LECTURE NOTES IN ELEMENTARY LINEAR ALGEBRA

1 Linear Equations and Matrices

1.1 Systems of Linear Equations

Definition 1.1. The equation

$$b = a_1x_1 + a_2x_2 + \ldots + a_nx_n \tag{1}$$

which expresses b in terms of the variables $x_1, x_2, \ldots, x_n$ and the constants $a_1, a_2, \ldots, a_n$ is called a **linear equation**. The solution to this linear equation is the sequence $s_1, s_2, \ldots, s_n$ such that Equation (1) is satisfied if $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ are substituted.

Definition 1.2. A **linear system** of m linear equations and n unknowns is a set of m linear equations each in n unknowns. This is written as

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \end{array}$$

A solution to the system above is a sequence $s_1, s_2, \ldots, s_n$ such that each equation in the system is satisfied if $x_1, x_2, \ldots, x_n$ are substituted respectively by $s_1, s_2, \ldots, s_n$.

If the system above has no solution it is called **inconsistent**, otherwise it is **consistent**. A special case of the linear system above is when $b_1 = b_2 = \ldots = b_n = 0$. This is called the **homogeneous system**. A solution to the homogenous system is when $x_1 = x_2 = \ldots = x_n = 0$. This solution is called the **trivial solution**. Another possible solution to the homogenous systems is when at least one of the x_i's is nonzero. This solution is called a **nontrivial solution**.

Definition 1.3. Two linear systems are equivalent if they have exactly the same solutions.

1.2 Matrices

Definition 1.4. An $m \times n$ **matrix** is a rectangular array of numbers arranged in m rows and n columns, denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

The element a_{ij} of A is called the (i, j) entry or (i, j)th element of A. This is the entry in the *ith* row and *jth* column of A. The matrix A can also be denoted by $[a_{ij}]$. If $m = n$, then A is called a **square matrix** of order n. The entries a_{ii}, $i = 1, 2, \ldots, n$ are called the **main diagonal entries** of the square matrix A.

Definition 1.5. Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if $a_{ij} = b_{ij}$, for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Matrix Operations

Definition 1.6. (Matrix Addition) If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices then $A + B = C$, where $C = [c_{ij}]$ is another $m \times n$ matrix defined by

$$c_{ij} = a_{ij} + b_{ij}.$$

Definition 1.7. (Scalar Matrix) If $A = [a_{ij}]$ is an $m \times n$ matrix and r is a real number, then the **scalar multiple** of A by r, is an $m \times n$ matrix $C = [c_{ij}]$ defined by $c_{ij} = ra_{ij}$.

Definition 1.8. (Matrix Multiplication) If $A = [a_{ij}]$ is an $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix, then the product of A by B, $AB = C = [c_{ij}]$, is an $m \times n$ matrix where for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$.

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

Definition 1.9. (Transpose of a Matrix) If $A = [a_{ij}]$ is an $m \times n$ matrix, then the **transpose** of A, $A' = [a'_{ij}]$, is an $n \times m$ matrix, where

$$a'_{ij} = a_{ji}.$$

1.3 Algebraic Properties of Matrix Operations

Theorem 1.1. *Let A and B be $m \times n$ matrices then*

$$A + B = B + A.$$

Theorem 1.2. *If A, B and C are $m \times n$ matrices, then*

$$A + (B + C) = (A + B) + C.$$

Theorem 1.3. *There exists a unique $m \times n$ matrix $\mathbf{O}$ such that*

$$A + \mathbf{O} = \mathbf{O} + A = A,$$

for any $m \times n$ matrix A.

The matrix $\mathbf{O}$ is called the **zero matrix**.

Theorem 1.4. *Given any $m \times n$ matrix A, there exists a unique $m \times n$ matrix B, such that $A + B = \mathbf{O}$.*

The matrix B above is called the **negative** of A, denoted by $-A$.

Theorem 1.5. *If A is an $m \times n$ matrix, B is an $n \times p$ matrix and C is a $p \times q$ matrix, then*

$$A(BC) = (AB)C.$$

Theorem 1.6. *1. If A and B are $m \times n$ matrices and C is an $n \times p$ matrix, then*

$$(A + B)C = AC + BC.$$

2. If C is an $m \times n$ matrix and A and B are $n \times p$ matrices, then

$$C(A + B) = CA + CB.$$

Theorem 1.7. *Let $r, s \in \mathbb{R}$, A be an $m \times n$ matrix and B be $n \times p$ matrix, then*

1. *$r(sA) = (rs)A = s(rA)$;*

2. *$A(rB) = r(AB)$.*

Theorem 1.8. *Let $r, s \in \mathbb{R}$ and A be an $m \times n$ matrix. Then,*

$$(r + s)A = rA + sA.$$

Theorem 1.9. *Let $r \in \mathbb{R}$, A and B be $m \times n$ matrices. Then,*

$$r(A + B) = rA + rB.$$

Theorem 1.10. *If A is an $m \times n$ matrix, then $(A')' = A$.*

Theorem 1.11. *1. If $r \in \mathbb{R}$ and A is an $m \times n$ matrix, then $(rA)' = rA'$.*

2. If A and B are $m \times n$ matrices, then $(A + B)' = A' + B'$.

3. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then $(AB)' = B'A'$.

1.4 Special Types of Matrices

Definition 1.10. An $n \times n$ matrix $A = [a_{ij}]$ is called a **diagonal matrix** if $a_{ij} = 0$ whenever $i \neq j$.

Definition 1.11. A **scalar matrix** is a diagonal matrix whose main diagonal elements are equal.

Definition 1.12. The $n \times n$ scalar matrix $A = [a_{ij}]$, where $a_{ii} = 1$ is called the **identity matrix** of order n. This matrix is denoted by I_n.

We note the following:

1. If A is an $m \times n$ matrix, then $I_m A = A$ and $AI_n = A$.

2. If A is a scalar matrix for some scalar r, then $A = rI_n$.

Also, if A is an $n \times n$ matrix and p, q are nonnegative integers, then

1. $A^0 = I_n$;

2. $A^p = A \cdot A \cdot \ldots \cdot A$ (A multiplied p times with itself);

3. $A^p A^q = A^{p+q}$

4. $(A^p)^q = A^{pq}$;

5. $(AB)^p = A^p B^p$, only when $AB = BA$.

Definition 1.13. Let $A = [a_{ij}]$ be an $n \times n$ matrix.

1. If $a_{ij} = 0$, whenever $i > j$, then A is **upper triangular** matrix;

2. If $a_{ij} = 0$, whenever $i < j$, then A is **lower triangular** matrix;

Definition 1.14. A matrix A is **symmetric** if $A' = A$.

Definition 1.15. A matrix A is **skew symmetric** if $A' = -A$.

Theorem 1.12. *If A is an $n \times n$ matrix, then $A = S + K$, where S is symmetric and K is skew symmetric. Moreover, this decomposition is unique.*

Definition 1.16. A **submatrix** of A is a matrix obtained after deleting some of the rows and/or columns of A.

1.5 Nonsingular Matrices

Definition 1.17. An $n \times n$ matrix A is called **nonsingular** or **invertible** if there exists an $n \times n$ matrix B such that

$$AB = I_n = BA.$$

Otherwise, A is called **singular** or **noninvertible**. The matrix B is called the **inverse** of A and is denoted by A^{-1}.

