Example 8.6. Let G be the graph given in Figure 29. We want to find its clique graph.

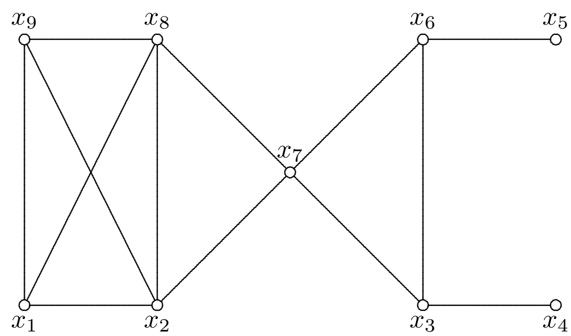

Figure 29: A graph G

The maximal cliques of G are given in Figure 30

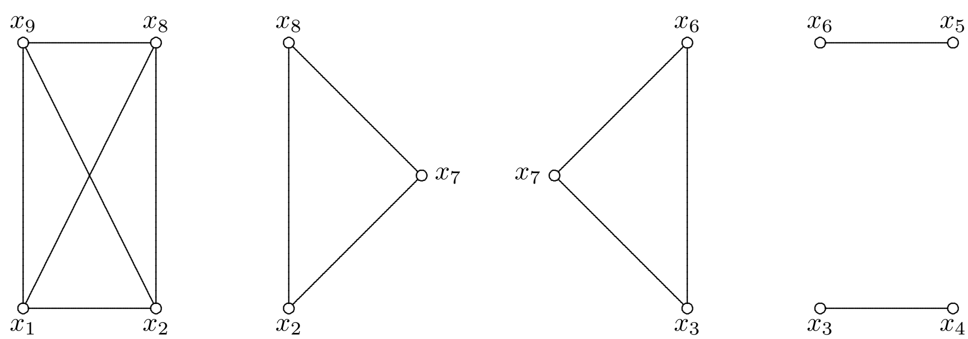

Figure 30: The maximal cliques of G

Thus, $S = \{S_1, S_2, S_3, S_4, S_5\}$, where $S_1 = \{x_1, x_2, x_8, x_9\}$, $S_2 = \{x_2, x_7, x_8\}$, $S_3 = \{x_7, x_3, x_6\}$, $S_4 = \{x_3, x_4\}$ and $S_5 = \{x_6, x_5\}$. Thus, letting $S_i = v_i$, the clique graph of G is given Figure 31.

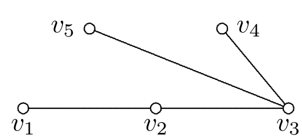

Figure 31: The clique graph of G

9 Digraphs and the Duality Principle

9.1 Digraphs

Definition 9.1. A *digraph* D is an ordered pair $D = (V, A)$, where $V = V(D)$ is a nonempty set of elements called *vertices* and $A = A(D)$ is a subset of $V(D) \times V(D)$. Thus, the elements of $A(D)$ are ordered pairs of elements of $V(D)$ and these are called *arcs*. The *order* of D is $|V(D)|$ and the *size* of D is $|A(D)$.

Example 9.1. Let D be a digraph with $V(D) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $A(D) = \{(x_1, x_2), (x_2, x_3), (x_2, x_4), (x_3, x_5), (x_4, x_1), (x_6, x_3)\}$. A pictorial representation of D is given in Figure 32.

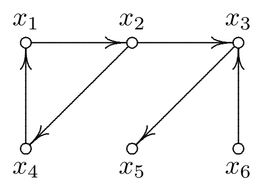

Figure 32: An example of a digraph

Definition 9.2. Let D be a digraph of order n with $V(G) = \{x_1, x_2, \ldots, x_n\}$, then the $n \times n$ matrix, denoted by $\mathcal{A}(D) = [a_{ij}]$, defined by

$$a_{ij} = \begin{cases} 1 \text{ if } (x_i, x_j) \in A(D) \\ 0 \text{ if } (x_i, x_j) \notin A(D) \end{cases}$$

is called the *adjacency matrix* of D.

Definition 9.3. Let D be a digraph with adjacency matrix $\mathcal{A}(D)$. It $\det \mathcal{A}(D) = 0$, then D is a *singular digraph*, otherwise D is nonsingular.

Example 9.2. Give the adjacency matrix of the digraph given in Figure 32. Is this digraph singular or nonsingular?

