Contents

Chapter 1

Real Analysis

(1.5)
$\mathit{OrderTrichotomy}[<, S] := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$
$\mathit{OrderTransitivity}[<, S] := \forall_{x,y,z \in S}\big((x < y \land y < z) \implies x < z\big)$
$\mathit{Order}[<, S] := (\mathit{OrderTrichotomy}[<, S]) \land (\mathit{OrderTransitivity}[<, S])$

(1.7)
$\mathit{BoundedAbove}[E, S, <] := (\mathit{Order}[<, S]) \land (E \subset S) \land \Big(\exists_{\beta \in S}\forall_{x \in E}(x \leq \beta)\Big)$
$\mathit{BoundedBelow}[E, S, <] := (\mathit{Order}[<, S]) \land (E \subset S) \land \Big(\exists_{\beta \in S}\forall_{x \in E}(\beta \leq x)\Big)$
$\mathit{UpperBound}[\beta, E, S, <] := (\mathit{Order}[<, S]) \land (E \subset S) \land \big(\beta \in S \land \forall_{x \in E}(x \leq \beta)\big)$
$\mathit{LowerBound}[\beta, E, S, <] := (\mathit{Order}[<, S]) \land (E \subset S) \land \big(\beta \in S \land \forall_{x \in E}(\beta \leq x)\big)$

(1.8)
$\mathit{LUB}[\alpha, E, S, <] := (\mathit{UpperBound}[\alpha, E, S, <]) \land \Big(\forall_{\gamma}(\gamma < \alpha \implies \neg \mathit{UpperBound}[\gamma, E, S, <])\Big)$
$\mathit{GLB}[\alpha, E, S, <] := (\mathit{LowerBound}[\alpha, E, S, <]) \land \Big(\forall_{\beta}(\alpha < \beta \implies \neg \mathit{LowerBound}[\beta, E, S, <])\Big)$

(1.10)
$\mathit{LUBProperty}[S, <] := \forall_{E}\Big(\big((\emptyset \neq E \subset S) \land (\mathit{BoundedAbove}[E, S, <]) \implies \exists_{\alpha \in S}(\mathit{LUB}[\alpha, E, S, <])\big)\Big)$
$\mathit{GLBProperty}[S, <] := \forall_{E}\Big(\big((\emptyset \neq E \subset S) \land (\mathit{BoundedBelow}[E, S, <]) \implies \exists_{\alpha \in S}(\mathit{GLB}[\alpha, E, S, <])\big)\Big)$

(1.11)
$\mathit{LUBPropertyImpliesGLBProperty} := \mathit{LUBProperty}[S, <] \implies \mathit{GLBProperty}[S, <]$

(1) $\mathit{LUBProperty}[S, <] \implies \ldots$ — wts: 2

(1.1) $(\emptyset \neq B \subset S \land \mathit{BoundedBelow}[B, S, <]) \implies \ldots$ — wts: 1.2

(1.1.1) $\mathit{Order}[<, S] \land \exists_{\delta' \in S}(\mathit{LowerBound}[\delta', B, S, <])$ — from: *BoundedBelow*, 1.1

(1.1.2) $|B| = 1 \implies \ldots$ — wts: 1.1.3

(1.1.2.1) $\exists_{u'}(u' \in B) \;\blacksquare\; u := \mathit{choice}(\{u' : u' \in B\}) \;\blacksquare\; B = \{u\}$ — from: 1.1.2

(1.1.2.2) $\mathit{GLB}[u, B, S, <] \;\blacksquare\; \exists_{\epsilon_0 \in S}(\mathit{GLB}[\epsilon_0, B, S, <])$

(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S}(\mathit{GLB}[\epsilon_0, B, S, <])$

(1.1.4) $|B| \neq 1 \implies \ldots$ — wts: 1.1.5

(1.1.4.1) $\forall_{E}\big((\emptyset \neq E \subset S \land \mathit{BoundedAbove}[E, S, <]) \implies \exists_{\alpha \in S}(\mathit{LUB}[\alpha, E, S, <])\big)$ — from: *LUBProperty*, 1

(1.1.4.2) $L := \{s \in S : \mathit{LowerBound}[s, B, S, <]\}$

(1.1.4.3) $|B| > 1 \land \mathit{OrderTrichotomy}[<, S] \;\blacksquare\; \exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$ — from: *Order*, 1.1.1 wts: 1.1.4.7

(1.1.4.4) $b_1 := \mathit{choice}\Big(\{b_1' \in B : \exists_{b_0' \in B}(b_0' < b_1')\}\Big) \;\blacksquare\; \neg \mathit{LowerBound}[b_1, B, S, <]$ — from: 1.1.4.2

(1.1.4.5) $b_1 \notin L \;\blacksquare\; L \subset S$

(1.1.4.6) $\delta := \mathit{choice}(\{\delta' \in S : \mathit{LowerBound}[\delta', B, S, <]\}) \;\blacksquare\; \delta \in L \;\blacksquare\; \emptyset \neq L$ — from: 1.1.1

(1.1.4.7) $\emptyset \neq L \subset S$ — from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y \in L}(\mathit{LowerBound}[y_0, B, S, <]) \;\blacksquare\; \forall_{y \in L}\forall_{x \in B}(y_0 \leq x)$ — from: *LowerBound*, 1.1.4.2 wts: 1.1.4.10

(1.1.4.9) $\forall_{x \in B}\Big(x \in S \wedge \forall_{y \in L}(y_0 \le x)\Big) \;\blacksquare\; \forall_{x \in B}(UpperBound[x, L, S, <])$ from: $UpperBound$

(1.1.4.10) $\exists_{x \in S}(UpperBound[x, L, S, <]) \;\blacksquare\; BoundedAbove[L, S, <]$

(1.1.4.11) $\emptyset \ne L \subset S \wedge BoundedAbove[L, S, <]$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha' \in S}(LUB[\alpha', L, S, <]) \;\blacksquare\; \alpha := choice\Big(\{\alpha' \in S : (LUB[\alpha', L, S, <])\}\Big)$ from: 1.1.4.1 wts: 1.1.4.21

(1.1.4.13) $\forall_x(x \in B \implies UpperBound[x, L, S, <])$ from: 1.1.4.9 wts: 1.1.4.17

(1.1.4.14) $\forall_x(\neg UpperBound[x, L, S, <] \implies x \notin B)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$ wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound[\gamma, L, S, <] \;\blacksquare\; \gamma \notin B$ from: LUB, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B \;\blacksquare\; \gamma \in B \implies \gamma \ge \alpha$

(1.1.4.17) $\forall_{\gamma \in B}(\alpha \le \gamma) \;\blacksquare\; LowerBound[\alpha, B, S, <]$ from: $LowerBound$

(1.1.4.18) $\alpha < \beta \implies \ldots$ wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y \in L}(y_0 \le \alpha < \beta) \;\blacksquare\; \forall_{y \in L}(y_0 \ne \beta)$ from: LUB, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta \notin L \;\blacksquare\; \neg LowerBound[\beta, B, S, <]$ from: 1.1.4.2

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound[\beta, B, S, <] \;\blacksquare\; \forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.20) $LowerBound[\alpha, B, S, <] \wedge \forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB[\alpha, B, S, <] \;\blacksquare\; \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.5) $|B| \ne 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.6) $\Big(|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])\Big) \wedge \Big(|B| \ne 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])\Big)$ from: 1.1.3, 1.1.5

(1.1.7) $(|B| = 1 \vee |B| \ne 1) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <]) \;\blacksquare\; \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.2) $(\emptyset \ne B \subset S \wedge BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.3) $\forall_B\Big((\emptyset \ne B \subset S \wedge BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])\Big)$

(1.4) $GLBProperty[S, <]$

(2) $LUBProperty[S, <] \implies GLBProperty[S, <]$

(1.12)

$$Field[F, +, *] := \exists_{\mathbb{0}, \mathbb{1} \in F} \forall_{x,y,z \in F} \left(\begin{array}{l} x + y \in F \quad \wedge \quad x * y \in F \quad \wedge \\ x + y = y + x \quad \wedge \quad x * y = y * x \quad \wedge \\ (x + y) + z = x + (y_0 + z) \quad \wedge \quad (x * y) * z = x * (y_0 * z) \quad \wedge \\ \mathbb{1} \ne \mathbb{0} \quad \wedge \quad x * (y_0 + z) = (x * y) + (x * z) \quad \wedge \\ \mathbb{0} + x = x \quad \wedge \quad \mathbb{1} * x = x \quad \wedge \\ \exists_{-x \in F}\big(x + (-x) = \mathbb{0}\big) \wedge \Big(x \ne \mathbb{0} \implies \exists_{1/x \in F}\big(x * (1/x) = \mathbb{1}\big)\Big) \end{array} \right)$$

************************ $(Field[F, +, *] \wedge x, y, z \in F) \implies \ldots$ ************************

(1.14)

$AdditiveCancellation := (x + y = x + z) \implies y = z$

(1) $y = \mathbb{0} + y = \big(x + (-x)\big) + y = \big((-x) + x\big) + y = (-x) + (x + y) = \ldots$ from: $Field$

(2) $(-x) + (x + z) = \big((-x) + x\big) + z = \big(x + (-x)\big) + z = \mathbb{0} + z = z$ from: $Field$

$AdditiveIdentityUniqueness := (x + y = x) \implies y = \mathbb{0}$

(1) $x + y = x = \mathbb{0} + x = x + \mathbb{0}$ from: $Field$

(2) $y = 0$ from: $AdditiveCancellation$

$AdditiveInverseUniqueness := (x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$ from: $Field$

(2) $y = -x$ from: $AdditiveCancellation$

$DoubleNegative := x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x \;\blacksquare\; \mathbb{0} = (-x) + x$ from: $Field$

from: *AdditiveInverseUniqueness*

(2) $x = -(-x)$

(1.15)

$\textcolor{red}{MultiplicativeCancellation} := (x \neq 0 \wedge x * y = x * z) \implies y = z$ —

$\textcolor{red}{MultiplicativeIdentityUniqueness} := (x \neq 0 \wedge x * y = x) \implies y = \mathbb{1}$ —

$\textcolor{red}{MultiplicativeInverseUniqueness} := (x \neq 0 \wedge x * y = \mathbb{1}) \implies y = 1/x$ —

$\textcolor{red}{DoubleReciprocal} := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)

$\textcolor{red}{Domination} := \mathbb{0} * x = \mathbb{0}$

from: *Field*

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x \;\blacksquare\; \mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$

from: *AdditiveIdentityUniqueness*

(2) $\mathbb{0} * x = \mathbb{0}$

$\textcolor{red}{NonDomination} := (x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies \ldots$

(1.1) $(x * y = 0) \implies \ldots$

from: *Field*, *Domination*, 1, 1.1

(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = \big(x * (1/x)\big) * \big(y * (1/y)\big) = (x * y) * \big((1/x) * (1/y)\big) = \mathbb{0} * \big((1/x) * (1/y)\big) = \mathbb{0}$

from: *Field*

(1.1.2) $\mathbb{1} = \mathbb{0} \wedge \mathbb{1} \neq \mathbb{0} \;\blacksquare\; \perp$

(1.2) $(x * y = 0) \implies \perp \;\blacksquare\; x * y \neq 0$

(2) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq 0$

$\textcolor{red}{NegationCommutativity} := (-x) * y = -(x * y) = x * (-y)$

from: *Field*, *Domination*
wts: 2

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0} \;\blacksquare\; x * y + (-x) * y = \mathbb{0}$

from: *AdditiveInverseUniqueness*

(2) $(-x) * y = -(x * y)$

from: *Field*, *Domination*
wts: 4

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * \mathbb{0} = \mathbb{0} \;\blacksquare\; x * y + x * (-y) = \mathbb{0}$

from: *AdditiveInverseUniqueness*

(4) $x * (-y) = -(x * y)$

from: 2, 4

(5) $(-x) * y = -(x * y) = x * (-y)$

$\textcolor{red}{NegativeMultiplication} := (-x) * (-y) = x * y$

from: *NegationCommutativity*, *DoubleNegative*

(1) $(-x) * (-y) = -\big(x * (-y)\big) = -\big(-(x * y)\big) = x * y$

(1.17)

$$\textcolor{red}{OrderedField}[F, +, *, <] := \left(\begin{array}{l} \textcolor{cyan}{Field}[F, +, *] \quad \wedge \quad \textcolor{cyan}{Order}[<, F] \quad \wedge \\ \forall_{x,y,z \in F} (y_0 < z \implies x + y < x + z) \;\wedge \\ \forall_{x,y \in F} \big((x > \mathbb{0} \wedge y > \mathbb{0}) \implies x * y > \mathbb{0}\big) \end{array} \right)$$

****************************** $(\textcolor{cyan}{OrderedField}[F, +, *, <] \wedge x, y, z \in F) \implies \ldots$ ******************************

(1.18)

$\textcolor{red}{NegationOnOrder} := x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \ldots$

from: *OrderedField*

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x \;\blacksquare\; \mathbb{0} > -x \;\blacksquare\; -x < \mathbb{0}$

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \ldots$

from: *OrderedField*

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x \;\blacksquare\; \mathbb{0} < x \;\blacksquare\; x > \mathbb{0}$

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

from: 2, 4

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \wedge -x < \mathbb{0} \implies x > \mathbb{0} \;\blacksquare\; x > \mathbb{0} \iff -x < \mathbb{0}$

$\textcolor{red}{PositiveFactorPreservesOrder} := (x > \mathbb{0} \wedge y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \wedge y < z) \implies \ldots$

from: *OrderedField*

(1.1) $(-y) + z > (-y) + y = \mathbb{0} \;\blacksquare\; z + (-y) = \mathbb{0}$

from: *OrderedField*

(1.2) $x * \big(z + (-y)\big) > \mathbb{0} \;\blacksquare\; x * z + x * (-y) > \mathbb{0}$

(1.3) $x * z = \mathbb{0} + x * z = \big(x * y + -(x * y)\big) + x * z = \big(x * y + x * (-y)\big) + x * z = \dots$ from: *Field*, *NegationCommutativity*

(1.4) $x * y + \big(x * z + x * (-y)\big) > x * y + \mathbb{0} = x * y$ from: *Field*, 1.2

(1.5) $x * z > x * y$ from: 1.3, 1.4

(2) $(x > \mathbb{0} \land y < z) \implies x * z > x * y$

NegativeFactorFlipsOrder $:= (x < \mathbb{0} \land y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \land y < z) \implies \dots$

(1.1) $-x > \mathbb{0}$ from: *NegationOnOrder*

(1.2) $(-x) * y < (-x) * z$ ▮ $\mathbb{0} = x * y + (-x) * y < x * y + (-x) * z$ ▮ $\mathbb{0} < x * y + (-x) * z$ from: *PositiveFactorPreservesOrder*

(1.3) $\mathbb{0} < (-x) * (-y + z)$ ▮ $\mathbb{0} > x * (-y + z)$ ▮ $\mathbb{0} > -(x * y) + x * z$ from: *NegationOnOrder*

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \land y < z) \implies x * y > x * z$

SquareIsPositive $:= (x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ from: *OrderedField*

(2) $(x < \mathbb{0}) \implies \dots$

(2.1) $-x > \mathbb{0}$ ▮ $x * x = (-x) * (-x) > \mathbb{0}$ ▮ $x * x > \mathbb{0}$ from: *NegationOnOrder*, *OrderedField*, *NegativeMultiplication*

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \lor x < \mathbb{0}) \implies x * x > \mathbb{0}$ ▮ $x \neq \mathbb{0} \implies x * x > \mathbb{0}$ from: *OrderTrichotomy*, 1, 3

OneIsPositive $:= \mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0}$ ▮ $\mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ from: *Field*, *SquareIsPositive*

ReciprocationOnOrder $:= (0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \dots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0}$ ▮ $x * (1/x) > 0$ from: *Field*, *OneIsPositive*

