A Proof of a Conjecture of Ohba

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Abstract: We prove a conjecture of Ohba that says that every graph G on at most $2\chi(G)+1$ vertices satisfies $\chi_\ell(G)=\chi(G)$. © 2014 Wiley Periodicals, Inc. J. Graph Theory 79: 86–102, 2015

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1. INTRODUCTION

List coloring is a variation on classical graph coloring. An instance of list coloring is obtained by assigning to each vertex v of a graph G a list $L(v)$ of *available colors*. An *acceptable coloring* for L is a proper coloring f of G such that $f(v)\in L(v)$ for all

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$v \in V(G)$. When an acceptable coloring for L exists, we say that G is L-colorable. The *list chromatic number* χ_ℓ is defined in analogy to the chromatic number:

$$\chi_\ell(G) = \min\{k : G \text{ is } L\text{-colorable whenever } |L(v)| \geq k \text{ for all } v \in V(G)\}.$$

List coloring was introduced independently by Vizing [28] and Erdős, Rubin and Taylor [4] and researchers have devoted a considerable amount of energy toward its study ever since (see e.g. [1, 27, 19, 29]).

A graph G has an ordinary k-coloring precisely if it has an acceptable coloring for L where $L(v) = \{1, 2, \ldots, k\}$ for all $v \in V(G)$. Therefore, the following bound is immediate:

$$\chi \leq \chi_\ell.$$

At first glance, one might expect the reverse inequality to hold as well. It would seem that having smaller intersection between color lists could only make it easier to map adjacent vertices to different colors. However, this intuition is misleading; in reality, a lack of shared colors can have the opposite effect. In fact, there are bipartite graphs with arbitrarily large list chromatic number, as was shown in the original article of Erdős et al. [4].

On the other hand, there are many special graph classes whose elements are conjectured to satisfy $\chi_\ell = \chi$; such graphs are said to be *chromatic-choosable* [22]. Probably the most well known problem in this area is the List Coloring Conjecture, which claims that every line graph is chromatic-choosable. This was independently formulated by many different researchers, including Albertson and Collins, Gupta, and Vizing (see [8]). Galvin [5] proved that line graphs of bipartite graphs are chromatic-choosable, and Kahn [11] proved that every line graph satisfies $\chi_\ell \leq (1 + o(1))\chi$. Other classes of graphs that have been conjectured to be chromatic-choosable include claw-free graphs [6], total graphs [2], and squares of graphs [17]; the last of these conjectures was recently disproved by Kim and Park [14]. In fact, it was shown independently in [13] and [15] that there does not exist an integer k such that G^k is chromatic-choosable for all G, answering a question of Zhu [30].

In [22], Ohba proved that $\chi_\ell(G + K_n) = \chi(G + K_n)$ for any graph G and sufficiently large n, where $G + H$ denotes the join of G and H. In their original article [4], Erdős et al. proved that the complete multipartite graph $K_{2,2,\ldots,2}$ is chromatic-choosable and the same was proved for $K_{3,2,2,\ldots,2}$ by Gravier and Maffray in [7]. This article concerns a conjecture of Ohba [22], which implies the last three results.

Conjecture 1.1 (Ohba [22]). *If $|V(G)| \leq 2\chi(G) + 1$, then G is chromatic-choosable.*

Infinite families of graphs satisfying $|V(G)| = 2\chi(G) + 2$ and $\chi_\ell(G) > \chi(G)$ are exhibited in [3] and so Ohba's Conjecture is best possible with respect to the bound on $|V(G)|$.

It is easy to see that the operation of adding an edge between vertices in different color classes of a $\chi(G)$-coloring does not increase χ or decrease χ_ℓ. It follows that Ohba's Conjecture is true for all graphs if and only if it is true for complete multipartite graphs. Thus, we can restate Ohba's Conjecture as follows.

Conjecture 1.2 (Ohba [22]). *If G is a complete k-partite graph on at most $2k + 1$ vertices, then $\chi_\ell(G) = k$.*

This conjecture has attracted a great deal of interest and substantial evidence has been amassed for it. This evidence mainly comes in two flavors: replacing $2k + 1$ with a smaller function of k, or restricting to graphs whose stability number is bounded above by a fixed constant.

Theorem 1.2. *Let G be a complete k-partite graph. If any of the following are true, then G is chromatic-choosable.*

(a) $|V(G)| \leq k + \sqrt{2k}$ *(Ohba [22]);*
(b) $|V(G)| \leq \frac{5}{3}k - \frac{4}{3}$ *(Reed and Sudakov [25]);*
(c) $|V(G)| \leq (2 - \varepsilon)k - n_0(\varepsilon)$ *for some function* n_0 *of* $\varepsilon \in (0, 1)$. *(Reed and Sudakov [24]).*[1]

Definition 1.3. *A maximal stable set of a complete multipartite graph is called a part.*

Theorem 1.4. *Let G be a complete k-partite graph on at most* $2k + 1$ *vertices and let* α *be the size of the largest part of G. If* $\alpha \leq 5$, *then G is chromatic-choosable (Kostochka et al. [16]; Shen et al. [26] proved the result for* $\alpha \leq 3$*).*

In this article, we prove Ohba's Conjecture. We divide the argument into three main parts. In Section 2, we show how a special type of proper nonacceptable coloring of G can be modified to yield an acceptable coloring for L. Then in Section 3, we argue that, under certain conditions, it is possible to find a coloring of this type. Finally, in Section 4, we complete the proof by showing that if Ohba's Conjecture is false, then there exists a counterexample that satisfies the conditions described in Section 3.

