Mathematical Logic Notes

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================== META Relation / Rel(R, S): $R \subseteq S$ and for any r_1, r_2, if $r_1 = r_2$, then $r_1 \in R$ iff $r_2 \in R$ Function / Func(f, X, Y): For any $x_1, x_2 \in X$, if $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \in f$ and $x_1 = x_2$, then $y_1 = y_2$ If $Function/Func(f, X, Y)$, then $Rel(f, X \times Y)$ Injection / Inj(f, X, Y): $Func(f, X, Y)$ and for any $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$ Surjection / Sur(f, X, Y): $Func(f, X, Y)$ and for any $y \in Y$, there exists $x \in X$, $y = f(x)$ Bijection / Bij(f, X, Y): $Inj(f, X, Y)$ and $Sur(f, X, Y)$

COUNTABLE SETS

IMPLICATION EQUIVALENCES

Implication definition - If A, then B – Not A or B

Implication over conjunction - If A and B, then C – If A, then if B, then C

Contraposition - If A, then B – If not B, then not A

Metaproof by contradictory - If not C, then CONTR – If TAUT, then C – C ==================

Chapter 1

Structures and Languages

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1.1 Languages

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1.1.1 (Definition) First-order Alphabet

- The first-order alphabet ($\mathcal{L}$) is a tuple of collections of symbols that consists:
– Connectives: $\vee, \neg$
– Quantifier: $\forall$
– Variables: $Var = \left\{ \underset{i\in\mathbb{N}}{\boxed{v_i}} \right\}$
– Equality: $\equiv$
– Constants $Const$
– Functions: $Func = \left\{ \underset{i\in\mathbb{N}}{\boxed{f : Arity(f) = i}} \right\}$
– Relations: $Rel = \left\{ \underset{i\in\mathbb{N}}{\boxed{P : Arity(P) = i}} \right\}$
– FOS: $\vee, \neg, \forall, \equiv$

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1.2 Terms and Formulas

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1.2.1 (Definition) Term

- The term t of the language $\mathcal{L}$ ($t \in Term(\mathcal{L})$) iff t is a non-empty finite string and it satisfies exactly one of the following:
– $t :\equiv v$ and $v \in Var$
– $t :\equiv c$ and $c \in Const$
– $t :\equiv f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}$ and $\left\{ \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}} \right\} \subseteq Term(\mathcal{L})^*$ and $f \in Func$
- Terms encode the objects or nouns in the language

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1.2.2 (Definition) Formula

- The formula ϕ of the language $\mathcal{L}$ ($\phi \in Form(\mathcal{L})$) iff ϕ is a non-empty finite string and it satisfies exactly one of the following:
– $\phi :\equiv \equiv rs$ and $\{r, s\} \subseteq Term(\mathcal{L})$
– $\phi :\equiv R \overset{Arity(R)}{\underset{i=1}{\boxed{t_i}}}$ and $\left\{ \overset{Arity(R)}{\underset{i=1}{\boxed{t_i}}} \right\} \subseteq Term(\mathcal{L})$ and $R \in Pred$

– $\phi :\equiv \neg\alpha$ and $\alpha \in Form(\mathcal{L})^*$
– $\phi :\equiv \vee\alpha\beta$ and $\{\alpha, \beta\} \subseteq Form(\mathcal{L})^*$
– $\phi :\equiv \forall v\alpha$ and $\alpha \in Form(\mathcal{L})^*$ and $v \in Var$
- Formulas encode the statements or assertions in the language
- Non-recursive definitions are called atomic formulas ($\phi \in AF(\mathcal{L})$)

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1.2.3 (Definition) Scope

- The the scope of the quantifier $scope(\phi, \alpha) :\equiv \alpha$ if $\phi :\equiv \forall v\alpha$
- The symbols in α lies within the scope of $\forall$

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1.3 Induction and Recursion

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1.3.1 (Definition) Definition by recursion

- The set S is the (recursively defined) closure of the set J under the set of rules Q ($Cl(S, J, Q)$) iff S is the smallest set that satisfies all of the following:
– $J \subseteq S$
– For any $R \in Q$, for any $\left\langle \overset{ArityR(R)-1}{\underset{i=1}{\boxed{s_i}}}, s \right\rangle \in R$, if $\left\{ \overset{ArityR(R)-1}{\underset{i=1}{\boxed{s_i}}} \right\} \subseteq S$, then $s \in S$
- In recusive definitions, the '$x \in X$ iff $P(x)$' qualifier is logically equivalent to the '$X \subseteq \{y : P(y)\}$' qualifier because
– For any z, $\widetilde{P(z)}$ iff $z \notin X$ as well
– Therefore X has to be the smallest set that satisfies P

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1.3.2 (Metatheorem) Proof by induction on structure

- If $J \subseteq \{x : P(x)\}$ and for any $R \in Q$, for any $\left\langle \overset{ArityR(R)-1}{\underset{i=1}{\boxed{s_i}}}, s \right\rangle \in R$, (if $\left\{ \overset{ArityR(R)-1}{\underset{i=1}{\boxed{s_i}}} \right\} \subseteq \{x : P(x)\}$, then $s \in \{x : P(x)\}$), then $S_{J,Q} \subseteq \{x : P(x)\}$
- Proof: $S_{J,Q} \subseteq \{x : P(x)\}$ from (definition of $S_{J,Q}$: satisfies the qualifier of smallest set)

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1.3.3 (Metatheorem) Proof by induction on complexity

- If $J \subseteq \{x : P(x)\}$ and (if $stage(J, Q, n) \subseteq \{x : P(x)\}$, then $stage(J, Q, n+1) \subseteq \{x : P(x)\}$), then $S_{J,Q} \subseteq \{x : P(x)\}$
- BACKLOG: PROPERLY DEFINE STAGE AND SAY STAGE = CLOSURE AND PROOF!!!

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1.3.4 (Definition) Initial segment

- The string s is an initial segment of the string t ($IS(s, t)$) iff there exists the string $u :\not\equiv \epsilon$, $t :\equiv su$

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1.3.5 (Metatheorem) Initial segments of terms

- For any $s \in Term(\mathcal{L})$, for any $t \in Term(\mathcal{L})$, $\widetilde{IS(s, t)}$
- Proof:
– $Term(\mathcal{L})_J \subseteq \left\{ s \in Term(\mathcal{L})_J : (\text{ for any } t \in Term(\mathcal{L})), \left(\widetilde{IS(s, t)}\right) \right\}$
— If $s :\equiv x \in Var \cup Const$, then
—- If $t :\equiv z \in Var \cup Const$, then $\widetilde{IS(s, t)}$ from
—— If $IS(s, t)$, then
——— $t :\equiv su$
——— $x :\equiv zu$

—— $x :\equiv z$
—— $u :\equiv \epsilon$
—— $u :\not\equiv \epsilon$
—— CONTRADICTION —- If $t :\equiv f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}$, then $\widetilde{IS(s,t)}$ from
—— If $IS(s,t)$, then
—— $t :\equiv su$
—— $f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}} :\equiv xu$
—— $f :\equiv x$
—— $f :\not\equiv x$
—— CONTRADICTION – $Term(\mathcal{L})_Q$ closed in $\left\{ s \in Term(\mathcal{L})_J : (\text{ for any } t \in Term(\mathcal{L})), \left(\widetilde{IS(s,t)}\right) \right\}$
— If $s :\equiv f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}$ and $f \in Func$ and $\left\{ \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}} \right\} \subseteq Term(\mathcal{L})$ and
— For any $t_i \in \left\{ \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}} \right\}$, for any $r \in Term(\mathcal{L})$, $\widetilde{IS(t_i, r)}$, then
—- If $t :\equiv z \in Var \cup Const$, then $\widetilde{IS(s,t)}$ from
—— If $IS(s,t)$, then
—— $t :\equiv su$
¡[1] (HYP: $IS(s,t)$)¿ —— $z :\equiv f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}} u$
¡[2] (HYP) on [1]¿ —— $z :\equiv f$
¡[3] (DEF: Alphabet, String Concat) on [2]¿ —— $z :\not\equiv f$
¡[4] (DEF: Alphabet) on [3]¿ —— CONTRADICTION [3, 4]
—- If $t :\equiv f' \overset{Arity(f)}{\underset{i=1}{\boxed{t'_i}}} \in Term(\mathcal{L})$, then
—— If $IS(s,t)$, then
—— $t :\equiv su$
¡[1] (HYP: $IS(s,t)$)¿ —— $f' \overset{Arity(f)}{\underset{i=1}{\boxed{t'_i}}} :\equiv f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}} u$
¡[2] (HYP) on [1]¿ —— $f' :\equiv f$
¡[3] (DEF: Alphabet, String Concat) on [2]¿ —— $\overset{Arity(f)}{\underset{i=1}{\boxed{t'_i}}} :\equiv \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}} u$
¡[4] (DEF: String Concat) on [3]¿ —— For $i \in \mathbb{N}_1^{Arity(f)}$, $t'_i :\equiv t_i$ from
¡[5] (Induction) on [4]¿ ——- If $t'_i :\not\equiv t_i$, then
——— $IS(t_i, t'_i)$
¡[5.1] (HYP: $t'_i :\not\equiv t_i$) on [4]¿ ——— $\widetilde{IS(t_i, t'_i)}$
¡[5.2] (HYP: $\widetilde{IS(t_i, r)}$)¿ ——— CONTRADICTION [5.1, 5.2]
—— $\overset{Arity(f)}{\underset{i=1}{\boxed{t'_i}}} :\equiv \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}$
¡[6] (DEF: String Concat) on [5]¿ —— $u :\equiv \epsilon$
¡[7] (DEF: String Concat) on [6]¿ —— $u :\not\equiv \epsilon$
¡[8] (HYP: $IS(s,t)$)¿ —— CONTRADICTION [7, 8]
—— $\widetilde{IS(s,t)}$
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1.3.6 (Metatheorem) Unique readability of terms

- For any $t \in Term(\mathcal{L})$, it satisfies exactly one of the following:
– $t :\equiv v \in Var$ and v is unique
– $t :\equiv c \in Const$ and c is unique

– $t :\equiv f \underset{i=1}{\overset{Arity(f)}{\boxed{t_i}}}$ and $f \in Func$ is unique and for any $i \in \left\{ \underset{i=1}{\overset{Arity(f)}{\boxed{i}}} \right\}$, $t_i \in Term(\mathcal{L})$ is unique
- Proof:
– If $t \in Var$, then variables are unique, then t is unique
– If $t \in Const$, then constants are unique, then t is unique
– If $t :\equiv f \underset{i=1}{\overset{Arity(f)}{\boxed{t_i}}}$, then
— If $t :\equiv f' \underset{i=1}{\overset{Arity(f')}{\boxed{t'_i}}}$, then
—- $f :\equiv f'$
—- $f \underset{i=1}{\overset{Arity(f)}{\boxed{t_i}}} :\equiv f \underset{i=1}{\overset{Arity(f)}{\boxed{t'_i}}}$
—- $\underset{i=1}{\overset{Arity(f)}{\boxed{t_i}}} :\equiv \underset{i=1}{\overset{Arity(f)}{\boxed{t'_i}}}$
—- If $Arity(f) = 1$ and $t_1 :\not\equiv t'_1$, then $IS(t_1, t'_1)$ or $IS(t'_1, t_1)$, then CONTRADICTION
—- If $Arity(f) > 1$ and for any $i \in \left\{ \underset{i=1}{\overset{n-1}{\boxed{i}}} \right\}$, $t_i :\equiv t'_i$ and $t_n :\not\equiv t'_n$, then $IS(t_n, t'_n)$ or $IS(t'_n, t_n)$, then CONTRADICTION
—- For any $i \in \left\{ \underset{i=1}{\overset{Arity(f)}{\boxed{i}}} \right\}$, $t_i :\equiv t'_i$

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1.3.7 (Metatheorem) Initial segments of formulas

- BACKLOG:

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1.3.8 (Metatheorem) Unique readability of formulas

- BACKLOG:

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1.3.9 (Definition) Language of Number theory

- $\mathcal{L}_{NT} = \{0, S, +, \cdot, E, <\}$ where:
– 0 is a constant symbol to be interpreted as 0
– S is a 1-arity function symbol to be interpreted as increment by 1
– $+$ is a 2-arity function symbol to be interpreted as addition
– Δ is a 2-arity function symbol to be interpreted as multiplication
– E is a 2-arity function symbol to be interpreted as exponentiation
– $<$ is a 2-arity relation symbol to be interpreted as less than

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1.4 Sentences

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1.4.1 (Definition) Free variable in a formula

- The variable v is free in the formula ϕ ($free(v, \phi)$) iff it satisfies some of the following:
– $\phi \in AF(\mathcal{L})$ and $occurs(v, \phi)$
– $\phi :\equiv \neg\alpha$ and $free(v, \alpha)$
– $\phi :\equiv \alpha \vee \beta$ and $free(v, \alpha)$ or $free(v, \beta)$
– $\phi :\equiv \forall w \alpha$ and $v :\not\equiv w$ and $free(v, \alpha)$

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1.4.2 (Definition) Sentence

- The $\phi \in Form(\mathcal{L})$ is a sentence ($\phi \in Sent(\mathcal{L})$) iff $\{x \in Var : free(x, \phi)\} = \emptyset$

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1.4.3 (Definition) Bound variable in a formula

- The variable v is bound in the formula ϕ ($bound(v, \phi)$) iff $occurs(v, phi)$ and $\widetilde{free(v}, \phi)$

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1.5 Structures

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1.5.1 (Definition) Structure

- The $\mathcal{L}$-structure $\mathfrak{A}$ of the language $\mathcal{L}$ ($Struct(\mathfrak{A}, \mathcal{L})$) is the tuple of:

– Universe: non-empty set A

– ConstI: for any $c \in Const$, $c^{\mathfrak{A}} \in A$

– FuncI: for any $f \in Func$, $f^{\mathfrak{A}} : A^{Arity(f)} \to A$

– RelI: for any $P \in Rel$, $P^{\mathfrak{A}} \subseteq A^{Arity(P)}$

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1.5.2 (Definition) Henkin structure

- The $\mathcal{L}$-structure $\mathfrak{A}$ is a Henkin structure iff it satisfies all of the following:

– $A = \left\{t \in Term(\mathcal{L}) : (\text{ for any } x \in Var), \left(\widetilde{occurs(x}, t)\right)\right\}$

– For any $c \in Const$, $c^{\mathfrak{A}} = c$

– For any $f \in Func$, for any $\left\{\underset{i=1}{\overset{Arity(f)}{\boxed{a_i}}}\right\} \subseteq A$, $f^{\mathfrak{A}}(\underset{i=1}{\overset{Arity(f)}{\boxed{a_i}}}) = f \underset{i=1}{\overset{Arity(f)}{\boxed{a_i}}}$

– For any $P \in Rel$, BACKLOG: not important

- The Henkin structure uses the syntactic elements as objects of the universe - useful for the Completeness theorem

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1.5.3 (Definition) Isomorphic structures

- The $\mathcal{L}$-structure $\mathfrak{A}$ is isomorphic to the $\mathcal{L}$-structure $\mathfrak{B}$ ($\mathfrak{A} \cong \mathfrak{B}$) iff there exists a function $i : A \to B$ and $Bij(i)$ and it satisfies all of the following:

– For any $c \in Const$, $i(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$

– For any $f \in Func$, for any $\left\{\underset{i=1}{\overset{Arity(f)}{\boxed{a_i}}}\right\} \subseteq A$, $i(f^{\mathfrak{A}}(\underset{i=1}{\overset{Arity(f)}{\boxed{a_i}}})) = f^{\mathfrak{B}}(\underset{i=1}{\overset{Arity(f)}{\boxed{i(a_i)}}})$

– For any $P \in Rel$, for any $\left\{\underset{i=1}{\overset{Arity(P)}{\boxed{a_i}}}\right\} \subseteq A$, $\underset{i=1}{\overset{Arity(P)}{\boxed{a_i}}} \in P^{\mathfrak{A}}$ iff $\underset{i=1}{\overset{Arity(P)}{\boxed{i(a_i)}}} \in P^{\mathfrak{B}}$

- i preserves structure by way of operations in $\mathfrak{A}$ have corresponding equivalent operations in $\mathfrak{B}$

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1.5.4 (Definition) Equivalence relation

- The relation R on the set S is an $EqRel(R, S)$ iff it satisfies all of the following:

– For any $a \in S$, aRa

– For any $\{a, b\} \subseteq S$, if aRb, then bRa

– For any $\{a, b, c\} \subseteq S$, if aRb and bRc, then aRc

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1.5.5 (Metatheorem) Isomorphic structure equivalence

- $EqRel(\cong, \{\mathfrak{X} : Struct(\mathfrak{X}, \mathcal{L}\}))$
- Proof:
– For any $\mathcal{L}$-structure $\mathfrak{A}$, then
— $j : A \to A$ and for any $a \in A$, $j(a) = a$
— BACKLOG: j satisfies $\mathfrak{A} \cong \mathfrak{B}$
– For any $\mathcal{L}$-structures $\{\mathfrak{A}, \mathfrak{B}\}$, then
— If $\mathfrak{A} \cong \mathfrak{B}$, then
—- There exists $i_{A,B}$, $i_{A,B}$ satisfies $\mathfrak{A} \cong \mathfrak{B}$
—- BACKLOG: ${i_{A,B}}^{-1}$ satisfies $\mathfrak{B} \cong \mathfrak{A}$
– For any $\mathcal{L}$-structure $\{\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\}$, then
— If $\mathfrak{A} \cong \mathfrak{B}$ and $\mathfrak{B} \cong \mathfrak{C}$, then
—- There exists $i_{A,B}$, $i_{A,B}$ satisfies $\mathfrak{A} \cong \mathfrak{B}$
—- There exists $i_{B,C}$, $i_{B,C}$ satisfies $\mathfrak{B} \cong \mathfrak{C}$
—- BACKLOG: $iB, C \circ i_{A,B}$ satisfies $\mathfrak{A} \cong \mathfrak{C}$
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1.6 Truth in a Structure

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1.6.1 (Definition) Variable-universe assignment function

- The function s is a variable-universe assignment function into the $\mathcal{L}$-structure $\mathfrak{A}$ iff $s : Var \to A$
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1.6.2 (Definition) Term-universe assignment function

- The function $\overline{s}$ is the function generated from the variable-universe assignment function s iff $\overline{s} : Term(\mathcal{L}) \to A$ and it satisfies all of the following:
– If $t :\equiv x \in Var$, then $\overline{s}(t) = \overline{s}(x) = s(x)$
– If $t :\equiv c \in Const$, then $\overline{s}(t) = \overline{s}(c) = c^{\mathfrak{A}}$
– If $t :\equiv f \underset{i=1}{\overset{Arity(f)}{\boxed{t_i}}}$, then $\overline{s}(t) = \overline{s}(f \underset{i=1}{\overset{Arity(f)}{\boxed{t_i}}}) = f^{\mathfrak{A}}(\underset{i=1}{\overset{Arity(f)}{\boxed{\overline{s}(t_i)}}})$
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1.6.3 (Definition) Modification of variable-universe assignment function

- The function $s[x|a]$ is an x-modification of the variable-universe assignment function s iff $x \in Var$ and $a \in A$ and it satisfies all of the following:
– If $v :\not\equiv x$, then $s[x|a](v) = s(v)$
– If $v :\equiv x$, then $s[x|a](v) = s[x|a](x) = a$
- The mapping of x is fixed to a
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1.6.4 (Definition) Relative truth to assignment

- The $\mathcal{L}$-structure $\mathfrak{A}$ satisfies the formula ϕ with the variable-universe assignment function s ($\mathfrak{A} \vDash \phi[s]$) iff it satisfies all of the following:
– If $\phi :\equiv\equiv rt$, then $\overline{s}(r) = \overline{s}(t)$
– If $\phi :\equiv P \underset{i=1}{\overset{Arity(P)}{\boxed{t_i}}}$, then $\left\langle \underset{i=1}{\overset{Arity(P)}{\boxed{\overline{s}(t_i)}}} \right\rangle \in P^{\mathfrak{A}}$
– If $\phi :\equiv \neg\alpha$, then $\mathfrak{A} \nvDash \alpha[s]$
– If $\phi :\equiv \vee\alpha\beta$, then $\mathfrak{A} \vDash \alpha[s]$ or $\mathfrak{A} \vDash \beta[s]$
– If $\phi :\equiv \forall x\alpha$, then for any $a \in A$, $\mathfrak{A} \vDash \alpha[s[x|a]]$
- The $\mathcal{L}$-structure $\mathfrak{A}$ satisfies the set of formulas Γ with the variable-universe assignment function s ($\mathfrak{A} \vDash \Gamma[s]$) if for any $\gamma \in \Gamma$, $\mathfrak{A} \vDash \gamma[s]$
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1.6.5 (Metatheorem) Variable assignment determines term assignment

- If s_1 and s_2 are variable-universe assignment functions into the $\mathcal{L}$-structure $\mathfrak{A}$ and for any $t \in Term(\mathcal{L})$, for any $v \in \{x \in Var : occurs(x,t)\}$, $s_1(v) = s_2(v)$, then $\overline{s_1}(t) = \overline{s_2}(t)$
- Proof:
– $Term(\mathcal{L})_J \subseteq \{t \in Term(\mathcal{L}) :$ if ((for any $v \in \{x \in Var : occurs(x,t)\}$), $(s_1(v) = s_2(v))$), then $(\overline{s_1}(t) = \overline{s_2}(t))\}$
— If (for any $v \in \{x \in Var : occurs(x,t)\}$), $(s_1(v) = s_2(v))$, then —- If $t :\equiv v \in Var$, then
—– $\overline{s_1}(v) = \overline{s_2}(v)$
—– $\overline{s_1}(t) = \overline{s_2}(t)$
—- If $t :\equiv c \in Const$, then
—– $c^{\mathfrak{A}} = c^{\mathfrak{A}}$
—– $\overline{s_1}(c) = \overline{s_2}(c)$
—– $\overline{s_1}(t) = \overline{s_2}(t)$
– $Term(\mathcal{L})_Q$ closed in $\{t \in Term(\mathcal{L}) :$ if ((for any $v \in \{x \in Var : occurs(x,t)\}$), $(s_1(v) = s_2(v))$), then $(\overline{s_1}(t) = \overline{s_2}(t))\}$
— If $t :\equiv f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}$ and $f \in Func$ and $\left\{ \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}} \right\} \subseteq Term(\mathcal{L})$ and
— for any $t_i \in \left\{ \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}} \right\}$, if ((for any $v \in \{x \in Var : occurs(x,t_i)\}$), $(s_1(v) = s_2(v))$), then $(\overline{s_1}(t_i) = \overline{s_2}(t_i))$, then
—- For any $t_i \in \left\{ \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}} \right\}$, $\{x \in Var : occurs(x,t_i)\} \subseteq \{x \in Var : occurs(x,t)\}$
—- If for any $v \in \{x \in Var : occurs(x,t)\}$, $s_1(v) = s_2(v)$, then
—– For any $t_i \in \left\{ \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}} \right\}$, $\overline{s_1}(t_i) = \overline{s_2}(t_i)$
—– $\left\langle \overset{Arity(f)}{\underset{i=1}{\boxed{\overline{s_1}(t_i)}}} \right\rangle = \left\langle \overset{Arity(f)}{\underset{i=1}{\boxed{\overline{s_2}(t_i)}}} \right\rangle$
—– $f^{\mathfrak{A}}(\overset{Arity(f)}{\underset{i=1}{\boxed{\overline{s_1}(t_i)}}}) = f^{\mathfrak{A}}(\overset{Arity(f)}{\underset{i=1}{\boxed{\overline{s_2}(t_i)}}})$
—– $\overline{s_1}(f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}) = \overline{s_2}(f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}})$
—– $\overline{s_1}(t) = \overline{s_2}(t)$

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1.6.6 (Metatheorem) Free variable assignment determines relative truth

- If s_1 and s_2 are variable-universe assignment functions into the $\mathcal{L}$-structure $\mathfrak{A}$ and $\phi \in Form(\mathcal{L})$ and for any $v \in \{x \in Var : free(x,\phi)\}$, $s_1(v) = s_2(v)$, then $Form(\mathcal{L}) \subseteq \{\phi \in Form(\mathcal{L}) : \mathfrak{A} \vDash \phi[s_1]($ iff $)\mathfrak{A} \vDash \phi[s_2]\}$
- Proof:
– $Form(\mathcal{L})_J \subseteq \{\phi \in Form(\mathcal{L}) :$ if ((for any $v \in \{x \in Var : free(x,\phi)\}$), $(s_1(v) = s_2(v))$), then $(\mathfrak{A} \vDash \phi[s_1]($ iff $)\mathfrak{A} \vDash \phi[s_2])\}$
— If $\phi :\equiv\equiv rt$, then
—- $\{x \in Var : free(x,\phi)\} = \{x \in Var : occurs(x,\phi)\}$
—- $\overline{s_1}(r) = \overline{s_2}(r)$
—- $\overline{s_1}(t) = \overline{s_2}(t)$
—- $\overline{s_1}(r) = \overline{s_1}(t)$ iff $\overline{s_2}(r) = \overline{s_2}(t)$
—- $\mathfrak{A} \vDash \phi[s_1]$ iff $\mathfrak{A} \vDash \phi[s_2]$
— If $\phi :\equiv P \overset{Arity(P)}{\underset{i=1}{\boxed{t_i}}}$, then
—- $\{x \in Var : free(x,\phi)\} = \{x \in Var : occurs(x,\phi)\}$
—- For any $t_i \in \left\{ \overset{Arity(P)}{\underset{i=1}{\boxed{t_i}}} \right\}$, $\overline{s_1}(t_i) = \overline{s_2}(t_i)$
—- $\left\langle \overset{Arity(P)}{\underset{i=1}{\boxed{\overline{s_1}(t_i)}}} \right\rangle = \left\langle \overset{Arity(P)}{\underset{i=1}{\boxed{\overline{s_2}(t_i)}}} \right\rangle$
—- $\left\langle \overset{Arity(P)}{\underset{i=1}{\boxed{\overline{s_1}(t_i)}}} \right\rangle \in P^{\mathfrak{A}}$ iff $\left\langle \overset{Arity(P)}{\underset{i=1}{\boxed{\overline{s_2}(t_i)}}} \right\rangle \in P^{\mathfrak{A}}$

—- $\mathfrak{A} \vDash \phi[s_1]$ iff $\mathfrak{A} \vDash \phi[s_2]$
– $Form(\mathcal{L})_Q$ closed in $\{\phi \in Form(\mathcal{L}) :$ if ((for any $v \in \{x \in Var : free(x, \phi)\}$), $(s_1(v) = s_2(v))$), then $(\mathfrak{A} \vDash \phi[s_1]($ iff $)\mathfrak{A} \vDash \phi[s_2])\}$
— If $\phi :\equiv \neg\alpha$ and $\alpha \in Form(\mathcal{L})$ and
— if ((for any $v \in \{x \in Var : free(x, \alpha)\}$), $(s'_1(v) = s'_2(v))$), then $(\mathfrak{A} \vDash \alpha[s'_1]($ iff $)\mathfrak{A} \vDash \alpha[s'_2])$, then
—- $\{x \in Var : free(x, \alpha)\} \subseteq \{x \in Var : free(x, \phi)\}$
—- If (for any $v \in \{x \in Var : free(x, \phi)\}$), $(s_1(v) = s_2(v))$, then
—— $\mathfrak{A} \vDash \alpha[s_1]$ iff $\mathfrak{A} \vDash \alpha[s_2]$
—— $\mathfrak{A} \nvDash \alpha[s_1]$ iff $\mathfrak{A} \nvDash \alpha[s_2]$
—— $\mathfrak{A} \vDash \neg\alpha[s_1]$ iff $\mathfrak{A} \vDash \neg\alpha[s_2]$
—— $\mathfrak{A} \vDash \phi[s_1]$ iff $\mathfrak{A} \vDash \phi[s_2]$
— If $\phi :\equiv \vee\alpha\beta$ and $\{\alpha, \beta\} \subseteq Form(\mathcal{L})$ and
— if ((for any $v \in \{x \in Var : free(x, \alpha)\}$), $(s'_1(v) = s'_2(v))$), then $(\mathfrak{A} \vDash \alpha[s'_1]($ iff $)\mathfrak{A} \vDash \alpha[s'_2])$ and
— if ((for any $v \in \{x \in Var : free(x, \beta)\}$), $(s''_1(v) = s''_2(v))$), then $(\mathfrak{A} \vDash \beta[s''_1]($ iff $)\mathfrak{A} \vDash \beta[s''_2])$, then
—- $\{x \in Var : free(x, \alpha)\} \subseteq \{x \in Var : free(x, \phi)\}$
—- $\{x \in Var : free(x, \beta)\} \subseteq \{x \in Var : free(x, \phi)\}$
—- If (for any $v \in \{x \in Var : free(x, \phi)\}$), $(s_1(v) = s_2(v))$, then
—— $\mathfrak{A} \vDash \alpha[s_1]$ iff $\mathfrak{A} \vDash \alpha[s_2]$
—— $\mathfrak{A} \vDash \beta[s_1]$ iff $\mathfrak{A} \vDash \beta[s_2]$
—— $(\mathfrak{A} \vDash \alpha[s_1]$ or $\mathfrak{A} \vDash \beta[s_1])$ iff $(\mathfrak{A} \vDash \alpha[s_2]$ or $\mathfrak{A} \vDash \beta[s_2])$
—— $\mathfrak{A} \vDash \vee\alpha\beta[s_1]$ iff $\mathfrak{A} \vDash \vee\alpha\beta[s_2]$
—— $\mathfrak{A} \vDash \phi[s_1]$ iff $\mathfrak{A} \vDash \phi[s_2]$
— If $\phi :\equiv \forall z\alpha$ and $z \in Var$ and $\alpha \in Form(\mathcal{L})$ and
— if ((for any $v \in \{x \in Var : free(x, \alpha)\}$), $(s'_1(v) = s'_2(v))$), then $(\mathfrak{A} \vDash \alpha[s'_1]($ iff $)\mathfrak{A} \vDash \alpha[s'_2])$, then
—- $\{x \in Var : free(x, \alpha)\} \subseteq \{x \in Var : free(x, \phi) \cup \{z\}\}$
—- If (for any $v \in \{x \in Var : free(x, \phi)\}$), $(s_1(v) = s_2(v))$, then —— For any $a \in A$, for any $v \in \{x \in Var : free(x, \alpha)\}$, $s_1[z|a](v) = s_2[z|a](v)$
—— For any $a \in A$, $\mathfrak{A} \vDash \alpha[s_1[z|a]]$ iff for any $a \in A$, $\mathfrak{A} \vDash \alpha[s_2[z|a]]$
—— $\mathfrak{A} \vDash \phi[s_1]$ iff $\mathfrak{A} \vDash \phi[s_2]$
==

1.6.7 (Metatheorem) Sentences have fixed truth

- If $\sigma \in Sent(\mathcal{L})$ and $\mathfrak{A}$ is an $\mathcal{L}$-structure, then for any variable-universe assignment functions s, $\mathfrak{A} \vDash \sigma[s]$ or for any variable-universe assignment functions s', $\mathfrak{A} \nvDash \sigma[s']$
- Proof:
– $\{x \in Var : free(x, \sigma)\} = \emptyset$
– For any variable-universe assignment functions s_1 and s_2, $\mathfrak{A} \vDash \sigma[s_1]$ iff $\mathfrak{A} \vDash \sigma[s_2]$
==

1.6.8 (Definition) Structure models formula

- The $\mathcal{L}$-structure $\mathfrak{A}$ models $\phi \in Form(\mathcal{L})$ $(\mathfrak{A} \vDash \phi)$ iff for any variable-universe assignment function s, $\mathfrak{A} \vDash \phi[s]$
- The $\mathcal{L}$-structure $\mathfrak{A}$ models $\Phi \subseteq Form(\mathcal{L})$ $(\mathfrak{A} \vDash \Phi)$ iff for any $\phi \in \Phi$, $\mathfrak{A} \vDash \phi$
==

1.6.9 (Definition) Abbreviations

- BACKLOG: $\wedge$, $\implies$, $\iff$, $\exists xQ(x)$, $(\forall P(x))Q(x)$, $(\exists P(x))Q(x)$
==