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \land x * (1/x) > 0 \implies \bot$ ▮ $1/x > \mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0}$ ▮ $y * (1/y) > 0$ from: *Field*, *OneIsPositive*

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \land y * (1/y) > 0 \implies \bot$ ▮ $1/y > \mathbb{0}$ from: *NegativeFactorFlipsOrder*, 1

(1.5) $(1/x) * (1/y) > \mathbb{0}$ from: *OrderedField*

(1.6) $\mathbb{0} < 1/y = \big((1/x) * (1/y)\big) * x < \big((1/x) * (1/y)\big) * y = 1/x$ from: *OrderedField*, 1, 1.4, 1.5

(1.19)
OrderedFieldQ $:=$ *OrderedField*$[\mathbb{Q}, +, *, <]$ —

Subfield$[K, F, +, *] :=$ *Field*$[F, +, *] \land K \subset F \land$ *Field*$[K, +, *]$
OrderedSubfield$[K, F, +, *, <] :=$ *OrderedField*$[F, +, *, <] \land K \subset F \land$ *OrderedField*$[K, +, *, <]$
CutI$[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$
CutII$[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
CutIII$[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} :$ *CutI*$[\alpha] \land$ *CutII*$[\alpha] \land$ *CutIII*$[\alpha]\}$

CutCorollaryI $:= (\alpha \in \mathbb{R} \land p \in \alpha \land q \in \mathbb{Q} \land q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \land p \in \alpha \land q \in \mathbb{Q} \land q \notin \alpha) \implies \dots$

(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ from: *CutII*, 1

(1.2) $q < p \implies q \in \alpha$ ▮ $q \notin \alpha \implies q \geq p$ from: 1

(1.3) $(q \notin \alpha) \implies \dots$

(1.3.1) $q \geq p$ from: 1.2

(1.3.2) $(q = p) \implies (p \in \alpha \land p \notin \alpha) \implies \bot$ ▮ $q \neq p$ from: 1, 1.3

(1.3.3) $q \geq p \wedge q \neq p \;\blacksquare\; p < q$

from: 1

(1.4) $q \notin \alpha \implies p < q \;\blacksquare\; p < q$

(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

$CutCorollaryII := (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$

from: $CutII$, 1

(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$

from: 1, 1.1

(1.2) $s \in \alpha \implies \big(r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)\big) \;\blacksquare\; s \in \alpha \implies r \in \alpha$

from: 1, 1.2

(1.3) $r \notin \alpha \implies s \notin \alpha \;\blacksquare\; s \notin \alpha$

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$OrderTrichotomyOfR := OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$

from: $<_{\mathbb{R}}$, 1.1

(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$

(1.1.2) $\exists_{p'}(p' \in \alpha \wedge p' \notin \beta) \;\blacksquare\; p := choice(\{p' : p' \in \alpha \wedge p' \notin \beta\})$

(1.1.3) $q \in \beta \implies \ldots$

(1.1.3.1) $p, q \in \mathbb{Q}$

from: $CutCorollaryI$

(1.1.3.2) $q < p$

from: $CutII$

(1.1.3.3) $q \in \alpha$

(1.1.4) $q \in \beta \implies q \in \alpha$

(1.1.5) $\forall_{q \in \beta}(q \in \alpha) \;\blacksquare\; \beta \subseteq \alpha$

(1.1.6) $\beta \subset \alpha \;\blacksquare\; \beta <_{\mathbb{R}} \alpha$

(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \;\blacksquare\; (\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

$OrderTransitivityOfR := OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a \in \alpha}(a \in \beta) \wedge \forall_{b \in \beta}(b \in \gamma)$

(1.1.3) $\forall_{a \in \alpha}(\alpha \in \gamma) \;\blacksquare\; \alpha \subset \gamma \;\blacksquare\; \alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}}\big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma\big)$

(4) $OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

$OrderOfR := Order[<_{\mathbb{R}}, \mathbb{R}]$

from: $OrderTrichotomyR$, $OrderTransitivityR$
wts:

$LUBPropertyOfR := LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset \;\blacksquare\; \exists_{\alpha}(\alpha \in A) \;\blacksquare\; \alpha_0 := choice(\{\alpha : \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset \;\blacksquare\; \exists_{a}(a \in \alpha_0) \;\blacksquare\; a_0 := choice(\{a : a \in \alpha_0\}) \;\blacksquare\; a_0 \in \gamma \;\blacksquare\; \gamma \neq \emptyset$

(1.4) $BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}] \;\blacksquare\; \exists_{\beta}(UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.5) $\beta_0 := choice(\{\beta : UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}]\})$

(1.6) $UpperBound[\beta_0, A, \mathbb{R}, <_{\mathbb{R}}] \;\blacksquare\; \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \beta_0) \;\blacksquare\; \forall_{\alpha \in A}(\alpha \subseteq \beta_0) \;\blacksquare\; \forall_{\alpha \in A}\forall_{a \in \alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \;\blacksquare\; \forall_{a \in \gamma}(a \in \beta_0) \;\blacksquare\; \gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q} \;\blacksquare\; \gamma \subseteq \beta_0 \subset \mathbb{Q} \;\blacksquare\; \gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q} \;\blacksquare\; CutI[\gamma]$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.10.1) $p \in \gamma \;\blacksquare\; \exists_{\alpha \in A}(p \in \alpha) \;\blacksquare\; \alpha_1 := choice(\{\alpha \in A : p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \;\blacksquare\; q \in \alpha_1 \;\blacksquare\; q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \;\blacksquare\; \forall_{p \in \gamma}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma) \;\blacksquare\; CutII[\gamma]$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha \in A}(p \in \alpha) \;\blacksquare\; \alpha_2 := choice(\{\alpha \in A : p \in \alpha\})$

(1.12.2) $\alpha_2 \in \mathbb{R} \;\blacksquare\; CutII[\alpha_2] \;\blacksquare\; \exists_{r \in \alpha_2}(p < r) \;\blacksquare\; r_0 := choice(\{r \in \alpha_2 : p < r\})$

(1.12.3) $r_0 \in \alpha_2 \;\blacksquare\; r_0 \in \gamma$

(1.12.4) $p < r_0 \;\blacksquare\; p < r_0 \wedge r_0 \in \gamma \;\blacksquare\; \exists_{r \in \gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r \in \gamma}(p < r) \;\blacksquare\; \forall_{p \in \gamma}\exists_{r \in \gamma}(p < r) \;\blacksquare\; CutIII[\gamma]$

(1.14) $CutI[\gamma] \wedge CutII[\gamma] \wedge CutIII[\gamma] \;\blacksquare\; \gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha \in A}(\alpha \subseteq \gamma) \;\blacksquare\; \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma)$

(1.16) $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R} \;\blacksquare\; UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.17) $\delta <_{\mathbb{R}} \gamma \implies \ldots$

(1.17.1) $\delta \subset \gamma \;\blacksquare\; \exists_{s}(s \in \gamma \wedge s \notin \delta) \;\blacksquare\; s_0 := choice(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$

(1.17.2) $s_0 \in \gamma \;\blacksquare\; \exists_{\alpha \in A}(s_0 \in \alpha) \;\blacksquare\; \alpha_3 := choice(\{\alpha \in A : s_0 \in \alpha\})$

(1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta \;\blacksquare\; \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \ldots$

(1.17.4.1) $\alpha_3 \subseteq \delta \;\blacksquare\; \forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \;\blacksquare\; \neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4.2) $\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \;\blacksquare\; \perp$

(1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \perp \;\blacksquare\; \delta <_{\mathbb{R}} \alpha_3 \;\blacksquare\; \exists_{\alpha \in A}(\delta <_{\mathbb{R}} \alpha) \;\blacksquare\; \exists_{\alpha \in A}\big(\neg(\alpha \leq_{\mathbb{R}} \delta)\big)$

(1.17.6) $\neg\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \delta) \;\blacksquare\; \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}]) \;\blacksquare\; \forall_{\delta}(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.19) $UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \wedge \forall_{\delta}(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.20) $LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \;\blacksquare\; \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(3) $\forall_{A}\Big((\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])\Big) \;\blacksquare\; LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

$+_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$
$0_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 0\}$

$ZeroInR := 0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}} \;\blacksquare\; \emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q} \;\blacksquare\; CutI[0_{\mathbb{R}}]$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}} \;\blacksquare\; \forall_{x \in 0_{\mathbb{R}}}\forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}}) \;\blacksquare\; CutII[0_{\mathbb{R}}]$

(3) $y := x/2 \;\blacksquare\; (x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y) \;\blacksquare\; \forall_{x \in 0_{\mathbb{R}}}\exists_{y \in 0_{\mathbb{R}}}(x < y) \;\blacksquare\; CutIII[0_{\mathbb{R}}]$

(4) $CutI[0_{\mathbb{R}}] \wedge CutII[0_{\mathbb{R}}] \wedge CutIII[0_{\mathbb{R}}] \;\blacksquare\; 0_{\mathbb{R}} \in \mathbb{R}$

$FieldAdditionClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha) ; \exists_b(b \in \beta)$ ▌ $a_0 := choice(\{a : a \in \alpha\}) ; b_0 := choice(\{b : b \in \beta\})$ ▌ $a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha) ; \exists_y(y_0 \notin \beta)$ ▌ $x_0 := choice(\{x : x \notin \alpha\}) ; y_0 := choice(\{y : y \notin \beta\})$

(1.5) $\forall_{r\in\alpha}(r < x_0) ; \forall_{s\in\beta}(s < y_0)$ ▌ $\forall_{r\in\alpha}\forall_{s\in\beta}(r + s < x_0 + y_0)$ ▌ $x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q}$ ▌ $CutI[\alpha +_{\mathbb{R}} \beta]$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \land q \in \mathbb{Q} \land q < p) \implies \ldots$

(1.7.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s)$ ▌ $(r_0, s_0) := choice\big((r, s) \in \alpha \times \beta : p = r + s\big)$

(1.7.2) $q < p = r_0 + s_0$ ▌ $(q - s_0) < r_0$ ▌ $(q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta$ ▌ $q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta$ ▌ $q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \land q \in \mathbb{Q} \land q < p) \implies q \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{p\in\alpha+_{\mathbb{R}}\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta)$ ▌ $CutII[\alpha +_{\mathbb{R}} \beta]$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s)$ ▌ $(r_1, s_1) := choice\big(\{(r, s) \in \alpha \times \beta : p = r + s\}\big)$

(1.9.2) $r_1 \in \alpha$ ▌ $\exists_{t\in\alpha}(r_1 < t)$ ▌ $t_0 := choice(\{t \in \alpha : r_1 < t\})$

(1.9.3) $s_1 \in \beta$ ▌ $t + s_1 \in \alpha +_{\mathbb{R}} \beta \land p = r_1 + s_1 < t + s_1$ ▌ $\exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$ ▌ $\forall_{p\in\alpha+_{\mathbb{R}}\beta}\exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$ ▌ $CutIII[\alpha +_{\mathbb{R}} \beta]$

(1.11) $CutI[\alpha +_{\mathbb{R}} \beta] \land CutII[\alpha +_{\mathbb{R}} \beta] \land CutIII[\alpha +_{\mathbb{R}} \beta]$ ▌ $\alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R}\big)$

$FieldAdditionCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s : r \in \alpha \land s \in \beta\} = \{s + r : s \in \beta \land r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

$FieldAdditionAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)\big)$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c : a \in \alpha \land b \in \beta \land c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) : a \in \alpha \land b \in \beta \land c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

$FieldAdditionIdentityOfR := (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \implies \ldots$

(1.1.1) $s < 0$ ▌ $r + s < r + 0 = r$ ▌ $r + s < r$ ▌ $r + s \in \alpha$

(1.2) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \implies r + s \in \alpha$ ▌ $\forall_{r\in\alpha}\forall_{s\in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \land s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▌ $\forall_{p\in\alpha+_{\mathbb{R}}0_{\mathbb{R}}}(p \in \alpha)$ ▌ $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r\in\alpha}(p < r)$ ▌ $r_2 := choice(\{r \in \alpha : p < r\})$

(1.4.2) $p < r_2$ ▌ $p - r_2 < r_2 - r_2 = 0$ ▌ $(p - r_2) < 0$ ▌ $(p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha$ ▌ $p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▌ $p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▌ $\forall_{p\in\alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▌ $\alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \land \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▌ $0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

$FieldAdditionInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} : \exists_{r>0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q}$ ▌ $\exists_{s\in\mathbb{Q}}(s \notin \alpha)$ ▌ $s_0 := choice(\{s : s \notin \alpha\})$ ▌ $p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha$ ▌ $-p_0 - 1 \notin \alpha$ ▌ $\exists_{r>0}(-p_0 - r \notin \alpha)$ ▌ $p_0 \in \beta$

(1.4) $\emptyset \neq \alpha$ ▌ $\exists_{q\in\alpha}$ ▌ $q_0 := choice(\{q \in \mathbb{Q} : q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha$ ▌ $-(-q_0) - r = q_0 - r < q_0$ ▌ $-(-q_0) - r < q_0$ ▌ $-(-q_0) - r \in \alpha$

(1.6) $\forall_{r>0}\big(-(-q_0) - r \in \alpha\big)$ ▌ $\neg\exists_{r>0}\big(-(-q_0) - r \notin \alpha\big)$ ▌ $-q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q}$ ▌ $CutI[\beta]$

(1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.8.1) $p \in \beta$ ▌ $\exists_{r>0}(-p-r \notin \alpha)$ ▌ $r_0 := choice(\{r > 0 : -p-r \notin \alpha\})$

(1.8.2) $q < p$ ▌ $-p-r < -q-r$

(1.8.3) $-q-r \notin \alpha$ ▌ $q \in \beta$

(1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta$ ▌ $\forall_{p\in\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \beta)$ ▌ $CutII[\beta]$

(1.10) $p \in \beta \implies \ldots$

(1.10.1) $p \in \beta$ ▌ $\exists_{r>0}(-p-r \notin \alpha)$ ▌ $r_1 := choice(\{r > 0 : -p-r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0$ ▌ $r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p$ ▌ $t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -\big(p + (r_1/2)\big) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha$ ▌ $-t_0 - (r_1/2) \notin \alpha$ ▌ $\exists_{r>0}(-t_0 - r \notin \alpha)$ ▌ $t_0 \in \beta$

(1.10.7) $t_0 > p \wedge t_0 \in \beta$ ▌ $\exists_{t\in\beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t\in\beta}(p < t)$ ▌ $\forall_{p\in\beta}\exists_{t\in\beta}(p < t)$ ▌ $CutIII[\beta]$

(1.12) $CutI[\beta] \wedge CutII[\beta] \wedge CutIII[\beta]$ ▌ $\beta \in \mathbb{R}$

(1.13) $(r \in \alpha \wedge s \in \beta) \implies \ldots$

(1.13.1) $s \in \beta$ ▌ $\exists_{t>0}(-s-t \notin \alpha)$ ▌ $t_1 := choice(\{t > 0 : -s-t \notin \alpha\})$ ▌ $-s - t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha$ ▌ $-s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha$ ▌ $r < -s$ ▌ $r + s < 0$ ▌ $r + s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}}$ ▌ $\forall_{(r,s)\in\alpha\times\beta}(r + s \in 0_{\mathbb{R}})$ ▌ $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \ldots$

(1.15.1) $v < 0$ ▌ $w_0 := -v/2$ ▌ $w > 0$

(1.15.2) $\exists_{n\in\mathbb{Z}}\big(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\big)$ ▌ $n_0 := choice\big(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\}\big)$ from: $ARCHIMEDEANPROPERTYOFQ + LUB$???

