# Preliminaries from Set Theory and Logic

Part 2

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MTH541M - Bridging Course for Real Analysis / Advanced Calculus

## Axiom 1

*Let $r \in \mathbb{R}$, and let $S \subseteq \mathbb{R}$ such that $S \neq \emptyset$.*

1. *[The Trichotomy Law.] Exactly one of the statements $r > 0$, $r < 0$, $r = 0$ is true. [i.e., Any real number is either positive, negative or zero, where the 'or' is an exclusive disjunction.]*
2. *[The Completeness Axiom.] If there exists $u \in \mathbb{R}$ such that $u$ is an upper bound of $S$, then there exists $s \in \mathbb{R}$ such that $s = \sup S$. [i.e., Any nonempty subset of $\mathbb{R}$ that has an upper bound has a least upper bound.]*
3. *[The Well-ordering Principle.] If $S \subseteq \mathbb{N}$, then there exists $m \in S$ such that $m = \min S$. [i.e., Any nonempty subset of $\mathbb{N}$ has a least element.]*

An alternative statement of the Trichotomy Law is that, given $a, b \in \mathbb{R}$, exactly one of the statements $a > b$, $a < b$, $a = b$ is true. This is easily deduced from how we stated the law in Axiom 1 by setting $r = a - b$. This is the reason why the relations $\not<$, $\not>$, $\not\leq$, $\not\geq$ are each equivalent to $\geq$, $\leq$, $>$, $<$, respectively.

## Corollary 2

*Let* $\varepsilon > 0$. *Then for any* $r \in \mathbb{R}$, *the following statements hold.*

1. *[Archimedean property of* $\mathbb{R}$.*] There exists* $n \in \mathbb{N}$ *such that* $r < n\varepsilon$.
2. *[Existence of the floor function.] There exists a unique* $\lfloor r \rfloor \in \mathbb{Z}$ *such that*

$$\lfloor r \rfloor \leq r < \lfloor r \rfloor + 1. \tag{1}$$

3. *[Denseness of* $\mathbb{Q}$ *in* $\mathbb{R}$.*]There exists* $q \in \mathbb{Q}$ *such that* $-\varepsilon < r - q < \varepsilon$.

# Proof of the Archimedean Property of $\mathbb{R}$

Consider $\varepsilon > 0$ as fixed. Our goal here is to prove

$$\forall r \in \mathbb{R} \quad \exists n \in \mathbb{N} \ [r < n\varepsilon]. \tag{2}$$

Suppose otherwise. That is,

$$\exists r \in \mathbb{R} \quad \forall n \in \mathbb{N} \ [r \geq n\varepsilon]. \tag{3}$$

Let us collect all numbers $n\varepsilon$ with $n \in \mathbb{N}$ into the set

$$S_\varepsilon := \{n\varepsilon \ : \ n \in \mathbb{N}\}. \tag{4}$$

Since $\varepsilon = 1 \cdot \varepsilon \in S_\varepsilon$, we find that $S_\varepsilon \neq \emptyset$. Also, (3) asserts that there exists $r \in \mathbb{R}$ such that $r$ is an upper bound of $S_\varepsilon$. By the completeness axiom, there exists $s \in \mathbb{R}$ such that $s = \sup S_\varepsilon$. Since $\varepsilon > 0$, the number $s - \varepsilon$ is lower than the supremum $s$ of $S_\varepsilon$, and so $s - \varepsilon$ is not an upper bound of $S_\varepsilon$. This means that there exists an element of $S_\varepsilon$ that is not bounded above by ($\not\leq$) the number $s - \varepsilon$. This element of $S_\varepsilon$, according to (4), is equal to $m\varepsilon$ for some $m \in \mathbb{N}$.

# Proof of the Archimedean Property of $\mathbb{R}$

Thus,

$$\begin{aligned} m\varepsilon &> s - \varepsilon, \\ (m+1)\varepsilon &> s, \end{aligned}$$

where $(m+1)\varepsilon \in S_\varepsilon$ because $m+1 \in \mathbb{N}$. This contradicts the fact that $s$ is an upper bound of $S_\varepsilon$. Therefore, (2) is true.

