Contents

Chapter 1

Real Analysis

(1.5)
$OrderTrichotomy[<, S] := \forall_{x,y \in S}(x < y \vee x = y \vee y < x)$
$OrderTransitivity[<, S] := \forall_{x,y,z \in S}((x < y \wedge y < z) \implies x < z)$
$Order[<, S] := (OrderTrichotomy[<, S]) \wedge (OrderTransitivity[<, S])$

(1.7)
$BoundedAbove[E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge (\exists_{\beta \in S} \forall_{x \in E}(x \le \beta))$
$BoundedBelow[E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge (\exists_{\beta \in S} \forall_{x \in E}(\beta \le x))$
$UpperBound[\beta, E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E}(x \le \beta))$
$LowerBound[\beta, E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E}(\beta \le x))$

(1.8)
$LUB[\alpha, E, S, <] := (UpperBound[\alpha, E, S, <]) \wedge (\forall_\gamma(\gamma < \alpha \implies \neg UpperBound[\gamma, E, S, <]))$
$GLB[\alpha, E, S, <] := (LowerBound[\alpha, E, S, <]) \wedge (\forall_\beta(\alpha < \beta \implies \neg LowerBound[\beta, E, S, <]))$

(1.10)
$LUBProperty[S, <] := \forall_E(((\emptyset \ne E \subset S) \wedge (BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <])))$
$GLBProperty[S, <] := \forall_E(((\emptyset \ne E \subset S) \wedge (BoundedBelow[E, S, <]) \implies \exists_{\alpha \in S}(GLB[\alpha, E, S, <])))$

(1.11)
$LUBPropertyImpliesGLBProperty := LUBProperty[S, <] \implies GLBProperty[S, <]$

(1) $LUBProperty[S, <] \implies \dots$		wts: 2
(1.1) $(\emptyset \ne B \subset S \wedge BoundedBelow[B, S, <]) \implies \dots$		wts: 1.2
(1.1.1) $Order[<, S] \wedge \exists_{\delta' \in S}(LowerBound[\delta', B, S, <])$		from: *BoundedBelow*, 1.1
(1.1.2) $|B| = 1 \implies \dots$		wts: 1.1.3
(1.1.2.1) $\exists_{u'}(u' \in B)$ ▍ $u := choice(\{u' : u' \in B\})$ ▍ $B = \{u\}$		from: 1.1.2
(1.1.2.2) $GLB[u, B, S, <]$ ▍ $\exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$		
(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$		
(1.1.4) $|B| \ne 1 \implies \dots$		wts: 1.1.5
(1.1.4.1) $\forall_E((\emptyset \ne E \subset S \wedge BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <]))$		from: *LUBProperty*, 1
(1.1.4.2) $L := \{s \in S : LowerBound[s, B, S, <]\}$		
(1.1.4.3) $|B| > 1 \wedge OrderTrichotomy[<, S]$ ▍ $\exists_{b_1' \in B} \exists_{b_0' \in B}(b_0' < b_1')$		from: *Order*, 1.1.1 wts: 1.1.4.7
(1.1.4.4) $b_1 := choice(\{b_1' \in B : \exists_{b_0' \in B}(b_0' < b_1')\})$ ▍ $\neg LowerBound[b_1, B, S, <]$		from: 1.1.4.2
(1.1.4.5) $b_1 \notin L$ ▍ $L \subset S$		
(1.1.4.6) $\delta := choice(\{\delta' \in S : LowerBound[\delta', B, S, <]\})$ ▍ $\delta \in L$ ▍ $\emptyset \ne L$		from: 1.1.1
(1.1.4.7) $\emptyset \ne L \subset S$		from: 1.1.4.5, 1.1.4.6
(1.1.4.8) $\forall_{y \in L}(LowerBound[y_0, B, S, <])$ ▍ $\forall_{y \in L} \forall_{x \in B}(y_0 \le x)$		from: *LowerBound*, 1.1.4.2 wts: 1.1.4.10
(1.1.4.9) $\forall_{x \in B}(x \in S \wedge \forall_{y \in L}(y_0 \le x))$ ▍ $\forall_{x \in B}(UpperBound[x, L, S, <])$		from: *UpperBound*
(1.1.4.10) $\exists_{x \in S}(UpperBound[x, L, S, <])$ ▍ $BoundedAbove[L, S, <]$		

(1.1.4.11) $\emptyset \neq L \subset S \wedge Bounded\,Above[L, S, <]$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha' \in S}(LUB[\alpha', L, S, <]) \;\blacksquare\; \alpha := choice(\{\alpha' \in S : (LUB[\alpha', L, S, <])\})$ from: 1.1.4.1 / wts: 1.1.4.21

(1.1.4.13) $\forall_x(x \in B \implies Upper\,Bound[x, L, S, <])$ from: 1.1.4.9 / wts: 1.1.4.17

(1.1.4.14) $\forall_x(\neg Upper\,Bound[x, L, S, <] \implies x \notin B)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$ wts: 1.1.4.16

   (1.1.4.15.1) $\neg Upper\,Bound[\gamma, L, S, <] \;\blacksquare\; \gamma \notin B$ from: LUB, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B \;\blacksquare\; \gamma \in B \implies \gamma \geq \alpha$

(1.1.4.17) $\forall_{\gamma \in B}(\alpha \leq \gamma) \;\blacksquare\; Lower\,Bound[\alpha, B, S, <]$ from: Lower Bound

(1.1.4.18) $\alpha < \beta \implies \ldots$ wts: 1.1.4.19

   (1.1.4.18.1) $\forall_{y \in L}(y_0 \leq \alpha < \beta) \;\blacksquare\; \forall_{y \in L}(y_0 \neq \beta)$ from: LUB, 1.1.4.12, 1.1.4.18

   (1.1.4.18.2) $\beta \notin L \;\blacksquare\; \neg Lower\,Bound[\beta, B, S, <]$ from: 1.1.4.2

(1.1.4.19) $\alpha < \beta \implies \neg Lower\,Bound[\beta, B, S, <] \;\blacksquare\; \forall_{\beta \in S}(\alpha < \beta \implies \neg Lower\,Bound[\beta, B, S, <])$

(1.1.4.20) $Lower\,Bound[\alpha, B, S, <] \wedge \forall_{\beta \in S}(\alpha < \beta \implies \neg Lower\,Bound[\beta, B, S, <])$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB[\alpha, B, S, <] \;\blacksquare\; \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.5) $|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.6) $(|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])) \wedge (|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <]))$ from: 1.1.3, 1.1.5

(1.1.7) $(|B| = 1 \vee |B| \neq 1) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <]) \;\blacksquare\; \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.2) $(\emptyset \neq B \subset S \wedge Bounded\,Below[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.3) $\forall_B((\emptyset \neq B \subset S \wedge Bounded\,Below[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <]))$

(1.4) $GLB\,Property[S, <]$

(2) $LUB\,Property[S, <] \implies GLB\,Property[S, <]$

(1.12)

$$Field[F, +, *] := \exists_{0,1 \in F} \forall_{x,y,z \in F} \begin{pmatrix} x + y \in F & \wedge & x * y \in F & \wedge \\ x + y = y + x & \wedge & x * y = y * x & \wedge \\ (x + y) + z = x + (y_0 + z) & \wedge & (x * y) * z = x * (y_0 * z) & \wedge \\ 1 \neq 0 & \wedge & x * (y_0 + z) = (x * y) + (x * z) & \wedge \\ 0 + x = x & \wedge & 1 * x = x & \wedge \\ \exists_{-x \in F}(x + (-x) = 0) \wedge (x \neq 0 \implies \exists_{1/x \in F}(x * (1/x) = 1)) \end{pmatrix}$$

     ********************** $(Field[F, +, *] \wedge x, y, z \in F) \implies \ldots$ **********************

(1.14)

$AdditiveCancellation := (x + y = x + z) \implies y = z$

(1) $y = 0 + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \ldots$ from: Field

(2) $(-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = 0 + z = z$ from: Field

$AdditiveIdentityUniqueness := (x + y = x) \implies y = 0$

(1) $x + y = x = 0 + x = x + 0$ from: Field

(2) $y = 0$ from: AdditiveCancellation

$AdditiveInverseUniqueness := (x + y = 0) \implies y = -x$

(1) $x + y = 0 = x + (-x)$ from: Field

(2) $y = -x$ from: AdditiveCancellation

$DoubleNegative := x = -(-x)$

(1) $0 = x + (-x) = (-x) + x \;\blacksquare\; 0 = (-x) + x$ from: Field

(2) $x = -(-x)$ from: AdditiveInverseUniqueness

(1.15)

$MultiplicativeCancellation := (x \neq 0 \wedge x * y = x * z) \implies y = z$ —

$MultiplicativeIdentityUniqueness := (x \neq 0 \wedge x * y = x) \implies y = 1$ —

$MultiplicativeInverseUniqueness := (x \neq 0 \wedge x * y = 1) \implies y = 1/x$ —

$DoubleReciprocal := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)

$Domination := 0 * x = 0$

(1) $0 * x = (0 + 0) * x = 0 * x + 0 * x$ ▋ $0 * x = 0 * x + 0 * x$

from: *Field*

(2) $0 * x = 0$

from: *AdditiveIdentityUniqueness*

$NonDomination := (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$

(1) $(x \neq 0 \wedge y \neq 0) \implies \ldots$

 (1.1) $(x * y = 0) \implies \ldots$

  (1.1.1) $1 = 1 * 1 = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = 0 * ((1/x) * (1/y)) = 0$

from: *Field*, *Domination*, 1, 1.1

  (1.1.2) $1 = 0 \wedge 1 \neq 0$ ▋ $\bot$

from: *Field*

 (1.2) $(x * y = 0) \implies \bot$ ▋ $x * y \neq 0$

(2) $(x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$

$NegationCommutativity := (-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = 0 * y = 0$ ▋ $x * y + (-x) * y = 0$

from: *Field*, *Domination*
wts: 2

(2) $(-x) * y = -(x * y)$

from: *AdditiveInverseUniqueness*

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * 0 = 0$ ▋ $x * y + x * (-y) = 0$

from: *Field*, *Domination*
wts: 4

(4) $x * (-y) = -(x * y)$

from: *AdditiveInverseUniqueness*

(5) $(-x) * y = -(x * y) = x * (-y)$

from: 2, 4

$NegativeMultiplication := (-x) * (-y) = x * y$

(1) $(-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$

from: *NegationCommutativity*, *DoubleNegative*

(1.17)

$$OrderedField[F, +, *, <] := \begin{pmatrix} Field[F, +, *] \quad \wedge \quad Order[<, F] \quad \wedge \\ \forall_{x,y,z \in F}(y_0 < z \implies x + y < x + z) \wedge \\ \forall_{x,y \in F}((x > 0 \wedge y > 0) \implies x * y > 0) \end{pmatrix}$$

*********************** $(OrderedField[F, +, *, <] \wedge x, y, z \in F) \implies \ldots$ ***********************

(1.18)

$NegationOnOrder := x > 0 \iff -x < 0$

(1) $x > 0 \implies \ldots$

 (1.1) $0 = (-x) + x > (-x) + 0 = -x$ ▋ $0 > -x$ ▋ $-x < 0$

from: *OrderedField*

(2) $x > 0 \implies -x < 0$

(3) $-x < 0 \implies \ldots$

 (3.1) $0 = x + (-x) < x + 0 = x$ ▋ $0 < x$ ▋ $x > 0$

from: *OrderedField*

(4) $-x < 0 \implies x > 0$

(5) $x > 0 \implies -x < 0 \wedge -x < 0 \implies x > 0$ ▋ $x > 0 \iff -x < 0$

from: 2, 4

$PositiveFactorPreservesOrder := (x > 0 \wedge y < z) \implies x * y < x * z$

(1) $(x > 0 \wedge y < z) \implies \ldots$

 (1.1) $(-y) + z > (-y) + y = 0$ ▋ $z + (-y) = 0$

from: *OrderedField*

 (1.2) $x * (z + (-y)) > 0$ ▋ $x * z + x * (-y) > 0$

from: *OrderedField*

 (1.3) $x * z = 0 + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \ldots$

from: *Field*, *NegationCommutativity*

 (1.4) $x * y + (x * z + x * (-y)) > x * y + 0 = x * y$

from: *Field*, 1.2

 (1.5) $x * z > x * y$

from: 1.3, 1.4

(2) $(x > 0 \land y < z) \implies x * z > x * y$

$NegativeFactorFlipsOrder := (x < 0 \land y < z) \implies x * y > x * z$

(1) $(x < 0 \land y < z) \implies \dots$

(1.1) $-x > 0$ from: *NegationOnOrder*

(1.2) $(-x) * y < (-x) * z$ ▮ $0 = x * y + (-x) * y < x * y + (-x) * z$ ▮ $0 < x * y + (-x) * z$ from: *PositiveFactorPreservesOrder*

(1.3) $0 < (-x) * (-y + z)$ ▮ $0 > x * (-y + z)$ ▮ $0 > -(x * y) + x * z$ from: *NegationOnOrder*

(1.4) $x * y > x * z$

(2) $(x < 0 \land y < z) \implies x * y > x * z$

$SquareIsPositive := (x \neq 0) \implies x * x > 0$

(1) $(x > 0) \implies x * x > 0$ from: *OrderedField*

(2) $(x < 0) \implies \dots$

(2.1) $-x > 0$ ▮ $x * x = (-x) * (-x) > 0$ ▮ $x * x > 0$ from: *NegationOnOrder*, *OrderedField*, *NegativeMultiplication*

(3) $(x < 0) \implies x * x > 0$

(4) $x \neq 0 \implies (x > 0 \lor x < 0) \implies x * x > 0$ ▮ $x \neq 0 \implies x * x > 0$ from: *OrderTrichotomy*, 1, 3

$OneIsPositive := 1 > 0$

(1) $1 \neq 0$ ▮ $1 = 1 * 1 > 0$ from: *Field*, *SquareIsPositive*

$ReciprocationOnOrder := (0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \dots$

(1.1) $x * (1/x) = 1 > 0$ ▮ $x * (1/x) > 0$ from: *Field*, *OneIsPositive*

(1.2) $1/x < 0 \implies x * (1/x) < 0 \land x * (1/x) > 0 \implies \bot$ ▮ $1/x > 0$ from: *NegativeFactorFlipsOrder*, 1

(1.3) $y * (1/y) = 1 > 0$ ▮ $y * (1/y) > 0$ from: *Field*, *OneIsPositive*

(1.4) $1/y < 0 \implies y * (1/y) < 0 \land y * (1/y) > 0 \implies \bot$ ▮ $1/y > 0$ from: *NegativeFactorFlipsOrder*, 1

(1.5) $(1/x) * (1/y) > 0$ from: *OrderedField*

(1.6) $0 < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$ from: *OrderedField*, 1, 1.4, 1.5

(1.19)

$OrderedFieldQ := OrderedField[\mathbb{Q}, +, *, <]$ —

$Subfield[K, F, +, *] := Field[F, +, *] \land K \subset F \land Field[K, +, *]$
$OrderedSubfield[K, F, +, *, <] := OrderedField[F, +, *, <] \land K \subset F \land OrderedField[K, +, *, <]$
$CutI[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$
$CutII[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
$CutIII[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} : CutI[\alpha] \land CutII[\alpha] \land CutIII[\alpha]\}$