For the proper nonacceptable colorings we consider in Section 2, if v is colored with a color c not on $L(v)$, then we insist that no other vertex is colored with c, and that c appears on the lists of many vertices. This helps us to prove, in Section 2, that we can modify such a coloring to obtain an acceptable coloring for L, as there are at least k colors that can be used on v and many vertices on which c can be used.

In Sections 3 and 4, we combine Hall's Theorem and various counting arguments to prove that such colorings exist for a minimal counterexample to Ohba's Conjecture. In the rest of this section, we provide some properties of such a minimal counterexample, which will help us do so. For one, we show that for any minimal counterexample G, if G is not L-colorable, then the total number of colors in the union of the lists of L is at most $2k$. This upper bound on the number of colors, foreshadowed in earlier work of Kierstead [12] and Reed and Sudakov [24, 25], is crucial in that it implies the existence of colors that appear in the lists of many vertices, which our approach requires.

A variant of Ohba's Conjecture for on-line list coloring has been proposed [10, 18]. We remark that this problem remains open since our approach, with its heavy reliance on Hall's Theorem, does not apply to the on-line variant.

1.1. Properties of a Minimal Counterexample

Throughout the rest of the article we assume, to obtain a contradiction, that Ohba's Conjecture is false and we let G be a minimal counterexample in the sense that G is

[1] Reed and Sudakov [24] actually prove that there is a function $n_1(\varepsilon)$ such that if $n_1(\varepsilon) \leq |V(G)| \leq (2 - \varepsilon)k$, then G is chromatic-choosable. The original statement is equivalent to our formulation here.

a complete k-partite graph on at most $2k+1$ vertices such that $\chi_\ell(G) > k$ and Ohba's Conjecture is true for all graphs on fewer than $|V(G)|$ vertices. Throughout the rest of the article, L will be a list assignment of G such that $|L(v)| \geq k$ for all $v \in V(G)$ and G is not L-colorable. Define $C := \bigcup_{v \in V(G)} L(v)$ to be the set of all colors.

Let us illustrate some properties of a minimal counterexample. To begin, suppose that for a nonempty set $A \subseteq V(G)$ there is a proper coloring $g : A \to C$ such that

$$g(v) \in L(v) \text{ for all } v \in A.$$

Such a mapping is called an *acceptable partial coloring* for L. Define $G' := G - A$ and $L'(v) := L(v) - g(A)$ for each $v \in V(G')$. If for some $\ell \geq 0$ the following inequalities hold, then we can obtain an acceptable coloring of G' for L' by minimality of G:

$$|V(G')| \leq 2(k-\ell)+1, \tag{1}$$

$$\chi(G') \leq k-\ell, \text{ and} \tag{2}$$

$$|L'(v)| \geq k-\ell \text{ for all } v \in V(G'). \tag{3}$$

However, such a coloring would extend to an acceptable coloring for L by coloring A with g, contradicting the assumption that G is not L-colorable. Thus, no such A and g can exist.

This argument can be applied to show that every part P of G containing at least two elements must have $\bigcap_{v \in P} L(v) = \emptyset$. Otherwise, if $c \in \bigcap_{v \in P} L(v)$, then the set $A := P$ and function $g(v) := c$ for all $v \in A$ would satisfy (1), (2), and (3) for $\ell = 1$, a contradiction. We have proven:

Lemma 1.5. *If P is a part of G such that $|P| \geq 2$, then $\bigcap_{v \in P} L(v) = \emptyset$.*

Many of our results are best understood by viewing an instance of list coloring in terms of a special bipartite graph. Let B be the bipartite graph with bipartition $(V(G), C)$ where each $v \in V(G)$ is joined to the colors of $L(v)$. For $x \in V(G) \cup C$, we let $N_B(x)$ denote the neighborhood of x in B. Clearly a matching in B corresponds to an acceptable partial coloring for L where each color is assigned to at most one vertex. Recall the classical theorem of Hall [9] that characterizes the sets in bipartite graphs that can be saturated by a matching.

Theorem 1.6 (Hall's Theorem [9]). *Let B be a bipartite graph with bipartition (X, Y) and let $S \subseteq X$. Then there is a matching M in B that saturates S if and only if $\left|N_B(S')\right| \geq |S'|$ for every $S' \subseteq S$.*

In the next proposition, we use the minimality of G and Hall's Theorem to show that B has a matching of size $|C|$. Slightly different forms of this result appear in the works of Kierstead [12] and Reed and Sudakov [25].

Proposition 1.7. *There is a matching in B that saturates C.*

Proof. Suppose to the contrary that no such matching exists. Then there is a set $T \subseteq C$ such that $|N_B(T)| < |T|$ by Hall's Theorem. Suppose further that T is minimal with respect to this property. Now, for some $c \in T$ let us define $S := T - c$ and $A := N_B(S)$. By our choice of T, we observe that $\left|N_B(S')\right| \geq |S'|$ for every subset S' of S. Thus, by Hall's Theorem there is a matching M in B that saturates S. Moreover, we have

$$|A| \geq |S| = |T| - 1 \geq |N_B(T)| \geq |A|.$$

This proves that $|A| = |S|$ and, since $N_B(T)$ is nonempty, it follows that A is nonempty. In particular, M must also saturate A. Let $g : A \to S$ be the bijection that maps each vertex of A to the color that it is matched to under M. Then clearly g is an acceptable partial coloring for L. Since $A = N_B(S)$, every $v \in V(G) - A$ must have $L(v) \cap g(A) = L(v) \cap S = \emptyset$. Thus, A and g satisfy (1), (2), and (3) for $\ell = 0$, a contradiction. □

Let us rephrase the above proposition into a form which we will apply in the rest of this section.