We consider three cases for $r$, according to the Trichotomy Law. If $r = 0$, then the integer $\lfloor r \rfloor := 0$ satisfies (1). For the case $r > 0$, we use 1 with $\varepsilon = 1$. Then the set

$$N_r := \{n \in \mathbb{N} \ : \ r < n\}, \tag{5}$$

is a non-empty subset of $\mathbb{N}$. By the Well-ordering Principle, there exists $m_r \in N_r$ such that $m_r$ is the least element of $N_r$. Then any number lower cannot be in $N_r$. That is,

$$x < m_r \implies x \notin N_r. \tag{6}$$

Also, from (5), we get

$$x \in N_r \iff r < x, \tag{7}$$

$$x \notin N_r \iff r \geq x, \tag{8}$$

and so (6) becomes

$$x < m_r \implies r \geq x. \tag{9}$$

# Proof of the Existence of the Floor Function

Because $m_r - 1 < m_r$ and $m_r \in N_r$, we can use (9) and (7), respectively, to obtain

$$m_r - 1 \leq r < m_r. \tag{10}$$

To see that this $m_r$ is unique, suppose we have some $m'_r \in \mathbb{N}$ such that

$$m'_r - 1 \leq r < m'_r. \tag{11}$$

Suppose we have an integer $x \geq m_r$. Because $r < m_r$ from (10), we have $x > r$, and so by (7), $x \in N_r$. That is,

$$x \geq m_r \implies x \in N_r. \tag{12}$$

The condition $m'_r - 1 \leq r$ from (11), using (8), implies $m'_r - 1 \notin N_r$, and by contraposition of (12), we have $m'_r - 1 < m_r$.

# Proof of the Existence of the Floor Function

We claim that $m_r' \leq m_r$. Otherwise, we have both of the conditions $m_r' - 1 < m_r$ and $m_r' > m_r$, which by some manipulation, lead to

$$0 < m_r' - m_r < 1,$$

which gives us the contradiction that there is an integer $m_r' - m_r$ between 0 and 1. Therefore, $m_r' \leq m_r$.
Using $r < m_r'$ from (11) and (7), we have $m_r' \in N_r$. Since $m_r$ is the least element of $N_r$, we have $m_r' \geq m_r$. Therefore, $m_r' = m_r$.
By (10), the integer $\lfloor r \rfloor := m_r - 1$ satisfies (1). Finally, we look at the case $r < 0$. The reasoning is similar to that in the previous case, except that instead of $N_r$ in (5), we use the set

$$M_r := \{n \in \mathbb{N} \ : \ n \geq -r\}.$$

# Proof of the Existence of the Floor Function

If $n_r = \min M_r$, then

$$n_r - 1 < -r \leq n_r. \tag{13}$$

The existence and uniqueness proofs for $n_r$ are similar to those for $m_r$ of the previous case.
By manipulating (13), we get $-n_r \leq r < -n_r + 1$, where we simply take $\lfloor r \rfloor := -n_r \in \mathbb{Z}$.

# Proof of the Denseness of $\mathbb{Q}$ in $\mathbb{R}$

By the Archimedean Property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $1 < n\varepsilon$. Later in the proof, we use the consequents $\frac{1}{n} < \varepsilon$ and $-\varepsilon < -\frac{1}{n}$. Let $m := \lfloor nr \rfloor$. Then:

$$\begin{aligned}
m \leq &\quad nr &< m+1,\\
\frac{m}{n} \leq &\quad r &< \frac{m}{n} + \frac{1}{n},\\
\frac{m}{n} - \frac{1}{n} < \frac{m}{n} \leq &\quad r &< \frac{m}{n} + \frac{1}{n},\\
\frac{m}{n} - \frac{1}{n} < &\quad r &< \frac{m}{n} + \frac{1}{n},\\
-\frac{1}{n} < &\quad r - \tfrac{m}{n} &< \frac{1}{n},\\
-\varepsilon < -\frac{1}{n} < &\quad r - \tfrac{m}{n} &< \frac{1}{n} < \varepsilon.
\end{aligned}$$