$CutCorollaryI := (\alpha \in \mathbb{R} \land p \in \alpha \land q \in \mathbb{Q} \land q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \land p \in \alpha \land q \in \mathbb{Q} \land q \notin \alpha) \implies \dots$

(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ from: *CutII*, 1

(1.2) $q < p \implies q \in \alpha$ ▮ $q \notin \alpha \implies q \geq p$ from: 1

(1.3) $(q \notin \alpha) \implies \dots$

(1.3.1) $q \geq p$ from: 1.2

(1.3.2) $(q = p) \implies (p \in \alpha \land p \notin \alpha) \implies \bot$ ▮ $q \neq p$ from: 1, 1.3

(1.3.3) $q \geq p \land q \neq p$ ▮ $p < q$

(1.4) $q \notin \alpha \implies p < q$ ▮ $p < q$ from: 1

(2) $(\alpha \in \mathbb{R} \land p \in \alpha \land q \in \mathbb{Q} \land q \notin \alpha) \implies p < q$

$CutCorollary11$ $:= (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \dots$

  (1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ from: *Cut11*, 1

  (1.2) $s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha))$ ▌ $s \in \alpha \implies r \in \alpha$ from: 1, 1.1

  (1.3) $r \notin \alpha \implies s \notin \alpha$ ▌ $s \notin \alpha$ from: 1, 1.2

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}} [\alpha, \beta]$ $:= \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$OrderTrichotomyOfR$ $:= OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \dots$

  (1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \dots$

    (1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$ from: $<_{\mathbb{R}}$, 1.1

    (1.1.2) $\exists_{p'} (p' \in \alpha \wedge p' \notin \beta)$ ▌ $p := choice(\{p' : p' \in \alpha \wedge p' \notin \beta\})$

    (1.1.3) $q \in \beta \implies \dots$

      (1.1.3.1) $p, q \in \mathbb{Q}$

      (1.1.3.2) $q < p$ from: *CutCorollary1*

      (1.1.3.3) $q \in \alpha$ from: *Cut11*

    (1.1.4) $q \in \beta \implies q \in \alpha$

    (1.1.5) $\forall_{q \in \beta} (q \in \alpha)$ ▌ $\beta \subseteq \alpha$

    (1.1.6) $\beta \subset \alpha$ ▌ $\beta <_{\mathbb{R}} \alpha$

  (1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

  (1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$ ▌ $(\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

  (1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

  (1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

  (1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

  (1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}} (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) *$OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$*

$OrderTransitivityOfR$ $:= OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$

  (1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \dots$

    (1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

    (1.1.2) $\forall_{a \in \alpha} (a \in \beta) \wedge \forall_{b \in \beta} (b \in \gamma)$

    (1.1.3) $\forall_{a \in \alpha} (\alpha \in \gamma)$ ▌ $\alpha \subset \gamma$ ▌ $\alpha <_{\mathbb{R}} \gamma$

  (1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}} ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(4) *$OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$*

$OrderOfR$ $:= Order[<_{\mathbb{R}}, \mathbb{R}]$ from: *OrderTrichotomyR, OrderTransitivityR*
 wts:

$LUBPropertyOfR$ $:= LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \dots$

  (1.1) $\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A} (p \in \alpha)\}$

  (1.2) $A \neq \emptyset$ ▌ $\exists_\alpha (\alpha \in A)$ ▌ $\alpha_0 := choice(\{\alpha : \alpha \in A\})$

  (1.3) $\alpha_0 \neq \emptyset$ ▌ $\exists_a (a \in \alpha_0)$ ▌ $a_0 := choice(\{a : a \in \alpha_0\})$ ▌ $a_0 \in \gamma$ ▌ $\gamma \neq \emptyset$

  (1.4) *$BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]$* ▌ $\exists_\beta (UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.5) $\beta_0 := choice(\{\beta : UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}]\})$

(1.6) $UpperBound[\beta_0, A, \mathbb{R}, <_{\mathbb{R}}]$ ▌ $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \beta_0)$ ▌ $\forall_{\alpha \in A}(\alpha \subseteq \beta_0)$ ▌ $\forall_{\alpha \in A}\forall_{a \in \alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma$ ▌ $\forall_{a \in \gamma}(a \in \beta_0)$ ▌ $\gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q}$ ▌ $\gamma \subseteq \beta_0 \subset \mathbb{Q}$ ▌ $\gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q}$ ▌ $CutI[\gamma]$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

     (1.10.1) $p \in \gamma$ ▌ $\exists_{\alpha \in A}(p \in \alpha)$ ▌ $\alpha_1 := choice(\{\alpha \in A : p \in \alpha\})$

     (1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p$ ▌ $q \in \alpha_1$ ▌ $q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma$ ▌ $\forall_{p \in \gamma}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma)$ ▌ $CutII[\gamma]$

(1.12) $p \in \gamma \implies \ldots$

     (1.12.1) $\exists_{\alpha \in A}(p \in \alpha)$ ▌ $\alpha_2 := choice(\{\alpha \in A : p \in \alpha\})$

     (1.12.2) $\alpha_2 \in \mathbb{R}$ ▌ $CutII[\alpha_2]$ ▌ $\exists_{r \in \alpha_2}(p < r)$ ▌ $r_0 := choice(\{r \in \alpha_2 : p < r\})$

     (1.12.3) $r_0 \in \alpha_2$ ▌ $r_0 \in \gamma$

     (1.12.4) $p < r_0$ ▌ $p < r_0 \wedge r_0 \in \gamma$ ▌ $\exists_{r \in \gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r \in \gamma}(p < r)$ ▌ $\forall_{p \in \gamma}\exists_{r \in \gamma}(p < r)$ ▌ $CutIII[\gamma]$

(1.14) $CutI[\gamma] \wedge CutII[\gamma] \wedge CutIII[\gamma]$ ▌ $\gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha \in A}(\alpha \subseteq \gamma)$ ▌ $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma)$

(1.16) $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R}$ ▌ $UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.17) $\delta <_{\mathbb{R}} \gamma \implies \ldots$

     (1.17.1) $\delta \subset \gamma$ ▌ $\exists_s(s \in \gamma \wedge s \notin \delta)$ ▌ $s_0 := choice(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$

     (1.17.2) $s_0 \in \gamma$ ▌ $\exists_{\alpha \in A}(s_0 \in \alpha)$ ▌ $\alpha_3 := choice(\{\alpha \in A : s_0 \in \alpha\})$

     (1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta$ ▌ $\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

     (1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \ldots$

        (1.17.4.1) $\alpha_3 \subseteq \delta$ ▌ $\forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta)$ ▌ $\neg \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

        (1.17.4.2) $\neg \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$ ▌ $\bot$

     (1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \bot$ ▌ $\delta <_{\mathbb{R}} \alpha_3$ ▌ $\exists_{\alpha \in A}(\delta <_{\mathbb{R}} \alpha)$ ▌ $\exists_{\alpha \in A}(\neg(\alpha \leq_{\mathbb{R}} \delta))$

     (1.17.6) $\neg\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \delta)$ ▌ $\neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$ ▌ $\forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.19) $UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \wedge \forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.20) $LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$ ▌ $\exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(3) $\forall_A((\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]))$ ▌ $LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

$+_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$
$0_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 0\}$

$ZeroInR := 0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}}$ ▌ $\emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q}$ ▌ $CutI[0_{\mathbb{R}}]$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}}$ ▌ $\forall_{x \in 0_{\mathbb{R}}}\forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}})$ ▌ $CutII[0_{\mathbb{R}}]$

(3) $y := x/2$ ▌ $(x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $\forall_{x \in 0_{\mathbb{R}}}\exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $CutIII[0_{\mathbb{R}}]$

(4) $CutI[0_{\mathbb{R}}] \wedge CutII[0_{\mathbb{R}}] \wedge CutIII[0_{\mathbb{R}}]$ ▌ $0_{\mathbb{R}} \in \mathbb{R}$

$FieldAdditionClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

     (1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

     (1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

     (1.3) $\exists_a(a \in \alpha) ; \exists_b(b \in \beta)$ ▌ $a_0 := choice(\{a : a \in \alpha\}) ; b_0 := choice(\{b : b \in \beta\})$ ▌ $a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

     (1.4) $\exists_x(x \notin \alpha) ; \exists_y(y_0 \notin \beta)$ ▌ $x_0 := choice(\{x : x \notin \alpha\}) ; y_0 := choice(\{y : y \notin \beta\})$

     (1.5) $\forall_{r \in \alpha}(r < x_0) ; \forall_{s \in \beta}(s < y_0)$ ▌ $\forall_{r \in \alpha}\forall_{s \in \beta}(r + s < x_0 + y_0)$ ▌ $x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

     (1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q}$ ▌ $CutI[\alpha +_{\mathbb{R}} \beta]$

(1.7) $(p \in \alpha +_\mathbb{R} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$

  (1.7.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▌ $(r_0, s_0) := choice((r, s) \in \alpha \times \beta : p = r + s)$

  (1.7.2) $q < p = r_0 + s_0$ ▌ $(q - s_0) < r_0$ ▌ $(q - s_0) \in \alpha$

  (1.7.3) $s_0 \in \beta$ ▌ $q = (q - s_0) + s_0 \in \alpha +_\mathbb{R} \beta$ ▌ $q \in \alpha +_\mathbb{R} \beta$

(1.8) $(p \in \alpha +_\mathbb{R} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_\mathbb{R} \beta$ ▌ $\forall_{p \in \alpha +_\mathbb{R} \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_\mathbb{R} \beta)$ ▌ $CutII[\alpha +_\mathbb{R} \beta]$

(1.9) $p \in \alpha \implies \dots$

  (1.9.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▌ $(r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta : p = r + s\})$

  (1.9.2) $r_1 \in \alpha$ ▌ $\exists_{t \in \alpha}(r_1 < t)$ ▌ $t_0 := choice(\{t \in \alpha : r_1 < t\})$

  (1.9.3) $s_1 \in \beta$ ▌ $t + s_1 \in \alpha +_\mathbb{R} \beta \wedge p = r_1 + s_1 < t + s_1$ ▌ $\exists_{r \in \alpha +_\mathbb{R} \beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r \in \alpha +_\mathbb{R} \beta}(p < r)$ ▌ $\forall_{p \in \alpha +_\mathbb{R} \beta} \exists_{r \in \alpha +_\mathbb{R} \beta}(p < r)$ ▌ $CutIII[\alpha +_\mathbb{R} \beta]$

(1.11) $CutI[\alpha +_\mathbb{R} \beta] \wedge CutII[\alpha +_\mathbb{R} \beta] \wedge CutIII[\alpha +_\mathbb{R} \beta]$ ▌ $\alpha +_\mathbb{R} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_\mathbb{R} \beta) \in \mathbb{R})$

$FieldAdditionCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_\mathbb{R} \beta = \beta +_\mathbb{R} \alpha)$

(1) $\alpha +_\mathbb{R} \beta = \{r + s : r \in \alpha \wedge s \in \beta\} = \{s + r : s \in \beta \wedge r \in \alpha\} = \beta +_\mathbb{R} \alpha$

$FieldAdditionAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma))$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \dots$

  (1.1) $(\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \{(a + b) + c : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \dots$

  (1.2) $\{a + (b + c) : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma)$

$FieldAdditionIdentityOfR := (\alpha \in \mathbb{R}) \implies 0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \dots$

  (1.1) $(r \in \alpha \wedge s \in 0_\mathbb{R}) \implies \dots$

   (1.1.1) $s < 0$ ▌ $r + s < r + 0 = r$ ▌ $r + s < r$ ▌ $r + s \in \alpha$

  (1.2) $(r \in \alpha \wedge s \in 0_\mathbb{R}) \implies r + s \in \alpha$ ▌ $\forall_{r \in \alpha} \forall_{s \in 0_\mathbb{R}}(r + s \in \alpha)$

  (1.3) $(r \in \alpha \wedge s \in 0_\mathbb{R}) \iff (r + s \in \alpha +_\mathbb{R} 0_\mathbb{R})$ ▌ $\forall_{p \in \alpha +_\mathbb{R} 0_\mathbb{R}}(p \in \alpha)$ ▌ $\alpha +_\mathbb{R} 0_\mathbb{R} \subseteq \alpha$

  (1.4) $p \in \alpha \implies \dots$

   (1.4.1) $\exists_{r \in \alpha}(p < r)$ ▌ $r_2 := choice(\{r \in \alpha : p < r\})$

   (1.4.2) $p < r_2$ ▌ $p - r_2 < r_2 - r_2 = 0$ ▌ $(p - r_2) < 0$ ▌ $(p - r_2) \in 0_\mathbb{R}$

   (1.4.3) $r_2 \in \alpha$ ▌ $p = r_2 + (p - r_2) \in \alpha +_\mathbb{R} 0_\mathbb{R}$ ▌ $p \in \alpha +_\mathbb{R} 0_\mathbb{R}$

  (1.5) $p \in \alpha \implies p \in \alpha +_\mathbb{R} 0_\mathbb{R}$ ▌ $\forall_{p \in \alpha}(p \in \alpha +_\mathbb{R} 0_\mathbb{R})$ ▌ $\alpha \subseteq \alpha +_\mathbb{R} 0_\mathbb{R}$

  (1.6) $\alpha +_\mathbb{R} 0_\mathbb{R} \subseteq \alpha \wedge \alpha \subseteq \alpha +_\mathbb{R} 0_\mathbb{R}$ ▌ $0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

$FieldAdditionInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}}(\alpha +_\mathbb{R} (-\alpha) = 0_\mathbb{R})$

(1) $\alpha \in \mathbb{R} \implies \dots$

  (1.1) $\beta := \{p \in \mathbb{Q} : \exists_{r > 0}(-p - r \notin \alpha)\}$

  (1.2) $\alpha \subset \mathbb{Q}$ ▌ $\exists_{s \in \mathbb{Q}}(s \notin \alpha)$ ▌ $s_0 := choice(\{s : s \notin \alpha\})$ ▌ $p_0 := -s_0 - 1$

  (1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha$ ▌ $-p_0 - 1 \notin \alpha$ ▌ $\exists_{r > 0}(-p_0 - r \notin \alpha)$ ▌ $p_0 \in \beta$

  (1.4) $\emptyset \neq \alpha$ ▌ $\exists_{q \in \alpha}$ ▌ $q_0 := choice(\{q \in \mathbb{Q} : q \in \alpha\})$

  (1.5) $r > 0 \implies \dots$

   (1.5.1) $q_0 \in \alpha$ ▌ $-(-q_0) - r = q_0 - r < q_0$ ▌ $-(-q_0) - r < q_0$ ▌ $-(-q_0) - r \in \alpha$

  (1.6) $\forall_{r > 0}(-(-q_0) - r \in \alpha)$ ▌ $\neg \exists_{r > 0}(-(-q_0) - r \notin \alpha)$ ▌ $-q_0 \notin \beta$

  (1.7) $\emptyset \neq \beta \subset \mathbb{Q}$ ▌ $CutI[\beta]$

  (1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \dots$

   (1.8.1) $p \in \beta$ ▌ $\exists_{r > 0}(-p - r \notin \alpha)$ ▌ $r_0 := choice(\{r > 0 : -p - r \notin \alpha\})$

   (1.8.2) $q < p$ ▌ $-p - r < -q - r$

   (1.8.3) $-q - r \notin \alpha$ ▌ $q \in \beta$

  (1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta$ ▌ $\forall_{p \in \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \beta)$ ▌ $CutII[\beta]$