1.6.10 (Metatheorem) Semantics of abbreviations

- BACKLOG: show they are semantically expected ==

1.7 Logical Implication

==

1.7.1 (Definition) Logical implication

- The set of formulas Δ logically implies the set of formulas Γ ($\Delta \vDash \Gamma$) iff for any $\mathcal{L}$-structure $\mathfrak{A}$, if $\mathfrak{A} \vDash \Delta$, then $\mathfrak{A} \vDash \Gamma$
- $\Delta \vDash \gamma$ abbreviates $\Delta \vDash \{\gamma\}$

==

1.7.2 (Definition) Valid formula

- The formula ϕ is valid ($\vDash \phi$) iff $\emptyset \vDash \phi$

==

1.7.3 (Metatheorem) Variables self-equiv are valid

- For any $v \in Var$, then $\vDash \equiv vv$
– For any structure $\mathfrak{A}$, for any variable-universe assignment funtion s,
– $s(v) = \overline{s}(v)$
– $\overline{s}(v) = \overline{s}(v)$
– $\mathfrak{A} \vDash (\equiv vv)[s]$

==

1.7.4 (Definition) Universal closure

- The universal closure of $\phi \in Form(\mathcal{L})$ with the free variables $\underset{i=1}{\overset{n}{\boxed{v_i}}}$ satisfies $UC(\phi) :\equiv \underset{i=1}{\overset{n}{\boxed{\forall v_i}}}\phi$
- The universal closures of $\Phi \subseteq Form(\mathcal{L})$ satisfies $UC(\Phi) = \{UC(\phi) : \phi \in \Phi\}$

==

1.7.5 (Metatheorem) Universal closure preserves validity

- For any $\phi \in Form(\mathcal{L})$, for any $x \in Var$, for any structure $\mathfrak{A}$, $\mathfrak{A} \vDash \phi$ iff $\mathfrak{A} \vDash \forall x\phi$
– If $\mathfrak{A} \vDash \phi$, then
— For any variable-universe assignment function s, $\mathfrak{A} \vDash \phi[s]$
— For any $a \in A$, $\mathfrak{A} \vDash \phi[s[x|a]]$
— $\mathfrak{A} \vDash \forall x\phi$
– If $\vDash \forall x\phi$, then
— For any variable-universe assignment function s, $\mathfrak{A} \vDash (\forall x\phi)[s]$
— For any $a \in A$, $\mathfrak{A} \vDash \phi[s[x|a]]$
— $\mathfrak{A} \vDash \phi[s[x|s(x)]]$
— $\mathfrak{A} \vDash \phi[s]$

==

1.7.6 (Metatheorem) Logical equivalence

- ϕ has a logical equivalence to ψ iff $\vDash (\phi \implies \psi)$ and $\vDash (\phi \implies \psi)$
- ϕ has a weak logical equivalence to ψ iff $\phi \vDash \psi$ and $\psi \vDash \phi$

==

1.7.7 (Metatheorem) Strong logical equivalence property

- If $\vDash (\phi \implies \psi)$, then $\phi \vDash \psi$
- Proof:
– If $\vDash (\phi \implies \psi)$, then
— For any structure $\mathfrak{A}$,
—- For any variable-universe assignment function s,
—– $\mathfrak{A} \vDash (\phi \implies \psi)[s]$
—– If $\mathfrak{A} \vDash \phi[s]$, then $\mathfrak{A} \vDash \psi[s]$
—- If (for any variable-universe assignment function s_1, $\mathfrak{A} \vDash \phi[s_1]$), then
—– For any variable-universe assignment function s_2,
—— $\mathfrak{A} \vDash \phi[s_2]$
¡HYP¿ —— If $\mathfrak{A} \vDash \phi[s_2]$, then $\mathfrak{A} \vDash \psi[s_2]$
—— $\mathfrak{A} \vDash \psi[s_2]$

—– For any variable-universe assignment function s_2, $\mathfrak{A} \models \psi[s_2]$
—- If $\mathfrak{A} \models \phi$, then $\mathfrak{A} \models \psi$
— $\phi \models \psi$
==

1.7.8 (Metatheorem) Weak logical equivalence property

- Not (If $\phi \models \psi$, then $\models (\phi \implies \psi)$)
- Equivallently, $\phi \models \psi$ and $\not\models (\phi \implies \psi)$
- Proof by counter-example:
– Let $\phi :\equiv (x < y)$ and $\psi :\equiv (z < w)$
– For any structure $\mathfrak{A}$,
— If $\mathfrak{A} \models (x < y)$, then
—- For any variable-universe assignment function s_1, $\mathfrak{A} \models (x < y)[s_1]$
—- $<^{\mathfrak{A}} = A \times A$
—- For any variable-universe assignment function s_2, $\mathfrak{A} \models (z < w)[s_2]$
—- $\mathfrak{A} \models (z < w)$
– $(x < y) \models (z < w)$
– Let $\mathfrak{N} = \langle \mathbb{N}, <_{std} \rangle$
– $\mathfrak{N} \not\models (x < y) \implies (z < w)[s[x|0][y|1][w|0][z|1]]$
– $\mathfrak{N} \not\models (x < y) \implies (z < w)$
– $\not\models (x < y) \implies (z < w)$
– $\not\models \phi \implies \psi$
==

1.8 Substitutions and Substitutability

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1.8.1 (Definition) Substitution in a term

- The term $|u|_t^x$ is the term u with the variable x replaced by the term t iff it satisfies some of the following:
– If $u :\equiv y \in Var$ and $y \neq x$, then $|u|_t^x :\equiv |y|_t^x :\equiv y$
– If $u :\equiv x$, then $|u|_t^x :\equiv |x|_t^x :\equiv t$
– If $u :\equiv c \in Const$, then $|u|_t^x :\equiv |c|_t^x :\equiv c$
– If $u :\equiv f \underset{i=1}{\overset{Arity(f)}{\boxed{u_i}}}$, then $|u|_t^x :\equiv \left| f \underset{i=1}{\overset{Arity(f)}{\boxed{u_i}}} \right|_t^x :\equiv f \underset{i=1}{\overset{Arity(f)}{\boxed{|u_i|_t^x}}}$
==

1.8.2 (Definition) Substitution in a formula

- The formula $|\phi|_t^x$ is the formula ϕ with the variable x replaced by the term t iff it satisfies some of the following:
– If ϕ is atomic
— If $\phi :\equiv\equiv u_1 u_2$, then $|\phi|_t^x :\equiv |\equiv u_1 u_2|_t^x :\equiv\equiv |u_1|_t^x |u_2|_t^x$
— If $\phi :\equiv P \underset{i=1}{\overset{Arity(P)}{\boxed{u_i}}}$, then $|\phi|_t^x :\equiv \left| P \underset{i=1}{\overset{Arity(P)}{\boxed{u_i}}} \right|_t^x :\equiv P \underset{i=1}{\overset{Arity(P)}{\boxed{|u_i|_t^x}}}$
– If ϕ is not atomic
— If $\phi :\equiv \neg\alpha$, then $|\phi|_t^x :\equiv |\neg\alpha|_t^x :\equiv \neg|\alpha|_t^x$
— If $\phi :\equiv \vee\alpha\beta$, then $|\phi|_t^x :\equiv |\vee\alpha\beta|_t^x :\equiv \vee|\alpha|_t^x|\beta|_t^x$
— If $\phi :\equiv \forall y\alpha$, then
—- If $y :\equiv x$, then $|\phi|_t^x :\equiv |\forall y\alpha|_t^x :\equiv \forall y\alpha$
—- If $y :\not\equiv x$, then $|\phi|_t^x :\equiv |\forall y\alpha|_t^x :\equiv \forall y|\alpha|_t^x$
==

1.8.3 (Definition) Substitutable term

- The term t is substitutable for the variable x in the formula ϕ ($Subbable(t, x, \phi)$) iff it satisfies some of the following:
– ϕ is atomic
– $\phi :\equiv \neg\alpha$ and $Subbable(t, x\alpha)$

– $\phi :\equiv \vee\alpha\beta$ and $Subbable(t, x\alpha)$ and $Subbable(t, x\beta)$
– $\phi :\equiv \forall y\alpha$ and it satisfies some of the following:
— $\widetilde{free(x, \phi)}$
— $\widetilde{occurs(y, t)}$ and $Subbable(t, x\alpha)$
- Identifies if the substitution preserves the context of the variables; i.e., bound variables stay bound, free variables stay free
- Some operations will not be permitted even though substitution is always defined
==

1.8.4 (Metatheorem) Closed terms are subbable

- If the term t is closed, then $Subbable(t, x, \phi)$
- If ϕ atomic, done
- If $\phi :\equiv \neg\alpha$ and if t is closed, then $Subbable(t, x, \alpha)$
– If t is closed, then
— $Subbable(t, x, \alpha)$
— $Subbable(t, x, \phi)$
- If $\phi :\equiv \vee\alpha\beta$ and if t is closed, then $Subbable(t, x, \alpha)$ and if t is closed, then $Subbable(t, x, \beta)$
– If t is closed, then
— $Subbable(t, x, \alpha)$
— $Subbable(t, x, \beta)$
— $Subbable(t, x, \phi)$
- If $\phi :\equiv \forall y\alpha$, and if t is closed, then $Subbable(t, x, \alpha)$
– If t is closed, then
— $Subbable(t, x, \alpha)$
— $\widetilde{occurs(y, t)}$
— $Subbable(t, x, \phi)$
==

1.8.5 (Metatheorem) Variables are self-subabble

- $Subbable(x, x, \phi)$
– If ϕ atomic, done
– If $\phi :\equiv \neg\alpha$ and $Subbable(x, x, \alpha)$, then $Subbable(x, x, \phi)$
– If $\phi :\equiv \vee\alpha\beta$ and $Subbable(x, x, \alpha)$ and $Subbable(x, x, \beta)$, then $Subbable(x, x, \phi)$
– If $\phi :\equiv \forall y\alpha$ and $Subbable(x, x, \alpha)$, then $Subbable(x, x, \phi)$
— If $y :\equiv x$, then $\widetilde{free(x, \phi)}$, $Subbable(x, x, \phi)$
— If $y :\not\equiv x$, then $\widetilde{occurs(y, t)}$, then $Subbable(x, x, \phi)$
==

1.8.6 (Metatheorem) Substitutions of non-free variables is the identity

- If $\widetilde{free(x, \phi)}$, then $|\phi|^x_t :\equiv \phi$
– If ϕ is atomic, then
— If $\widetilde{free(x, \phi)}$, $\widetilde{occurs(x, \phi)}$, then sub is identity (BACKLOG: not proven)
– If ϕ is not atomic, then
— If $\phi :\equiv \neg\alpha$ and if $\widetilde{free(x, \alpha)}$, then $|\alpha|^x_t :\equiv \alpha$, then
—- If $\widetilde{free(x, \phi)}$, then
—– $\widetilde{free(x, \alpha)}$
—– $|\alpha|^x_t :\equiv \alpha$
—– $|\phi|^x_t :\equiv |\neg\alpha|^x_t :\equiv \neg|\alpha|^x_t :\equiv \neg\alpha :\equiv \phi$
— If $\phi :\equiv \vee\alpha\beta$, BACKLOG: do
— If $\phi :\equiv \forall y\alpha$, BACKLOG: do
==

1.8.7 (Metatheorem) Subbable is decidable

- BACKLOG: do ==

Chapter 2

Deductions

==

2.1 Deductions

==

2.1.1 (Definition) Meta-restrictions for deduction

- Λ is the set of formulas that are logical axioms
- Σ is the set of formulas that are non-logical axioms
- R_I is the set of relations that are rules of inferences
- In order to do this, we will impose the following restrictions on our logical axioms and rules of inference:
– 1. (Logical) axioms are decidable
– 2. Rules of inference are decidable
– 3. Rules of inference have finite inputs
– 4. (Logical) axiom are valid
– 5. Our rules of inference will preserve truth. For any $\langle \Gamma, \phi \rangle \in R_I$, $\Gamma \vDash \phi$
- (1-3) States that each step must be checkable and computable in finite time
- (4-5) States that each step is valid
==

2.1.2 (Definition) Deduction

- The finite sequence $\left\langle \boxed{\phi_i}_{i=1}^{n} \right\rangle$ is a deduction from the non-logical axioms Σ ($\Sigma \vdash \left\langle \boxed{\phi_i}_{i=1}^{n} \right\rangle$) iff $n \in \mathbb{N}$ and for any $1 \leq i \leq n$, it satisfies some of the following:
– $\phi_i \in \Lambda$
– $\phi_i \in \Sigma$
– There exists $R \in R_I$, $\langle \Gamma, \phi_i \rangle \in R$ and $\Gamma \subseteq \left\{ \boxed{\phi_j}_{j=1}^{i-1} \right\}$
- $\Sigma \vdash \phi_n$ abbreviates $\Sigma \vdash \left\langle \boxed{\phi_i}_{i=1}^{n} \right\rangle$
==

2.1.3 (Metatheorem) Top-down definition equivalence of deduction

- $Thm_\Sigma = \{\phi \in Form(\mathcal{L}) : \Sigma \vdash \phi\} = Cl(\Lambda \cup \Sigma, R_I)$
- Proof:
– $Cl(\Lambda \cup \Sigma, R_I) \subseteq Thm_\Sigma$
— If $\phi \in \Lambda \cup \Sigma$ then
—- $\Sigma \vdash \langle \phi \rangle$
—- $\Sigma \vdash \phi$
—- $\phi \in Thm_\Sigma$

— If there exists $R \in R_I$, $\langle \Gamma, \phi \rangle \in R$ and $\Gamma \subseteq Thm_\Sigma$, then
—- $\Sigma \vdash \langle \Gamma \rangle$
—- $\Sigma \vdash \langle \Gamma, \phi \rangle$
—- $\Sigma \vdash \phi$
—- $\phi \in Thm_\Sigma$
– $Thm_\Sigma \subseteq Cl(\Lambda \cup \Sigma, R_I)$
— If $\phi_i \in Thm_\Sigma$, then
—- If $i = 1$, then
—– $\Sigma \vdash \langle \phi_i \rangle$
—– $\phi_i \in \Lambda \cup \Sigma$
—– $\phi_i \in Cl(\Lambda \cup \Sigma, R_I)$
—- If $i > 1$ and $\left\{ \overset{i-1}{\underset{j=1}{\boxed{\phi_j}}} \right\} \subseteq Cl(\Lambda \cup \Sigma, R_I)$, then
—– If $\phi_i \in \Lambda \cup \Sigma$, then $\phi_i \in Cl(\Lambda \cup \Sigma, R_I)$
—– If there exists $R \in R_I$, $\langle \Gamma, \phi_i \rangle \in R$ and $\Gamma \subseteq \left\{ \overset{i-1}{\underset{j=1}{\boxed{\phi_j}}} \right\}$, then
—— $\Gamma \subseteq Cl(\Lambda \cup \Sigma, R_I)$
—— $\phi_i \in Cl(\Lambda \cup \Sigma, R_I)$
==

2.2 Logical Axioms

- Λ is the collection of all logical axioms
==

2.2.1 (Definition) Equality axioms

- E1: For any $v \in Var$, $\equiv vv \in \Lambda$
- E2: For any $f \in Func$, $((\wedge \overset{Arity(f)}{\underset{i=1}{\boxed{\equiv x_i y_i}}}) \implies (f(\overset{Arity(f)}{\underset{i=1}{\boxed{x_i}}}) \equiv f(\overset{Arity(f)}{\underset{i=1}{\boxed{y_i}}}))) \in \Lambda$
- E3: For any $P \in Rel \cup \{\equiv\}$, $((\wedge \overset{Arity(P)}{\underset{i=1}{\boxed{\equiv x_i y_i}}}) \implies (P(\overset{Arity(P)}{\underset{i=1}{\boxed{x_i}}}) \implies P(\overset{Arity(P)}{\underset{i=1}{\boxed{y_i}}}))) \in \Lambda$
- E2 and E3 allows equal parameters to be swapped
==

2.2.2 (Definition) Quantifier axioms

- Q1: For any $Subbable(t, x, \phi)$, $((\forall x \phi) \implies |\phi|_t^x) \in \Lambda$
- Q2: For any $Subbable(t, x, \phi)$, $(|\phi|_t^x \implies (\exists x \phi)) \in \Lambda$
- Q1 and Q2 use the *Subbable* qualifier to preserve the nature of the variables
==

2.2.3 (Metatheorem) Logical axioms are decidable

- BACKLOG: (Equality axioms are decidable + Quantifier axioms are decidable) = Λ are decidable
==

2.3 Rules of Inference

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2.3.1 (Definition) Propositional formula

- The propositional formula ϕ of the language $\mathcal{L}$ ($\phi \in Prop(\mathcal{L})$) iff $\phi \in Form(\mathcal{L})$ and it satisfies some of the following:
– $\phi \in AF(\mathcal{L})$
– $\phi :\equiv \forall x \alpha$

– $\phi :\equiv \neg\alpha$ and $\alpha \in Prop(\mathcal{L})^*$
– $\phi :\equiv \vee\alpha\beta$ and $\{\alpha, \beta\} \subseteq Prop(\mathcal{L})^*$
- Non-recursive definitions are called propositional variables ($\phi \in PV(\mathcal{L})$)
==

2.3.2 (Definition) Truth assignment

- The variable-truth assignment v is the function $v : Prop(\mathcal{L})_J \to \{\perp, \top\}$
- The formula-truth assignment $\overline{v}$ of the variable-truth assignment v is the function $\overline{v} : Prop(\mathcal{L}) \to \{\perp, \top\}$ and it satisfies some of the following:
– $\phi \in Prop(\mathcal{L})_J$ and $\overline{v}(\phi) = v(\phi)$
– $\phi \in Prop$ and $\phi :\equiv \neg\alpha$ and
— If $\overline{v}(\alpha) = \perp$, then $\overline{v}(\phi) = \top$
— If $\overline{v}(\alpha) = \top$, then $\overline{v}(\phi) = \perp$
– $\phi \in Prop$ and $\phi :\equiv \vee\alpha\beta$ and
— If $\overline{v}(\alpha) = \perp$ and $\overline{v}(\beta) = \perp$, then $\overline{v}(\phi) = \perp$
— If $\overline{v}(\alpha) = \perp$ and $\overline{v}(\beta) = \top$, then $\overline{v}(\phi) = \top$
— If $\overline{v}(\alpha) = \top$ and $\overline{v}(\beta) = \perp$, then $\overline{v}(\phi) = \top$
— If $\overline{v}(\alpha) = \top$ and $\overline{v}(\beta) = \top$, then $\overline{v}(\phi) = \top$
- The set of formulas Φ is true for the variable-truth assignment v ($\overline{v}^*(\Phi) = \top$) iff for any $\phi \in \Phi$, $\overline{v}(\phi) = \top$
==

2.3.3 (Metatheorem) Formulas are propositional

- $Form(\mathcal{L}) = Prop(\mathcal{L})$
- Proof:
– $Prop(\mathcal{L}) \subseteq Form(\mathcal{L})$ from definition
– $Form(\mathcal{L}) \subseteq Prop(\mathcal{L})$
— If $\phi \in AF(\mathcal{L})$, then $\phi \in Prop(\mathcal{L})$
— If $\phi \notin AF(\mathcal{L})$, then
—- If $\phi :\equiv \forall x\alpha$ and $\alpha \in Prop(\mathcal{L})$, then $\phi \in Prop(\mathcal{L})$
—- If $\phi :\equiv \neg\alpha$ and $\alpha \in Prop(\mathcal{L})$, then $\phi \in Prop(\mathcal{L})$
—- If $\phi :\equiv \vee\alpha\beta$ and $\{\alpha, \beta\} \subseteq Prop(\mathcal{L})$
==

2.3.4 (Definition) Propositional consequence

- The formula ϕ is a propositional consequence of the set of formulas Γ ($\Gamma \vDash_{PC} \phi$) iff for any variable-truth assignment v, if $\overline{v}^*(\Gamma) = \top$, then $\overline{v}(\phi) = \top$
- The formula ϕ is a tautology iff $\emptyset \vDash_{PC} \phi$
- $\vDash_{PC} \phi$ abbreviates $\emptyset \vDash_{PC} \phi$
==

2.3.5 (Metatheorem) Deduction theorem for PL

- $\left\{ \overset{n}{\underset{i=1}{\boxed{\gamma_i}}} \right\} \vDash_{PC} \phi$ iff $\vDash_{PC} (\wedge \overset{n}{\underset{i=1}{\boxed{\gamma_i}}}) \implies \phi$
– If $n = 1$, then
— If $\gamma_1 \vDash_{PC} \phi$, then
—- For any variable-truth assignment v,
—– If $\overline{v}(\gamma_1) = \top$, then $\overline{v}(\phi) = \top$
—– If $\overline{v}(\gamma_1) = \top$, $\overline{v}(\gamma_1 \implies \phi) = \top$
—– If $\overline{v}(\gamma_1) = \perp$, $\overline{v}(\gamma_1 \implies \phi) = \top$
—- $\vDash_{PC} \gamma_1 \implies \phi$
— If $\vDash_{PC} \gamma_1 \implies \phi$, then
—- For any variable-truth assignment v,
—– If $\overline{v}(\gamma_1) = \top$, then $\overline{v}(\phi) = \top$
—- $\gamma_1 \vDash_{PC} \phi$
— $\gamma_1 \vDash_{PC} \phi$ iff $\vDash \gamma_1 \implies \phi$

– If $n > 1$ and $\left\{\underset{i=1}{\overset{n-1}{\boxed{\gamma_i}}}\right\} \vDash_{PC} \phi$ iff $\vDash_{PC} (\wedge \underset{i=1}{\overset{n-1}{\boxed{\gamma_i}}}) \implies \phi$, then
— $\left\{\underset{i=1}{\overset{n-1}{\boxed{\gamma_i}}}\right\} \cup \gamma_n$ ... ditto basis step arguments
— Proof: TODO: from truth tables and definitions

==

2.3.6 (Definition) PC rules of inference

- PC: If $\Gamma \vDash_{PC} \phi$, then $\langle \Gamma, \phi \rangle \in R_I$
- This allows tautologies to be immediately useable in deductions

==

2.3.7 (Definition) QR rules of inference

- QR1: If $\widetilde{free(x, \psi)}$, then $\langle \{\psi \implies \phi\}, \psi \implies (\forall x \phi) \rangle \in R_I$
- QR2: If $\widetilde{free(x, \psi)}$, then $\langle \{\phi \implies \psi\}, (\exists x \phi) \implies \psi \rangle \in R_I$
- The qualifier $\widetilde{free(x, \psi)}$ is used to denote that there are no assumptions about x in ψ

==

2.3.8 (Metatheorem) Rules of inferences are decidable

- BACKLOG: (PC rules are decidable + QR axioms are decidable) = Λ are decidable

==

2.3.9 (Metatheorem) Tautologies and models have similar shapes

- For any $\mathcal{L}$-structure $\mathfrak{A}$, for any variable-universe assignment function s, for any $\phi \in Form(\mathcal{L})$, if for any $p \in \{p \in PV(\mathcal{L}) : occurs(p, \phi)\}$
$v(p_i) = \top$ iff $\mathfrak{A} \vDash p_i[s]$, then $\overline{v}(\phi) = \top$ iff $\mathfrak{A} \vDash \phi[s]$
– If $\phi \in PV(\mathcal{L})$, then $\overline{v}(\phi) = v(\phi) = \top$ iff $\mathfrak{A} \vDash \phi[s]$
– If $\phi :\equiv \neg\alpha$ and (if for any $p \in \{p \in PV(\mathcal{L}) : occurs(p, \alpha)\}$, $v(p_i) = \top$ iff $\mathfrak{A} \vDash p_i[s]$, then $\overline{v}(\alpha) = \top$ iff $\mathfrak{A} \vDash \alpha[s]$), then
— If for any $p \in \{p \in PV(\mathcal{L}) : occurs(p, \phi)\}$, $v(p_i) = \top$ iff $\mathfrak{A} \vDash p_i[s]$, then
—- $\{p \in PV(\mathcal{L}) : occurs(p, \alpha)\} \subseteq \{p \in PV(\mathcal{L}) : occurs(p, \phi)\}$
—- $\overline{v}(\alpha) = \top$ iff $\mathfrak{A} \vDash \alpha[s]$
—- $\overline{v}(\alpha) = \bot$ iff $\mathfrak{A} \nvDash \alpha[s]$
—- $\overline{v}(\neg\alpha) = \top$ iff $\mathfrak{A} \vDash (\neg\alpha)[s]$
—- $\overline{v}(\phi) = \top$ iff $\mathfrak{A} \vDash \phi[s]$
– If $\phi :\equiv \vee\alpha\beta$ and (if for any $p \in \{p \in PV(\mathcal{L}) : occurs(p, \alpha)\}$, $v(p_i) = \top$ iff $\mathfrak{A} \vDash p_i[s]$, then $\overline{v}(\alpha) = \top$ iff $\mathfrak{A} \vDash \alpha[s]$) and (if for any $p \in \{p \in PV(\mathcal{L}) : occurs(p, \beta)\}$, $v(p_i) = \top$ iff $\mathfrak{A} \vDash p_i[s]$, then $\overline{v}(\beta) = \top$ iff $\mathfrak{A} \vDash \beta[s]$), then
— If for any $p \in \{p \in PV(\mathcal{L}) : occurs(p, \phi)\}$, $v(p_i) = \top$ iff $\mathfrak{A} \vDash p_i[s]$, then
—- $\{p \in PV(\mathcal{L}) : occurs(p, \alpha)\} \subseteq \{p \in PV(\mathcal{L}) : occurs(p, \phi)\}$
—- $\{p \in PV(\mathcal{L}) : occurs(p, \beta)\} \subseteq \{p \in PV(\mathcal{L}) : occurs(p, \phi)\}$
—- $\overline{v}(\alpha) = \top$ iff $\mathfrak{A} \vDash \alpha[s]$
—- $\overline{v}(\beta) = \top$ iff $\mathfrak{A} \vDash \beta[s]$
—- ($\overline{v}(\alpha) = \top$ or $\overline{v}(\beta) = \top$) iff ($\mathfrak{A} \vDash \alpha[s]$ or $\mathfrak{A} \vDash \beta[s]$)
—- $\overline{v}(\vee\alpha\beta) = \top$ iff $\mathfrak{A} \vDash (\vee\alpha\beta)[s]$
—- $\overline{v}(\phi) = \top$ iff $\mathfrak{A} \vDash \phi[s]$

==

2.3.10 (Metatheorem) Tautologies are valid

- If $\vDash_{PC} \phi$, then $\vDash \phi$
– If $\phi \in PV(\mathcal{L})$, then $\nvDash_{PC} \phi$
– If $\phi \notin PV(\mathcal{L})$, then
— If $\vDash_{PC} \phi$, then
—- For any $\mathcal{L}$-structure $\mathfrak{A}$, for any variable-universe assignment function s,
—– For any $p \in \{p \in PV(\mathcal{L}) : occurs(p, \phi)\}$, $v*(p) = \top$ iff $\mathfrak{A} \vDash p[s]$
—– $\overline{v*}(\phi) = \top$ iff $\mathfrak{A} \vDash \phi[s]$
—– $\overline{v*}(\phi) = \top$

—— $\mathfrak{A} \vDash \phi[s]$
—- $\vDash \phi$
==

2.4 Soundness

- Preserve truth: if $\vdash$, then $\vDash$
==

2.4.1 (Metatheorem) Logical axioms are valid

- If $\phi \in \Lambda$, then $\vDash \phi$
– If $\phi \in E1$, then
— $\phi :\equiv \equiv vv$
— $\vDash \equiv vv$ ¡Variables self-equiv are valid¿
— $\vDash \phi$
– If $\phi \in E2$, then
— $\phi :\equiv (\wedge \underset{i=1}{\overset{Arity(f)}{\boxed{\equiv x_i y_i}}}) \implies (\equiv f(\underset{i=1}{\overset{Arity(f)}{\boxed{x_i}}}) f(\underset{i=1}{\overset{Arity(f)}{\boxed{y_i}}}))$
— For any structure $\mathfrak{A}$, for any variable-universe assignment s,
—- If $\mathfrak{A} \vDash (\wedge \underset{i=1}{\overset{Arity(f)}{\boxed{\equiv x_i y_i}}})[s]$, then
—— For any $i \in \left\{ \underset{i=1}{\overset{Arity(f)}{\boxed{i}}} \right\}$, $\mathfrak{A} \vDash (\equiv x_i y_i)[s]$
—— For any $i \in \left\{ \underset{i=1}{\overset{Arity(f)}{\boxed{i}}} \right\}$, $s(x_i) = \overline{s}(x_i) = \overline{s}(y_i) = s(y_i)$
—— $f^{\mathfrak{A}}(\underset{i=1}{\overset{Arity(f)}{\boxed{\overline{s}(x_i)}}}) = f^{\mathfrak{A}}(\underset{i=1}{\overset{Arity(f)}{\boxed{\overline{s}(y_i)}}})$
—— $\overline{s}(f \underset{i=1}{\overset{Arity(f)}{\boxed{x_i}}}) = \overline{s}(f \underset{i=1}{\overset{Arity(f)}{\boxed{y_i}}})$
—— $\mathfrak{A} \vDash (\equiv f(\underset{i=1}{\overset{Arity(f)}{\boxed{x_i}}}) f(\underset{i=1}{\overset{Arity(f)}{\boxed{y_i}}}))[s]$
—- $\mathfrak{A} \vDash ((\wedge \underset{i=1}{\overset{Arity(f)}{\boxed{\equiv x_i y_i}}}) \implies (\equiv f(\underset{i=1}{\overset{Arity(f)}{\boxed{x_i}}}) f(\underset{i=1}{\overset{Arity(f)}{\boxed{y_i}}})))[s]$
— $\vDash (\wedge \underset{i=1}{\overset{Arity(f)}{\boxed{\equiv x_i y_i}}}) \implies (\equiv f(\underset{i=1}{\overset{Arity(f)}{\boxed{x_i}}}) f(\underset{i=1}{\overset{Arity(f)}{\boxed{y_i}}}))$
— $\vDash \phi$
– If $\phi \in E3$, then
— $\phi :\equiv (\wedge \underset{i=1}{\overset{Arity(R)}{\boxed{\equiv x_i y_i}}}) \implies (\equiv R(\underset{i=1}{\overset{Arity(R)}{\boxed{x_i}}}) R(\underset{i=1}{\overset{Arity(R)}{\boxed{y_i}}}))$
— For any structure $\mathfrak{A}$, for any variable-universe assignment s,
—- If $\mathfrak{A} \vDash (\wedge \underset{i=1}{\overset{Arity(f)}{\boxed{\equiv x_i y_i}}})[s]$, then
—— For any $i \in \left\{ \underset{i=1}{\overset{Arity(f)}{\boxed{i}}} \right\}$, $\mathfrak{A} \vDash (\equiv x_i y_i)[s]$
—— For any $i \in \left\{ \underset{i=1}{\overset{Arity(f)}{\boxed{i}}} \right\}$, $s(x_i) = \overline{s}(x_i) = \overline{s}(y_i) = s(y_i)$
—— $\left\langle \underset{i=1}{\overset{Arity(f)}{\boxed{\overline{s}(x_i)}}} \right\rangle \in R^{\mathfrak{A}}$ iff $\left\langle \underset{i=1}{\overset{Arity(f)}{\boxed{\overline{s}(y_i)}}} \right\rangle \in R^{\mathfrak{A}}$