Corollary 1.8. *There is an injective function $h : C \to V(G)$ such that $c \in L(h(c))$ for all $c \in C$.*

The following is a simple, yet useful, consequence of this result.

Corollary 1.9. $|C| < |V(G)| \leq 2k + 1$.

Proof. If $|V(G)| \leq |C|$, then the injective function $h : C \to V(G)$ from Corollary 1.8 would be a bijection. The inverse of h would be an acceptable coloring for L since each color $c \in C$ would appear on exactly one vertex of G for which c is available. This contradicts the assumption that G is not L-colorable. □

Corollary 1.10. *If there are $u, v \in V(G)$ such that $L(u) \cap L(v) = \emptyset$, then $L(u) \cup L(v) = C$ and $|C| = 2k$.*

Proof. Since the list of every vertex has size k, if two lists $L(u)$ and $L(v)$ are disjoint, then $|C| \geq |L(u) \cup L(v)| \geq 2k$. However, by Corollary 1.9 we have $|C| < |V(G)| \leq 2k + 1$ and so it must be the case that $L(u) \cup L(v) = C$ and $|C| = 2k$. □

By Corollary 1.9, the difference between $|V(G)|$ and $|C|$ is always positive. Throughout the rest of the article, it will be useful for us to keep track of this quantity.

Definition 1.11. $\gamma := |V(G)| - |C|$.

We conclude this section with two other useful consequences of Proposition 1.7.

Corollary 1.12. $|V(G)| = 2k + 1$.

Proof. If G has a part of size 2, then, by Lemma 1.5, the lists of the two vertices it contains are disjoint. Hence by Corollaries 1.9 and 1.10, $|C| = 2k$ and $|V(G)| = 2k + 1$.

Otherwise G does not contain any part of size two and so G must contain a singleton part, say $\{v\}$.[2] If $|V(G)| \leq 2k$, then we can obtain an acceptable coloring by using an arbitrary color of $L(v)$ to color v and applying minimality of G. The result follows. □

Proposition 1.13. *If $f : V(G) \to C$ is a proper coloring, then there is a proper surjective coloring $g : V(G) \to C$ such that for every $v \in V(G)$, either*

(a) *$g(v) \in L(v)$, or*
(b) *$g^{-1}(g(v)) \subseteq f^{-1}(f(v))$ and $g(v) = f(v)$.*

Proof. Let $h : C \to V(G)$ be a function as in Corollary 1.8. Given a proper coloring $g : V(G) \to C$ and a color $c \in C$, we say that g *agrees* with h at c if $g(h(c)) = c$.

[2]Otherwise, all parts of G would contain at least 3 elements and so $3k \leq |V(G)| \leq 2k + 1$, which would imply $k \leq 1$. Ohba's Conjecture is trivially true in this case.

Now, let $g : V(G) \to C$ be a proper coloring in which every vertex $v \in V(G)$ satisfies either (a) or (b) and, subject to this, the number of colors $c \in C$ at which g agrees with h is maximized. We show that g is surjective. Otherwise, let $c' \in C - g(V(G))$ be arbitrary and define a coloring $g' : V(G) \to C$ as follows:

$$g'(v) = \begin{cases} c' & \text{if } v = h(c'), \\ g(v) & \text{otherwise.} \end{cases}$$

Clearly g' is proper since g does not map any vertex to c'. Moreover, g' agrees with h at c' and at every color at which g agrees with h. Let us show that every vertex v of G satisfies either (a) or (b) for g', which will contradict our choice of g and complete the proof.

In the case that $v = h(c')$, then we have $g'(v) = c' \in L(v)$ and so (a) is satisfied for v. Now, suppose that $v \neq h(c')$. If $g'(v) \in L(v)$, then v satisfies (a) and we are done. So, we assume $g'(v) \notin L(v)$. By definition of g', we have that $g'(v) = g(v)$, which implies that $g'(v) = g(v) = f(v)$ by our choice of g. Since every vertex $w \neq h(c')$ satisfies $g'(w) = g(w) \neq c'$, we see that

$$g'^{-1}(g'(v)) = g^{-1}(g(v)) - h(c') \subseteq f^{-1}(f(v))$$

and so (b) is satisfied for v. The result follows. □

2. BAD COLORINGS WITH GOOD PROPERTIES

In this section, we show that certain types of nonacceptable colorings can be modified to produce acceptable colorings. The following definitions describe the types of colorings that we are interested in. Say that a vertex $v \in V(G)$ is a *singleton* if $\{v\}$ is a part of G. Recall that $\gamma = |V(G)| - |C| > 0$.

Definition 2.1. *A color $c \in C$ is said to be*

- *globally frequent if it appears in the lists of at least $k + 1$ vertices of G.*
- *frequent among singletons if it appears in the lists of at least γ singletons of G.*

If c is either globally frequent or frequent among singletons, then we say that c is frequent.

Definition 2.2. *A proper coloring $f : V(G) \to C$ is said to be near-acceptable for L if for every vertex $v \in V(G)$ either*

- *$f(v) \in L(v)$, or*
- *$f(v)$ is frequent and $f^{-1}(f(v)) = \{v\}$.*

Suppose that f is a proper coloring of G and let $V_f := \{f^{-1}(c) : c \in C\}$ be the set of color classes under f. Generalizing the construction of B in Section 1.1, we define B_f to be a bipartite graph with bipartition (V_f, C) where each color class $f^{-1}(c) \in V_f$ is joined to the colors of $\bigcap_{v \in f^{-1}(c)} L(v)$.[3] A matching in B_f corresponds to a partial acceptable

[3]One can think of this construction as taking the graph G and collapsing each color class of f into a single vertex. Each collapsed vertex is then assigned a list that is the intersection of the lists of all vertices in its corresponding color class.

coloring for L whose color classes are contained in V_f. We use this observation to prove the following.

