) := \exists_{0,1 \in F}\forall_{x,y,z \in F} \begin{cases} x + y \in F & \land & x * y \in F & \land \\ x + y = y + x & \land & x * y = y * x & \land \\ (x + y) + z = x + (y + z) & \land & (x * y) * z = x * (y * z) & \land \\ 1 \neq 0 & \land & x * (y + z) = (x * y) + (x * z) & \land \\ 0 + x = x & \land & 1 * x = x & \land \\ \exists_{-x \in F}\big(x + (-x) = 0\big) \land \Big(x \neq 0 \implies \exists_{1/x \in F}\big(x * (1/x) = 1\big)\Big) \end{cases}$$

$\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast$ $\Big(Field(F, +, *) \land x, y, z \in F\Big) \implies \ldots$ $\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast$

(1.14)

$AdditiveCancellation$	$(x + y = x + z) \implies y = z$

(1) $y = 0 + y = \big(x + (-x)\big) + y = \big((-x) + x\big) + y = (-x) + (x + y) = \ldots$

(2) $(-x) + (x + z) = \big((-x) + x\big) + z = \big(x + (-x)\big) + z = 0 + z = z$

$AdditiveIdentityUniqueness$	$(x + y = x) \implies y = 0$

(1) $x + y = x = 0 + x = x + 0$

(2) $y = 0$

$\boxed{AdditiveInverseUniqueness}$ $(x + y = \mathbb{0}) \implies y = -x$

from: *Field*

(1) $x + y = \mathbb{0} = x + (-x)$

from: *AdditiveCancellation*

(2) $y = -x$

$\boxed{DoubleNegative}$ $x = -(-x)$

from: *Field*

(1) $\mathbb{0} = x + (-x) = (-x) + x$ ▌ $\mathbb{0} = (-x) + x$

from: *AdditiveInverseUniqueness*

(2) $x = -(-x)$

(1.15)

$\boxed{MultiplicativeCancellation}$ $(x \neq \mathbb{0} \wedge x * y = x * z) \implies y = z$ —

$\boxed{MultiplicativeIdentityUniqueness}$ $(x \neq \mathbb{0} \wedge x * y = x) \implies y = \mathbb{1}$ —

$\boxed{MultiplicativeInverseUniqueness}$ $(x \neq \mathbb{0} \wedge x * y = \mathbb{1}) \implies y = 1/x$ —

$\boxed{DoubleReciprocal}$ $(x \neq \mathbb{0}) \implies x = 1/(1/x)$ —

(1.16)

$\boxed{Domination}$ $\mathbb{0} * x = \mathbb{0}$

from: *Field*

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x$ ▌ $\mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$

from: *AdditiveIdentityUniqueness*

(2) $\mathbb{0} * x = \mathbb{0}$

$\boxed{NonDomination}$ $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies \ldots$

  (1.1) $(x * y = \mathbb{0}) \implies \ldots$

from: *Field*, *Domination*, 1, 1.1

    (1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = \left(x * (1/x)\right) * \left(y * (1/y)\right) = (x * y) * \left((1/x) * (1/y)\right) = \mathbb{0} * \left((1/x) * (1/y)\right) = \mathbb{0}$

from: *Field*

    (1.1.2) $\mathbb{1} = \mathbb{0} \wedge \mathbb{1} \neq \mathbb{0}$ ▌ $\bot$

  (1.2) $(x * y = \mathbb{0}) \implies \bot$ ▌ $x * y \neq \mathbb{0}$

(2) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

$\boxed{NegationCommutativity}$ $(-x) * y = -(x * y) = x * (-y)$

from: *Field*, *Domination*
wts: 2

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0}$ ▌ $x * y + (-x) * y = \mathbb{0}$

from: *AdditiveInverseUniqueness*

(2) $(-x) * y = -(x * y)$

from: *Field*, *Domination*
wts: 4

(3) $x * y + x * (-y) = x * (y + -y) = x * \mathbb{0} = \mathbb{0}$ ▌ $x * y + x * (-y) = \mathbb{0}$

from: *AdditiveInverseUniqueness*

(4) $x * (-y) = -(x * y)$

from: 2, 4

(5) $(-x) * y = -(x * y) = x * (-y)$

$\boxed{NegativeMultiplication}$ $(-x) * (-y) = x * y$

from: *NegationCommutativity*, *DoubleNegative*

(1) $(-x) * (-y) = -\left(x * (-y)\right) = -\left(-(x * y)\right) = x * y$

(1.17)