(1.10) $p \in \beta \implies \ldots$

(1.10.1) $p \in \beta$ ▮ $\exists_{r>0}(-p - r \notin \alpha)$ ▮ $r_1 := choice(\{r > 0 : -p - r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0$ ▮ $r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p$ ▮ $t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha$ ▮ $-t_0 - (r_1/2) \notin \alpha$ ▮ $\exists_{r>0}(-t_0 - r \notin \alpha)$ ▮ $t_0 \in \beta$

(1.10.7) $t_0 > p \wedge t_0 \in \beta$ ▮ $\exists_{t\in\beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t\in\beta}(p < t)$ ▮ $\forall_{p\in\beta}\exists_{t\in\beta}(p < t)$ ▮ $CutIII[\beta]$

(1.12) $CutI[\beta] \wedge CutII[\beta] \wedge CutIII[\beta]$ ▮ $\beta \in \mathbb{R}$

(1.13) $(r \in \alpha \wedge s \in \beta) \implies \ldots$

(1.13.1) $s \in \beta$ ▮ $\exists_{t>0}(-s - t \notin \alpha)$ ▮ $t_1 := choice(\{t > 0 : -s - t \notin \alpha\})$ ▮ $-s - t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha$ ▮ $-s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha$ ▮ $r < -s$ ▮ $r + s < 0$ ▮ $r + s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}}$ ▮ $\forall_{(r,s)\in\alpha\times\beta}(r + s \in 0_{\mathbb{R}})$ ▮ $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \ldots$

(1.15.1) $v < 0$ ▮ $w_0 := -v/2$ ▮ $w > 0$

(1.15.2) $\exists_{n\in\mathbb{Z}}(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha)$ ▮ $n_0 := choice(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$ from: *ARCHIMEDEANPROPERTYOFQ + LUB???*

(1.15.3) $p_0 := -(n_0 + 2)w_0$ ▮ $-p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha$ ▮ $-p_0 - w_0 \notin \alpha$ ▮ $p_0 \in \beta$

(1.15.4) $n_0 w_0 \in \alpha \wedge p_0 \in \beta$ ▮ $n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta$ ▮ $\forall_{v\in0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta)$ ▮ $0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$ ▮ $\alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$ ▮ $\exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

$*_{\mathbb{R}}[\alpha, \beta] :=$ —
$1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$1IsNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ —
$1InR := 1_{\mathbb{R}} \in \mathbb{R}$ —
$FieldMultiplicationClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R})$ —
$FieldMultiplicationCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ —
$FieldMultiplicationAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma))$ —
$FieldMultiplicationIdentityOfR := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ —
$FieldMultiplicationInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha\in\mathbb{R}}(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}})$ —

$FieldDistributativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ —

$FieldWithR := Field[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}]$ —
$OrderedFieldWithR := OrderedField[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$
$QROrderedSubfieldOfR := OrderedSubfield[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —
$QIsomorphicToQR := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ —
$CompletenessOfR := \exists_{\mathbb{R}}(LUBProperty[\mathbb{R}, <_{\mathbb{R}}] \wedge OrderedSubfield[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}])$ —

(1.20)
$ArchimedeanPropertyOfR := \forall_{x,y\in\mathbb{R}}(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y))$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx : n \in \mathbb{N}^+\}$ ▮ $(\emptyset \neq A \subset \mathbb{R}) \wedge (a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a))$

(1.2) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n\in\mathbb{N}^+}(nx > y)$ ▮ $\forall_{n\in\mathbb{N}^+}(nx \leq y)$ ▮ $UpperBound[y_0, A, \mathbb{R}, <]$ ▮ $BoundedAbove[A, \mathbb{R}, <]$

(1.2.2) $CompletenessOfR$ ▮ $LUBProperty[\mathbb{R}, <]$

(1.2.3) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (BoundedAbove[A, \mathbb{R}, <])$ ▮ $\exists_{\alpha\in\mathbb{R}}(LUB[\alpha, A, \mathbb{R}, <])$ . . .

(1.2.4) $\ldots \alpha_0 := choice(\{\alpha \in \mathbb{R} : LUB[\alpha, A, \mathbb{R}, <]\})$ ▮ $LUB[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0$ ▮ $\alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge (LUB[\alpha_0, A, \mathbb{R}, <])$ ▮ $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$ ▮ $\exists_{c \in A}(\alpha_0 - x < c)$ $\ldots$

(1.2.8) $\ldots c_0 := choice(\{c \in A : \alpha_0 - x < c\})$ ▮ $(c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$ ▮ $\exists_{m \in \mathbb{N}^+}(mx = c_0)$ $\ldots$

(1.2.10) $\ldots m_0 := choice(\{m \in \mathbb{N}^+ : mx = c_0\})$ ▮ $(m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0)$ ▮ $\alpha_0 - x < c_0 = m_0 x$ ▮ $\alpha_0 < m_0 x + x$ ▮ $\alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+$ ▮ $m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$ ▮ $(m_0 + 1)x \in A$

(1.2.14) $(\alpha_0 < (m_0 + 1)x) \wedge ((m_0 + 1)x \in A)$ ▮ $\exists_{c \in A}(\alpha_0 < c)$

(1.2.15) $LUB[\alpha_0, A, \mathbb{R}, <]$ ▮ $UpperBound[\alpha_0, A, \mathbb{R}, <]$ ▮ $\forall_{c \in A}(c \leq \alpha_0)$ ▮ $\neg\exists_{c \in A}(c > \alpha_0)$ ▮ $\neg\exists_{c \in A}(\alpha_0 < c)$

(1.2.16) $(\exists_{c \in A}(\alpha_0 < c)) \wedge (\neg\exists_{c \in A}(\alpha_0 < c))$ ▮ $\bot$

(1.3) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \bot$ ▮ $\exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y)$ ▮ $\forall_{x,y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

$QDenseInR := \forall_{x,y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y$ ▮ $(0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $ArchimedeanPropertyOfR \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R})$ ▮ $\exists_{n \in \mathbb{N}^+}(n(y - x) > 1)$ $\ldots$

(1.3) $\ldots n_0 := choice(\{n \in \mathbb{N}^+ : n(y - x) > 1\})$ ▮ $(n_0 \in \mathbb{N}^+) \wedge (n_0(y - x) > 1)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R})$ ▮ $n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R})$ ▮ $\exists_{m \in \mathbb{N}^+}(m(1) > n_0 x)$ $\ldots$

(1.6) $\ldots m_1 := choice(\{m \in \mathbb{N}^+ : m(1) > n_0 x\})$ ▮ $(m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R})$ ▮ $\exists_{m \in \mathbb{N}^+}(m(1) > -n_0 x)$ $\ldots$

(1.8) $\ldots m_2 := choice(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\})$ ▮ $(m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x)$ ▮ $-m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+$ ▮ $|m_1 - (-m_2)| \geq 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge (|m_1 - (-m_2)| \geq 2)$ ▮ $\exists_{m \in \mathbb{Z}}((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m))$ $\ldots$

(1.12) $\ldots m_0 := choice(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\})$ ▮ $(-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$

(1.13) $(n_0(y - x) > 1) \wedge (m_0 - 1 \leq n_0 x < m_0)$ ▮ $n_0 x < m_0 \leq 1 + n_0 x < n_0 y$ ▮ $n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y)$ ▮ $x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z}$ ▮ $m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y)$ ▮ $\exists_{p \in \mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y)$ ▮ $\forall_{x,y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1.21)
$RootLemma := (0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a) \sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b$ ▮ $b/a > 1$

(1.3) $b/a > 1$ ▮ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq \sum_{i=1}^{n}(b^{n-i}a^{i-1}(b/a)^{i-1}) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$ ▮ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b - a) \sum_{i=1}^{n}(b^{n-i}a^{i-1}) \leq (b - a)nb^{n-1}$ ▮ $b^n - a^n \leq (b - a)nb^{n-1}$

(2) $(0 < a < b) \implies (b^n - a^n \leq (b - a)nb^{n-1})$

$RootExistenceInR := \forall_{0 < x \in \mathbb{R}}\forall_{0 < n \in \mathbb{Z}}\exists!_{0 < y \in \mathbb{R}}(y_0{}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\}$ ▮ $t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$

(1.2) $t_0 := x/(1 + x)$ ▮ $(t_0 = x/(1 + x)) \wedge (t_0 \in \mathbb{R})$

(1.3) $0 < x$ ▮ $0 < x < 1 + x$ ▮ $t_0 = x/(1 + x) > 0$ ▮ $t_0 > 0$

(1.4) $1 = (1+x)/(1+x) > x/(1+x) = t_0$ ▮ $1 > t_0$

(1.5) $(t_0 > 0) \wedge (1 > t_0)$ ▮ $0 < t_0 < 1$

(1.6) $(0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1)$ ▮ $t_0{}^n \leq t_0$

(1.7) $0 < x$ ▮ $x > x/(1+x) = t_0$ ▮ $x > t_0$

(1.8) $(t_0{}^n \leq t_0) \wedge (x > t_0)$ ▮ $t_0{}^n < x$

(1.9) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge (t_0{}^n < x)$ ▮ $t_0 \in E$ ▮ $\emptyset \neq E$

(1.10) $t_1 := choice(\{t \in \mathbb{R} : t > 1 + x\})$ ▮ $(t_1 \in \mathbb{R}) \wedge (t_1 > 1 + x)$

(1.11) $x > 0$ ▮ $t_1 > 1 + x > 1$ ▮ $t_1 > 1$ ▮ $t_1{}^n \geq t_1$

(1.12) $(t_1{}^n \geq t_1) \wedge (t_1 > 1 + x) \wedge (1 > 0)$ ▮ $t_1{}^n \geq t_1 > 1 + x > x$ ▮ $t_1{}^n > x$

(1.13) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_1{}^n > x)$ ▮ $t_1 \notin E$ ▮ $E \subset \mathbb{R}$

(1.14) $(\emptyset \neq E) \wedge (E \subset \mathbb{R})$ ▮ $\emptyset \neq E \subset \mathbb{R}$

(1.15) $t \in E \implies \ldots$

(1.15.1) $(t \in E) \wedge (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x))$ ▮ $t^n < x$

(1.15.2) $(t_1{}^n > x) \wedge (t^n < x)$ ▮ $t^n < x < t_1{}^n$ ▮ $t < t_1$

(1.16) $t \in E \implies t < t_1$ ▮ $\forall_{t \in E}(t \leq t_1)$ ▮ *UpperBound*$[t_1, E, \mathbb{R}, <]$ ▮ *BoundedAbove*$[E, \mathbb{R}, <]$

(1.17)   *CompletenessOf R* ▮ *LUBProperty*$[\mathbb{R}, <]$

(1.18) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (BoundedAbove[E, \mathbb{R}, <])$ ▮ $\exists_{y \in \mathbb{R}}(LUB[y, E, \mathbb{R}, <])$ . . .

(1.19)   . . . $y_0 := choice(\{y \in \mathbb{R} : LUB[y, E, \mathbb{R}, <]\})$ ▮ $LUB[y_0, E, \mathbb{R}, <]$

(1.20) $(LUB[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0)$ ▮ $0 < t_0 \leq y_0 \in \mathbb{R}$ ▮ $0 < y_0 \in \mathbb{R}$

(1.21) $y_0{}^n < x \implies \ldots$

(1.21.1) $k_0 := \frac{x - y_0{}^n}{n(y_0+1)^{n-1}}$ ▮ $k_0 \in \mathbb{R}$

(1.21.2) $y_0{}^n < x$ ▮ $0 < x - y_0{}^n$

(1.21.3) $(n > 0) \wedge (y_0 > 0)$ ▮ $0 < n(y_0+1)^{n-1}$

(1.21.4) $(0 < x - y_0{}^n) \wedge (0 < n(y_0+1)^{n-1})$ ▮ $0 < \frac{x-y_0{}^n}{n(y_0+1)^{n-1}} = k_0$ ▮ $0 < k_0$

(1.21.5) $(0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R})$ ▮ $0 < min(1, k_0) \in \mathbb{R}$

(1.21.6)   *QDenseInR* $\wedge (0, min(1, k_0) \in \mathbb{R}) \wedge (0 < min(1, k_0))$ ▮ $\exists_{h \in \mathbb{Q}}(0 < h < min(1, k_0))$ . . .

(1.21.7)   . . . $h_0 := choice(\{h \in \mathbb{Q} : 0 < h < min(1, k_0)\})$ ▮ $(0 < h_0 < 1) \wedge (h_0 < k_0 = \frac{x-y_0{}^n}{n(y_0+1)^{n-1}})$

(1.21.8) $(y_0 > 0) \wedge (h_0 > 0)$ ▮ $0 < y_0 < y_0 + h_0$

(1.21.9)   *RootLemma* $\wedge (0 < y_0 < y_0 + h_0)$ ▮ $(y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + h_0)^{n-1}$

(1.21.10) $h_0 < 1$ ▮ $h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$

(1.21.11) $((y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + h_0)^{n-1}) \wedge (h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1})$ ▮ $(y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + 1)^{n-1}$

(1.21.12) $(0 < n(y_0 + 1)^{n-1}) \wedge (h_0 < k_0 = \frac{x-y_0{}^n}{n(y_0+1)^{n-1}})$ ▮ $h_0 n(y_0 + 1)^{n-1} < x - y_0{}^n$

(1.21.13) $((y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + 1)^{n-1}) \wedge (h_0 n(y_0 + 1)^{n-1} < x - y_0{}^n)$ ▮ $(y_0 + h_0)^n - y_0{}^n < x - y_0{}^n$ ▮ $(y_0 + h_0)^n < x$

(1.21.14) $(y_0 + h_0)^n - y_0{}^n < x - y_0{}^n$ ▮ $(y_0 + h_0)^n < x$

(1.21.15) $(0 < y_0 \mathbb{R}) \wedge (0 < h_0 < \mathbb{R})$ ▮ $0 < y_0 < y_0 + h_0 \in \mathbb{R}$

(1.21.16) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ((y_0 + h_0)^n < x) \wedge (0 < y_0 + h_0 \in \mathbb{R})$ ▮ $(y_0 + h_0)^n \in E$

(1.21.17) $((y_0 + h_0)^n \in E) \wedge (y_0 < y_0 + h_0)$ ▮ $\exists_{e \in E}(y_0 < e)$

(1.21.18) $LUB[y_0, E, \mathbb{R}, <]$ ▮ *UpperBound*$[y_0, E, \mathbb{R}, <]$ ▮ $\forall_{e \in E}(e \leq y_0)$ ▮ $\neg\exists_{e \in E}(e > y_0)$

(1.21.19) $(\exists_{e \in E}(e > y_0)) \wedge (\neg\exists_{e \in E}(e > y_0))$ ▮ $\perp$

(1.22) $y_0{}^n < x \implies \perp$ ▮ $y_0{}^n \geq x$

(1.23) $y_0{}^n > x \implies \ldots$

(1.23.1) $k_1 := \frac{y_0{}^n - x}{n y_0{}^{n-1}}$ ▮ $(k_1 \in \mathbb{R}) \wedge (k_1 n y_0{}^{n-1} = y_0{}^n - x)$