Lemma 2.3. *If there is a near-acceptable coloring for L, then there is an acceptable coloring for L.*

Proof. Suppose that there is a near-acceptable coloring f for L. A matching in B_f that saturates V_f would indicate an acceptable coloring for L with the same color classes as f. Therefore, we assume that no such matching exists. By Hall's Theorem, there is a set $S \subseteq V_f$ such that $\left|N_{B_f}(S)\right| < |S|$.

Since $\left|N_{B_f}(S)\right| < |S|$ there must be a color class $f^{-1}(c^*) \in S$ such that $c^* \notin N_{B_f}(S)$. In particular, we have $c^* \notin N_{B_f}(f^{-1}(c^*))$. It follows that there is a vertex v such that $f(v) = c^*$ and $c^* \notin L(v)$. Since f is near-acceptable for L, we must have that c^* is frequent and $f^{-1}(c^*) = \{v\}$.

Case 1. *c^* is globally frequent.*

Since $f^{-1}(c^*) = \{v\} \in S$, we have that $N_{B_f}(S) \supseteq N_{B_f}(f^{-1}(c^*)) = L(v)$ and so $\left|N_{B_f}(S)\right| \geq k$. This implies that

$$|S| > \left|N_{B_f}(S)\right| \geq k. \tag{4}$$

However, since $c^* \notin N_{B_f}(S)$, every color class of S must contain a vertex whose list does not contain c^*. Since c^* is globally frequent, there are at most

$$|V(G)| - (k+1) \leq k$$

such vertices. Thus, $|S| \leq k$, contradicting (4) and so the result holds in this case.

Case 2. *c^* is frequent among singletons.*

In order to complete the proof in this case, we impose some additional conditions on f and S.

- By Proposition 1.13, we may assume that f maps surjectively to C.
- We can assume that S is chosen to be a set that maximizes $|S| - \left|N_{B_f}(S)\right|$ over all subsets of V_f.

By our choice of S, for any $T \subseteq V_f - S$ we must have $\left|N_{B_f}(T) - N_{B_f}(S)\right| \geq |T|$. It follows, by Hall's Theorem, that

$$\text{there is a matching } M \text{ in } B_f - N_{B_f}(S) \text{ saturating } V_f - S. \tag{5}$$

If x is a singleton such that $c^* \in L(x)$, then $\{x\}$ is a color class of f that cannot be in S since f is proper and $c^* \notin N_{B_f}(S)$. Since c^* is frequent among singletons, we get that

$$V_f - S \textit{ contains at least } \gamma \textit{ singleton parts of } G. \tag{6}$$

The proof follows immediately from the following claim, which will be used again later in the article.

Claim 2.4. *Suppose that $f : V(G) \to C$ is surjective. If* (5) *and* (6) *hold, then there is an acceptable coloring for L.*

Proof. We let ℓ be the number of color classes under f with more than one element. Define A to be the union of the color classes of f which either are not in S or contain more than one element and define $G' := G - A$. We will find a partial acceptable coloring g of A satisfying (1), (2), and (3) for this definition of ℓ.

Since A contains all color classes of f with more than one element, we have that A contains at least 2ℓ vertices and so (1) holds for ℓ. By (6), we get that $\chi(G - A) \le k - \gamma$. The fact that f is surjective implies $\ell \le \gamma$ and so (2) holds for ℓ. In order to show that (3) holds, we will insist that our partial acceptable coloring satisfies the following:

$$g(A) \text{ contains at most } \ell \text{ colors of } N_{B_f}(S). \tag{$*$}$$

For every vertex $w \in V(G) - A$, we see that $\{w\}$ is a color class of f contained in S; hence $L(w) \subseteq N_B(S)$. Thus, if $(*)$ holds, then (3) holds for ℓ.

Therefore, to obtain a contradiction, we need only show that there is a partial acceptable coloring of A satisfying $(*)$. To begin we note that the matching M in (5) defines a partial acceptable coloring h for the set A_1 of vertices in the color classes of $V_f - S$ using only colors in $C - N_{B_f}(S)$. We also note that f is a partial acceptable coloring when restricted to the vertices in the color classes in S with more than one element, since if $f(v) \notin L(v)$ then $f^{-1}(f(v)) = \{v\}$. We define g so that it agrees with h on A_1 and agrees with f on the rest of A. It is a partial acceptable coloring because, by definition, $g(A_1)$ is disjoint from $N_{B_f}(S)$ while $g(A - A_1)$ is contained in $N_{B_f}(S)$. Furthermore, since $A - A_1$ consists of the union of at most ℓ color classes of f, we see that $(*)$ holds for g and we are done. □

3. CONSTRUCTING NEAR-ACCEPTABLE COLORINGS

In the previous section, we saw that finding an acceptable coloring for L is equivalent to finding a near-acceptable coloring for L. However, in practice it can be much easier to construct a near-acceptable coloring than an acceptable coloring. In constructing a near-acceptable coloring, we need not worry about whether a frequent color c is available for a vertex v, provided that v is the only vertex to be colored with c. In this section, we exploit this flexibility to prove the following result.

Lemma 3.1. *If C contains at least k frequent colors, then there is a near-acceptable coloring for L.*

Proof. Let F be a set of k frequent colors and assume, to the contrary, that there does not exist a near-acceptable coloring for L. Our goal is to construct a near-acceptable coloring for L by applying a three phase greedy procedure. In the first phase, choose a subset $V_1 \subseteq V(G)$ and an acceptable partial coloring $f_1 : V_1 \to C - F$ such that V_1 contains as many vertices as possible and, subject to this, V_1 contains vertices from as many parts as possible. Before moving on, we prove the following claim.