Theorem 1.13. *The inverse of a matrix, if it exists is unique.*

Theorem 1.14. *If A and B are both nonsingular $n \times n$ matrices, then AB is nonsingular and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Corollary 1.14.1. *If $A_1, A_2, \ldots, A_n$ are $n \times n$ nonsingular matrices, then*

$$(A_1 A_2 \cdots A_n)^{-1} = (A_n)^{-1}(A_{n-1})^{-1} \cdots (A_2)^{-1}(A_1)^{-1}.$$

Theorem 1.15. *If A is nonsingular, then A^{-1} is nonsingular and*

$$(A^{-1})^{-1} = A.$$

Theorem 1.16. *If A is a nonsingular matrix, then A' is nonsingular and $(A')^{-1} = (A^{-1})'$.*

1.6 Echelon Form of a Matrix

Definition 1.18. An $m \times n$ matrix A is said to be in **reduced row echelon form(RREF)** if it satisfies the following properties:

1. All rows consisting entirely of zeros, if any, are at the bottom of the matrix.

2. The first nonzero entry in each row that does not consists entirely of zeros is a 1, called the *leading entry* of its row.

3. If rows i and $i+1$ are two successive rows that do not consists entirely of zeros, then the leading entry of row $i+1$ is to the right of the leading entry of row i.

4. If column contains a leading entry of some row, then all other entries in that column are zeros.

If matrix A satisfies only conditions (1), (2) and (3) above, then A is said to be in row echelon form. A similar definition can be given to define reduced column echelon form and column echelon form.

Definition 1.19. An **elementary row(column) operation** on a matrix A is any one of the following operations:

1. **Type I**: Interchange rows (columns) i and j, $i \neq j$ of A;

2. **Type II**: Multiply row (column) i of A by a nonzero constant c;

3. **Type III**: Add c, $c \neq 0$ times row (column) i of A to row (column) j of A, $i \neq j$.

Definition 1.20. An $m \times n$ matrix A is said to be **row(column) equivalent** to an $m \times n$ matrix B, if B can be obtained by applying a finite sequence of elementary row operations to A.

We note the following:

1. Every matrix is row(column) equivalent to itself.

2. If A is row(column) equivalent to B, then B is row(column) equivalent to A.

Theorem 1.17. *Every nonzero matrix $m \times n$ matrix A is row(column) equivalent to a matrix in reduced row (column) echelon form.*

Procedure in transforming a matrix in RREF:

1. Find the first (counting from left to right) column in A not all of whose entries are zeros. This column is called the *pivotal column.*

2. Identify the first (counting from top to bottom) nonzero entry in the pivotal column. This entry is called the *pivot* and the row where the pivot is located is called the *pivotal row.*

3. Perform a Type I row operation on A by interchanging the pivotal row and the first row. Call this new matrix A_1
4. Perform a Type II row operation on A_i, by multiplying the first row of A_1 by the reciprocal of the pivot. Thus the leading entry of the first row is now a 1. Call this new matrix A_2.
5. Perform a Type III row operation by adding multiples of the first row of A_2 to all the other row to make all the entries in the pivotal column, except the entry where the pivot is located, equal to zero. Call this new matrix A_3.
6. Identify matrix B as the $(m-1) \times n$ submatrix of A_3, by ignoring the first row of A_3. Repeat step 1 to 5 above, to B until a matrix in RREF is obtained.

Example 1.1. Find a matrix in rref that is row equivalent to the matrix

$$A = \begin{bmatrix} 5 & 9 & 1 & 6 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 1 & 2 & -1 & 1 \end{bmatrix}.$$

Theorem 1.18. *Let $AX = B$ and $CX = D$ be two linear systems each with m equations in n unknowns. If the augmented matrices $[A|B]$ and $[C|D]$ are row equivalent, then the linear systems are row equivalent, that is, they have exactly the same solutions.*

Corollary 1.18.1. *If the $m \times n$ matrices A and B are row equivalent, the homogeneous systems $AX = O$ and $BX = O$ are equivalent.*

1.7 Solving linear systems

Example 1.2. Solve the following systems of linear equations using the Gauss-Jordan reduction method.

1. $$\begin{array}{ccccccc} x & + & y & + & z & = & 0 \\ x & + & y & & & = & 3 \\ & & y & + & z & = & 1 \end{array}$$

2. $$\begin{array}{ccccccccc} w & + & 2x & + & 3y & + & z & = & 8 \\ w & + & 3x & + & & & z & = & 7 \\ w & + & 2x & + & & & z & = & 3 \end{array}$$

3. $$\begin{array}{ccccccc} x & - & 2y & + & 3z & = & 4 \\ 2x & - & y & - & 3z & = & 5 \\ 3x & + & & & z & = & 2 \\ 3x & - & 3y & & & = & 7 \end{array}$$

Theorem 1.19. *A homogeneous system $AX = O$ of m linear equations in n unknowns always has a nontrivial solution if $m < n$.*

Example 1.3. Find the solution set of the homogeneous system:

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0 \\ x_1 & & & & & + & x_4 & = & 0. \\ x_1 & + & 2x_2 & + & x_3 & & & = & 0 \end{array}$$

1.8 Elementary Matrices and Finding A^{-1}

Definition 1.21. An $n \times n$ **elementary matrix of Type I, Type II or Type III** is a matrix obtained from the identity matrix I_n by performing a single elementary row or elementary column operation of Type I, Type II or Type III respectively.

Theorem 1.20. *Let A be an $m \times n$ matrix and let an elementary row (column) operation of type I, type II, or type III be performed on A to yield B. Let E be the elementary matrix obtained from I_m (I_n) by performing the same elementary row(column) operation as was performed on A. Then, $B = EA$ ($B = AE$).*

Theorem 1.21. *If A and B are $m \times n$ matrices, then A is row (column) equivalent to B if and only if $B = E_k E_{k-1} \cdots E_2 E_1 A$ ($E = AE_1 E_2 \cdots E_{k-1} E_k$), where $E_1, E_2, \ldots, E_k$ are elementary matrices.*

Theorem 1.22. *An elementary matrix E is nonsingular and its inverse is an elementary matrix of the same type.*

Lemma 1.23. *Let A be an $m \times n$ matrix and let the homogeneous system $AX = O$ have only the trivial solution $X = O$. Then, A is row equivalent to I_n.*

Theorem 1.24. *A is nonsingular if and only if A is a product of elementary matrices.*

Corollary 1.24.1. *A is nonsingular if and only if A is row equivalent to I_n.*

Theorem 1.25. *The homogeneous system of n linear equations in n unknowns $AX = O$ has a nontrivial solution if and only if A is singular.*

The following statements are equivalent for an $n \times n$ matrix.

1. A is nonsingular;
2. $AX = O$ has only the trivial solution;
3. A is row(column) equivalent to I_r;
4. The linear system $AX = B$ has a unique solution for any n matrix B.
5. A is a product of elementary matrices.

Example 1.4. Find the inverse of the following matrices, if it exists:

1. $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 1 \\ 0 & -2 & 4 \end{bmatrix}$. 2. $A = \begin{bmatrix} 3 & 0 & -6 \\ 1 & 1 & 1 \\ -2 & 0 & 4 \end{bmatrix}$.

2

Theorem 1.26. *An $n \times n$ matrix A is singular if and only if A is row equivalent to a matrix B that has a row zeros.*

Theorem 1.27. *If A and B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$. Thus, $B = A^{-1}$.*

PROBLEM SET NO. 1

Answer the following questions completely.

1. Given the matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}; C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix};$$

$$D = \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}; E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}; F = \begin{bmatrix} -1 & 2 \\ 0 & 4 \\ 3 & 5 \end{bmatrix}.$$

   Evaluate the following matrices, if possible,

   (a) $DA + B'$;
   (b) EC; CE;
   (c) $EB + F$; $FC + D$
   (d) $A(BD)$; $(AB)D$;
   (e) $A(C - 3E)$

2. Consider the following linear system:

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & 3x_2 & - & 3x_3 & + & x_4 & + & x_5 & = & 7 \\ 3x_1 & & & + & 2x_3 & & & + & 3x_5 & = & -2 \\ 2x_1 & + & 3x_2 & & & - & 4x_4 & & & = & 3 \\ & & & & x_3 & + & x_4 & + & x_5 & = & 5 \end{array}.$$

   (a) Write this linear system in matrix form;
   (b) What is the coefficient matrix?;
   (c) Find the augmented matrix.