(1.23.2) $(0 < x) \wedge (0 < n \in \mathbb{Z})$ ▮ $y_0{}^n - x < y_0{}^n \leq n y_0{}^n$ ▮ $y_0{}^n - x < n y_0{}^n$

(1.23.3) $y_0{}^n - x < n y_0{}^n$ ▮ $k_1 = \frac{y_0{}^n - x}{n y_0{}^{n-1}} < \frac{n y_0{}^n}{n y_0{}^{n-1}} = y_0$ ▮ $k_1 < y_0$

(1.23.4) $y_0{}^n > x$ ▮ $0 < y_0{}^n - x$

(1.23.5) $(n > 0) \wedge (y_0 > 0)$ ▮ $0 < n y_0{}^{n-1}$

(1.23.6) $(0 < y_0{}^n - x) \wedge 0 < (n y_0{}^{n-1})$ ▮ $0 < \frac{y_0{}^n - x}{n y_0{}^{n-1}} = k_1$ ▮ $0 < k_1$

(1.23.7) $(k_1 < y_0) \wedge (0 < k_1)$ ▮ $(0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$

(1.23.8) $t \geq y_0 - k_1 \implies \ldots$

(1.23.8.1) $t \geq y_0 - k_1$ ▮ $t^n \geq (y_0 - k_1)^n$ ▮ $-t^n \leq -(y_0 - k_1)^n$ ▮ $y_0{}^n - t^n \leq y_0{}^n - (y_0 - k_1)^n$

(1.23.8.2) $Root\,Lemma \wedge (0 < y_0 - k_1 < y_0)$ ▮ $y_0{}^n - (y_0 - k_1)^n < k_1 n y_0{}^{n-1}$

(1.23.8.3) $(y_0{}^n - t^n \leq y_0{}^n - (y_0 - k_1)^n) \wedge (y_0{}^n - (y_0 - k_1)^n < k_1 n y_0{}^{n-1})$ ▮ $y_0{}^n - t^n < k_1 n y_0{}^{n-1}$

(1.23.8.4) $(k_1 n y_0{}^{n-1} = y_0{}^n - x) \wedge (y_0{}^n - t^n < k_1 n y_0{}^{n-1})$ ▮ $y_0{}^n - t^n < y_0{}^n - x$ ▮ $-t^n < -x$ ▮ $t^n > x$

(1.23.8.5) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t^n > x)$ ▮ $t \notin E$

(1.23.9) $t \geq y_0 - k_1 \implies t \notin E$ ▮ $t \in E \implies t < y_0 - k_1$ ▮ $\forall_{t \in E}(t \leq y_0 - k_1)$ ▮ $Upper\,Bound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.10) $(LUB[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0))$ ▮ $\neg Upper\,Bound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.11) $(Upper\,Bound[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg Upper\,Bound[y_0 - k_1, E, \mathbb{R}, <])$ ▮ $\perp$

(1.24) $y_0{}^n > x \implies \perp$ ▮ $y_0{}^n \leq x$

(1.25) $Order[\mathbb{R}, <]$ ▮ $OrderTrichotomy[\mathbb{R}, <]$

(1.26) $(OrderTrichotomy[\mathbb{R}, <]) \wedge (y_0{}^n \geq x) \wedge (y_0{}^n \leq x)$ ▮ $y_0{}^n = x$

(1.27) $(y_0{}^n = x) \wedge (y_0 \in \mathbb{R})$ ▮ $\exists_{y \in \mathbb{R}}(y^n = x)$

(1.28) $y_1, y_2 := choice(\{y \in \mathbb{R} : y^n = x\})$

(1.29) $y_1 \neq y_2 \implies \ldots$

(1.29.1) $(OrderTrichotomy[\mathbb{R}, <]) \wedge (y_1 \neq y_2)$ ▮ $(y_1 < y_2) \vee (y_2 < y_1) \ldots$

(1.29.2) $\ldots (x = y_1{}^n < y_2{}^n = x) \vee (x = y_2{}^n < y_1{}^n = x)$ ▮ $(x < x) \vee (x > x)$ ▮ $\perp \vee \perp$ ▮ $\perp$

(1.30) $y_1 \neq y_2 \implies \perp$ ▮ $y_1 = y_2$ ▮ $\forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b)$

(1.31) $(\exists_{y \in \mathbb{R}}(y^n = x)) \wedge (\forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b))$ ▮ $\exists!_{y \in \mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}}(y^n = x)$ ▮ $\forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}}(y_0{}^n = x)$

$Root\,Existence\,In\,RCorollary := \forall_{0 < a \in \mathbb{R}} \forall_{0 < b \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}}((ab)^{1/n} = a^{1/n} b^{1/n})$ —

$$Extended\,Real\,System[\bar{\mathbb{R}}, +, *, <] := \left\{ \begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \wedge x - \infty = -\infty \wedge \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \wedge \\ (x > 0) \implies (x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty) \wedge \\ (x < 0) \implies (x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty) \end{array} \right.$$

$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$
$+_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d \rangle$
$*_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c \rangle$

$Field\,C := Field[\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}}]$ —
$RSubfield\,C := Subfield[\mathbb{R}, \mathbb{C}, +, *]$ —

$i := \langle 0, 1 \rangle \in \mathbb{C}$

$i\,Property := i^2 = -1$ —
$C\,Property := (a, b \in \mathbb{R}) \implies (\langle a, b \rangle = a + bi)$ —

$Conjugate[\overline{a + bi}] := a - bi$

$Conjugate\,Properties := (w, z \in \mathbb{C}) \implies \ldots$ —

(1) $\overline{z + w} = \overline{z} + \overline{w}$

(2) $\overline{z * w} = \overline{z} * \overline{w}$

(3) $Re(z) = (1/2)(z + \overline{z}) \wedge Im(z) = (1/2)(z - \overline{z})$

(4) $0 \leq z * \overline{z} \in \mathbb{R}$

$AbsoluteValue\,C[|z|] = (z * \overline{z})^{1/2}$
$AbsoluteValue\,Properties := (z, w \in \mathbb{C}) \implies \ldots$ —

(1) 123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$

$Function(f, X, Y) := X \neq \emptyset \neq Y \land Relation(f, X \times Y) \land \forall_{x \in X} \exists!_{y \in Y}((x, y) \in f)$

$(Function(f, X, Y) \land A \subseteq X \land B \subseteq Y) \implies \ldots$

(1)	$Domain(f) := X; Codomain(f) := Y$
(2)	$Image(f, A) := \{f(a) : a \in A\}; Preimage(f, B) := \{a : f(a) \in B\}$
(3)	$Range(f) := Image(Domain(f))$

$Injective(f, X, Y) := Function(f, X, Y) \land \forall_{x_1, x_2 \in X}(x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$

$Surjective(f, X, Y) := Function(f, X, Y) \land \forall_{y \in Y} \exists_{x \in X}(y_0 = f(x))$

$Bijective(f, X, Y) := Injective(f, X, Y) \land Surjective(f, X, Y)$

$SurjectiveEquivalent := (Range(f) = Codomain(f)) \implies Surjective(f)$

$(Function(f, X, Y) \land Function(g, Y, Z)) \implies (f \circ g)(x) := f(g(x)); Function(f \circ g, X, Z)$

$Propertiesof Functions := (Function(f, A, B) \land Function(g, B, C) \land Function(h, C, D)) \implies \ldots$

(1)	$h \circ (g \circ f) = (h \circ g) \circ f$
(2)	$(Injective(f) \land Injective(g)) \implies Injective(g \circ f)$
(3)	$(Surjective(f) \land Surjective(g)) \implies Surjective(g \circ f)$
(4)	$(Bijective(f, A, B)) \implies \exists_{f^{-1}}(Function(f^{-1}, B, A) \land \forall_{a \in A}(f^{-1}(f(a)) = a) \land \forall_{b \in B}(f(f^{-1}(b)) = b))$

$|(a, b) := a, b \in \mathbb{Z} \land a \neq 0 \land \exists_{c \in \mathbb{Z}}(b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \ldots$

(1)	$(a|b) \implies a|bc$
(2)	$(a|b \land b|c) \implies a|c$
(3)	$(a|b \land b|c) \implies a|(bx + cy)$
(4)	$(a|b \land b|a) \implies a = \pm b$
(5)	$(a|b \land a > 0 \land b > 0) \implies (a \leq b)$
(6)	$(a|b) \iff (m \neq 0 \land ma|mb)$

$DivisionAlgorithm := (a, b \in \mathbb{Z} \land a > 0) \implies \exists!_{q, r \in \mathbb{Z}}(b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \land a : b \land a : c$

$GCD(a, b, c) := CD(a, b, c) \land \forall_d((d : b \land d : c) \implies d : a)$

$GCDEquivalent := 123123$

Chapter 3

Linear Algebra

3.1 Matrix Operations and Special Matrices

$Matrix[A, m, n] := [a_{i,j}]_{m \times n} :=$ m rows, n columns of real numbers
$\mathcal{M}_{m,n} := \{A : Matrix[A, m, n]\}$
$O_{m,n} := (Matrix[O, m, n]) \wedge (a_{i,j} = 0)$
$Square[A, n] := Matrix[A, n, n]$
$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$
$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$
$Diagonal[A, n] := (Square[A, n]) \wedge (i \neq j \implies a_{i,j} = 0)$
$Scalar[A, n, k] := (Diagonal[A, n]) \wedge (a_{i,i} = k)$
$I_n := Scalar[I, n, 1]$

$+(A, B) := ((Matrix[A, m, n]) \wedge (Matrix[B, m, n])) \implies (A + B = [a_{i,j} + b_{i,j}]_{m \times n})$
$*(r, A) := ((r \in \mathbb{R}) \wedge (Matrix[A, m, n])) \implies (r * A = [ra_{i,j}]_{m \times n})$
$*(A, B) := ((Matrix[A, m, p]) \wedge (Matrix[B, p, n])) \implies (A * B = \left[\sum_{k=1}^{p}(a_{i,k} b_{k,j})\right]_{m \times n})$
$^T[A] := (Matrix[A, m, n]) \implies (A^T = [a_{j,i}]_{n \times m})$

$AddCom := \forall_{A,B \in \mathcal{M}}(A + B = B + A)$

(1)	$A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C \in \mathcal{M}}((A + B) + C = A + (B + C))$

(1)	$(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A \in \mathcal{M}} \exists!_{O \in \mathcal{M}}(A + O = A = O + A)$

(1)	$A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$
(2)	$A + O_1 = A = A + O_2 \; \blacksquare \; O_1 = O_2$

$AddInv := \forall_{A \in \mathcal{M}} \exists!_{(-A) \in \mathcal{M}}(A + (-A) = O = (-A) + A)$

(1)	$A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$
(2)	$A + (-A_1) = O = A + (-A_2) \; \blacksquare \; -A_1 = -A_2 \; \blacksquare \; A_1 = A_2$

$MulAssoc := \forall_{A,B,C \in \mathcal{M}}((A * B) * C = A * (B * C))$

(1)	$(A * B) * C = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,j})\right] * C = \left[\sum_{k_2=1}^{p_2}(\sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,k_2})c_{k_2,j})\right] = \left[\sum_{k_2=1}^{p_2} \sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,k_2} c_{k_2,j})\right] = \ldots$
(2)	$\ldots \left[\sum_{k_1=1}^{p_1} \sum_{k_2=1}^{p_2}(a_{i,k_1} b_{k_1,k_2} c_{k_2,j})\right] = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1} \sum_{k_2=1}^{p_2}(b_{k_1,k_2} c_{k_2,j}))\right] = \ldots = A * (B * C)$

$MulId := \forall_{A : Square[A,n]}(A * I_n = A = I_n * A)$

(1) $A * I_n = \left[\sum_{k=1}^n \left(a_{i,k} \left(\begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \right) \right) \right] = [a_{i,j}] = A$

(2) $TODO = A$

$ScalAssoc := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} (r(sA) = (rs)A = s(rA))$

(1) $r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$

(2) $(rs)A = [rsa_{i,j}]$

(3) $s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$

$TransCancel := \forall_{A \in \mathcal{M}} (A = (A^T)^T)$

(1) $A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$

$ScalMulCom := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} ((rA) * B = r(A * B) = A * (rB))$

(1) $(rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[\sum_{k=1}^p (ra_{i,k} b_{k,j}) \right] = r(A * B)$

(2) $A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[\sum_{k=1}^p (a_{i,k} rb_{k,j}) \right] = \left[\sum_{k=1}^p (ra_{i,k} b_{k,j}) \right] = r(A * B)$

$ScalDistLeft := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} ((r + s)A = rA + sA)$

(1) TODO

$ScalDistRight := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} (r(A + B) = rA + rB)$

(1) TODO

$MulDistRight := \forall_{A,B,C \in \mathcal{M}} ((A + B) * C = A * C + B * C)$

(1) $(A + B) * C = [a_{i,j} + b_{i,j}] * C = \left[\sum_{k=1}^p ((a_{i,k} + b_{i,k}) c_{k,j}) \right] = \dots$

(2) $\dots \left[\sum_{k=1}^p (a_{i,k} c_{k,j} + b_{i,k} c_{k,j}) \right] = \left[\sum_{k=1}^p (a_{i,k} c_{k,j}) \right] + \left[\sum_{k=1}^p (b_{i,k} c_{k,j}) \right] = A * C + B * C$

$MulDistLeft := \forall_{A,B,C \in \mathcal{M}} (C * (A + B) = C * A + C * B)$

(1) TODO

$TransAddDist := \forall_{A,B \in \mathcal{M}} ((A + B)^T = A^T + B^T)$

(1) TODO

$TransMulDist := \forall_{A,B \in \mathcal{M}} ((A * B)^T = B^T * A^T)$

(1) $(A * B)^T = \left[\sum_{k=1}^p (a_{i,k} b_{k,j}) \right]^T = \left[\sum_{k=1}^p (a_{j,k} b_{k,i}) \right] = \left[\sum_{k=1}^p (b_{k,i} a_{j,k}) \right] = \left[\sum_{k=1}^p (b_{i,k}^T a_{k,j}^T) \right] = B^T * A^T$

$Sym[A] := A = A^T$
$SkewSym[A] := A = -A^T$
$Invertible[A] := (Square[A, n]) \wedge (\exists_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A))$

$SymGen := \forall_{A \in \mathcal{M}} (Sym[A + A^T])$

(1) $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$SkewSymGen := \forall_{A \in \mathcal{M}} (SkewSym[A - A^T])$

(1) $-(A - A^T)^T = -(A^T - (A^T)^T) = -(A^T - A) = (A - A^T)$

$SymDecomp := \forall_{A \in \mathcal{M}} \exists!_{B:Sym[B]} \exists!_{C:SkewSym[C]} (A = B + C)$

(1)	$B := (1/2) * (A + A^T) \, ; \, C := (1/2) * (A - A^T)$
(2)	$SymGen[B] \wedge SkewSymGen[C]$
(3)	$A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$
(4)	$(1/2) * (A_1 + A_1{}^T) = (1/2) * (A_2 + A_2{}^T)$ ▎ $A_1 = A_2$
(5)	$(1/2) * (A_3 - A_3{}^T) = (1/2) * (A_4 - A_4{}^T)$ ▎ $A_3 = A_4$

$InvId := \forall_{A:Invertible[A]}(\exists!_{A^{-1} \in \mathcal{M}}(A * A^{-1} = I_n = A^{-1} * A))$