Claim 3.2. *Every part of size two contains a vertex of V_1.*

Proof. Suppose that $P = \{u, v\}$ is a part of size 2 such that $P \cap V_1 = \emptyset$. Then $L(u) \cap L(v) = \emptyset$ by Lemma 1.5, and so by Corollary 1.10 we see that $L(u) \cup L(v) = C$ and $|C| = 2k$. In particular, the image of f_1 must contain every color $c \in C - F$ for, if not, we could use c to color one of u or v, increasing the size of V_1. Since $|C| = 2k$, we have $|C - F| = k$ which implies that $|V_1| \ge k$.

If $|V_1| \ge k + 1$, then $|V(G) - V_1| \le |V(G)| - (k + 1) = k = |F|$, and we can obtain a near-acceptable coloring for L by mapping the vertices of $V(G) - V_1$ injectively to F. So, we must have $|V_1| = k$ and that every color of $C - F$ is used by f_1 on a unique vertex of V_1. Since neither vertex of P is in V_1 and G has precisely k parts, there must be a part $Q \neq P$ containing at least two vertices of V_1, say x and y. However, since $L(u) \cup L(v) = C$, we can uncolor x and use its color to color one of u or v, which maintains the number of colored vertices and increases the number of parts with a colored vertex. This contradicts our choice of V_1 and completes the proof of the claim. □

For each part P let $R_P := P - V_1$, the set of vertices that are not colored by f_1. Label the parts of G as $P_1, \ldots, P_k$ so that $\left|R_{P_1}\right| \geq \ldots \geq \left|R_{P_k}\right|$. The second phase of our coloring procedure is described as follows. For each part P_i, in turn, we try to color R_{P_i} with a frequent color that has not yet been used and is available for every vertex of R_{P_i}. We terminate this phase when we reach an index $i + 1$ for which this is not possible. Define

$$V_2 := \bigcup_{j=1}^{i} R_{P_j}, \text{ and}$$

$$V_3 := \bigcup_{j=i+1}^{k} R_{P_j}$$

That is, V_2 is the set of vertices colored by the second phase and V_3 is the set of vertices that must be colored in the third. If $i = k$, then we have obtained an acceptable coloring for L after the first two phases and we are done. So, we assume that $i < k$ and that there is no frequent color that has not yet been used and is available for every vertex of $R_{P_{i+1}}$.

Let U denote the set of colors of F that have not been used in the second phase. We observe that $|U| = k - i$. If $|V_3| \leq k - i$, then in the third phase we simply map V_3 injectively into U, thereby obtaining a near-acceptable coloring for L. So, we can assume that $|V_3| \geq k - i + 1$. In particular, this implies $\left|R_{P_{i+1}}\right| \geq 2$ by our choice of ordering. Thus, again by our choice of ordering, we get that $|V_2| \geq \left|R_{P_{i+1}}\right| i \geq 2i$. Since $|V(G)| = 2k + 1$ and $|V_3| \geq k - i + 1$, we get

$$|V_1 \cup V_2| = |V(G)| - |V_3| \leq (2k + 1) - (k - i + 1) = k + i.$$

It follows that $|V_1| \leq k + i - |V_2| \leq k + i - 2i = k - i$.

Let us show that $|V_1|$ is exactly $k - i$. To do so, we use the fact that every color in U is absent from $L(v)$ for at least one $v \in R_{P_{i+1}}$. Since there are k colors available for each vertex of $R_{P_{i+1}}$ and exactly k colors in F, these absences imply that the colors of $C - F$ must appear at least $|U| = k - i$ times in the lists of vertices of $R_{P_{i+1}}$. Now if a color $c \in C - F$ is available for $j > 0$ vertices of $R_{P_{i+1}}$, then:

(i) c was not used to color any vertex of P_{i+1}. Otherwise, we could use c to color those j vertices of $R_{P_{i+1}}$ for which it is available. This would contradict our choice of V_1.

(ii) at least j vertices are colored with c in the first phase. Otherwise, we could uncolor the vertices that were colored with c and use c to color j vertices of $R_{P_{i+1}}$ instead, again contradicting our choice of V_1.

Thus, since the colors of $C - F$ appear at least $k - i$ times in the lists of vertices in $R_{P_{i+1}}$, by (i) and (ii) we have that at least $k - i$ vertices of $V(G) - P_{i+1}$ were colored in the first phase; that is, $|V_1| \geq k - i$. Since we have already proven that $|V_1| \leq k - i$, this implies that

$$|V_1| = k - i, \text{ and} \tag{7}$$

$$V_1 \cap P_{i+1} = \emptyset. \tag{8}$$

Recall that, by Claim 3.2, every part of size two intersects V_1. So, by (7), we must have $\left|R_{P_{i+1}}\right| = |P_{i+1}| \geq 3$. By our choice of ordering, this implies that $|V_2| \geq \left|R_{P_{i+1}}\right| i \geq 3i$, and so $|V_1| \leq k + i - 3i = k - 2i$. Thus, by (6), we must have $k - 2i \geq k - i$ which implies $i = 0$.

Therefore, we have $|V_1| = k$ and $|V_2| = 0$, which implies that $U = F$, $|V_3| = k + 1$, and $R_{P_1} = P_1$ by (7). At this point, our goal is to show that there exists a color $c \in F$ that is available for two vertices $u, v \in P_1$. If such a color exists, then, in the third phase of our procedure, we simply color u and v with c and map the vertices of $V_3 - \{u, v\}$ to $F - c$ bijectively to obtain a near-acceptable coloring for L.