—– If $\left\langle \boxed{\overline{s}(x_i)}_{i=1}^{Arity(f)} \right\rangle \in R^{\mathfrak{A}}$, then $\left\langle \boxed{\overline{s}(y_i)}_{i=1}^{Arity(f)} \right\rangle \in R^{\mathfrak{A}}$
—– If $\mathfrak{A} \models (R(\boxed{x_i}_{i=1}^{Arity(R)}))[s]$, then $\mathfrak{A} \models (R(\boxed{y_i}_{i=1}^{Arity(R)}))[s]$
—– $\mathfrak{A} \models (R(\boxed{x_i}_{i=1}^{Arity(R)}) \implies R(\boxed{y_i}_{i=1}^{Arity(R)}))[s]$
—- $\mathfrak{A} \models ((\wedge\boxed{\equiv x_i y_i}_{i=1}^{Arity(f)}) \implies (R(\boxed{x_i}_{i=1}^{Arity(R)}) \implies R(\boxed{y_i}_{i=1}^{Arity(R)})))[s]$
— $\models (\wedge\boxed{\equiv x_i y_i}_{i=1}^{Arity(f)}) \implies (R(\boxed{x_i}_{i=1}^{Arity(R)}) \implies R(\boxed{y_i}_{i=1}^{Arity(R)}))$
— $\models \phi$
– If $\phi \in Q1$, then
— $\phi :\equiv (\forall x\phi) \implies |\phi|_t^x$ and $Subbable(t, x, \phi)$
— For any structure $\mathfrak{A}$, for any variable-universe assignment s,
—- If $\mathfrak{A} \models (\forall x\phi)[s]$, then
—– For any $a \in A$, $\mathfrak{A} \models \phi[s[x|a]]$
—– $\mathfrak{A} \models \phi[s[x|\overline{s}(t)]]$
¡NEW THEOREM¿ —– $\mathfrak{A} \models (|\phi|_t^x)[s]$
—- $\mathfrak{A} \models ((\forall x\phi) \implies |\phi|_t^x)[s]$
— $\models (\forall x\phi) \implies |\phi|_t^x$
— $\models \phi$
– If $\phi \in Q2$, then
— $\phi :\equiv |\phi|_t^x \implies (\exists x\phi)$ and $Subbable(t, x, \phi)$
— For any structure $\mathfrak{A}$, for any variable-universe assignment s,
—- If $\mathfrak{A} \models (|\phi|_t^x)[s]$, then
—– $\mathfrak{A} \models \phi[s[x|\overline{s}(t)]]$
¡NEW THEOREM¿ —– $\overline{s}(t) \in A$
—– There exists $a \in A$, $\mathfrak{A} \models \phi[s[x|a]]$
—– $\mathfrak{A} \models (\exists x\phi)[s]$
—- $\mathfrak{A} \models (|\phi|_t^x \implies (\exists x\phi))[s]$
— $\models |\phi|_t^x \implies (\exists x\phi)$
— $\models \phi$
==

2.4.2 (Metatheorem) Rules of inference are closed under validity

- If $\langle \Gamma, \phi \rangle \in R_I$, then $\Gamma \models \phi$
– If $\langle \Gamma, \phi \rangle \in PC$, then
— $\Gamma \models_{PC} \phi$
— $\models_{PC} (\wedge\boxed{\gamma}_{\gamma\in\Gamma}) \implies \phi$
¡NEW THEOREM¿ — $\models (\wedge\boxed{\gamma}_{\gamma\in\Gamma}) \implies \phi$
— If $\models \Gamma$, then
—- For any $\gamma \in \Gamma$, $\models \gamma$
—- $\models (\wedge\boxed{\gamma}_{\gamma\in\Gamma})$
—- $\models \phi$
— $\Gamma \models \phi$
– If $\langle \Gamma, \phi \rangle \in QR1$, then
— $\phi :\equiv \alpha \implies (\forall x\beta)$ and $\Gamma = \{\alpha \implies \beta\}$ and $\widetilde{free(x, \alpha)}$
— For any structure $\mathfrak{A}$, if $\mathfrak{A} \models \Gamma$, then
—- $\mathfrak{A} \models \alpha \implies \beta$
—- For any variable-universe assignment s, $\mathfrak{A} \models (\alpha \implies \beta)[s]$
—- For any variable-universe assignment s',
—– If $\mathfrak{A} \models \alpha[s']$, then
—— For any $a \in A$,

——- $\mathfrak{A} \models (\alpha \implies \beta)[s'[x|a]]$
——- If $\mathfrak{A} \models \alpha[s'[x|a]]$, then $\mathfrak{A} \models \beta[s'[x|a]]$
——- $\mathfrak{A} \models \alpha[s']$ iff $\mathfrak{A} \models \alpha[s'[x|a]]$
¡NOT FREE IN ALPHA¿ ——- $\mathfrak{A} \models \beta[s'[x|a]]$
—— $\mathfrak{A} \models (\forall x\beta)[s']$
—– $\mathfrak{A} \models (\alpha \implies \forall x\beta)[s']$
—- $\mathfrak{A} \models \alpha \implies \forall x\beta$
—- $\mathfrak{A} \models \phi$
— $\Gamma \models \phi$
– If $\langle \Gamma, \phi \rangle \in QR2$, then
— $\phi :\equiv (\exists x\beta) \implies \alpha$ and $\Gamma = \{\beta \implies \alpha\}$ and $\widetilde{free(x, \alpha)}$
— For any structure $\mathfrak{A}$, if $\mathfrak{A} \models \Gamma$, then
—- $\mathfrak{A} \models \beta \implies \alpha$
—- For any variable-universe assignment s, $\mathfrak{A} \models (\beta \implies \alpha)[s]$
—- For any variable-universe assignment s',
—– If $\mathfrak{A} \models (\exists x\beta)[s']$, then
—— There exists $a \in A$,
——- $\mathfrak{A} \models (\beta \implies \alpha)[s'[x|a]]$
——- If $\mathfrak{A} \models \beta[s'[x|a]]$, then $\mathfrak{A} \models \alpha[s'[x|a]]$
——- $\mathfrak{A} \models \beta[s'[x|a]]$
——- $\mathfrak{A} \models \alpha[s'[x|a]]$
——- $\mathfrak{A} \models \alpha[s'[x|a]]$ iff $\mathfrak{A} \models \alpha[s']$
¡NOT FREE IN ALPHA¿ ——- $\mathfrak{A} \models \alpha[s']$
—— $\mathfrak{A} \models \alpha[s']$
—– $\mathfrak{A} \models ((\exists x\beta) \implies \alpha)[s']$
—- $\mathfrak{A} \models (\exists x\beta) \implies \alpha$
—- $\mathfrak{A} \models \phi$
— $\Gamma \models \phi$
==

2.4.3 (Definition) Soundness

- If $\Sigma \vdash \phi$, then $\Sigma \models \phi$
==

2.4.4 (Metatheorem) Soundness of First-order Logic

- If $\Sigma \vdash \phi$, then $\Sigma \models \phi$
- $\{\phi : \Sigma \vdash \phi\} \subseteq \{\phi : \Sigma \models \phi\}$
– If $\phi \in \Lambda$, then $\models \phi$, then $\Sigma \models \phi$
– If $\phi \in \Sigma$, then $\Sigma \models \phi$
– If $\langle \Gamma, \phi \rangle \in R_I$ and $\Gamma \subseteq \{\phi : \Sigma \models \phi\}$, then
— $\Sigma \models \Gamma$
— $\Gamma \models \phi$
— $\Sigma \models \phi$
- Brain dead syntactic manipulation corresponding to truth
==

2.5 Two Technical Lemmas

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2.5.1 (Metatheorem) Substitution and modification identity on assignments

- $\overline{s}(|u|^x_t) = \overline{s[x|\overline{s}(t)]}(u)$
– If $u \in Var$ and $u :\equiv x$, then
— $\overline{s}(|x|^x_t) = \overline{s}(t) = \overline{s[x|\overline{s}(t)]}(x)$
— $\overline{s}(|u|^x_t) = \overline{s[x|\overline{s}(t)]}(u)$
– If $u \in Var$ and $u :\equiv y :\not\equiv x$, then

— $\overline{s}(|y|^x_t) = \overline{s}(y) = \overline{s[x|\overline{s}(t)]}(y)$
— $\overline{s}(|u|^x_t) = \overline{s[x|\overline{s}(t)]}(u)$
– If $u \in Const$ and $u :\equiv c$, then
— $\overline{s}(|c|^x_t) = \overline{s}(c) = c^{\mathfrak{A}}$
— $\overline{s}(|u|^x_t) = \overline{s[x|\overline{s}(t)]}(u)$
– If $u :\equiv f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}$ and $\left\{\overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}\right\} \subseteq \left\{r : \overline{s}(|r|^x_t) = \overline{s[x|\overline{s}(t)]}(r)\right\}$, then
— $f^{\mathfrak{A}}(\overset{Arity(f)}{\underset{i=1}{\boxed{\overline{s}(|t_i|^x_t)}}}) = f^{\mathfrak{A}}(\overset{Arity(f)}{\underset{i=1}{\boxed{\overline{s[x|\overline{s}(t)]}(t_i)}}})$
— $\overline{s}(\left|f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}\right|^x_t) = \overline{s}(f \overset{Arity(f)}{\underset{i=1}{\boxed{|t_i|^x_t}}}) = \overline{s[x|\overline{s}(t)]}(f \overset{Arity(f)}{\underset{i=1}{\boxed{(t_i)}}})$
— $\overline{s}(|u|^x_t) = \overline{s[x|\overline{s}(t)]}(u)$
==

2.5.2 (Metatheorem) Substitution and modification identity on models

- If $Subbable(t, x, \phi)$, then $\mathfrak{A} \models |\phi|^x_t[s]$ iff $\mathfrak{A} \models \phi[s[x|\overline{s}(t)]]$
– If $\phi :\equiv\equiv u_1 u_2$, then
— $\overline{s}(|u_1|^x_t) = \overline{s}(|u_2|^x_t)$ iff $\overline{s[x|\overline{s}(t)]}(u_1) = \overline{s[x|\overline{s}(t)]}(u_2)$
— $\mathfrak{A} \models (|\equiv u_1 u_2|^x_t)[s]$ iff $\mathfrak{A} \models (\equiv |u_1|^x_t |u_2|^x_t)[s]$ iff $\mathfrak{A} \models (\equiv u_1 u_2)[s[x|\overline{s}(t)]]$
— $\mathfrak{A} \models |\phi|^x_t[s]$ iff $\mathfrak{A} \models \phi[s[x|\overline{s}(t)]]$
– If $\phi :\equiv R \overset{Arity(R)}{\underset{i=1}{\boxed{u_i}}}$, then
— $\left\langle \overset{Arity(R)}{\underset{i=1}{\boxed{\overline{s}(|u_i|^x_t)}}} \right\rangle \in R^{\mathfrak{A}}$ iff $\left\langle \overset{Arity(R)}{\underset{i=1}{\boxed{\overline{s[x|\overline{s}(t)]}(u_i)}}} \right\rangle \in R^{\mathfrak{A}}$
— $\mathfrak{A} \models (\left|R \overset{Arity(R)}{\underset{i=1}{\boxed{u_i}}}\right|^x_t)[s]$ iff $\mathfrak{A} \models (R \overset{Arity(R)}{\underset{i=1}{\boxed{|u_i|^x_t}}})[s]$ iff $\mathfrak{A} \models (R \overset{Arity(R)}{\underset{i=1}{\boxed{u_i}}})[s[x|\overline{s}(t)]]$
— $\mathfrak{A} \models |\phi|^x_t[s]$ iff $\mathfrak{A} \models \phi[s[x|\overline{s}(t)]]$
– If $\phi :\equiv \neg\alpha$ and $\{\alpha\} \subseteq \{\gamma :$ if $(Subbable(t, x, \gamma))$, then $(\mathfrak{A} \models |\gamma|^x_t[s]($ iff $)\mathfrak{A} \models \gamma[s[x|\overline{s}(t)]])\}$, then
— $Subbable(t, x, \alpha)$
— $\mathfrak{A} \models |\alpha|^x_t[s]$ iff $\mathfrak{A} \models \alpha[s[x|\overline{s}(t)]]$
— $\mathfrak{A} \not\models |\alpha|^x_t[s]$ iff $\mathfrak{A} \not\models \alpha[s[x|\overline{s}(t)]]$
— $\mathfrak{A} \models (|\neg\alpha|^x_t)[s]$ iff $\mathfrak{A} \models (\neg|\alpha|^x_t)[s]$ iff $\mathfrak{A} \models (\neg\alpha)[s[x|\overline{s}(t)]]$
— $\mathfrak{A} \models |\phi|^x_t[s]$ iff $\mathfrak{A} \models \phi[s[x|\overline{s}(t)]]$
– If $\phi :\equiv \vee\alpha\beta$ and $\{\alpha, \beta\} \subseteq \{\gamma :$ if $(Subbable(t, x, \gamma))$, then $(\mathfrak{A} \models |\gamma|^x_t[s]($ iff $)\mathfrak{A} \models \gamma[s[x|\overline{s}(t)]])\}$
— $Subbable(t, x, \alpha)$
— $\mathfrak{A} \models |\alpha|^x_t[s]$ iff $\mathfrak{A} \models \alpha[s[x|\overline{s}(t)]]$
— $Subbable(t, x, \beta)$
— $\mathfrak{A} \models |\beta|^x_t[s]$ iff $\mathfrak{A} \models \beta[s[x|\overline{s}(t)]]$
— $(\mathfrak{A} \models |\alpha|^x_t[s]$ or $\mathfrak{A} \models |\beta|^x_t[s])$ iff $(\mathfrak{A} \models \alpha[s[x|\overline{s}(t)]]$ or $\mathfrak{A} \models \beta[s[x|\overline{s}(t)]])$
— $\mathfrak{A} \models (|\vee\alpha\beta|^x_t)[s]$ iff $\mathfrak{A} \models (\vee|\alpha|^x_t|\beta|^x_t)[s]$ iff $\mathfrak{A} \models (\vee\alpha\beta)[s[x|\overline{s}(t)]]$
— $\mathfrak{A} \models |\phi|^x_t[s]$ iff $\mathfrak{A} \models \phi[s[x|\overline{s}(t)]]$
– If $\phi :\equiv \forall y\alpha$ and $\{\alpha\} \subseteq \{\gamma :$ if $(Subbable(t, x, \gamma))$, then $(\mathfrak{A} \models |\gamma|^x_t[s]($ iff $)\mathfrak{A} \models \gamma[s[x|\overline{s}(t)]])\}$, then
— If $y :\equiv x$, then
—- $\mathfrak{A} \models |\forall y\alpha|^x_t[s]$ iff $\mathfrak{A} \models (\forall y\alpha)[s]$
¡DEF SUB¿ —- $\mathfrak{A} \models (\forall y\alpha)[s]$ iff $\mathfrak{A} \models (\forall y\alpha)[s[x|\overline{s}(t)]]$
¡THM AGREE ALL FREE¿ —- $\mathfrak{A} \models |\forall y\alpha|^x_t[s]$ iff $\mathfrak{A} \models (\forall y\alpha)[s[x|\overline{s}(t)]]$
—- $\mathfrak{A} \models |\phi|^x_t[s]$ iff $\mathfrak{A} \models \phi[s[x|\overline{s}(t)]]$
— If $y \underset{\sim}{:\not\equiv x}$, then
—- If $free(x, \phi)$, then
—— $\mathfrak{A} \models |\forall y\alpha|^x_t[s]$ iff $\mathfrak{A} \models (\forall y\alpha)[s]$
¡Substitutions of non-free variables is the identity¿ —— $\mathfrak{A} \models (\forall y\alpha)[s]$ iff $\mathfrak{A} \models (\forall y\alpha)[s[x|\overline{s}(t)]]$
¡THM AGREE ALL FREE¿ —— $\mathfrak{A} \models |\forall y\alpha|^x_t[s]$ iff $\mathfrak{A} \models (\forall y\alpha)[s[x|\overline{s}(t)]]$
—— $\mathfrak{A} \models |\phi|^x_t[s]$ iff $\mathfrak{A} \models \phi[s[x|\overline{s}(t)]]$

—- If $\widetilde{occurs}(y,t)$ and $Subbable(t, x\alpha)$, then
—– For any $a \in A$, $\mathfrak{A} \vDash |\alpha|^x_t[(s[y|a])]$ iff $\mathfrak{A} \vDash \alpha[(s[y|a])[x|\overline{s}(t)]]$
¡IH WHERE s=s[y—a]¿ —– $\mathfrak{A} \vDash |\forall y\alpha|^x_t[s]$ iff $\mathfrak{A} \vDash (\forall y|\alpha|^x_t)[s]$ iff $\mathfrak{A} \vDash (\forall y\alpha)[s[x|\overline{s}(t)]]$
—– $\mathfrak{A} \vDash |\phi|^x_t[s]$ iff $\mathfrak{A} \vDash \phi[s[x|\overline{s}(t)]]$
==

2.6 Properties of Our Deductive System

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2.6.1 (Metatheorem) equiv is an equivalence relation

- For any $\{x, y, z\} \in Var$,
– $\vdash x \equiv x$
– $\vdash x \equiv y \implies y \equiv x$
– $\vdash (x \equiv y \wedge y \equiv z) \implies x \equiv z$
- Proof:
– $\vdash x \equiv x$
¡E1¿ – $\vdash x \equiv y \implies y \equiv x$
— $((x \equiv y) \wedge (x \equiv x)) \implies ((x \equiv x) \implies (y \equiv x))$
¡E3¿ — $x \equiv x$
¡E1¿ — $(x \equiv y) \implies ((x \equiv x) \implies (y \equiv x))$
¡PC¿ — $(x \equiv y) \implies y \equiv x$
¡PC¿ – $\vdash (x \equiv y \wedge y \equiv z) \implies (x \equiv z)$
— $(x \equiv x \wedge y \equiv z) \implies ((x \equiv y) \implies (x \equiv z))$
¡E3¿ — $x \equiv x$
¡E1¿ — $(y \equiv z) \implies ((x \equiv y) \implies (x \equiv z))$
¡PC¿ — $(y \equiv z \wedge x \equiv y) \implies (x \equiv z)$
¡PC¿ — $(x \equiv y \wedge y \equiv z) \implies (x \equiv z)$
¡PC¿ ==

2.6.2 (Metatheorem) Universal closure preserves deductiblity

- $\Sigma \vdash \phi$ iff $\Sigma \vdash \forall x\phi$
– If $\Sigma \vdash \phi$, then
— $\Sigma \vdash \phi$
— $\Sigma \vdash ((\forall z(z \equiv z)) \vee \neg(\forall z(z \equiv z))) \implies \phi$
¡PC¿ — $\Sigma \vdash ((\forall z(z \equiv z)) \vee \neg(\forall z(z \equiv z))) \implies \forall x\phi$
¡QR1¿ — $\Sigma \vdash ((\forall z(z \equiv z)) \vee \neg(\forall z(z \equiv z)))$
¡PC¿ — $\Sigma \vdash \forall x\phi$
¡PC¿ – If $\Sigma \vdash \forall x\phi$, then
— $\Sigma \vdash \forall x\phi$
— $\Sigma \vdash \forall x\phi \implies |\phi|^x_x$
¡Q1¿ — $\Sigma \vdash |\phi|^x_x$
¡PC¿ — $\Sigma \vdash \phi$
- $((\forall z(z \equiv z)) \vee \neg(\forall z(z \equiv z)))$ is a closed formula that is tautological
- Keep structures + variable-universe assignment functions in mind when interpreting universal closure deductions
- We can replace axioms with all sentences without changing the strength of the deductive system
==

2.6.3 (Metatheorem) Universal closure preserves strength of axioms

- $\Sigma \vdash \phi$ iff $UC(\Sigma) \vdash \phi$
- Proof:
– If $\Sigma \vdash \phi$, then — $UC(\Sigma) \vdash \Sigma$
— $UC(\Sigma) \vdash \phi$
– If $UC(\Sigma) \vdash \phi$, then
— $\Sigma \vdash UC(\Sigma)$
— $\Sigma \vdash \phi$

- We can universally close the set of formulas Σ and it will deduce the same as $UC(\Sigma)$ sentences

===

2.6.4 (Metatheorem) Deduction theorem

- If θ is a sentence, then $\Sigma \cup \{\theta\} \vdash \phi$ iff $\Sigma \vdash \theta \implies \phi$
– If $\Sigma \vdash \theta \implies \phi$, then
— $\Sigma \cup \{\theta\} \vdash \theta \implies \phi$
— $\Sigma \cup \{\theta\} \vdash \theta$
— $\Sigma \cup \{\theta\} \vdash \phi$
¡PC¿ – $\{\alpha : \Sigma \cup \{\theta\} \vdash \alpha\} \subseteq \{\alpha : \Sigma \vdash \theta \implies \alpha\}$
— If $\alpha \in \Lambda$, then
—- $\vdash \alpha$
—- $\Sigma \vdash \theta \implies \alpha$
¡PC¿ — If $\alpha \in \Sigma$, then
—- $\Sigma \vdash \alpha$
—- $\Sigma \vdash \theta \implies \alpha$
¡PC¿ — If $\alpha :\equiv \theta$, then
—- $\vdash \theta \implies \theta$
¡PC¿ —- $\Sigma \vdash \theta \implies \alpha$ — If $\langle \Gamma, \alpha \rangle \in PC$ and for any $\gamma \in \Gamma$, $\Sigma \vdash \theta \implies \gamma$, then
—- $\Sigma \implies \Gamma$
—- $\Sigma \implies \alpha$
¡PC¿ — If $\langle \Gamma, \alpha \rangle \in QR1$ and for any $\gamma \in \Gamma$, $\Sigma \vdash \theta \implies \gamma$, then
—- $\Gamma = \{\rho \implies \tau\}$
—- $\alpha :\equiv \rho \implies \forall x \tau$
—- $free(x, \rho)$
—- $\Sigma \vdash \theta \implies (\rho \implies \tau)$
—- $\Sigma \vdash (\theta \wedge \rho) \implies \tau$
¡PC¿ —- $free(x, \theta)$
—- $free(x, \theta \wedge \rho)$
—- $\Sigma \vdash (\theta \wedge \rho) \implies \forall x \tau$
¡QR1¿ —- $\Sigma \vdash \theta \implies (\rho \implies \forall x \tau)$
¡PC¿ —- $\Sigma \vdash \phi$
— If $\langle \Gamma, \alpha \rangle \in QR2$ and for any $\gamma \in \Gamma$, $\Sigma \vdash \theta \implies \gamma$, then
—- $\Gamma = \{\tau \implies \rho\}$
—- $\alpha :\equiv \exists x \tau \implies \rho$
—- $free(x, \rho)$
—- $\Sigma \vdash \theta \implies (\tau \implies \rho)$
—- $\Sigma \vdash (\theta \wedge \tau) \implies \rho$
¡PC¿ —- $\Sigma \vdash (\tau \wedge \theta) \implies \rho$
¡PC¿ —- $\Sigma \vdash \tau \implies (\theta \implies \rho)$
¡PC¿ —- $free(x, \theta)$
—- $free(x, \theta \wedge \rho)$
—- $\Sigma \vdash \exists x \tau \implies (\theta \implies \rho)$
¡QR2¿ —- $\Sigma \vdash (\exists x \tau \wedge \theta) \implies \rho$
¡PC¿ —- $\Sigma \vdash (\theta \wedge \exists x \tau) \implies \rho$
¡PC¿ —- $\Sigma \vdash \theta \implies (\exists x \tau \implies \rho)$
¡PC¿ —- $\Sigma \vdash \theta \implies \alpha$
– If $\Sigma \cup \{\theta\} \vdash \phi$, then $\Sigma \vdash \theta \implies \phi$

===

2.6.5 (Metatheorem) Proof by contradiction

– If $\Sigma \vdash \phi$, then
— $\Sigma \cup \{\neg\phi\} \vdash \phi$
— $\Sigma \cup \{\neg\phi\} \vdash \neg\phi$
— $\Sigma \cup \{\neg\phi\} \vdash ((\forall z(z \equiv z)) \wedge \neg(\forall z(z \equiv z)))$
¡PC¿ – If $\Sigma \cup \{\neg\phi\} \vdash ((\forall z(z \equiv z)) \wedge \neg(\forall z(z \equiv z)))$, then
— $\Sigma \vdash \neg\phi \implies ((\forall z(z \equiv z)) \wedge \neg(\forall z(z \equiv z)))$

¡Deduction theorem¿ — $\Sigma \vdash (\neg(\forall z(z \equiv z)) \vee (\forall z(z \equiv z))) \implies \phi$
¡PC¿ — $\Sigma \vdash \phi$
¡PC¿ ==

2.6.6 (Metatheorem) Strong to weak quantification

- If P is a 1-ary relation symbol, then $\vdash (\forall x(P(x))) \implies (\exists x(P(x)))$
- Proof:
– $\neg((\forall x(P(x))) \implies (\exists x(P(x)))) \vdash \neg(\neg(\forall x(P(x))) \vee (\exists x(P(x))))$
¡PC¿ – $\neg(\neg(\forall x(P(x))) \vee (\exists x(P(x)))) \vdash (\forall x(P(x)) \wedge \neg(\exists x(P(x)))$
¡PC¿ – $(\forall x(P(x)) \wedge \neg(\exists x(P(x))) :\equiv (\forall x(P(x)) \wedge \neg(\neg\forall x(\neg P(x)))$
– $(\forall x(P(x)) \wedge \neg(\neg\forall x(\neg P(x))) \vdash (\forall x(P(x)) \wedge (\forall x(\neg P(x)))$
¡PC¿ – $(\forall x(P(x)) \wedge (\forall x(\neg P(x)) \vdash ((\forall z(z \equiv z)) \wedge \neg(\forall z(z \equiv z)))$
¡PC¿ – $\neg((\forall x(P(x))) \implies (\exists x(P(x)))) \vdash ((\forall z(z \equiv z)) \wedge \neg(\forall z(z \equiv z)))$ ================================

2.6.7 (Metatheorem) Quantifier switcheroni

- $\forall x \forall y P(x, y) \vdash \forall y \forall x P(x, y)$
- Proof:
– $\vdash \forall x \forall y P(x, y) \implies \forall y \forall x P(x, y)$
¡PC¿ – $\forall x \forall y P(x, y) \vdash \forall y \forall x P(x, y)$
¡Deduction theorem¿ ==

2.6.8 (Metatheorem) Quantifier combineroni

- $\vdash (\forall x(P(x)) \wedge \forall x(Q(x))) \implies \forall x(P(x) \wedge Q(x))$
¡PC¿ ==

2.7 Non-logical Axioms

- The non-logical axioms characterizes the behavior of a specific theory
- Non-logical axioms have to be decidable as well
==

2.7.1 (Definition) Weak number theory

- The non-logical axioms of Number theory $N \subseteq Form(\mathcal{L}_{NT})$ consists of:
– $\forall x \neg(Sx \equiv 0)\forall\forall$
– $\forall x \forall y(Sx \equiv Sy \implies x \equiv y)$
– $\forall x(x + 0 \equiv 0)$
– $\forall x \forall y(x + Sy \equiv S(x + y))$
– $\forall x(x \centerdot 0 \equiv 0)$
– $\forall x \forall y(x \centerdot Sy \equiv (x \centerdot y) + x)$
– $\forall x(xE0 \equiv S0)$
– $\forall x \forall y(xE(Sy) \equiv (xEy) \centerdot x)$
– $\forall x(\neg x < 0)$
– $\forall x \forall y(x < Sy \iff (x < y \vee x \equiv y))$
– $\forall x \forall y(x < y \vee x \equiv y \vee y < x)$
- $\hat{a} :\equiv \overleftarrow{a}$
==

2.7.2 (Metatheorem) Weak number theory theorems

- For any natural numbers a, b,
– If $a = b$, then $N \vdash \hat{a} \equiv \hat{b}$
– If $a \neq b$, then $N \vdash \neg(\hat{a} \equiv \hat{b})$
– If $a < b$, then $N \vdash \hat{a} < \hat{b}$
– BACKLOG: ... - BACKLOG: Proof:
==

2.7.3 (Metatheorem) Weakness of weak number theory 1

- $N \nvdash \neg(x < x)$
- BACKLOG: p.298 Construct a structure $\mathfrak{A}$ that satisfies $\mathfrak{A} \models N$ and $\mathfrak{A} \not\models \forall x \neg(x < x)$

==

2.7.4 (Metatheorem) Weakness of weak number theory 2

- $N \nvdash (x + y) \equiv (y + x)$
- BACKLOG: p.298 Construct a structure $\mathfrak{A}$ that satisfies $\mathfrak{A} \models N$ and $\mathfrak{A} \not\models (x + y) \equiv (y + x)$

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Chapter 3

Completeness and Compactness

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3.1 Naively

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3.1.1 (Definition) Completeness

- If $\Sigma \models \phi$, then $\Sigma \vdash \phi$

==

3.2 Completeness

==

3.2.1 (Definition) Contradictory sentence

- The sentence $\overleftarrow{\perp} :\equiv ((\forall z(z \equiv z)) \wedge \neg(\forall z(z \equiv z)))$
- For any language $\mathcal{L}$, $\overleftarrow{\perp} \in Sent(\mathcal{L})$

==

3.2.2 (Definition) Inconsistent and unsatisfiable

- The set of formulas Σ is inconsistent iff $\Sigma \vdash \overleftarrow{\perp}$
- The set of formulas Σ is consistent iff $\Sigma \nvdash \overleftarrow{\perp}$
- The set of formulas Σ is unsatisfiable iff $\Sigma \models \overleftarrow{\perp}$
- The set of formulas Σ is satisfiable iff $\Sigma \not\models \overleftarrow{\perp}$

==

3.2.3 (Metatheorem) Contradiction has no model

- $\mathfrak{A} \not\models \overleftarrow{\perp}$
- Proof:

– If $\mathfrak{A} \models \overleftarrow{\perp}$, then
— $\mathfrak{A} \models ((\forall z(z \equiv z)) \wedge \neg(\forall z(z \equiv z)))$
— $\mathfrak{A} \models (\forall z(z \equiv z))$ and $\mathfrak{A} \models \neg(\forall z(z \equiv z))$
¡Definition¿ — $\mathfrak{A} \not\models (\forall z(z \equiv z))$
¡Definition¿ — Not $\mathfrak{A} \models (\forall z(z \equiv z))$
¡Definition¿ — $\mathfrak{A} \models (\forall z(z \equiv z))$ and not $\mathfrak{A} \models (\forall z(z \equiv z))$
— CONTR
– $\mathfrak{A} \not\models ((\forall z(z \equiv z)) \wedge \neg(\forall z(z \equiv z)))$
– $\mathfrak{A} \not\models \overleftarrow{\perp}$

==

3.2.4 (Metatheorem) Unsatisfiable equivalence

- $\Sigma \vDash \overleftarrow{\bot}$ iff for any $\mathfrak{A}$, $\mathfrak{A} \nvDash \Sigma$
- Proof:
– $\Sigma \vDash \overleftarrow{\bot}$ iff
— For any $\mathfrak{A}$, $\mathfrak{A} \nvDash \Sigma$ iff
— For any $\mathfrak{A}$, if $\mathfrak{A} \vDash \Sigma$, then $\mathfrak{A} \vDash \overleftarrow{\bot}$ iff
— For any $\mathfrak{A}$, no $\mathfrak{A} \vDash \Sigma$ or $\mathfrak{A} \vDash \overleftarrow{\bot}$ iff
— For any $\mathfrak{A}$, $\mathfrak{A} \nvDash \Sigma$
- $\Sigma \nvDash \overleftarrow{\bot}$ iff there exists $\mathfrak{A}$, $\mathfrak{A} \vDash \Sigma$
==