(1)	$A^{-1}{}_1 = A^{-1}{}_1 * I_n = A^{-1}{}_1 * (A * A^{-1}{}_2) = (A^{-1}{}_1 * A) * A^{-1}{}_2 = I_n * A^{-1}{}_2 = A^{-1}{}_2$

$InvCancel := \forall_{A:Invertible[A]}((A^{-1})^{-1} = A)$

(1)	$(A * A^{-1})^{-1} = I_n{}^{-1} = I_n$
(2)	$(A^{-1})^{-1} * A^{-1} = I_n$ ▎ $(A^{-1})^{-1} = I_n * A = A$

$InvDist := \forall_{A:Invertible[A]} \forall_{B:Invertible[B]}((A * B)^{-1} = B^{-1} * A^{-1})$

(1)	$(A * B) * (A * B)^{-1} = I$ ▎ $B * (A * B)^{-1} = A^{-1}$ ▎ $(A * B)^{-1} = B^{-1} * A^{-1}$

$InvTrans := \forall_{A:Invertible[A]}((A^T)^{-1} = (A^{-1})^T)$ ▎ $\Leftarrow$

(1)	$A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I$ ▎ $(A^{-1})^T = (A^T)^{-1}$

3.2 Elementary Matrices on Invertibility and Systems of Linear Equations

$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$
$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$
$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X(Sol[X, A, B])$
$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$
$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$

$HomoSysProps := (Sys[A, O]) \implies \ldots$

(1)	$u_0 := O \, ; \, u_1 := choice(\{X \in \mathcal{M} | X \neq O\}) \, ; \, k := choice(\mathbb{R})$
(2)	$TrivSol[u_0, A]$
(3)	$(NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$
(4)	$(TrivSol[\vec{X}, A]) \implies (TrivSol[LC(\vec{X}), A])$

$ElemMat[E] := (E = Swap[I_n, i, j]) \vee (Scale_*(I_n, i, c)) \vee (Combine_*(I_n, i, c, j))$
$ElemMatProd[E^*] := \exists_{\langle E \rangle}(\forall_{E_i \in E^*}(ElemMat[E_i]) \wedge (E^* = \Pi_{E_i \in E^*}(E_i)))$
$RowEquiv[A, B] := \exists_{E^*}((ElemMatProd[E^*]) \wedge (B = E^* * A))$

$ElemMatInv := \forall_{E \in \mathcal{M}}((ElemMat[E]) \implies (Invertible[E]))$

(1)	$E - RowSwap[E] \implies TODO \, ; \, E - RowScale_*(E) \implies TODO \, ; \, E - RowCombine_*(E) \implies TODO$

$ElemMatProdInv := \forall_{E^*}((ElemMatProd[E^*]) \implies (Invertible[E^*]))$

(1)	TODO

$RowEquivSys := \forall_{A,B,C,D,X \in \mathcal{M}}(((Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]])) \implies (Sol[X, A, B] \iff Sol[X, C, D]))$

(1)	$\exists_{E^*:ElemMatProd[E^*]}([CD] = E^* * [AB])$
(2)	$(E^* * A = C) \wedge (E^* * B = D)$
(3)	$Sol[Y, A, B] \implies \ldots$

(3.1)	$A * Y = B$
(3.2)	$C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D$ ▮ $Sol[Y, C, D]$
(4)	$Sol[Y, A, B] \implies Sol[Y, C, D]$
(5)	$(A = (E^*)^{-1} * C) \wedge (B = (E^*)^{-1} * D)$
(6)	$Sol[Z, C, D] \implies \ldots$
(6.1)	$C * Z = D$
(6.2)	$A * Z = ((E^*)^{-1} * C) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$
(7)	$Sol[Z, C, D] \implies Sol[Z, A, B]$
(8)	$Sol[X, A, B] \iff Sol[X, C, D]$

$RowEquivHomoSysSol := \forall_{A,C,X \in \mathcal{M}}((RowEquiv[A, C]) \implies ((Sol[X, A, O]) \iff (Sol[X, C, O])))$

(1)	Set $B = D = O$

$RREF[A] := (A \in \mathcal{M}) \wedge \begin{cases} \text{All zero rows are at the bottom of the matrix.} & \wedge \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} \wedge \\ \text{The leading entry in any nonzero row is 1.} & \wedge \\ \text{All entries in the column above and below a leading 1 are zero.} & \wedge \end{cases}$

$GaussJordanElim := \forall_{A \in \mathcal{M}} \exists!_{B \in \mathcal{M}}((RREF[B]) \wedge (RowEquiv[A, B]))$

(1)	Hit A with $ElemMat$'s until it becomes B
(2)	$(B = E^* * A) \wedge (RREF[B])$

$HasZero[A] := (Matrix(A, m, n)) \wedge (\exists_{i \leq m}(A_{i,:} = O))$

$HasZeroNonInvertible := \forall_{A \in \mathcal{M}}((HasZero[A]) \implies (\neg Invertible[A]))$

(1)	$i := choice(\{i \leq m | A_{i,:} = O\})$
(2)	$(B \in \mathcal{M}) \implies \ldots$
(2.1)	$(A * B)_{i,:} = O \neq I_{n_{i,:}}$ ▮ $A * B \neq I_n$
(3)	$(B \in \mathcal{M}) \implies (A * B \neq I_n)$ ▮ $\forall_{B \in \mathcal{M}}(A * B \neq I_n)$ ▮ $\neg Invertible[A]$

$InvIffRowEquivI := \forall_{A \in \mathcal{M}}((Invertible[A]) \iff (RowEquiv[A, I_n]))$

(1)	$(Invertible[A]) \implies \ldots$
(1.1)	$(RREF[B]) \wedge (RowEquiv[A, B])$
(1.2)	$B = E^* * A$
(1.3)	$(Invertible[E^*]) \wedge (Invertible[A])$ ▮ $Invertible[B]$
(1.4)	$Invertible[B]$ ▮ $\neg HasZero[B]$
(1.5)	$(RREF[B]) \wedge (\neg HasZero[B])$ ▮ $B = I_n$
(1.6)	$RowEquiv[A, I_n]$
(2)	$(Invertible[A]) \implies (RowEquiv[A, I_n])$
(3)	$(RowEquiv[A, I_n]) \implies \ldots$
(3.1)	$I_n = E^* * A$ ▮ $(E^*)^{-1} = A$
(3.2)	$A^{-1} = E^*_{DescSort}$ ▮ $Invertible[A]$
(4)	$(RowEquiv[A, I_n]) \implies (Invertible[A])$
(5)	$(Invertible[A]) \iff (RowEquiv[A, I_n])$

$RowEquivIIIffTrivSol := \forall_{A \in \mathcal{M}}((RowEquiv[A, I_n]) \iff (\forall_X((X = O) \iff (Sol[X, A, O]))))$

(1)	$(RowEquiv[A, I_n]) \implies \ldots$
(1.1)	$RowEquiv[A, I_n]$ ▮ $Invertible[A]$
(1.2)	$(Sol[X, A, O]) \implies \ldots$
(1.2.1)	$A * X = O$ ▮ $X = A^{-1} * O = O$ ▮ $X = O$
(1.3)	$(Sol[X, A, O]) \implies (X = O)$
(1.4)	$(X = O) \implies (Sol[X, A, O])$

(1.5)	$(X = O) \iff (Sol[X, A, O])$ ▌ $\forall_X((X = O) \iff (Sol[X, A, O]))$

(2)	$(RowEquiv[A, I_n]) \implies (\forall_X((X = O) \iff (Sol[X, A, O])))$
(3)	$(\forall_X((X = O) \iff (Sol[X, A, O]))) \implies \dots$

(3.1)	$(RREF[B]) \wedge (RowEquiv[A, B])$
(3.2)	$Sol[X, B, O]$
(3.3)	$(B \neq I_n) \implies \dots$

(3.3.1)	$(\exists_{Y \neq X}(Sol[Y, B, O]))$
(3.3.2)	$Sol[Y, A, O]$ ▌ $Y = X$
(3.3.3)	$(Y \neq X) \wedge (Y = X)$ ▌ $\perp$

(3.4)	$(B \neq I_n) \implies \perp$ ▌ $B = I_n$
(3.5)	$(RowEquiv[A, B]) \wedge (B = I_n)$ ▌ $RowEquiv[A, I_n]$

(4)	$(\forall_X((X = O) \iff (Sol[X, A, O]))) \implies (RowEquiv[A, I_n])$
(5)	$(RowEquiv[A, I_n]) \iff (\forall_X((X = O) \iff (Sol[X, A, O])))$

$InvIffUniqSol := \forall_{A \in \mathcal{M}}((Invertible[A]) \iff (\forall_{B \in \mathcal{M}}\exists!_{X \in \mathcal{M}}(Sol[X, A, B])))$

(1)	$(Invertible[A] \wedge B \in \mathcal{M}) \implies \dots$

(1.1)	$(Invertible[A]) \wedge (Sys[A, B])$
(1.2)	$(X = A^{-1} * B) \iff (Sol[X, A, B])$ ▌ $\exists!_{X \in \mathcal{M}}(Sol[X, A, B])$

(2)	$(\forall_{B \in \mathcal{M}}\exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies \dots$

(2.1)	$X_i := choice(\{X_i | Sol[X_i, A, I_{n:,i}]\})$
(2.2)	$A * [X_1 \dots X_n] = [(A * X_1) \dots (A * X_n)] = [I_{n:,1} \dots I_{n:,n}] = I_n$
(2.3)	$A^{-1} = [X_1 \dots X_n]$

(3)	$(\forall_{B \in \mathcal{M}}\exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies (Invertible[A])$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}} \left\{ \begin{array}{l} (Invertible[A]) \iff \\ (RowEquiv[A, I_n]) \iff \\ (\forall_X((X = O) \iff (Sol[X, A, O]))) \iff \\ (\forall_{B \in \mathcal{M}}\exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \end{array} \right\}$$

3.3 Vector Spaces

$$VectorSpace[V, +, *] := \exists_{O \in V}\forall_{\alpha, \beta \in \mathbb{R}}\forall_{u,v,w \in V} \left\{ \begin{array}{l} (u + v \in V) \wedge (u + v = v + u) \wedge ((u + v) + w = u + (v + w)) \wedge \\ (u + O = u) \wedge (\exists_{-u \in V}(u + (-u) = O)) \wedge \\ (\alpha * u \in V) \wedge (\alpha * (\beta * u) = (\alpha\beta) * u) \wedge (1 * u = u) \wedge \\ (\alpha * (u + v) = (\alpha * u) + (\alpha * v)) \wedge ((\alpha + \beta) * u = (\alpha * u) + (\beta * u)) \end{array} \right\}$$

$ZeroVectorUniq := \forall_{O',v \in V}((v + O' = v) \implies (O' = O))$

(1)	$O' = O' + O = O + O' = O$ ▌ $O' = O$

$AddInvUnique := \forall_{-v',v \in V}((v + -v' = O) \implies (-v' = -v))$

(1)	$-v' = -v' + O = -v' + (v + -v) = (-v' + v) + -v = (v + -v') + -v = O + -v = -v$ ▌ $-v' = -v$

$AddInvGen := \forall_{v \in V}((-1) * v = -v)$

(1)	$v + (-1) * v = (1 - 1) * v = 0 * v = O$ ▌ $(-1) * v = -v$

$ZeroVectorGenLeft := \forall_{v \in V}(0 * v = O)$

(1)	$0 * v = (0 + 0) * v = (0 * v) + (0 * v)$ ▌ $O = 0 * v$

$ZeroVectorGenRight := \forall_{r \in \mathbb{R}}(r * O = O)$

(1)	$r * O = r * (O + O) = (r * O) + (r * O)$ ▌ $O = r * O$

$ZeroVectorEquiv := \forall_{r\in\mathbb{R}}\forall_{v\in V}((r*v=O) \iff ((v=O)\vee(r=0)))$

(1) $(ZeroVectorGenLeft) \wedge (ZeroVectorGenRight)$ ▋ $((v=O)\vee(r=0)) \implies (r*v=O))$

(2) $(r*v=O) \implies \ldots$

  (2.1) $(r\neq 0) \implies \ldots$

    (2.1.1) $r\neq 0$ ▋ $r^{-1}\in\mathbb{R}$

    (2.1.2) $ZeroVectorGenRight$ ▋ $O=r^{-1}*O=r^{-1}*(r*v)=(r^{-1}r)*v=1*v=v$ ▋ $O=v$

  (2.2) $(r\neq 0) \implies (v=O)$ ▋ $(r=0)\vee(r\neq 0)$ ▋ $(r=0)\vee(v=O)$

(3) $(r*v=O) \implies ((r=0)\vee(v=O))$

(4) $(r*v=O) \iff ((r=0)\vee(v=O))$

3.4 Subspaces and Special Subspaces

$Subspace[S,V,+,*] := (VectorSpace[V,+,*]) \wedge (\emptyset \neq S \subseteq V) \wedge (VectorSpace[S,+,*])$

$SubspaceEquiv := \forall_{V,S}\left(\begin{array}{c} (VectorSpace[V,+,*]) \implies \\ ((Subspace[S,V,+,*]) \iff ((\emptyset\neq S\subseteq V) \wedge (\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S)))) \end{array} \right)$

(1) $(Subspace[S,V,+,*]) \implies \ldots$

  (1.1) $Subspace[S,V,+,*]$ ▋ $\emptyset\neq S\subseteq V$

  (1.2) $VectorSpace[S,+,*]$ ▋ $(\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))$

  (1.3) $(\emptyset\neq S\subseteq V) \wedge (\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))$

(2) $(Subspace[S,V,+,*]) \implies ((\emptyset\neq S\subseteq V) \wedge (\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S)))$

(3) $((\emptyset\neq S\subseteq V) \wedge (\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))) \implies \ldots$

  (3.1) $((\alpha,\beta\in\mathbb{R}) \wedge (\emptyset\neq S) \wedge (u,v,w\in S)) \implies \ldots$

    (3.1.1) $u,v\in V$ ▋ $u+v=v+u$

    (3.1.2) $u,v,w\in V$ ▋ $(u+v)+w=u+(v+w)$

    (3.1.3) $(ZeroVectorGenLeft) \wedge (u\in S)$ ▋ $0*u=O\in S$

    (3.1.4) $u\in V$ ▋ $u+O=u$

    (3.1.5) $(AddInvGen) \wedge (u\in S)$ ▋ $(-1)*u=-u\in S$

    (3.1.6) $u\in V$ ▋ $\alpha*(\beta*u)=(\alpha\beta)*u$

    (3.1.7) $u\in V$ ▋ $1*u=u$

    (3.1.8) $u,v\in V$ ▋ $\alpha*(u+v)=(\alpha*u)+(\alpha*v)$

    (3.1.9) $u\in V$ ▋ $(\alpha+\beta)*u=(\alpha*u)+(\beta*u)$

(4) $((\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))) \implies (Subspace[S,V,+,*])$

(5) $(Subspace[S,V,+,*]) \iff ((\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S)))$

$SumSubContains := \forall_{A,B,V}\left(\begin{array}{c} ((Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \wedge (SetSum[A+B,A,B,V,+,*])) \implies \\ ((Subspace[A+B,V,+,*]) \wedge (A,B\subseteq A+B)) \end{array} \right)$

(1) $(Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*])$ ▋ $(O\in A) \wedge (O\in B)$