Recall that the number of times the colors of $C - F$ appear in lists of vertices in P_1 is at most the cardinality of V_1, which is exactly k. Since $|P_1| \geq 3$ and each list has size at least k, the colors of F must appear at least $2k$ times in the lists of vertices in P_1. Since $|F| = k$, there is a color in F (in fact, many) that is available for at least two vertices in P_1. This completes the proof. □

4. ADDING COLORS TO THE LISTS

From now on, we impose an additional requirement on our list assignment L for which there is no acceptable coloring. We insist that it is maximal in the sense that increasing the size of any list makes it possible to find an acceptable coloring. That is, for any $v \in V(G)$ and $c \in C - L(v)$, if we define $L^*(v) = L(v) \cup \{c\}$ and $L^*(u) = L(u)$ for all $u \neq v$, then there is an acceptable coloring for L^*. Given this property, it is straightforward to prove that every frequent color is available for every singleton.

Lemma 4.1. *If $c \in C$ is frequent, then $c \in L(v)$ for every singleton v of G.*

Proof. Otherwise, add c to the list of v. Since L is maximal, there is an acceptable coloring for this modified list assignment. Since G is not L-colorable, this coloring must use c to color v and, since v is a singleton, v is the only vertex colored with c. Therefore, this coloring is a near-acceptable coloring for L and so by Lemma 2.3 it follows that G is L-colorable, a contradiction. □

Recall by Lemmas 2.3 and 3.1 that there are fewer than k frequent colors. We show now that this implies that there are at least γ singletons.

Definition 4.2. *Let b denote the number of nonsingleton parts of G.*

Proposition 4.3. *G contains at least γ singletons.*

Proof. Suppose to the contrary that the number of singletons, namely $k - b$, is less than γ. Let F' denote the set of all globally frequent colors. Then each color of

$C - F'$ is available for at most k vertices, and each color of F' is available for at most $|V(G)| - b \leq 2k + 1 - b$ vertices by Lemma 1.5. Thus,

$$k|V(G)| \leq \sum_{v \in V(G)} |L(v)| = \sum_{c \in C} |N_B(c)|$$
$$\leq k|C - F'| + (2k + 1 - b)|F'| = k|C| + (k + 1 - b)|F'|.$$

In other words,

$$|F'| \geq \frac{k(|V(G)| - |C|)}{k + 1 - b} = \frac{k\gamma}{k + 1 - b}. \tag{9}$$

Now, since we are assuming that $k + 1 - b \leq \gamma$, we obtain $|F'| \geq k$ by the above inequality. However, this contradicts the fact that there are fewer than k frequent colors. Thus, G must contain at least γ singletons. □

Corollary 4.4. $\gamma + b \leq k$.

Proof. By Proposition 4.3, we have $k - b \geq \gamma$, which implies the result. □

Corollary 4.5. *A color $c \in C$ is frequent if and only if it is available for every singleton.*

Proof. By Proposition 4.3, any color that is available for every singleton is automatically frequent among singletons, and therefore frequent. The reverse implication follows from Lemma 4.1. □

By Lemma 3.1 and Corollary 4.5, to obtain the desired contradiction, we need only show that there are k colors that are available for every singleton. In fact, as the next result shows, it is enough to prove that there are at least b such colors.

Lemma 4.6. *There are fewer than b frequent colors.*

Proof. Suppose that there are at least b frequent colors and let $A_b = \{c_1, c_2, \ldots, c_b\}$ be a set of b such colors. By Lemma 4.1, all such colors are available for every singleton. Label the singletons of G as $v_{b+1}, v_{b+2}, \ldots, v_k$. For each $i \in \{b + 1, b + 2, \ldots, k\}$, in turn, choose a color $c_i \in L(v_i) - A_{i-1}$ greedily and define $A_i := A_{i-1} \cup \{c_i\}$. Let L' be a list assignment of G defined by

$$L'(v) := \begin{cases} A_k & \text{if } v \text{ is a singleton,} \\ L(v) & \text{otherwise.} \end{cases}$$

Clearly L' assigns each singleton the same list of k colors. Hence, there are at least k frequent colors under L' and so by Lemmas 2.3 and 3.1 there is an acceptable coloring f' for L'. We use this to construct an acceptable coloring f for L, contradicting the fact that G is not L-colorable.

We let $\mathcal{S}$ denote the set of all singletons and for each $v \in V(G) - \mathcal{S}$ we set $f(v) = f'(v)$. We note that $f'(\mathcal{S})$ is a set of exactly $k - b$ colors of A_k disjoint from $f'(V(G) - \mathcal{S}) = f(V(G) - \mathcal{S})$. We let

$$\mathcal{S}' := \{v_i \in \mathcal{S} : c_i \in f'(\mathcal{S})\}.$$

We note that $|\mathcal{S}'| = k - b - |f'(\mathcal{S}) \cap A_b|$ and hence $|\mathcal{S} - \mathcal{S}'| = |f'(\mathcal{S}) \cap A_b|$. For each $v_i \in \mathcal{S}'$, we set $f(v_i) = c_i$. We arbitrarily choose a bijection $\pi : \mathcal{S} - \mathcal{S}' \to f'(S) \cap A_b$ and set $f(v) = \pi(v)$ for every $v \in \mathcal{S} - \mathcal{S}'$. Since each color in A_b is available for every singleton, we are done. □

Before moving on to the final counting arguments, we establish a few simple consequences of Proposition 4.3.