Definition 9.4. Let $a = (u, v)$, be an arc in a digraph D, then u is said to be *adjacent to* v and v is said to be *adjacent from* u, u is called the *initial vertex* of a and v is called the *terminal vertex* of a. An arc of the form (v, v) is called a *loop*.

Definition 9.5. Let $D = (V(D), A(D))$ be a digraph. The *out-neighbors* of $x \in V(D)$ denoted by $N^+(x)$ is defined as

$$N^+(x) = \{y \in V(D) | (x, y) \in A(D)\}.$$

The *in-neighbors* of x, denoted by $N^-(x)$ is defined as

$$N^-(x) = \{y \in V(D) | (y, x) \in A(D)\}.$$

The cardinality of $N^+(x)$ is called the *out-degree* of x, denoted by $od(x)$, and the cardinality of $N^-(x)$ is called the *in-degree* of x, denoted by $id(x)$.

Thus, $N^+(x)$ is the set of all vertices which are adjacent from x and $N^-(x)$ is the set of all vertices which are adjacent to x.

Example 9.3. With reference to the digraph in Example 32, the out-neighbors of the vertex x_3 is the set $N^+(x_3) = \{x_5\}$ and the in-neighbors of x_3 is the set $N^-(x_3) = \{x_2, x_6\}$. Thus, $od(x_3) = 1$ and $id(x_3) = 2$.

Definition 9.6. If $|N^+(x)| = |N^-(x)| = r, \forall (x) \in V(D)$, we then say that the digraph D is *r-regular*.

Definition 9.7. Suppose D is a digraph and $x \in V(D)$.

1. If $|N^+(x)| > 0$ and $|N^-(x)| = 0$, then x is called a *source*;
2. If $x \in V(D)$ with $|N^-(x)| > 0$ and $|N^+(x)| = 0$, then x is called a *sink*;
3. If $od(v) = 1$ and $id(v) = 1$, then x is called a *carrier*;
4. If $od(v) = 0$ and $id(v) = 0$, then x is called an *isolated vertex* ;

Remark 9.1. Note that if a digraph has a source or a sink or an isolated then this digraph is singular since its corresponding adjacency matrix will have a column or a row of zeroes.

Theorem 9.1. (The First Theorem of digraph theory) *Let D be a digraph of size m and order n, with vertex set $V(D) = \{x_1, x_2, \ldots, x_n\}$. Then*

$$\sum_{i=1}^{n} od(x_i) = \sum_{i=1}^{n} id(x_i) = m.$$

Proof: Each arc (x_i, x_j) contributes one unit to the outdegree of x_i and one unit to the indegree of x_j, and that there are m arcs.

Definition 9.8. Let D be a digraph. A *directed walk* of length n in D is a sequence of vertices $< x_1, x_2, \ldots, x_n, x_{n+1} >$ such that (x_i, x_{i+1}) is an arc for each $i = 1, 2, \ldots, n$. A directed walk is called a *directed path* if $x_1, x_2, \ldots, x_n, x_{n+1}$ are distinct. A directed walk is said to be *closed* if $x_1 = x_{n+1}$. A closed directed walk is called a *circuit* if $(n+1) > 1$ and $< x_1, x_2, \ldots, x_n >$ is a directed path. The *order of a directed walk* is the number of distinct vertices. A *spanning directed walk* is a directed walk that contains all the vertices of D. A *spanning directed path* is a directed path that contains all the vertices of D.

Definition 9.9. Let D be a digraph. A *semiwalk* of length n in D is a sequence of vertices $< x_1, x_2, \ldots, x_n, x_{n+1} >$ such that either (x_i, x_{i+1}) or (x_{i+1}, x_i) is an arc for each $i = 1, 2, \ldots, n$. A *semipath*, *semicircuit* are defined appropriately.

Definition 9.10. Let D be a digraph. If there is a directed path from u to v, then we say that v is *reachable from* u. The distance $d(u, v)$ from u to v is the length of the shortest directed path from u to v.

Theorem 9.2. *Let D be a digraph. If D contains a $u-v$ directed walk of length k, the D contains a $u-v$ directed path of length at most k.*

Proof: Let W be a $u-v$ directed walk of minimum length. If $W=<x_1,x_2,\ldots,x_r>$, where $u=x_1$ and $v=x_r$, then $r\leq k$. If the vertices of W are unique, then W is a path of length at most k. Otherwise, there exists vertices $x_i,x_j\in W$, such that $x_i=x_j$ and $i<j$. Delete the vertices $x_{i+1},\ldots,x_j$ to obtain a walk W'. We continue this process until we obtain a $u-v$ walk, say W^* wherein all the vertices in it is unique, Thus W^* is a $u-v$ path and of length at most k. $\square$

Definition 9.11. Let D be a digraph.