Since $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, we have $q := \frac{m}{n} \in \mathbb{Q}$. Therefore, $-\varepsilon < r - q < \varepsilon$. $\square$

The special case $\varepsilon = 1$ in Corollary 2 [Part 1] is often called the *Axiom of Archimedes*, albeit occuring as a theorem instead of an axiom in the standard literature. The Archimedean property of $\mathbb{R}$ is a staple in the analysis of real-valued functions on subsets of $\mathbb{R}$, i.e., a staple in Real Analysis. It plays crucial roles in the proofs of a good number of theorems. A theorem with this characteristic is sometimes referred to as a *principle*; hence, the name *Archimedean Principle* in some standard Analysis texts. There is some awkwardness here since the name is very close to the law of buoyancy in physics, or Archimedes' Principle, which of course has nothing to do with a theoretical study of the number system $\mathbb{R}$. We choose instead the, also standard, *Archimedean property* of $\mathbb{R}$, as how it is called in formal constructions of $\mathbb{R}$.

The precise notion of 'denseness' belongs to the study of the standard topology in $\mathbb{R}$. A customary alternative way of stating said denseness property is *"Between any two distinct real numbers is a rational number."* This can be easily deduced from our statement of Corollary 2 [Part 3]: if $a, b$ are distinct real numbers, we assume without loss of generality that $a < b$, and use $r = \frac{b+a}{2}$ and $\varepsilon = \frac{b-a}{2} > 0$.

# The Floor, Ceiling and Absolute Value Functions

Corollary 2 [Part 2] establishes the existence of the *floor function* $x \mapsto \lfloor x \rfloor$, which is an integer-valued function on $\mathbb{R}$, a function $\mathbb{R} \to \mathbb{Z}$. We also call $x \mapsto \lfloor x \rfloor$ the *greatest integer function*, where by 'greatest' we mean the greatest integer *below* the input $x$. We have a corresponding notion, which is the *ceiling function* or *least integer function* $x \mapsto \lceil x \rceil$, where

$$\lceil x \rceil := \begin{cases} x, & x \in \mathbb{Z}, \\ \lfloor x \rfloor + 1, & x \notin \mathbb{Z}, \end{cases}$$

with the corresponding property $\lceil x \rceil - 1 < x \leq \lceil x \rceil$ for any $x \in \mathbb{R}$.

The *absolute value* function $x \mapsto |x|$ is defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases} \tag{14}$$

We immediately have

$$\begin{aligned} x \geq 0 &\implies |x| = x \geq 0, \\ x < 0 &\implies -x > 0 \implies |x| > 0 \implies |x| \geq 0, \end{aligned}$$

and so the function $x \mapsto |x|$ always has non-negative real values, i.e., $x \mapsto |x|$ is a *non-negative function*.

# Fundamental Properties of Absolute Value

## Proposition 3

*Let* $a, b \in \mathbb{R}$.

1. $|ab| = |a| \cdot |b|$ *[i.e., the absolute value function is multiplicative].*
2. $\pm a \leq |a|$.
3. $\forall \varepsilon > 0 \ \ [|x| < \varepsilon \iff -\varepsilon < x < \varepsilon]$.
4. $\forall \varepsilon > 0 \ \ [|x| \leq \varepsilon \iff -\varepsilon \leq x \leq \varepsilon]$.
5. *If* $\pm a \leq b$ *and* $b \geq 0$, *then* $|a| \leq b$.
6. *If* $|a - b| < \varepsilon$ *for any* $\varepsilon > 0$, *then* $a = b$.
7. *If* $a \leq b + \varepsilon$ *for any* $\varepsilon > 0$, *then* $a \leq b$.