Corollary 4.7. *G contains no part of size 2.*

Proof. If G contains a part $P = \{u, v\}$, then $L(u) \cap L(v) = \emptyset$ by Lemma 1.5. By Corollary 1.10, this implies that $|C| = 2k$ and so $\gamma = 1$. Thus, by Proposition 4.3, G contains a singleton v and every color of $L(v)$ is frequent among singletons since $\gamma = 1$. This contradicts the fact that there are fewer than k frequent colors, and proves that G contains no part of size two. □

Corollary 4.8. $b \leq \frac{k+1}{2}$.

Proof. Since G contains no parts of size 2, we see that G consists of $k - b$ singletons and b parts of size at least three. Therefore

$$(k - b) + 3b \leq |V(G)| \leq 2k + 1$$

which implies that $b \leq \frac{k+1}{2}$. □

Recall from the proof of Proposition 4.3 that the set F' of globally frequent colors satisfies

$$\frac{k\gamma}{k + 1 - b} \leq |F'| \leq b - 1 < \frac{k}{2}$$

by (9), Lemma 4.6 and Corollary 4.8. This implies that

$$2\gamma < k + 1 - b. \tag{10}$$

We will apply this inequality later on in the proof.

4.1. A Bit of Counting

The final step is to apply a counting argument to show that there are at least b colors that are frequent among singletons, which would contradict Lemma 4.6 and complete the proof of Ohba's Conjecture. In order to do so, we will find a fairly large set X of singletons such that $N_B(X)$ is fairly small. This implies that the average number of singletons for which a color in $N_B(X)$ is available is large. Using this, we will show that there are at least b colors in $N_B(X)$ which are available for γ singletons, which gives us the desired contradiction. We begin with the following proposition.

Proposition 4.9. *Let c^* be a color that is not available for every singleton. Then there is a set X (depending on c^*) of singletons such that*

(a) $|X| \geq k + 1 - b - \gamma$, *and*
(b) $\left|\bigcup_{v \in X} L(v)\right| \leq 2k - |N_B(c^*)|$.

Proof. To prove this result, we modify a coloring that is almost acceptable as in the proof of Lemma 2.3. To begin, we let x be a singleton such that $c^* \notin L(x)$ and define a list assignment L^* by

$$L^*(v) = \begin{cases} L(x) \cup \{c^*\} & \text{if } v = x, \\ L(v) & \text{otherwise.} \end{cases}$$

Then, since L is maximal there is an acceptable coloring f for L^*. Clearly $f^{-1}(c^*) = \{x\}$ and for every $v \in V(G) - x$ we have $f(v) \in L(v)$. By Proposition 1.13, we can also assume that f is surjective. If there is a matching in B_f that saturates V_f, then G is L-colorable. Thus, we obtain a set $S \subseteq V_f$ such that $\left|N_{B_f}(S)\right| < |S|$. We can choose S to maximize $|S| - \left|N_{B_f}(S)\right|$. By this choice, we have that (5) is satisfied.

Since $\left|N_{B_f}(S)\right| < |S|$ and c is adjacent to $f^{-1}(c)$ in B_f for every $c \neq c^*$ it must be the case that $f^{-1}(c^*) \in S$ and $c^* \notin N_{B_f}(S)$. Since $c^* \notin N_{B_f}(S)$, every color class of S must contain a vertex whose list does not contain c^*. Thus, we have

$$\left|N_{B_f}(S)\right| < |S| \leq |V(G)| - \left|N_B(c^*)\right| \leq 2k + 1 - \left|N_B(c^*)\right|.$$

Now, define X to be the set of all singletons of G whose color classes under f belong to S. Then $\bigcup_{v \in X} L(v) \subseteq N_{B_f}(S)$, and so (b) holds.

Finally, if X contains fewer than $k + 1 - b - \gamma$ singletons, then there are at least γ singletons y such that $\{y\} \in V_f - S$. That is, (6) is satisfied. However, if this were the case, then we would obtain an acceptable coloring for L by Claim 2.4. Thus, (a) must hold. □

We let c^* be a color that is not frequent (equivalently, not available for every singleton), and subject to this maximizes $|N_B(c^*)|$. Let X be a set of singletons as in Proposition 4.9.

Let Z be a set of $b - 1$ colors that appear most often in the lists of vertices of X and define $Y := N_B(X) - Z$. That is, $|Z| = b - 1$ and if $c_1 \in Z$ and $c_2 \in Y$, then $|N_B(c_1) \cap X| \geq |N_B(c_2) \cap X|$. We can further assume that every frequent color appears in Z, since by Lemma 4.1 these colors appear in the list of every vertex of X and by Lemma 4.6 there are at most $b - 1$ of them. Finally, let $c' \in Y$ such that $\left|N_B(c') \cap X\right|$ is maximized. Our goal is to prove that $\left|N_B(c') \cap X\right| \geq \gamma$, contradicting the fact that Z contains every frequent color. The next proposition provides one way of doing this.

Definition 4.10. $\beta := k - |N_B(c^*)|$.

Remark 4.11. Since c^* is not frequent, we have $\beta \geq 0$.