- D is *strongly connected* or *strong*, if every two vertices are mutually reachable.
- D is *unilaterally connected* or *unilateral* if for any two vertices at least one is reachable from the other.
- D is *weakly connected* or *weak* if every two points is joined by a semi-path.

Remark 9.2. Clearly every strong digraph is unilateral and every unilateral digraph is weak. A graph is *disconnected* if it is not even weak. The trivial digraph is vacuously strong.

Remark 9.3. Given a digraph D. If all the arcs (x_i,x_j) of D is replaced by the edge x_ix_j, then, the resulting graph is called the *underlying graph* of D. We note that multiple arcs, if there are any are replaced by an single edge.

Example 9.4. Consider the digraph D is Figure 32. Its corresponding underlying graph is given in Figure 33

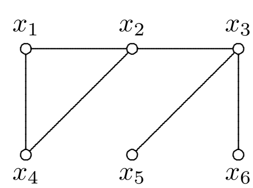

Figure 33: The underlying graph G of the digraph D

Remark 9.4. We note that if the underlying graph G of a digraph D is connected, then G is weakly connected. Thus, the digraph given in Figure 32 is weakly connected. However, if this digraph strongly connected? It is not, because x_5 is reachable from x_1 but x_1 is not reachable from x_5. However, the digraph given is actually unilateral. Verify this!

Example 9.5. The digraph in Figure 34 is a strongly connected digraph.

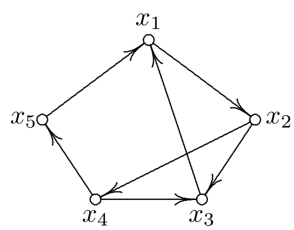

Figure 34: A strongly connected digraph

Theorem 9.3. *A digraph D is strong if and only if D has a closed spanning directed walk.*

Proof: A trivial digraph is always strong. Thus, we assume that D is nontrivial.

($\Rightarrow$) Suppose D is strong. Let $V(D) = \{x_1, x_2, \ldots, x_{n-1}, x_n\}$. Since D is strong, there is a directed path $x_i - x_{i+1}$ for $i = 1, 2, \ldots, n-1$. Also, let P_n be a directed path $v_n - v_1$. If we join these directed paths, from end to end, we form a closed spanning directed walk in D.

($\Leftarrow$) Suppose D contains a closed spanning directed walk $W =< x_1, x_x, \ldots, x_n >$, where $x_1 = x_n = w$. We need to show that for every pair of vertices u, v in D, there is an $u - v$ directed path and a $v - u$ directed path. Let $i < j$, then in W, there is a vertex $x_i = u$ and another vertex $x_j = v$. Thus, there is W_1 a $u - v$ directed walk in D, that is $W_1 =< u = x_i, x_{i=1}, x_{i+2}, \ldots, v = x_j >$. Also, there is a $v - u$ directed walk W_2, that is $W_2 =< x_j = v, x_{j+1} \ldots, x_n, x_1, \ldots, x_i = u >$. Using Thm. 9.2, W_1 contains a $u - v$ directed path and W_2 contains a $v - u$ directed path. Therefore, D is strong. $\square$

9.2 Directional Duality Principle

Definition 9.12. Let D be a digraph, its *converse* denoted by D' has the same vertices as D and the arc (u, v) is in D' if and only if the arc (v, u) is in D.

Example 9.6. The converse of the graph in Figure 32 is given in Figure 35.

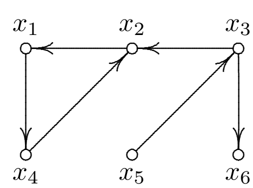

Figure 35: The converse D' of the digraph D in Figure 32

In forming the converse of a digraph, the direction every arc of D is reversed. We note that the indegree and outdegree of a vertex are reversed as well. Concepts like this which are concerned with direction of arcs are related by the *"Principle of Directional Duality"*.

Definition 9.13. Let P be a statement about digraphs. The statement P' obtained from P by changing every directional concept is called the directional converse of P. The **directional duality principle** states that if P is a theorem on digraphs then P' is also a theorem on digraphs.