# Proof of $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ [|ab| = |a| \cdot |b|]$

We consider three cases for the value of $ab$ according to the Trichotomy Law, and by (14), we have

$$ab = 0 \implies |ab| = 0, \tag{15}$$
$$ab > 0 \implies |ab| = ab, \tag{16}$$
$$ab < 0 \implies |ab| = -ab. \tag{17}$$

If $ab = 0$, then one of $a, b$ is zero, and by (14), so is one of $|a|$, $|b|$. Then $|a| \cdot |b| = 0$. Combining this with (15), we get the desired equation. If $ab > 0$, then either $a, b > 0$ or $a, b < 0$. Using (14), we have either $|a| \cdot |b| = ab$ or $|a| \cdot |b| = (-a)(-b) = ab$. By (16), we have the desired equation. Finally, if $ab < 0$, then one of $a, b$ is negative, and $|a| \cdot |b|$ is either $(-a)b$ or $a(-b)$, which both reduce into the right-hand side of (17), and so the desired equation follows.

# Proof of $\forall a \in \mathbb{R}\ [\pm a \leq |a|]$

By the Trichotomy Law, we have either $a \geq 0$ or $a < 0$. For each of such cases, we have the following consequences.

$$
\begin{aligned}
a \geq 0 \quad \Longrightarrow \quad & -a \leq 0, && (18) \\
& -a \leq 0 \leq a = |a|, && (19) \\
a < 0 \quad \Longrightarrow \quad & -a > 0, && (20) \\
& a < 0 < -a = |a|. && (21)
\end{aligned}
$$

We note here that the inequalities in (19) are the result of combining the inequalities in (18) by transitivity, while the equality in (19) is based on the definition of absolute value for the case indicated by the left-hand side of (18). Also, (21) was obtained from (20) in an analogous manner. Now, comparing $-a$ and $|a|$ according to (19), we have $-a \leq |a|$ and the equality in (19) implies $a \leq |a|$. Therefore, $\pm a \leq |a|$ for the case $a \geq 0$. In an analogous manner, we also conclude $\pm a \leq |a|$ from (21) for the case $a < 0$.

# Proof of $\forall \varepsilon > 0 \; [|x| < \varepsilon \iff -\varepsilon < x < \varepsilon]$ and $\forall \varepsilon > 0 \; [|x| \leq \varepsilon \iff -\varepsilon \leq x \leq \varepsilon]$

Let $\varepsilon > 0$. We aim here for a 'simultaneous' proof of the two properties, so we let $\lhd$ be any of the order relations $<$, $\leq$. Our goal is to prove

$$
\begin{aligned}
|x| \lhd \varepsilon &\iff -\varepsilon \lhd x \lhd \varepsilon, \\
&\iff -\varepsilon \lhd x \;\wedge\; x \lhd \varepsilon, \\
|x| \lhd \varepsilon &\iff -x \lhd \varepsilon \wedge x \lhd \varepsilon.
\end{aligned}
$$

For necessity, assume $|x| \lhd \varepsilon$. Using 2, we have $\pm x \leq |x| \lhd \varepsilon$. Thus, $\pm x \lhd \varepsilon$, which means $-x \lhd \varepsilon$ and $x \lhd \varepsilon$. For sufficiency, assume $-x \lhd \varepsilon$ and $x \lhd \varepsilon$. But since $-x$ and $x$ are the only possible values of $|x|$, we have $|x| \lhd \varepsilon$ in any case.

# Proof of " If $\pm a \leq b$ and $b \geq 0$, then $|a| \leq b$."

If $b = 0$, then we cannot have $a > 0$ or $a < 0$. Otherwise, one of $\pm a$ is positive, contradicting the assumption $\pm a \leq b = 0$. Thus, the only possibility is $a = 0$, and immediately, $|a| = 0 \leq b$. Suppose $b > 0$. The condition $\pm a \leq b$ means $a \leq b$ and $-a \leq b$, or equivalently, $-b \leq a$. Thus, $-b \leq a \leq b$. Therefore, $|a| \leq b$.

# Proof of "If $|a - b| < \varepsilon$ for any $\varepsilon > 0$, then $a = b$."

We prove this by contraposition. If $a \neq b$, then $a - b \neq 0$, which by the Trichotomy Law, implies either $a - b > 0$ or $a - b < 0$. By the definition of absolute value, we have $|a - b| > 0$ in any of the two cases. Take $\varepsilon := \frac{|a-b|}{2} > 0$, for which $|a - b| \geq \varepsilon$.