$OrderedField(F, +, *, <) := \left(\begin{array}{l} Field(F, +, *) \quad \wedge \quad Order(<, F) \quad \wedge \\ \forall_{x,y,z \in F}(y < z \implies x + y < x + z) \quad \wedge \\ \forall_{x,y \in F}\left((x > \mathbb{0} \wedge y > \mathbb{0}) \implies x * y > \mathbb{0}\right) \end{array} \right)$

*********************** $\left(OrderedField(F, +, *, <) \wedge x, y, z \in F\right) \implies \ldots$ ***********************

(1.18)

$\boxed{NegationOnOrder}$ $x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > 0 \implies \ldots$

   (1.1) $0 = (-x) + x > (-x) + 0 = -x$ ▌ $0 > -x$ ▌ $-x < 0$ from: *Ordered Field*

(2) $x > 0 \implies -x < 0$

(3) $-x < 0 \implies \ldots$

   (3.1) $0 = x + (-x) < x + 0 = x$ ▌ $0 < x$ ▌ $x > 0$ from: *Ordered Field*

(4) $-x < 0 \implies x > 0$

(5) $x > 0 \implies -x < 0 \land -x < 0 \implies x > 0$ ▌ $x > 0 \iff -x < 0$ from: 2, 4

▐ *PositiveFactorPreservesOrder* ▌ $(x > 0 \land y < z) \implies x * y < x * z$

(1) $(x > 0 \land y < z) \implies \ldots$

   (1.1) $(-y) + z > (-y) + y = 0$ ▌ $z + (-y) = 0$ from: *Ordered Field*

   (1.2) $x * (z + (-y)) > 0$ ▌ $x * z + x * (-y) > 0$ from: *Ordered Field*

   (1.3) $x * z = 0 + x * z = \big(x * y + -(x * y)\big) + x * z = \big(x * y + x * (-y)\big) + x * z = \ldots$ from: *Field*, *NegationCommutativity*

   (1.4) $x * y + \big(x * z + x * (-y)\big) > x * y + 0 = x * y$ from: *Field*, 1.2

   (1.5) $x * z > x * y$ from: 1.3, 1.4

(2) $(x > 0 \land y < z) \implies x * z > x * y$

▐ *NegativeFactorFlipsOrder* ▌ $(x < 0 \land y < z) \implies x * y > x * z$

(1) $(x < 0 \land y < z) \implies \ldots$

   (1.1) $-x > 0$ from: *NegationOnOrder*

   (1.2) $(-x) * y < (-x) * z$ ▌ $0 = x * y + (-x) * y < x * y + (-x) * z$ ▌ $0 < x * y + (-x) * z$ from: *PositiveFactorPreservesOrder*

   (1.3) $0 < (-x) * (-y + z)$ ▌ $0 > x * (-y + z)$ ▌ $0 > -(x * y) + x * z$ from: *NegationOnOrder*

   (1.4) $x * y > x * z$

(2) $(x < 0 \land y < z) \implies x * y > x * z$

▐ *SquareIsPositive* ▌ $(x \neq 0) \implies x * x > 0$

(1) $(x > 0) \implies x * x > 0$ from: *Ordered Field*

(2) $(x < 0) \implies \ldots$

   (2.1) $-x > 0$ ▌ $x * x = (-x) * (-x) > 0$ ▌ $x * x > 0$ from: *NegationOnOrder*, *Ordered Field*, *NegativeMultiplication*

(3) $(x < 0) \implies x * x > 0$

(4) $x \neq 0 \implies (x > 0 \lor x < 0) \implies x * x > 0$ ▌ $x \neq 0 \implies x * x > 0$ from: *OrderTrichotomy*, 1, 3

▐ *OneIsPositive* ▌ $1 > 0$

(1) $1 \neq 0$ ▌ $1 = 1 * 1 > 0$ from: *Field*, *SquareIsPositive*

▐ *ReciprocationOnOrder* ▌ $(0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \ldots$

   (1.1) $x * (1/x) = 1 > 0$ ▌ $x * (1/x) > 0$ from: *Field*, *OneIsPositive*

   (1.2) $1/x < 0 \implies x * (1/x) < 0 \land x * (1/x) > 0 \implies \bot$ ▌ $1/x > 0$ from: *NegativeFactorFlipsOrder*, 1

   (1.3) $y * (1/y) = 1 > 0$ ▌ $y * (1/y) > 0$ from: *Field*, *OneIsPositive*

   (1.4) $1/y < 0 \implies y * (1/y) < 0 \land y * (1/y) > 0 \implies \bot$ ▌ $1/y > 0$ from: *NegativeFactorFlipsOrder*, 1