3.2.5 (Metatheorem) Completeness of First-order Logic: Proof lemma schema

- Prove: (I) If $UC(\Sigma) \nvdash \overleftarrow{\bot}$, then there exists $\mathfrak{A}$, $\mathfrak{A} \vDash UC(\Sigma)$
- Corollaries: $\Sigma \vDash \phi$, then $\Sigma \vdash \phi$
– If $UC(\Sigma) \nvdash \overleftarrow{\bot}$,
— There exists $\mathfrak{A}$, $\mathfrak{A} \vDash UC(\Sigma)$ and $\mathfrak{A} \nvDash \overleftarrow{\bot}$
¡Contradiction has no model¿ — Not for any $\mathfrak{A}$, if $\mathfrak{A} \vDash UC(\Sigma)$, then $\mathfrak{A} \vDash \overleftarrow{\bot}$
¡Definition¿ — $UC(\Sigma) \nvDash \overleftarrow{\bot}$
¡Definition¿ – If $UC(\Sigma) \nvdash \overleftarrow{\bot}$, then $UC(\Sigma) \nvDash \overleftarrow{\bot}$
¡Abbreviate¿ – If $UC(\Sigma) \vDash \overleftarrow{\bot}$, then $UC(\Sigma) \vdash \overleftarrow{\bot}$
¡Contraposition¿ – $UC(\Sigma) \vDash \overleftarrow{\bot}$ iff
— For any $\mathfrak{A}$, if $\mathfrak{A} \vDash UC(\Sigma)$, then $\mathfrak{A} \vDash \overleftarrow{\bot}$ iff
¡Definition¿ — For any $\mathfrak{A}$, if $\mathfrak{A} \vDash \Sigma$ or $\mathfrak{A} \vDash \overleftarrow{\bot}$ iff
¡Universal closure preserves validity¿ — $\Sigma \vDash \overleftarrow{\bot}$
¡Definition¿ – $UC(\Sigma) \vDash \overleftarrow{\bot}$ iff $\Sigma \vDash \overleftarrow{\bot}$
¡Abbreviate¿ – If $\Sigma \vDash \overleftarrow{\bot}$, then $UC(\Sigma) \vdash \overleftarrow{\bot}$
¡Equivalence¿ – $UC(\Sigma) \vdash \overleftarrow{\bot}$ iff $\Sigma \vdash \overleftarrow{\bot}$
¡Universal closure preserves strength of axioms¿ – If $\Sigma \vDash \overleftarrow{\bot}$, then $\Sigma \vdash \overleftarrow{\bot}$
¡Equivalence¿ – If $\Sigma \vDash \phi$, then
— For any $\mathfrak{A}$,
—- If $\mathfrak{A} \vDash \Sigma$, then $\mathfrak{A} \vDash \phi$
—- If $\mathfrak{A} \vDash \Sigma \cup \{\neg\phi\}$, then
—– $\mathfrak{A} \vDash \Sigma$
—– $\mathfrak{A} \vDash \phi$
—– $\mathfrak{A} \vDash \neg\phi$
—– $\mathfrak{A} \nvDash \phi$
—– $\mathfrak{A} \vDash \phi$ and $\mathfrak{A} \nvDash \phi$
—– $\mathfrak{A} \vDash \phi$ and not $\mathfrak{A} \vDash \phi$
—– CONTR
—- $\mathfrak{A} \nvDash \Sigma \cup \{\neg\phi\}$
— For any $\mathfrak{A}$, $\mathfrak{A} \nvDash \Sigma \cup \{\neg\phi\}$
¡Abbreviate¿ — $\Sigma \cup \{\neg\phi\} \vDash \overleftarrow{\bot}$
¡Unsatisfiable equivalence¿ — $\Sigma \cup \{\neg\phi\} \vdash \overleftarrow{\bot}$
— $\Sigma \vdash \phi$
¡Proof by contradiction¿ – If $\Sigma \vDash \phi$, then $\Sigma \vdash \phi$
¡Abbreviate¿ ==

3.2.6 (Definition) Henkin theory for countable language

- A theory with added constants and axioms to make it easier to model with a universe of variable free terms
- $\Sigma' \subseteq Sent(\mathcal{L}')$ is the Henkin theory of $\Sigma \subseteq Sent(\mathcal{L})$ iff – $\mathcal{L}'$ construction: language with Henkin constants
— $\mathcal{L}_0 = \mathcal{L}$
— $\mathcal{L}_{i+1} = \mathcal{L}_i \cup^{Const} \left\{c_{(i,j)} : j \in \mathbb{N}\right\}$
— $\mathcal{L}' = \cup^{Const}_{i\in\mathbb{N}} \mathcal{L}_i$
– $\hat{\Sigma}$ construction: theory with Henkin axioms

— $\Sigma_0 = \Sigma$
— $H_{i+1} = \left\{ (\exists x \theta_j \implies |\theta_j|^x_{c_{(i,j)}}) : \exists x \theta_j \in Sent(\mathcal{L}_i) \right\}$
— $\Sigma_{i+1} = \Sigma_i \cup H_{i+1}$
— $\hat{\Sigma} = \cup_{i \in \mathbb{N}} \Sigma_i$
– Σ' construction: theory with chosen enumerated axioms
— $\Sigma^0 = \hat{\Sigma}$
— $\alpha_i \in Sent(\mathcal{L}')$
— $\Sigma^{i+1} = \Sigma^i \cup \{\alpha_i\}$ iff $\Sigma^i \cup \{\alpha_i\} \nvdash \overleftarrow{\bot}$
— $\Sigma^{i+1} = \Sigma^i \cup \{\neg\alpha_i\}$ iff $\Sigma^i \cup \{\alpha_i\} \vdash \overleftarrow{\bot}$
— $\Sigma' = \cup_{i \in \mathbb{N}} \Sigma^i$
==

3.2.7 (Definition) Deduction language notation

- $\Sigma \vdash_{\mathcal{L}} \phi$ abbreviates $\phi \in Cl(\Sigma \cup \Lambda(\mathcal{L}), RI(\mathcal{L})))$
==

3.2.8 (Metatheorem) Expansion by Henkin constants preserves consistency

- If $\Sigma \subseteq Sent(\mathcal{L})$, then if $\Sigma \nvdash_{\mathcal{L}} \overleftarrow{\bot}$, then $\Sigma \nvdash_{\mathcal{L}'} \overleftarrow{\bot}$
- Proof:
– If $\Sigma \subseteq Sent(\mathcal{L})$ and $\Sigma \nvdash_{\mathcal{L}} \overleftarrow{\bot}$, then
— If $\Sigma \vdash_{\mathcal{L}'} \overleftarrow{\bot}$, then
—- There exists D', D' has the smallest number n of added Henkin constants that satisfies $\overleftarrow{\bot} \in D'$
—- If $n = 0$, then
—— $\Sigma \vdash_{\mathcal{L}'} \overleftarrow{\bot}$ iff $\Sigma \vdash_{\mathcal{L}} \overleftarrow{\bot}$
—— $\Sigma \vdash_{\mathcal{L}} \overleftarrow{\bot}$
—— $\Sigma \nvdash_{\mathcal{L}} \overleftarrow{\bot}$
—— Not $\Sigma \vdash_{\mathcal{L}} \overleftarrow{\bot}$ and $\Sigma \vdash_{\mathcal{L}} \overleftarrow{\bot}$
—— CONTR
—- If $n > 0$, then
—— There exists c, c is an added constant that occurs in D'
—— There exists v, v is a variable that does not occur in D'
¡INFINITE VARS¿ —— There exists D, $D = \left\langle \boxed{d_i : |d_i|^v_c :\equiv d'_i}_{i=1}^{|D'|} \right\rangle$
—— For any $d_i \in \left\{ \boxed{d_i}_{i=1}^{|D|} \right\}$,
——– If $d'_i \in \Lambda$, then
——– If $d'_i \in E1 \cup E2 \cup E3$, then
——— $d'_i :\equiv d_i$
——— $d_i \in \Sigma$
——– If $d'_i \in Q1$, then
——— $d'_i :\equiv ((\forall x \phi') \implies |\phi'|^x_t)$
——— $Subbable(t, x, \phi')$
——— $Subbable(t, x, \phi)$
——— $d_i :\equiv ((\forall x \phi) \implies |\phi|^x_t)$
——— $d_i \in Q1$
——– If $d'_i \in Q2$, then proof isomorphic to $d'_i \in Q1$
——– If $d'_i \in \Sigma$, then
——– $d'_i :\equiv d_i$
——– $d_i \in \Sigma$
——– If $\langle \Gamma', d'_i \rangle \in R_I$, then
——– If $\langle \Gamma', d'_i \rangle \in PR$, then
——– $\Gamma' \vDash_{PC} d'_i$
——– $\Gamma \vDash_{PC} d_i$
——– $\langle \Gamma, d_i \rangle \in PR$
——– If $\langle \Gamma', d'_i \rangle \in QR1$, then

—— $\Gamma' = \{\psi' \implies \phi'\}$
—— $\Gamma' \subseteq \left\{\overset{i-1}{\underset{j=1}{\boxed{d'_j}}}\right\}$
—— $d'_i = \psi' \implies (\forall x \phi')$
—— $free(x, \psi')$
—— $free(x, \psi)$
—— $\Gamma = \{\psi \implies \phi\}$
—— $\Gamma \subseteq \left\{\overset{i-1}{\underset{j=1}{\boxed{d_j}}}\right\}$
—— $d_i = \psi \implies (\forall x \phi)$
—— $\langle \Gamma, d_i \rangle \in QR1$
——- If $\langle \Gamma', d'_i \rangle \in QR2$, then proof isomorphic to $\langle \Gamma', d'_i \rangle \in QR1$
—– D has $n-1$ added constants for a deduction of $\overleftarrow{\perp}$
—– $n \leq n-1$
—– CONTR
— $\Sigma \nvdash_{\mathcal{L}'} \overleftarrow{\perp}$
¡Metaproof by contradiction¿ – If $\Sigma \subseteq Sent(\mathcal{L})$, then if $\Sigma \nvdash_{\mathcal{L}} \overleftarrow{\perp}$, then $\Sigma \nvdash_{\mathcal{L}'} \overleftarrow{\perp}$
¡Implication over conjunction¿ ===

3.2.9 (Metatheorem) Expansion by Henkin axioms preserves consistency

- If $\Sigma \subseteq Sent(\mathcal{L}')$, then if $\Sigma \nvdash \overleftarrow{\perp}$, then $\hat{\Sigma} \nvdash \overleftarrow{\perp}$
- Proof:
– If $\Sigma \subseteq Sent(\mathcal{L}')$ and $\Sigma \nvdash \overleftarrow{\perp}$, then
— If $\hat{\Sigma} \vdash \overleftarrow{\perp}$, then
—- There exists n, n is the smallest number of added Henkin axioms for any deduction of $\overleftarrow{\perp}$
—- There exists H and α, $|H \cup \{\alpha\}| = n$ and $\Sigma \cup H \cup \{\alpha\} \vdash \overleftarrow{\perp}$
—- There exists v, v is a variable that does not occur in Σ
¡INFINITE VARS¿ —- There exists c, $\alpha :\equiv \exists x \phi \implies |\phi|^x_c$
—- $\Sigma \cup H \vdash \neg \alpha$
¡Proof by contradiction¿ —- $\Sigma \cup H \vdash \neg(\exists x \phi \implies |\phi|^x_c)$
—- $\Sigma \cup H \vdash (\exists x \phi \wedge \neg|\phi|^x_c)$
¡PC¿ —- $\Sigma \cup H \vdash \exists x \phi$
¡PC¿ —– $\Sigma \cup H \vdash \neg \forall x \neg \phi$
—- $\Sigma \cup H \vdash \neg|\phi|^x_c$
¡PC¿ —– $\Sigma \cup H \vdash \neg|\phi|^x_c$
—– $\Sigma \cup H \vdash \neg|\phi|^x_z$
—– $\Sigma \cup H \vdash \neg \forall z |\phi|^x_z$
—– $Subbable(z, x, \neg|\phi|^x_z)$
—– $\vdash (\forall z \neg|\phi|^x_z) \implies |\neg|\phi|^x_z|^z_x$
¡Q1¿ —– $\Sigma \cup H \vdash |\neg|\phi|^x_z|^z_x$
¡PC¿ —– $\Sigma \cup H \vdash \neg \phi$
¡PC¿ —– $\Sigma \cup H \vdash \forall x \neg \phi$
¡PC¿ —- $\Sigma \cup H \vdash (\neg \forall x \neg \phi) \wedge (\forall x \neg \phi)$
¡PC¿ —- $\Sigma \cup H \vdash \overleftarrow{\perp}$
¡PC¿ —- $|H| = n-1$
—- $n \leq n-1$
—- CONTR
— $\hat{\Sigma} \nvdash \overleftarrow{\perp}$
¡Metaproof by contradiction¿ – If $\Sigma \subseteq Sent(\mathcal{L}')$, then if $\Sigma \nvdash \overleftarrow{\perp}$, then $\hat{\Sigma} \nvdash \overleftarrow{\perp}$
¡Implication over conjunction¿ ===

3.2.10 (Metatheorem) Consistency from below

- If for any $i \in \mathbb{N}$, $\Sigma_i \nvdash \overleftarrow{\perp}$ and $\Sigma_i \subseteq \Sigma_{i+1}$, then $\cup_{i \in \mathbb{N}} \Sigma_i \nvdash \overleftarrow{\perp}$
- Proof:

– If for any $i \in \mathbb{N}$, $\Sigma_i \nvdash \overleftarrow{\bot}$ and $\Sigma_i \subseteq \Sigma_{i+1}$, then
— If $\cup_{i\in\mathbb{N}}\Sigma_i \vdash \overleftarrow{\bot}$, then
—- There exists $k \in \mathbb{N}$, $\Sigma_k \vdash \overleftarrow{\bot}$
¡DEDUCTIONS ARE FINITE CHOOSE SUFFICIENTLY LARGE K¿ —- $\Sigma_k \nvdash \overleftarrow{\bot}$
—- $\Sigma_k \vdash \overleftarrow{\bot}$ and not $\Sigma_k \vdash \overleftarrow{\bot}$
—- CONTR
— $\cup_{i\in\mathbb{N}}\Sigma_i \nvdash \overleftarrow{\bot}$
¡Metaproof by contradiction¿ ==

3.2.11 (Metatheorem) Consistency step

- If $\Sigma \nvdash \overleftarrow{\bot}$, then if $\Sigma \cup \{\alpha\} \vdash \overleftarrow{\bot}$, then $\Sigma \cup \{\neg\alpha\} \nvdash \overleftarrow{\bot}$
- Proof:
– If $\Sigma \nvdash \overleftarrow{\bot}$, then
— If $\Sigma \cup \{\alpha\} \vdash \overleftarrow{\bot}$ and $\Sigma \cup \{\neg\alpha\} \vdash \overleftarrow{\bot}$, then
—- $\Sigma \vdash \alpha \implies \overleftarrow{\bot}$
¡Deduction theorem¿ —- $\Sigma \vdash \neg\alpha \implies \overleftarrow{\bot}$
¡Deduction theorem¿ —- $\left\langle \alpha \implies \overleftarrow{\bot}, \neg\alpha \implies \overleftarrow{\bot}, \overleftarrow{\bot} \right\rangle \in PC$
—- $\Sigma \vdash \overleftarrow{\bot}$
¡PC¿ —- $\Sigma \vdash \overleftarrow{\bot}$ and not $\Sigma \vdash \overleftarrow{\bot}$
—- CONTR
— Not $(\Sigma \cup \{\alpha\} \vdash \overleftarrow{\bot}$ and $\Sigma \cup \{\neg\alpha\} \vdash \overleftarrow{\bot})$
¡Metaproof by contradiction¿ — If $\Sigma \cup \{\alpha\} \vdash \overleftarrow{\bot}$, then $\Sigma \cup \{\neg\alpha\} \nvdash \overleftarrow{\bot}$
¡Implication definition¿ - If $\Sigma \nvdash \overleftarrow{\bot}$, then if $\Sigma \cup \{\neg\alpha\} \vdash \overleftarrow{\bot}$, then $\Sigma \cup \{\alpha\} \nvdash \overleftarrow{\bot}$
==

3.2.12 (Metatheorem) Expansion by chosen enumerated axioms preserves consistency

- If $\Sigma \subseteq Sent(\mathcal{L}')$, then if $\hat{\Sigma} \nvdash \overleftarrow{\bot}$, then $\Sigma' \nvdash \overleftarrow{\bot}$
- Proof:
– If $\Sigma \subseteq Sent(\mathcal{L}')$, then
— If $k = 0$, $\Sigma^k = \Sigma^0 = \hat{\Sigma} \nvdash \overleftarrow{\bot}$
— If $k > 0$ and $\Sigma^k \nvdash \overleftarrow{\bot}$, then
—- If $\Sigma^{k+1} = \Sigma^k \cup \{\alpha_k\}$, then
—– $\Sigma^{k+1} = \Sigma^k \cup \{\alpha_k\} \nvdash \overleftarrow{\bot}$
—- If $\Sigma^{k+1} = \Sigma^k \cup \{\neg\alpha_k\}$, then
—– $\Sigma^k \cup \{\alpha_k\} \vdash \overleftarrow{\bot}$
—– $\Sigma^k \cup \{\neg\alpha_k\} \nvdash \overleftarrow{\bot}$
¡Consistency step¿ —- $\Sigma^{k+1} \nvdash \overleftarrow{\bot}$
— For any $k \in \mathbb{N}$, $\Sigma_k \nvdash \overleftarrow{\bot}$
¡Induction¿ — For any $k \in \mathbb{N}$, $\Sigma_k \subseteq \Sigma_{k+1}$
— For any $k \in \mathbb{N}$, $\Sigma_k \subseteq \Sigma_{k+1}$ and $\Sigma_k \nvdash \overleftarrow{\bot}$
— $\Sigma' = \cup_{i\in\mathbb{N}}\Sigma_i \nvdash \overleftarrow{\bot}$
¡Consistency from below¿ ==

3.2.13 (Metatheorem) Expansion by chosen enumerated axioms is deductively closed

- If $\phi \in Sent(\mathcal{L}')$, then $\phi \in \Sigma'$ iff $\Sigma' \vdash \phi$
- Proof:
– If $\phi \in \Sigma'$, then $\Sigma' \vdash \phi$
¡Definition¿ – If $\Sigma' \vdash \phi$, then
— There exists i, $\Sigma^i \vdash \phi$
¡DEDUCTIONS ARE FINITE¿ — $\Sigma^i \nvdash \overleftarrow{\bot}$
¡Expansion by chosen enumerated axioms preserves consistency¿ — $\Sigma^i \cup \neg\phi \vdash \overleftarrow{\bot}$
¡Proof by contradiction¿ — $\Sigma^i \cup \phi \nvdash \overleftarrow{\bot}$
¡Consistency step¿ — $\Sigma^{i+1} = \Sigma^i \cup \{\phi\}$

¡Definition¿ — $\phi \in \Sigma^{i+1}$
— $\phi \in \Sigma'$
¡Definition¿ ==

3.2.14 (Metatheorem) Expansion by chosen enumerated axioms is maximal

- If $\phi \in Sent(\mathcal{L}')$, then $\phi \in \Sigma'$ iff $\neg\phi \notin \Sigma'$
- Proof:
– If $\phi \in Sent(\mathcal{L}')$, then
— $\phi \in \Sigma'$ iff
—- $\Sigma' \vdash \phi$ iff
¡Expansion by chosen enumerated axioms is deductively closed¿ —- $\Sigma' \nvdash \neg\phi$ iff
¡Expansion by chosen enumerated axioms preserves consistency¿ —- $\neg\phi \notin \Sigma'$
¡Expansion by chosen enumerated axioms is deductively closed¿ — $\phi \in \Sigma'$ iff $\neg\phi \notin \Sigma'$
¡Abbreviate¿ ==

3.2.15 (Definition) VFT

- $VFT(\mathcal{L}') = \left\{t \in Term(\mathcal{L}') : (\text{ for any } v \in Var), \left(\widetilde{occurs(v,t)}\right)\right\}$
==

3.2.16 (Definition) VFTS relation

- $\langle t_1, t_2\rangle \in \sim \subseteq VFT(\mathcal{L}')^2$ iff $t_1 \equiv t_2 \in \Sigma'$
==

3.2.17 (Metatheorem) VFTS is an equivalence relation

- $\sim$ is an equivalence relation on $VFT(\mathcal{L}')^2$
- Proof:
– $t_1 \sim t_1$
— $\Sigma' \vdash x \equiv x$
¡E1¿ — $\Sigma' \vdash \forall x(x \equiv x)$
¡Universal closure preserves deductiblity¿ — $Subbable(t_1, x, x \equiv x)$
¡Definition¿ — $\Sigma' \vdash \forall x(x \equiv x) \implies |x \equiv x|^x_{t_1}$
¡Q1¿ — $\Sigma' \vdash |x \equiv x|^x_{t_1}$
¡PC¿ — $\Sigma' \vdash t_1 \equiv t_1$
¡Definition¿ — $t_1 \equiv t_1 \in \Sigma'$
¡Expansion by chosen enumerated axioms is deductively closed¿ — $t_1 \sim t_1$
¡Definition¿ – If $t_1 \sim t_2$, then $t_2 \sim t_1$
— If $t_1 \sim t_2$, then
—- $t_1 \equiv t_2 \in \Sigma'$
¡Definition¿ —- $\Sigma' \vdash t_1 \equiv t_2$
¡Expansion by chosen enumerated axioms is deductively closed¿ —- $\vdash t_1 \equiv t_2 \implies t_2 \equiv t_1$
¡equiv is an equivalence relation¿ —- $\langle t_1 \equiv t_2, t_1 \equiv t_2 \implies t_2 \equiv t_1, t_2 \equiv t_1\rangle \in PC$
—- $\Sigma' \vdash t_2 \equiv t_1$
¡PC¿ —- $t_2 \equiv t_1 \in \Sigma'$
¡Expansion by chosen enumerated axioms is deductively closed¿ —- $t_2 \sim t_1$
¡Definition¿ – If $t_1 \sim t_2$ and $t_2 \sim t_3$, then $t_1 \sim t_3$
— If $t_1 \sim t_2$ and $t_2 \sim t_3$, then
—- $t_1 \equiv t_2 \in \Sigma'$
¡Definition¿ —- $t_2 \equiv t_3 \in \Sigma'$
¡Definition¿ —- $\Sigma' \vdash t_1 \equiv t_2$
¡Expansion by chosen enumerated axioms is deductively closed¿ —- $\Sigma' \vdash t_2 \equiv t_3$
¡Expansion by chosen enumerated axioms is deductively closed¿ —- $\vdash (\vdash t_1 \equiv t_2 \wedge t_2 \equiv t_3) \implies t_1 \equiv t_3$
¡equiv is an equivalence relation¿ —- $\langle t_1 \equiv t_2, t_2 \equiv t_3, (t_1 \equiv t_2 \wedge t_2 \equiv t_3) \implies t_1 \equiv t_3, t_1 \equiv t_3\rangle \in PC$
—- $\Sigma' \vdash t_1 \equiv t_3$
¡PC¿ —- $t_1 \equiv t_3 \in \Sigma'$
¡Expansion by chosen enumerated axioms is deductively closed¿ —- $t_1 \sim t_3$
¡Definition¿ ==

3.2.18 (Definition) VFT in Sigma' equivalence class

- $[t]^{\sim} = \{s \in VFT(\mathcal{L}') : t \sim s\}$
==

3.2.19 (Definition) Henkin universe

- $A' = \{[t] : t \in VFT(\mathcal{L}')\}$
==

3.2.20 (Definition) Henkin ConstI

- $ConstI'$ is for any $c \in Const(\mathcal{L}')$, $c^{\mathfrak{A}'} = [c]$
==

3.2.21 (Definition) Henkin FuncI

- $FuncI'$ is for any $f \in Func(\mathcal{L}')$, $f^{\mathfrak{A}'}(\overset{Arity(f)}{\underset{i=1}{\boxed{[t_i]}}}) = [f \overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}]$
==

3.2.22 (Metatheorem) Henkin FuncI is a function

- $Func(f^{\mathfrak{A}'}, A'^{Arity(f)}, A')$
- Proof:

– For any $\left\{\overset{Arity(f)}{\underset{i=1}{\boxed{[t_i]}}}\right\}, \left\{\overset{Arity(f)}{\underset{i=1}{\boxed{[t'_i]}}}\right\} \subseteq A'$,

— If $\left\langle\overset{Arity(f)}{\underset{i=1}{\boxed{[t_i]}}}\right\rangle = \overset{Arity(f)}{\underset{i=1}{\boxed{[t'_i]}}}$, then

—- $\vdash (\wedge\overset{Arity(f)}{\underset{i=1}{\boxed{x_i \equiv y_i}}}) \implies (f(\overset{Arity(f)}{\underset{i=1}{\boxed{x_i}}}) \equiv f(\overset{Arity(f)}{\underset{i=1}{\boxed{y_i}}}))$

¡E2¿ —- $E :\equiv (\wedge\overset{Arity(f)}{\underset{i=1}{\boxed{x_i \equiv y_i}}}) \implies (f(\overset{Arity(f)}{\underset{i=1}{\boxed{x_i}}}) \equiv f(\overset{Arity(f)}{\underset{i=1}{\boxed{y_i}}}))$

—- $\vdash \overset{Arity(f)}{\underset{i=1}{\boxed{\forall x_i}}}\overset{Arity(f)}{\underset{i=1}{\boxed{\forall y_i}}} E$

¡Universal closure preserves deductiblity¿ —- For any t_i, t'_i, $Subbable(t_i, x_i, E)$ and $Subbable(t'_i, y_i, E)$

¡Definition¿ —- $\vdash \overset{Arity(f)}{\underset{i=1}{\boxed{\forall x_i}}}\overset{Arity(f)}{\underset{i=1}{\boxed{\forall y_i}}} E \implies \left| |E|^{\overset{Arity(f)}{\underset{i=1}{\boxed{x_i}}}}_{\overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}} \right|^{\overset{Arity(f)}{\underset{i=1}{\boxed{y_i}}}}_{\overset{Arity(f)}{\underset{i=1}{\boxed{t'_i}}}}$

¡Q1¿ —- $\vdash \left| |E|^{\overset{Arity(f)}{\underset{i=1}{\boxed{x_i}}}}_{\overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}} \right|^{\overset{Arity(f)}{\underset{i=1}{\boxed{y_i}}}}_{\overset{Arity(f)}{\underset{i=1}{\boxed{t'_i}}}}$

¡PC¿ —- $\vdash (\wedge\overset{Arity(f)}{\underset{i=1}{\boxed{t_i \equiv t'_i}}}) \implies (f(\overset{Arity(f)}{\underset{i=1}{\boxed{t_i}}}) \equiv f(\overset{Arity(f)}{\underset{i=1}{\boxed{t'_i}}}))$

¡Definition¿ —- $\vdash (\wedge\overset{Arity(f)}{\underset{i=1}{\boxed{t_i \equiv t'_i}}})$

—- $\vdash (f(\underset{i=1}{\overset{Arity(f)}{\boxed{t_i}}}) \equiv f(\underset{i=1}{\overset{Arity(f)}{\boxed{t'_i}}}))$
¡PC¿ —- $(f(\underset{i=1}{\overset{Arity(f)}{\boxed{t_i}}}) \equiv f(\underset{i=1}{\overset{Arity(f)}{\boxed{t'_i}}})) \in \Sigma'$
¡Expansion by chosen enumerated axioms is deductively closed¿ —- $(f(\underset{i=1}{\overset{Arity(f)}{\boxed{t_i}}}) \sim f(\underset{i=1}{\overset{Arity(f)}{\boxed{t'_i}}}))$
¡Definition¿ —- $[f(\underset{i=1}{\overset{Arity(f)}{\boxed{t_i}}})] = [f(\underset{i=1}{\overset{Arity(f)}{\boxed{t'_i}}})]$
¡Definition¿ — If $\left\langle \underset{i=1}{\overset{Arity(f)}{\boxed{[t_i]}}} \right\rangle = \underset{i=1}{\overset{Arity(f)}{\boxed{[t'_i]}}}$, then $[f(\underset{i=1}{\overset{Arity(f)}{\boxed{t_i}}})] = [f(\underset{i=1}{\overset{Arity(f)}{\boxed{t'_i}}})]$
¡Abbreviate¿ ==

3.2.23 (Definition) Henkin RelI

- $RelI'$ is for any $P \in Rel(\mathcal{L}')$, $\left\langle \underset{i=1}{\overset{Arity(P)}{\boxed{[t_i]}}} \right\rangle \in P^{\mathfrak{A}'}$ iff $P \underset{i=1}{\overset{Arity(P)}{\boxed{t_i}}} \in \Sigma'$
==

3.2.24 (Metatheorem) Henkin RelI is a relation

- $Rel(P^{\mathfrak{A}}, A'^{Arity(P)})$
- Proof:
– If $\left\langle \underset{i=1}{\overset{Arity(P)}{\boxed{[t_i]}}} \right\rangle = \left\langle \underset{i=1}{\overset{Arity(P)}{\boxed{[t'_i]}}} \right\rangle$, then
— $P \underset{i=1}{\overset{Arity(P)}{\boxed{t_i}}} \in \Sigma'$ iff $P \underset{i=1}{\overset{Arity(P)}{\boxed{t'_i}}} \in \Sigma'$
— $\left\langle \underset{i=1}{\overset{Arity(P)}{\boxed{[t_i]}}} \right\rangle \in P^{\mathfrak{A}'}$ iff $\left\langle \underset{i=1}{\overset{Arity(P)}{\boxed{[t'_i]}}} \right\rangle \in P^{\mathfrak{A}'}$
¡Definition¿ ==

3.2.25 (Definition) Henkin structure

- $\mathfrak{A}'$ is the $\mathcal{L}'$-structure $\langle A', ConstI', FuncI', RelI' \rangle$
==

3.2.26 (Metatheorem) Henkin structure models Henkin theory: Proof lemma schema

- Prove: (I) If $\sigma' \in Sent(\mathcal{L}')$, then $\sigma' \in \Sigma'$ iff $\mathfrak{A}' \vDash \sigma'$
- Corollaries: $\mathfrak{A}' \vDash \Sigma'$
– For any $\sigma' \in \Sigma'$,
— $\Sigma' \vdash \sigma'$
¡Definition¿ — $\Sigma' \vdash UC(\sigma')$
¡Universal closure preserves deductiblity¿ — $UC(\sigma') \in \Sigma'$
¡Expansion by chosen enumerated axioms is deductively closed¿ — $UC(\sigma') \in Sent(\mathcal{L}')$
¡Definition¿ — $\mathfrak{A}' \vDash UC(\sigma')$
¡(I)¿ — $\mathfrak{A}' \vDash \sigma'$
¡Universal closure preserves validity¿ – For any $\sigma' \in \Sigma'$, $\mathfrak{A}' \vDash \sigma'$
¡Abbreviate¿ – $\mathfrak{A}' \vDash \Sigma'$
¡Definition¿ ==

3.2.27 (Metatheorem) VFT-universe assignment in Henkin structure

- For any $t \in VFT(\mathcal{L}')$, for any variable-universe assignment s of $\mathfrak{A}'$, $\overline{s}(t) = [t]$
- Proof:

– For any $t \in VFT(\mathcal{L}')$,
— If $t :\equiv c$, then
—- $c \in Const(\mathcal{L}')$
¡Definition¿ —- $\overline{s}(t) =$
—– $\overline{s}(c) =$
—– $c^{\mathfrak{A}'} =$
¡Definition¿ —– $[c] =$
¡Definition¿ —– $[t]$
—- $\overline{s}(t) = [t]$
¡Abbreviate¿ — If $t :\equiv f \underset{i=1}{\overset{Arity(f)}{\boxed{c_i}}}$ and $\left\{ \underset{i=1}{\overset{Arity(f)}{\boxed{c_i}}} \right\} \subseteq \{z : \overline{s}(z) = [z]\}$, then
—- $\left\{ \underset{i=1}{\overset{Arity(f)}{\boxed{c_i}}} \right\} \subseteq Const(\mathcal{L}')$
¡Definition¿ —- $\overline{s}(t) =$
—– $\overline{s}(f \underset{i=1}{\overset{Arity(f)}{\boxed{c_i}}}) =$
—– $f^{\mathfrak{A}'}(\underset{i=1}{\overset{Arity(f)}{\boxed{\overline{s}(c_i)}}}) =$
¡Definition¿ —– $f^{\mathfrak{A}'}(\underset{i=1}{\overset{Arity(f)}{\boxed{[c_i]}}}) =$
¡Inductive hypothesis¿ —– $[f \underset{i=1}{\overset{Arity(f)}{\boxed{c_i}}}] =$
¡Definition¿ —– $[t]$
—- $\overline{s}(t) = [t]$
¡Definition¿ – For any $t \in VFT(\mathcal{L}')$, for any s, $\overline{s}(t) = [t]$
¡Induction¿ ==

3.2.28 (Metatheorem) Henkin structure models Henkin theory: Lemma (I)