3. Write the following linear system in matrix form

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

4. If $A = [a_{ij}]$ is an $n \times n$ matrix, then the **trace** of A, denoted by $Tr(A)$ is defined as the sum of the main diagonal elements of A, that is, $Tr(A) = \sum_{i=1}^{n} a_{ii}$. For example, with matrix C in question no. 1 above, $Tr(C) = 3 + 1 + 3 = 7$ and $Tr(E) = 2 + 1 + 1 = 4$. Let $c \in \mathbb{R}$. Prove the following:

(a) $Tr(cA) = cTr(A)$; (d) $Tr(A^T) = Tr(A)$;
(b) $Tr(A + B) = Tr(A) + Tr(B)$;
(c) $Tr(AB) = Tr(BA)$; (e) $Tr(A^T A) \geq 0$.

5. Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

   (a) Determine a simple expression for A^2; A^3.

   (b) Conjecture the form of a simple expression for A^k, k is a positive integer. Then, prove this conjecture.

   (c) Determine the reduced row echelon form of A.

6. Let
$$A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}; B = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}; C = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}.$$

   (a) Find $(AB^T)C$; $(B^T A)C$.

   (b) Explain why $(AB^T)C = (B^T A)C$.

7. Let X_1 and X_2 be solutions of the homogeneous linear system $AX = O$. Show that $rX_1 + sX_2$ is a solution, where $r, s \in \mathbb{R}$.

8. Let
$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 6 & -3 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

   We note that both A and B are upper triangular matrices. Verify that $A + B$ and AB are both upper triangular matrices too.

9. Let A be any $m \times n$ matrix. Show that AA^T and $A^T A$ are symmetric.

10. Let A be a diagonal matrix with nonzero main diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$. Prove that A^{-1} exists and that it is also a diagonal matrix with main diagonal entries, $1/a_{11}, 1/a_{22}, \ldots, 1/a_{nn}$.

11. Given the linear system
$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & 2x_3 & + & 3x_4 & = & 13 \\ x_1 & - & 2x_2 & + & x_3 & + & x_4 & = & 8 \\ 3x_1 & + & x_2 & + & x_3 & - & x_4 & = & 1 \end{array}.$$

   Find the solution set of this system using the Gauss-Jordan reduction method.

2 Real Vector Spaces

2.1 Vector Spaces and Subspaces

Definition 2.1. A nonempty set V together with two defined operations $\oplus$ and $\odot$ is a **vector space** if it satisfies the following properties:

1. Under vector addition $\oplus$.
   (a) Closure Property: $\alpha \oplus \beta \in V$, $\forall \alpha, \beta \in V$
   (b) Commutativity Property: $\alpha \oplus \beta = \beta \oplus \alpha$, $\forall \alpha, \beta \in V$.
   (c) Associativity Property: $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$, $\forall \alpha, \beta, \gamma \in V$.
   (d) Existence of the zero vector: $\forall \alpha \in V$, $\exists \theta \in V$, such that $\alpha \oplus \theta = \alpha = \theta \oplus \alpha$.
   (e) Existence of the Inverse under $\oplus$: $\forall \alpha \in V, \exists \beta \in V$ such that $\alpha \oplus \beta = \theta$.
2. Under scalar multiplication $\odot$:
   (a) Closure: $r \odot \alpha \in V$, $\forall r \in \mathbb{R}, \forall \alpha \in V$.
   (b) $r \odot (\alpha \oplus \beta) = (r \odot \alpha) \oplus (r \odot \beta)$, $\forall r \in \mathbb{R}, \forall \alpha, \beta \in V$.
   (c) $(r + s) \odot \alpha = (r \odot \alpha) \oplus (s \odot \alpha)$ $\forall r, s \in \mathbb{R}, \forall \alpha \in V$.
   (d) $r \odot (s \odot \alpha) = (rs) \odot \alpha$, $\forall r, s \in \mathbb{R}, \forall \alpha \in V$.
   (e) $1 \odot \alpha = \alpha$, $\forall \alpha \in V$.

Example 2.1. Let $V = \mathbb{R}$ and vector addition is ordinary addition of real numbers and scalar multiplication is ordinary multiplication of real numbers is a vector space.

Example 2.2. Let V be the set of all $m \times n$ matrices, $M_{m,n}$ and vector addition is matrix addition and scalar multiplication is scalar multiplication of matrices is a vector space.

Example 2.3. Let $V = P_n$, is the set of all polynomials of degree less than or equal to n. Define $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $g(x) = b_0 + b1x + b_2x^2 + \ldots + a_nx^n$. Let

$$f(x) \oplus g(x) = f(x) + g(x) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+\ldots+(a_n+b_n)x^n,$$

and

$$r \odot f(x) = rf(x) = (ra_0) + (ra_1)x + (ra_2)x^2 + \ldots + (ra_n)x^n.$$

This is a vector space.

Example 2.4. Let $V = \mathbb{R}^3$. Let vector addition be defined by

$$(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2),$$

and scalar multiplication is defined by

$$r \odot (x, y, z) = (x, 1, z).$$

This is NOT a vector space because 2b, 2c and 2e are not satisfied.

Example 2.5. Let $V = \mathbb{R}^+$. Let vector addition be defined as

$$\alpha \oplus \beta = \alpha\beta,$$

and scalar multiplication be defined by

$$r \odot \alpha = \alpha^r.$$

Is this a vector space?

Theorem 2.1. *If V is a vector space, then*

1. $0 \odot \alpha = \theta, \forall \alpha \in V$;
2. $r \odot \theta = \theta, \forall r \in \mathbb{R}$;
3. *If* $r \odot \alpha = \theta$, *then either* $r = 0$ *or* $\alpha = \theta$;
4. $(-1) \odot \alpha = -\alpha, \forall \alpha \in V$.

2.2 Subspaces

Definition 2.2. Let V be a vector space. Let W be a subset of V. If W is a vector space with respect to the operations defined in V, then W is called a **subspace** of V.

Remark 2.1. The set V and the set $\{\theta\}$ are subspaces of any vector space V. These are called the **trivial subspaces** of V.

Theorem 2.2. *Let V be a vector space with operations $\oplus$ and $\odot$, and let W be a nonempty subset of V. Then, W is a subspace of V if and only if the following conditions hold:*

1. $\alpha \oplus \beta \in W, \forall \alpha, \beta \in W$;
2. $r \odot \alpha \in W, \forall r \in \mathbb{R}, \forall \alpha \in W$.

Example 2.6. Let $V = \mathbb{R}^3 = \{(a, b, c) | a, b, c \in \mathbb{R}\}$. Let $W = \{(a, b, 0) | a, b \in \mathbb{R}\}$. Clearly, $W \subseteq V$. Moreover, W is a subspace of V.

Example 2.7. Let $V = \mathbb{R}^3 = \{(a, b, c) | a, b, c \in \mathbb{R}\}$. Let $W = \{(a, b, 1) | a, b \in \mathbb{R}\}$. Clearly, $W \subseteq V$. However, W is NOT a subspace of V.

Example 2.8. Let $W \subseteq \mathbb{R}_n$ consisting of all solutions of the homogeneous system $AX = 0$, where A is $m \times n$. Thus W consists of all $n \times 1$ matrix X such that $AX = 0$ is satisfied. Verify if W is a subspace of V.

Remark 2.2. The solution set $AX = B$, where $B \neq 0$ is NOT a subspace of $V = \mathbb{R}_n$.

Remark 2.3. From now on, we will denote $\alpha \oplus \beta$ by $\alpha + \beta$ and $r \odot \alpha$ by $r\alpha$.