The following theorems illustrate the directional duality principle.

Theorem 9.4. *A finite digraph which does not contain any circuit has at least one vertex with zero outdegree.*

Theorem 9.5. *A finite digraph which does not contain any circuit has at least one vertex with zero indegree.*

It will be enough to establish only one of the two theorems because of the directional duality principle.

10 Some Operations on Digraphs

We now extend our definition of the operations on graphs defined in the last chapter to digraphs. We note that the extension is done by replacing every occurrence of edge xy in the graph by an arc (x, y) in the corresponding digraph.

10.1 Sum of Digraphs

Definition 10.1. Let D_1 and D_2 be digraphs. Then, the *sum* of D_1 and D_2, denoted by $D_1 + D_2$ is a digraph with $V(D_1 + D_2) = V(D_1) \cup V(D_2)$ and $A(D_1 + D_2) = A(D_1) \cup A(D_2) \cup \{(x, y) |\, x \in V(D_1),\, y \in V(D_2)\}$.

Example 10.1. Consider the digraphs $\vec{P}_2^*$ and $\vec{C}_3^*$ with

$$V(\vec{P}_2^*) = \{x_1, x_2\}; \quad A(\vec{P}_2^*) = \{(x_1, x_2)\}$$

and $\vec{C}_3^*$ with

$$V(\vec{C}_3^*) = \{y_1, y_2, y_3\}; \quad A(\vec{C}_3^*) = \{(y_1, y_2), (y_2, y_3), (y_3, y_1)\},$$

respectively. Then,

$$V(\vec{P}_2^* + \vec{C}_3^*) = \{x_1, x_2, y_1, y_2, y_3\}$$

and

$$\begin{aligned} A(\vec{P}_2^* + \vec{C}_3^*) \quad = \quad & \{(x_1, x_2), (y_1, y_2), (y_2, y_3), (y_3, y_1), (x_1, y_1), (x_1, y_2), (x_1, y_3), \\ & (x_2, y_1), (x_2, y_2), (x_2, y_3)\}. \end{aligned}$$

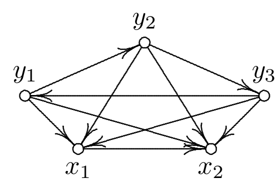

Figure 36: $P_2 + C_3$

We note that the adjacency matrix of $D_1 + D_2$, where D_1 and D_2 are digraphs with corresponding adjacency matrices $\mathcal{A}(D_1)$ and $\mathcal{A}(D_2)$ of orders n and m respectively is

$$\mathcal{A}(D_1 + D_2) = \begin{bmatrix} \mathcal{A}(D_1) & \mathbf{J_{n,m}} \\ \mathbf{O_{m,n}} & \mathcal{A}(D_2) \end{bmatrix},$$

where $\mathbf{J_{n,m}}$ is the $n \times m$ matrix of $1's$ and $\mathbf{O_{m,n}}$ is the $m \times n$ matrix of $0's$.

Remark 10.1. We will call the digraph $\vec{C}_3^*$ with

$$V(\vec{C}_3^*) = \{y_1, y_2, y_3\}; \quad A(\vec{C}_3^*) = \{(y_1, y_2), (y_2, y_3), (y_3, y_1)\},$$

the *circuit* of order 3. In general, the *circuit of order* n is digraph $\vec{C}_n^*$, with vertex set

$$V(\vec{C}_3^*) = \{y_1, y_2, \ldots, y_n\},$$

and arc set

$$A(\vec{C}_3^*) = \{(y_1, y_2), (y_2, y_3), \ldots, (y_i, y_{i+1}), \ldots, (y_n, y_1)\}.$$

10.2 Cartesian Product of Digraphs

Definition 10.2. Given digraphs D_1 and D_2, we define the *cartesian product* of D_1 and D_2 as the digraph $D_1 \times D_2$, with vertex set $V(D_1) \times V(D_2)$ and where (a, b) is adjacent to (c, d) if either $[a = c$ and $(b, d) \in A(D_2)]$ or $[b = d$ and $(a, c) \in A(D_1)]$.

