ADV CAL 1 Exam 1

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1.) Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent. Prove $\lim_{n\to\infty}(a_n - b_n) = a - b$, where $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$.

Suppose $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converges to a and b respectively. Let $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, $|a_n - a| < \varepsilon/2$, since $\varepsilon/2 > 0$. Similarly, there exists $N_2 \in \mathbb{N}$ such that $|b_n - b| < \varepsilon/2$. Let $N = \max(N_1, N_2)$. It follows that for any $n \geq N$,

$$|a_n - a| < \varepsilon/2 \quad \text{and} \quad |b_n - b| < \varepsilon/2$$

By combining the two inequalities,

$$\begin{aligned}
\varepsilon = \varepsilon/2 + \varepsilon/2 &> |a_n - a| + |b_n - b| \\
&= |a_n - a| + |b - b_n| && \text{(Symmetry of the distance function)} \\
&\geq |(a_n - a) + (b - b_n)| && \text{(Minkowski Inequality)} \\
&= |(a_n - b_n) - (a - b)|.
\end{aligned}$$

Therefore, $\lim_{n\to\infty}(a_n - b_n) = a - b$. ∎

2.) Prove that a nonempty set of negative integers has a largest element.

Let S be a nonempty set of negative integers. Let $S' = \{-s \mid s \in S\}$. Since S' is a nonempty subset of the integers, there exists $m' \in S'$ such that $m' \leq s'$, for $s' \in S'$. It follows that $-m' \in S$, Furthermore $-m' \geq s$, for $s \in S$. ∎

3.) Prove that a nonempty subset of real numbers that has a lower bound has a greatest upper bound.

Suppose S is a nonempty subset of the real numbers that has a lower bound. Let L be the set of all lower bounds of S. By the hypothesis, $L \neq \emptyset$. Observe that all elements of S are upper bounds of L. It follows that L is bounded above. Thus, there exists $z \in \mathbb{R}$ where z is the supremum of L.

since z is the least upper bound of L

For any $x < z$, x cannot be an upper bound of L^. Furthermore, $x \notin S$ since S are upper bounds of L. Taking the contrapositive, $x \in S$ implies $x \geq z$. In other words, z is a lower bound of S.

For any $x > z$, x cannot be a lower bound since z is an upper bound of L. Therefore, z is a lower bound and there are no larger lower bounds. ■

4.a) Prove or disprove: (S has an upper bound) → (S has a largest element)
Consider $S = \{x \in \mathbb{R} \mid x < 1\}$. Clearly S is a non-empty subset of $\mathbb{R}$ with an upper bound of 1, but S has no largest element. So, it is not true. ■

4.b.) Prove or disprove: (S has a largest element) → (S has an upper bound)
Let a be the largest element of S. By definition, $a \leq s$ implies $s = a$, for any $s \in S$. For any $b \in S$, either $a \leq b$ or $a > b$, by trichotomy. If $a \leq b$, then $a = b$. If $a > b$, then $a > b$ vacuously. In either case, $a \geq b$, for any $b \in S$. Therefore, a is an upper bound of S. So it's true. ■

5.) The result is not a sequence since $\text{dom}(f) = \{a_n \in \mathbb{R} \mid n \in \mathbb{N}\}$ is a subset of $\mathbb{R}$ that is not necessarily equal to $\mathbb{N}$. ■