Proposition 4.12. *If $\beta \leq 2(k + 1 - b - 2\gamma)$, then $\left|N_B(c') \cap X\right| \geq \gamma$.*

Proof. Since $|Z| = b - 1$, every vertex $v \in X$ must satisfy $|L(v) \cap Y| \geq k - (b - 1) = k + 1 - b$. Thus, by our choice of c',

$$|Y| \left|N_B(c') \cap X\right| \geq \sum_{c \in Y} |N_B(c) \cap X| = \sum_{v \in X} |L(v) \cap Y| \geq |X|(k + 1 - b).$$

Combining this inequality with the two bounds of Proposition 4.9 gives

$$\begin{aligned} \left|N_B(c') \cap X\right| &\geq \frac{|X|(k+1-b)}{|Y|} = \frac{|X|(k+1-b)}{|N_B(X)| + 1 - b} \\ &\geq \frac{(k+1-b-\gamma)(k+1-b)}{2k - |N_B(c^*)| + 1 - b} = \frac{(k+1-b-\gamma)(k+1-b)}{\beta + k + 1 - b} \end{aligned} \tag{11}$$

However, if $\beta \leq 2(k + 1 - b - 2\gamma)$ then, by (10) and Remark 4.11, we have $0 \leq \beta\gamma < (k + 1 - b - 2\gamma)(k + 1 - b)$. So, multiplying top and bottom by $\gamma > 0$, the right hand side of (11) is at least

$$\frac{\gamma(k+1-b-\gamma)(k+1-b)}{(k+1-b-2\gamma)(k+1-b) + \gamma(k+1-b)} = \gamma.$$

The result follows. □

Together with Proposition 4.12, the following proposition completes the proof of Ohba's Conjecture (with a factor of four to spare).

Proposition 4.13. $\beta < \frac{1}{2}(k + 1 - b - 2\gamma)$.

Proof. Let F denote the set of all frequent colors. Recall that c^* was chosen to maximize $|N_B(c^*)|$ over all colors that are not frequent. Thus, each color $c \notin F$ must have $|N_B(c)| \leq |N_B(c^*)|$. Moreover, by Lemma 4.6 there are at most $b - 1$ frequent colors and by Lemma 1.5 every color $c \in C$ satisfies $|N_B(c)| \leq |V(G)| - b$. We have $|V(G)| = 2k + 1$ by Corollary 1.12 and so,

$$(2k+1)k \leq \sum_{v \in V(G)} |L(v)| = \sum_{c \in C} |N_B(c)| \leq |C - F| \left|N_B(c^*)\right| + |F|(2k + 1 - b)$$

$$\leq (2k + 1 - \gamma - b + 1) \left|N_B(c^*)\right| + (b - 1)(2k + 1 - b)$$

since $|C| = |V(G)| - \gamma$ and $|F| \leq b - 1$ by Lemma 4.6. Substituting $|N_B(c^*)| = k - \beta$ and rearranging, we obtain

$$\begin{aligned}(2k + 2 - \gamma - b)\beta &\leq (-\gamma - b + 1)k + (b - 1)(2k + 1 - b)\\ &= (b - 1)(k + 1 - b) - k\gamma\end{aligned}$$

By Corollary 4.4 we have $\gamma + b < 2k + 2$. Thus, we can divide both sides of the above inequality by $(2k + 2 - \gamma - b)$ to obtain

$$\beta \leq \frac{(b-1)(k+1-b) - k\gamma}{2k + 2 - \gamma - b} < \frac{\frac{1}{2}k(k+1-b-2\gamma)}{2k+2-\gamma-b}$$

since $b - 1 < \frac{k}{2}$ by Corollary 4.8. Now, by Corollary 4.4, we obtain

$$\frac{\frac{1}{2}k(k+1-b-2\gamma)}{2k+2-\gamma-b} \leq \frac{\frac{1}{2}k(k+1-b-2\gamma)}{k+2} < \frac{1}{2}(k+1-b-2\gamma)$$

which implies the result. □

This completes the proof of Ohba's Conjecture.

5. Conclusion

We conclude the article by mentioning some subsequent work and open problems for future study. In general, one may ask the following: *given a function $f(k) > 2k + 1$, what is the best bound on $\chi_\ell(G)$for k-chromatic graphs on at most $f(k)$ vertices?* By applying the result of the current article, Noel et al. [21] have solved this problem for every function f such that $f(k) \leq 3k$ and $f(k) - k$ is even for all k. Their main result is the following strengthening of Ohba's Conjecture, which holds for all graphs G:

$$\chi_\ell(G) \leq \max\left\{\chi(G), \left\lceil \frac{|V(G)| + \chi(G) - 1}{3} \right\rceil\right\}. \tag{12}$$

Ohba [23] proved that equality holds in (12) whenever G is a complete multipartite graph in which every part has size 1 or 3, generalizing a result of Kierstead [12] for $K_{3,3,\ldots,3}$. Putting this together, we see that the list chromatic number of a graph on at most $3\chi(G)$ vertices is bounded above by the list chromatic number of the complete $\chi(G)$-partite graph in which every part has size 3. Noel [20] has conjectured that a similar result holds for graphs on at most $m\chi(G)$ vertices for every fixed integer m.

Conjecture 5.1 (Noel [20]). *For $m, k \geq 2$, let G be a k-chromatic graph on at most mk vertices. Then $\chi_\ell(G)$ is bounded above by the list chromatic number of the complete k-partite graph in which every part has size m.*

Noel [20] also proposed the following, more general, conjecture.

Conjecture 5.2 (Noel [20]). *For $n \geq k \geq 2$, there exists a graph $G_{n,k}$ such that*

- *$G_{n,k}$ is a complete k-partite graph on n vertices,*
- *$\alpha\left(G_{n,k}\right) = \left\lceil \frac{n}{k} \right\rceil$, and*
- *if G is a k-chromatic graph on n vertices, then $\chi_\ell(G) \leq \chi_\ell\left(G_{n,k}\right)$.*

Another, rather ambitious, problem could be to characterize all complete k-partite graphs with $\chi_\ell(G) = k$. Short of this, it would be interesting to characterize all such graphs on at most $f(k)$ vertices for some function f that is larger than $2k + 1$. In [20], it is conjectured that if G is a complete k-partite graph on $2k + 2$ vertices, then $\chi_\ell(G) > k$ if and only if k is even and either every part of G has size 1 or 3, or every part of G has size 2 or 4. In [3] it was proved that such graphs are not chromatic-choosable.

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