Example 10.2. Consider the digraphs $\vec{P}_2^*$ and $\vec{C}_3^*$ as given in Example 10.1. The cartesian product of $\vec{P}_2^*$ and $\vec{C}_3^*$, $\vec{P}_2^* \times \vec{C}_3^*$ has as its vertex set

$$V(\vec{P}_2^* \times \vec{C}_3^*) = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_1), (x_2, y_2), (x_2, y_3)\}$$

and arc set

$$\begin{aligned} A(\vec{P}_2^* \times \vec{C}_3^*) \quad = \quad & \{((x_1, y_1), (x_1, y_2)), ((x_1, y_2), (x_1, y_3)), ((x_1, y_3), (x_1, y_1)), ((x_1, y_1), (x_2, y_1)), \\ & ((x_1, y_2), (x_2, y_2)), ((x_1, y_3), (x_2, y_3)), ((x_2, y_1), (x_2, y_2)), ((x_2, y_2), (x_2, y_3)), \\ & ((x_2, y_3), (x_2, y_1))\}. \end{aligned}$$

A pictorial representation of $\vec{P}_2^* \times \vec{C}_3^*$ is given in Figure 37.

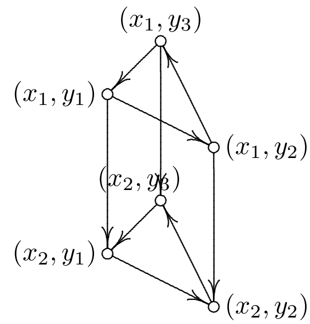

Figure 37: $P_2 \times C_3$

10.3 Composition of Digraphs

Definition 10.3. Let D_1 and D_2 be digraphs. The *composition* of D_1 and D_2, denoted by $D_1 \circ D_2$, is the digraph with $V(D_1 \circ D_2) = V(D_1) \times V(D_2)$ and there is an arc from vertex (a, b) to vertex (c, d) if and only if either (1.) $a = c$ and $(b, d) \in A(D_2)$ or (2.) $(a, c) \in A(D_1)$.

Example 10.3. Consider the circuit $\vec{C}_4^*$ with $V(\vec{C}_4^*) = \{x_1, x_2, x_3, x_4\}$ and the digraph $\vec{P}_2^*$ with $V(\vec{P}_2^*) = \{y_1, y_2\}$ and $A((\vec{P}_2^*) = \{(y_1, y_2)\}$. Then the composition, $\vec{C}_4^* \circ \vec{P}_2^*$ has vertex set

$$V(\vec{C}_4^* \circ \vec{P}_2^*) = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2), (x_4, y_1), (x_4, y_2)\}$$

and

$$\begin{aligned} A(\vec{C}_4^* \circ \vec{P}_2^*) \quad = \quad & \{((x_1, y_1), (x_1, y_2)), ((x_2, y_1), (x_2, y_2)), ((x_3, y_1), (x_3, y_2)), \\ & ((x_4, y_1), (x_4, y_2)), ((x_1, y_1), (x_2, y_1)), ((x_1, y_1), (x_2, y_2)), \\ & ((x_1, y_2), (x_2, y_1)), ((x_1, y_2), (x_2, y_2)), ((x_2, y_1), (x_3, y_1)), \\ & ((x_2, y_1), (x_3, y_2)), ((x_2, y_2), (x_3, y_1)), ((x_2, y_2), (x_3, y_2)), \\ & ((x_3, y_1), (x_4, y_1)), ((x_3, y_1), (x_4, y_2)), ((x_3, y_2), (x_4, y_1)), \\ & ((x_3, y_2), (x_4, y_2)), ((x_4, y_1), (x_1, y_1)), ((x_4, y_1), (x_1, y_2)), \\ & ((x_4, y_2), (x_1, y_1)), ((x_4, y_2), (x_1, y_2))\}. \end{aligned}$$

A pictorial representation of $\vec{C}_4^* \circ \vec{P}_2^*$ is given in Figure 38

10.4 Conjunction of Digraphs

The following definition was taken from [3].

Definition 10.4. Let D_1 and D_2 be digraphs. The *conjunction* of D_1 and D_2, denoted by $D_1 \wedge D_2$, is a digraph with $V(D_1 \wedge D_2) = V(D_1) \times V(D_2)$ and $E(D_1 \wedge D_2) = \{((x_1, x_2), (y_1, y_2)) \mid (x_1, y_1) \in E(D_1) \text{ and } (x_2, y_2) \in E(D_2)\}$.

