Integration of Rational Functions

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MTH202A

Motivation

- **Big idea:** we show how to integrate any rational function (a ratio of polynomials) by expressing it as a **sum of simpler fractions**, called **partial fractions** , that we already know how to integrate.
- **Illusration:** Observe that

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

Hence,

$$\int \frac{x+5}{x^2+x-2} = \int \left(\frac{2}{x-1} - \frac{1}{x+2} \right) \, dx$$
$$= 2\ln|x-1| - \ln|x+2| + C$$

Method of Partial Fractions

Let $f(x) = \dfrac{P(x)}{Q(x)}$ where P and Q are polynomials. We want to express f as a sum of simpler fractions provided that the degree of P is less than the degree of Q (that is, the rational function is **proper**).

If f is **improper** (i.e., $\deg(P) \geq \deg(Q)$), then we divide P by Q (via long division) until a remainder R is obtained such that $\deg(R) < \deg(Q)$. The division statement is

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where S and R are also polynomials.

Example

Example 1: Evaluate the following integral.

$$\int \frac{x^3 + x}{x - 1} \, dx$$

Method of Partial Fractions

Express the **proper** rational function $\dfrac{R(x)}{Q(x)}$ as a sum of partial fractions.

Case 1: The denominator $Q(x)$ is a product of distinct linear factors.

If $Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$, then there exist constants $A_1, A_2, \ldots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}.$$

Examples

Example 2: $\displaystyle\int \frac{4y^2 - 7y - 12}{y(y+2)(y-3)}\, dy$

Example 3: $\displaystyle\int \frac{x^3 - 4x - 10}{x^2 - x - 6}\, dx$

Case 2: The denominator $Q(x)$ is a product of linear factors, some of which are repeated.

If the linear factor $(ax+b)^r$ occurs in the factorization of $Q(x)$, then the corresponding partial fractions

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}$$

will be used.

Example 4: $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$

Example 5: $\displaystyle\int \frac{3x^2 + 1}{(x+1)(x-5)^2}\, dx$

Exercises

Evaluate the given integral.

1. $\displaystyle\int \frac{4}{x^2+5x-14}\,dx$
2. $\displaystyle\int \frac{w^2+7w}{(w+2)(w-1)(w-4)}\,dw$
3. $\displaystyle\int \frac{8}{3x^3+7x^2+4x}\,dx$
4. $\displaystyle\int \frac{4x-11}{x^3-9x^2}\,dx$
5. $\displaystyle\int \frac{2x}{(x-2)^2(x+2)}\,dx$

Case 3: The denominator $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then the decomposition will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c},$$

where A and B are constants to be determined.

Example 6: $\displaystyle\int \frac{dx}{x(x^2 + 5)}$

Example 7: $\displaystyle\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}\, dx$

Case 4: The denominator $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition.

Example 8: $\displaystyle\int \frac{dx}{x(x^2+1)^2}$

Exercises

Evaluate the given integral.

1. $\displaystyle\int \frac{dx}{(x^2+1)(x^2+4)}$
2. $\displaystyle\int \frac{w^4+1}{w^3+9w}\, dw$
3. $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\, dx$

References

- Stewart, J., *Calculus: Early Transcendentals* (6th ed.)
- Dawkins, P. *Pauls Online Math Notes* Link: http://tutorial.math.lamar.edu
- Mendelson, Elliott. *3,000 Solved Problems in Calculus*.