- If $\sigma' \in Sent(\mathcal{L}')$, then $\sigma' \in \Sigma'$ iff $\mathfrak{A}' \vDash \sigma'$
- Proof:
– If $\sigma' \in Sent(\mathcal{L}')$, then
— If $\sigma' :\equiv t_1 \equiv t_2$, then
—- $\{t_1, t_2\} \subseteq VFT(\mathcal{L}')$
—- $\sigma' \in \Sigma'$ iff
—– $t_1 \equiv t_2 \in \Sigma'$ iff
—– $t_1 \sim t_2$ iff
¡Definition¿ —– $[t_1] = [t_2]$ iff
¡Definition¿ —– For any s, $\overline{s}(t_1) = \overline{s}(t_2)$ iff
¡Definition¿ —– For any s, $\mathfrak{A}' \vDash (t_1 \equiv t_2)[s]$
¡Definition¿ —– $\mathfrak{A}' \vDash t_1 \equiv t_2$ iff
—– $\mathfrak{A}' \vDash \sigma$
—- $\sigma' \in \Sigma'$ iff $\mathfrak{A}' \vDash \sigma'$
¡Abbreviate¿ — If $\sigma' :\equiv P \underset{i=1}{\overset{Arity(P)}{\boxed{t_i}}}$, then
—- $\left\{ \underset{i=1}{\overset{Arity(P)}{\boxed{t_i}}} \right\} \subseteq VFT(\mathcal{L}')$
—- $\sigma \in \Sigma'$ iff
—– $P \underset{i=1}{\overset{Arity(P)}{\boxed{t_i}}} \in \Sigma'$ iff
—– $\left\langle \underset{i=1}{\overset{Arity(P)}{\boxed{[t_i]}}} \right\rangle \in P^{\mathfrak{A}'}$ iff
¡Definition¿ —– $\mathfrak{A}' \vDash P \underset{i=1}{\overset{Arity(P)}{\boxed{t_i}}}$

¡Definition¿ —— $\mathfrak{A}' \vDash \sigma'$
—- $\sigma' \in \Sigma'$ iff $\mathfrak{A}' \vDash \sigma'$
¡Abbreviate¿ — If $\sigma' :\equiv \neg\alpha$ and $\{\alpha\} \subseteq \{\zeta : \zeta \in \Sigma'(\text{ iff })\mathfrak{A}' \vDash \zeta\}$, then
—- $\sigma' \in \Sigma'$ iff
—— $\neg\alpha \in \Sigma'$ iff
—— $\alpha \notin \Sigma'$ iff
¡Expansion by chosen enumerated axioms is maximal¿ —— $\mathfrak{A}' \nvDash \alpha$ iff
¡Inductive hypothesis¿ —— $\mathfrak{A}' \vDash \neg\alpha$ iff
¡Definition¿ —— $\mathfrak{A}' \vDash \sigma'$
—- $\sigma' \in \Sigma'$ iff $\mathfrak{A}' \vDash \sigma'$
¡Abbreviate¿ — If $\sigma' :\equiv \alpha \vee \beta$ and $\{\alpha, \beta\} \subseteq \{\zeta : \zeta \in \Sigma'(\text{ iff })\mathfrak{A}' \vDash \zeta\}$, then
—- $\mathfrak{A}' \vDash \sigma'$ iff
—— $\mathfrak{A}' \vDash \alpha \vee \beta$ iff
—— $\mathfrak{A}' \vDash \alpha$ or $\mathfrak{A}' \vDash \beta$ iff
¡Definition¿ —— $\alpha \in \Sigma'$ or $\beta \in \Sigma'$ iff
¡Inductive hypothesis¿ —— $\Sigma' \vdash \alpha$ or $\Sigma' \vdash \beta$ iff
¡Expansion by chosen enumerated axioms is deductively closed¿ —— $\Sigma' \vdash \alpha \vee \beta$ iff
¡PC¿ —— $\alpha \vee \beta \in \Sigma'$ iff
¡Expansion by chosen enumerated axioms is deductively closed¿ —— $\sigma' \in \Sigma'$
—- $\sigma' \in \Sigma'$ iff $\mathfrak{A}' \vDash \sigma'$
¡Abbreviate¿ — If $\sigma' :\equiv \forall x\alpha$ and $Stage(Comp(\sigma') - 1) \subseteq \{\zeta : \zeta \in \Sigma'(\text{ iff })\mathfrak{A}' \vDash \zeta\}$, then
—- If $\sigma' \in \Sigma'$, then
—— $\forall x\alpha \in \Sigma'$
—— $\Sigma' \vdash \forall x\alpha$
¡Expansion by chosen enumerated axioms is deductively closed¿ —— For any $t \in VFT(\mathcal{L}')$,
—— $Subbable(t, x\alpha)$
¡Definition¿ —— $\vdash \forall x\alpha \implies |\alpha|^x_t$
¡Q1¿ —— $\langle \forall x\alpha, \forall x\alpha \implies |\alpha|^x_t, |\alpha|^x_t \rangle \in PC$
—— $\Sigma' \cup \vdash |\alpha|^x_t$
¡PC¿ —— $|\alpha|^x_t \in \Sigma'$
¡Expansion by chosen enumerated axioms is deductively closed¿ —— $\mathfrak{A}' \vDash |\alpha|^x_t$
¡Inductive hypothesis¿ —— For any $t \in VFT(\mathcal{L}')$, $\mathfrak{A}' \vDash |\alpha|^x_t$
¡Abbreviate¿ —— For any variable-universe assignment s, for any $[t] \in A'$,
—— $t \in VFT(\mathcal{L}')$
¡Definition¿ —— $\mathfrak{A}' \vDash |\alpha|^x_t$
—— $Subbable(t, x, \alpha)$
¡Definition¿ —— $\mathfrak{A}' \vDash \alpha[s[x|\overline{s}(t)]]$
¡Substitution and modification identity on models¿ —— $\overline{s}(t) = [t]$
¡VFT-universe assignment in Henkin structure¿ —— $\mathfrak{A}' \vDash \alpha[s[x|[t]]]$
—— For any variable-universe assignment s, for any $[t] \in A'$, $\mathfrak{A}' \vDash \alpha[s[x|[t]]]$
¡Abbreviate¿ —— For any variable-universe assignment s, $\mathfrak{A}' \vDash (\forall x\alpha)[s]$
¡Definition¿ —— $\mathfrak{A}' \vDash \sigma'$
—- If $\sigma' \in \Sigma'$, then $\mathfrak{A}' \vDash \sigma'$
¡Abbreviate¿ —- If $\sigma' \notin \Sigma'$, then
—— $\forall x\alpha \notin \Sigma'$
—— $\neg\forall x\alpha \in \Sigma'$
¡Expansion by chosen enumerated axioms is maximal¿ —— $\exists x\neg\alpha \in \Sigma'$
¡Definition¿ —— There exists $c_{(i,j)}$, $(\exists x\neg\alpha \implies |\neg\alpha|^x_{c_{(i,j)}}) \in \Sigma'$
¡Definition¿ —— $\left\langle \exists x\neg\alpha, \exists x\neg\alpha \implies |\neg\alpha|^x_{c_{(i,j)}}, |\neg\alpha|^x_{c_{(i,j)}} \right\rangle \in PC$ —— $\Sigma' \vdash |\neg\alpha|^x_{c_{(i,j)}}$
¡PC¿ —— $|\neg\alpha|^x_{c_{(i,j)}} \in \Sigma'$
¡Expansion by chosen enumerated axioms is deductively closed¿ —— $\mathfrak{A}' \vDash |\neg\alpha|^x_{c_{(i,j)}}$
¡Inductive hypothesis¿ —— There exists s, there exists $[t] \in A'$,
—— $\mathfrak{A}' \vDash |\neg\alpha|^x_{c_{(i,j)}}[s]$
¡Definition¿ —— $Subbable(c_{(i,j)}, x, \neg\alpha)$
¡Definition¿ —— $\mathfrak{A}' \vDash (\neg\alpha)[s[x|\overline{s}(c_{(i,j)})]]$
¡Substitution and modification identity on models¿ —— $\overline{s}(c_{(i,j)} = [c_{(i,j)}]$
¡VFT-universe assignment in Henkin structure¿ —— $\mathfrak{A}' \vDash \neg\alpha[s[x|[c_{(i,j)}]]]$
—— $\mathfrak{A}' \nvDash \alpha[s[x|[c_{(i,j)}]]]$
¡Definition¿ —— $[t] = [c_{(i,j)}]$

—— $\mathfrak{A}' \not\models \alpha[s[x|[t]]]$
—– There exists s, there exists $[t] \in A'$, $\mathfrak{A}' \not\models \alpha[s[x|[t]]]$
¡Abbreviate¿ —– There exists s, not for any $[t] \in A'$, $\mathfrak{A}' \models \alpha[s[x|[t]]]$
—– There exists s, not $\mathfrak{A}' \models (\forall x \in \alpha)[s]$
—– Not for any s, $\mathfrak{A}' \models (\forall x \in \alpha)[s]$
—– $\mathfrak{A}' \not\models (\forall x \in \alpha)$
—– $\mathfrak{A}' \not\models \sigma'$
—- If $\sigma' \notin \Sigma'$, then $\mathfrak{A}' \not\models \sigma'$
¡Abbreviate¿ —- If $\mathfrak{A}' \models \sigma'$, then $\sigma' \in \Sigma'$
¡Contraposition¿ —- $\sigma' \in \Sigma'$ iff $\mathfrak{A}' \models \sigma'$
¡Conjunction¿ — $\sigma' \in \Sigma'$ iff $\mathfrak{A}' \models \sigma'$
¡Induction¿ – If $\sigma' \in Sent(\mathcal{L}')$, $\sigma' \in \Sigma'$ iff $\mathfrak{A}' \models \sigma'$
¡Abbreviate¿ ==

3.2.29 (Definition) Structure reduct to a language

- The $\mathcal{L}$-structure $\mathfrak{A}^+ \restriction_{\mathcal{L}}$ is the reduct of the $\mathcal{L}^+$-structure $\mathfrak{A}^+$ iff
– $\mathcal{L}$ is the restriction on constants of $\mathcal{L}^+$
– $Universe(\mathfrak{A}^+ \restriction_{\mathcal{L}}) = Universe(\mathfrak{A}^+)$
– $ConstI(\mathfrak{A}^+ \restriction_{\mathcal{L}})$ is for any $c \in Const(\mathcal{L})$, $c^{\mathfrak{A}^+ \restriction_{\mathcal{L}}} = c^{\mathfrak{A}^+}$
– $FuncI(\mathfrak{A}^+ \restriction_{\mathcal{L}})$ is for any $f \in Func(\mathcal{L})$, $f^{\mathfrak{A}^+ \restriction_{\mathcal{L}}} = f^{\mathfrak{A}^+}$
– $RelI(\mathfrak{A}^+ \restriction_{\mathcal{L}})$ is for any $P \in Rel(\mathcal{L})$, $P^{\mathfrak{A}^+ \restriction_{\mathcal{L}}} = P^{\mathfrak{A}^+}$
==

3.2.30 (Metatheorem) Henkin structure reduct models consistent theory: Proof lemma schema

- Prove: (I) If $\sigma \in Sent(\mathcal{L})$, then $\sigma \in \Sigma'$ iff $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
- Corollaries: $\mathfrak{A}' \restriction_{\mathcal{L}} \models \Sigma$
– For any $\sigma \in \Sigma$,
— $\sigma \in \Sigma'$
¡Definition¿ — $\Sigma' \vdash \sigma$
¡Definition¿ — $\Sigma' \vdash UC(\sigma)$
¡Universal closure preserves deductiblity¿ — $UC(\sigma) \in \Sigma'$
¡Expansion by chosen enumerated axioms is deductively closed¿ — $UC(\sigma) \in Sent(\mathcal{L})$
¡Definition¿ — $\mathfrak{A}' \restriction_{\mathcal{L}} \models UC(\sigma)$
¡(I)¿ — $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
¡Universal closure preserves validity¿ – For any $\sigma \in \Sigma$, $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
¡Abbreviate¿ – $\mathfrak{A}' \restriction_{\mathcal{L}} \models \Sigma$
¡Definition¿ ==

3.2.31 (Metatheorem) VFT-universe assignment in Henkin structure reduct

- For any $t \in VFT(\mathcal{L})$, for any variable-universe assignment s of $\mathfrak{A}' \restriction_{\mathcal{L}}$, $\overline{s}(t) = [t]$
- Proof:
– For any $t \in VFT(\mathcal{L})$,
— If $t :\equiv c$, then
—- $c \in Const(\mathcal{L})$
¡Definition¿ —- $\overline{s}(t) =$
—– $\overline{s}(c) =$
—– $c^{\mathfrak{A}' \restriction_{\mathcal{L}}} =$
¡Definiton¿ —– $c^{\mathfrak{A}'} =$
¡Definition¿ —– $[c] =$
¡Definition¿ —– $[t]$
—- $\overline{s}(t) = [t]$
¡Abbreviate¿ — If $t :\equiv f \underset{i=1}{\overset{Arity(f)}{\boxed{c_i}}}$ and $\left\{ \underset{i=1}{\overset{Arity(f)}{\boxed{c_i}}} \right\} \subseteq \{z : \overline{s}(z) = [z]\}$, then
—- $\left\{ \underset{i=1}{\overset{Arity(f)}{\boxed{c_i}}} \right\} \subseteq Const(\mathcal{L})$

¡Definition¿ —- $\overline{s}(t) =$
—— $\overline{s}(f \boxed{c_i}_{i=1}^{Arity(f)}) =$
—— $f^{\mathfrak{A}' \restriction_{\mathcal{L}}}(\boxed{\overline{s}(c_i)}_{i=1}^{Arity(f)}) =$
¡Definition¿ —— $f^{\mathfrak{A}' \restriction_{\mathcal{L}}}(\boxed{[c_i]}_{i=1}^{Arity(f)}) =$
¡Inductive hypothesis¿ —— $f^{\mathfrak{A}'}(\boxed{[c_i]}_{i=1}^{Arity(f)}) =$
¡Definition¿ —— $[f \boxed{c_i}_{i=1}^{Arity(f)}] =$
¡Definition¿ —— $[t]$
—- $\overline{s}(t) = [t]$
¡Abbreviate¿ – For any $t \in VFT(\mathcal{L})$, for any variable-universe assignment s of $\mathfrak{A}' \restriction_{\mathcal{L}}$, $\overline{s}(t) = [t]$
¡Induction¿ ==

3.2.32 (Metatheorem) Henkin structure reduct models consistent theory: Lemma (I)

- If $\sigma \in Sent(\mathcal{L})$, then $\sigma \in \Sigma'$ iff $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
- Proof:
– If $\sigma \in Sent(\mathcal{L})$, then
— If $\sigma :\equiv t_1 \equiv t_2$, then
—- $\{t_1, t_2\} \subseteq VFT(\mathcal{L})$
—- $\sigma \in \Sigma'$ iff
—— $t_1 \equiv t_2 \in \Sigma'$ iff
—— $t_1 \sim t_2$ iff
¡Definition¿ —— $[t_1] = [t_2]$ iff
¡Definition¿ —— For any s, $\overline{s}(t_1) = \overline{s}(t_2)$ iff
¡Definition¿ —— For any s, $\mathfrak{A}' \restriction_{\mathcal{L}} \models (t_1 \equiv t_2)[s]$
¡Definition¿ —— $\mathfrak{A}' \restriction_{\mathcal{L}} \models t_1 \equiv t_2$ iff
—— $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
—- $\sigma \in \Sigma'$ iff $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
¡Abbreviate¿ — If $\sigma :\equiv P \boxed{t_i}_{i=1}^{Arity(P)}$, then
—- $\left\{ \boxed{t_i}_{i=1}^{Arity(P)} \right\} \subseteq VFT(\mathcal{L})$
—- $\sigma \in \Sigma'$ iff
—— $P \boxed{t_i}_{i=1}^{Arity(P)} \in \Sigma'$ iff
—— $\left\langle \boxed{[t_i]}_{i=1}^{Arity(P)} \right\rangle \in P^{\mathfrak{A}'}$ iff
¡Definition¿ —— $\mathfrak{A}' \models P \boxed{t_i}_{i=1}^{Arity(P)}$ iff
¡Definition¿ —— $\mathfrak{A}' \restriction_{\mathcal{L}} \models P \boxed{t_i}_{i=1}^{Arity(P)}$ iff
¡Definition¿ —— $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
—- $\sigma \in \Sigma'$ iff $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
¡Abbreviate¿ — If $\sigma :\equiv \neg\alpha$ and $\{\alpha\} \subseteq \{\zeta : \zeta \in \Sigma'(\text{ iff })\mathfrak{A}' \restriction_{\mathcal{L}} \models \zeta\}$, then
—- $\sigma \in \Sigma'$ iff
—— $\neg\alpha \in \Sigma'$ iff
—— $\alpha \notin \Sigma'$ iff
¡Expansion by chosen enumerated axioms is maximal¿ —— $\mathfrak{A}' \restriction_{\mathcal{L}} \not\models \alpha$ iff
¡Inductive hypothesis¿ —— $\mathfrak{A}' \restriction_{\mathcal{L}} \models \neg\alpha$ iff

¡Definition¿ —— $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
—- $\sigma \in \Sigma'$ iff $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
¡Abbreviate¿ — If $\sigma :\equiv \alpha \vee \beta$ and $\{\alpha, \beta\} \subseteq \{\zeta : \zeta \in \Sigma'(\text{ iff })\mathfrak{A}' \restriction_{\mathcal{L}} \models \zeta\}$, then
—- $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$ iff
—— $\mathfrak{A}' \restriction_{\mathcal{L}} \models \alpha \vee \beta$ iff
—— $\mathfrak{A}' \restriction_{\mathcal{L}} \models \alpha$ or $\mathfrak{A}' \restriction_{\mathcal{L}} \models \beta$ iff
¡Definition¿ —— $\alpha \in \Sigma'$ or $\beta \in \Sigma'$ iff
¡Inductive hypothesis¿ —— $\Sigma' \vdash \alpha$ or $\Sigma' \vdash \beta$ iff
¡Expansion by chosen enumerated axioms is deductively closed¿ —— $\Sigma' \vdash \alpha \vee \beta$ iff
¡PC¿ —— $\alpha \vee \beta \in \Sigma'$ iff
¡Expansion by chosen enumerated axioms is deductively closed¿ —— $\sigma \in \Sigma'$
—- $\sigma \in \Sigma'$ iff $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
¡Abbreviate¿ — If $\sigma :\equiv \forall x\alpha$ and $Stage(Comp(\sigma) - 1) \subseteq \{\zeta : \zeta \in \Sigma'(\text{ iff })\mathfrak{A}' \restriction_{\mathcal{L}} \models \zeta\}$, then
—- If $\sigma \in \Sigma'$, then
—— $\forall x\alpha \in \Sigma'$
—— $\Sigma' \vdash \forall x\alpha$
¡Expansion by chosen enumerated axioms is deductively closed¿ —— For any $t \in VFT(\mathcal{L})$,
——— $Subbable(t, x\alpha)$
¡Definition¿ ——— $\vdash \forall x\alpha \implies |\alpha|^x_t$
¡Q1¿ ——— $\langle \forall x\alpha, \forall x\alpha \implies |\alpha|^x_t, |\alpha|^x_t \rangle \in PC$
——— $\Sigma' \cup \vdash |\alpha|^x_t$
¡PC¿ ——— $|\alpha|^x_t \in \Sigma'$
¡Expansion by chosen enumerated axioms is deductively closed¿ ——— $\mathfrak{A}' \restriction_{\mathcal{L}} \models |\alpha|^x_t$
¡Inductive hypothesis¿ —— For any $t \in VFT(\mathcal{L})$, $\mathfrak{A}' \restriction_{\mathcal{L}} \models |\alpha|^x_t$
¡Abbreviate¿ —— For any variable-universe assignment s, for any $[t] \in A'$,
——— $t \in VFT(\mathcal{L})$
¡Definition¿ ——— $\mathfrak{A}' \restriction_{\mathcal{L}} \models |\alpha|^x_t$
——— $Subbable(t, x, \alpha)$
¡Definition¿ ——— $\mathfrak{A}' \restriction_{\mathcal{L}} \models \alpha[s[x|\overline{s}(t)]$
¡Substitution and modification identity on models¿ ——— $\overline{s}(t) = [t]$
¡VFT-universe assignment in Henkin structure reduct¿ ——— $\mathfrak{A}' \restriction_{\mathcal{L}} \models \alpha[s[x|[t]]]$
—— For any variable-universe assignment s, for any $[t] \in A'$, $\mathfrak{A}' \restriction_{\mathcal{L}} \models \alpha[s[x|[t]]]$
¡Abbreviate¿ —— For any variable-universe assignment s, $\mathfrak{A}' \restriction_{\mathcal{L}} \models (\forall x\alpha)[s]$
¡Definition¿ —— $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma'$
¡Definition¿ —- If $\sigma \in \Sigma'$, then $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma'$
¡Abbreviate¿ —- If $\sigma \notin \Sigma'$, then
—— $\forall x\alpha \notin \Sigma'$
—— $\neg\forall x\alpha \in \Sigma'$
¡Expansion by chosen enumerated axioms is maximal¿ —— $\exists x\neg\alpha \in \Sigma'$
¡Definition¿ —— There exists $c_{(i,j)}$, $(\exists x\neg\alpha \implies |\neg\alpha|^x_{c_{(i,j)}}) \in \Sigma'$
¡Definition¿ —— $\left\langle \exists x\neg\alpha, \exists x\neg\alpha \implies |\neg\alpha|^x_{c_{(i,j)}}, |\neg\alpha|^x_{c_{(i,j)}} \right\rangle \in PC$ —— $\Sigma' \vdash |\neg\alpha|^x_{c_{(i,j)}}$
¡PC¿ —— $|\neg\alpha|^x_{c_{(i,j)}} \in \Sigma'$
¡Expansion by chosen enumerated axioms is deductively closed¿ —— $\mathfrak{A}' \restriction_{\mathcal{L}} \models |\neg\alpha|^x_{c_{(i,j)}}$
¡Inductive hypothesis¿ —— There exists s, there exists $[t] \in A'$,
——— $\mathfrak{A}' \restriction_{\mathcal{L}} \models |\neg\alpha|^x_{c_{(i,j)}}[s]$
¡Definition¿ ——— $Subbable(c_{(i,j)}, x, \neg\alpha)$
¡Definition¿ ——— $\mathfrak{A}' \restriction_{\mathcal{L}} \models (\neg\alpha)[s[x|\overline{s}(c_{(i,j)}))]]$
¡Substitution and modification identity on models¿ ——— $\overline{s}(c_{(i,j)} = [c_{(i,j)}]$
¡VFT-universe assignment in Henkin structure reduct¿ ——— $\mathfrak{A}' \restriction_{\mathcal{L}} \models \neg\alpha[s[x|[c_{(i,j)}]]]$
——— $\mathfrak{A}' \restriction_{\mathcal{L}} \not\models \alpha[s[x|[c_{(i,j)}]]]$
¡Definition¿ ——— $[t] = [c_{(i,j)}]$
——— $\mathfrak{A}' \restriction_{\mathcal{L}} \not\models \alpha[s[x|[t]]]$
—— There exists s, there exists $[t] \in A'$, $\mathfrak{A}' \restriction_{\mathcal{L}} \not\models \alpha[s[x|[t]]]$
¡Abbreviate¿ —— There exists s, not for any $[t] \in A'$, $\mathfrak{A}' \restriction_{\mathcal{L}} \models \alpha[s[x|[t]]]$
—— There exists s, not $\mathfrak{A}' \restriction_{\mathcal{L}} \models (\forall x \in \alpha)[s]$
—— Not for any s, $\mathfrak{A}' \restriction_{\mathcal{L}} \models (\forall x \in \alpha)[s]$
—— $\mathfrak{A}' \restriction_{\mathcal{L}} \not\models (\forall x \in \alpha)$
—— $\mathfrak{A}' \restriction_{\mathcal{L}} \not\models \sigma'$
—- If $\sigma' \notin \Sigma'$, then $\mathfrak{A}' \restriction_{\mathcal{L}} \not\models \sigma'$

¡Abbreviate¿ —- If $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma'$, then $\sigma' \in \Sigma'$
¡Contraposition¿ —- $\sigma' \in \Sigma'$ iff $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma'$
¡Conjunction¿ — $\sigma \in \Sigma'$ iff $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
¡Induction¿ – If $\sigma \in Sent(\mathcal{L})$, $\sigma \in \Sigma'$ iff $\mathfrak{A}' \restriction_{\mathcal{L}} \models \sigma$
¡Abbreviate¿ ===

3.2.33 (Metatheorem) Completeness of First-order Logic: Lemma (I)

- If $UC(\Sigma) \nvdash \overleftarrow{\perp}$, then there exists $\mathfrak{A}$, $\mathfrak{A} \models UC(\Sigma)$
- Proof:
– If $UC(\Sigma) \nvdash \overleftarrow{\perp}$, then
— $UC(\Sigma) \subseteq Sent(\mathcal{L})$
— $UC(\Sigma)'$ is consistent, deductively closed, maximal
— $\mathfrak{B}' \models UC(\Sigma)'$
¡Henkin structure models Henkin theory¿ — $\mathfrak{B}' \restriction_{\mathcal{L}} \models UC(\Sigma)$
¡Henkin structure reduct models consistent theory¿ — $\mathfrak{A} = \mathfrak{B}' \restriction_{\mathcal{L}}$
— There exists $\mathfrak{A}$, $\mathfrak{A} \models UC(\Sigma)$
– If $UC(\Sigma) \nvdash \overleftarrow{\perp}$, then there exists $\mathfrak{A}$, $\mathfrak{A} \models UC(\Sigma)$
¡Abbreviate¿ ===

3.2.34 (Metatheorem) Completeness for uncountable language

- Countable language assumption only affects Henkin theory construction - TODO VERIFY: ANNOTATIONS!!! - If $\mathcal{L}$ is uncountable, then
– $\mathcal{L}'$ is uncountable
– $\hat{\Sigma}$ is uncountable
– Σ' is uncountable
TODO: FIX WHY COUNTABLE - $\Sigma_{all} = \left\{ \hat{\Sigma} \cup \Sigma_{ext} : \hat{\Sigma} \cup \Sigma_{ext} \nvdash \overleftarrow{\perp} \right\}$
- $Poset(\Sigma_{all}, \subseteq)$
- For any T, if $T \subseteq \Sigma_{all}$ and $Woset(T, \subseteq)$, then there exists Σ_{ub}, $UB(\Sigma_{ub}, T, \hat{\Sigma}, \subseteq)$
– $\Sigma_{ub} = \hat{\Sigma} \cup \underset{t \in T}{\boxed{\Sigma^t_{ext}}}$
- There exists Σ_{max}, $Max(\Sigma_{max}, \Sigma_{all}, \subseteq)$
¡Zorn's lemma¿ - Σ_{max} is consistent, deductively closed, maximal
- $\mathfrak{A}_{max} \models \Sigma_{max}$
- $\mathfrak{A}_{max} \restriction_{\mathcal{L}} \models \Sigma$
===

3.2.35 (Metatheorem) Contradiction explosion

- If $\Gamma \models \overleftarrow{\perp}$, then $\Gamma \models \phi$
- Proof:
– If $\Gamma \models \overleftarrow{\perp}$, then
— $\overleftarrow{\perp} \models_{PC} \phi$
— $\overleftarrow{\perp} \vdash \phi$
¡PC¿ — $\Gamma \vdash \phi$
===

3.3 Compactness

===

3.3.1 (Metatheorem) Compactness theorem

- $\Sigma \not\models \overleftarrow{\perp}$ iff for any Γ, if $\Gamma \subseteq \Sigma$ and $Finite(\Gamma)$, then $\Gamma \not\models \overleftarrow{\perp}$
- Proof:
– If $\Sigma \not\models \overleftarrow{\perp}$, then
— There exists $\mathfrak{A}$, $\mathfrak{A} \models \Sigma$

— For any Γ, if $\Gamma \subseteq \Sigma$ and $Finite(\Gamma)$, then
—- $\mathfrak{A} \models \Gamma$
¡Definition¿ —- $\Gamma \not\models \overleftarrow{\bot}$
¡Definition¿ — For any Γ, if $\Gamma \subseteq \Sigma$ and $Finite(\Gamma)$, then $\Gamma \not\models \overleftarrow{\bot}$
¡Abbreviate¿ – If $\Sigma \models \overleftarrow{\bot}$, then
— $\Sigma \vdash \overleftarrow{\bot}$
¡Completeness theorem¿ — There exists Σ_{fin}, $\Sigma_{fin} \subseteq \Sigma$ and $Finite(\Sigma_{fin})$ and $\Sigma_{fin} \vdash \overleftarrow{\bot}$
¡DEDUCTIONS ARE FINITE¿ — $\Sigma_{fin} \models \overleftarrow{\bot}$
¡Soundness theorem¿ — $\Gamma = \Sigma_{fin}$
— There exists Γ, ($\Gamma \subseteq \Sigma$ and $Finite(\Gamma)$) and $\Gamma \models \overleftarrow{\bot}$
— Not for any Γ, not (($\Gamma \subseteq \Sigma$ and $Finite(\Gamma)$) and $\Gamma \models \overleftarrow{\bot}$)
— Not for any Γ, not ($\Gamma \subseteq \Sigma$ and $Finite(\Gamma)$) or not $\Gamma \models \overleftarrow{\bot}$
— Not for any Γ, if $\Gamma \subseteq \Sigma$ and $Finite(\Gamma)$, then not $\Gamma \models \overleftarrow{\bot}$
— Not for any Γ, if $\Gamma \subseteq \Sigma$ and $Finite(\Gamma)$, then $\Gamma \not\models \overleftarrow{\bot}$
– If $\Sigma \models \overleftarrow{\bot}$, then not for any Γ, if $\Gamma \subseteq \Sigma$ and $Finite(\Gamma)$, then $\Gamma \not\models \overleftarrow{\bot}$
¡Abbreviate¿ – If for any Γ, if $\Gamma \subseteq \Sigma$ and $Finite(\Gamma)$, then $\Gamma \not\models \overleftarrow{\bot}$, then $\Sigma \not\models \overleftarrow{\bot}$
¡Contraposition¿ – $\Sigma \not\models \overleftarrow{\bot}$ iff for any Γ, if $\Gamma \subseteq \Sigma$ and $Finite(\Gamma)$, then $\Gamma \not\models \overleftarrow{\bot}$ ¡Conjunction¿ ==========================

3.3.2 (Metatheorem) Logical implication takes finite hypotheses

- $\Sigma \models \phi$ iff there exists Σ_{fin}, $Finite(\Sigma_{fin})$ and $\Sigma_{fin} \subseteq \Sigma$ and $\Sigma_{fin} \models \phi$
- Proof:
– $\Sigma \models \phi$ iff
— $\Sigma \vdash \phi$ iff
¡Completeness theorem, Soundness theorem¿ — There exists Σ_{fin}, $Finite(\Sigma_{fin})$ and $\Sigma_{fin} \subseteq \Sigma$ and $\Sigma_{fin} \vdash \phi$ iff
¡DEDUCTION ARE FINITE¿ — There exists Σ_{fin}, $Finite(\Sigma_{fin})$ and $\Sigma_{fin} \subseteq \Sigma$ and $\Sigma_{fin} \models \phi$ iff
¡Soundness theorem, Completeness theorem¿ ==

3.3.3 (Definition) Theory of a structure

- The theory of the $\mathcal{L}$-structure $\mathfrak{A}$ is $Th(\mathfrak{A}) = \{\phi \in \mathcal{L} : \mathfrak{A} \models \phi\}$
==

3.3.4 (Definition) Elementary equivalent structures

- The $\mathcal{L}$-structures $\mathfrak{A}, \mathfrak{B}$ are elementary equivalent ($\mathfrak{A} =_E \mathfrak{B}$) iff $Th(\mathfrak{A}) = Th(\mathfrak{B})$
==

3.4 Substructures and the Lowenheim-Skolem theorems

==

3.4.1 (Definition) Function restriction

- The function $f \restriction_A : A \to C$ is a restriction of the function $f : A \cup B \to C$ iff
– For any $a \in A$, $f \restriction_A (a) = f(a)$
==