(1.15.3) $p_0 := -(n_0 + 2)w_0$ ▌ $-p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha$ ▌ $-p_0 - w_0 \notin \alpha$ ▌ $p_0 \in \beta$

(1.15.4) $n_0 w_0 \in \alpha \wedge p_0 \in \beta$ ▌ $n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{v\in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta)$ ▌ $0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$ ▌ $\alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$ ▌ $\exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

$*_{\mathbb{R}}[\alpha, \beta] :=$ —
$1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$1IsNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ —
$1InR := 1_{\mathbb{R}} \in \mathbb{R}$ —
$FieldMultiplicationClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R}\big)$ —
$FieldMultiplicationCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ —
$FieldMultiplicationAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)\big)$ —
$FieldMultiplicationIdentityOfR := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ —
$FieldMultiplicationInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha\in\mathbb{R}}\big(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}\big)$ —

$FieldDistributativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ —

$FieldWithR := Field[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}]$ —
$OrderedFieldWithR := OrderedField[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$
$QROrderedSubfieldOfR := OrderedSubfield[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —
$QIsomorphicToQR := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ —
$CompletenessOfR := \exists_{\mathbb{R}}(LUBProperty[\mathbb{R}, <_{\mathbb{R}}] \wedge OrderedSubfield[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}])$ —

(1.20)
$ArchimedeanPropertyOfR := \forall_{x,y\in\mathbb{R}}\big(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y)\big)$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx : n \in \mathbb{N}^+\} \;\blacksquare\; (\emptyset \neq A \subset \mathbb{R}) \wedge \big(a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)\big)$

(1.2) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \;\blacksquare\; \forall_{n \in \mathbb{N}^+}(nx \leq y) \;\blacksquare\; UpperBound[y_0, A, \mathbb{R}, <] \;\blacksquare\; BoundedAbove[A, \mathbb{R}, <]$

(1.2.2) $CompletenessOfR \;\blacksquare\; LUBProperty[\mathbb{R}, <]$

(1.2.3) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (BoundedAbove[A, \mathbb{R}, <]) \;\blacksquare\; \exists_{\alpha \in \mathbb{R}}(LUB[\alpha, A, \mathbb{R}, <]) \;\ldots$

(1.2.4) $\ldots \alpha_0 := choice(\{\alpha \in \mathbb{R} : LUB[\alpha, A, \mathbb{R}, <]\}) \;\blacksquare\; LUB[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0 \;\blacksquare\; \alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge (LUB[\alpha_0, A, \mathbb{R}, <]) \;\blacksquare\; \neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <] \;\blacksquare\; \exists_{c \in A}(\alpha_0 - x < c) \;\ldots$

(1.2.8) $\ldots c_0 := choice(\{c \in A : \alpha_0 - x < c\}) \;\blacksquare\; (c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge \big(a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)\big) \;\blacksquare\; \exists_{m \in \mathbb{N}^+}(mx = c_0) \;\ldots$

(1.2.10) $\ldots m_0 := choice(\{m \in \mathbb{N}^+ : mx = c_0\}) \;\blacksquare\; (m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0) \;\blacksquare\; \alpha_0 - x < c_0 = m_0 x \;\blacksquare\; \alpha_0 < m_0 x + x \;\blacksquare\; \alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+ \;\blacksquare\; m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge \big(a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)\big) \;\blacksquare\; (m_0 + 1)x \in A$

(1.2.14) $\big(\alpha_0 < (m_0 + 1)x\big) \wedge \big((m_0 + 1)x \in A\big) \;\blacksquare\; \exists_{c \in A}(\alpha_0 < c)$

(1.2.15) $LUB[\alpha_0, A, \mathbb{R}, <] \;\blacksquare\; UpperBound[\alpha_0, A, \mathbb{R}, <] \;\blacksquare\; \forall_{c \in A}(c \leq \alpha_0) \;\blacksquare\; \neg\exists_{c \in A}(c > \alpha_0) \;\blacksquare\; \neg\exists_{c \in A}(\alpha_0 < c)$

(1.2.16) $\big(\exists_{c \in A}(\alpha_0 < c)\big) \wedge \big(\neg\exists_{c \in A}(\alpha_0 < c)\big) \;\blacksquare\; \bot$

(1.3) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \bot \;\blacksquare\; \exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y) \;\blacksquare\; \forall_{x, y \in \mathbb{R}}\big(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y)\big)$

$QDenseInR := \forall_{x, y \in \mathbb{R}}\Big(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y)\Big)$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y \;\blacksquare\; (0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $ArchimedeanPropertyOfR \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R}) \;\blacksquare\; \exists_{n \in \mathbb{N}^+}\big(n(y - x) > 1\big) \;\ldots$

(1.3) $\ldots n_0 := choice\big(\{n \in \mathbb{N}^+ : n(y - x) > 1\}\big) \;\blacksquare\; (n_0 \in \mathbb{N}^+) \wedge \big(n_0(y - x) > 1\big)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R}) \;\blacksquare\; n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R}) \;\blacksquare\; \exists_{m \in \mathbb{N}^+}\big(m(1) > n_0 x\big) \;\ldots$

(1.6) $\ldots m_1 := choice\big(\{m \in \mathbb{N}^+ : m(1) > n_0 x\}\big) \;\blacksquare\; (m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R}) \;\blacksquare\; \exists_{m \in \mathbb{N}^+}\big(m(1) > -n_0 x\big) \;\ldots$

(1.8) $\ldots m_2 := choice\big(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\}\big) \;\blacksquare\; (m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x) \;\blacksquare\; -m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+ \;\blacksquare\; |m_1 - (-m_2)| \geq 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge \big(|m_1 - (-m_2)| \geq 2\big) \;\blacksquare\; \exists_{m \in \mathbb{Z}}\big((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\big) \;\ldots$

(1.12) $\ldots m_0 := choice\big(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\}\big) \;\blacksquare\; (-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$

(1.13) $\big(n_0(y - x) > 1\big) \wedge (m_0 - 1 \leq n_0 x < m_0) \;\blacksquare\; n_0 x < m_0 \leq 1 + n_0 x < n_0 y \;\blacksquare\; n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y) \;\blacksquare\; x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z} \;\blacksquare\; m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y) \;\blacksquare\; \exists_{p \in \mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y) \;\blacksquare\; \forall_{x, y \in \mathbb{R}}\Big(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y)\Big)$

(1.21)
$RootLemma := (0 < a < b) \implies \Big(b^n - a^n \leq (b - a)nb^{n-1}\Big)$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b \;\blacksquare\; b/a > 1$

(1.3) $b/a > 1 \;\blacksquare\; \sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}\left(b^{n-i}a^{i-1}(b/a)^{i-1}\right) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1} \;\blacksquare\; \sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b-a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le (b-a)nb^{n-1} \;\blacksquare\; b^n - a^n \le (b-a)nb^{n-1}$

(2) $(0 < a < b) \implies \left(b^n - a^n \le (b-a)nb^{n-1}\right)$

$\mathit{RootExistenceInR} := \forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}({y_0}^n = x)$

(1) $(0 < x \in \mathbb{R} \land 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} : t > 0 \land t^n < x\} \;\blacksquare\; t \in E \iff (t \in \mathbb{R} \land t > 0 \land t^n < x)$

(1.2) $t_0 := x/(1+x) \;\blacksquare\; \left(t_0 = x/(1+x)\right) \land (t_0 \in \mathbb{R})$

(1.3) $0 < x \;\blacksquare\; 0 < x < 1 + x \;\blacksquare\; t_0 = x/(1+x) > 0 \;\blacksquare\; t_0 > 0$

(1.4) $1 = (1+x)/(1+x) > x/(1+x) = t_0 \;\blacksquare\; 1 > t_0$

(1.5) $(t_0 > 0) \land (1 > t_0) \;\blacksquare\; 0 < t_0 < 1$

(1.6) $(0 < n \in \mathbb{Z}) \land (0 < t_0 < 1) \;\blacksquare\; {t_0}^n \le t_0$

(1.7) $0 < x \;\blacksquare\; x > x/(1+x) = t_0 \;\blacksquare\; x > t_0$

(1.8) $({t_0}^n \le t_0) \land (x > t_0) \;\blacksquare\; {t_0}^n < x$

(1.9) $\left(t \in E \iff (t \in \mathbb{R} \land t > 0 \land t^n < x)\right) \land (t_0 \in \mathbb{R}) \land (t_0 > 0) \land ({t_0}^n < x) \;\blacksquare\; t_0 \in E \;\blacksquare\; \emptyset \ne E$

(1.10) $t_1 := \mathit{choice}(\{t \in \mathbb{R} : t > 1 + x\}) \;\blacksquare\; (t_1 \in \mathbb{R}) \land (t_1 > 1 + x)$

(1.11) $x > 0 \;\blacksquare\; t_1 > 1 + x > 1 \;\blacksquare\; t_1 > 1 \;\blacksquare\; {t_1}^n \ge t_1$

(1.12) $({t_1}^n \ge t_1) \land (t_1 > 1 + x) \land (1 > 0) \;\blacksquare\; {t_1}^n \ge t_1 > 1 + x > x \;\blacksquare\; {t_1}^n > x$

(1.13) $\left(t \in E \iff (t \in \mathbb{R} \land t > 0 \land t^n < x)\right) \land ({t_1}^n > x) \;\blacksquare\; t_1 \notin E \;\blacksquare\; E \subset \mathbb{R}$

(1.14) $(\emptyset \ne E) \land (E \subset \mathbb{R}) \;\blacksquare\; \emptyset \ne E \subset \mathbb{R}$

(1.15) $t \in E \implies \ldots$

(1.15.1) $(t \in E) \land \left(t \in E \iff (t \in \mathbb{R} \land t > 0 \land t^n < x)\right) \;\blacksquare\; t^n < x$

(1.15.2) $({t_1}^n > x) \land (t^n < x) \;\blacksquare\; t^n < x < {t_1}^n \;\blacksquare\; t < t_1$

(1.16) $t \in E \implies t < t_1 \;\blacksquare\; \forall_{t\in E}(t \le t_1) \;\blacksquare\; \mathit{UpperBound}[t_1, E, \mathbb{R}, <] \;\blacksquare\; \mathit{BoundedAbove}[E, \mathbb{R}, <]$

(1.17) $\mathit{CompletenessOfR} \;\blacksquare\; \mathit{LUBProperty}[\mathbb{R}, <]$

(1.18) $(\mathit{LUBProperty}[\mathbb{R}, <]) \land (\emptyset \ne E \subset \mathbb{R}) \land (\mathit{BoundedAbove}[E, \mathbb{R}, <]) \;\blacksquare\; \exists_{y\in\mathbb{R}}(\mathit{LUB}[y, E, \mathbb{R}, <]) \;\ldots$

(1.19) $\ldots y_0 := \mathit{choice}(\{y \in \mathbb{R} : \mathit{LUB}[y, E, \mathbb{R}, <]\}) \;\blacksquare\; \mathit{LUB}[y_0, E, \mathbb{R}, <]$

(1.20) $(\mathit{LUB}[y_0, E, \mathbb{R}, <]) \land (t_0 \in E) \land (t_0 > 0) \;\blacksquare\; 0 < t_0 \le y_0 \in \mathbb{R} \;\blacksquare\; 0 < y_0 \in \mathbb{R}$

(1.21) ${y_0}^n < x \implies \ldots$

(1.21.1) $k_0 := \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} \;\blacksquare\; k_0 \in \mathbb{R}$

(1.21.2) ${y_0}^n < x \;\blacksquare\; 0 < x - {y_0}^n$

(1.21.3) $(n > 0) \land (y_0 > 0) \;\blacksquare\; 0 < n(y_0 + 1)^{n-1}$

(1.21.4) $(0 < x - {y_0}^n) \land \left(0 < n(y_0+1)^{n-1}\right) \;\blacksquare\; 0 < \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} = k_0 \;\blacksquare\; 0 < k_0$

(1.21.5) $(0 < 1 \in \mathbb{R}) \land (0 < k_0 \in \mathbb{R}) \;\blacksquare\; 0 < \mathit{min}(1, k_0) \in \mathbb{R}$

(1.21.6) $\mathit{QDenseInR} \land \left(0, \mathit{min}(1, k_0) \in \mathbb{R}\right) \land \left(0 < \mathit{min}(1, k_0)\right) \;\blacksquare\; \exists_{h\in\mathbb{Q}}\left(0 < h < \mathit{min}(1, k_0)\right) \;\ldots$

(1.21.7) $\ldots h_0 := \mathit{choice}\left(\{h \in \mathbb{Q} : 0 < h < \mathit{min}(1, k_0)\}\right) \;\blacksquare\; (0 < h_0 < 1) \land \left(h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}\right)$

(1.21.8) $(y_0 > 0) \land (h_0 > 0) \;\blacksquare\; 0 < y_0 < y_0 + h_0$

(1.21.9) $\mathit{RootLemma} \land (0 < y_0 < y_0 + h_0) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}$

(1.21.10) $h_0 < 1 \;\blacksquare\; h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$

(1.21.11) $\left((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}\right) \land \left(h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}\right) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}$

(1.21.12) $\left(0 < n(y_0 + 1)^{n-1}\right) \land \left(h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}\right) \;\blacksquare\; h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n$

(1.21.13) $\left((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}\right) \land \left(h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n\right) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \;\blacksquare\; (y_0 + h_0)^n < x$

(1.21.14) $(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \;\blacksquare\; (y_0 + h_0)^n < x$

(1.21.15) $(0 < y_0\mathbb{R}) \land (0 < h_0 < \mathbb{R}) \;\blacksquare\; 0 < y_0 < y_0 + h_0 \in \mathbb{R}$

(1.21.16) $\left(t \in E \iff (t \in \mathbb{R} \land t > 0 \land t^n < x)\right) \land \left((y_0 + h_0)^n < x\right) \land (0 < y_0 + h_0 \in \mathbb{R}) \;\blacksquare\; (y_0 + h_0)^n \in E$

(1.21.17) $\big((y_0+h_0)^n \in E\big) \wedge (y_0 < y_0 + h_0) \blacksquare \exists_{e\in E}(y_0 < e)$

(1.21.18) $\mathit{LUB}[y_0, E, \mathbb{R}, <] \blacksquare \mathit{UpperBound}[y_0, E, \mathbb{R}, <] \blacksquare \forall_{e\in E}(e \le y_0) \blacksquare \neg\exists_{e\in E}(e > y_0)$

(1.21.19) $\big(\exists_{e\in E}(e > y_0)\big) \wedge \big(\neg\exists_{e\in E}(e > y_0)\big) \blacksquare \perp$

(1.22) ${y_0}^n < x \implies \perp \blacksquare {y_0}^n \ge x$

(1.23) ${y_0}^n > x \implies \ldots$

(1.23.1) $k_1 := \frac{{y_0}^n - x}{n{y_0}^{n-1}} \blacksquare (k_1 \in \mathbb{R}) \wedge (k_1 n {y_0}^{n-1} = {y_0}^n - x)$

(1.23.2) $(0 < x) \wedge (0 < n \in \mathbb{Z}) \blacksquare {y_0}^n - x < {y_0}^n \le n{y_0}^n \blacksquare {y_0}^n - x < n{y_0}^n$

(1.23.3) ${y_0}^n - x < n{y_0}^n \blacksquare k_1 = \frac{{y_0}^n - x}{n{y_0}^{n-1}} < \frac{n{y_0}^n}{n{y_0}^{n-1}} = y_0 \blacksquare k_1 < y_0$

(1.23.4) ${y_0}^n > x \blacksquare 0 < {y_0}^n - x$

(1.23.5) $(n > 0) \wedge (y_0 > 0) \blacksquare 0 < n{y_0}^{n-1}$

(1.23.6) $(0 < {y_0}^n - x) \wedge 0 < (n{y_0}^{n-1}) \blacksquare 0 < \frac{{y_0}^n - x}{n{y_0}^{n-1}} = k_1 \blacksquare 0 < k_1$

(1.23.7) $(k_1 < y_0) \wedge (0 < k_1) \blacksquare (0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$