A simple way of constructing subspaces in a vector space is as follows: Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of n vectors in V. Let W be set of vectors α in V of the form:

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n,$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$. Then W is a subspace of V. We denote W as *span* S.

2.3 Linear Independence

Definition 2.3. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of vectors in a vector space V. A vector $\alpha \in V$ is a **linear combination** of the vectors in S if

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n,$$

for some real numbers $a_1, a_2, \ldots, a_n$.

Example 2.9. In $V = \mathbb{R}^4$, let $\alpha_1 = (1, 2, 1, -1)$; $\alpha_2 = (1, 0, 2, -3)$ and $\alpha_3 = (1, 1, 0, -2)$. Is the vector $\alpha = (2, 1, 5, -5)$ a linear combination of α_1, α_2 and α_3?

Example 2.10. In $V = \mathbb{R}^3$, let $\alpha_1 = (1, 2, -1)$ and $\alpha_2 = (1, 0, -1)$. Is the vector $\alpha = (1, 0, 2)$ a linear combination of α_1 and α_2?

Definition 2.4. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a set of vectors in a vector space V. The set S **spans** V or V is **spanned by** S if every vector in V is a linear combination of the vectors in S. The set S is called a **spanning set** of V.

Example 2.11. Let $V = \mathbb{R}^3$. Let $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 3)$ and $\alpha_3 = (1, 2, 0)$. Is the set $S = \{\alpha_1, \alpha_2, \alpha_3\}$ a spanning set of $V = \mathbb{R}^3$?

Example 2.12. Let $V = P_2$. Do the set $S = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = x^2 + 2x + 1$ and $\alpha_2 = x^2 + 2$ span $V = P_2$?

Definition 2.5. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of distinct vectors in a vector space V. Then, S is said to be **linearly dependent** if there exists constants, $a_1, a_2, \ldots, a_k$ not all zero such that

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = \theta. \tag{2}$$

Otherwise, S is **linearly independent**. That is, S is linearly independent if (2) holds only when

$$a_1 = a_2 = \ldots = a_n = 0.$$

Remark 2.4. (2) is always satisfied if we choose $a_1 = a_2 = \ldots = a_k = 0$. What is important then is find out if there exists scalars $a_1, a_2, \ldots, a_k$ not all of which are zeros such that (2) is satisfied.

Example 2.13. Let $V = \mathbb{R}^3$ and $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Is S linearly independent?

Remark 2.5. If $V = \mathbb{R}^n$ and

$$S = \{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)\},$$

where if $\alpha_i \in S$, then the ith component of α_i is 1 and is zero elsewhere. This set S is linearly independent.

Example 2.14. Let $V = P_2$, $S = \{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = x^2 + x + 2$, $\alpha_2 = 2x^2 + x$ and $\alpha_3 = 3x^2 + 2x + 2$. Is S linearly independent?

Theorem 2.3. *Let S_1 and S_2 be finite subsets of a vector space V and $S_1 \subseteq S_2$. Then,*

1. *If S_1 is linearly dependent, then so is S_2;*
2. *If S_2 is linearly independent, then so is S_1.*

Remark 2.6. If $S = \{\theta\}$, then S is linearly dependent. If $S = \{\alpha\}$, where $\alpha \neq \theta$, then S is linearly independent.

Theorem 2.4. *Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of nonzero vectors in a vector space V. Then, S is linearly dependent if and only if one of the vectors α_j is a linear combination of the preceding vectors in S.*

Remark 2.7. It can also be shown that S is linearly dependent if and only if there is a vector $\alpha \in S$ which is a linear combination of all the other vectors in S.

2.4 Basis and Dimension

Definition 2.6. A set of vectors $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is called a **basis** for V if

1. S spans V;
2. S is linearly independent.

Example 2.15. Let $V = \mathbb{R}^3$ and $S = \{(1,0,0), (0,1,0), (0,0,1)\}$. Is S a basis for $\mathbb{R}^3$?

Example 2.16. Let $V = P_2$ and $S = \{x^2, x, 1\}$. Is S a basis for V?

Example 2.17. Let $V = M_{2,2}$ and

$$S = \{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\}.$$

Is S a basis for V?

Theorem 2.5. *If $S = \{\alpha_2, \alpha_2, \ldots, \alpha_k\}$ is a basis for a vector space V, then every vector $\alpha \in V$, can be written in one and only one way as a linear combination of the vectors in S.*

Theorem 2.6. *If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a set of nonzero vectors spanning a vector space V, then S contains a basis T for V.*

Remark 2.8. Thm. 2.6 guarantees the existence of a subset T of a spanning set S of V which is a basis for V. To find this subset T of S, we do the following:

1. Form the equation $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = \theta$.
2. Form the augmented matrix associated with the equation above and transform this to RREF.

3. The vectors corresponding to the columns with the leading $1's$ form a basis for V.

Example 2.18. Let W be a subspace of $\mathbb{R}^3$ spanned by

$$S = \{(1,2,2),(3,2,1),(11,10,7),(7,6,4)\}.$$

Find a basis for W.

Theorem 2.7. *If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis for a vector space V and $T = \{\beta_1, \beta_2, \ldots, \beta_r\}$ be a linearly independent set of vectors in V, then $r \leq n$.*

Corollary 2.7.1. *If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $T = \{\beta_1, \beta_2, \ldots, \beta_r\}$ are bases for the vector space V, then $r = n$.*

Definition 2.7. The **dimension** of a nonzero vector space V, denoted by $\dim V$ is the number of vectors in a basis for V. The dimension of the set $\{\theta\}$ is zero.

Example 2.19. Find the dimension of the subspace W of $\mathbb{R}^4$ defined by

$$W = \{(a+b, a-b, b+c, -a+b) | a, b, c \in \mathbb{R}\}.$$

Example 2.20. Find a basis for the solution set of the homogenous system:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 3 & 0 & 3 & 1 \\ 1 & 1 & 2 & 2 & 1 \\ 3 & 5 & 0 & 6 & 2 \\ 2 & 3 & 2 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Remark 2.9. Let A be an $m \times n$ matrix. The **nullity** of A is the dimension of the null space of A, that is, the dimension of the solution space of the homogeneous system $AX = O$.

Corollary 2.7.2. *If a vector space V has dimension n, then the largest independent set of vectors in V contains n vectors and is a basis for V.*

Corollary 2.7.3. *If a vector space V has dimension n, then the smallest set of vectors in V that spans V contains n vectors and is a basis for V.*

Corollary 2.7.4. *If a vector space V has dimension n, then any set of $m > n$ vectors in V must be linearly dependent.*

Corollary 2.7.5. *If a vector space V has dimension n, then any set of $m < n$ vectors in V can not span V.*

Remark 2.10. If a vector space V has a basis consisting of a finite number of vectors, then V is called a **finite-dimensional vector space**. Otherwise, V is called an **infinite dimensional vector space**.

Theorem 2.8. *If S is a linearly independent set of vectors in a finite-dimensional vector space V, then there is basis T for V, which contains S.*

Example 2.21. Find a basis for $\mathbb{R}^4$, that includes the vectors in S, where

$$S = \{(1, 0, 1, 0), (0, 1, -1, 0)\}.$$

Theorem 2.9. *Let V be an n-dimensional vector space.*

1. *If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly independent, then S is a basis for V;*
2. *If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ spans V, then S is a basis for V;*

Definition 2.8. Let S be a set of vectors in a vector space V. A subset T of S is called a **maximal independent subset** of S if T is a linearly independent set of vectors in S and if there is no linearly independent subset of S having more vectors than T does.

Theorem 2.10. *Let S be a finite subset of a vector space V that spans V. A maximal independent subset T of S is a basis for V.*

2.5 Rank of a Matrix

Definition 2.9. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix},$$

be an $m \times n$ matrix. The rows of A considered as vectors in $\mathbb{R}^n$ span a subspace of $\mathbb{R}^n$ called the **row space** of A. Similarly, the columns of A, considered as vectors in $\mathbb{R}^m$ span a subspace of $\mathbb{R}^m$ called the **column space** of A.