   (1.5) $(1/x) * (1/y) > 0$ from: *Ordered Field*

   (1.6) $0 < 1/y = \big((1/x) * (1/y)\big) * x < \big((1/x) * (1/y)\big) * y = 1/x$ from: *Ordered Field*, 1, 1.4, 1.5

(1.19)

$\boxed{Ordered\,Field\,Q}$ $\quad Ordered\,Field(\mathbb{Q}, +, *, <)$ $\qquad$ ——

$Subfield(K, F, +, *) := Field(F, +, *) \wedge K \subset F \wedge Field(K, +, *)$

$Ordered\,Subfield(K, F, +, *, <) := Ordered\,Field(F, +, *, <) \wedge K \subset F \wedge Ordered\,Field(K, +, *, <)$

$Cut\,I(\alpha) := \emptyset \neq \alpha \subset \mathbb{Q}$

$Cut\,I\,I(\alpha) := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha)$

$Cut\,I\,I\,I(\alpha) := \forall_{p \in \alpha} \exists_{r \in \alpha}(p < r)$

$\mathbb{R} := \{\alpha \in \mathbb{Q} | Cut\,I(\alpha) \wedge Cut\,I\,I(\alpha) \wedge Cut\,I\,I\,I(\alpha)\}$

$\boxed{CutCorollaryI}$ $\quad (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \ldots$	
$\quad$ (1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}}(q' < p' \implies q' \in \alpha)$	from: $Cut\,II, 1$
$\quad$ (1.2) $q < p \implies q \in \alpha \;\blacksquare\; q \notin \alpha \implies q \geq p$	from: 1
$\quad$ (1.3) $(q \notin \alpha) \implies \ldots$	
$\quad\quad$ (1.3.1) $q \geq p$	from: 1.2
$\quad\quad$ (1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \bot \;\blacksquare\; q \neq p$	from: 1, 1.3
$\quad\quad$ (1.3.3) $q \geq p \wedge q \neq p \;\blacksquare\; p < q$	
$\quad$ (1.4) $q \notin \alpha \implies p < q \;\blacksquare\; p < q$	from: 1
(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$	

$\boxed{CutCorollaryI\,I}$ $\quad (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$	
$\quad$ (1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}}(r' < s' \implies r' \in \alpha)$	from: $Cut\,II, 1$
$\quad$ (1.2) $s \in \alpha \implies \big(r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)\big) \;\blacksquare\; s \in \alpha \implies r \in \alpha$	from: 1, 1.1
$\quad$ (1.3) $r \notin \alpha \implies s \notin \alpha \;\blacksquare\; s \notin \alpha$	from: 1, 1.2
(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$	

$<_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$\boxed{OrderTrichotomyR}$ $\quad OrderTrichotomy(\mathbb{R}, <_{\mathbb{R}})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$	
$\quad$ (1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$	
$\quad\quad$ (1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$	from: $<_{\mathbb{R}}, 1.1$
$\quad\quad$ (1.1.2) $\exists_{p'}(p' \in \alpha \wedge p' \notin \beta) \;\blacksquare\; p := choice(\{p' | p' \in \alpha \wedge p' \notin \beta\})$	
$\quad\quad$ (1.1.3) $q \in \beta \implies \ldots$	
$\quad\quad\quad$ (1.1.3.1) $p, q \in \mathbb{Q}$	
$\quad\quad\quad$ (1.1.3.2) $q < p$	from: $CutCorollaryI$
$\quad\quad\quad$ (1.1.3.3) $q \in \alpha$	from: $Cut\,II$
$\quad\quad$ (1.1.4) $q \in \beta \implies q \in \alpha$	
$\quad\quad$ (1.1.5) $\forall_{q \in \beta}(q \in \alpha) \;\blacksquare\; \beta \subseteq \alpha$	
$\quad\quad$ (1.1.6) $\beta \subset \alpha \;\blacksquare\; \beta <_{\mathbb{R}} \alpha$	
$\quad$ (1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$	
$\quad$ (1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \;\blacksquare\; (\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$	
$\quad$ (1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$	
$\quad$ (1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$	
$\quad$ (1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$	
$\quad$ (1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$	
(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$	

(3) $\forall_{\alpha,\beta\in\mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4)   *OrderTrichotomy*$(\mathbb{R}, <_{\mathbb{R}})$

OrderTransitivityR   *OrderTransitivity*$(\mathbb{R}, <_{\mathbb{R}})$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$