(1.23.8) $t \ge y_0 - k_1 \implies \ldots$

(1.23.8.1) $t \ge y_0 - k_1 \blacksquare t^n \ge (y_0 - k_1)^n \blacksquare -t^n \le -(y_0 - k_1)^n \blacksquare {y_0}^n - t^n \le {y_0}^n - (y_0 - k_1)^n$

(1.23.8.2) $\mathit{RootLemma} \wedge (0 < y_0 - k_1 < y_0) \blacksquare {y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}$

(1.23.8.3) $\big({y_0}^n - t^n \le {y_0}^n - (y_0 - k_1)^n\big) \wedge \Big({y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}\Big) \blacksquare {y_0}^n - t^n < k_1 n {y_0}^{n-1}$

(1.23.8.4) $(k_1 n {y_0}^{n-1} = {y_0}^n - x) \wedge ({y_0}^n - t^n < k_1 n {y_0}^{n-1}) \blacksquare {y_0}^n - t^n < {y_0}^n - x \blacksquare -t^n < -x \blacksquare t^n > x$

(1.23.8.5) $\big(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\big) \wedge (t^n > x) \blacksquare t \notin E$

(1.23.9) $t \ge y_0 - k_1 \implies t \notin E \blacksquare t \in E \implies t < y_0 - k_1 \blacksquare \forall_{t\in E}(t \le y_0 - k_1) \blacksquare \mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.10) $\big(\mathit{LUB}[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)\big) \blacksquare \neg\mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.11) $(\mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg\mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \blacksquare \perp$

(1.24) ${y_0}^n > x \implies \perp \blacksquare {y_0}^n \le x$

(1.25) $\mathit{Order}[\mathbb{R}, <] \blacksquare \mathit{OrderTrichotomy}[\mathbb{R}, <]$

(1.26) $(\mathit{OrderTrichotomy}[\mathbb{R}, <]) \wedge ({y_0}^n \ge x) \wedge ({y_0}^n \le x) \blacksquare {y_0}^n = x$

(1.27) $({y_0}^n = x) \wedge (y_0 \in \mathbb{R}) \blacksquare \exists_{y\in\mathbb{R}}(y^n = x)$

(1.28) $y_1, y_2 := \mathit{choice}(\{y \in \mathbb{R} : y^n = x\})$

(1.29) $y_1 \ne y_2 \implies \ldots$

(1.29.1) $(\mathit{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_1 \ne y_2) \blacksquare (y_1 < y_2) \veebar (y_2 < y_1) \ \ldots$

(1.29.2) $\ldots (x = {y_1}^n < {y_2}^n = x) \veebar (x = {y_2}^n < {y_1}^n = x) \blacksquare (x < x) \veebar (x > x) \blacksquare \perp \veebar \perp \blacksquare \perp$

(1.30) $y_1 \ne y_2 \implies \perp \blacksquare y_1 = y_2 \blacksquare \forall_{a,b\in\mathbb{R}}\big((a^n = x \wedge b^n = x) \implies a = b\big)$

(1.31) $\Big(\exists_{y\in\mathbb{R}}(y^n = x)\Big) \wedge \Big(\forall_{a,b\in\mathbb{R}}\big((a^n = x \wedge b^n = x) \implies a = b\big)\Big) \blacksquare \exists!_{y\in\mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y\in\mathbb{R}}(y^n = x) \blacksquare \forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}({y_0}^n = x)$

$\mathit{RootExistenceInRCorollary} := \forall_{0<a\in\mathbb{R}}\forall_{0<b\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\Big((ab)^{1/n} = a^{1/n}b^{1/n}\Big)$ —

$$\mathit{ExtendedRealSystem}[\bar{\mathbb{R}}, +, *, <] := \left(\begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \ \wedge \ x - \infty = -\infty \ \wedge \ \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \ \wedge \\ (x > 0) \implies \big(x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty\big) \wedge \\ (x < 0) \implies \big(x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty\big) \end{array}\right)$$

$\mathbb{C} := \{\langle a, b\rangle \in \mathbb{R} \times \mathbb{R}\}$
$+_{\mathbb{C}}[\langle a, b\rangle, \langle c, d\rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d\rangle$
$*_{\mathbb{C}}[\langle a, b\rangle, \langle c, d\rangle] := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c\rangle$

$\mathit{FieldC} := \mathit{Field}[\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}}]$ —
$\mathit{RSubfieldC} := \mathit{Subfield}[\mathbb{R}, \mathbb{C}, +, *]$ —

$i := \langle 0, 1\rangle \in \mathbb{C}$

$\mathit{iProperty} := i^2 = -1$ —
$\mathit{CProperty} := (a, b \in \mathbb{R}) \implies (\langle a, b\rangle = a + bi)$ —

$Conjugate[\overline{a+bi}] := a - bi$

$ConjugateProperties := (w, z \in \mathbb{C}) \implies \ldots$ —

(1)	$\overline{z+w} = \overline{z} + \overline{w}$
(2)	$\overline{z*w} = \overline{z} * \overline{w}$
(3)	$Re(z) = (1/2)(z+\overline{z}) \wedge Im(z) = (1/2)(z-\overline{z})$
(4)	$0 \leq z * \overline{z} \in \mathbb{R}$

$AbsoluteValueC[|z|] = (z * \overline{z})^{1/2}$
$AbsoluteValueProperties := (z, w \in \mathbb{C}) \implies \ldots$ —

(1)	123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y} \big((x, y) \in f\big)$

$\big(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y\big) \implies \ldots$
(1) $Domain(f) := X; Codomain(f) := Y$
(2) $Image(f, A) := \{f(a) : a \in A\}; Preimage(f, B) := \{a : f(a) \in B\}$
(3) $Range(f) := Image\big(Domain(f)\big)$

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X} \big(x_1 \neq x_2 \implies f(x_1) \neq f(x_2)\big)$
$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X} \big(y_0 = f(x)\big)$
$Bijective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$
$SurjectiveEquivalent := \big(Range(f) = Codomain(f)\big) \implies Surjective(f)$

$\big(Function(f, X, Y) \wedge Function(g, Y, Z)\big) \implies (f \circ g)(x) := f\big(g(x)\big); Function(f \circ g, X, Z)$

$PropertiesofFunctions := \big(Function(f, A, B) \wedge Function(g, B, C) \wedge Function(h, C, D)\big) \implies \ldots$
(1) $h \circ (g \circ f) = (h \circ g) \circ f$
(2) $\big(Injective(f) \wedge Injective(g)\big) \implies Injective(g \circ f)$
(3) $\big(Surjective(f) \wedge Surjective(g)\big) \implies Surjective(g \circ f)$
(4) $\big(Bijective(f, A, B)\big) \implies \exists_{f^{-1}} \Bigg(Function(f^{-}1, B, A) \wedge \forall_{a \in A} \Big(f^{-1}\big(f(a)\big) = a\Big) \wedge \forall_{b \in B} \Big(f\Big(f^{-1}(b)\Big) = b\Big)\Bigg)$

$|(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}} (b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \ldots$
(1) $(a\|b) \implies a\|bc$
(2) $(a\|b \wedge b\|c) \implies a\|c$
(3) $(a\|b \wedge b\|c) \implies a\|(bx + cy)$
(4) $(a\|b \wedge b\|a) \implies a = \pm b$
(5) $(a\|b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$
(6) $(a\|b) \iff (m \neq 0 \wedge ma\|mb)$

$DivisionAlgorithm := (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}} (b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \wedge a : b \wedge a : c$
$GCD(a, b, c) := CD(a, b, c) \wedge \forall_d \big((d : b \wedge d : c) \implies d : a\big)$
$GCDEquivalent := 123123$

Chapter 3

Linear Algebra

3.1 Matrix Operations and Special Matrices

$Matrix[A,m,n] := [a_{i,j}]_{m\times n}$:= m rows, n columns of real numbers
$\mathcal{M}_{m,n} := \{A : Matrix[A,m,n]\}$
$O_{m,n} := (Matrix[O,m,n]) \wedge (a_{i,j} = 0)$
$Square[A,n] := Matrix[A,n,n]$
$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$
$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$
$Diagonal[A,n] := (Square[A,n]) \wedge (i \neq j \implies a_{i,j} = 0)$
$Scalar[A,n,k] := (Diagonal[A,n]) \wedge (a_{i,i} = k)$
$I_n := Scalar[I,n,1]$

$+(A,B) := \big((Matrix[A,m,n]) \wedge (Matrix[B,m,n])\big) \implies (A + B = [a_{i,j} + b_{i,j}]_{m\times n})$
$*\,(r,A) := \big((r \in \mathbb{R}) \wedge (Matrix[A,m,n])\big) \implies (r * A = [ra_{i,j}]_{m\times n})$
$*\,(A,B) := \big((Matrix[A,m,p]) \wedge (Matrix[B,p,n])\big) \implies \left(A * B = \left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]_{m\times n}\right)$
${}^T[A] := (Matrix[A,m,n]) \implies (A^T = [a_{j,i}]_{n\times m})$

$AddCom := \forall_{A,B\in\mathcal{M}}(A + B = B + A)$

(1) $A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C\in\mathcal{M}}\big((A + B) + C = A + (B + C)\big)$

(1) $(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A\in\mathcal{M}}\exists!_{O\in\mathcal{M}}(A + O = A = O + A)$

(1) $A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$

(2) $A + O_1 = A = A + O_2 \;\blacksquare\; O_1 = O_2$

$AddInv := \forall_{A\in\mathcal{M}}\exists!_{(-A)\in\mathcal{M}}\big(A + (-A) = O = (-A) + A\big)$

(1) $A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$

(2) $A + (-A_1) = O = A + (-A_2) \;\blacksquare\; -A_1 = -A_2 \;\blacksquare\; A_1 = A_2$

$MulAssoc := \forall_{A,B,C\in\mathcal{M}}\big((A * B) * C = A * (B * C)\big)$

(1) $(A * B) * C = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,j})\right] * C = \left[\sum_{k_2=1}^{p_2}\left(\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2})c_{k_2,j}\right)\right] = \left[\sum_{k_2=1}^{p_2}\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right] = \ldots$

(2) $\ldots\left[\sum_{k_1=1}^{p_1}\sum_{k_2=1}^{p_2}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right] = \left[\sum_{k_1=1}^{p_1}\left(a_{i,k_1}\sum_{k_2=1}^{p_2}(b_{k_1,k_2}c_{k_2,j})\right)\right] = \ldots = A * (B * C)$

$MulId := \forall_{A:Square[A,n]}(A * I_n = A = I_n * A)$

(1) $A * I_n = \left[\sum_{k=1}^{n}\left(a_{i,k}\left(\begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}\right)\right)\right] = [a_{i,j}] = A$

(2) $TODO = A$

$ScalAssoc := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} \big(r(sA) = (rs)A = s(rA)\big)$

(1) $r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$

(2) $(rs)A = [rsa_{i,j}]$

(3) $s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$

$TransCancel := \forall_{A \in \mathcal{M}} \big(A = (A^T)^T\big)$

(1) $A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$

$ScalMulCom := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} \big((rA) * B = r(A * B) = A * (rB)\big)$

(1) $(rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[\sum_{k=1}^{p}(ra_{i,k}b_{k,j})\right] = r(A * B)$

(2) $A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[\sum_{k=1}^{p}(a_{i,k}rb_{k,j})\right] = \left[\sum_{k=1}^{p}(ra_{i,k}b_{k,j})\right] = r(A * B)$

$ScalDistLeft := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} \big((r + s)A = rA + sA\big)$

(1) TODO

$ScalDistRight := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} \big(r(A + B) = rA + rB\big)$

(1) TODO

$MulDistRight := \forall_{A,B,C \in \mathcal{M}} \big((A + B) * C = A * C + B * C\big)$

(1) $(A + B) * C = [a_{i,j} + b_{i,j}] * C = \left[\sum_{k=1}^{p}\Big((a_{i,k} + b_{i,k})c_{k,j}\Big)\right] = \ldots$

(2) $\ldots \left[\sum_{k=1}^{p}(a_{i,k}c_{k,j} + b_{i,k}c_{k,j})\right] = \left[\sum_{k=1}^{p}(a_{i,k}c_{k,j})\right] + \left[\sum_{k=1}^{p}(b_{i,k}c_{k,j})\right] = A * C + B * C$

$MulDistLeft := \forall_{A,B,C \in \mathcal{M}} \big(C * (A + B) = C * A + C * B\big)$

(1) TODO

$TransAddDist := \forall_{A,B \in \mathcal{M}} \big((A + B)^T = A^T + B^T\big)$

(1) TODO

$TransMulDist := \forall_{A,B \in \mathcal{M}} \big((A * B)^T = B^T * A^T\big)$

(1) $(A * B)^T = \left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]^T = \left[\sum_{k=1}^{p}(a_{j,k}b_{k,i})\right] = \left[\sum_{k=1}^{p}(b_{k,i}a_{j,k})\right] = \left[\sum_{k=1}^{p}(b^T_{i,k}a^T_{k,j})\right] = B^T * A^T$

$Sym[A] := A = A^T$
$SkewSym[A] := A = -A^T$
$Invertible[A] := (Square[A, n]) \wedge \left(\exists_{A^{-1} \in \mathcal{M}}(A * A^{-1} = I_n = A^{-1} * A)\right)$

$SymGen := \forall_{A \in \mathcal{M}}(Sym[A + A^T])$

(1) $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$SkewSymGen := \forall_{A \in \mathcal{M}}(SkewSym[A - A^T])$

(1) $-(A - A^T)^T = -\big(A^T - (A^T)^T\big) = -(A^T - A) = (A - A^T)$

$SymDecomp := \forall_{A \in \mathcal{M}} \exists!_{B:Sym[B]} \exists!_{C:SkewSym[C]} (A = B + C)$

(1) $B := (1/2) * (A + A^T)$; $C := (1/2) * (A - A^T)$

(2) $SymGen[B] \wedge SkewSymGen[C]$

(3) $A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$

(4) $(1/2) * (A_1 + A_1{}^T) = (1/2) * (A_2 + A_2{}^T) \;\blacksquare\; A_1 = A_2$

(5) $(1/2) * (A_3 - A_3{}^T) = (1/2) * (A_4 - A_4{}^T) \;\blacksquare\; A_3 = A_4$

$InvId := \forall_{A:Invertible[A]} \Big(\exists!_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A) \Big)$

(1) $A^{-1}{}_1 = A^{-1}{}_1 * I_n = A^{-1}{}_1 * (A * A^{-1}{}_2) = (A^{-1}{}_1 * A) * A^{-1}{}_2 = I_n * A^{-1}{}_2 = A^{-1}{}_2$

$InvCancel := \forall_{A:Invertible[A]} \Big((A^{-1})^{-1} = A \Big)$

(1) $(A * A^{-1})^{-1} = I_n{}^{-1} = I_n$

(2) $(A^{-1})^{-1} * A^{-1} = I_n \;\blacksquare\; A^{-1})^{-1} = I_n * A = A$

$InvDist := \forall_{A:Invertible[A]} \forall_{B:Invertible[B]} \Big((A * B)^{-1} = B^{-1} * A^{-1} \Big)$

(1) $(A * B) * (A * B)^{-1} = I \;\blacksquare\; B * (A * B)^{-1} = A^{-1} \;\blacksquare\; (A * B)^{-1} = B^{-1} * A^{-1}$

$InvTrans := \forall_{A:Invertible[A]} \Big((A^T)^{-1} = (A^{-1})^T \Big) \;\blacksquare \Leftarrow$

(1) $A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I \;\blacksquare\; (A^{-1})^T = (A^T)^{-1}$

3.2 Elementary Matrices on Invertibility and Systems of Linear Equations

$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$
$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$
$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X (Sol[X, A, B])$
$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$
$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$

$HomoSysProps := (Sys[A, O]) \implies \ldots$

(1) $u_0 := O$; $u_1 := choice(\{X \in \mathcal{M} | X \neq O\})$; $k := choice(\mathbb{R})$