3.4.2 (Definition) Substructure

- The $\mathcal{L}$-structure $\mathfrak{A}$ is a substructure of the $\mathcal{L}$-structure $\mathfrak{B}$ ($\mathfrak{A} \subseteq \mathfrak{B}$) iff
– $A \subseteq B$ and
– For any $c \in Const$, $c^{\mathfrak{A}} = c^{\mathfrak{B}}$ and
– For any $f \in Func$, $f^{\mathfrak{A}} = f^{\mathfrak{B}} \restriction_{A^{Arity(f)}}$ and
– For any $P \in Rel$, $P^{\mathfrak{A}} = P^{\mathfrak{B}} \cap A^{Arity(P)}$ and
– $\mathfrak{A}$ is an $\mathcal{L}$-structure
==

3.4.3 (Metatheorem) Stronger substructure

- If $\emptyset \neq A \subset B$ and for any $c \in Const$, $c^{\mathfrak{B}} \in A$ and for any $f \in Func$, $f^{\mathfrak{B}} \restriction_{A^{Arity(f)}}: A^{Arity(f)} \to A$, then $\mathfrak{A}_{A,\mathfrak{B}} \subseteq \mathfrak{B}$
- Proof: definition
==

3.4.4 (Definition) Elementary substructure

- The $\mathcal{L}$-structure $\mathfrak{A}$ is an elementary substructure of the $\mathcal{L}$-structure $\mathfrak{B}$ ($\mathfrak{A} \prec \mathfrak{B}$) iff
– $\mathfrak{A} \subseteq \mathfrak{B}$ and
– For any $\phi \in Form(\mathcal{L})$, for any $s : Var \to A$, $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
- ===

3.4.5 (Metatheorem) Elementary substructure property

- If $\mathfrak{A} \prec \mathfrak{B}$, then for any $\phi \in Sent(\mathcal{L})$, $\mathfrak{A} \vDash \phi$ iff $\mathfrak{B} \vDash \phi$
– Proof:
— If $\mathfrak{A} \prec \mathfrak{B}$, then
—- For any $\chi \in Form(\mathcal{L})$, for any $s : Var \to A$, $\mathfrak{A} \vDash \chi[s]$ iff $\mathfrak{B} \vDash \chi[s]$
¡Definition¿ —- $\phi \in Form(\mathcal{L})$
—- For any $s : Var \to A$, $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
—- $\mathfrak{A} \vDash \phi$ iff
—— For any $s : Var \to A$, $\mathfrak{A} \vDash \phi[s]$ iff
¡Definition¿ —— For any $s : Var \to A$, $\mathfrak{B} \vDash \phi[s]$ iff
—— For any $s : Var \to B$, $\mathfrak{B} \vDash \phi[s]$ iff
¡Sentences have fixed truth¿ —— $\mathfrak{B} \vDash \phi$
¡Definition¿ —- $\mathfrak{A} \vDash \phi$ iff $\mathfrak{B} \vDash \phi$ ¡Abbreviate¿ ==================================

3.4.6 (Metatheorem) Stronger elementary substructure

- If ($\mathfrak{A} \subseteq \mathfrak{B}$ and for any $\gamma \in Form(\mathcal{L})$, for any $s : Var \to A$, if $\mathfrak{B} \vDash (\exists x \gamma)[s]$, then there exists $a \in A$, $\mathfrak{B} \vDash \gamma[s[x|a]]$), then $\mathfrak{A} \prec \mathfrak{B}$
- Proof:
– If ($\mathfrak{A} \subseteq \mathfrak{B}$ and for any $\gamma \in Form(\mathcal{L})$, for any $s : Var \to A$, if $\mathfrak{B} \vDash (\exists x \gamma)[s]$, then there exists $a \in A$, $\mathfrak{B} \vDash \gamma[s[x|a]]$), then
— $\mathfrak{A} \subseteq \mathfrak{B}$
¡Hypothesis¿ — $A \subseteq B$ ¡(1)¿
¡Definition¿ — If $s : Var \to A$, then $s : Var \to B$ ¡(2)¿
¡Definition¿ — If $P \in Rel$, then $P^{\mathfrak{A}} = P^{\mathfrak{B}} \cap A^{Arity(P)}$ ¡(3)¿
¡Definition¿ — For any $\gamma \in Form(\mathcal{L})$, for any $s : Var \to A$, if $\mathfrak{B} \vDash (\exists x \gamma)[s]$, then there exists $a \in A$, $\mathfrak{B} \vDash \gamma[s[x|a]]$ ¡(4)¿
¡Hypothesis¿ — If $\phi :\equiv t_1 \equiv t_2$, then
—- For any $s : Var \to A$,
—— $\mathfrak{A} \vDash \phi[s]$ iff
——- $\mathfrak{A} \vDash (t_1 \equiv t_2)[s]$ iff
¡Definition¿ ——- $\overline{s}(t_1) = \overline{s}(t_2)$ iff
¡Definition¿ ——- $\mathfrak{B} \vDash (t_1 \equiv t_2)[s]$ iff
¡(2)¿ ——- $\mathfrak{B} \vDash \phi[s]$
—— $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
¡Abbreviate¿ —- For any $s : Var \to A$, $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
¡Abbreviate¿ — If $\phi :\equiv P \overset{Arity(P)}{\underset{i=1}{\boxed{t_i}}}$, then
—- For any $s : Var \to A$,
—— $\mathfrak{A} \vDash \phi[s]$ iff
——- $\mathfrak{A} \vDash (P \overset{Arity(P)}{\underset{i=1}{\boxed{t_i}}})[s]$ iff
¡Definition¿ ——- $\left\langle \overset{Arity(P)}{\underset{i=1}{\boxed{\overline{s}(t_i)}}} \right\rangle \in P^{\mathfrak{A}}$
¡Definition¿ ——- $\left\langle \overset{Arity(P)}{\underset{i=1}{\boxed{\overline{s}(t_i)}}} \right\rangle \in P^{\mathfrak{B}}$

¡(3)¿ —— $\mathfrak{B} \vDash (P \overset{Arity(P)}{\underset{i=1}{\boxed{t_i}}})[s]$ iff
¡Definition¿ —— $\mathfrak{B} \vDash \phi[s]$
¡Definition¿ —– $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
¡Abbreviate¿ —- For any $s : Var \to A$, $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
¡Abbreviate¿ — If $\phi :\equiv \neg\alpha$ and $\{\alpha\} \subseteq \{\zeta : ($ for any $s : Var \to A)$, $(\mathfrak{A} \vDash \zeta[s]($ iff $)\mathfrak{B} \vDash \zeta[s])\}$, then
—- For any $s : Var \to A$,
—– $\mathfrak{A} \vDash \phi[s]$ iff
—— $\mathfrak{A} \vDash (\neg\alpha)[s]$ iff
¡Definition¿ —— $\mathfrak{A} \nvDash \alpha[s]$ iff
¡Definition¿ —— $\mathfrak{B} \nvDash \alpha[s]$ iff
¡Inductive hypothesis¿ —— $\mathfrak{B} \vDash (\neg\alpha)[s]$ iff
¡Definition¿ —— $\mathfrak{B} \vDash \phi[s]$
¡Definition¿ —– $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
¡Abbreviate¿ —- For any $s : Var \to A$, $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
¡Abbreviate¿ — If $\phi :\equiv \alpha \vee \beta$ and $\{\alpha, \beta\} \subseteq \{\zeta : ($ for any $s : Var \to A)$, $(\mathfrak{A} \vDash \zeta[s]($ iff $)\mathfrak{B} \vDash \zeta[s])\}$, then
—- For any $s : Var \to A$,
—– $\mathfrak{A} \vDash \phi[s]$ iff
—— $\mathfrak{A} \vDash (\alpha \vee \beta)[s]$ iff
¡Definition¿ —— $\mathfrak{A} \vDash \alpha[s]$ or $\mathfrak{A} \vDash \beta[s]$ iff
¡Definition¿ —— $\mathfrak{B} \vDash \alpha[s]$ or $\mathfrak{B} \vDash \beta[s]$ iff
¡Inductive hypothesis¿ —— $\mathfrak{B} \vDash (\alpha \vee \beta)[s]$ iff
¡Definition¿ —— $\mathfrak{B} \vDash \phi[s]$
¡Definition¿ —– $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
¡Abbreviate¿ —- For any $s : Var \to A$, $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
¡Abbreviate¿ — If $\phi :\equiv \exists x\alpha$ and $\{\alpha\} \subseteq \{\zeta : ($ for any $s : Var \to A)$, $(\mathfrak{A} \vDash \zeta[s]($ iff $)\mathfrak{B} \vDash \zeta[s])\}$, then
—- For any $s : Var \to A$,
—– If $\mathfrak{A} \vDash \phi[s]$, then
—— $\mathfrak{A} \vDash (\exists x\alpha)[s]$
—— There exists $a \in A$, $\mathfrak{A} \vDash \alpha[s[x|a]]$
¡Definition¿ —— $\mathfrak{B} \vDash \alpha[s[x|a]]$
¡Inductive hypothesis¿ —— $a \in B$
¡(I)¿ —— There exists $a \in B$, $\mathfrak{B} \vDash \alpha[s[x|a]]$
¡Conjunction¿ —— $\mathfrak{B} \vDash (\exists x\alpha)[s]$
¡Definition¿ —— $\mathfrak{B} \vDash \phi[s]$
—– If $\mathfrak{A} \vDash \phi[s]$, then $\mathfrak{B} \vDash \phi[s]$
¡Abbreviate¿ —– If $\mathfrak{B} \vDash \phi[s]$, then
—— There exists $a \in A$, $\mathfrak{B} \vDash \alpha[s[x|a]]$
¡(4)¿ —— $\mathfrak{A} \vDash \alpha[s[x|a]]$
¡Inductive hypothesis¿ —— There exists $a \in A$, $\mathfrak{A} \vDash \alpha[s[x|a]]$
¡Conjunction¿ —— $\mathfrak{A} \vDash (\exists x\alpha)[s]$
¡Definition¿ —— $\mathfrak{A} \vDash \phi[s]$
—– If $\mathfrak{B} \vDash \phi[s]$, then $\mathfrak{A} \vDash \phi[s]$
¡Abbreviate¿ —– $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
¡Conjunction¿ —- For any $s : Var \to A$, $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
¡Abbreviate¿ — For any $s : Var \to A$, $\mathfrak{A} \vDash \phi[s]$ iff $\mathfrak{B} \vDash \phi[s]$
¡Induction¿ — $\mathfrak{A} \prec \mathfrak{B}$
¡Definition¿ ==

3.4.7 (Definition) TODO Countable/finite/infinite notations

- $Finite(X)$ iff $|X| \in \mathbb{N}$
- $Infinite(X)$ iff not $Finite(X)$
- $Countable(X)$ iff there exists f, $Bij(f, X, \mathbb{N})$
- $Countable_L(\mathcal{L})$ iff $Countable(Form(\mathcal{L}))$
- $Countable_S(\mathfrak{A})$ iff $Countable(Universe(\mathfrak{A}))$
- Cardinal = set cardinality
==

3.4.8 (Metatheorem) Downward Lowenheim-Skolem theorem

- If $Countable_L(\mathcal{L})$ and $\mathfrak{B}$ is an $\mathcal{L}$-structure, then there exists $\mathfrak{A}$, $\mathfrak{A} \prec \mathfrak{B}$ and $Countable_S(\mathfrak{A})$
- Proof: TODO ABSTRACTED
==

3.4.9 (Metatheorem) PLACEHOLDER

- If κ is an infinite cardinal and there exists $\mathfrak{A}$, $\mathfrak{A} \vDash \Sigma$ and $Infinite_S(\mathfrak{A})$, then there exists $\mathfrak{B}$, $\mathfrak{B} \vDash \Sigma$ and $|B| \geq \kappa$
- Proof: TODO ABSTRACTED
==

3.4.10 (Metatheorem) PLACEHOLDER

- If κ is an infinite cardinal and $Countable_L(\mathcal{L})$ and $\Sigma \subseteq Form(\mathcal{L})$ and there exists $\mathfrak{A}$, $\mathfrak{A} \vDash \Sigma$ and $Infinite_S(\mathfrak{A})$, then there exists $\mathfrak{B}$, $\mathfrak{B} \vDash \Sigma$ and $|B| = \kappa$
- Proof: TODO ABSTRACTED
==

3.4.11 (Metatheorem) PLACEHOLDER

- If $Infinite_S(\mathfrak{A})$, then not there exists Σ, $\mathfrak{B} \vDash \Sigma$ iff $\mathfrak{A} \cong \mathfrak{B}$
- Proof: TODO ABSTRACTED
==

3.4.12 (Metatheorem) Upward Lowenheim-Skolem theorem

- If $Countable_L(\mathcal{L})$ and $Infinite_S(\mathfrak{A})$ and κ is a cardinal, then there exists $\mathfrak{B}$, $\mathfrak{A} \prec \mathfrak{B}$ and $|B| \geq \kappa$
- Proof: TODO ABSTRACTED
==

Chapter 4

Incompleteness From Two Points of View

==

4.1 Introduction

- $\mathcal{L}$ is cool and all, but how about $\mathcal{L}_{NT}$ and $\mathfrak{N}$?
- Can we find some way for any $\phi \in Form(\mathcal{L}_{NT})$, if $\mathfrak{N} \models \phi$, then $\Sigma \vdash \phi$ (complete) such that Σ is consistent and decidable?

==

4.1.1 (Definition) Axiomatic completeness

- Σ is axiomatically complete iff for any $\sigma \in Form(\mathcal{L})$, $\Sigma \vdash \sigma$ or $\Sigma \vdash \neg\sigma$

==

4.1.2 (Definition) Axiomatization

- Σ is an axiomatization of $Th(\mathfrak{A})$ iff for any $\sigma \in Th(\mathfrak{A})$, $\Sigma \vdash \sigma$
- Promise: Given any complete, consistent, and decidable axiomatization for $\mathfrak{N}$ (Σ), we are going to find a sentence σ such that $\mathfrak{N} \models \sigma$ but $\Sigma \not\vdash \sigma$

==

4.2 Complexity of Formulas

- We will find this Godel sentence via complexity of formulas

==

4.2.1 (Definition) Bounded quantifiers

- If $occurs(x, t)$, then the following are bounded quantifiers:
– $(\forall x \leq t)\phi :\equiv \forall x(x \leq t \implies t)$
– $(\exists x \leq t)\phi :\equiv \exists x(x \leq t \land t)$

==

4.2.2 (Definition) Sigma-formulas

- Σ_{Form} is defined as the smallest set of $\mathcal{L}_{NT}$ formulas that contains:
– Atomic formulas
– If $\alpha \in \Sigma_{Form}$, then $\neg\alpha \in \Sigma_{Form}$
– If $\{\alpha, \beta\} \subseteq \Sigma_{Form}$, then $\{\alpha \lor \beta, \alpha \land \beta\} \subseteq \Sigma_{Form}$
– If $\alpha \in \Sigma_{Form}$ and $occurs(x, t)$, then $\{(\forall x < t)\alpha, (\exists x < t)\alpha\} \subseteq \Sigma_{Form}$
– If $\alpha \in \Sigma_{Form}$ and $x \in Var$, then $\exists x\alpha \in \Sigma_{Form}$
- There are closed under bounded quantification + unbounded existential quantification
- These are not complicated enough to establish incompleteness

==

4.2.3 (Definition) Pi-formulas

- Π_{Form} is defined as the smallest set of $\mathcal{L}_{NT}$ formulas that contains:
– Atomic formulas
– If $\alpha \in \Sigma_{Form}$, then $\neg\alpha \in \Sigma_{Form}$
– If $\{\alpha, \beta\} \subseteq \Sigma_{Form}$, then $\{\alpha \vee \beta, \alpha \wedge \beta\} \subseteq \Sigma_{Form}$
– If $\alpha \in \Sigma_{Form}$ and $occurs(x, t)$, then $\{(\forall x < t)\alpha, (\exists x < t)\alpha\} \subseteq \Sigma_{Form}$
– If $\alpha \in \Sigma_{Form}$ and $x \in Var$, then $\forall x\alpha \in \Sigma_{Form}$
- There are closed under bounded quantification + unbounded universal quantification
- These are complicated enough to establish incompleteness

==

4.2.4 (Definition) Delta-formulas

- $\Delta_{Form} = \Sigma_{Form} \cap \Pi_{Form}$

==

TODO: REMARKS, EXERCISES

4.3 The Roadmap to Incompleteness

- Key idea: use numbers to encode deductions, then construct a self-reference paradoxical deduction
- It is easy to encode, decode, validate numbers into deductions and vice versa
- Promise: fix our coding scheme, prove that the coding is nice, use the coding scheme in order to construct the formula σ, and then prove that σ is both true and not provable

==

4.4 An Alternate Route

- Instead of looking at formulas and deductions, we can look at computations
- In this route, we will still encode computations are numbers

==

4.5 How to Code a Sequence of Numbers

- We will use prime numbers with non-zero exponents

==

4.5.1 Prime number function

- The function $p : \mathbb{N} \to \mathbb{N}$ is defined as $p(k)$ is the kth prime number
- $p(0) = 1, p(1) = 2, p(2) = 3, p(3) = 4, ..., p_i = p(i)$

==

4.5.2 Set of finite sequences of natural numbers

- The set $\mathbb{N}^{<\mathbb{N}}$ is the set of all finite sequences of natural numbers

==

4.5.3 Encoding function

- The encoding function $enc : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is defined as:
– If $k > 0$, then $enc(\overset{k}{\underset{i=1}{\boxed{a_i}}}) = \Pi_{i=1}^{k}(p_i^{a_i+1})$
– Otherwise, then $enc() = 1$

==

4.5.4 Code numbers

- The set code numbers C is defined as $C = \{enc(s) : s \in \mathbb{N}^{<\mathbb{N}}\}$
- This is easy to check

==

4.5.5 Decoding function

- The decoding function $dec : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$ is defined as:
– If $a \in C$, then
— There exists $\overset{k}{\underset{i=1}{\boxed{a_i}}}$, $a = enc(\overset{k}{\underset{i=1}{\boxed{a_i}}})$
¡Fundamental theorem of arithmetic + Definition¿ — $dec(a) = \left\langle \overset{k}{\underset{i=1}{\boxed{a_i}}} \right\rangle$
– Otherwise, then $dec(a) = \langle\rangle$

==

4.5.6 Length function

- The length function $len : \mathbb{N} \to \mathbb{N}$ is defined as:
– If $a \in C$, then
— There exists $\overset{k}{\underset{i=1}{\boxed{a_i}}}$, $a = enc(\overset{k}{\underset{i=1}{\boxed{a_i}}})$
¡Fundamental theorem of arithmetic + Definition¿ — $len(a) = k$
– Otherwise, then $len(a) = 0$
- The Fundamental theorem of arithmetic ensures that for any positive integer, there exists is a unique prime factorization

==

4.5.7 Index function

- The index function $idx : \mathbb{N}^2 \to \mathbb{N}$ is defined as:
– If $a \in C$, then
— There exists $\overset{k}{\underset{i=1}{\boxed{a_i}}}$, $a = enc(\overset{k}{\underset{i=1}{\boxed{a_i}}})$
¡Fundamental theorem of arithmetic + Definition¿ — If $1 \leq i \leq k$, then $idx(a, i) = a_i$
— Otherwise, $idx(a, i) = 0$
– Otherwise, then $idx(a, i) = 0$

==

4.5.8 Concatenate function

- The concatenate function $cat : \mathbb{N}^2 \to \mathbb{N}$ is defined as:
– If $a \in C$ and $b \in C$, then
— There exists $\overset{k}{\underset{i=1}{\boxed{a_i}}}$, $a = enc(\overset{k}{\underset{i=1}{\boxed{a_i}}})$
¡Fundamental theorem of arithmetic + Definition¿ — There exists $\overset{k}{\underset{i=1}{\boxed{b_i}}}$, $b = enc(\overset{k}{\underset{i=1}{\boxed{b_i}}})$
¡Fundamental theorem of arithmetic + Definition¿ — $cat(a, b) = enc(\overset{k_a}{\underset{i=1}{\boxed{a_i}}}), \overset{k_b}{\underset{i=1}{\boxed{b_i}}}))$
– Otherwise, then $cat(a, b) = 0$

==

4.6 An Old Friend

- N is strong enough to prove every true sentence in Σ_{Form}, but it is not strong enough to prove every true sentence in Π_{Form}
- Proof: TODO ABSTRACTED

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4.6.1 (Definition) Goden numbering function

- $GN : String(\mathcal{L}_{NT}) \to \mathbb{N}$ is defined as:
– If $s \in Form(\mathcal{L}_{NT})$ and $s :\equiv \neg\alpha$, then $GN(s) = enc(1, GN(\alpha))$
– If $s \in Form(\mathcal{L}_{NT})$ and $s :\equiv \alpha \vee \beta$, then $GN(s) = enc(3, GN(\alpha), GN(\beta))$
– If $s \in Form(\mathcal{L}_{NT})$ and $s :\equiv \forall v_i \alpha$, then $GN(s) = enc(5, GN(v_i), GN(\alpha))$
– If $s \in Form(\mathcal{L}_{NT})$ and $s :\equiv\equiv t_1 t_2$, then $GN(s) = enc(7, GN(t_1), GN(t_2))$
– If $s \in Form(\mathcal{L}_{NT})$ and $s :\equiv< t_1 t_2$, then $GN(s) = enc(19, GN(t_1), GN(t_2))$
– If $s \in Term(\mathcal{L}_{NT})$ and $s :\equiv S(t)$, then $GN(s) = enc(11, GN(t))$
– If $s \in Term(\mathcal{L}_{NT})$ and $s :\equiv +t_1 t_2$, then $GN(s) = enc(13, GN(t_1), GN(t_2))$
– If $s \in Term(\mathcal{L}_{NT})$ and $s :\equiv \cdot t_1 t_2$, then $GN(s) = enc(15, GN(t_1), GN(t_2))$
– If $s \in Term(\mathcal{L}_{NT})$ and $s :\equiv E t_1 t_2$, then $GN(s) = enc(17, GN(t_1), GN(t_2))$
– If $s \in Var(\mathcal{L}_{NT})$ and $s :\equiv v_i$, then $GN(s) = enc(2i)$
– If $s \in Const(\mathcal{L}_{NT})$ and $s :\equiv 0$, then $GN(s) = enc(9)$
– Otherwise, $GN(s) = 3$
==

Chapter 5

Computability Theory

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5.1 The Origin of Computability Theory

- Computability theory formalizes the notion of algorithms and computations
- The goal is to create formal models of computation and study its limitations
- Several models of note: Herbrand-Godel equations, Church's lambda-calculus, Kleene recursion, Turing machines
- It's easy to see that if a function is computable in these models, then it is computable in the real-world, but the converse is not so clear
- Turing machines model computation similar to how we do computations in the real-world, so maybe the converse holds (Church-Turing thesis)
- All models mentioned induce the same class of computable functions

==

5.2 The Basics

- We will use Kleene recursion because it is easy to use in proofs

==

5.2.1 (Definition) Computable functions

- The set of computable functions μ is defined by:
– Zero function: If $\mathcal{O} : \emptyset \to \{0\}$ and $O() = 0$, then $O \in \mu$
– Successor function: If $S : \mathbb{N} \to \mathbb{N}$ and $S(x) = x + 1$, then $S \in \mu$
– Projection function: If $1 \leq i \leq n$ and $\mathcal{I}_i^n : \mathbb{N}^n \to \mathbb{N}$ and $\mathcal{I}_i^n(\overset{n}{\underset{j=1}{\boxed{x_j}}}) = x_i$, then $\mathcal{I}_i^n \in \mu$
– Composition: If $h : \mathbb{N}^m \to \mathbb{N}$ and for any $i \in \left\{\overset{n}{\underset{j=1}{\boxed{j}}}\right\}$, $g_i : \mathbb{N}^n \to \mathbb{N}$ and $\left\{h, \overset{n}{\underset{i=1}{\boxed{g_i}}}\right\} \subseteq \mu$ and $f : \mathbb{N}^n \to \mathbb{N}$ and $f(\overset{n}{\underset{j=1}{\boxed{x_j}}}) = h(\overset{m}{\underset{i=1}{\boxed{g_i(\overset{n}{\underset{j=1}{\boxed{x_j}}})}}})$, then $f \in \mu$
– Primitive recursion: If $g : \mathbb{N}^n \to \mathbb{N}$ and $h : \mathbb{N}^{n+2} \to \mathbb{N}$ and $\{g, h\} \subseteq \mu$ and $f : \mathbb{N}^{n+1} \to \mathbb{N}$ and $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, 0) = g(\overset{n}{\underset{i=1}{\boxed{x_i}}})$ and $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y + 1) = h(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y, f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y))$, then $f \in \mu$
– Minimalization: If ($g : \mathbb{N}^{n+1} \to \mathbb{N}$ and $g \in \mu$ and $\mu_{UBS}(g) : \mathbb{N}^n \to \mathbb{N}$ and if (there exists z, $g(\overset{n}{\underset{i=1}{\boxed{x_i}}}, z) = 0$ and for any $z_- < z$, $g(\overset{n}{\underset{i=1}{\boxed{x_i}}}, z_-) \neq 0$), then $\mu_{US}(g)(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = z$), then $\mu_{US} \in \mu$
- Projection and composition can simulate arbitrary function arities
- Minimalization is also called unbounded search and it can possibly be undefined which introduces partial functions

- Partial functions are important in computability theory
- When we claim that an algorithm computes a partial function $f : \mathbb{N}^n \to \mathbb{N}$, we claim that $f(\underset{i=1}{\overset{n}{\boxed{x_i}}})$ is defined iff the algorithm terminates on the inputs and returns the correct output

==

5.2.2 (Definition) Primitive recursive functions

- The set of primitive recursive PR is defined by the definition of computable functions without Minimalization

==

5.2.3 (Definition) Characteristic function

- The characteristic function $\chi_{A(\square)} : \mathbb{N}^n \to \{0, 1\}$ for $A \subseteq \mathbb{N}^n$ and $n > 1$ is defined as:
– If $\left\langle \underset{i=1}{\overset{n}{\boxed{x_i}}} \right\rangle \in A$, then $\chi_{A(\square)}(\underset{i=1}{\overset{n}{\boxed{x_i}}}) = 0$
– If $\left\langle \underset{i=1}{\overset{n}{\boxed{x_i}}} \right\rangle \notin A$, then $\chi_{A(\square)}(\underset{i=1}{\overset{n}{\boxed{x_i}}}) = 1$
- $\square$ is a place holder or an abbreviation for exactly the same input arguments if it is defined

==

5.2.4 (Definition) Computable set/relation

- The set/relation A is computable iff its characteristic function $\chi_{A(\square)}$ is computable
- The set/relation A is primitive recursive iff its characteristic function $\chi_{A(\square)}$ is recursive

==

5.2.5 (Metatheorem) Constant function is primitive recursive

- The constant function $c_i^n(\underset{j=1}{\overset{n}{\boxed{x_j}}}) = i$ is primitive recursive

- Proof:
– If $i = 0$, then
— $c_0^n(\underset{j=1}{\overset{n}{\boxed{x_j}}}) = 0 = \mathcal{O}()$
¡Zero function¿ — $c_0^n \in PR$
¡Composition¿ – If $i > 0$ and $c_i^n \in PR$, then
— $S \in PR$
¡Successor function¿ — $c_{i+1}^n(\underset{j=1}{\overset{n}{\boxed{x_j}}}) = S(c_i^n(\underset{j=1}{\overset{n}{\boxed{x_j}}}))$
— $c_{i+1}^n(\underset{j=1}{\overset{n}{\boxed{x_j}}}) \in PR$
¡Composition¿ – $c_i^n \in PR$
¡Induction¿ – $c_i^n(\underset{i=1}{\overset{n}{\boxed{x_i}}}) = i$
¡Definition¿ - The approach is not a construction via primitive recursion because i is not treated as a function argument

==

5.2.6 (Metatheorem) Standard addition, multiplication, exponentiaion are primitive recursive

- The functions $+, \cdot, E$ from the standard number theory $(\mathcal{N})$ are primitive recursive
- $+ \in PR$
- Proof:
– $I_1^1(x) = x$
– $I_1^1 \in PR$
¡Projection function¿ – $S \in PR$
¡Successor function¿ – $S_1^3(x, y, z) = S(I_1^3(x, y, z))$

– $S_1^3 \in PR$
¡Composition¿ – $+(x, 0) = I_1^1(x)$
– $+(x, y+1) = S_1^3(x, y, +(x, y))$
– $+ \in PR$
¡Primitive recursion¿ - $\cdot \in PR$
- Proof:
– $c_0^1(x) = 0$
¡Definition¿ – $c_0^1 \in PR$
¡Constant function is primitive recursive¿ – $+ \in PR$
¡Standard addition, multiplication, exponentiaion are primitive recursive¿ – $+_1^3(x, y, z) = +(I_1^3(x, y, z), I_3^3(x, y, z))$
– $+_1^3 \in PR$
¡Composition¿ – $\cdot(x, 0) = c_0^1(x)$
– $\cdot(x, y+1) = +_1^3(x, y, \cdot(x, y))$
– $\cdot \in PR$
¡Primitive recursion¿ - $E \in PR$
- Proof:
– $c_1^1(x) = 1$
¡Definition¿ – $c_1^1 \in PR$
¡Constant function is primitive recursive¿ – $\cdot \in PR$
¡Standard addition, multiplication, exponentiaion are primitive recursive¿ – $\cdot_1^3(x, y, z) = \cdot(I_1^3(x, y, z), I_3^3(x, y, z))$
– $\cdot_1^3 \in PR$
¡Composition¿ – $E(x, 0) = c_1^1(x)$
– $E(x, y+1) = \cdot_1^3(x, y, E(x, y))$
– $E \in PR$
¡Primitive recursion¿ ==

5.2.7 (Metatheorem) Modified subtraction is primitive recursive

- The modified subtraction function $\dot{-}$ is defined as:
– If $y > x$, then $x \dot{-} y = 0$
– If $y \not> x$, then $x \dot{-} y = x - y$
- $\dot{-} \in PR$
- Proof:
– $\mathcal{O} \in PR$
– $I_1^2 \in PR$
– $P(0) = \mathcal{O}()$
– $P(y+1) = I_1^2(y, P(y))$
– $P \in PR$
¡Primitive recursion¿ – $I_1^1 \in PR$
– $P_1^3(x, y, z) = P(I_1^3(x, y, z))$
– $P_1^3 \in PR$
¡Composition¿ – $\dot{-}(x, 0) = I_1^1(x)$
– $\dot{-}(x, y+1) = P_1^3(x, y, \dot{-}(x, y))$
– $\dot{-} \in PR$
¡Primitive recursion¿ ==

5.2.8 (Metatheorem) Standard logic connectives are closed under the primitive recursion

- The relations $\neg, \vee$ from the standard propositional logic ($\mathcal{PL}$) are closed under primitive recursion
- For any $\{\chi_{U(\square)}, \chi_{V(\square)}\} \subseteq PR$, $\{\chi_{\neg U(\square)}, \chi_{U(\square) \vee V(\square)}\} \subseteq PR$
- Proof:
– For any $\chi_{U(\square)} \in PR$,
— $\dot{-} \in PR$
¡Modified subtraction is primitive recursive¿ — $Conj(x) = \dot{-}(c_1^1(x), I_1^1(x))$
— $Conj \in PR$
¡Composition¿ — $\chi_{\neg U(\square)}(\boxed{x_i}_{i=1}^{Arity(U)}) = Conj(\chi_{U(\square)}(\boxed{x_i}_{i=1}^{Arity(U)}))$
— $\chi_{\neg U(\square)} \in PR$
– For any $\{\chi_{U(\square)}, \chi_{V(\square)}\} \subseteq PR$,
— $\cdot \in PR$

¡Standard addition, multiplication, exponentiaion are primitive recursive¿ — $\chi'_{U(\Box)}(\overset{Arity(U)}{\underset{i=1}{\boxed{r_i}}}, \overset{Arity(V)}{\underset{i=1}{\boxed{s_i}}}) = \chi_{U(\Box)}(\overset{Arity(U)}{\underset{j=1}{\boxed{I_j^{Arity(U)+Arity(V)}(\boxed{r}}}}$

