

0.1 Problem Set 1

1.) Prove Proposition 6: For any $a, b \in \mathbb{R}$, $\left(\begin{array}{l} a \vee b = \frac{1}{2}(a + b + |a - b|) \\ a \wedge b = \frac{1}{2}(a + b - |a - b|) \end{array} \right)$.

Let $a, b \in \mathbb{R}$. Suppose $a \geq b$. Clearly, $a \vee b = a$ and $a \wedge b = b$. Furthermore, $|a - b| = a - b$ since $a - b \geq 0$. Thus,

$$a \vee b = a = \frac{1}{2}(a + b + (a - b)) = \frac{1}{2}(a + b + |a - b|) \quad (1)$$

$$a \wedge b = b = \frac{1}{2}(a + b - (a - b)) = \frac{1}{2}(a + b - |a - b|). \quad (2)$$

Suppose $a < b$. Clearly, $a \vee b = b$ and $a \wedge b = a$. Furthermore, $|a - b| = -(a - b)$ since $a - b < 0$. Thus,

$$a \vee b = b = \frac{1}{2}(a + b + (-(a - b))) = \frac{1}{2}(a + b + |a - b|) \quad (3)$$

$$a \wedge b = a = \frac{1}{2}(a + b - (-(a - b))) = \frac{1}{2}(a + b - |a - b|). \quad (4)$$

In either case, $a \vee b = \frac{1}{2}(a + b + |a - b|)$ and $a \wedge b = \frac{1}{2}(a + b - |a - b|)$ holds. ■

2.) Prove Proposition 7: For any $a, b, r \in \mathbb{R}$, $\left(\begin{array}{l} a \vee b = b \vee a \\ a \wedge b = b \wedge a \\ (a \wedge b \leq r \leq a \vee b) \implies ((|r - a| \leq |a - b|) \wedge (r - b \leq |a - b|)) \end{array} \right)$.

Let $a, b, r \in \mathbb{R}$. The first two statements immediately follow by applying the commutativity of real numbers and $|a - b| = |-(a - b)| = |b - a|$ to Proposition 6.

Suppose $a \wedge b \leq r \leq a \vee b$. Without loss of generality, let $a \geq b$. Thus,

$$b \leq r \leq a \quad (5)$$

$$r - a \leq 0 \quad (6)$$

$$b - r \leq 0 \quad (7)$$

$$b - a \leq 0 \quad (8)$$

From (5), $r - a \geq b - a$. This along with (6) and (8) implies $|r - a| = -(r - a) \leq -(b - a) = |b - a|$.

From (5), $r - b \leq a - b$. This along with (7) and (8) implies $|r - b| = r - b \leq a - b = |a - b|$. ■