# Proof of "If $a \leq b + \varepsilon$ for any $\varepsilon > 0$, then $a \leq b$."

We also use contraposition here. If $a > b$, then $a - b > 0$, and furthermore, $\varepsilon := \frac{a-b}{2} > 0$. Consequently, $a - b > \varepsilon$, and so $a > b + \varepsilon$.

**Corollary 4**

*Let $a, b, c \in \mathbb{R}$.*

1. $|a|^2 = |a^2| = a^2$.
2. $|a| = \sqrt{a^2}$.
3. $|a| \leq \sqrt{a^2 + b^2}$.
4. $|a - b| = 0 \iff a = b$.
5. $|a - b| = |b - a|$.
6. $|a + b| \leq |a| + |b|$.
7. $|a - b| \leq |a - c| + |c - b|$.
8. $||a| - |b|| \leq |a - b|$.

# Proof of $\forall a \in \mathbb{R} \ \left[|a|^2 = |a^2| = a^2\right]$.

For the first equality, we use multiplicativity of absolute value:

$$|a|^2 = |a| \cdot |a| = |a \cdot a| = |a^2|.$$

For the second equality, we note that in any case under the Trichotomy Law, $a^2 \geq 0$. From the definition of absolute value, $|a^2| = a^2$.

# Proof of $\forall a \in \mathbb{R}\ \left[|a| = \sqrt{a^2}\right]$.

The desired equation is immediate if $a = 0$. Thus, we suppose $a \neq 0$, where both $a^2$ and $|a|$ are positive, and so is $|a| + \sqrt{a^2}$. From $|a|^2 = a^2$ in the previous item, we have $\left(|a| - \sqrt{a^2}\right)\left(|a| + \sqrt{a^2}\right) = 0$, both sides of which can be divided by $|a| + \sqrt{a^2} > 0$, and hence, $|a| = \sqrt{a^2}$.

# Proof of $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \left[|a| \leq \sqrt{a^2 + b^2}\right]$.

If both $a$ and $b$ are zero, then both sides of the desired inequality are zero, and we are done. Thus, we suppose at least one of $a$, $b$ is nonzero so that $\sqrt{a^2 + b^2} + |a|$ is positive. From $|a|^2 = a^2$ in 1 and since $b^2 \geq 0$, we have $|a|^2 \leq a^2 + b^2$, and we now have the inequality $0 \leq \left(\sqrt{a^2 + b^2} - |a|\right)\left(\sqrt{a^2 + b^2} + |a|\right)$. We divide both sides by $\sqrt{a^2 + b^2} + |a| > 0$, and obtain $|a| \leq \sqrt{a^2 + b^2}$.

# Proof of $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ [|a-b| = 0 \iff a = b]$, or the *Identity of Indiscernibles* in $\mathbb{R}$

For necessity, if $|a-b| = 0$, then $|a-b| < \varepsilon$ for any $\varepsilon > 0$. Thus, $a = b$. For sufficiency, if $a = b$, then $|a-b| = |a-a| = |0| = 0$.

# Proof of $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \ [|a-b| = |b-a|]$, or the *Symmetry* of the *Distance Function* $(a, b) \mapsto |a-b|$ in $\mathbb{R}$

$$|a-b| = |(-1)(b-a)| = |-1| \cdot |b-a| = 1 \cdot |b-a| = |b-a|.$$

# Proof of $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \ [|a+b| \leq |a|+|b|]$, or the *Minkowski inequality* in $\mathbb{R}$

For the case $a = 0 = b$, mere substitution to $|a+b|$ and to $|a|+|b|$ gives us the desired inequality. Suppose then that one of $a$, $b$ is nonzero. Then one of $|a+b|$, $|a|$, $|b|$ is positive, and we have

$$|a+b| + |a| + |b| > 0. \tag{22}$$

Using Proposition 3 [Part 2],

$$\begin{aligned} ab &\leq |ab|, \\ 2ab &\leq 2|ab|, \\ a^2 + 2ab + b^2 &\leq a^2 + 2|ab| + b^2. \end{aligned} \tag{23}$$