  (1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \dots$

    (1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

    (1.1.2) $\forall_{a\in\alpha}(a \in \beta) \wedge \forall_{b\in\beta}(b \in \gamma)$

    (1.1.3) $\forall_{a\in\alpha}(\alpha \in \gamma)$ ▮ $\alpha \subset \gamma$ ▮ $\alpha <_{\mathbb{R}} \gamma$

  (1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(3) $\forall_{\alpha,\beta,\gamma\in\mathbb{R}}\big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(4)   *OrderTransitivity*$(\mathbb{R}, <_{\mathbb{R}})$

OrderR   *Order*$(<_{\mathbb{R}}, \mathbb{R})$

LUBPropertyR   *LUBProperty*$(\mathbb{R}, <_{\mathbb{R}})$

(1) $\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \dots$

  (1.1) $\gamma := \{p \in \mathbb{Q} | \exists_{\alpha\in A}(p \in \alpha)\}$

  (1.2) $A \neq \emptyset$ ▮ $\exists_{\alpha}(\alpha \in A)$ ▮ $\alpha_0 := choice(\{\alpha | \alpha \in A\})$

  (1.3) $\alpha_0 \neq \emptyset$ ▮ $\exists_a(a \in \alpha_0)$ ▮ $a_0 := choice(\{a | a \in \alpha_0\})$ ▮ $a_0 \in \gamma$ ▮ $\gamma \neq \emptyset$

  (1.4)   *BoundedAbove*$(A, \mathbb{R}, <_{\mathbb{R}})$ ▮ $\exists_{\beta}\big(UpperBound(\beta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

  (1.5) $\beta_0 := choice(\{\beta | UpperBound(\beta, A, \mathbb{R}, <_{\mathbb{R}})\})$

  (1.6)   *UpperBound*$(\beta_0, A, \mathbb{R}, <_{\mathbb{R}})$ ▮ $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \beta_0)$ ▮ $\forall_{\alpha\in A}(\alpha \subseteq \beta_0)$ ▮ $\forall_{\alpha\in A}\forall_{a\in\alpha}(a \in \beta_0)$

  (1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma$ ▮ $\forall_{a\in\gamma}(a \in \beta_0)$ ▮ $\gamma \subseteq \beta_0$

  (1.8) $\beta_0 \subset \mathbb{Q}$ ▮ $\gamma \subseteq \beta_0 \subset \mathbb{Q}$ ▮ $\gamma \subseteq \mathbb{Q}$

  (1.9) $\emptyset \neq \gamma \subset \mathbb{Q}$ ▮ *CutI*(γ)

  (1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$

    (1.10.1) $p \in \gamma$ ▮ $\exists_{\alpha\in A}(p \in \alpha)$ ▮ $\alpha_1 := choice(\{\alpha \in A | p \in \alpha\})$

    (1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p$ ▮ $q \in \alpha_1$ ▮ $q \in \gamma$

  (1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma$ ▮ $\forall_{p\in\gamma}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \gamma)$ ▮ *CutII*(γ)

  (1.12) $p \in \gamma \implies \dots$

    (1.12.1) $\exists_{\alpha\in A}(p \in \alpha)$ ▮ $\alpha_2 := choice(\{\alpha \in A | p \in \alpha\})$

    (1.12.2) $\exists_{r\in\alpha_2}(p < r)$ ▮ $r_0 := choice(\{r \in \alpha_2 | p < r\})$

    (1.12.3) $r_0 \in \alpha_2$ ▮ $r_0 \in \gamma$

    (1.12.4) $p < r_0$ ▮ $p < r_0 \wedge r_0 \in \gamma$ ▮ $\exists_{r\in\gamma}(p < r)$

  (1.13) $p \in \gamma \implies \exists_{r\in\gamma}(p < r)$ ▮ $\forall_{p\in\gamma}\exists_{r\in\gamma}(p < r)$ ▮ *CutIII*(γ)

  (1.14)   *CutI*$(\gamma) \wedge CutII(\gamma) \wedge CutIII(\gamma)$ ▮ $\gamma \in \mathbb{R}$

  (1.15) $\forall_{\alpha\in A}(\alpha \subseteq \gamma)$ ▮ $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma)$

  (1.16) $\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R}$ ▮ *UpperBound*$(\gamma, A, \mathbb{R}, <_{\mathbb{R}})$