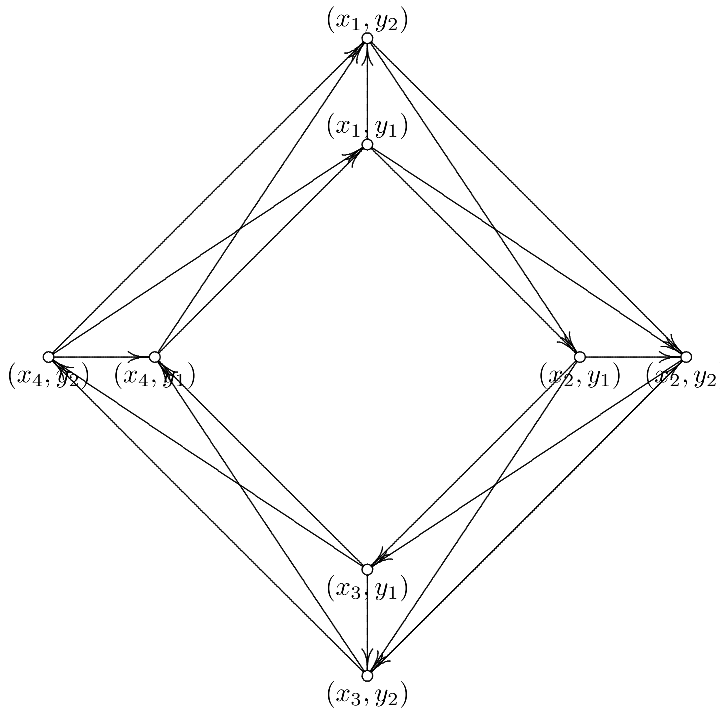

Figure 38: The composition $C_4 \circ P_2$

Example 10.4. Consider the circuit $\vec{C}_4^*$ with $V(\vec{C}_4^*) = \{x_1, x_2, x_3, x_4\}$ and the digraph $\vec{P}_2^*$ with $V(\vec{P}_2^*) = \{y_1, y_2\}$ and $A((\vec{P}_2^*) = \{(y_1, y_2)\}$. Then the conjunction, $\vec{C}_4^* \wedge \vec{P}_2^*$ has vertex set

$V(\vec{C}_4^* \circ \vec{P}_2^*) = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2), (x_4, y_1), (x_4, y_2)\}$

and

$A(\vec{C}_4^* \circ \vec{P}_2^*) = \{((x_1, y_1), (x_2, y_2)), ((x_2, y_1), (x_3, y_2)), ((x_3, y_1), (x_4, y_2)), ((x_4, y_1), (x_1, y_2))\}.$

A pictorial representation of $\vec{C}_4^* \circ \vec{P}_2^*$ is given in Figure 39

Observe that $\mathcal{A}(C_3 \wedge C_4)$ can be expressed as the Kronecker product of two matrices, that is,

$$\mathcal{A}(C_3 \wedge C_4) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \mathcal{A}(C_4) \otimes \mathcal{A}(C_3).$$

In general, if D_1 and D_2 are digraphs with adjacency matrices $\mathcal{A}(D_1)$ and $\mathcal{A}(D_2)$ respectively, then