Theorem 2.11. *If A and B are two $m \times n$ row (column) equivalent matrices, then the row (column) spaces of A and B are equal.*

Example 2.22. Find a basis for the subspace W of rn^5 that is spanned by $S = \{\alpha_1, \alpha_2, \alpha_3\}$, where

$$\alpha_1 = (1, 2, -1, 0, 1); \alpha_2 = (2, 1, -1, 1, 1); \alpha_3 = (3, 4, 2, 2, 1).$$

Definition 2.10. The dimension of the row(column) space of A is called the **row(column) rank** of A.

Theorem 2.12. *The row rank and the column rank of an $m \times n$ matrix A are equal.*

Since the row rank and the column rank of the matrix A are equal, we will now refer to it as the **rank** of the matrix A.

Theorem 2.13. *If A is an $m \times n$ matrix, then rank A + nullity $A = n$.*

Theorem 2.14. *If A is an $n \times n$ matrix, then rank $A = n$ if and only if A is row equivalent to I_n.*

Corollary 2.14.1. *Let A be an $n \times n$ matrix. A is nonsingular if and only if rank $A = n$.*

Corollary 2.14.2. *Let A be an $n \times n$ matrix. The homogeneous system $AX = O$ has a nontrivial solution if and only if rank $A < n$.*

Corollary 2.14.3. *Let A be an $n \times n$ matrix. The linear system $AX = B$ has a unique solution for every $n \times 1$ matrix B if and only if rank $A = n$.*

Remark 2.11. The following statements are equivalent:

1. A is nonsingular;
2. $AX = O$ has only the trivial solution;
3. A is row(column) equivalent to I_r;
4. The linear system $AX = B$ has a unique solution for any n matrix B.
5. A is a product of elementary matrices.
6. A has rank n.
7. The nullity of A is zero.
8. The rows of A form a linearly independent set of vectors in $\mathbb{R}^n$.
9. The columns of A form a linearly independent set of vectors in $\mathbb{R}^n$.

PROBLEM SET NO. 2

Answer the following questions completely.

1. Let V be the set of all positive real numbers. Define $\alpha \oplus \beta = \alpha\beta - 1$ and $c \odot \alpha = \alpha$. For example, $5 \oplus 6 = (5)(6) - 1 = 29$ and $3 \odot 7 = 7$. Is V a vector space or not? Support your answer.
2. Let $V = \mathbb{R}$. Define $\alpha \oplus \beta = 2\alpha - \beta$ and $c \odot \alpha = c\alpha$. For example, $3 \oplus 5 = 2(3) - 5 = 1$ and $3 \odot 5 = 3(5) = 15$. Is V a vector space or not? Support your answer.
3. Let $V = M_{n,n}$, the set of all $n \times n$ matrix. Let W be the set of all diagonal $n \times n$ matrices. Is W a subspace of V? Support your answer.
4. Let $V = M_{n,n}$, the set of all $n \times n$ matrix. Let W be the set of all upper triangular $n \times n$ matrices. Is W a subspace of V? Support your answer.
5. Find a basis for the subspace W of $M_{3,3}$ consisting of all symmetric matrices. What is the $\dim W$?
6. Find a basis for the subspace W of $M_{3,3}$ consisting of all diagonal matrices. What is the $\dim W$?

7. Find a basis for the subspace $W = (a+c, a-b, b+c, -a+b)|a, b, c \in \mathbb{R}\}$ of $V = \mathbb{R}^4$. What is $\dim W$?

8. Suppose $S = \{\alpha_1, \alpha_2, \alpha_3\}$ is a linearly independent set of vectors in a vector space V. Prove that $T = \{\beta_1, \beta_2, \beta_3\}$ is also linearly independent set if $\beta_1 = \alpha_1 + \alpha_2$, $\beta_2 = \alpha_1 + \alpha_3$ and $\beta_3 = \alpha_2 + \alpha_3$.

9. Let A be an $n \times n$ matrix and λ be a scalar. Show that the set W consisting of all vectors $\alpha \in \mathbb{R}^n$ such that $A\alpha = \lambda\alpha$ is a subspace of $\mathbb{R}^n$.

10. Let W_1 and W_2 be subspaces of a vector space V. Let $W_1 + W_2$ be the set of all vectors $\alpha \in V$ such that $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$. Show that $W_1 + W_2$ is a subspace of V.

3 Linear Transformation and Matrices

3.1 Definition and Examples

Definition 3.1. Let W and V be vector spaces. A function $L : V \to W$ is called a **linear transformation** of V into W if

1. $L(\alpha + \beta) = L(\alpha) + L(\beta)$, $\forall \alpha, \beta \in V$;

2. $L(r\alpha) = rL(\alpha)$, $\forall \alpha \in V$ and $\forall r \in \mathbb{R}$.

If $V = W$, then the linear transformation L is called a **linear operator** on V.

Example 3.1. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$L(x, y, z) = (x, y).$$

Show that L is a linear transformation.

Example 3.2. Let $L : P_2 \to P_1$ be defined by

$$L(at^2 + bt + c) = 2at + b.$$

Show that L is a linear transformation.

Example 3.3. Let $L : \mathbb{R}_n \to \mathbb{R}_n$, where $\mathbb{R}_n$ is the set of all $n \times 1$ matrices, be defined by

$$L(\alpha) = A\alpha,$$

where A is an $n \times n$ matrix. Is L a linear transformation?

Example 3.4. Let $L : \mathbb{R} \to \mathbb{R}$, be defined by

$$L(\alpha) = 5\alpha + 2.$$

Is L a linear transformation?

Remark 3.1. An alternative way to show that $L : V \to W$ is a linear transformation is to show that $\forall \alpha, \beta \in V$ and $\forall r, s \in \mathbb{R}$,

$$L(r\alpha + s\beta) = rL(\alpha) + sL(\beta).$$

Theorem 3.1. *Let $L : V \to W$ be a linear transformation. Then,*

1. $L(\theta_V) = \theta_W$*;*
2. $L(\alpha - \beta) = L(\alpha) - L(\beta)$*,* $\forall \alpha, \beta \in V$*.*

Theorem 3.2. *Let $L : V \to W$ be a linear transformation of an n-dimensional vector space V into a vector space W. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for V, then $L(\alpha)$ is completely determined by $\{L(\alpha_1), L(\alpha_2), \ldots, L(\alpha_n)\}$.*

3.2 The Kernel and Range of a Linear Transformation

Definition 3.2. A linear transformation $L : V \to W$ is called **one-to-one** if $L(\alpha_1) = L(\alpha_2)$ implies that $\alpha_1 = \alpha_2$, $\alpha_1, \alpha_2 \in V$.

Example 3.5. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(x, y, z) = (x, y)$. Is L a one-to-one linear transformation?

Example 3.6. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(a, b, c) = (ra, rb, rc)$, $r \in \mathbb{R}$. Is L a one-to-one linear transformation?

Definition 3.3. Let $L : V \to W$ be a linear transformation of a vector space V into a vector space W. The **kernel of L**, denoted by $\ker L$ is the subset of V consisting of all elements of V such that $L(\alpha) = \theta_W$. In other words, $\ker L = \{\alpha \in V | L(\alpha) = \theta_W\}$.

Example 3.7. Let $L\mathbb{R}^3 \to \mathbb{R}^2$ defined by $L(x, y, z) = (x, y)$. Find $\ker L$.

Theorem 3.3. *Let $L : V \to W$ be a linear transformation of a vector space V into a vector space W. Then,*

1. $\ker L$ *is a subspace of* V*;*
2. L *is a one-to-one function if and only if* $\ker L = \{\theta_V\}$*.*

Example 3.8. Let $L : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $L(a, b, c, d) = (a+b, c+d, a+c)$. Find the $\ker L$ and a basis for $\ker L$.