(2) $(SetSum[A+B,A,B,V,+,*]) \wedge (O\in A) \wedge (O\in B)$ ▋ $O\in A+B$ ▋ $\emptyset\neq A+B$

(3) $(Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*])$ ▋ $A+B\subseteq V$ ▋ $\emptyset\neq A+B\subseteq V$

(4) $(u,v\in A+B) \implies \ldots$

  (4.1) $(\exists_{a_1\in A}\exists_{b_1\in B}(u=a_1+b_1)) \wedge (\exists_{a_2\in A}\exists_{b_2\in B}(v=a_2+b_2))$

  (4.2) $u+v=(a_1+b_1)+(a_2+b_2)=(a_1+a_2)+(b_1+b_2)$

  (4.3) $(a_1+a_2\in A) \wedge (b_1+b_2\in B)$ ▋ $u+v\in A+B$

(5) $(u,v\in A+B) \implies (u+v\in A+B)$ ▋ $\forall_{u,v\in A+B}(u+v\in A+B)$

(6) $((r\in\mathbb{R}) \wedge (v\in A+B)) \implies \ldots$

  (6.1) $\exists_{a\in A}\exists_{b\in B}(v=a+b)$

  (6.2) $r*v=r*(a+b)=r*a+r*b$

  (6.3) $(r*a\in A) \wedge (r*b)\in B$ ▋ $r*v\in A+B$

(7) $((r\in\mathbb{R}) \wedge (v\in A+B)) \implies (r*v\in A+B)$ ▋ $\forall_{r\in\mathbb{R}}\forall_{v\in A+B}(r*v\in A+B)$

(8)	$(\emptyset \neq A + B \subseteq V) \wedge (\forall_{u,v \in A+B}(u + v \in A + B)) \wedge (\forall_{r \in \mathbb{R}} \forall_{v \in A+B}(r * v \in A + B))$ ▌ $Subspace[A + B, V, +, *]$
(9)	$(\forall_{a \in A}(a + O) = a) \wedge (O \in B)$ ▌ $A \subseteq A + B$
(10)	$(\forall_{b \in B}(b + O) = b) \wedge (O \in A)$ ▌ $B \subseteq A + B$
(11)	$(A, B \subseteq A + B) \wedge (Subspace[A + B, V, +, *])$

$$SumSubMinContains := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])) \implies \\ (\forall_C((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies (A + B \subseteq C)) \end{array} \right)$$

(1)	$SumSub$ ▌ $(A, B \subseteq A + B) \wedge (Subspace[A + B, V, +, *])$
(2)	$((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies \ldots$
(2.1)	$(s \in A + B) \implies \ldots$
(2.1.1)	$\exists_{a \in A} \exists_{b \in B}(s = a + b)$
(2.1.2)	$(A, B \subseteq C)$ ▌ $a, b \in C$
(2.1.3)	$Subspace[C, V, +, *]$ ▌ $s = a + b \in C$
(2.2)	$(s \in A + B) \implies (s \in C)$ ▌ $A + B \subseteq C$
(3)	$((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies (A + B \subseteq C)$

$NullSpace[N, A, m, n] := (Matrix[A, m, n]) \wedge (N = \{x \in \mathbb{R}^n | A * x = O\})$
$RowSpace[R, A, m, n] := (Matrix[A, m, n]) \wedge (R = \{x^T * A \in \mathbb{R}^n | x \in \mathbb{R}^m\})$
$ColSpace[C, A, m, n] := (Matrix[A, m, n]) \wedge (C = \{A * x \in \mathbb{R}^m | x \in \mathbb{R}^n\})$

$NullSubspace := (NullSpace[N, A, m, n]) \implies (Subspace[N, \mathbb{R}^n, +, *])$

(1)	TODO

$RowSubspace := (RowSpace[R, A, m, n]) \implies (Subspace[R, \mathbb{R}^n, +, *])$

(1)	TODO

$ColSubspace := (ColSpace[C, A, m, n]) \implies (Subspace[C, \mathbb{R}^m, +, *])$

(1)	TODO

3.5 Linear Combination, Linear Span, Linear Independence

$LinComb[c, U, K, V, +, *] := (VectorSpace[V, +, *]) \wedge (n \in \mathbb{N}) \wedge (U \in V^n) \wedge (K \in \mathbb{R}^n) \wedge (c = \sum_{i=1}^n (k_i * u_i))$

$$LinSpan[S', S, V, +, *] := \left(\begin{array}{l} (VectorSpace[V, +, *]) \wedge (S \in V^n) \wedge ((S = \emptyset) \implies (S' = \{O\})) \wedge \\ ((S \neq \emptyset) \implies (S' = \{c \in V | \exists_{K \in \mathbb{R}^n}(LinComb[c, S, K, V, +, *])\})) \end{array} \right)$$

$LinSpanSubContains := \forall_{S',S,V,+,*}((LinSpan[S', S, V, +, *]) \implies ((Subspace[S', V, +, *]) \wedge (S \subseteq S')))$

(1)	$(S = \emptyset) \implies (S' = \{O\}) \implies (\emptyset \neq S')$
(2)	$(S \neq \emptyset) \implies (LinComb[O, S, \{0\}^n, V, +, *]) \implies (O \in S') \implies (\emptyset \neq S')$
(3)	$((S = \emptyset) \vee (S \neq \emptyset)) \implies (\emptyset \neq S')$ ▌ $\emptyset \neq S'$
(4)	$LinSpan[S', S, V, +, *]$ ▌ $S' \subseteq V$ ▌ $\emptyset \neq S' \subseteq V$
(5)	$(a, b \in S') \implies \ldots$
(5.1)	$(\exists_{K \in \mathbb{R}^n}(LinComb[a, S, K, V, +, *])) \wedge (\exists_{L \in \mathbb{R}^n}(LinComb[b, S, L, V, +, *]))$
(5.2)	$a + b = \sum_{i=1}^n (k_i * s_i) + \sum_{i=1}^n (l_i * s_i) = \sum_{i=1}^n ((k_i + l_i) * s_i)$ ▌ $a + b = \sum_{i=1}^n ((k_i + l_i) * s_i)$
(5.3)	$\langle k_i + l_i | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{N}^n$
(5.4)	$(a + b = \sum_{i=1}^n ((k_i + l_i) * s_i)) \wedge (\langle k_i + l_i | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{N}^n)$ ▌ $\exists_{M \in \mathbb{N}^n}(a + b = \sum_{i=1}^n (m_i * s_i))$
(5.5)	$\exists_{M \in \mathbb{N}^n}(LinComb[a + b, S, M, V, +, *])$ ▌ $a + b \in S'$
(6)	$(a, b \in S') \implies (a + b \in S')$ ▌ $\forall_{a,b \in S'}(a + b \in S')$
(7)	$((r \in \mathbb{R}) \wedge (u \in S')) \implies \ldots$
(7.1)	$\exists_{K \in \mathbb{R}^n}(LinComb[u, S, K, V, +, *])$ ▌ $\exists_{K \in \mathbb{R}^n}(u = \sum_{i=1}^n (k_i * s_i))$
(7.2)	$r * u = r * \sum_{i=1}^n (k_i * s_i) = \sum_{i=1}^n (r * (k_i * s_i)) = \sum_{i=1}^n (rk_i) * s_i$ ▌ $r * u = \sum_{i=1}^n (rk_i) * s_i$
(7.3)	$\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n$

(7.4) $(\sum_{i=1}^{n}(rk_i * s_i)) \wedge (\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n)$

(7.5) $\exists_{M \in \mathbb{R}^n}(r * u = \sum_{i=1}^{n}(m_i * s_i))$ ▌ $\exists_{M \in \mathbb{R}^n}(LinComb[r * u, S, M, V, +, *])$ ▌ $r * u \in S'$

---

(8) $((r \in \mathbb{R}) \wedge (u \in S')) \implies (r * u \in S')$ ▌ $\forall_{r \in \mathbb{R}}\forall_{u \in S'}(r * u \in S')$

---

(9) $(SubspaceEquiv) \wedge (\emptyset \neq S' \subseteq V) \wedge (\forall_{a,b \in S'}(a + b \in S')) \wedge (\forall_{r \in \mathbb{R}}\forall_{u \in S'}(r * u \in S'))$ ▌ $Subspace[S', V, +, *]$

---

(10) $(s_j \in S) \implies \dots$

(10.1) $K := \langle \left\{ \begin{array}{ll} 1 & i = j \\ 0 & i \neq j \end{array} \right. | \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n$ ▌ $\sum_{i=1}^{n}(k_i * s_i) = s_j$

(10.2) $\dots$ ▌ $\exists_{K \in \mathbb{R}^n}(LinComb[s_j, S, K, V, +, *])$ ▌ $s_j \in S'$

---

(11) $(s_j \in S) \implies (s_j \in S')$ ▌ $\forall_{x \in S}(x \in S')$ ▌ $S \subseteq S'$

---

(12) $(Subspace[S', V, +, *]) \wedge (S \subseteq S')$

$LinSpanSubMinContains := \forall_{S',S,V,+,*}((LinSpan[S', S, V, +, *]) \implies (\forall_W(((Subspace[W, V, +, *]) \wedge (S \subseteq W)) \implies (S' \subseteq W))))$

(1) $(s' \in S') \implies \dots$

(1.1) $\exists_{K \in \mathbb{R}^n}(LinComb[s', S, K, V, +, *])$ ▌ $s' = \sum_{i=1}^{n}(k_i * s_i)$

(1.2) $(S \subseteq W) \wedge (Subspace[W, V, +, *])$ ▌ $s' = \sum_{i=1}^{n}(k_i * s_i) \in W$

---

(2) $(s' \in S') \implies (s' \in W)$ ▌ $S' \subseteq W$

$LinInd[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (S \in V^n) \wedge ((S \neq \emptyset) \implies (\forall_{K \in \mathbb{R}^n}((LinComb[O, S, K, V, +, *]) \implies (K = \{0\}^n))))$

$ZeroDependent := (O \in S) \implies (\neg LinInd[S, V, +, *])$

---

(1) $K := \langle \left\{ \begin{array}{ll} 1 & u_i = O \\ 0 & u_i \neq O \end{array} \right. | (1 \leq i \leq n) \wedge (i \in \mathbb{N})\rangle$ ▌ $K \in \mathbb{R}^n$

---

(2) $(LinComb[O, S, K, V, +, *]) \wedge (K \neq \{O\}^n)$ ▌ $\neg LinInd[S, V, +, *]$

$SingletonNonZeroIndependent := (v \neq O) \implies (LinInd[\langle v\rangle, V, +, *])$

---

(1) $(r * v = O) \iff ((r = 0) \vee (v \neq O))$

---

(2) $v \neq O$ ▌ $r = 0$

---

(3) $\forall_{r \in \mathbb{R}}((r * v = O) \implies (r = 0))$

$SubIndependent := \forall_{V,A,B}(((VectorSpace[V, +, *]) \wedge (A \subseteq B \in V^m)) \implies ((LinInd[B, V, +, *]) \implies (LinInd[A, V, +, *])))$

(1) $(LinComb[O, A, K, V, +, *]) \implies \dots$

(1.1) $L := \langle \left\{ \begin{array}{ll} 1 & j \leq n \\ 0 & j > n \end{array} \right. | (1 \leq j \leq m \wedge (j \in \mathbb{N})\rangle$ ▌ $L \in \mathbb{R}^m$

(1.2) $A \subseteq B$ ▌ $\forall_{n \geq j \in \mathbb{N}}(a_j = b_j)$

(1.3) $\forall_{n \geq j \in \mathbb{N}}(a_j = b_j)$ ▌ $\sum_{i=1}^{n}(k_i * a_i)) = \sum_{i=1}^{n}(k_i * a_i)) + O = \sum_{j=1}^{m}(l_j * b_j))$

(1.4) $LinComb[O, A, K, V, +, *]$ ▌ $O = \sum_{i=1}^{n}(k_i * a_i)$

(1.5) $O = \sum_{i=1}^{n}(k_i * a_i) = \sum_{j=1}^{m}(l_j * b_j)$ ▌ $LinComb[O, B, L, V, +, *]$

(1.6) $(LinInd[B, V, +, *]) \wedge (LinComb[O, B, L, V, +, *])$ ▌ $L = \{0\}^m$

(1.7) $(\forall_{n \geq j \in \mathbb{N}}(a_j = b_j)) \wedge (L = \{0\}^m)$ ▌ $\forall_{n \geq j \in \mathbb{N}}(k_j * a_j = l_j * b * j = l_j * a_j)$ ▌ $K = \{0\}^n$

---

(2) $(LinComb[O, A, K, V, +, *]) \implies (K = \{0\}^n)$ ▌ $\forall_{K \in \subseteq \mathbb{R}^n}((LinComb[O, A, K, V, +, *]) \implies (K = \{0\}^n))$ ▌ $LinInd[A, V, +, *]$

$SuperDependent := \forall_{V,A,B}(((VectorSpace[V, +, *]) \wedge (A \subseteq B \subseteq V)) \implies ((\neg LinInd[A, V, +, *]) \implies (\neg LinInd[B, V, +, *])))$

(1)   TODO : A has a non trivial solution ▌ use the same non trivial solution in combination with B and L

$LinIndEquiv := \forall_{U,V}((LinInd[U, V, +, *]) \iff (\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])))$

(1) $\Gamma' = \Gamma \setminus \{j\}$

(2) $(\neg LinInd[U, V, +, *]) \implies \ldots$

(2.1) $(\exists_{\Gamma \in \mathbb{R}^{|U|}}((\sum(\gamma_i * u_i) = O) \wedge (\Gamma \neq \{0\}^{|U|})))$

(2.2) $\exists_{\gamma_k \in \Gamma}(\gamma_k \neq 0)$

(2.3) $\sum(\gamma_i' * u_i) = \sum(\gamma_i * u_i) - \gamma_k * u_k = -\gamma_k * u_j$

(2.4) $u_k = (-1/\gamma_k)(\sum(\gamma_i' * u_i)) = \sum((-\gamma_i'/\gamma_k) * u_i)$ ▌ $\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])$

(3) $(\neg LinInd[U, V, +, *]) \implies (\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *]))$

(4) $(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])) \implies (LinInd[U, V, +, *])$

(5) $(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])) \implies \ldots$

(5.1) $\exists_{j \in U}(j = \sum(\gamma_i' * u_i))$

(5.2) $\Gamma := \Gamma' \cup \{-1\}$

(5.3) $(\sum(\gamma_i * u_i) = \sum(\gamma_i' * u_i) + (-1) * \gamma_j = O) \wedge (\Gamma \neq \{0\}^n)$ ▌ $\neg LinInd[U, V, +, *]$

(6) $(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])) \implies (\neg LinInd[U, V, +, *])$

(7) $(LinInd[U, V, +, *]) \implies (\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *]))$

(8) $(LinInd[U, V, +, *]) \iff (\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *]))$

$LinIndSuperspace := \forall_{U,V}((Subspace[U, V]) \implies (\forall_W((LinInd[W, U, +, *]) \implies (LinInd[W, V, +, *]))))$

(1) $(\neg LinInd[W, V, +, *]) \implies \ldots$

(1.1) $\exists_{j \in W}(LinComb[j, W \setminus \{j\}, +, *])$ ▌ $\neg LinInd[W, U, +, *]$