(2) $TrivSol[u_0, A]$

(3) $(NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$

(4) $(TrivSol[\overrightarrow{X}, A]) \implies \Big(TrivSol[LC(\overrightarrow{X}), A] \Big)$

$ElemMat[E] := (E = Swap[I_n, i, j]) \vee \big(Scale_*(I_n, i, c) \big) \vee \big(Combine_*(I_n, i, c, j) \big)$

$ElemMatProd[E^*] := \exists_{\langle E \rangle} \Big(\forall_{E_i \in E^*} (ElemMat[E_i]) \wedge \Big(E^* = \Pi_{E_i \in E^*} (E_i) \Big) \Big)$

$RowEquiv[A, B] := \exists_{E^*} \big((ElemMatProd[E^*]) \wedge (B = E^* * A) \big)$

$ElemMatInv := \forall_{E \in \mathcal{M}} \big((ElemMat[E]) \implies (Invertible[E]) \big)$

(1) $E - RowSwap[E] \implies TODO$; $E - RowScale_*(E) \implies TODO$; $E - RowCombine_*(E) \implies TODO$

$ElemMatProdInv := \forall_{E^*} \big((ElemMatProd[E^*]) \implies (Invertible[E^*]) \big)$

(1) TODO

$RowEquivSys := \forall_{A,B,C,D,X \in \mathcal{M}} \Big(\big((Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]]) \big) \implies (Sol[X, A, B] \iff Sol[X, C, D]) \Big)$

(1) $\exists_{E^*:ElemMatProd[E^*]} ([CD] = E^* * [AB])$

(2)	$(E^* * A = C) \wedge (E^* * B = D)$
(3)	$Sol[Y, A, B] \implies \ldots$
(3.1)	$A * Y = B$
(3.2)	$C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D$ ▮ $Sol[Y, C, D]$
(4)	$Sol[Y, A, B] \implies Sol[Y, C, D]$
(5)	$\left(A = (E^*)^{-1} * C\right) \wedge \left(B = (E^*)^{-1} * D\right)$
(6)	$Sol[Z, C, D] \implies \ldots$
(6.1)	$C * Z = D$
(6.2)	$A * Z = \left((E^*)^{-1} * C\right) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$
(7)	$Sol[Z, C, D] \implies Sol[Z, A, B]$
(8)	$Sol[X, A, B] \iff Sol[X, C, D]$

$$RowEquivHomoSysSol := \forall_{A,C,X \in \mathcal{M}}\Big((RowEquiv[A, C]) \implies \big((Sol[X, A, O]) \iff (Sol[X, C, O])\big)\Big)$$

(1)	Set $B = D = O$

$$RREF[A] := (A \in \mathcal{M}) \wedge \left(\begin{array}{l} \text{All zero rows are at the bottom of the matrix.} \wedge \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} \wedge \\ \text{The leading entry in any nonzero row is 1.} \wedge \\ \text{All entries in the column above and below a leading 1 are zero.} \wedge \end{array}\right)$$

$$GaussJordanElim := \forall_{A \in \mathcal{M}} \exists!_{B \in \mathcal{M}}\big((RREF[B]) \wedge (RowEquiv[A, B])\big)$$

(1)	Hit A with $ElemMat$'s until it becomes B
(2)	$(B = E^* * A) \wedge (RREF[B])$

$$HasZero[A] := \big(Matrix(A, m, n)\big) \wedge \big(\exists_{i \le m}(A_{i,:} = O)\big)$$

$$HasZeroNonInvertible := \forall_{A \in \mathcal{M}}\big((HasZero[A]) \implies (\neg Invertible[A])\big)$$

(1)	$i := choice(\{i \le m \mid A_{i,:} = O\})$
(2)	$(B \in \mathcal{M}) \implies \ldots$
(2.1)	$(A * B)_{i,:} = O \neq I_{n i,:}$ ▮ $A * B \neq I_n$
(3)	$(B \in \mathcal{M}) \implies (A * B \neq I_n)$ ▮ $\forall_{B \in \mathcal{M}}(A * B \neq I_n)$ ▮ $\neg Invertible[A]$

$$InvIffRowEquivI := \forall_{A \in \mathcal{M}}\big((Invertible[A]) \iff (RowEquiv[A, I_n])\big)$$

(1)	$(Invertible[A]) \implies \ldots$
(1.1)	$(RREF[B]) \wedge (RowEquiv[A, B])$
(1.2)	$B = E^* * A$
(1.3)	$(Invertible[E^*]) \wedge (Invertible[A])$ ▮ $Invertible[B]$
(1.4)	$Invertible[B]$ ▮ $\neg HasZero[B]$
(1.5)	$(RREF[B]) \wedge (\neg HasZero[B])$ ▮ $B = I_n$
(1.6)	$RowEquiv[A, I_n]$
(2)	$(Invertible[A]) \implies (RowEquiv[A, I_n])$
(3)	$(RowEquiv[A, I_n]) \implies \ldots$
(3.1)	$I_n = E^* * A$ ▮ $(E^*)^{-1} = A$
(3.2)	$A^{-1} = E^*_{DescSort}$ ▮ $Invertible[A]$
(4)	$(RowEquiv[A, I_n]) \implies (Invertible[A])$
(5)	$(Invertible[A]) \iff (RowEquiv[A, I_n])$

$$RowEquivIIffTrivSol := \forall_{A \in \mathcal{M}}\Big((RowEquiv[A, I_n]) \iff \Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big)\Big)$$

(1)	$(RowEquiv[A, I_n]) \implies \ldots$
(1.1)	$RowEquiv[A, I_n]$ ▮ $Invertible[A]$

(1.2) $(Sol[X, A, O]) \implies \ldots$

(1.2.1) $A * X = O \blacksquare X = A^{-1} * O = O \blacksquare X = O$

(1.3) $(Sol[X, A, O]) \implies (X = O)$

(1.4) $(X = O) \implies (Sol[X, A, O])$

(1.5) $(X = O) \iff (Sol[X, A, O]) \blacksquare \forall_X \big((X = O) \iff (Sol[X, A, O])\big)$

(2) $(RowEquiv[A, I_n]) \implies \Big(\forall_X \big((X = O) \iff (Sol[X, A, O])\big)\Big)$

(3) $\Big(\forall_X \big((X = O) \iff (Sol[X, A, O])\big)\Big) \implies \ldots$

(3.1) $(RREF[B]) \wedge (RowEquiv[A, B])$

(3.2) $Sol[X, B, O]$

(3.3) $(B \neq I_n) \implies \ldots$

(3.3.1) $\Big(\exists_{Y \neq X}(Sol[Y, B, O])\Big)$

(3.3.2) $Sol[Y, A, O] \blacksquare Y = X$

(3.3.3) $(Y \neq X) \wedge (Y = X) \blacksquare \perp$

(3.4) $(B \neq I_n) \implies \perp \blacksquare B = I_n$

(3.5) $(RowEquiv[A, B]) \wedge (B = I_n) \blacksquare RowEquiv[A, I_n]$

(4) $\Big(\forall_X \big((X = O) \iff (Sol[X, A, O])\big)\Big) \implies (RowEquiv[A, I_n])$

(5) $(RowEquiv[A, I_n]) \iff \Big(\forall_X \big((X = O) \iff (Sol[X, A, O])\big)\Big)$

$$InvIffUniqSol := \forall_{A \in \mathcal{M}} \Big((Invertible[A]) \iff \big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}} (Sol[X, A, B])\big)\Big)$$

(1) $(Invertible[A] \wedge B \in \mathcal{M}) \implies \ldots$

(1.1) $(Invertible[A]) \wedge (Sys[A, B])$

(1.2) $(X = A^{-1} * B) \iff (Sol[X, A, B]) \blacksquare \exists!_{X \in \mathcal{M}}(Sol[X, A, B])$

(2) $\big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}} (Sol[X, A, B])\big) \implies \ldots$

(2.1) $X_i := choice(\{X_i | Sol[X_i, A, I_{n:,i}]\})$

(2.2) $A * [X_1 \ldots X_n] = [(A * X_1) \ldots (A * X_n)] = [I_{n:,1} \ldots I_{n:,n}] = I_n$

(2.3) $A^{-1} = [X_1 \ldots X_n]$

(3) $\big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}} (Sol[X, A, B])\big) \implies (Invertible[A])$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}} \left(\begin{array}{ll} (Invertible[A]) & \iff \\ (RowEquiv[A, I_n]) & \iff \\ \Big(\forall_X \big((X = O) \iff (Sol[X, A, O])\big)\Big) & \iff \\ \big(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}} (Sol[X, A, B])\big) & \end{array} \right)$$

3.3 Vector Spaces

$$VectorSpace[V, +, *] := \exists_{O \in V} \forall_{\alpha, \beta \in \mathbb{R}} \forall_{u,v,w \in V} \left(\begin{array}{l} (u + v \in V) \ \wedge \ (u + v = v + u) \ \wedge \ \big((u + v) + w = u + (v + w)\big) \ \wedge \\ (u + O = u) \quad \wedge \quad \Big(\exists_{-u \in V}\big(u + (-u) = O\big)\Big) \quad \wedge \\ (\alpha * u \in V) \ \wedge \ \big(\alpha * (\beta * u) = (\alpha\beta) * u\big) \ \wedge \ (1 * u = u) \ \wedge \\ \big(\alpha * (u + v) = (\alpha * u) + (\alpha * v)\big) \wedge \big((\alpha + \beta) * u = (\alpha * u) + (\beta * u)\big) \end{array} \right)$$

$$ZeroVectorUniq := \forall_{O', v \in V} \big((v + O' = v) \implies (O' = O)\big)$$

(1) $O' = O' + O = O + O' = O \blacksquare O' = O$

$$AddInvUnique := \forall_{-v', v \in V} \big((v + -v' = O) \implies (-v' = -v)\big)$$

(1) $-v' = -v' + O = -v' + (v + -v) = (-v' + v) + -v = (v + -v') + -v = O + -v = -v \blacksquare -v' = -v$

$$AddInvGen := \forall_{v \in V} \big((-1) * v = -v\big)$$

(1) $v + (-1) * v = (1-1) * v = 0 * v = O \blacksquare (-1) * v = -v$

$ZeroVectorGenLeft := \forall_{v \in V}(0 * v = O)$

(1) $0 * v = (0+0) * v = (0 * v) + (0 * v) \blacksquare O = 0 * v$

$ZeroVectorGenRight := \forall_{r \in \mathbb{R}}(r * O = O)$

(1) $r * O = r * (O + O) = (r * O) + (r * O) \blacksquare O = r * O$

$ZeroVectorEquiv := \forall_{r \in \mathbb{R}} \forall_{v \in V} \Big((r * v = O) \iff \big((v = O) \vee (r = 0) \big) \Big)$

(1) $(ZeroVectorGenLeft) \wedge (ZeroVectorGenRight) \blacksquare \big((v = O) \vee (r = 0)\big) \implies (r * v = O))$

(2) $(r * v = O) \implies \ldots$

(2.1) $(r \neq 0) \implies \ldots$

(2.1.1) $r \neq 0 \blacksquare r^{-1} \in \mathbb{R}$

(2.1.2) $ZeroVectorGenRight \blacksquare O = r^{-1} * O = r^{-1} * (r * v) = (r^{-1} r) * v = 1 * v = v \blacksquare O = v$

(2.2) $(r \neq 0) \implies (v = O) \blacksquare (r = 0) \vee (r \neq 0) \blacksquare (r = 0) \vee (v = O)$

(3) $(r * v = O) \implies \big((r = 0) \vee (v = O)\big)$

(4) $(r * v = O) \iff \big((r = 0) \vee (v = O)\big)$

3.4 Subspaces and Special Subspaces

$Subspace[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (\emptyset \neq S \subseteq V) \wedge (VectorSpace[S, +, *])$

$$SubspaceEquiv := \forall_{V,S} \left(\begin{array}{l} (VectorSpace[V, +, *]) \qquad\qquad\qquad\qquad \implies \\ \Big((Subspace[S, V, +, *]) \iff \Big((\emptyset \neq S \subseteq V) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big) \Big) \Big) \end{array} \right)$$

(1) $(Subspace[S, V, +, *]) \implies \ldots$

(1.1) $Subspace[S, V, +, *] \blacksquare \emptyset \neq S \subseteq V$

(1.2) $VectorSpace[S, +, *] \blacksquare \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)$

(1.3) $(\emptyset \neq S \subseteq V) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)$

(2) $(Subspace[S, V, +, *]) \implies \Big((\emptyset \neq S \subseteq V) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big) \Big)$

(3) $\Big((\emptyset \neq S \subseteq V) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big) \Big) \implies \ldots$

(3.1) $\big((\alpha, \beta \in \mathbb{R}) \wedge (\emptyset \neq S) \wedge (u, v, w \in S)\big) \implies \ldots$

(3.1.1) $u, v \in V \blacksquare u + v = v + u$

(3.1.2) $u, v, w \in V \blacksquare (u + v) + w = u + (v + w)$

(3.1.3) $(ZeroVectorGenLeft) \wedge (u \in S) \blacksquare 0 * u = O \in S$

(3.1.4) $u \in V \blacksquare u + O = u$

(3.1.5) $(AddInvGen) \wedge (u \in S) \blacksquare (-1) * u = -u \in S$

(3.1.6) $u \in V \blacksquare \alpha * (\beta * u) = (\alpha\beta) * u$

(3.1.7) $u \in V \blacksquare 1 * u = u$

(3.1.8) $u, v \in V \blacksquare \alpha * (u + v) = (\alpha * u) + (\alpha * v)$

(3.1.9) $u \in V \blacksquare (\alpha + \beta) * u = (\alpha * u) + (\beta * u)$

(4) $\Big(\big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big) \Big) \implies (Subspace[S, V, +, *])$

(5) $(Subspace[S, V, +, *]) \iff \Big(\big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big) \Big)$

$$SumSubContains := \forall_{A,B,V} \left(\begin{array}{l} \big((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])\big) \implies \\ \big((Subspace[A + B, V, +, *]) \wedge (A, B \subseteq A + B)\big) \end{array} \right)$$

(1) $(Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \;\blacksquare\; (O \in A) \wedge (O \in B)$

(2) $(SetSum[A+B,A,B,V,+,*]) \wedge (O \in A) \wedge (O \in B) \;\blacksquare\; O \in A+B \;\blacksquare\; \emptyset \neq A+B$

(3) $(Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \;\blacksquare\; A+B \subseteq V \;\blacksquare\; \emptyset \neq A+B \subseteq V$

(4) $(u,v \in A+B) \implies \ldots$

  (4.1) $\left(\exists_{a_1 \in A}\exists_{b_1 \in B}(u = a_1 + b_1)\right) \wedge \left(\exists_{a_2 \in A}\exists_{b_2 \in B}(v = a_2 + b_2)\right)$

  (4.2) $u + v = (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$

  (4.3) $(a_1 + a_2 \in A) \wedge (b_1 + b_2 \in B) \;\blacksquare\; u + v \in A + B$

(5) $(u,v \in A+B) \implies (u+v \in A+B) \;\blacksquare\; \forall_{u,v \in A+B}(u+v \in A+B)$

(6) $\left((r \in \mathbb{R}) \wedge (v \in A+B)\right) \implies \ldots$

  (6.1) $\exists_{a \in A}\exists_{b \in B}(v = a + b)$

  (6.2) $r * v = r * (a+b) = r * a + r * b$

  (6.3) $(r * a \in A) \wedge (r * b) \in B \;\blacksquare\; r * v \in A + B$

(7) $\left((r \in \mathbb{R}) \wedge (v \in A+B)\right) \implies (r * v \in A+B) \;\blacksquare\; \forall_{r \in \mathbb{R}}\forall_{v \in A+B}(r * v \in A+B)$