Definition 3.4. Let $L : V \to W$ be a linear transformation of a vector space V into a vector space W. Then, the **range of L** or **image of V under L**, denoted by *Range* L, consists of all vectors in W that are images of the vectors in V under L. Thus,

$$\textit{Range } L = \{\beta \in W | \exists \alpha \in V \text{ where } L(\alpha) = \beta\}.$$

Moreover, the linear transformation is called **onto** if *Range* $L = W$, i.e., $\forall \beta \in W$ has a preimage $\alpha \in V$.

Theorem 3.4. Let $L : V \to W$ be a linear transformation of a vector space V into a vector space W. Then, the **range of L** is a subspace of W.

Example 3.9. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L(a_1, a_2, a_3) = (a_1 + a_3, a_1 + a_2 + 2a_3, 2a_1 + a_2 + 3a_3).$$

Is L onto? Find a basis for *Range* L. Find a basis for $\ker L$.

Theorem 3.5. *If $L : V \to W$ be a linear transformation of an n-dimensional vector space V into a vector space W, then*

$$\dim \ker L + \dim \ \textit{Range } L = \dim V.$$

Corollary 3.5.1. *If $L : V \to W$ be a linear transformation of a vector space V into a vector space W and $\dim V = \dim W$, then*

1. *If L is one-to-one, then it is onto;*
2. *If L is onto, then its one-to-one.*

Definition 3.5. A linear transformation $L : V \to W$ be a linear transformation of a vector space V into a vector space W is called **invertible** if it is an invertible function, that is, there exists a unique function $L^{-1} : W \to V$ such that $L \circ L^{-1} = I_W$ and $L^{-1} \circ L = I_V$. I_V and I_W are the identity linear transformations on V and W respectively.

Theorem 3.6. *A linear transformation $L : V \to W$ be a linear transformation of a vector space V into a vector space W is invertible if and only if L is one-to-one and onto. Moreover, L^{-1} is a linear transformation and $(L^{-1})^{-1} = L$.*

Example 3.10. Let

$$L(a_1, a_2, a_3) = (a_1 + a_2 + a_3, 2a_1 + 2a_2 + a_3, a_2 + a_3).$$

Find L^{-1}.

Theorem 3.7. *A linear transformation $L : V \to W$ is one-to-one if and only if the image of every linearly independent set of vectors in V is linearly independent set of vectors in W.*

3.3 Matrix of Linear Transformation

Definition 3.6. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis of an n-dimensional vector space V, then $\forall \alpha \in V$,

$$\alpha = a_1\alpha_2 + a_2\alpha_2 + \ldots + a_n\alpha_n,$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$. Then the vector,

$$[\alpha]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

is called the **coordinate vector of α with respect to the ordered basis S**.

Example 3.11. Let $V = P_1$. If $S = \{t, 1\}$ and $T = \{t+1, t-1\}$, then find the coordinate vector of α with respect to S and then with respect to T, where $\alpha = 5t - 2$.

Example 3.12. Let $V = \mathbb{R}^3$ and $S = \{(1,1,0), (2,0,1), (0,1,2)\}$. Find $\left[(1,1,-5)\right]_S$.

Theorem 3.8. Let $L : V \to W$ be a linear transformation. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be an ordered basis for V and $T = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be an ordered basis for W. Then, there exists an $m \times n$ matrix A whose jth column is the coordinate vector of $L(\alpha_j)$ with respect to T, i.e., $\left[L(\alpha_j)\right]$ and satisfying the property that

$$\left[L(\alpha)\right]_T = A \left[\alpha\right]_S.$$

A is called the **matrix representation of L with respect to the ordered bases S and T**.

Example 3.13. Let $L : P_2 \to P_1$, defined by $L(p(t)) = p'(t)$, and consider the ordered bases $S = \{t^2, t, 1\}$ and $T = \{t, 1\}$ for P_2 and P_1 respectively. Find A, the matrix representation of the linear transformation L.

Example 3.14. Let $L : \mathbb{R}_3 \to \mathbb{R}_2$, be defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find the matrix representation A of L with respect to S and T if

1. S is the natural basis for $\mathbb{R}_3$ and T is the natural basis for $\mathbb{R}_2$;
2. $S = \{(1,1,0)^T, (0,1,1)^T, (0,0,1)^T\}$ and $T = \{(1,2)^T, (1,3)^T\}$.

3.4 Similarity

Suppose $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $T = \{\beta_1, \beta_2, \ldots, \beta_n\}$ are two ordered bases for an n-dimensional vector space V. We want to show the relationship between the coordinate vectors of the same vector α with respect to the bases S and T. Suppose $\alpha \in V$, then $\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n$ and

$$[\alpha]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We can find $[\alpha]_T$, in the following way:

$$\begin{aligned}[\alpha]_T &= [c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n]_T \\ &= [c_1\alpha_1]_T + [c_2\alpha_2]_T + \ldots + [c_n\alpha_n]_T \\ &= c_1[\alpha_1]_T + c_2[\alpha_2]_T + \ldots + c_n[\alpha_n]_T\end{aligned}$$

Let the coordinate vector α_j, $j = 1, 2, \ldots, n$ with respect to T be denoted by

$$[\alpha_j] = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}.$$

Then,

$$\begin{aligned}[\alpha]_T &= c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \ldots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \\ &= P[\alpha]_S\end{aligned}$$

The matrix P given above is called the **transition matrix from S to T**. We note that the matrix P is always a nonsingular matrix. Thus,

$$[\alpha]_S = P^{-1}[\alpha]_T,$$

that is the transition matrix from T to S is P^{-1} and its jth column is $[\beta_j]_S$.

Example 3.15. Let $V = \mathbb{R}_3$ and let

$$S = \left\{ \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix} \right\} \quad \text{and} \quad T = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

be ordered bases for $\mathbb{R}_3$. Find the transition matrix of S to T and the transition matrix from T to S.

Theorem 3.9. *Let $L : V \to W$ be a linear transformation of an n-dimensional vector space V into an m-dimensional vector space W. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $S'\{\alpha'_1, \alpha'_2, \ldots, \alpha'_n\}$ be ordered bases for V with transition matrix P from S to S'. Let $T = \{\beta_1, \beta_2, \ldots, \beta_m\}$ and $T' = \{\beta'_1, \beta'_2, \ldots, \beta'_m\}$ be ordered bases for W with transition matrix Q from T to T'. If A is the matrix representation of L with respect to S and T, then $Q^{-1}AP$ is the matrix representation of L with respect to S' and T'.*

Example 3.16. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L(x, y, z) = (x + z, x - z)$ Let $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $S' = \{(1, 1, 0), (0, 1, 1), (0, 0, 1)\}$ be ordered bases for $\mathbb{R}^3$ and $T = \{(1, 0), (0, 1)\}$ and $T' = \{(1, 1), (1, 3)\}$ be ordered bases for $\mathbb{R}^2$.

1. Find the transition matrix from S to S';
2. Find the transition matrix from T to T';
3. Find the matrix representation for L with respect to the ordered bases S and T;
4. Find the matrix representation for L with respect to the ordered bases S' and T';

Corollary 3.9.1. *Let $L : V \to V$ be a linear operator on an n-dimensional vector space V. Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $S' = \{\alpha'_1, \alpha'_2, \ldots, \alpha'_n\}$ be ordered bases for V with transition matrix P from S' to S. If A is the matrix representation for L with respect to S, then $P^{-1}AP$ is the matrix representation of L with respect to S'.*

Definition 3.7. Let $L : V \to W$ be a linear transformation. The *rank* of L is the rank of any matrix representing L.