  (1.17) $\delta <_{\mathbb{R}} \gamma \implies \dots$

    (1.17.1) $\delta \subset \gamma$ ▮ $\exists_s(s \in \gamma \wedge s \notin \delta)$ ▮ $s_0 := choice(\{s \in \mathbb{Q} | s \in \gamma \wedge s \notin \delta\})$

    (1.17.2) $s_0 \in \gamma$ ▮ $\exists_{\alpha\in A}(s_0 \in \alpha)$ ▮ $\alpha_3 := choice(\{\alpha \in A | s_0 \in \alpha\})$

    (1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta$ ▮ $\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

    (1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \dots$

      (1.17.4.1) $\alpha_3 \subseteq \delta$ ▮ $\forall_{s\in\mathbb{Q}}(s \in \alpha_3 \implies s \in \delta)$ ▮ $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

      (1.17.4.2) $\neg\exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s\in\mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$ ▮ $\bot$

    (1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \bot$ ▮ $\delta <_{\mathbb{R}} \alpha_3$ ▮ $\exists_{\alpha\in A}(\delta <_{\mathbb{R}} \alpha)$ ▮ $\exists_{\alpha\in A}\big(\neg(\alpha \leq_{\mathbb{R}} \delta)\big)$

    (1.17.6) $\neg\forall_{\alpha\in A}(\alpha \leq_{\mathbb{R}} \delta)$ ▮ $\neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}}))$ ▌ $\forall_\delta \big(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.19) $UpperBound(\gamma, A, \mathbb{R}, <_{\mathbb{R}}) \wedge \forall_\delta \big(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(1.20) $LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})$ ▌ $\exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(2) $\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)$

(3) $\forall_A \Big(\big(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove(A, \mathbb{R}, <_{\mathbb{R}})\big) \implies \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <_{\mathbb{R}})\big)\Big)$ ▌ $LUBProperty(\mathbb{R}, <_{\mathbb{R}})$

$+_{\mathbb{R}}(\alpha, \beta) := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$
$0_{\mathbb{R}} := \{x \in \mathbb{Q} | x < 0\}$

$0R$	$0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}}$ ▌ $\emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q}$ ▌ $CutI(0_{\mathbb{R}})$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}}$ ▌ $\forall_{x \in 0_{\mathbb{R}}} \forall_{y \in \mathbb{Q}}(y < x \implies y \in 0_{\mathbb{R}})$ ▌ $CutII(0_{\mathbb{R}})$

(3) $y := x/2$ ▌ $(x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $\forall_{x \in 0_{\mathbb{R}}} \exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $CutIII(0_{\mathbb{R}})$

(4) $CutI(0_{\mathbb{R}}) \wedge CutII(0_{\mathbb{R}}) \wedge CutIII(0_{\mathbb{R}})$ ▌ $0_{\mathbb{R}} \in \mathbb{R}$

$Field + R$	$Field^+(\mathbb{R})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s | r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha) ; \exists_b(b \in \beta)$ ▌ $a_0 := choice(\{a | a \in \alpha\}) ; b_0 := choice(\{b | b \in \beta\})$ ▌ $a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha) ; \exists_y(y \notin \beta)$ ▌ $x_0 := choice(\{x | x \notin \alpha\}) ; y_0 := choice(\{y | y \notin \beta\})$

(1.5) $\forall_{r \in \alpha}(r < x_0) ; \forall_{s \in \beta}(s < y_0)$ ▌ $\forall_{r \in \alpha} \forall_{s \in \beta}(r + s < x_0 + y_0)$ ▌ $x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q}$ ▌ $CutI(\alpha +_{\mathbb{R}} \beta)$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.7.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▌ $(r_0, s_0) := choice((r, s) \in \alpha \times \beta | p = r + s)$

(1.7.2) $q < p = r_0 + s_0$ ▌ $(q - s_0) < r_0$ ▌ $(q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta$ ▌ $q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta$ ▌ $q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta)$ ▌ $CutII(\alpha +_{\mathbb{R}} \beta)$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▌ $(r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta | p = r + s\})$

(1.9.2) $r_1 \in \alpha$ ▌ $\exists_{t \in \alpha}(r_1 < t)$ ▌ $t_0 := choice(\{t \in \alpha | r_1 < t\})$

(1.9.3) $s_1 \in \beta$ ▌ $t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1$ ▌ $\exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} \beta} \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▌ $CutIII(\alpha +_{\mathbb{R}} \beta)$

(1.11) $CutI(\alpha +_{\mathbb{R}} \beta) \wedge CutII(\alpha +_{\mathbb{R}} \beta) \wedge CutIII(\alpha +_{\mathbb{R}} \beta)$ ▌ $\alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