(8) $(\emptyset \neq A+B \subseteq V) \wedge \left(\forall_{u,v \in A+B}(u+v \in A+B)\right) \wedge \left(\forall_{r \in \mathbb{R}}\forall_{v \in A+B}(r * v \in A+B)\right) \;\blacksquare\; Subspace[A+B,V,+,*]$

(9) $\left(\forall_{a \in A}(a + O) = a\right) \wedge (O \in B) \;\blacksquare\; A \subseteq A+B$

(10) $\left(\forall_{b \in B}(b + O) = b\right) \wedge (O \in A) \;\blacksquare\; B \subseteq A+B$

(11) $(A,B \subseteq A+B) \wedge (Subspace[A+B,V,+,*])$

$$SumSubMinContains := \forall_{A,B,V}\left(\begin{array}{l} \left((Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \wedge (SetSum[A+B,A,B,V,+,*])\right) \implies \\ \left(\forall_C\left((Subspace[C,V,+,*]) \wedge (A,B \subseteq C)\right) \implies (A+B \subseteq C)\right) \end{array}\right)$$

(1) $SumSub \;\blacksquare\; (A,B \subseteq A+B) \wedge (Subspace[A+B,V,+,*])$

(2) $\left((Subspace[C,V,+,*]) \wedge (A,B \subseteq C)\right) \implies \ldots$

  (2.1) $(s \in A+B) \implies \ldots$

    (2.1.1) $\exists_{a \in A}\exists_{b \in B}(s = a + b)$

    (2.1.2) $(A,B \subseteq C) \;\blacksquare\; a,b \in C$

    (2.1.3) $Subspace[C,V,+,*] \;\blacksquare\; s = a + b \in C$

  (2.2) $(s \in A+B) \implies (s \in C) \;\blacksquare\; A+B \subseteq C$

(3) $\left((Subspace[C,V,+,*]) \wedge (A,B \subseteq C)\right) \implies (A+B \subseteq C)$

$NullSpace[N,A,m,n] := (Matrix[A,m,n]) \wedge (N = \{x \in \mathbb{R}^n | A * x = O\})$
$RowSpace[R,A,m,n] := (Matrix[A,m,n]) \wedge (R = \{x^T * A \in \mathbb{R}^n | x \in \mathbb{R}^m\})$
$ColSpace[C,A,m,n] := (Matrix[A,m,n]) \wedge (C = \{A * x \in \mathbb{R}^m | x \in \mathbb{R}^n\})$

$NullSubspace := (NullSpace[N,A,m,n]) \implies (Subspace[N,\mathbb{R}^n,+,*])$

(1) TODO

$RowSubspace := (RowSpace[R,A,m,n]) \implies (Subspace[R,\mathbb{R}^n,+,*])$

(1) TODO

$ColSubspace := (ColSpace[C,A,m,n]) \implies (Subspace[C,\mathbb{R}^m,+,*])$

(1) TODO

3.5 Linear Combination, Linear Span, Linear Independence

$LinComb[c,U,K,V,+,*] := (VectorSpace[V,+,*]) \wedge (n \in \mathbb{N}) \wedge (U \in V^n) \wedge (K \in \mathbb{R}^n) \wedge \left(c = \sum_{i=1}^n (k_i * u_i)\right)$

$$LinSpan[S',S,V,+,*] := \left(\begin{array}{l} (VectorSpace[V,+,*]) \wedge (S \in V^n) \wedge \left((S = \emptyset) \implies (S' = \{O\})\right) \wedge \\ \left((S \neq \emptyset) \implies \left(S' = \{c \in V | \exists_{K \in \mathbb{R}^n}(LinComb[c,S,K,V,+,*])\}\right)\right) \end{array}\right)$$

$LinSpanSubContains := \forall_{S',S,V,+,*}\Big((LinSpan[S',S,V,+,*]) \implies \big((Subspace[S',V,+,*]) \wedge (S \subseteq S')\big)\Big)$

(1) $(S = \emptyset) \implies (S' = \{O\}) \implies (\emptyset \neq S')$

(2) $(S \neq \emptyset) \implies (LinComb[O,S,\{0\}^n,V,+,*]) \implies (O \in S') \implies (\emptyset \neq S')$

(3) $\big((S = \emptyset) \vee (S \neq \emptyset)\big) \implies (\emptyset \neq S') \blacksquare\ \emptyset \neq S'$

(4) $LinSpan[S',S,V,+,*] \blacksquare\ S' \subseteq V \blacksquare\ \emptyset \neq S' \subseteq V$

(5) $(a,b \in S') \implies \ldots$

(5.1) $\big(\exists_{K\in\mathbb{R}^n}(LinComb[a,S,K,V,+,*])\big) \wedge \big(\exists_{L\in\mathbb{R}^n}(LinComb[b,S,L,V,+,*])\big)$

(5.2) $a + b = \sum_{i=1}^n (k_i * s_i) + \sum_{i=1}^n (l_i * s_i) = \sum_{i=1}^n \big((k_i + l_i) * s_i\big) \blacksquare\ a + b = \sum_{i=1}^n \big((k_i + l_i) * s_i\big)$

(5.3) $\langle k_i + l_i | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{N}^n$

(5.4) $\Big(a + b = \sum_{i=1}^n \big((k_i + l_i) * s_i\big)\Big) \wedge (\langle k_i + l_i | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{N}^n) \blacksquare\ \exists_{M\in\mathbb{N}^n}\big(a + b = \sum_{i=1}^n (m_i * s_i)\big)$

(5.5) $\exists_{M\in\mathbb{N}^n}(LinComb[a+b,S,M,V,+,*]) \blacksquare\ a + b \in S'$

(6) $(a,b \in S') \implies (a + b \in S') \blacksquare\ \forall_{a,b\in S'}(a + b \in S')$

(7) $\big((r \in \mathbb{R}) \wedge (u \in S')\big) \implies \ldots$

(7.1) $\exists_{K\in\mathbb{R}^n}(LinComb[u,S,K,V,+,*]) \blacksquare\ \exists_{K\in\mathbb{R}^n}\big(u = \sum_{i=1}^n (k_i * s_i)\big)$

(7.2) $r * u = r * \sum_{i=1}^n (k_i * s_i) = \sum_{i=1}^n \big(r * (k_i * s_i)\big) = \sum_{i=1}^n (rk_i) * s_i) \blacksquare\ r * u = \sum_{i=1}^n (rk_i) * s_i)$

(7.3) $\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n$

(7.4) $\big(\sum_{i=1}^n (rk_i) * s_i)\big) \wedge (\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n)$

(7.5) $\exists_{M\in\mathbb{R}^n}\big(r * u = \sum_{i=1}^n (m_i * s_i)\big) \blacksquare\ \exists_{M\in\mathbb{R}^n}(LinComb[r * u,S,M,V,+,*]) \blacksquare\ r * u \in S'$

(8) $\big((r \in \mathbb{R}) \wedge (u \in S')\big) \implies (r * u \in S') \blacksquare\ \forall_{r\in\mathbb{R}}\forall_{u\in S'}(r * u \in S')$

(9) $(SubspaceEquiv) \wedge (\emptyset \neq S' \subseteq V) \wedge \big(\forall_{a,b\in S'}(a + b \in S')\big) \wedge \big(\forall_{r\in\mathbb{R}}\forall_{u\in S'}(r * u \in S')\big) \blacksquare\ Subspace[S',V,+,*]$

(10) $(s_j \in S) \implies \ldots$

(10.1) $K := \langle\left(\begin{cases} 1 & i = j \\ 0 & i \neq j\end{cases}\right) | \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n \blacksquare\ \sum_{i=1}^n (k_i * s_i) = s_j$

(10.2) $\ldots \blacksquare\ \exists_{K\in\mathbb{R}^n}(LinComb[s_j,S,K,V,+,*]) \blacksquare\ s_j \in S'$

(11) $(s_j \in S) \implies (s_j \in S') \blacksquare\ \forall_{x\in S}(x \in S') \blacksquare\ S \subseteq S'$

(12) $(Subspace[S',V,+,*]) \wedge (S \subseteq S')$

$LinSpanSubMinContains := \forall_{S',S,V,+,*}\Big((LinSpan[S',S,V,+,*]) \implies \Big(\forall_W\big(((Subspace[W,V,+,*]) \wedge (S \subseteq W)\big) \implies (S' \subseteq W)\Big)\Big)$

(1) $(s' \in S') \implies \ldots$

(1.1) $\exists_{K\in\mathbb{R}^n}(LinComb[s',S,K,V,+,*]) \blacksquare\ s' = \sum_{i=1}^n (k_i * s_i)$

(1.2) $(S \subseteq W) \wedge (Subspace[W,V,+,*]) \blacksquare\ s' = \sum_{i=1}^n (k_i * s_i) \in W$

(2) $(s' \in S') \implies (s' \in W) \blacksquare\ S' \subseteq W$

$LinInd[S,V,+,*] := (VectorSpace[V,+,*]) \wedge (S \in V^n) \wedge \Big((S \neq \emptyset) \implies \Big(\forall_{K\in\mathbb{R}^n}\big((LinComb[O,S,K,V,+,*]) \implies (K = \{0\}^n)\big)\Big)\Big)$

$ZeroDependent := (O \in S) \implies (\neg LinInd[S,V,+,*])$

(1) $K := \langle\left(\begin{cases} 1 & u_i = O \\ 0 & u_i \neq O\end{cases}\right) | (1 \leq i \leq n) \wedge (i \in \mathbb{N})\rangle \blacksquare\ K \in \mathbb{R}^n$

(2) $(LinComb[O,S,K,V,+,*]) \wedge (K \neq \{O\}^n) \blacksquare\ \neg LinInd[S,V,+,*]$

$SingletonNonZeroIndependent := (v \neq O) \implies (LinInd[\langle v\rangle,V,+,*])$

(1) $(r * v = O) \iff \big((r = 0) \vee (v \neq O)\big)$

(2) $v \neq O \blacksquare\ r = 0$

(3) $\forall_{r\in\mathbb{R}}\big((r * v = O) \implies (r = 0)\big)$

$$SubIndependent := \forall_{V,A,B}\Big(\big((VectorSpace[V,+,*]) \wedge (A \subseteq B \in V^m)\big) \implies \big((LinInd[B,V,+,*]) \implies (LinInd[A,V,+,*])\big)\Big)$$

(1) $(LinComb[O,A,K,V,+,*]) \implies \dots$

(1.1) $L := \left\langle \left(\begin{cases} 1 & j \le n \\ 0 & j > n \end{cases} \right) \middle| (1 \le j \le m \wedge (j \in \mathbb{N})\right\rangle \blacksquare\ L \in \mathbb{R}^m$

(1.2) $A \subseteq B \ \blacksquare\ \forall_{n \ge j \in \mathbb{N}}(a_j = b_j)$

(1.3) $\forall_{n \ge j \in \mathbb{N}}(a_j = b_j) \ \blacksquare\ \sum_{i=1}^{n}(k_i * a_i)) = \sum_{i=1}^{n}(k_i * a_i)) + O = \sum_{j=1}^{m}(l_j * b_j))$

(1.4) $LinComb[O,A,K,V,+,*] \ \blacksquare\ O = \sum_{i=1}^{n}(k_i * a_i)$

(1.5) $O = \sum_{i=1}^{n}(k_i * a_i) = \sum_{j=1}^{m}(l_j * b_j) \ \blacksquare\ LinComb[O,B,L,V,+,*]$

(1.6) $(LinInd[B,V,+,*]) \wedge (LinComb[O,B,L,V,+,*]) \ \blacksquare\ L = \{0\}^m$

(1.7) $\big(\forall_{n \ge j \in \mathbb{N}}(a_j = b_j)\big) \wedge (L = \{0\}^m) \ \blacksquare\ \forall_{n \ge j \in \mathbb{N}}(k_j * a_j = l_j * b * j = l_j * a_j) \ \blacksquare\ K = \{0\}^n$

(2) $(LinComb[O,A,K,V,+,*]) \implies (K = \{0\}^n) \ \blacksquare\ \forall_{K \in \subseteq \mathbb{R}^n}\big((LinComb[O,A,K,V,+,*]) \implies (K = \{0\}^n)\big) \ \blacksquare\ LinInd[A,V,+,*]$

$$SuperDependent := \forall_{V,A,B}\Big(\big((VectorSpace[V,+,*]) \wedge (A \subseteq B \subseteq V)\big) \implies \big((\neg LinInd[A,V,+,*]) \implies (\neg LinInd[B,V,+,*])\big)\Big)$$

(1) TODO : A has a non trivial solution $\blacksquare$ use the same non trivial solution in combination with B and L

$$LinIndEquiv := \forall_{U,V}\Big((LinInd[U,V,+,*]) \iff \big(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])\big)\Big)$$

(1) $\Gamma' = \Gamma \setminus \{j\}$

(2) $(\neg LinInd[U,V,+,*]) \implies \dots$

(2.1) $\Big(\exists_{\Gamma \in \mathbb{R}^{|U|}}\Big(\big(\sum(\gamma_i * u_i) = O\big) \wedge (\Gamma \neq \{0\}^{|U|})\Big)\Big)$

(2.2) $\exists_{\gamma_k \in \Gamma}(\gamma_k \neq 0)$

(2.3) $\sum(\gamma_i' * u_i) = \sum(\gamma_i * u_i) - \gamma_k * u_k = -\gamma_k * u_j$

(2.4) $u_k = (-1/\gamma_k)\Big(\sum(\gamma_i' * u_i)\Big) = \sum\Big((-\gamma_i'/\gamma_k) * u_i\Big) \ \blacksquare\ \exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])$

(3) $(\neg LinInd[U,V,+,*]) \implies \big(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])\big)$

(4) $\big(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])\big) \implies (LinInd[U,V,+,*])$

(5) $\big(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])\big) \implies \dots$

(5.1) $\exists_{j \in U}\big(j = \sum(\gamma_i' * u_i)\big)$

(5.2) $\Gamma := \Gamma' \cup \{-1\}$

(5.3) $\Big(\sum(\gamma_i * u_i) = \sum(\gamma_i' * u_i) + (-1) * \gamma_j = O\Big) \wedge (\Gamma \neq \{0\}^n) \ \blacksquare\ \neg LinInd[U,V,+,*]$

(6) $\big(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])\big) \implies (\neg LinInd[U,V,+,*])$

(7) $(LinInd[U,V,+,*]) \implies \big(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])\big)$

(8) $(LinInd[U,V,+,*]) \iff \big(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])\big)$

$$LinIndSuperspace := \forall_{U,V}\Big((Subspace[U,V]) \implies \big(\forall_W\big((LinInd[W,U,+,*]) \implies (LinInd[W,V,+,*])\big)\big)\Big)$$

(1) $(\neg LinInd[W,V,+,*]) \implies \dots$

(1.1) $\exists_{j \in W}(LinComb[j, W \setminus \{j\}, +, *]) \ \blacksquare\ \neg LinInd[W,U,+,*]$

(1.2) $(LinInd[W,U,+,*]) \wedge (\neg LinInd[W,U,+,*]) \ \blacksquare\ \bot$

(2) $(\neg LinInd[W,V,+,*]) \implies \bot \ \blacksquare\ LinInd[W,V,+,*]$

$$Spans[S,V,+,*] := LinSpan[V,S,V,+,*]$$
$$FinDim[V,+,*] := \exists_{S \in V^n}(Spans[S,V,+,*])$$

$$LinDepLemma := \forall_{S,V}\left(\begin{array}{l}(\neg LinInd[S,V,+,*]) \qquad\qquad\qquad\qquad \implies \\ \exists_{j\in\mathbb{N}_{1,n}}\Big((s_j \in LinSpan[P_1, S_{1,j-1}, V,+,*]) \wedge (LinSpan[P_2,S,V,+,*] = LinSpan[P_3, S\setminus\{s_j\}, V,+,*])\Big)\end{array}\right)$$