Theorem 3.10. *Let $L : V \to W$ be a linear transformation. Then, rank $L = \dim$ range L.*

Remark 3.2. Recall Thm 3.5, i.e.,

$$\dim \ker L + \dim \ \textit{Range}\ L = \dim V,$$

can be restated

$$\textit{nullity}\ L + \textit{rank}\ L = \dim V,$$

where $L : V \to W$ be a linear transformation and $\dim V = n$.

Definition 3.8. If A and B are $n \times n$ matrices, we say that B is **similar** to A if there is a nonsingular matrix P such that $B = P^{-1}AP$.

Theorem 3.11. *Let V be any n-dimensional vector space and let A and B be any $n \times n$ matrices. Then, A and B are similar if and only if A and B represent the same linear transformation $L : V \to W$ with respect to different ordered bases for V.*

Theorem 3.12. *If A and B are similar $n \times n$ matrices, then rank $A =$ rank B.*

PROBLEM SET NO. 3

Answer the following questions completely. These questions were lifted from Exercise 4.1, pp. 251 to 252; Exercise 42., pp. 264 to 266; Exercise 4.3, pp. 274 to 276 of the text.

1. Consider the function $L : M_{3,4} \to M_{2,4}$ defined by

$$L(A) = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -3 \end{bmatrix} A,$$

where $A \in M_{3,4}$.

   (a) Find $L\left(\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 0 & 2 & 3 \\ 4 & 1 & -2 & 1 \end{bmatrix}\right)$

   (b) Show that L is a linear transformation.

2. Let $L : \mathbb{R}_3 \to \mathbb{R}_2$ be a linear transformation wherein $L(1,0,0) = (2,-4)$, $L(0,1,0) = (3,5)$ and $L(0,0,1) = (2,3)$.

   (a) What is $L(1,-2,3)$?

   (b) What is $L(a,b,c)$?

3. Let V and W be vector spaces. The function $O : V \to W$ defined by

$$O(\alpha) = \theta_W, \alpha \in V,$$

is called the *zero linear transformation*.

   (a) Show that O is a linear transformation.

   (b) Suppose $\dim V = n$ and $\dim W = m$, show that the matrix representation of O with respect to any ordered bases for V and W is the zero $m \times n$ matrix, $O_{m,n}$.

4. Let $I : V \to V$ be defined by $I(\alpha) = \alpha$, $\forall \alpha \in V$. I is called the *identity operator* in V.

   (a) Show that I is a linear transformation.

   (b) Suppose $\dim V = n$. Prove that the matrix representation of I is the identity matrix of order n, I_n.

5. Let $L : V \to W$ be a linear transformation from a vector space V into W. The *image* of the subspace V_1 of V is

$$L(V_1) = \{\beta \in W | \beta = L(\alpha) \text{ where } \exists \alpha \in V\}.$$

The *preimage* of the subspace W_1 of W is

$$L^{-1}(W_1) = \{\alpha \in V | L(\alpha) \in W_1\}.$$

Show that

(a) $L(V_1)$ is a subspace of V;

(b) $L^{-1}(W_1)$ is a subspace of W.

6. Let $\mathbb{R}_4 \to \mathbb{R}_2$ be the linear transformation defined by

$$L(a_1, a_2, a_3, a_4) = (a_1 + a_3, a_2 + a_4).$$

   (a) Is $(2, 3, -2, 3)$ in $Ker\ L$? Is $(4, -2, -4, 2)$ in $Ker\ L$?

   (b) Is $(1, 2)$ in $Range\ L$? Is $(0, 0)$ in $Range\ L$?

   (c) Find $Ker\ L$.

   (d) Find a basis for $Range\ L$.

7. Let $L : P_2 \to \mathbb{R}_2$ defined by

$$L(at^2 + bt + c) = (a, b).$$

   (a) Find a basis for $Ker\ L$;

   (b) Find a basis for $Range\ L$.

8. Let $L : V \to W$ be a linear transformation.

   (a) Show that $\dim Range\ L \leq \dim V$;

   (b) Prove that if L is onto, then $\dim W \leq \dim V$.

9. Let $L : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation defined by

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 & 3 & -1 \\ 1 & 0 & 0 & 2 & -1 \\ 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

   (a) Find a basis for and the dimension of the $Ker\ L$;

   (b) Find a basis for and the dimension of the $Range\ L$.

10. Let $L : \mathbb{R}_4 \to \mathbb{R}_3$ be defined by

$$L(x_1, x_2, x_3, x_4) = (x_1, x_2 + x_3, x_3 + x_4).$$

   Let S and T be the natural bases for $\mathbb{R}_4$ and $\mathbb{R}_3$ respectively. Let $S' = \{(1, 0, 0, 1), (0, 0, 0, 1), (1, 1, 0, 0), (0, 1, 1, 0)\}$ and $T' = \{(1, 1, 0), (0, 1, 0), (1, 0, 1)\}$.

   (a) Find the matrix representation of L with respect to S and T;

   (b) Find the matrix representation of L with respect to S" and T";

   (c) Find $L(2, 1, -1, 3)$ using the matrices in (a.) and (b.) above.

11. Let $L : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ -1 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Let S and T be the natural bases for $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively, and consider the ordered bases

$$S' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\},$$

and

$$T' = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

of $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively. Find the matrix representation of L with respect to

(a) S and T

(b) S' and T'

12. Let $L : M_{2,2} \to M_{2,2}$ be defined by

$$L(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A, \text{ where } A \in M_{2,2}.$$

Consider the ordered bases $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, and $T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$, of $M_{2,2}$. Find the matrix representation of L with respect to the ordered bases

(a) S and T;

(b) T and S.

4 Determinants

4.1 Definition

Definition 4.1. Let $S = \{1, 2, \ldots, n\}$ be a set of integers from 1 to n arranged in ascending order. A rearrangement $j_1 j_2 \cdots j_n$ of the elements of S is called a **permutation** of S. Thus, a permutation is a one-to-one mapping of S onto itself.

Definition 4.2. A permutation $j_1 j_2 \cdots j_n$ of S is said to have an **inversion** if a larger integer j_r precedes a smaller integer j_s.

Definition 4.3. A permutation is called **even** if the number of inversions in it is even, otherwise the permutation is called **odd**.

Remark 4.1. If $n \geq 2$, S_n will have $\frac{n!}{2}$ even and $\frac{n!}{2}$ odd permutations. Furthermore, if we interchange two numbers in $j_1 j_2 \cdots j_n$, we either increase or decrease the number of inversions by an odd number.

Definition 4.4. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **determinant** function denoted by $\det A$ or $|A|$ is defined by

$$\det A = \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n}, \tag{3}$$

where the summation is over all permutations $j_1 j_2 \cdots j_n$ of the set $S = \{1, 2, \ldots, n\}$. The sign in Equation 3 is positive if the permutation $j_1 j_2 \cdots j_n$ is even and negative if it is odd.

4.2 Properties of Determinants

All the matrices considered in this section are square matrices of order $n \times n$.

Theorem 4.1. *If A is matrix, then* $\det A = \det A^T$.

Theorem 4.2. *If matrix B results from A by interchanging two rows (columns) of A, then* $\det A = -\det B$.

Theorem 4.3. *If two rows (columns) of A are equal, then* $\det A = 0$.

Theorem 4.4. *If a row (column) of A consists entirely of zeros, then* $\det A = 0$.

Theorem 4.5. *If B is obtained from A by multiplying a row (column) of A by a scalar c, then* $\det B = c \det A$.

Theorem 4.6. *If B is obtained from A by adding each element of the rth row (column) of A, c times the corresponding sth row (column), $r \neq s$ of A, then* $\det B = \det A$.

Theorem 4.7. *If $A = [a_{ij}]$ is an upper (a lower) triangular matrix, then* $\det A = a_{11} a_{22} \cdots a_{nn}$.