# Proof of $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ [|a+b| \leq |a| + |b|]$, or the *Minkowski inequality* in $\mathbb{R}$

Using Proposition 3 [Part 1] and Corollary 4 [Part 1] on the right-hand side of (23),

$$\begin{aligned} a^2 + 2ab + b^2 &\leq |a|^2 + 2|a| \cdot |b| + |b|^2, \\ (a+b)^2 &\leq (|a| + |b|)^2, \\ |a+b|^2 = (a+b)^2 &\leq (|a| + |b|)^2, \\ |a+b|^2 &\leq (|a| + |b|)^2, \\ |a+b|^2 - (|a| + |b|)^2 &\leq 0, \\ (|a+b| + |a| + |b|)(|a+b| - |a| - |b|) &\leq 0. \end{aligned} \tag{24}$$

Because of (22), we can divide both sides of (24) by $|a+b| + |a| + |b| > 0$ to obtain

$$|a+b| - |a| - |b| \leq 0,$$

which leads to the desired inequality.

# Proof of $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \forall c \in \mathbb{R}\ [|a-b| \leq |a-c| + |c-b|]$, or the *Triangle Inequality* in $\mathbb{R}$

Use the Minkowski Inequality as stated before, with $a - c$ instead of $a$ and with $c - b$ instead of $b$.

# Proof of $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ [||a| - |b|| \leq |a - b|]$, or the *Reverse Triangle Inequality* in $\mathbb{R}$

If $a = b$, then both $||a| - |b||$ and $|a - b|$ reduce to zero, and we get the desired inequality. Thus, we assume henceforth that $a \neq b$. By the Identity of Indiscernibles in $\mathbb{R}$, we have $|a - b| \neq 0$, and by the non-negativity of absolute values, we have

$$|a - b| > 0. \tag{25}$$

The trick in this proof consists of the trivial equations

$$\begin{aligned} a &= (a - b) + b, \\ b &= (b - a) + a, \end{aligned}$$

from which, with the use of 7, we obtain

$$\begin{aligned} |a| = |(a - b) + b| &\leq |a - b| + |b|, \\ |b| = |(b - a) + a| &\leq |b - a| + |a| = |a - b| + |a|. \end{aligned}$$

# Proof of $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ [||a| - |b|| \leq |a - b|]$, or the *Reverse Triangle Inequality* in $\mathbb{R}$

That is,

$$\begin{aligned} |a| &\leq |a - b| + |b|, \\ |b| &\leq |a - b| + |a|, \end{aligned}$$

which further imply

$$\begin{aligned} |a| - |b| &\leq |a - b|, & (26) \\ |b| - |a| &\leq |a - b|. & (27) \end{aligned}$$

Multiplying both sides of (27) by $-1$ and combining with (26), we obtain

$$-|a - b| \leq |a| - |b| \leq |a - b|,$$

on which we use (25) and Proposition 34 to obtain
$||a| - |b|| \leq |a - b|$.

# The Lattice Operations in $\mathbb{R}$

**Definition 5**

For any $a, b \in \mathbb{R}$, we define $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. We shall refer to $\wedge$ and $\vee$ as the *lattice operations* in $\mathbb{R}$.

We note here that there should be no confusion with the lattice operations and logical conjunction and disjunction, since the former are operations on real numbers, while the latter are operations on statements. Also, the lattice operations are performed on two and only two real numbers at a time. We shall not bother with extending them to more than two real numbers.

# Properties of Lattice Operations

## Proposition 6

*For any* $a, b \in \mathbb{R}$,

$$\begin{aligned} a \vee b &= \frac{1}{2}(a + b + |a - b|), \\ a \wedge b &= \frac{1}{2}(a + b - |a - b|). \end{aligned}$$

## Proposition 7

*For any* $a, b \in \mathbb{R}$,

$$\begin{aligned} a \vee b &= b \vee a, \\ a \wedge b &= b \wedge a, \\ a \wedge b \leq r \leq a \vee b &\implies |r - a| \leq |a - b|, \\ & \quad\;\; |r - b| \leq |a - b|. \end{aligned}$$

Problem Set 1: Prove Propositions 6 and 7.