Answer the following questions completely.

1. Given the following matrices:

$$A = \begin{bmatrix} 4 & 1 & 0 & 2 \\ -1 & 4 & -3 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 5 & 4 \\ 3 & -6 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 2 & -2 \\ 4 & 0 & 5 \\ 6 & -4 & 3 \\ 5 & -3 & -1 \end{bmatrix}.$$

   Find the following matrices if it exists

   (a) AC;
   (b) AB;
   (c) BA;
   (d) $B^T - 3A$;
   (e) C^2

2. Give different examples of matrices of size 5×5 of the following types: a lower triangular matrix; an upper triangular matrix; a scalar matrix; a diagonal matrix; an identity matrix.

3. Give an example of a 4×5 matrix. With this matrix, successively perform a type I; type II then a type III row operations. Specify the operation you are performing in each step.

4. Find the solution set of the homogenous system:

$$\begin{aligned} 2w - x + 3y - 2z &= 0 \\ x + 2y + 3z &= 0 \end{aligned}$$

5. Given the matrices

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 \\ -1 & -2 & 0 & 1 \\ 0 & -1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}.$$

   (a) Find the inverse of A.

   (b) Given $B = \begin{bmatrix} -1 \\ 4 \\ 3 \\ 2 \end{bmatrix}$, find the solution set of the linear system $AX = B$.

   (c) Find the solution set of the homogeneous system $AX = O$, where O is the 4×1 zero matrix.

6. Prove the following statements:

   (a) Let A and B be $m \times p$ matrices and C is a $p \times n$ matrix. Show that

$$(A + B)C = AC + BC.$$

   (b) Given the linear system $AX = B$, where $B \neq O$. If X_1 is a solution to the given linear system and X_2 is a solution to the associated homogenous system $AX = O$, then $X_1 + X_2$ is also a solution to $AX = B$.

   (c) Show that if A is any square matrix, then AA^T is symmetric.

   (d) Let A be a diagonal matrix with nonzero main diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$. Prove that A^{-1} exists and that it is also a diagonal matrix with main diagonal entries,

$$1/a_{11}, 1/a_{22}, \ldots, 1/a_{nn}.$$