(1) $\neg LinInd[S,V,+,*] \;\blacksquare\; \exists_{K\in\mathbb{R}^n}\big((LinComb[O,S,K,V,+,*]) \wedge (K \neq \{0\}^n)\big)$

(2) $\exists_{j\in\mathbb{N}_{1,n}}\Big((k_j \neq 0) \wedge \Big(\forall_{i\in\mathbb{N}_{1,n}}\big((i > j) \implies (k_i = 0)\big)\Big)\Big)$

(3) $s_j = (-1/k_j)\sum_{i=1}^{j-1}(k_i * s_i) = \sum_{i=1}^{j-1}\Big((-k_i/k_j) * s_i\Big)$

(4) $\langle -k_i/k_j | i \in \mathbb{N}_{1,j-1}\rangle \in \mathbb{R}^{j-1}$

(5) $\exists_{M\in\mathbb{R}^{j-1}}(LinComb[s_j, S_{1,j-1}, M, V,+,*]) \;\blacksquare\; s_j \in LinSpan[P_1, S_{1,j-1}, V,+,*]$

(6) $(v \in P_2) \iff (v \in LinSpan[P_2,S,V,+,*]) \iff \ldots$

(7) $\ldots\big(v = \sum_{i=1}^{n}(k_i * s_i)\big) = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^{n}(k_i * s_i)) + k_j * s_j = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^{n}(k_i * s_i)) + k_j * \sum_{i=1}^{j-1}\Big((-k_i/k_j) * s_i\Big) \iff \ldots$

(8) $(v \in LinSpan[P_3, S\setminus\{s_j\}, V,+,*]) \iff (v \in P_3) \;\blacksquare\; (v \in P_2) \iff (v \in P_3) \;\blacksquare\; P_2 = P_3$

$$LinIndLengthLeqSpan := \forall_{L,S}\Big(\big((LinInd[L,V,+,*]) \wedge (Spans[S,V,+,*])\big) \implies (|L| \leq |S|)\Big)$$

(1) TODO : form $B = L \cup S$, remove dependent elements in S such that $(Spans[B,V,+,*]) \wedge (|B| = |S|)$ by $LinDepLemma$, $|L| \leq |B| = |S|$

(2) $\forall_{l_i\in L} \cdots$

(2.1) $l_i \in V \;\blacksquare\; LinComb[l_i, S, K, V,+,*] \;\blacksquare\; \neg LinInd[\langle l_i\rangle \cup S, V,+,*]$

(2.2) $LinDepLemma \;\blacksquare\; \exists_{j\in\mathbb{N}_{1,n}}(LinSpan[V,S,V,+,*] = LinSpan[V, S\setminus\{s_j\}, V,+,*])$

(2.3) $B := \langle l_i\rangle \cup S\setminus\{s_j\} \;\blacksquare\; |B| = 1 + |S| - 1 = |S|$

(3) $|L| \leq |B| = |S| \;\blacksquare\; |L| \leq |S|$

$$FinSubSpace := \forall_{U,V}\Big(\big((Subspace[U,V,+,*]) \wedge (FinDim[V,+,*])\big) \implies (FinDim[U,+,*])\Big)$$

(1) TODO : take $Spans[S,V,+,*]$, remove all $s_j \in S$ such that $U \subseteq LinSpan[S\setminus\{s_j\}] \;\blacksquare\; S' = S\setminus\{s_j\}$ $(LinSpan[U, S', V,+,*]) \wedge (|S'| \leq |S|) \;\blacksquare\; FinDim[U,+,*]$

(2) $FinDim[V,+,*] \;\blacksquare\; \exists_{S\in V^n}(Spans[S,V,+,*])$

(3) $\forall_{(u_j\in U)\wedge(\neg LinSpan[U,S,V,+,*])} \cdots$

(3.1) $\neg LinSpan[U,S,V,+,*] \;\blacksquare\; \exists_{u_j\in U}(\neg LinComb[u_j, S_{1,j-1}, K_{1,j-1}, V,+,*])$

(3.2) $B := S_{1,j-1} \;\blacksquare\; |B| = |S| - 1 < |S|$

(4) $LinSpan[U,B,V,+,*] \;\blacksquare\; \exists_{B\in V^M}(Spans[B,U,+,*]) \;\blacksquare\; FinDim[U,+,*$

3.6 Bases and Dimensions

$$Basis[S,V,+,*] := (Spans[S,V,+,*]) \wedge (LinInd[S,V,+,*])$$

$$BasisEquiv := \forall_{S,V}\big((Basis[S,V,+,*]) \iff (\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])\big)$$

(1) $(Basis[S,V,+,*]) \implies \ldots$

(1.1) $(v \in V) \implies \ldots$

(1.1.1) $Basis[S,V,+,*] \;\blacksquare\; Spans[V,S,+,*] \;\blacksquare\; \exists_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])$

(1.1.2) $\big((K_1, K_2 \in \mathbb{R}^n) \wedge (LinComb[v,S,K_1,V,+,*]) \wedge (LinComb[v,S,K_2,V,+,*])\big) \implies \ldots$

(1.1.2.1) $\big(v = \sum(k_{1i} * s_i)\big) \wedge \big(v = \sum(k_{2i} * s_i)\big)$

(1.1.2.2) $O = v - v = \sum(k_{1i} * s_i) - \sum(k_{2i} * s_i) = \sum\big((k_{1i} - k_{2i}) * s_i\big)$

(1.1.2.3) $L := \langle k_{1i} - k_{2i} | i \in \mathbb{N}_{i=1}^{n}\rangle \in \mathbb{R}^n$

(1.1.2.4) $(LinInd[S,V,+,*]) \wedge (LinComb[O,S,L,V,+,*]) \;\blacksquare\; L = \{0\}^n \;\blacksquare\; K_2 = K_1$

(1.1.3) $\big((K_1, K_2 \in \mathbb{R}^n) \wedge (LinComb[v,S,K_1,V,+,*]) \wedge (LinComb[v,S,K_2,V,+,*])\big) \implies (K_1 = K_2)$

(1.1.4) $\forall_{K_1,K_2\in\mathbb{R}^n}\big((LinComb[v,S,K_1,V,+,*]) \wedge (LinComb[v,S,K_2,V,+,*]) \implies (K_1 = K_2)\big)$

(1.1.5) $\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])$

(1.2) $(v \in V) \implies \big(\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])\big)$

(2) $(Basis[S,V,+,*]) \implies \big(\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])\big)$

(3) $\big(\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])\big) \implies \ldots$

(3.1) $\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*]) \blacksquare \forall_{v\in V}\exists_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*]) \blacksquare Spans[S,V,+,*]$

(3.2) $O \in V \blacksquare \exists!_{K\in\mathbb{R}^n}(LinComb[O,S,K,V,+,*])$

(3.3) $(K \neq \{0\}^n) \implies \big(\neg\exists!_{K\in\mathbb{R}^n}(LinComb[O,S,K,V,+,*])\big) \implies \perp \blacksquare K = \{0\}^n$

(3.4) $\big(\exists!_{K\in\mathbb{R}^n}(LinComb[O,S,K,V,+,*])\big) \wedge (K = \{0\}^n) \blacksquare LinInd[S,V,+,*]$

(3.5) $(Spans[S,V,+,*]) \wedge (LinInd[S,V,+,*]) \blacksquare Basis[S,V,+,*]$

(4) $\big(\forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,S,K,V,+,*])\big) \implies (Basis[S,V,+,*])$

$SpanReduceBasis := \forall_{S,V}\Big(Spans[S,V,+,*] \implies \exists_B\big((B \subseteq S) \wedge (Basis[B,V,+,*])\big)\Big)$

(1) TODO : Remove all dependent $s_j \in S \blacksquare (S' = S \setminus \{s_j\}) \wedge (LinSpan[S'] = LinSpan[S])$ until $LinInd[S'] \blacksquare Basis[S']$

$FinDimBasis := \forall_V\Big((FinDim[V,+,*]) \implies \big(\exists_B(Basis[B,V,+,*])\big)\Big)$

(1) $FinDim[V,+,*] \blacksquare \exists_{S\in V^n}(Spans[S,V,+,*])$

(2) $(SpanReduceBasis) \wedge (Spans[S,V,+,*]) \blacksquare \exists_B(Basis[B,V,+,*])$

$LinIndExpandBasis := \forall_{L,V}\Big(LinInd[L,V,+,*] \implies \exists_B\big((L \subseteq B) \wedge (Basis[B,V,+,*])\big)\Big)$

(1) TODO : $FinDimBasis \blacksquare \exists_A(Basis[A,V,+,*])$, form $B = L \cup A \blacksquare Span[B]$,
use $SpanReduceBasis$ call it B', $(L \subseteq B') \wedge (Basis[B'])$

$BasisLinearIndCard := \forall_{S,T,V}\Big(\big((Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*])\big) \implies (|T| \leq |S|)\Big)$

(1) $(Basis[S,V,+,*]) \implies \ldots$

(1.1) $(|T| > |S|) \implies \ldots$

(1.1.1) $(Spans[S,V,+,*]) \wedge (T \subseteq V) \blacksquare t_{1\ldots t_j} = \sum(\gamma_i * s * i)\ \ldots$

(1.1.2) $\ldots t_j = \sum(\gamma_i' * t_i) \blacksquare \neg LinInd[T,V,+,*]$

(1.2) $(|T| > |S|) \implies (\neg LinInd[T,V,+,*]) \blacksquare (LinInd[T,V,+,*]) \implies (|T| \leq |S|)$

(2) $\big((Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*])\big) \implies (|T| \leq |S|)$

$BasisCard := \forall_{S,T,V}\Big(\big((Basis[S,V,+,*]) \wedge (Basis[T,V,+,*])\big) \implies (|T| = |S|)\Big)$

(1) $Basis[S,V,+,*] \blacksquare LinInd[S,V,+,*]$

(2) $(Basis[T,V,+,*]) \wedge (LinInd[S,V,+,*]) \blacksquare |S| \leq |T|$

(3) $Basis[T,V,+,*] \blacksquare LinInd[T,V,+,*]$

(4) $(Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*]) \blacksquare |T| \leq |S|$

(5) $(|S| \leq |T|) \wedge (|T| \leq |S|) \blacksquare |T| = |S|$

$Dim[d,V,+,*] := \big(\exists_B(Basis[B,V,+,*])\big) \wedge \big((V = \{O\}) \implies (d = 0)\big) \wedge \big((V \neq \{O\}) \implies (d = |B|)\big)$

$LinIndLengthDim := \forall_{U,V}\Big(\big((LinInd[U,V,+,*]) \wedge (Dim[|U|,V,+,*])\big) \implies (Basis[U,V,+,*])\Big)$

(1) $(LinIndExpandBasis) \wedge (LinInd[U,V,+,*]) \blacksquare \exists_B\big((U \subseteq B) \wedge (Basis[B,V,+,*])\big)$

(2) $(BasisCard) \wedge (Dim[|U|,V,+,*]) \wedge (Basis[B,V,+,*]) \blacksquare |B| = |U| \blacksquare B = U \blacksquare Basis[U,V,+,*]$

$SpanLengthDim := \forall_{U,V}\Big(\big((Spans[U,V,+,*]) \wedge (Dim[|U|,V,+,*])\big) \implies (Basis[U,V,+,*])\Big)$

(1) $(SpanReduceBasis) \wedge (Spans[U,V,+,*]) \blacksquare \exists_B\big((B \subseteq U) \wedge (Basis[B,V,+,*])\big)$

(2) $(BasisCard) \wedge (Dim[|U|,V,+,*]) \wedge (Basis[B,V,+,*]) \blacksquare |B| = |U| \blacksquare B = U \blacksquare Basis[U,V,+,*]$

$LinDepLengthDim := \forall_{U,V}\Big(\big((U \subseteq V) \wedge (|U| > Dim[V])\big) \implies (\neg LinInd[U,V,+,*])\Big)$

(1) Contrapositive of $BasisLinearIndCard$

$LinDepLengthDim := \forall_{U,V}\Big(\big((U \subseteq V) \wedge (|U| < Dim[V])\big) \implies (\neg Spans[U,V,+,*])\Big)$

(1) Suppose $Spans[U,V,+,*]$, $B = SpanReduceBasis[U]$ to form a basis, $(|B| \leq |U| < Dim[V]) \wedge |B| = Dim[V]$ ▮ $\bot$

(2) $\neg Spans[U,V,+,*]$

3.7 Rank

$Nullity[n,A] := (NullSpace[N,A]) \wedge (Dim[n,N,+,*])$
$Rank[r,A,m,n] := (Matrix[A,m,n]) \wedge (RowSpace[R,A,m,n]) \wedge (Dim[r,R,A,+,*])$

$RowRankEqColRank := \forall_A(TODO)$

(1) TODO

$RankNullity := \forall_A\big((Matrix[A,m,n]) \implies (Rank[A] + Nullity[A] = n)\big)$

(1) TODO

$RankInv := \forall_A\Big((Matrix[A,m,n]) \implies \big((Rank[A] = n) \iff (Inv[A])\big)\Big)$

(1) TODO

$RankNonTrivialSol := \Big(\exists_X\big((A * X = O) \wedge (X \neq O)\big)\Big) \iff (Rank[A] < n)$

(1) TODO

$RankUniqueSol := \big(\forall_{B\in\mathcal{M}}\exists!_{X\in\mathcal{M}}(Sol[X,A,B])\big) \iff (Rank[A] = n)$

(1) TODO

$$SquareTheorems_{()} := \forall_{A\in\mathcal{M}}\left(\begin{array}{ll} (Invertible[A]) & \iff \\ (RowEquiv[A,I_n]) & \iff \\ \Big(\forall_X\big((X = O) \iff (Sol[X,A,O])\big)\Big) & \iff \\ \big(\forall_{B\in\mathcal{M}}\exists!_{X\in\mathcal{M}}(Sol[X,A,B])\big) & \iff \\ (Rank[A] = n) & \iff \\ (Nullity[A] = 0) & \iff \\ \big(\text{The rows form a linearly independent set of vectors (to get full rank)}\big) & \iff \\ \big(\text{The columns form a linearly independent set of vectors (to get full rank)}\big) & \iff \end{array}\right)$$

3.8 Linear Transformations

$$LinTrans[L,V,+_v,*_v,W,+_w,*_w] := \left(\begin{array}{l} (Function[f,V,W]) \wedge (VectorSpace[V,+_v,*_v]) \wedge (VectorSpace[W,+_w,*_w]) \wedge \\ \Big(\forall_{\alpha,\beta\in V}\big(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta)\big)\Big) \wedge \Big(\forall_{r\in\mathbb{R}}\forall_{\alpha\in V}\big(L(r *_v \alpha) = r *_w L(\alpha)\big)\Big) \end{array}\right)$$

$LinOp[L,V,+_v,*_v] := LinTrans[L,V,+_v,*_v,V,+_v,*_v]$
$\mathcal{L}[V,W] := \{L | LinTrans[L,V,+_v,*_v,W,+_w,*_w]\}$

$ZeroMapsToZero := \forall_{L,V,W}\Big((LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \implies \big(L(O_v) = O_w\big)\Big)$

(1) $L(O_v) = L(O_v +_v O_v) = L(O_v) +_w L(O_v)$

(2) $O_w = L(O_v) - L(O_v) = L(O_v)$

$SplitAddInv := \forall_{L,V,W}\Bigg((LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \implies \Big(\forall_{\alpha,\beta\in V}\big(L(\alpha -_v \beta) = L(\alpha) -_w L(\beta)\big)\Big)\Bigg)$

(1) $L(\alpha-\beta)=L\big(\alpha+(-\beta)\big)=L(\alpha)+L(-\beta)=L(\alpha)+(-1)*L(\beta)=L(\alpha)-L(\beta)$

$$BasisDomainInduceLinTrans:=\forall_{V,W}\left(\begin{array}{l}\big((Basis[A,V,+_v,*_v])\wedge(B\subseteq W)\wedge(n=|B|=|A|)\wedge(VectorSpace[W,+_w,*_w])\big)\implies\\ \Big(\exists!_T\Big((LinTrans[T,V,+_v,*_v,W,+_w,*_w])\wedge\Big(\forall_{i\in\mathbb{N}_{1,n}}\big(T(a_i)=b_i\big)\Big)\Big)\Big)\end{array}\right)$$