(1.2) $(LinInd[W, U, +, *]) \wedge (\neg LinInd[W, U, +, *])$ ▌ $\bot$

(2) $(\neg LinInd[W, V, +, *]) \implies \bot$ ▌ $LinInd[W, V, +, *]$

$Spans[S, V, +, *] := LinSpan[V, S, V, +, *]$
$FinDim[V, +, *] := \exists_{S \in V^n}(Spans[S, V, +, *])$

$LinDepLemma := \forall_{S,V}\left(\begin{array}{l} (\neg LinInd[S, V, +, *]) \\ \exists_{j \in \mathbb{N}_{1,n}}((s_j \in LinSpan[P_1, S_{1,j-1}, V, +, *]) \wedge (LinSpan[P_2, S, V, +, *] = LinSpan[P_3, S \setminus \{s_j\}, V, +, *])) \end{array} \right)$ $\implies$

(1) $\neg LinInd[S, V, +, *]$ ▌ $\exists_{K \in \mathbb{R}^n}((LinComb[O, S, K, V, +, *]) \wedge (K \neq \{0\}^n))$

(2) $\exists_{j \in \mathbb{N}_{1,n}}((k_j \neq 0) \wedge (\forall_{i \in \mathbb{N}_{1,n}}((i > j) \implies (k_i = 0))))$

(3) $s_j = (-1/k_j) \sum_{i=1}^{j-1}(k_i * s_i) = \sum_{i=1}^{j-1}((-k_i/k_j) * s_i)$

(4) $\langle -k_i/k_j | i \in \mathbb{N}_{1,j-1} \rangle \in \mathbb{R}^{j-1}$

(5) $\exists_{M \in \mathbb{R}^{j-1}}(LinComb[s_j, S_{1,j-1}, M, V, +, *])$ ▌ $s_j \in LinSpan[P_1, S_{1,j-1}, V, +, *]$

(6) $(v \in P_2) \iff (v \in LinSpan[P_2, S, V, +, *]) \iff \ldots$

(7) $\ldots (v = \sum_{i=1}^n(k_i * s_i)) = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^n(k_i * s_i)) + k_j * s_j = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^n(k_i * s_i)) + k_j * \sum_{i=1}^{j-1}((-k_i/k_j) * s_i)) \iff \ldots$

(8) $(v \in LinSpan[P_3, S \setminus \{s_j\}, V, +, *]) \iff (v \in P_3)$ ▌ $(v \in P_2) \iff (v \in P_3)$ ▌ $P_2 = P_3$

$LinIndLengthLeqSpan := \forall_{L,S}(((LinInd[L, V, +, *]) \wedge (Spans[S, V, +, *])) \implies (|L| \leq |S|))$

(1) TODO : form $B = L \cup S$, remove dependent elements in S such that
$(Spans[B, V, +, *]) \wedge (|B| = |S|)$ by $LinDepLemma$, $|L| \leq |B| = |S|$

(2) $\forall_{l_i \in L} \ldots$

(2.1) $l_i \in V$ ▌ $LinComb[l_i, S, K, V, +, *]$ ▌ $\neg LinInd[\langle l_i \rangle \cup S, V, +, *]$

(2.2) $LinDepLemma$ ▌ $\exists_{j \in \mathbb{N}_{1,n}}(LinSpan[V, S, V, +, *] = LinSpan[V, S \setminus \{s_j\}, V, +, *])$

(2.3) $B := \langle l_i \rangle \cup S \setminus \{s_j\}$ ▌ $|B| = 1 + |S| - 1 = |S|$

(3) $|L| \leq |B| = |S|$ ▌ $|L| \leq |S|$

$FinSubSpace := \forall_{U,V}(((Subspace[U, V, +, *]) \wedge (FinDim[V, +, *])) \implies (FinDim[U, +, *]))$

(1) TODO : take $Spans[S, V, +, *]$, remove all $s_j \in S$ such that $U \subseteq LinSpan[S \setminus \{s_j\}]$ ▌ $S' = S \setminus \{s_j\}$
$(LinSpan[U, S', V, +, *]) \wedge (|S'| \leq |S|)$ ▌ $FinDim[U, +, *]$

(2) $FinDim[V, +, *]$ ▮ $\exists_{S \in V^n}(Spans[S, V, +, *])$

(3) $\forall_{(u_j \in U) \wedge (\neg LinSpan[U, S, V, +, *])} \cdots$

   (3.1) $\neg LinSpan[U, S, V, +, *]$ ▮ $\exists_{u_j \in U}(\neg LinComb[u_j, S_{1,j-1}, K_{1,j-1}, V, +, *])$

   (3.2) $B := S_{1,j-1}$ ▮ $|B| = |S| - 1 < |S|$

(4) $LinSpan[U, B, V, +, *]$ ▮ $\exists_{B \in V^M}(Spans[B, U, +, *])$ ▮ $FinDim[U, +, *$

3.6   Bases and Dimensions

$Basis[S, V, +, *] := (Spans[S, V, +, *]) \wedge (LinInd[S, V, +, *])$

$BasisEquiv := \forall_{S, V}((Basis[S, V, +, *]) \iff (\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *]))$

(1) $(Basis[S, V, +, *]) \implies \ldots$

  (1.1) $(v \in V) \implies \ldots$

    (1.1.1) $Basis[S, V, +, *]$ ▮ $Spans[V, S, +, *]$ ▮ $\exists_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *])$

    (1.1.2) $((K_1, K_2 \in \mathbb{R}^n) \wedge (LinComb[v, S, K_1, V, +, *]) \wedge (LinComb[v, S, K_2, V, +, *])) \implies \ldots$

      (1.1.2.1) $(v = \sum(k_{1_i} * s_i)) \wedge (v = \sum(k_{2_i} * s_i))$

      (1.1.2.2) $O = v - v = \sum(k_{1_i} * s_i) - \sum(k_{2_i} * s_i) = \sum((k_{1_i} - k_{2_i}) * s_i)$

      (1.1.2.3) $L := \langle k_{1_i} - k_{2_i} | i \in \mathbb{N}_{i=1}^n \rangle \in \mathbb{R}^n$

      (1.1.2.4) $(LinInd[S, V, +, *]) \wedge (LinComb[O, S, L, V, +, *])$ ▮ $L = \{0\}^n$ ▮ $K_2 = K_1$

    (1.1.3) $((K_1, K_2 \in \mathbb{R}^n) \wedge (LinComb[v, S, K_1, V, +, *]) \wedge (LinComb[v, S, K_2, V, +, *])) \implies (K_1 = K_2)$

    (1.1.4) $\forall_{K_1, K_2 \in \mathbb{R}^n}((LinComb[v, S, K_1, V, +, *]) \wedge (LinComb[v, S, K_2, V, +, *]) \implies (K_1 = K_2))$

    (1.1.5) $\exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *])$

  (1.2) $(v \in V) \implies (\exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *]))$

(2) $(Basis[S, V, +, *]) \implies (\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *]))$

(3) $(\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *])) \implies \ldots$

  (3.1) $\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *])$ ▮ $\forall_{v \in V} \exists_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *])$ ▮ $Spans[S, V, +, *]$

  (3.2) $O \in V$ ▮ $\exists!_{K \in \mathbb{R}^n}(LinComb[O, S, K, V, +, *])$

  (3.3) $(K \neq \{0\}^n) \implies (\neg \exists!_{K \in \mathbb{R}^n}(LinComb[O, S, K, V, +, *])) \implies \bot$ ▮ $K = \{0\}^n$

  (3.4) $(\exists!_{K \in \mathbb{R}^n}(LinComb[O, S, K, V, +, *])) \wedge (K = \{0\}^n)$ ▮ $LinInd[S, V, +, *]$

  (3.5) $(Spans[S, V, +, *]) \wedge (LinInd[S, V, +, *])$ ▮ $Basis[S, V, +, *]$

(4) $(\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *])) \implies (Basis[S, V, +, *])$

$SpanReduceBasis := \forall_{S, V}(Spans[S, V, +, *] \implies \exists_B((B \subseteq S) \wedge (Basis[B, V, +, *])))$

(1) TODO : Remove all dependent $s_j \in S$ ▮ $(S' = S \setminus \{s_j\}) \wedge (LinSpan[S'] = LinSpan[S])$ until $LinInd[S']$ ▮ $Basis[S']$

$FinDimBasis := \forall_V((FinDim[V, +, *]) \implies (\exists_B(Basis[B, V, +, *])))$

(1) $FinDim[V, +, *]$ ▮ $\exists_{S \in V^n}(Spans[S, V, +, *])$

(2) $(SpanReduceBasis) \wedge (Spans[S, V, +, *])$ ▮ $\exists_B(Basis[B, V, +, *])$

$LinIndExpandBasis := \forall_{L, V}(LinInd[L, V, +, *] \implies \exists_B((L \subseteq B) \wedge (Basis[B, V, +, *])))$

(1) TODO : $FinDimBasis$ ▮ $\exists_A(Basis[A, V, +, *])$, form $B = L \cup A$ ▮ $Span[B]$,
use $SpanReduceBasis$ call it B', $(L \subseteq B') \wedge (Basis[B'])$

$BasisLinearIndCard := \forall_{S, T, V}(((Basis[S, V, +, *]) \wedge (LinInd[T, V, +, *])) \implies (|T| \leq |S|))$

(1) $(Basis[S, V, +, *]) \implies \ldots$

  (1.1) $(|T| > |S|) \implies \ldots$

    (1.1.1) $(Spans[S, V, +, *]) \wedge (T \subseteq V)$ ▮ $t_{1 \ldots t_j} = \sum(\gamma_i * s * i) \ldots$

    (1.1.2) $\ldots t_j = \sum(\gamma_i' * t_i)$ ▮ $\neg LinInd[T, V, +, *]$

  (1.2) $(|T| > |S|) \implies (\neg LinInd[T, V, +, *])$ ▮ $(LinInd[T, V, +, *]) \implies (|T| \leq |S|)$

(2) $((Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*])) \implies (|T| \leq |S|)$

$BasisCard := \forall_{S,T,V}(((Basis[S,V,+,*]) \wedge (Basis[T,V,+,*])) \implies (|T| = |S|))$

(1)	$Basis[S,V,+,*]$ ▮ $LinInd[S,V,+,*]$	
(2)	$(Basis[T,V,+,*]) \wedge (LinInd[S,V,+,*])$ ▮ $|S| \leq |T|$	
(3)	$Basis[T,V,+,*]$ ▮ $LinInd[T,V,+,*]$	
(4)	$(Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*])$ ▮ $|T| \leq |S|$	
(5)	$(|S| \leq |T|) \wedge (|T| \leq |S|)$ ▮ $|T| = |S|$	

$Dim[d,V,+,*] := (\exists_B(Basis[B,V,+,*])) \wedge ((V = \{O\}) \implies (d = 0)) \wedge ((V \neq \{O\}) \implies (d = |B|))$

$LinIndLengthDim := \forall_{U,V}(((LinInd[U,V,+,*]) \wedge (Dim[|U|,V,+,*])) \implies (Basis[U,V,+,*]))$

(1)	$(LinIndExpandBasis) \wedge (LinInd[U,V,+,*])$ ▮ $\exists_B((U \subseteq B) \wedge (Basis[B,V,+,*]))$
(2)	$(BasisCard) \wedge (Dim[|U|,V,+,*]) \wedge (Basis[B,V,+,*])$ ▮ $|B| = |U|$ ▮ $B = U$ ▮ $Basis[U,V,+,*]$

$SpanLengthDim := \forall_{U,V}(((Spans[U,V,+,*]) \wedge (Dim[|U|,V,+,*])) \implies (Basis[U,V,+,*]))$

(1)	$(SpanReduceBasis) \wedge (Spans[U,V,+,*])$ ▮ $\exists_B((B \subseteq U) \wedge (Basis[B,V,+,*]))$
(2)	$(BasisCard) \wedge (Dim[|U|,V,+,*]) \wedge (Basis[B,V,+,*])$ ▮ $|B| = |U|$ ▮ $B = U$ ▮ $Basis[U,V,+,*]$

$LinDepLengthDim := \forall_{U,V}(((U \subseteq V) \wedge (|U| > Dim[V])) \implies (\neg LinInd[U,V,+,*]))$

(1)	Contrapositive of $BasisLinearIndCard$

$LinDepLengthDim := \forall_{U,V}(((U \subseteq V) \wedge (|U| < Dim[V])) \implies (\neg Spans[U,V,+,*]))$

(1)	Suppose $Spans[U,V,+,*]$, $B = SpanReduceBasis[U]$ to form a basis, $(|B| \leq |U| < Dim[V]) \wedge |B| = Dim[V]$ ▮ $\bot$
(2)	$\neg Spans[U,V,+,*]$

3.7   Rank

$Nullity[n,A] := (NullSpace[N,A]) \wedge (Dim[n,N,+,*])$
$Rank[r,A,m,n] := (Matrix[A,m,n]) \wedge (RowSpace[R,A,m,n]) \wedge (Dim[r,R,A,+,*])$

$RowRankEqColRank := \forall_A(TODO)$

(1)	TODO

$RankNullity := \forall_A((Matrix[A,m,n]) \implies (Rank[A] + Nullity[A] = n))$

(1)	TODO

$RankInv := \forall_A((Matrix[A,m,n]) \implies ((Rank[A] = n) \iff (Inv[A])))$

(1)	TODO

$RankNonTrivialSol := (\exists_X((A * X = O) \wedge (X \neq O))) \iff (Rank[A] < n)$

(1)	TODO

$RankUniqueSol := (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X,A,B])) \iff (Rank[A] = n)$

(1)	TODO

$$SquareTheorems_() := \forall_{A \in \mathcal{M}} \left\{ \begin{array}{ll} (Invertible[A]) & \Longleftrightarrow \\ (RowEquiv[A, I_n]) & \Longleftrightarrow \\ (\forall_X((X = O) \iff (Sol[X, A, O]))) & \Longleftrightarrow \\ (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) & \Longleftrightarrow \\ (Rank[A] = n) & \Longleftrightarrow \\ (Nullity[A] = 0) & \Longleftrightarrow \\ \text{(The rows form a linearly independent set of vectors (to get full rank))} & \Longleftrightarrow \\ \text{(The columns form a linearly independent set of vectors (to get full rank))} & \Longleftrightarrow \end{array} \right\}$$

3.8 Linear Transformations

$$LinTrans[L, V, +_v, *_v, W, +_w, *_w] := \left(\begin{array}{l} (Function[f, V, W]) \wedge (VectorSpace[V, +_v, *_v]) \wedge (VectorSpace[W, +_w, *_w]) \wedge \\ (\forall_{\alpha, \beta \in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))) \quad \wedge \quad (\forall_{r \in \mathbb{R}} \forall_{\alpha \in V}(L(r *_v \alpha) = r *_w L(\alpha))) \end{array} \right)$$

$$LinOp[L, V, +_v, *_v] := LinTrans[L, V, +_v, *_v, V, +_v, *_v]$$

$$\mathcal{L}[V, W] := \{L | LinTrans[L, V, +_v, *_v, W, +_w, *_w]\}$$

$$ZeroMapsToZero := \forall_{L, V, W}((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies (L(O_v) = O_w))$$

(1)	$L(O_v) = L(O_v +_v O_v) = L(O_v) +_w L(O_v)$

(2)	$O_w = L(O_v) - L(O_v) = L(O_v)$

$$SplitAddInv := \forall_{L, V, W}((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies (\forall_{\alpha, \beta \in V}(L(\alpha -_v \beta) = L(\alpha) -_w L(\beta))))$$

