0.1 Problem Set 2

Suppose $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$ and $\lim_{n\to\infty}(c_n) \neq 0$ and for any $n \in \mathbb{N}$, $c_n \neq 0$.
1.) Prove $\lim_{n\to\infty}(c) = c$: Observe that $(c)_{n\in\mathbb{N}} = c(1)_{n\in\mathbb{N}}$. By Theorem 4, $\lim_{n\to\infty}(c) = \lim_{n\to\infty}(c * 1) = c \lim_{n\to\infty}(1)$. Since $\lim_{n\to\infty}(1) = 1$, $\lim_{n\to\infty}(c) = c * 1 = c$.
2.) Prove $\lim_{n\to\infty}(|a_n|) = |\lim_{n\to\infty}(a_n)|$: Let $\lim_{n\to\infty}(a_n) = a$ and let $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that for any $n \geq N$, $|a_n - a| < \epsilon$. By the reverse triangle inequality, $||a_n| - |a|| \leq |a_n - a|$. Furthermore $||a_n| - |a|| < \epsilon$, $n \geq N$. Thus $\lim_{n\to\infty}(|a_n|) = |a| = |\lim_{n\to\infty}(a_n)|$.
3.) ...

1.) Prove Proposition 6: For any $a, b \in \mathbb{R}$, $\begin{pmatrix} a \vee b = \frac{1}{2}(a+b+|a-b|) \\ a \wedge b = \frac{1}{2}(a+b-|a-b|) \end{pmatrix}$.

Let $a, b \in \mathbb{R}$. Suppose $a \geq b$. Clearly, $a \vee b = a$ and $a \wedge b = b$. Furthermore, $|a-b| = a-b$ since $a - b \geq 0$. Thus,

$$a \vee b = a = \frac{1}{2}(a+b+(a-b)) = \frac{1}{2}(a+b+|a-b|) \tag{1}$$

$$a \wedge b = b = \frac{1}{2}(a+b-(a-b)) = \frac{1}{2}(a+b-|a-b|). \tag{2}$$

Suppose $a < b$. Clearly, $a \vee b = b$ and $a \wedge b = a$. Furthermore, $|a-b| = -(a-b)$ since $a - b < 0$. Thus,

$$a \vee b = b = \frac{1}{2}(a+b+(-(a-b))) = \frac{1}{2}(a+b+|a-b|) \tag{3}$$

$$a \wedge b = a = \frac{1}{2}(a+b-(-(a-b))) = \frac{1}{2}(a+b-|a-b|). \tag{4}$$

In either case, $a \vee b = \frac{1}{2}(a+b+|a-b|)$ and $a \wedge b = \frac{1}{2}(a+b-|a-b|)$ holds. ■

2.) Prove Proposition 7: For any $a, b, r \in \mathbb{R}$, $\begin{pmatrix} a \vee b = b \vee a \\ a \wedge b = b \wedge a \\ (a \wedge b \leq r \leq a \vee b) \implies ((|r-a| \leq |a-b|) \wedge (r-b \leq |a-b|)) \end{pmatrix}$.

Let $a, b, r \in \mathbb{R}$. The first two statements immediately follow by applying the commutativity of real numbers and $|a-b| = |-(a-b)| = |b-a|$ to Proposition 6.

Suppose $a \wedge b \leq r \leq a \vee b$. Without loss of generality, let $a \geq b$. Thus,

$$b \leq r \leq a \tag{5}$$

$$r - a \leq 0 \tag{6}$$

$$b - r \leq 0 \tag{7}$$

$$b - a \leq 0 \tag{8}$$

From (5), $r - a \geq b - a$. This along with (6) and (8) implies $|r-a| = -(r-a) \leq -(b-a) = |b-a|$.
From (5), $r - b \leq a - b$. This along with (7) and (8) implies $|r-b| = r-b \leq a-b = |a-b|$. ■