(3) $\alpha +_{\mathbb{R}} \beta = \{r + s | r \in \alpha \wedge s \in \beta\} = \{s + r | s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

(4) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(4.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

(4.2) $\{a + (b + c) | a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(5) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(6) $\alpha \in \mathbb{R} \implies \ldots$

(6.1) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \ldots$

(6.1.1) $s < 0$ ▌ $r + s < r + 0 = r$ ▌ $r + s < r$ ▌ $r + s \in \alpha$

(6.2) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha$ ▌ $\forall_{r \in \alpha} \forall_{s \in 0_{\mathbb{R}}}(r + s \in \alpha)$

(6.3) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}}(p \in \alpha)$ ▌ $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(6.4) $p \in \alpha \implies \ldots$

(6.4.1) $\exists_{r \in \alpha}(p < r)$ ▌ $r_2 := choice(\{r \in \alpha | p < r\})$

(6.4.2)	$p < r_2$ ▮ $p - r_2 < r_2 - r_2 = 0$ ▮ $(p - r_2) < 0$ ▮ $(p - r_2) \in 0_{\mathbb{R}}$
(6.4.3)	$r_2 \in \alpha$ ▮ $p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▮ $p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$
(6.5)	$p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▮ $\forall_{p \in \alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▮ $\alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$
(6.6)	$\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▮ $0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$
(6.7)	$\alpha \in \mathbb{R} \implies \dots$
(6.7.1)	$\beta := \{p \in \mathbb{Q} \vert \exists_{r>0}(-p - r \notin \alpha)\}$
(6.7.2)	$\alpha \subset \mathbb{Q}$ ▮ $\exists_{s \in \mathbb{Q}}(s \notin \alpha)$ ▮ $s_0 := choice(\{s \vert s \notin \alpha\})$ ▮ $p_0 := -s_0 - 1$
(6.7.3)	$-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha$ ▮ $-p_0 - 1 \notin \alpha$ ▮ $\exists_{r>0}(-p_0 - r \notin \alpha)$ ▮ $p_0 \in \beta$
(6.7.4)	$\emptyset \neq \alpha$ ▮ $\exists_{q \in \alpha}$ ▮ $q_0 := choice(\{q \in \mathbb{Q} \vert q \in \alpha\})$
(6.7.5)	$r > 0 \implies \dots$
(6.7.5.1)	$q_0 \in \alpha$ ▮ $-(-q_0) - r = q_0 - r < q_0$ ▮ $-(-q_0) - r < q_0$ ▮ $-(-q_0) - r \in \alpha$
(6.7.6)	$\forall_{r>0}\big(-(-q_0) - r \in \alpha\big)$ ▮ $\neg\exists_{r>0}(-(-q_0) - r \notin \alpha)$ ▮ $-q_0 \notin \beta$
(6.7.7)	$\emptyset \neq \beta \subset \mathbb{Q}$ ▮ *CutI*(β)
(6.7.8)	$(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$
(6.7.8.1)	$p \in \beta$ ▮ $\exists_{r>0}(-p - r \notin \alpha)$ ▮ $r_0 := choice(\{r > 0 \vert -p - r \notin \alpha\})$
(6.7.8.2)	$q < p$ ▮ $-p - r < -q - r$
(6.7.8.3)	$-q - r \notin \alpha$ ▮ $q \in \beta$
(6.7.9)	$(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta$ ▮ $\forall_{p \in \beta}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \beta)$ ▮ *CutII*(β)
(6.7.10)	... page 30 CUT III
(6.8)	$\alpha \in \mathbb{R} \implies \exists_{-\alpha \in \mathbb{R}}\big(\alpha + (-\alpha) = 0_{\mathbb{R}}\big)$

ExistenceOfR	$\exists_{\mathbb{R}}\big(LUBProperty(\mathbb{R}, <) \wedge OrderedSubfield(\mathbb{Q}, \mathbb{R}, +, *, <)\big)$
(1)	123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y} ((x, y) \in f)$

$(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \dots$
(1) $Domain(f) := X; Codomain(f) := Y$
(2) $Image(f, A) := \{f(a) \| a \in A\}; Preimage(f, B) := \{a \| f(a) \in B\}$
(3) $Range(f) := Image(Domain(f))$

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X}(x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$
$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X}(y = f(x))$
$Bijective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$

TODO: Definition properties Surjective Equivalent: $(Range(f) = Codomain(f)) \implies Surjective(f)$