$$\mathcal{A}(D_1 \wedge D_2) = \mathcal{A}(D_2) \otimes \mathcal{A}(D_1).$$

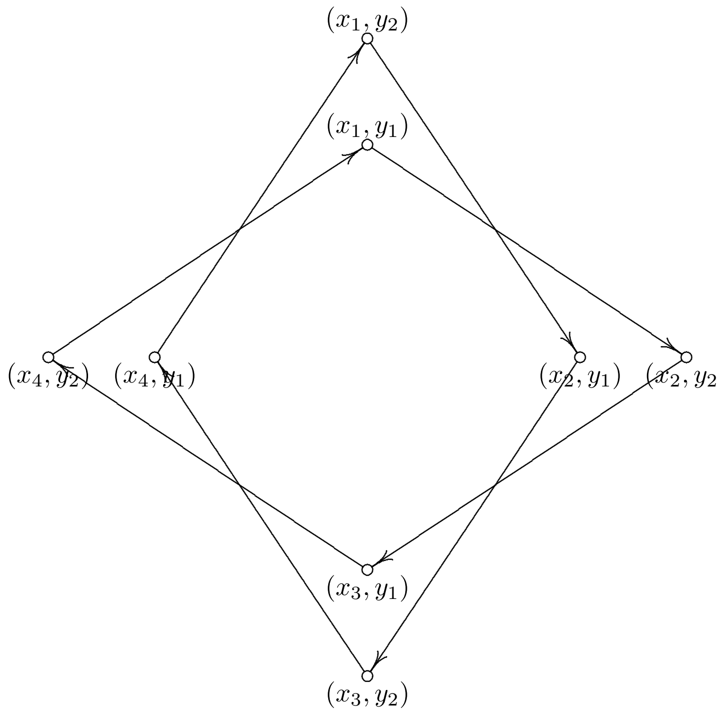

Figure 39: The conjunction $C_4 \wedge P_2$

11 Tournaments

Definition 11.1. An *orientation* of a graph $G = (V, E)$ is a mapping $\phi : E \to V \times V$ such that for each $[x, y] \in E$, either $\phi[x, y] = (x, y)$ or $\phi[x, y] = (y, x)$, but not both. The digraph $(V, \phi(E))$ is called an *orientation* of G or an oriented graph.

From this definition we can see that an oriented graph is a special type of a digraph. Moreover, the oriented graphs that we will consider are digraphs whose underlying graphs are simple. Also, it is possible that different orientations say, ϕ_1 and ϕ_2 of a graph give oriented graphs which are isomorphic to each other. In symbols, $(V, \phi_1(E)) \cong (V, \phi_2(E))$. If this situation occurs, we will consider the orientations to be equivalent, thus if $(V, \phi_1(E)) \cong (V, \phi_2(E))$, then ϕ_1 is equivalent to ϕ_2

Definition 11.2. A *tournament* is an orientation of a complete graph.

Any round-robin tournaments that do not allow draws can be represented by a tournament. Each player or team are represented by vertices and the arc for vertex x to y will indicate that in the game played by players x and y with x defeating y.

Definition 11.3. A tournament $T = (V(T), A(T))$ is said to be *transitive* if, whenever $(u, v) \in A(T)$ and $(v, w) \in A(T)$, then $(u, w) \in A(T)$.

Example 11.1. The tournament in Figure 40 is transitive.

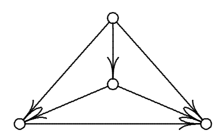

Figure 40: A transitive tournament

Theorem 11.1. *A tournament T is transitive if and only if T has no directed cycles.*

Proof:

$\{\Rightarrow\}$ Suppose T is transitive and has a directed cycle $C =< x_1, x_x, \ldots, x_k, x_1 >$, We note that $(x_k, x_1) \in A(T)$. If $k = 2$, then both (x_1, x_2) and (x_2, x_1) are arcs in T, which is not possible since T is a tournament. Thus, $k \geq 3$. Since in the directed cycle C, (x_1, x_2) and (x_2, x_3) are arcs in T, then (x_1, x_3), by transitivity is also an arc in T. Since (x_3, x_4) is in T, then (x_1, x_4) by transitivity is also an arc in T. Continuing in this manner, we can see that (x_1, x_k) is an arc in T. However, this is not possible since (x_k, x_1) is an arc in a tournament T.

$\{\Leftarrow\}$ Suppose T do not contain a directed cycle. We have to show that if (u, v) and (v, w) are arcs in T, then (u, w) is an arc in T If (w, u) is an arc in T, then the arcs $(u, v), (v, w)$ and (w, u) forms the directed cycle $< u, v, w, u >$. This contradicts the assumption and since T is an orientation of the complete graph, then (u, w) is an arc of T. Thus, T is transitive. $\square$

Theorem 11.2. *Every tournament has a spanning path.*

Proof: A spanning directed path is a directed path that contains all the vertices of T. Let $P =< x_1, x_2, \ldots, x_k >$ be the longest directed path in T. If P contains all the vertices in T, then P is a spanning directed path of T. If there exists another vertex in T not in P, say v, then the vertices (v, x_1) and (x_n, v) can not be in T, because otherwise there would be a path in T longer than P. However, since the underlying graph of T is a complete graph, then the arcs (x_1, v) and (v, x_n) must be in T. Thus, there is a vertex $v_i \in P$, such that (v_i, v) and (v, v_{i+j}) are both arcs in T. Then, we have a directed path $P' =< x_1, x_2, \ldots, x_i, v, x_{i+1}, \ldots, x_n >$. Clearly, P' is longer than P. This is a contradiction to our assumption. $\square$

Theorem 11.3. (Moon's Theorem) *Every strong tournament with order n has a circuit of length p, for $p = 3, 4, \ldots, n$.*

Corollary 11.3.1. *A tournament is strong if and only if it has a spanning cycle.*

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