(1)	$L(\alpha - \beta) = L(\alpha + (-\beta)) = L(\alpha) + L(-\beta) = L(\alpha) + (-1) * L(\beta) = L(\alpha) - L(\beta)$

$$BasisDomainInduceLinTrans := \forall_{V, W} \left(\begin{array}{l} ((Basis[A, V, +_v, *_v]) \wedge (B \subseteq W) \wedge (n = |B| = |A|) \wedge (VectorSpace[W, +_w, *_w])) \implies \\ (\exists!_T((LinTrans[T, V, +_v, *_v, W, +_w, *_w]) \wedge (\forall_{i \in \mathbb{N}_{1,n}}(T(a_i) = b_i)))) \end{array} \right)$$

(1)	$T(\sum_{i=1}^{n}(k_i * a_i)) := \sum_{i=1}^{n}(k_i * b_i)$

(2)	$(i \in \mathbb{N}_{1,n}) \implies \ldots$

(2.1)	$L := \left\langle \left\{ \begin{array}{ll} 1 & j = i \\ 0 & j \neq i \end{array} \right. \middle| j \in \mathbb{N}_{1,n} \right\rangle \quad\blacksquare\quad L \in \mathbb{R}^n$

(2.2)	$T(a_i) = T(\sum_{i=1}^{n}(l_i * a_i)) = \sum_{i=1}^{n}(l_i * b_i) = b_i \quad\blacksquare\quad T(a_i) = b_i$

(3)	$(i \in \mathbb{N}_{1,n}) \implies (T(a_i) = b_i) \quad\blacksquare\quad \forall_{i \in \mathbb{N}_{1,n}}(T(a_i) = b_i)$

(4)	$(BasisEquiv) \wedge (Basis[A, V, +_v, *_v]) \quad\blacksquare\quad \forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, A, K, V, +, *]) \quad\blacksquare\quad T : Span[A] \to Span[B] \quad\blacksquare\quad Function[T, V, W]$

(5)	$(\alpha, \beta \in V) \implies \ldots$

(5.1)	$(\exists_{K_\alpha}(LinComb[\alpha, A, K_\alpha, V, +_v, *_v])) \wedge (\exists_{K_\beta}(LinComb[\beta, A, K_\beta, V, +_v, *_v])) \quad\blacksquare\quad (\alpha = \sum_{i=1}^{n}(k_{\alpha i} * a_i)) \wedge (\beta = \sum_{i=1}^{n}(k_{\beta i} * a_i))$

(5.2)	$T(\alpha + \beta) = T(\sum_{i=1}^{n}(k_{\alpha i} * a_i) + \sum_{i=1}^{n}(k_{\beta i} * a_i)) = T(\sum_{i=1}^{n}((k_{\alpha i} + k_{\beta i}) * a_i)) = \sum_{i=1}^{n}((k_{\alpha i} + k_{\beta i}) * b_i) = \ldots$

(5.3)	$\ldots \sum_{i=1}^{n}(k_{\alpha i} * b_i) + \sum_{i=1}^{n}(k_{\beta i} * b_i) = T(\sum_{i=1}^{n}(k_{\alpha i} * a_i)) + T(\sum_{i=1}^{n}(k_{\beta i} * a_i)) = T(\alpha) + T(\beta)$

(6)	$(\alpha, \beta \in V) \implies (L(\alpha +_v \beta) = L(\alpha) +_w L(\beta)) \quad\blacksquare\quad \forall_{\alpha, \beta \in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))$

(7)	$((r \in \mathbb{R}) \wedge (\alpha \in V)) \implies \ldots$

(7.1)	$\exists_K(LinComb[\alpha, A, K, V, +_v, *_v]) \quad\blacksquare\quad \alpha = \sum_{i=1}^{n}(k_i * a_i)$

(7.2)	$L(r *_v \alpha) = L(r *_v \sum_{i=1}^{n}(k_i *_v a_i)) = L(\sum_{i=1}^{n}((rk_i) *_v a_i)) = \ldots$

(7.3)	$\ldots \sum_{i=1}^{n}((rk_i) *_w b_i) = r *_w \sum_{i=1}^{n}(k_i *_w b_i) = r *_w L(\sum_{i=1}^{n}(k_i *_v a_i)) = r *_w L(\alpha)$

(8)	$((r \in \mathbb{R}) \wedge (\alpha \in V)) \implies (L(r *_v \alpha) = r *_w L(\alpha)) \quad\blacksquare\quad \forall_{r \in \mathbb{R}} \forall_{\alpha \in V}(L(r *_v \alpha) = r *_w L(\alpha))$

(9)	$(\forall_{i \in \mathbb{N}_{1,n}}(T(a_i) = b_i)) \wedge (Function[T, V, W]) \wedge (\forall_{\alpha, \beta \in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))) \wedge (\forall_{r \in \mathbb{R}} \forall_{\alpha \in V}(L(r *_v \alpha) = r *_w L(\alpha))) \wedge \ldots$

(10)	$\ldots (VectorSpace[V, +_v, *_v]) \wedge (VectorSpace[W, +_w, *_w]) \quad\blacksquare\quad LinTrans[T, V, +_v, *_v, W, +_w, *_w]$

$$Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (ker_L = \{\alpha \in V | L(\alpha) = O_w\})$$

$$KerSub := \forall_{L, V, W}((Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w]) \implies (Subspace[ker_L, V, +_v, *_v]))$$

(1)	$ZeroMapsToZero \quad\blacksquare\quad L(O_v) = O_w \quad\blacksquare\quad O_v \in ker_L \quad\blacksquare\quad \emptyset \neq ker_L \quad\blacksquare\quad \emptyset \neq ker_L \subseteq V$

(2)	$(\alpha, \beta \in ker_L) \implies \ldots$

(2.1)	$(L(\alpha) = O_w) \wedge (L(\beta) = O_w)$

(2.2) $L(\alpha + \beta) = L(\alpha) + L(\beta) = O_w + O_w = O_w$ ▮ $L(\alpha + \beta) \in ker_L$

(3) $(\alpha, \beta \in ker_L) \implies (\alpha + \beta \in ker_L)$ ▮ $\forall_{\alpha, \beta \in ker_L}(\alpha + \beta \in ker_L)$

(4) $((r \in \mathbb{R}) \wedge (\alpha \in ker_L)) \implies \dots$

(4.1) $L(\alpha) = O_w$ ▮ $L(r * \alpha) = r * L(\alpha) = r * O_w = O_w$ ▮ $r * \alpha \in ker_L$

(5) $((r \in \mathbb{R}) \wedge (\alpha \in ker_L)) \implies (r * \alpha \in ker_L)$ ▮ $\forall_{r \in \mathbb{R}} \forall_{\alpha \in ker_L}(r * \alpha \in ker_L)$

(6) $(SubspaceEquiv) \wedge (\emptyset \neq ker_L \subseteq V) \wedge (\forall_{\alpha, \beta \in ker_L}(\alpha + \beta \in ker_L)) \wedge (\forall_{r \in \mathbb{R}} \forall_{\alpha \in ker_L}(r * \alpha \in ker_L))$ ▮ $Subspace[ker_L, V, +_v, *_v]$

$KerInjective := \forall_{L,V,W}((Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w]) \implies ((Injective[L, V, W]) \iff (ker_L = \{O_v\})))$

(1) $(Injective[L, V, W]) \implies \dots$

(1.1) $ZeroMapsToZero$ ▮ $L(O_v) = O_w$

(1.2) $O_v \in ker_L$ ▮ $\{O_v\} \subseteq ker_L$

(1.3) $(v \in ker_L) \implies \dots$

(1.3.1) $L(v) = O_w$

(1.3.2) $(Injective[L, V, W]) \wedge (L(O_v) = O_w)$ ▮ $O_v = v$

(1.4) $(v \in ker_L) \implies (v = O_v)$ ▮ $ker_L \subseteq \{O_v\}$

(1.5) $(\{O_v\} \subseteq ker_L) \wedge (ker_L \subseteq \{O_v\})$ ▮ $ker_L = \{O_v\}$

(2) $(Injective[L, V, W]) \implies (ker_L = \{O_v\})$

(3) $(ker_L = \{O_v\}) \implies \dots$

(3.1) $((u, v \in V) \wedge (L(u) = L(v))) \implies \dots$

(3.1.1) $O_w = L(u) - L(v) = L(u - v)$ ▮ $u - v \in ker_L$

(3.1.2) $ker_L = \{O_v\}$ ▮ $u - v = O_v$ ▮ $u = v$

(3.2) $((u, v \in V) \wedge (L(u) = L(v))) \implies (u = v)$ ▮ $\forall_{u,v \in V}((L(u) = L(v)) \implies (u = v))$ ▮ $Injective[L, V, W]$

(4) $(ker_L = \{O_v\}) \implies (Injective[L, V, W])$

(5) $(Injective[L, V, W]) \iff (ker_L = \{O_v\})$

$Rng[rng_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (rng_L = \{\beta \in W | \exists_{\alpha \in V}(\beta = L(\alpha))\})$

$RangeSub := \forall_{L,V,W}((Ran[rng_L, L, V, +_v, *_v, W, +_w, *_w]) \implies (Subspace[rng_L, W, +_w, *_w]))$

(1) $ZeroMapsToZero$ ▮ $O_w = L(O_v)$ ▮ $\exists_{\alpha \in V}(O_w = L(\alpha))$ ▮ $O_w \in rng_L$ ▮ $\emptyset \neq rng_L$ ▮ $\emptyset \neq rng_L \subseteq W$

(2) $(\alpha, \beta \in rng_L) \implies \dots$

(2.1) $(\exists_{u \in V}(\alpha = L(u))) \wedge (\exists_{v \in V}(\beta = L(v)))$

(2.2) $\alpha + \beta = L(u) + L(v) = L(u + v)$ ▮ $\exists_{w \in V}(\alpha + \beta = L(w))$ ▮ $\alpha + \beta \in rng_L$

(3) $(\alpha, \beta \in rng_L) \implies (\alpha + \beta \in rng_L)$ ▮ $\forall_{\alpha, \beta \in rng_L}(\alpha + \beta \in rng_L)$

(4) $((r \in \mathbb{R}) \wedge (\alpha \in rng_L)) \implies \dots$

(4.1) $\exists_{v \in V}(\alpha = L(v))$ ▮ $L(r * v) = r * L(v) = r * \alpha$ ▮ $\exists_{w \in V}(r * \alpha = L(w))$ ▮ $r * \alpha \in rng_L$

(5) $((r \in \mathbb{R}) \wedge (\alpha \in rng_L)) \implies (r * \alpha \in rng_L)$ ▮ $\forall_{r \in \mathbb{R}} \forall_{\alpha \in rng_L}(r * \alpha \in rng_L)$

(6) $(SubspaceEquiv) \wedge (\emptyset \neq rng_L \subseteq W) \wedge (\forall_{\alpha, \beta \in rng_L}(\alpha + \beta \in rng_L)) \wedge (\forall_{r \in \mathbb{R}} \forall_{\alpha \in rng_L}(r * \alpha \in rng_L))$ ▮ $Subspace[rng_L, W, +_w, *_w]$

$RankKer := \forall_{L,V,W}((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies (Dim[V] = Dim[ker_L] + Dim[rng_L]))$

(1) $\exists_U(Basis[U, ker_L, +_v, *_v])$ ▮ $Dim[ker_L] = |U|$

(2) $(LinIndSuperspace) \wedge (LinInd[U, ker_L, +_v, *_v])$ ▮ $LinInd[U, V, +_v, *_v]$

(3) $LinIndExpandBasis$ ▮ $\exists_B((U \subseteq B) \wedge (Basis[B, V, +_v, *_v]))$ ▮ $Dim[V] = |B|$

(4) $T := B \setminus U$ ▮ $B = U \cup T$

(5) $m := |U| \, ; \, n := |T| \, ; \, p := |B|$

(6) $L(T) := \langle L(t_i) | i \in \mathbb{N}_{1,n} \rangle \subseteq W^n$

(7) $(w \in W) \implies \dots$

(7.1) $\exists_{v \in V}(w = L(v))$

(7.2) $Basis[B, V, +_v, *_v]$ ▮ $\exists_{K \in \mathbb{R}^p}(v = \sum_{i=1}^{p}(k_i * b_i))$

(7.3) $v = \sum_{i=1}^{p}(k_i * b_i) = \sum_{i=1}^{m}(k_i * u_i) + \sum_{i=1}^{n}(\kappa_i * t_i)$

(7.4) $w = L(v) = L(\sum_{i=1}^{m}(k_i * u_i) + \sum_{i=1}^{n}(k_i * t_i)) = \sum_{i=1}^{m}(k_i * L(u_i)) + \sum_{i=1}^{n}(k_i * L(t_i)) = O_w + \sum_{i=1}^{n}(k_i * L(t_i)) = \sum_{i=1}^{n}(k_i * L(t_i))$

(7.5)	$\exists_K (LinComb[w, L(T), K, W, +_w, *_w])$

(8)	$(w \in W) \implies (\exists_K(LinComb[w, L(T), K, W, +_w, *_w]))$ ▌ $\forall_{w \in W}(\exists_K(LinComb[w, L(T), K, W, +_w, *_w]))$

(9)	$Spans[L(T), W, +_w, *_w]$

(10)	$((K \in \mathbb{R}^n) \wedge (LinComb[O_w, L(T), K, W, +_w, *_w])) \implies \ldots$

(10.1)	$O_w = \sum_{i=1}^n (k_i * L(t_i)) = L(\sum_{i=1}^n (k_i * t_i))$ ▌ $\sum_{i=1}^n (k_i * t_i) \in ker_L$

(10.2)	$Basis[U, ker_L, +_v, *_v]$ ▌ $\exists_{D \in \mathbb{R}^m}(\sum_{i=1}^n (k_i * t_i) = \sum_{i=1}^m (d_i * u_i))$

(10.3)	$(LinInd[T \cup U]) \wedge (\sum_{i=1}^n (k_i * t_i) = \sum_{i=1}^m (d_i * u_i))$ ▌ $K = D = \{O\}$ ▌ $K = \{O\}$

(11)	$((K \in \mathbb{R}^n) \wedge (LinComb[O_w, L(T), K, W, +_w, *_w])) \implies (K = \{O\})$

(12)	$\forall_{K \in \mathbb{R}^n}((LinComb[O_w, L(T), K, W, +_w, *_w]) \implies (K = \{O\}))$

(13)	$\forall_{K \in \mathbb{R}^n}((LinComb[O_w, L(T), K, W, +_w, *_w]) \iff (K = \{O\}))$ ▌ $LinInd[L(T), W, +_w, *_w]$

(14)	$Basis[L(T), W, +_w, *_w]$ ▌ $Dim[V] = |B| = |U| + |L(T)| = Dim[ker_L] + Dim[rng_L]$

$LTSurInj := \forall_{L,V,W}(((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (Surjective[L, V, W])) \implies (Injective[L, V, W]))$

(1)	TODO

$LTInjSur := \forall_{L,V,W}(((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (Injective[L, V, W])) \implies (Surjective[L, V, W]))$

(1)	TODO

$LTInjLinInd := \forall_{L,V,W}((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies ((Injective[L, V, W]) \iff (\text{LI In W are LI IN V})))$

(1)	TODO

TODO coordinatesb