(1) $T\big(\sum_{i=1}^{n}(k_i*a_i)\big):=\sum_{i=1}^{n}(k_i*b_i)$

(2) $(i\in\mathbb{N}_{1,n})\implies\ \ldots$

(2.1) $L:=\langle\left(\begin{cases}1 & j=i\\ 0 & j\neq i\end{cases}\right)|j\in\mathbb{N}_{1,n}\rangle\ \blacksquare\ L\in\mathbb{R}^n$

(2.2) $T(a_i)=T\big(\sum_{i=1}^{n}(l_i*a_i)\big)=\sum_{i=1}^{n}(l_i*b_i)=b_i\ \blacksquare\ T(a_i)=b_i$

(3) $(i\in\mathbb{N}_{1,n})\implies\big(T(a_i)=b_i\big)\ \blacksquare\ \forall_{i\in\mathbb{N}_{1,n}}\big(T(a_i)=b_i\big)$

(4) $(BasisEquiv)\wedge(Basis[A,V,+_v,*_v])\ \blacksquare\ \forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,A,K,V,+,*])\ \blacksquare\ T:Span[A]\to Span[B]\ \blacksquare\ Function[T,V,W]$

(5) $(\alpha,\beta\in V)\implies\ \ldots$

(5.1) $\big(\exists_{K_\alpha}(LinComb[\alpha,A,K_\alpha,V,+_v,*_v])\big)\wedge\big(\exists_{K_\beta}(LinComb[\beta,A,K_\beta,V,+_v,*_v])\big)\ \blacksquare\ (\alpha=\sum_{i=1}^{n}(k_{\alpha i}*a_i))\wedge\big(\beta=\sum_{i=1}^{n}(k_{\beta_i}*a_i)\big)$

(5.2) $T(\alpha+\beta)=T\Big(\sum_{i=1}^{n}(k_{\alpha i}*a_i)+\sum_{i=1}^{n}(k_{\beta_i}*a_i)\Big)=T\Big(\sum_{i=1}^{n}\Big((k_{\alpha i}+k_{\beta_i})*a_i\Big)\Big))=\sum_{i=1}^{n}\Big((k_{\alpha i}+k_{\beta_i})*b_i\Big)=\ \ldots$

(5.3) $\ldots\sum_{i=1}^{n}(k_{\alpha i}*b_i)+\sum_{i=1}^{n}(k_{\beta_i}*b_i)=T\big(\sum_{i=1}^{n}(k_{\alpha i}*a_i)\big)+T\Big(\sum_{i=1}^{n}(k_{\beta_i}*a_i)\Big)=T(\alpha)+T(\beta)$

(6) $(\alpha,\beta\in V)\implies\big(L(\alpha+_v\beta)=L(\alpha)+_w L(\beta)\big)\ \blacksquare\ \forall_{\alpha,\beta\in V}\big(L(\alpha+_v\beta)=L(\alpha)+_w L(\beta)\big)$

(7) $\big((r\in\mathbb{R})\wedge(\alpha\in V)\big)\implies\ \ldots$

(7.1) $\exists_K(LinComb[\alpha,A,K,V,+_v,*_v])\ \blacksquare\ \alpha=\sum_{i=1}^{n}(k_i*a_i)$

(7.2) $L(r*_v\alpha)=L\big(r*_v\sum_{i=1}^{n}(k_i*_v a_i)\big)=L\Big(\sum_{i=1}^{n}\big((rk_i)*_v a_i\big)\Big)=\ \ldots$

(7.3) $\ldots\sum_{i=1}^{n}\big((rk_i)*_w b_i\big)=r*_w\sum_{i=1}^{n}(k_i*_w b_i)=r*_w L\big(\sum_{i=1}^{n}(k_i*_v a_i)\big)=r*_w L(\alpha)$

(8) $\big((r\in\mathbb{R})\wedge(\alpha\in V)\big)\implies\big(L(r*_v\alpha)=r*_w L(\alpha)\big)\ \blacksquare\ \forall_{r\in\mathbb{R}}\forall_{\alpha\in V}\big(L(r*_v\alpha)=r*_w L(\alpha)\big)$

(9) $\Big(\forall_{i\in\mathbb{N}_{1,n}}\big(T(a_i)=b_i\big)\Big)\wedge(Function[T,V,W])\wedge\Big(\forall_{\alpha,\beta\in V}\big(L(\alpha+_v\beta)=L(\alpha)+_w L(\beta)\big)\Big)\wedge\Big(\forall_{r\in\mathbb{R}}\forall_{\alpha\in V}\big(L(r*_v\alpha)=r*_w L(\alpha)\big)\Big)\wedge\ \ldots$

(10) $\ldots(VectorSpace[V,+_v,*_v])\wedge(VectorSpace[W,+_w,*_w])\ \blacksquare\ LinTrans[T,V,+_v,*_v,W,+_w,*_w]$

$$Ker[ker_L,L,V,+_v,*_v,W,+_w,*_w]:=(LinTrans[L,V,+_v,*_v,W,+_w,*_w])\wedge\big(ker_L=\{\alpha\in V|L(\alpha)=O_w\}\big)$$

$$KerSub:=\forall_{L,V,W}\big((Ker[ker_L,L,V,+_v,*_v,W,+_w,*_w])\implies(Subspace[ker_L,V,+_v,*_v])\big)$$

(1) $ZeroMapsToZero\ \blacksquare\ L(O_v)=O_w\ \blacksquare\ O_v\in ker_L\ \blacksquare\ \emptyset\neq ker_L\ \blacksquare\ \emptyset\neq ker_L\subseteq V$

(2) $(\alpha,\beta\in ker_L)\implies\ \ldots$

(2.1) $\big(L(\alpha)=O_w\big)\wedge\big(L(\beta)=O_w\big)$

(2.2) $L(\alpha+\beta)=L(\alpha)+L(\beta)=O_w+O_w=O_w\ \blacksquare\ L(\alpha+\beta)\in ker_L$

(3) $(\alpha,\beta\in ker_L)\implies(\alpha+\beta\in ker_L)\ \blacksquare\ \forall_{\alpha,\beta\in ker_L}(\alpha+\beta\in ker_L)$

(4) $\big((r\in\mathbb{R})\wedge(\alpha\in ker_L)\big)\implies\ \ldots$

(4.1) $L(\alpha)=O_w\ \blacksquare\ L(r*\alpha)=r*L(\alpha)=r*O_w=O_w\ \blacksquare\ r*\alpha\in ker_L$

(5) $\big((r\in\mathbb{R})\wedge(\alpha\in ker_L)\big)\implies(r*\alpha\in ker_L)\ \blacksquare\ \forall_{r\in\mathbb{R}}\forall_{\alpha\in ker_L}(r*\alpha\in ker_L)$

(6) $(SubspaceEquiv)\wedge(\emptyset\neq ker_L\subseteq V)\wedge\Big(\forall_{\alpha,\beta\in ker_L}(\alpha+\beta\in ker_L)\Big)\wedge\Big(\forall_{r\in\mathbb{R}}\forall_{\alpha\in ker_L}(r*\alpha\in ker_L)\Big)\ \blacksquare\ Subspace[ker_L,V,+_v,*_v]$

$$KerInjective:=\forall_{L,V,W}\Big((Ker[ker_L,L,V,+_v,*_v,W,+_w,*_w])\implies\big((Injective[L,V,W])\iff(ker_L=\{O_v\})\big)\Big)$$

(1) $(Injective[L,V,W])\implies\ \ldots$

(1.1) $ZeroMapsToZero\ \blacksquare\ L(O_v)=O_w$

(1.2) $O_v\in ker_L\ \blacksquare\ \{O_v\}\subseteq ker_L$

(1.3) $(v\in ker_L)\implies\ \ldots$

(1.3.1) $L(v) = O_w$

(1.3.2) $(Injective[L,V,W]) \wedge \big(L(O_v) = O_w\big) \blacksquare O_v = v$

(1.4) $(v \in ker_L) \implies (v = O_v) \blacksquare ker_L \subseteq \{O_v\}$

(1.5) $(\{O_v\} \subseteq ker_L) \wedge (ker_L \subseteq \{O_v\}) \blacksquare ker_L = \{O_v\}$

(2) $(Injective[L,V,W]) \implies (ker_L = \{O_v\})$

(3) $(ker_L = \{O_v\}) \implies \dots$

(3.1) $\Big((u,v \in V) \wedge \big(L(u) = L(v)\big)\Big) \implies \dots$

(3.1.1) $O_w = L(u) - L(v) = L(u-v) \blacksquare u - v \in ker_L$

(3.1.2) $ker_L = \{O_v\} \blacksquare u - v = O_v \blacksquare u = v$

(3.2) $\Big((u,v \in V) \wedge \big(L(u) = L(v)\big)\Big) \implies (u = v) \blacksquare \forall_{u,v \in V}\Big(\big(L(u) = L(v)\big) \implies (u = v)\Big) \blacksquare Injective[L,V,W]$

(4) $(ker_L = \{O_v\}) \implies (Injective[L,V,W])$

(5) $(Injective[L,V,W]) \iff (ker_L = \{O_v\})$

$Rng[rng_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \wedge \Big(rng_L = \{\beta \in W \mid \exists_{\alpha \in V}\big(\beta = L(\alpha)\big)\}\Big)$

$RangeSub := \forall_{L,V,W}\big((Ran[rng_L, L, V, +_v, *_v, W, +_w, *_w]) \implies (Subspace[rng_L, W, +_w, *_w])\big)$

(1) $ZeroMapsToZero \blacksquare O_w = L(O_v) \blacksquare \exists_{\alpha \in V}\big(O_w = L(\alpha)\big) \blacksquare O_w \in rng_L \blacksquare \emptyset \neq rng_L \blacksquare \emptyset \neq rng_L \subseteq W$

(2) $(\alpha, \beta \in rng_L) \implies \dots$

(2.1) $\Big(\exists_{u \in V}\big(\alpha = L(u)\big)\Big) \wedge \Big(\exists_{v \in V}\big(\beta = L(v)\big)\Big)$

(2.2) $\alpha + \beta = L(u) + L(v) = L(u+v) \blacksquare \exists_{w \in V}\big(\alpha + \beta = L(w)\big) \blacksquare \alpha + \beta \in rng_L$

(3) $(\alpha, \beta \in rng_L) \implies (\alpha + \beta \in rng_L) \blacksquare \forall_{\alpha,\beta \in rng_L}(\alpha + \beta \in rng_L)$

(4) $\big((r \in \mathbb{R}) \wedge (\alpha \in rng_L)\big) \implies \dots$

(4.1) $\exists_{v \in V}\big(\alpha = L(v)\big) \blacksquare L(r * v) = r * L(v) = r * \alpha \blacksquare \exists_{w \in V}\big(r * \alpha = L(w)\big) \blacksquare r * \alpha \in rng_L$

(5) $\big((r \in \mathbb{R}) \wedge (\alpha \in rng_L)\big) \implies (r * \alpha \in rng_L) \blacksquare \forall_{r \in \mathbb{R}}\forall_{\alpha \in rng_L}(r * \alpha \in rng_L)$

(6) $(SubspaceEquiv) \wedge (\emptyset \neq rng_L \subseteq W) \wedge \Big(\forall_{\alpha,\beta \in rng_L}(\alpha + \beta \in rng_L)\Big) \wedge \Big(\forall_{r \in \mathbb{R}}\forall_{\alpha \in rng_L}(r * \alpha \in rng_L)\Big) \blacksquare Subspace[rng_L, W, +_w, *_w]$

$RankKer := \forall_{L,V,W}\big((LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \implies (Dim[V] = Dim[ker_L] + Dim[rng_L])\big)$ TODO

(1) $\exists_U(Basis[U, ker_L, +_v, *_v]) \blacksquare Dim[ker_L] = |U|$

(2) $(LinIndSuperspace) \wedge (LinInd[U, ker_L, +_v, *_v]) \blacksquare LinInd[U, V, +_v, *_v]$

(3) $LinIndExpandBasis \blacksquare \exists_B\big((U \subseteq B) \wedge (Basis[B, V, +_v, *_v])\big) \blacksquare Dim[V] = |B|$

(4) $T := B \setminus U \blacksquare B = U \cup T$

(5) $m := |U|\ ;\ n := |T|\ ;\ p := |B|$

(6) $L(T) := \langle L(t_i) | i \in \mathbb{N}_{1,n}\rangle \subseteq W^n$

(7) $(w \in W) \implies \dots$

(7.1) $\exists_{v \in V}\big(w = L(v)\big)$

(7.2) $Basis[B, V, +_v, *_v] \blacksquare \exists_{K \in \mathbb{R}^p}\Big(v = \sum_{i=1}^{p}(k_i * b_i)\Big)$

(7.3) $v = \sum_{i=1}^{p}(k_i * b_i) = \sum_{i=1}^{m}(k_i * u_i) + \sum_{i=1}^{n}(\kappa_i * t_i)$

(7.4) $w = L(v) = L\big(\sum_{i=1}^{m}(k_i * u_i) + \sum_{i=1}^{n}(k_i * t_i)\big) = \sum_{i=1}^{m}\big(k_i * L(u_i)\big) + \sum_{i=1}^{n}\big(k_i * L(t_i)\big) = O_w + \sum_{i=1}^{n}\big(k_i * L(t_i)\big) = \sum_{i=1}^{n}\big(k_i * L(t_i)\big)$

(7.5) $\exists_K\big(LinComb[w, L(T), K, W, +_w, *_w]\big)$

(8) $(w \in W) \implies \Big(\exists_K\big(LinComb[w, L(T), K, W, +_w, *_w]\big)\Big) \blacksquare \forall_{w \in W}\Big(\exists_K\big(LinComb[w, L(T), K, W, +_w, *_w]\big)\Big)$

(9) $Spans[L(T), W, +_w, *_w]$

(10) $\Big((K \in \mathbb{R}^n) \wedge \big(LinComb[O_w, L(T), K, W, +_w, *_w]\big)\Big) \implies \dots$

(10.1) $O_w = \sum_{i=1}^{n}\big(k_i * L(t_i)\big) = L\big(\sum_{i=1}^{n}(k_i * t_i)\big) \blacksquare \sum_{i=1}^{n}(k_i * t_i) \in ker_L$

(10.2) $Basis[U, ker_L, +_v, *_v] \blacksquare \exists_{D \in \mathbb{R}^m}\big(\sum_{i=1}^{n}(k_i * t_i) = \sum_{i=1}^{m}(d_i * u_i)\big)$

(10.3) $(LinInd[T \cup U]) \land \left(\sum_{i=1}^{n}(k_i * t_i) = \sum_{i=1}^{m}(d_i * u_i)\right) \blacksquare K = D = \{O\} \blacksquare K = \{O\}$

(11) $\Big((K \in \mathbb{R}^n) \land \left(LinComb[O_w, L(T), K, W, +_w, *_w]\right)\Big) \implies (K = \{O\})$

(12) $\forall_{K \in \mathbb{R}^n}\Big(\left(LinComb[O_w, L(T), K, W, +_w, *_w]\right) \implies (K = \{O\})\Big)$

(13) $\forall_{K \in \mathbb{R}^n}\Big(\left(LinComb[O_w, L(T), K, W, +_w, *_w]\right) \iff (K = \{O\})\Big) \blacksquare LinInd[L(T), W, +_w, *_w]$

(14) $Basis[L(T), W, +_w, *_w] \blacksquare Dim[V] = |B| = |U| + |L(T)| = Dim[ker_L] + Dim[rng_L]$

TODO INJ -> SUR, SUR -> INJ