Remark 4.2. It is clear that $\det I_n = 1$. Further,

1. If E_1 is a type 1 elementary matrix, then $\det E_1 = -1$;
2. If E_2 is a type 2 elementary matrix obtained by multiplying the rth row of I_n by c, then $\det E_2 = c$;
3. If E_3 is a type 3 elementary matrix, then $\det E_3 = 1$.

Lemma 4.8. *If E is an elementary matrix, then* $\det EA = \det E \det A$.

Theorem 4.9. *The matrix A is nonsingular if and only if* $\det A \neq 0$.

Corollary 4.9.1. *The rank $A = n$ if and only if* $\det A \neq 0$

Corollary 4.9.2. *The homogenous system $AX = O$ has a trivial solution if and only if* $\det A = 0$.

Theorem 4.10. *If A and B are $n \times n$ matrices* $\det AB = \det A \det B$.

Corollary 4.10.1. *If A is nonsingular, then* $\det A^{-1} = \frac{1}{\det A}$.

4.3 Cofactor Expansion

Definition 4.5. Let $A = [a_{ij}]$ and M_{ij} be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and the jth column of A. The determinant, $\det M_{ij}$ is called the **minor** of a_{ij}.

Definition 4.6. Let $A = [a_{ij}]$. The **cofactor** A_{ij} of a_{ij} is defined as $A_{ij} = (-1)^{i+j} \det M_{ij}$.

Theorem 4.11. *Let $A = [a_{ij}]$. Then,*

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in},$$

(expansion of $\det A$ *about the ith row); and*

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj},$$

(expansion of det *about the jth column).*

4.4 Inverse of a Matrix

Theorem 4.12. *If $A = [a_{ij}]$, then*

$$\begin{aligned} a_{i1}A_{k1} + a_{i2}A_{k2} + \ldots + a_{in}A_{kn} &= 0 \quad \text{for } i \neq k \\ a_{1j}A_{1k} + a_{2j}A_{2k} + \ldots + a_{nj}A_{nk} &= 0 \quad \text{for } j \neq k \end{aligned}$$

Remark 4.3. If we combine Theorems 4.11 and 4.12, then

$$\begin{aligned} a_{i1}A_{k1} + a_{i2}A_{k2} + \ldots + a_{in}A_{kn} &= \begin{cases} \det A & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases} \\ a_{1j}A_{1k} + a_{2j}A_{2k} + \ldots + a_{nj}A_{nk} &= \begin{cases} \det A & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \end{aligned}$$

Definition 4.7. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $n \times n$ matrix *adj* A, called the **adjoint** of A, is the matrix whose (i, j)th entry is the cofactor A_{ji} of a_{ji}. Thus,

$$adj\ A = \begin{bmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & \ldots & A_{nn} \end{bmatrix}.$$

Theorem 4.13. *If $A = [a_{ij}]$ is an $n \times n$ matrix, then*

$$A(adj\ A) = (adj\ A)A = (\det A)I_n.$$

Corollary 4.13.1. *If A is an $n \times n$ matrix and* $\det A \neq 0$, *then*

$$A^{-1} = \frac{1}{\det A}(adj\ A).$$

4.5 Cramer's Rule

Theorem 4.14. Cramer's rule *Let*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

be a linear system of n equations in n unknowns and let $A = [a_{ij}]$ be the coefficient matrix, so we can write the given system as $AX = B$, where $B = (b_1, B_2, \ldots, b_n)^T$. If $\det A \neq 0$, then the system has the unique solution:

$$x_1 = \frac{\det A_1}{\det A}; x_2 = \frac{\det A_2}{\det A}; \ldots ; x_n = \frac{\det A_n}{\det A},$$

where A_i is the matrix obtained from A by replacing the ith column by B.

Example 4.1. Using the Cramer's rule, find the solution of the linear system:

$$\begin{array}{ccccccc} 3x & + & y & - & 2z & = & -3 \\ 2x & + & 7y & + & 3z & = & 9 \\ 4x & - & 3y & - & z & = & 7 \end{array}$$

PROBLEM SET NO. 4

Answer the following questions completely. These questions were lifted from Exercise 5.2, pp. 312 to 314; Exercise 5.4., pp. 325 to 326; Exercise 5.5, p. 331 of the text.

1. Find the determinant of the following matrices:

   (a) $\begin{bmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 5 & 3 & 5 \end{bmatrix}$

2. Is $\det(AB) = \det(BA)$? Justify your answer.

3. If $\det(AB) = 0$, is either $\det A = 0$ or $\det B = 0$? Justify your answer.

4. Show that if c is a scalar and A is $n \times n$, then $\det(cA) = c^n \det A$.

5. Let $A = \begin{bmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{bmatrix}$. Find *adj* A and $\det A$.

6. Using the Cramer's rule, find the solution of the linear system:

$$\begin{array}{ccccccc} 2x & + & y & + & z & = & 6 \\ 3x & + & 2y & - & 2z & = & -2 \\ x & + & y & + & 2z & = & -4 \end{array}$$

5 Eigenvalues and Eigenvectors

5.1 Diagonalization

Definition 5.1. Let $L : V \to V$ be a linear transformation of an n-dimensional vector space V into itself. Then L is said to be **diagonalizable** or can be **diagonalized** if there exists a basis S for V such that L is represented with respect to S by a diagonal matrix D.

Example 5.1. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$L(x, y, z) = (2x - z, x + y - z, z).$$

Show that L is diagonalizable. Use the ordered basis $S = \{(1, 0, 1), (0, 1, 0), (1, 1, 0)\}$ of $\mathbb{R}^3$.

Definition 5.2. Let $L : V \to V$ be a linear transformation of an n-dimensional vector space V into itself. The real number λ is called an **eigenvalue** of L if there exists a nonzero vector $\alpha \in V$ such that

$$L(\alpha) = \lambda\alpha. \tag{4}$$

Every nonzero vector α satisfying Equation 4 is called an **eigenvector of L associated with the eigenvalue λ.**

Example 5.2. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(x, y) = (y, x)$. Then, the vector (a, a), $a \neq 0$ is an eigenvector associated with the eigenvalue $c = 1$, and the vector $(a, -a)$, $a \neq 0$ is an eigenvector associated with the eigenvalue $c = -1$.

Theorem 5.1. *Let $L : V \to V$ be a linear transformation of an n-dimensional vector space V into itself. Then, L is diagonalizable if and only if V has a basis S of eigenvectors of L. Moreover, if D is the diagonal matrix representing L with respect to S, then the entries of the main diagonal of D are the eigenvalues of L.*

Remark 5.1. We now formulate the notion of eigenvalue and eigenvector to any square matrix A of order n. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation defined by $A\alpha = c\alpha$, where c is a scalar $\alpha \in \mathbb{R}^n$, $\alpha \neq \theta$. We say that c is an **eigenvalue** of A and α is an **eigenvector** of A associated with c. Furthermore, we say that A is diagonalizable, or can be diagonalized if A is similar to a diagonal matrix D.

Theorem 5.2. *An $n \times n$ matrix A is similar to a diagonal matrix D if and only if $\mathbb{R}^n$ has a basis of eigenvectors of A. Moreover, the elements of the main diagonal of D are the eigenvalues of A.*

Definition 5.3. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then

$$\det(\lambda I_n - A),$$

is called the **characteristic polynomial** of A. The equation

$$f(\lambda) = \det(\lambda I_n - A) = 0,$$

is called the **characteristic equation** of A.

Theorem 5.3. *Let A be an $n \times n$ matrix. The eigenvalues of A are the real roots of the characteristic polynomial of A.*

Theorem 5.4. *An $n \times n$ matrix A is diagonalizable if all the roots of its characteristic polynomial are real and distinct.*

Example 5.3. Foe each of the following matrices, find all eigenvalues and determine whether its is diagonalizable or not.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Theorem 5.5. *Similar matrices have the same characteristic polynomial.*