— $\chi'_{U(\Box)} \in PR$

¡Composition¿ — $\chi'_{V(\Box)}(\overset{Arity(U)}{\underset{i=1}{\boxed{r_i}}}, \overset{Arity(V)}{\underset{i=1}{\boxed{s_i}}}) = \chi_{V(\Box)}(\overset{Arity(U)+Arity(V)}{\underset{j=Arity(U)+1}{\boxed{I_j^{Arity(U)+Arity(V)}(\overset{Arity(U)}{\underset{i=1}{\boxed{r_i}}}, \overset{Arity(V)}{\underset{i=1}{\boxed{s_i}}})}}})$

— $\chi'_{V(\Box)} \in PR$

¡Composition¿ — $\chi_{U(\Box)\vee V(\Box)}(\overset{Arity(U)}{\underset{i=1}{\boxed{r_i}}}, \overset{Arity(V)}{\underset{i=1}{\boxed{s_i}}}) = \cdot(\chi'_{U(\Box)}(\overset{Arity(U)}{\underset{i=1}{\boxed{r_i}}}, \overset{Arity(V)}{\underset{i=1}{\boxed{s_i}}}), \chi'_{V(\Box)}(\overset{Arity(U)}{\underset{i=1}{\boxed{r_i}}}, \overset{Arity(V)}{\underset{i=1}{\boxed{s_i}}}))$

— $\chi_{U(\Box)\vee V(\Box)} \in PR$

¡Composition¿ ==

5.2.9 (Metatheorem) Standard ordering relations are primitive recursive

- The relations $\chi_{\leq(\Box)}, \chi_{<(\Box)}, \chi_{=(\Box)}$ from the standard number theory ($\mathcal{N}$) are primitive recursive
- $\chi_{\leq(\Box)} \in PR$
- Proof:

– $\langle c_1^2, \dot{-}, +, \chi_{\neg\leq(\Box)}, \chi_{\leq(\Box)\wedge\leq(\Box)}\rangle \in PR$

¡Misc. theorems¿ – $\chi_{x\leq y}(x, y) = 1 \dot{-} ((y+1) \dot{-} x)$

¡Informal¿ – $\chi_{\leq(\Box)} \in PR$

- $\chi_{<(\Box)} \in PR$
- Proof:

– $\chi_{x<y}(x, y) = \chi_{\neg(y\leq x)}$

¡Informal¿ – $\chi_{<(\Box)} \in PR$

- $\chi_{<(\Box)} \in PR$
- Proof:

– $\chi_{x=y}(x, y) = \chi_{x\leq y \wedge y\leq x}(x, y)$

¡Informal¿ – $\chi_{=(\Box)} \in PR$

==

5.2.10 (Metatheorem) Bounded sums and products are closed under the primitive recursion

- If $f : \mathbb{N}^{n+1} \to \mathbb{N} \in PR$, then $Sum(f) : \mathbb{N}^{n+1} \to \mathbb{N} \subseteq PR$
- Proof:

– If $f \in PR$, then

— $Sum(f)(\overset{n}{\underset{i=1}{\boxed{x_i}}}, 0) = f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, 0)$

¡Informal¿ — $Sum(f)(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y+1) = f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y+1) + Sum(f)(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y)$

¡Informal¿ — $Sum(f) \in PR$

¡Primitive recursion¿ - If $f : \mathbb{N}^{n+1} \to \mathbb{N} \in PR$, then $Prod(f) : \mathbb{N}^{n+1} \to \mathbb{N} \subseteq PR$

- Proof:

– If $f \in PR$, then

— $Prod(f)(\overset{n}{\underset{i=1}{\boxed{x_i}}}, 0) = f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, 0)$

¡Informal¿ — $Prod(f)(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y+1) = f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y+1) \cdot Prod(f)(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y)$

¡Informal¿ — $Prod(f) \in PR$

¡Primitive recursion¿ ==

5.2.11 (Metatheorem) Bounded quantifiers are closed under the primitive recursion

- If $\chi_{P(\Box)} : \mathbb{N}^{n+1} \to \mathbb{N} \in PR$, then $\chi_{(\exists i\leq m)P(\Box)} : \mathbb{N}^{n+1} \to \mathbb{N} \in PR$
- Proof:

– $Prod_{\chi_{P(\Box)}} \in PR$

¡Bounded sums and products are closed under the primitive recursion¿ – $\chi_{(\exists i \leq m)P(\Box)}(\underset{j=1}{\overset{n}{\boxed{x_j}}}, m) = Prod_{\chi_{P(\Box)}}(\underset{j=1}{\overset{n}{\boxed{x_j}}}, m)$
¡Informal¿ – $\chi_{(\exists i \leq m)P(\Box)} \in PR$
¡Composition¿ - If $\chi_{P(\Box)} \in PR$, then $\chi_{(\forall i \leq m)P(\Box)} : \mathbb{N}^{n+1} \to \mathbb{N} \in PR$
- Proof:
– $\chi_{(\exists i \leq m)P(\Box)} \in PR$
¡Bounded quantifiers are closed under the primitive recursion¿ – $\chi_{\neg(\exists i \leq m)\neg P(\Box)} \in PR$
¡Standard logic connectives are closed under the primitive recursion¿ – $\chi_{(\forall i \leq m)P(\Box)}(\underset{j=1}{\overset{n}{\boxed{x_j}}}, m) = \chi_{\neg(\exists i \leq m)\neg P(\Box)}(\underset{j=1}{\overset{n}{\boxed{x_j}}}, y)$
¡Informal¿ – $\chi_{(\forall i \leq m)P(\Box)} \in PR$
¡Composition¿ ==

5.2.12 (Definition) Definition by cases

- The function $f : \mathbb{N}^n \to \mathbb{N}$ is defined by cases using the functions $h, g_1, g_2 : \mathbb{N}^n \to \mathbb{N}$ iff
– If $h(\underset{i=1}{\overset{n}{\boxed{x_i}}}) = 0$, then $f(\underset{i=1}{\overset{n}{\boxed{x_i}}}) = g_1(\underset{i=1}{\overset{n}{\boxed{x_i}}})$ and
– Otherwise, $f(\underset{i=1}{\overset{n}{\boxed{x_i}}}) = g_2(\underset{i=1}{\overset{n}{\boxed{x_i}}})$
==

5.2.13 (Metatheorem) Definition by cases is closed under the primitive recursion

- If $\{h, g_1, g_2\} \subseteq PR$ and f is defined by cases using h, g_1, g_2, then $f \in PR$
- Proof:
– If $\{h, g_1, g_2\} \subseteq PR$ and f is defined by cases using h, g_1, g_2, then
— $\{\chi_{h(\Box)=0}, Conj, \cdot, +\} \subseteq PR$
¡Misc. theorems¿ — $f(\underset{i=1}{\overset{n}{\boxed{x_i}}}) = Conj(\chi_{h(\Box)=0}(\underset{i=1}{\overset{n}{\boxed{x_i}}})) \cdot g_1(\underset{i=1}{\overset{n}{\boxed{x_i}}}) + \chi_{h(\Box)=0}(\underset{i=1}{\overset{n}{\boxed{x_i}}}) \cdot g_2(\underset{i=1}{\overset{n}{\boxed{x_i}}})$
¡Informal¿ — $f \in PR$
¡Composition¿ ==

5.2.14 (Definition) Bounded minimalization

- The function $\mu_{BS}(g) : \mathbb{N}^{n+1} \to \mathbb{N}$ is a bounded minimalization using the function $g : \mathbb{N}^{n+1} \to \mathbb{N}$ iff
– If there exists $i \leq y$, $g(\underset{j=1}{\overset{n}{\boxed{x_j}}}, i) = 0$ and for any $j < i$, $g(\underset{j=1}{\overset{n}{\boxed{x_j}}}, j) \neq 0$, then $\mu_{BS}(g)(\underset{j=1}{\overset{n}{\boxed{x_j}}}, y) = i$
– Otherwise, $\mu_{BS}(g)(\underset{j=1}{\overset{n}{\boxed{x_j}}}, y) = y + 1$
==

5.2.15 (Metatheorem) Bounded minimalization is closed under the primitive recursion

- If $g : \mathbb{N}^{n+1} \to \mathbb{N} \in PR$, then $\mu_{BS}(g) \in PR$
- Proof:
– If $g : \mathbb{N}^{n+1} \to \mathbb{N} \in PR$, then
— $\{\chi_{(\exists i \leq y)(g(\Box)=0)}, Sum(\chi_{(\exists i \leq y)(g(\Box)=0)})\} \subseteq PR$
¡Misc. theorems¿ — $\mu_{BS}(g)(\underset{j=1}{\overset{n}{\boxed{x_j}}}, y) = Sum(\chi_{(\exists i \leq y)(g(\Box)=0)})(\underset{j=1}{\overset{n}{\boxed{x_j}}}, y)$
¡Informal¿ — $\mu_{BS} \in PR$
¡Composition¿ - $\mu_{BS}(g)(\underset{j=1}{\overset{n}{\boxed{x_j}}}, y) = Sum(\chi_{(\exists i \leq y)(g(\Box)=0)})(\underset{j=1}{\overset{n}{\boxed{x_j}}}, y)$
- Proof:
– If there exists $i \leq y$, $g(\underset{j=1}{\overset{n}{\boxed{x_j}}}, i) = 0$ and for any $j < i$, $g(\underset{j=1}{\overset{n}{\boxed{x_j}}}, j) \neq 0$, then
— For any $a < i$, $\chi_{(\exists i \leq y)(g(\Box)=0)}(\underset{j=1}{\overset{n}{\boxed{x_j}}}, a) = 1$

— For any $i \leq b \leq y$, $\chi_{(\exists i \leq y)(g(\square)=0)}(\boxed{x_j}_{j=1}^{n}, b) = 0$
— $Sum(\chi_{(\exists i \leq y)(g(\square)=0)})(\boxed{x_i}_{i=1}^{n}, y) = \sum_{z=0}^{i-1}(1) + \sum_{z=i}^{y}(0) = i$
– Otherwise,
— $Sum(\chi_{(\exists i \leq y)(g(\square)=0)})(\boxed{x_i}_{i=1}^{n}, y) = \sum_{z=0}^{y}(1) = y + 1$
- Note that the occurrence of y in $\chi_{(\exists i \leq y)}$ also varies with $y \in Sum$
==

5.2.16 (Metatheorem) Prime number function is the primitive recursive

- The prime number function $p \in PR$
- Proof:
– $NotPrime(x)$ iff $\neg(2 \leq x \wedge (\forall y \leq x)(\forall z \leq x)((y+2) \cdot (z+2) \neq x))$
– $NumPrimesLeq(x) = Sum(\chi_{NotPrime(x)})(x)$
– $p(n)$ as definition by cases:
— If $I_1^1(n) = 0$, then $p(n) = 1$
— Otherwise, $p(n) = \mu_{BS}(\chi_{NumPrimesLeq(\square)=n})(2^{2^n})$
¡N-th prime is bounded by 2ˆ(2ˆn)¿ – $p \in PR$
¡Misc. theorems¿ ==

5.2.17 (Definition) Prime factor index function

- The prime factor index function π_i returns the exponent of the ith prime factor in its unique prime factorization
- $\pi_i(n)$ as definition by cases:
– If $\chi_{n \leq 1}(n) = 0$, $\pi_i(n) = 0$
– Otherwise, $\pi_i(n) = \mu_{BS}(\chi_{(\exists x \leq n)(x \cdot p(i)E\square=n) \wedge (\forall x \leq n)(x \cdot p(i)E(\square+1) \neq n)})(n)$
==

5.2.18 (Metatheorem) Prime factor index function is primitive recursive

- For any $i > 0$, $\pi_i : \mathbb{N} \to \mathbb{N} \in PR$
- Proof: all functions used are in PR or closed under PR
¡Misc. theorems¿ ==

5.2.19 (Metatheorem) SingleDec, length, isCodeFor functions are primitive recursive

- For any $\left\langle \boxed{a_i}_{i=1}^{n} \right\rangle$, there exists $a \in \mathbb{N}$, there are the following primitive recursive functions:
– $len(a) = n$
– $singleDec_j(a) = a_j$
– $\{len, singleDec_j\} \subseteq PR$
- $isCodeFor(a, \boxed{a_i}_{i=1}^{n})$ iff $len(a) = n$ and for any $1 \leq j \leq n$, $singleDec_j(a) = a_j$ and $\chi_{isCodeFor} \in PR$
- Proof: all utilized functions and relations of prime numbers are in PR or closed under PR
¡Misc. theorems¿ ==

5.2.20 (Metatheorem) IsCode, empty, singleEnc, concatenate functions are primitive recursive

- $isCode(a)$ iff there exists $\left\langle \boxed{a_i}_{i=1}^{len(a)} \right\rangle$, $isCodeFor(a, \left\langle \boxed{a_i}_{i=1}^{len(a)} \right\rangle)$ and $\chi_{isCode} \in PR$
- $len(empty()) = 0$ and $empty \in PR$
- If $len(a) = 1$, then there $singleEnc(a) = p(1)E(singleDec_1(a) + 1)$ and $singleEnc \in PR$
- If $isCodeFor(a, \boxed{a_i}_{i=1}^{n})$ and $isCodeFor(b, \boxed{b_j}_{j=1}^{m})$, then $isCodeFor(cat(a, b), \boxed{a_i}_{i=1}^{n}, \boxed{b_j}_{j=1}^{m})$ and $cat \in PR$
- Proof: all utilized functions and relations of prime numbers are in PR or closed under PR
¡Misc. theorems¿ ==

5.2.21 (Metatheorem) Enc, dec are primitive recursive

- $enc(\boxed{x_i}_{i=1}^{n}) = cat(\boxed{singleEnc(x_i)}_{i=1}^{n})$ and $enc \in PR$
- $dec(x) = \left\langle \boxed{singleDec_i(x)}_{i=1}^{n} \right\rangle$ and $dec \in PR$
- Alternative definitions could be formed using bounded products and the prime number function
- Proof: all utilized functions and relations of prime numbers are in PR or closed under PR

¡Misc. theorems¿ ===

5.2.22 (Metatheorem) Coding is monotonic

- For any $\left\langle \boxed{a_i}_{i=1}^{n} \right\rangle$, for any $1 \leq m \leq n$, $enc(\boxed{a_i}_{i=1}^{n}) < enc(\boxed{a_h}_{h=1}^{m-1}, a_m + 1, \boxed{a_t}_{t=m+1}^{n})$
- For any $\left\langle \boxed{a_i}_{i=1}^{n+1} \right\rangle$, $enc(\boxed{a_i}_{i=1}^{n}) < enc(\boxed{a_i}_{i=1}^{n+1})$
- These monotonicity properties guarantee that:
– All the numbers in the sequence encoded by the number x will be smaller than x
– The code for a subsequence of a sequence will be smaller than the code for the sequence itself
– This makes it easy to find primitive recursive definitions of predicates and functions dealing with encoded sequences
- Proof: TODO ABSTRACTED

===

5.2.23 (Metatheorem) Subbed Godel numbering function is the primitive recursive

- If $\phi \in Form(\mathcal{L}_{NT})$ and $free(x, \phi)$, then there exists $f_\phi(a) = GN(|\phi|\frac{x}{a})$ and $f_\phi \in PR$
- Proof:
– If $t :\equiv 0$, then $g_t(a) = enc(9)$
– If $t :\equiv v_i$, then $g_t(a) = enc(2i)$
– If $t :\equiv St_1$, then $g_t(a) = enc(11, g_{t_1}(a))$
– If $t :\equiv +t_1t_2$, then $g_t(a) = enc(13, g_{t_1}(a), g_{t_2}(a))$
– If $t :\equiv \cdot t_1t_2$, then $g_t(a) = enc(15, g_{t_1}(a), g_{t_2}(a))$
– If $t :\equiv Et_1t_2$, then $g_t(a) = enc(17, g_{t_1}(a), g_{t_2}(a))$
– If $\phi :\equiv\equiv t_1t_2$, then $f_\phi(a) = enc(7, g_{t_1}(a), g_{t_2}(a))$
– If $\phi :\equiv< t_1t_2$, then $f_\phi(a) = enc(19, g_{t_1}(a), g_{t_2}(a))$
– If $\phi :\equiv \neg\alpha$, then $f_\phi(a) = enc(1, f_\alpha(a))$
– If $\phi :\equiv \alpha \vee \beta$, then $f_\phi(a) = enc(3, f_\alpha(a), f_\beta(a))$
– If $\phi :\equiv \forall v_i\alpha$, then $f_\phi(a) = enc(5, g_{v_i}(a), f_\alpha(a))$
– $f_\phi = GN(|\phi|\frac{x}{\square})$

¡Induction¿ – $f_\phi \in PR$

¡Misc. theorems¿ ===

5.2.24 (Definition) Ackermann function

- The Ackermann function $A : \mathbb{N}^2 \to \mathbb{N}$ is defined as:
– $A(0, y) = y + 1$
– $A(x + 1, 0) = A(x, 1)$
– $A(x + 1, y + 1) = A(x, A(x + 1, y))$

===

5.2.25 (Definition) Majorization

- The function $h : \mathbb{N}^n \to \mathbb{N}$ is majorized by the function $f : \mathbb{N}^2 \to \mathbb{N}$ ($Majorized(h, f)$) iff there exists b, for any $\left\{ \boxed{a_i}_{i=1}^{n} \right\} \subseteq \mathbb{N}$,

$h(\boxed{a_i}_{i=1}^{n}) < f(b, max(\boxed{a_i}_{i=1}^{n}))$

===

5.2.26 (Metatheorem) Binary functions cannot majorize themselves

- $f : \mathbb{N}^2 \to \mathbb{N}$ and $Maj\widetilde{orized}(f, f)$
- Proof:
– If $Majorized(f, f)$, then
— There exists b, for any x, y, $f(x, y) < f(b, max(x, y))$
— $f(b, max(x, y, b)) < f(b, max(x, y, b)) = f(b, max(x, y, b))$
— CONTRADICTION
– $Maj\widetilde{orized}(f, f)$
¡Metaproof by contradiction¿ ==

5.2.27 (Definition) Majorized by the Ackermann function

- The set $\mathcal{A}$ is defined by $\mathcal{A} = \{h : Majorized(h, A)\}$
===

5.2.28 (Metatheorem) Primitive recursive functions are majorized by the Ackermann function

- $PR \subseteq \mathcal{A}$
- Proof: TODO: ABSTRACTED
https://planetmath.org/ackermannfunctionisnotprimitiverecursive – $a_{max} = max(\overset{k}{\underset{i=1}{\boxed{a_i}}})$
– If $f = \mathcal{O}$, then
— $f(a) = 0 < a + 1 = A(0, a_{max})$
— $f \in \mathcal{A}$
– If $f = S$, then
— $f(a) = a + 1 < a + 2 = A(1, a_{max})$
— $f \in \mathcal{A}$
– If $f = \mathcal{I}^m_j$, then
— $f(\overset{Arity(f)}{\underset{i=1}{\boxed{a_i}}}) = a_j \leq a_{max} < a_{max} + 1 = A(0, a_{max})$
— $f \in \mathcal{A}$
– If $f(\overset{Arity(f)}{\underset{i=1}{\boxed{a_i}}}) = h(\overset{Arity(h)}{\underset{j=1}{\boxed{g_j(\overset{Arity(f)}{\underset{i=1}{\boxed{a_i}}})}}})$ and $\left\{h, \overset{Arity(h)}{\underset{j=1}{\boxed{g_j}}}\right\} \subseteq \mathcal{A}$, then
— For any $1 \leq j \leq Arity(h)$, there exists r_{g_j}, $g_j(\overset{Arity(f)}{\underset{i=1}{\boxed{a_i}}}) < A(r_{g_j}, a_{max})$
¡Inductive hypothesis¿ — There exists r_h, $h(\overset{Arity(h)}{\underset{i=1}{\boxed{a_i}}}) < A(r_h, a_{max})$
¡Inductive hypothesis¿ — $f(\overset{Arity(f)}{\underset{i=1}{\boxed{a_i}}}) = h(\overset{Arity(h)}{\underset{j=1}{\boxed{g_j(\overset{Arity(f)}{\underset{i=1}{\boxed{a_i}}})}}}) < A(r_h, g_{j_{max}})$
¡Inductive hypothesis¿ — $A(r_h, g_{j_{max}}) < A(r_h, A(r_{g_j}, a_{max}))$
¡Monotonic property¿ — $A(r_h, A(r_{g_j}, a_{max})) < A(b, a_{max})$
¡Branch is primitive recurisve property¿ — $f \in \mathcal{A}$
– If $f ... primitiverecursion$, then $f \in \mathcal{A}$
– $PR \subseteq \mathcal{A}$
¡Induction¿ ==

5.2.29 (Metatheorem) Ackermann function is not primitive recursive

- $A \notin PR$
- Proof:
– If $f \in PR$, then $f \in \mathcal{A}$
¡Primitive recursive functions are majorized by the Ackermann function¿ – If $f \notin \mathcal{A}$, then $f \notin PR$

¡Contrapositive¿ – $A \notin \mathcal{A}$
¡Binary functions cannot majorize themselves¿ – $A \notin PR$
¡Conjunction¿ ==

5.2.30 (Definition) Computable index

- The natural number e is a computable index for the function f $(CI(e_f, f))$ iff:
– If $f = S$, then $e_f = enc(0)$
– If $f = \mathcal{I}_i^n$, then $e_f = enc(1, n, i)$
– If $f = \mathcal{O}$, then $e_f = enc(2)$
– If $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = h(\overset{Arity(h)}{\underset{j=1}{\boxed{g_j(\overset{n}{\underset{i=1}{\boxed{x_i}}})}}})$ and for any $1 \leq j \leq Arity(h)$, $CI(e_{g_j}, g_j)$ and $CI(e_h, h)$, then $e_f = enc(3, n, \overset{Arity(h)}{\underset{j=1}{\boxed{e_{g_j}}}}, e_h)$
– If $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, 0) = g(\overset{n}{\underset{i=1}{\boxed{x_i}}})$ and $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y+1) = h(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y, f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y))$ and $CI(e_g, g)$ and $CI(e_h, h)$, then $e_f = enc(4, n, e_g, e_h)$
– If $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = \mu_{US}(g)(\overset{n}{\underset{i=1}{\boxed{x_i}}})$ and $CI(e_g, g)$, then $e_f = enc(5, n, e_g)$
- e_f is like a computer program / source code for f
– ALTERNATIVE TRASH – $e_f = enc(0)$ and $f = S$
– $e_f = enc(1, n, i)$ and $f = \mathcal{I}_i^n$
– $e_f = enc(2)$ and $f = \mathcal{O}$
– $e_f = enc(3, n, \overset{Arity(h)}{\underset{j=1}{\boxed{e_{g_j}}}}, e_h)$ and $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = h(\overset{Arity(h)}{\underset{j=1}{\boxed{g_j(\overset{n}{\underset{i=1}{\boxed{x_i}}})}}})$ and for any $1 \leq j \leq Arity(h)$, $CI(e_{g_j}, g_j)$ and $CI(e_h, h)$
– $e_f = enc(4, n, e_g, e_h)$ and $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, 0) = g(\overset{n}{\underset{i=1}{\boxed{x_i}}})$ and $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y+1) = h(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y, f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y))$ and $CI(e_g, g)$ and $CI(e_h, h)$
– $e_f = enc(5, n, e_g)$ and $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = \mu_{US}(g)(\overset{n}{\underset{i=1}{\boxed{x_i}}})$ and $CI(e_g, g)$
==

5.2.31 (Metatheorem) Padding lemma

- If $f \in \mu$, then there exists E, $InfiniteSet(E)$ and for any $e \in E$, $CI(e, f)$
- Proof:
– By definition of computable index, $CI(e_f, f)$
– Let $I_1^1(f(x)) = f(x)$, so $CI(e_{I_1^1(f)}, f)$, and so on ...
– TODO ABSTRACTED
==

5.2.32 (Definition) Computations

- The collection of computations $\mathcal{C}$ is defined by:
– If $C = \langle\rangle$, then $C \in \mathcal{C}$
– If $C = \Gamma \cup \langle enc(e_S, a, b)\rangle$ and $\Gamma \in \mathcal{C}$ and
— $CI(e_S, S)$ and $b = S(a)$, then
— $C \in \mathcal{C}$
– If $C = \Gamma \cup \left\langle enc(e_{\mathcal{I}_i^n}, enc(\overset{n}{\underset{i=1}{\boxed{a_i}}}), b) \right\rangle$ and $\Gamma \in \mathcal{C}$ and
— $CI(e_{\mathcal{I}_i^n}, \mathcal{I}_i^n)$ and
— $1 \leq i \leq n$ and $b = a_i$, then
— $C \in \mathcal{C}$
– If $C = \Gamma \cup \langle enc(e_{\mathcal{O}}, enc(), 0)\rangle$ and $\Gamma \in \mathcal{C}$ and
— $CI(e_{\mathcal{O}}, \mathcal{O})$, then
— $C \in \mathcal{C}$
– If $C = \Gamma \cup \left\langle enc(e_f, enc(\overset{n}{\underset{i=1}{\boxed{a_i}}}), b) \right\rangle$ and $\Gamma \in \mathcal{C}$ and

— $f(\underset{i=1}{\overset{n}{\boxed{a_i}}}) = h(\underset{j=1}{\overset{Arity(h)}{\boxed{g_j(\underset{i=1}{\overset{n}{\boxed{a_i}}})}}})$ and $CI(e_f, f)$ and

— There exists $\left\{\underset{j=1}{\overset{Arity(h)}{\boxed{v_j}}}\right\} \subseteq \mathbb{N}$, (—- For any $1 \leq l \leq Arity(h)$, $enc(e_{g_l}, enc(\underset{i=1}{\overset{n}{\boxed{a_i}}}), v_l) \in \Gamma$ and

—- $enc(e_h, \underset{j=1}{\overset{Arity(h)}{\boxed{v_j}}}, b) \in \Gamma$), then

— $C \in \mathcal{C}$

– If $C = \Gamma \cup \left\langle enc(e_f, enc(\underset{i=1}{\overset{n}{\boxed{a_i}}}, c), b) \right\rangle$ and $\Gamma \in \mathcal{C}$ and

— $f(\underset{i=1}{\overset{n}{\boxed{a_i}}}, 0) = g(\underset{i=1}{\overset{n}{\boxed{a_i}}})$ and $f(\underset{i=1}{\overset{n}{\boxed{a_i}}}, y+1) = h(\underset{i=1}{\overset{n}{\boxed{a_i}}}, y, f(\underset{i=1}{\overset{n}{\boxed{a_i}}}, y))$ and $CI(e_f, f)$ and

— There exists $\left\{\underset{j=0}{\overset{c}{\boxed{v_j}}}\right\} \subseteq \mathbb{N}$, (

—- $enc(e_g, enc(\underset{i=1}{\overset{n}{\boxed{a_i}}}), v_0) \in \Gamma$ and

—- For any $1 \leq l \leq c$, $enc(e_h, enc(\underset{i=1}{\overset{n}{\boxed{a_i}}}, l, v_{l-1}), v_l))$, then

— $C \in \mathcal{C}$

– If $C = \Gamma \cup \left\langle e_f, enc(\underset{i=1}{\overset{n}{\boxed{a_i}}}), b \right\rangle$ and $\Gamma \in \mathcal{C}$ and

— $f(\underset{i=1}{\overset{n}{\boxed{a_i}}}) = \mu_{US}(g)(\underset{i=1}{\overset{n}{\boxed{a_i}}})$ and $CI(e_f, f)$ and

— There exists $\left\{\underset{j=0}{\overset{b}{\boxed{v_j}}}\right\} \subseteq \mathbb{N}$, (

—- For any $0 \leq l \leq b$, $enc(e_g, enc(\underset{i=1}{\overset{n}{\boxed{a_i}}}, l), v_l) \in \Gamma$ and

—- If $i < b$, then $v_i \neq 0$ and $v_b = 0$), then

— $C \in \mathcal{C}$

==

5.2.33 (Metatheorem) Computation iff computable indexable

- If $CI(e_f, f)$, then $f(\underset{i=1}{\overset{n}{\boxed{a_i}}}) = b$ iff there exists $\Gamma \in \mathcal{C}$, $enc(e_f, enc(\underset{i=1}{\overset{n}{\boxed{a_i}}}), b) \in \Gamma$
- Proof:

– If $CI(e_f, f)$, then

— If $e_f = enc(0)$, then

—- $f = S$

¡Definition¿ —- If $f(a) = b$, then

—– $f(a) = S(a) = a + 1 = b$

—– There exists $\Omega \in \mathcal{C}$

—– $\Gamma = \Omega \cup \langle e_f, enc(a), b \rangle \in \mathcal{C}$

¡Definition¿ —– There exists $\Gamma \in \mathcal{C}$, $enc(e_f, enc(\underset{i=1}{\overset{n}{\boxed{a_i}}}), b) \in \Gamma$

—- If there exists $\Gamma \in \mathcal{C}$, $enc(e_f, enc(\underset{i=1}{\overset{n}{\boxed{a_i}}}), b) \in \Gamma$, then

—– $b = a + 1$

¡Definition¿ —– $b = a + 1 = S(a) = f(a)$

—- $f(a) = b$ iff there exists $\Gamma \in \mathcal{C}$, $enc(e_f, enc(\underset{i=1}{\overset{n}{\boxed{a_i}}}), b) \in \Gamma$

¡Conjunction¿ — If $e_f = enc(1, n, i)$, then

—- $f = \mathcal{I}_i^n$

¡Definition¿ —- If $f(\overset{n}{\underset{i=1}{\boxed{a_i}}}) = b$, then

—- $f(\overset{n}{\underset{i=1}{\boxed{a_i}}}) = \mathcal{I}_i^n(\overset{n}{\underset{i=1}{\boxed{a_i}}}) = a_i = b$

—— There exists $\Omega \in \mathcal{C}$

—— $\Gamma = \Omega \cup \left\langle e_f, enc(\overset{n}{\underset{i=1}{\boxed{a_i}}}), b \right\rangle \in \mathcal{C}$

¡Definition¿ —— There exists $\Gamma \in \mathcal{C}$, $enc(e_f, enc(\overset{n}{\underset{i=1}{\boxed{a_i}}}), b) \in \Gamma$

—- If there exists $\Gamma \in \mathcal{C}$, $enc(e_f, enc(\overset{n}{\underset{i=1}{\boxed{a_i}}}), b) \in \Gamma$, then

—— $b = a_i$

¡Definition¿ —— $b = a_i = \mathcal{I}_i^n(\overset{n}{\underset{i=1}{\boxed{a_i}}}) = f(\overset{n}{\underset{i=1}{\boxed{a_i}}})$

—- $f(a) = b$ iff there exists $\Gamma \in \mathcal{C}$, $enc(e_f, enc(\overset{n}{\underset{i=1}{\boxed{a_i}}}), b) \in \Gamma$

¡Conjunction¿ — TODO ABSTRACTED
==

5.2.34 (Metatheorem) Computation iff computable indexable corollary

- If $CI(e_f, f)$, then $f(\overset{n}{\underset{i=1}{\boxed{a_i}}}) = b$ iff there exists $\Gamma \in \mathcal{C}$, $\Gamma = \Omega \cup \left\langle enc(e_f, enc(\overset{n}{\underset{i=1}{\boxed{a_i}}}), b) \right\rangle$
- Proof:

TODO: ABSTRACTED ==

5.2.35 (Notation) Indexed abbreviations

- $dec_{a,b}(t) = singleDec_b(singleDec_a(t))$

==

5.2.36 (Metatheorem) IsComputation is primitive recursive

- The predicate $isComputation$ is defined as $isComputation(t)$ iff $t = enc(\overset{k}{\underset{i=1}{\boxed{c_i}}})$ and $k \geq 1$ and $\left\langle \overset{k}{\underset{i=1}{\boxed{c_i}}} \right\rangle \in \mathcal{C}$
- $\chi_{isComputation} \in PR$
- Proof:

TODO: ABSTRACTED ==

5.2.37 (Metatheorem) T-predicate is primitive recursive

- The predicate $\mathcal{T}_n$ is defined as $\mathcal{T}_n(e, \overset{n}{\underset{i=1}{\boxed{x_i}}}, t)$ iff $isComputation(t)$ and $dec_{len(t),1}(t) = e$ and $len(dec_{len(t),2}(t)) = n$ and $\left\langle \overset{n}{\underset{i=1}{\boxed{dec_{len(t),2,i}(t)}}} \right\rangle = \left\langle \overset{n}{\underset{i=1}{\boxed{x_i}}} \right\rangle$
- $\mathcal{T}_n(e_f, \overset{n}{\underset{i=1}{\boxed{x_i}}}, t)$ states that the number t encodes an execution of the program given by the index e on the inputs $\overset{n}{\underset{i=1}{\boxed{x_i}}}$
- $\chi_{\mathcal{T}_n} \in PR$
- Proof:

TODO: ABSTRACTED ==

5.2.38 (Metatheorem) U is primitive recursive

- The function $\mathcal{U}$ is defined as $\mathcal{U}(t) = dec_{len(t),3}(t)$
- $\mathcal{U}(t)$ picks the output from the computation encoded by t
- $\mathcal{U} \in PR$
- Proof:

TODO: ABSTRACTED ==

5.2.39 (Metatheorem) Kleene's Normal Form theorem***

- For any $f \in \mu$, if $CI(e_f, f)$, then $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = \mathcal{U}(\mu_{US}(\chi_{\mathcal{T}_n})(e_f, \overset{n}{\underset{i=1}{\boxed{x_i}}}))$
- Proof:

TODO: ABSTRACTED – $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = b$ is defined iff

— There exists $\Gamma \in \mathcal{C}$, $\Gamma = \Omega \cup \left\langle enc(e_f, enc(\overset{n}{\underset{i=1}{\boxed{a_i}}}), b) \right\rangle$ iff

¡Computation iff computable indexable corollary¿ —- $\mu_{US}(\chi_{\mathcal{T}_n})(e_f, \overset{n}{\underset{i=1}{\boxed{x_i}}})$ is defined iff

—- $\mathcal{U}(\mu_{US}(\chi_{\mathcal{T}_n})(e_f, \overset{n}{\underset{i=1}{\boxed{x_i}}})) = b$

¡Definition¿ ==

5.2.40 (Definition) Computable function by index

- The e-th N-ary computable function $\{e\}^n$ is defined as $\{e\}^n(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = \mathcal{U}(\mu_{US}(\chi_{\mathcal{T}_n})(e, \overset{n}{\underset{i=1}{\boxed{x_i}}}))$
- If $CI(e, f)$, then $\{e\}^n = f$
- Otherwise, $\{e\}^n$ is undefined everywhere

==

5.2.41 (Metatheorem) Enumeration theorem***

- For any $f \in \mu$, there exists e, $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = \{e\}^n((\overset{n}{\underset{i=1}{\boxed{x_i}}}))$
- Proof:

– For any $f \in \mu$,

— There exists e, $CI(e, f)$

¡Padding lemma¿ — $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = \mathcal{U}(\mu_{US}(\chi_{\mathcal{T}_n})(e, \overset{n}{\underset{i=1}{\boxed{x_i}}}))$

¡Kleene's Normal Form theorem¿ — $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = \{e\}^n((\overset{n}{\underset{i=1}{\boxed{x_i}}}))$

¡Definition¿ - The function g is defined as $g(y, \overset{n}{\underset{i=1}{\boxed{x_i}}}) = \mathcal{U}(\mu_{US}(\chi_{\mathcal{T}_n})(y, \overset{n}{\underset{i=1}{\boxed{x_i}}}))$

- g outputs the computable function indexed by y
- $g \in \mu$ and for any $y \in \mathbb{N}$, $g(y, \overset{n}{\underset{i=1}{\boxed{x_i}}}) = \{y\}^n(\overset{n}{\underset{i=1}{\boxed{x_i}}})$
- Proof:

TODO: ABSTRACTED – $\mathcal{U} \in \mu$

¡U is primitive recursive¿ – $\chi_{\mathcal{T}_n} \in \mu$

¡T-predicate is primitive recursive¿ – $g \in \mu$

¡Misc. theorems¿ – For any $y \in \mathbb{N}$,

— $\{y\}^n(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = \mathcal{U}(\mu_{US}(\chi_{\mathcal{T}_n})(y, \overset{n}{\underset{i=1}{\boxed{x_i}}}))$

¡Definition¿ — $g(y, \overset{n}{\underset{i=1}{\boxed{x_i}}}) = \{y\}^n(\overset{n}{\underset{i=1}{\boxed{x_i}}})$

==

5.2.42 (Metatheorem) Universal function theorem

- The computable function u is defined as $u(y, enc(\overset{n}{\underset{i=1}{\boxed{x_i}}})) = \{y\}^1(enc(\overset{n}{\underset{i=1}{\boxed{x_i}}}))$
- u is the universal function
- For any $f \in \mu$, there exists $t \in \mathbb{N}$, $u(t, enc(\overset{n}{\underset{i=1}{\boxed{x_i}}})) = f(\overset{n}{\underset{i=1}{\boxed{x_i}}})$
- Proof:

TODO: ABSTRACTED – There exists $f_0 \in \mu$, $f_0(enc(\boxed{x_i}_{i=1}^{n})) = f(\boxed{x_i}_{i=1}^{n})$

¡Misc. theorems¿ – There exists y, $CI(y, f_0)$

– $u(y, enc(\boxed{x_i}_{i=1}^{n})) =$

— $\{y\}^1(enc(\boxed{x_i}_{i=1}^{n})) =$

¡Definition¿ — $\mathcal{U}(\mu_{US}(\chi_{\mathcal{T}_n})(y, enc(\boxed{x_i}_{i=1}^{n}))) =$

¡Definition¿ — $f_0(enc(\boxed{x_i}_{i=1}^{n})) =$

¡Kleene's Normal Form theorem¿ — $f(\boxed{x_i}_{i=1}^{n})$

==

5.2.43 (Metatheorem) Diagonal functions are non-computable

- For simplicity, consider functions that are only 1-ary
- The diagonal function d is defined as $d(i) \neq \{i\}^1(i)$
- $d \notin \mu$
- Proof: TODO ABSTRACTED

– For any $f \in \mu$,
— There exists e_f, $CI(e_f, f)$
— $d(e_f) \neq \{e_f\}^1(e_f)$
¡Definition¿ — $\{e_f\}^1(e_f) =$
—- $\mathcal{U}(\mu_{US}(\chi_{\mathcal{T}_n})(e_f, e_f)) =$
¡Definition¿ —- $f(e_f)$
¡Kleene's Normal Form theorem¿ — $d(e_f) \neq f(e_f)$
¡Conjunction¿ — $d \neq f$
– For any $f \in \mu$, $d \neq f$
¡Abbreviate¿ – $d \notin \mu$

==

5.2.44 (Metatheorem) Total diagonal functions are non-computable

- One simple total example of d^* can be defined as:

– If $\{x\}^1(x)$ is defined, then $d^*(x) = \{x\}^1(x) + 1$
– Otherwise, $d^*(x) = 0$

- $Total(d^*)$ and d^* satisfies the properties of the diagonal function, thus $d^* \notin \mu$

==

5.2.45 (Metatheorem) Undecidability of the Halting Problem

- The halting predicate H is defined as $H(y, x)$ iff $u(y, x)$ is defined
- $\chi_H \notin \mu$
- Proof:

– If $\chi_H \in \mu$, then
— If $\chi_H(x, x) = 0$, then $d'(x) = \{x\}^1(x) + 1$ and otherwise, $d'(x) = 0$
— $d' \in \mu$
¡Definition by cases are closed under primitive recursion¿ — $d' \notin \mu$
¡Total diagonal functions are non-computable¿ — CONTRADICTION !! – $\chi_H \notin \mu$
¡Metaproof by contradiction¿ ==

5.2.46 (Metatheorem) S-m-n theorem

- There exists $S_n^m \in PR$, $\{S_n^m(e, \boxed{x_i}_{i=1}^{n})\}^m(\boxed{y_j}_{j=1}^{m}) = \{e\}^{n+m}(\boxed{x_i}_{i=1}^{n}, \boxed{y_j}_{j=1}^{m})$
- TODO Something about the combination of two functions

- Proof: TODO ABSTRACTED
==

5.3 (Notation) Computability notations

- $\mathcal{T}(e, x, t)$ abbreviates $\mathcal{T}_1(e, x, t)$
- $\{e\}(x)$ abbreviates $\{e\}^1(x)$
- $\{e\}(x) = \{e\}^1(x) = \mathcal{U}(\mu_{US}(\chi_{\mathcal{T}_1})(e, x)) = \mathcal{U}(\mu_{US}(\chi_{\mathcal{T}})(e, x))$
- If $f \in \mu$ and $f : \mathbb{N} \to \mathbb{N}$, then $dom(f) = \{x \in \mathbb{N} :$ (there exists $y \in \mathbb{N}$), $(f(x) = y)\}$
- If $f \in \mu$ and $f : \mathbb{N} \to \mathbb{N}$, then $rng(f) = \{y \in \mathbb{N} :$ (there exists $x \in \mathbb{N}$), $(f(x) = y)\}$
==

5.3.1 (Definition) Semi-computable set

- The set A is semi-computable ($A \in SC$) iff there exists $f \in \mu$, $A = dom(f)$
- There exists an algorithm that confirms membership, but not necessarily decide membership
==

5.3.2 (Definition) Computably enumerable set

- The set A is computable enumerable ($A \in CE$) iff there exists $f \in \mu$, $Total(f)$ and $A = rng(f)$
- Alternative definition: $A \in CE$ iff $Finite(A)$ or there exists $f \in \mu$, $Bijection(f)$ and $Total(f)$ and $A = rng(f)$
- There exists an algorithm that can list down all the elements of the set
==

5.3.3 (Metatheorem) Equivalent definition for domain

- If $f \in \mu$ and $CI(e, f)$, then $dom(f) = \{x :$ (there exists t), $(\mathcal{T}(e, x, t))\}$
- Proof:
¡Kleene's Normal Form theorem¿ – If $x \in dom(f)$, then
— There exists $y \in \mathbb{N}$, $f(x) = y$
¡Definition¿ — There exists $\Gamma \in \mathcal{C}$, $\Gamma = \Omega \cup \langle enc(e, enc(x), y)\rangle$
¡Computation iff computable indexable corollary¿ — $t = enc(\boxed{\omega_i}_{i=1}^{|\Omega|}, enc(e, enc(x), y))$
¡IsComputation is primitive recursive¿ — There exists t, $\mathcal{T}(e, enc(x), t)$
¡T-predicate is primitive recursive¿ – $dom(f) \subseteq \{x :$ (there exists t), $(\mathcal{T}(e, x, t))\}$
– If $x \in \{x :$ (there exists t), $(\mathcal{T}(e, x, t))\}$, then
— $f(\boxed{x_i}_{i=1}^{n}) = \mathcal{U}(t) = y$
¡Kleene's Normal Form theorem¿ — There exists $y \in \mathbb{N}$, $f(x) = y$
— $x \in dom(f)$
¡Definition¿ – $\{x :$ (there exists t), $(\mathcal{T}(e, x, t))\} \subseteq dom(f)$
– $dom(f) = \{x :$ (there exists t), $(\mathcal{T}(e, x, t))\}$
==

5.3.4 (Metatheorem) Computable sets are semi-computable

- If $\chi_A \in \mu$, then $A \in SC$
- Proof:
– If $\chi_A \in \mu$, then
— $f(x) = \mu_{US}(\chi_{x \in A \wedge (\square = \square)})(x)$
— $f \in \mu$
¡Misc. theorems¿ — $dom(f) = A$
— There exists $f \in \mu$, $A = dom(f)$
— $A \in SC$
==

5.3.5 (Metatheorem) SC iff CE property

- If $A \subseteq \mathbb{N}$, then
– $A \in SC$ iff
– $A = \{\}$ or there exists $f \in PR$, $rng(f) = A$ iff
– $A \in CE$
- Proof: If $A \in SC$, then $A = \{\}$ or there exists $f \in PR$, $rng(f) = A$
– If $A \in SC$ and $A = \{\}$, then $A = \{\}$ or there exists $f \in PR$, $rng(f) = A$
– If $A \in SC$, and $A \neq \{\}$, then
— There exists f, $A = dom(f)$
¡Definition¿ — There exists e, $CI(e, f)$
— $A = \{x : (\text{ there exists } t), (\mathcal{T}(e, x, t))\}$ ¡(I)¿
¡Equivalent definition for domain¿ — There exists a, $a \in A$
— There exists g_a, if $\chi_{\mathcal{T}(e,singleDec_1(x),singleDec_2(x))} = 0$, then $g_a(x) = singleDec_1(x)$ and otherwise, $g_a(x) = a$ ¡(II)¿
— $g_a \in PR$
¡Misc. theorems¿ — If $b \in A$, then
—- There exists t, $\mathcal{T}(e, b, t)$
¡(I)¿ —- $g_a(enc(b, t)) = b$
¡(II)¿ —- $b \in rng(g_a)$
¡Definition¿ — $A \subseteq rng(g_a)$
— If $b \in rng(g_a)$, then
—- If $b = a$, then $b \in A$
—- If $b \neq a$, then
—– There exists t, $\mathcal{T}(e, b, t)$
¡(II)¿ —– $b \in A$
¡(I)¿ —- $b \in A$
¡Conjunction¿ — $rng(g_a) \subseteq A$
— $rng(g_a) = A$
¡Conjunction¿ — There exists $f \in PR$, $rng(f) = A$ or $A = \{\}$
– If $A \in SC$, then $A = \{\}$ or there exists $f \in PR$, $rng(f) = A$
¡Conjunction¿ - Proof: If $A = \{\}$ or there exists $f \in PR$, $rng(f) = A$, then $A \in CE$
– If $A = \{\}$ or there exists $f \in PR$, $rng(f) = A$, then
— If $Finite(A)$, then $\utilde{A \in} CE$
¡Definition¿ — If $\utilde{Finite}(A)$, then
—- There exists $NextHasOccurred_f$, if $\chi_{(\forall j \leq x)(f(j) \neq f(x+1))}(x) = 0$, then $NextHasOccurred_f = 1$ and otherwise, $NextHasOccurred_f$
0
—- $NextHasOccurred_f \in PR$
¡Misc. theorems¿ —- $NumOfUniqueOutputsLeq(n) = Sum(NextHasOccurred_f)(n) + 1$
—- $NumOfUniqueOutputsLeq \in PR$
¡Misc. theorems¿ —- There exists g, if $\mathcal{I}_1^1(x) = 0$, then $g(x) = f(0)$ and otherwise, $g(x) = f(\mu_{US}(\chi_{NumOfUniqueOutputsLeq(\square)-1=x}))$
¡(I)¿
—- $g \in \mu$
¡Misc. theorems¿ —- $Total(g)$ and $Bijection(g)$ and $rng(g) = A$ ¡(I)¿
—- There exists $g \in \mu$, $Bijection(f)$ and $Total(f)$ and $A = rng(f)$
—- $A \in CE$
— $A \in CE$
¡Conjunction¿ – If $A = \{\}$ or there exists $f \in PR$, $rng(f) = A$, then $A \in CE$
¡Conjunction¿ - Proof: If $A \in CE$, then $A \in SC$
– If $A \in CE$, then
— $Finite(A)$ or there exists $f \in \mu$, $Bijection(f)$ and $Total(f)$ and $A = rng(f)$
— If $Finite(A)$, then
—- There exists g, $f(x) = \mu_{US}(\chi_{(\bigvee_{i=1}^{|A|} \boxed{a_i = \square}) \wedge (x=\square)})(x)$
—- $f \in \mu$
¡Misc. theorems¿ —- $dom(g) = A$
—- There exists $g \in \mu$, $A = dom(g)$
—- $A \in SC$
— If $Finite(A)$, then $\utilde{A \in} SC$
¡Abbreviate¿ — If $\utilde{Finite}(A)$ and there exists $f \in \mu$, $Bijection(f)$ and $Total(f)$ and $A = rng(f)$, then

—- There exists g, $g(x) = \mu_{US}(\chi_{f(\square)=x})(x)$
—- $g \in \mu$
¡Misc. theorems¿ —- $dom(g) = A$
—- There exists $g \in \mu$, $dom(g) = A$
—- $A \in SC$
— If $Finite(A)$ and there exists $f \in \mu$, $Bijection(f)$ and $Total(f)$ and $A = rng(f)$, then $A \in SC$
¡Abbreviate¿ — $A \in SC$
¡Conjunction¿ - Proof: $A \in SC$ iff $A = \{\}$ or there exists $f \in PR$, $rng(f) = A$ iff $A \in CE$
¡Conjunction¿ ==

5.3.6 (Definition) N-complement

- The set $\bar{A}$ is the N-complement of the A iff $\bar{A} = \mathbb{N} \setminus A$
==

5.3.7 (Metatheorem) Computable iff CE property

- $\chi_A \in \mu$ iff $A \in CE$ and $\bar{A} \in CE$
- Proof:
– If $\chi_A \in \mu$, then
— $\chi_{\bar{A}} = Conj(\chi_A)$
— $\chi_{\bar{A}} \in \mu$
¡Misc. theorems¿ — $A \in SC$ and $\bar{A} \in SC$
¡Computable sets are semi-computable¿ — $A \in CE$ and $\bar{A} \in CE$
¡SC iff CE property¿ – If $\chi_A \in \mu$, then $A \in CE$ and $\bar{A} \in CE$
¡Abbreviate¿ – If $A \in CE$ and $\bar{A} \in CE$, then
— If $A = \{\}$, then
—- $\chi_A(x) = c_1^1(x) = 1$
—- $\chi_A \in \mu$
¡Misc. theorems¿ — If $\bar{A} = \{\}$, then
—- $\chi_A(x) = c_0^1(x) = 0$
—- $\chi_A \in \mu$
¡Misc. theorems¿ — If $A \neq \{\}$, then
—- There exists $f_0 \in PR$, $Total(f_0)$ and $rng(f_0) = A$ iff
¡SC iff CE property¿ —- There exists $f_1 \in PR$, $Total(f_1)$ and $rng(f_1) = \bar{A}$ iff
¡SC iff CE property¿ —- There exists $inFind$, $inFind(x) = \mu_{US}(\chi_{(f_0(\square)=x) \vee (f_1(\square)=x)})(x)$
—- $inFind \in \mu$
¡Misc. theorems¿ —- $Total(f_0)$ and $Total(f_1)$ and $rng(f) = rng(f_0) \cup rng(f_1) = \mathbb{N}$ ¡(I)¿
¡Disjunction¿ —- $Total(inFind)$
¡(I)¿ —- There exists χ, if $f_0(inFind(x)) = x$, then $\chi(x) = 0$, and otherwise $\chi(x) = 1$
—- $\chi \in \mu$
¡Misc. theorems¿ —- $\chi_A(x) = \chi(x) = 0$ iff $x \in A$
—- $\chi_A \in \mu$
— $\chi_A \in \mu$
¡Conjunction¿ – If $A \in CE$ and $\bar{A} \in CE$, then $\chi_A \in \mu$
¡Abbreviate¿ – $\chi_A \in \mu$ iff $A \in CE$ and $\bar{A} \in CE$
¡Conjunction¿ - The case $A = \{\}$ is required because a function can't be total if $rng(f) = \{\}$
==

5.3.8 (Metatheorem) Computable iff SC property

- $\chi_A \in \mu$ iff $A \in SC$ and $\bar{A} \in SC$
- Proof:
– $A \in SC$ and $\bar{A} \in SC$ iff $A \in CE$ and $\bar{A} \in CE$
¡SC iff CE property¿ – $A \in CE$ and $\bar{A} \in CE$ iff $\chi_A \in \mu$
¡Computable iff CE property¿ – $\chi_A \in \mu$ iff $A \in SC$ and $\bar{A} \in SC$
==

5.3.9 (Definition) Semi-computable set by index

- The e-th semi-computable set $\mathcal{W}_e$ is defined as $\mathcal{W}_e = dom(\{e\})$
==

5.3.10 (Metatheorem) SC iff SC indexed

- $A \in SC$ iff there exists e, $A = \mathcal{W}_e$
- Proof:
– $A \in SC$ iff
— There exists $f \in \mu$, $A = dom(f)$ iff
¡Definition¿ — There exists e, $A = dom(\{e\})$ iff
¡Enumeration theorem¿ — There exists e, $A = \mathcal{W}_e$ iff
¡Definition¿ ==

5.3.11 (Definition) K

- The set $\mathcal{K}$ is defined as $\mathcal{K} = \{x : x \in \mathcal{W}_x\}$
- $\mathcal{K}$ stands for kool
==

5.3.12 (Metatheorem) N-complement of K is not semi-computable

- $\bar{\mathcal{K}} \notin SC$
- Proof:
– If $\bar{\mathcal{K}} \in SC$, then
— There exists m, $\bar{\mathcal{K}} = \mathcal{W}_m$ ¡(I)¿
¡SC iff SC indexed¿ — $m \in \bar{\mathcal{K}}$ iff $m \in \mathcal{W}_m$
¡(I)¿ — $m \in \mathcal{W}_m$ iff $m \in \mathcal{K}$
¡Definition¿ — $m \in \mathcal{K}$ iff $m \notin \bar{\mathcal{K}}$
¡Definition¿ — $m \in \bar{\mathcal{K}}$ iff $m \notin \bar{\mathcal{K}}$
¡Conjunction¿ — CONTRADICTION !! – $\bar{\mathcal{K}} \in SC$
¡Metaproof by contradiction¿ ==

5.3.13 (Metatheorem) K is semi-computable

- $\mathcal{K} \in SC$
- Proof:
– $x \in \mathcal{K}$ iff
— $x \in \mathcal{W}_x$ iff
¡Definition¿ — $x \in dom(\{x\})$ iff
¡Definition¿ — There exists t, $\mathcal{T}(x, x, t)$
¡Equivalent definition for domain¿ – $x \in \mathcal{K}$ iff there exists t, $\mathcal{T}(x, x, t)$ ¡(I)¿
¡Abbreviate¿ – There exists f, $f(x) = \mu_{US}(\chi_{\mathcal{T}(x,x,\Box)})(x, x)$
– $f \in \mu$
¡Misc. theorems¿ – $dom(f) = \mathcal{K}$
¡(I)¿ – $\mathcal{K} \in SC$
==

5.3.14 (Metatheorem) K is not computable

- $\chi_{\mathcal{K}} \notin \mu$
- Proof:
– If $\chi_{\mathcal{K}} \in \mu$, then
— $\mathcal{K} \in SC$ and $\bar{\mathcal{K}} \in SC$
¡Computable iff SC property¿ — $\bar{\mathcal{K}} \notin SC$
¡N-complement of K is not semi-computable¿ — $\bar{\mathcal{K}} \in SC$ and $\bar{\mathcal{K}} \notin SC$
¡Conjunction¿ — CONTRADICTION !!
– $\chi_{\mathcal{K}} \notin \mu$
¡Metaproof by contradiction¿ ==

5.3.15 (Metatheorem) SC subset of N-complement of K contains a nonSC element

- For any e, if $\mathcal{W}_e \subseteq \bar{\mathcal{K}}$, then $e \in \bar{\mathcal{K}} \setminus \mathcal{W}_e$
- Proof:
– For any e, if $\mathcal{W}_e \subseteq \bar{\mathcal{K}}$, then
— If $a \in \mathcal{W}_e$, then $a \in \bar{\mathcal{K}}$ ¡(I)¿
¡Hypothesis¿ — $b \in \bar{\mathcal{K}}$ iff $b \notin \mathcal{W}_b$ ¡(II)¿
¡Definition¿ — If $e \in \mathcal{W}_e$, then
— $e \in \bar{\mathcal{K}}$
¡(I)¿ — $e \notin \mathcal{W}_e$
¡(II)¿ — CONTRADICTION !! — $e \notin \mathcal{W}_e$
¡Metaproof by contradiction¿ — $e \in \bar{\mathcal{K}}$
¡(II)¿ — $e \in \bar{\mathcal{K}} \setminus \mathcal{W}_e$
¡Conjunction¿ ==

5.3.16 (Metatheorem) Sigma formulas can emulate computable functions

- For any $f \in \mu$, there exists $\phi(\underset{i=1}{\overset{Arity(f)}{\boxed{x_i}}}, y) \in \Sigma_{Form}$, $f(\underset{i=1}{\overset{Arity(f)}{\boxed{a_i}}}) = b$ iff $\mathfrak{N} \vDash |\phi|^{\underset{i=1}{\overset{Arity(f)}{\boxed{x_i}}}, y}_{\underset{i=1}{\overset{Arity(f)}{\boxed{\overleftarrow{a_i}}}}, \overleftarrow{b}}$

¡TODO CLEANUP¿ - Proof:
– If $f = S$, then
— There exists $\phi(x, y) \in \Sigma_{Form}$, $\phi(x, y) :\equiv S(x) \equiv y$
¡Definition¿ — $f(a) = b$ iff
—- $b = S(a)$ iff
—- $b = a + 1$ iff
—- $\mathfrak{N} \vDash S(\overleftarrow{a}) \equiv \overleftarrow{b}$ iff
—- $\mathfrak{N} \vDash |\phi|^{x,y}_{\overleftarrow{a}, \overleftarrow{b}}$
— There exists $\phi(x, y) \in \Sigma_{Form}$, $f(a) = b$ iff $\mathfrak{N} \vDash |\phi|^{x,y}_{\overleftarrow{a}, \overleftarrow{b}}$
– If $f = \mathcal{I}^n_i$, then
— There exists $\phi(\underset{j=1}{\overset{n}{\boxed{x_j}}}, y) \in \Sigma_{Form}$, $\phi(\underset{j=1}{\overset{n}{\boxed{x_j}}}, y) :\equiv \wedge(\underset{j=1}{\overset{n}{\boxed{x_j \equiv x_j}}}) \wedge (x_i \equiv y)$
¡Definition¿ — $f(\underset{j=1}{\overset{n}{\boxed{a_j}}}) = b$ iff
—- $\mathcal{I}^n_i(\underset{j=1}{\overset{n}{\boxed{a_j}}}) = b$ iff
—- $b = a_i$ iff
—- $\mathfrak{N} \vDash \wedge(\underset{j=1}{\overset{n}{\boxed{x_j \equiv x_j}}}) \wedge (\overleftarrow{a_i} \equiv \overleftarrow{b})$ iff
—- $\mathfrak{N} \vDash |\phi|^{x,y}_{\overleftarrow{a}, \overleftarrow{b}}$
– If $f = \mathcal{O}$, then
— There exists $\phi(y) \in \Sigma_{Form}$, $\phi(y) :\equiv y \equiv 0$
¡Definition¿ — $f() = b$ iff
—- $b = \mathcal{O}()$ iff
—- $b = 0$ iff
—- $\mathfrak{N} \vDash \overleftarrow{b} \equiv \overleftarrow{0}$ iff
—- $\mathfrak{N} \vDash |\phi|^{x,y}_{\overleftarrow{a}, \overleftarrow{b}}$
– If $f(\underset{i=1}{\overset{n}{\boxed{x_i}}}) = h(\underset{j=1}{\overset{m}{\boxed{g_j(\underset{i=1}{\overset{n}{\boxed{x_i}}})}}})$ and

TODO CLEAN UP — For any $z \in \left\{ h, \underset{j=1}{\overset{m}{\boxed{g_j}}} \right\}$, there exists $\phi_z(\underset{i=1}{\overset{Arity(z)}{\boxed{x_i}}}, y) \in \Sigma_{Form}$,

— $z(\overset{Arity(z)}{\underset{i=1}{\boxed{a_{z,i}}}}) = b_z$ iff $\mathfrak{N} \models |\phi_z|^{\overset{Arity(z)}{\underset{i=1}{\boxed{x_i}}}, y}_{\overset{Arity(z)}{\underset{i=1}{\boxed{\overleftarrow{a_{z,i}}}}}, \overleftarrow{b_z}}$), then

—- There exists $\phi(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y) \in \Sigma_{Form}$, $\phi(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y) :\equiv \overset{m}{\underset{j=1}{\boxed{(\exists y_j)}}}(\wedge(\overset{m}{\underset{j=1}{\boxed{\phi_{g_j}(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y_j)}}}) \wedge \phi_h(\overset{m}{\underset{j=1}{\boxed{y_j}}}, y))$

¡Definition¿ —- $f(\overset{n}{\underset{i=1}{\boxed{a_i}}}) = b$ iff

—– $b = b_h = h(\overset{m}{\underset{j=1}{\boxed{b_{g_j}}}})$ and $\overset{m}{\underset{j=1}{\boxed{b_{g_j} = g_j(\overset{n}{\underset{i=1}{\boxed{a_i}}})}}}$ iff

¡Definition¿ —– $\mathfrak{N} \models |\phi_h|^{\overset{m}{\underset{i=1}{\boxed{x_i}}}, y}_{\overset{m}{\underset{i=1}{\boxed{\overleftarrow{a_{h,i}}}}}, \overleftarrow{b_h}}$ and $\overset{m}{\underset{j=1}{\boxed{\mathfrak{N} \models \left|\phi_{g_j}\right|^{\overset{n}{\underset{i=1}{\boxed{x_i}}}, y}_{\overset{n}{\underset{i=1}{\boxed{\overleftarrow{a_{g_j,i}}}}}, \overleftarrow{b_{g_j}}}}}}$ iff

¡Inductive hypothesis¿ —– $\mathfrak{N} \models \overset{m}{\underset{j=1}{\boxed{(\exists y_j)}}}(\wedge(\overset{m}{\underset{j=1}{\boxed{\phi_{g_j}(\overset{n}{\underset{i=1}{\boxed{\overleftarrow{a_i}}}}, y_j)}}}) \wedge \phi_h(\overset{m}{\underset{j=1}{\boxed{y_j}}}, \overleftarrow{b}))$ iff

¡Substitution and modification identity on models¿ —– $\mathfrak{N} \models |\phi|^{\overset{n}{\underset{i=1}{\boxed{x_i}}}, y}_{\overset{Arity(n)}{\underset{i=1}{\boxed{\overleftarrow{a_i}}}}, \overleftarrow{b}}$

– If $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, 0) = g(\overset{n}{\underset{i=1}{\boxed{x_i}}})$ and $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y+1) = h(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y, f(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y))$ and

— $a = (t)_i$ iff $\mathfrak{N} \models |IE|^{x,y,z}_{\overleftarrow{a}, \overleftarrow{i}, \overleftarrow{t}}$

¡TODO 5.6¿ — For any $z \in \{g, h\}$, there exists $\phi_z(\overset{Arity(z)}{\underset{i=1}{\boxed{x_{z,i}}}}, y_z) \in \Sigma_{Form}$,

— $z(\overset{Arity(z)}{\underset{i=1}{\boxed{a_{z,i}}}}) = b_z$ iff $\mathfrak{N} \models |\phi_z|^{\overset{Arity(z)}{\underset{i=1}{\boxed{x_{z,i}}}}, y_z}_{\overset{Arity(z)}{\underset{i=1}{\boxed{\overleftarrow{a_{z,i}}}}}, \overleftarrow{b_z}}$), then

—- There exists $\phi(\overset{n}{\underset{i=1}{\boxed{x_i}}}, z, y) \in \Sigma_{Form}$, $\phi(\overset{n}{\underset{i=1}{\boxed{x_i}}}, z, y) :\equiv \exists t(IE(y, S(y), t) \wedge$

$\exists y_0(IE(y_0, S(0), t) \wedge |\phi_g|^{\overset{n}{\underset{i=1}{\boxed{x_{g,i}}}}, y_g}_{\overset{n}{\underset{i=1}{\boxed{x_i}}}, y_0}$) $\wedge$

$(\forall i < z)(\exists u, v)(IE(u, S(i), t) \wedge IE(v, S(S(i)), t) \wedge |\phi_h|^{\overset{n+2}{\underset{i=1}{\boxed{x_{g,i}}}}, y_h}_{\overset{n}{\underset{i=1}{\boxed{x_i}}}, i, v, u}$))

¡Definition¿ —- $f(\overset{n}{\underset{i=1}{\boxed{a_i}}}, c+1) = b$ iff

—– $b = h(\overset{n}{\underset{i=1}{\boxed{a_i}}}, c, f(\overset{n}{\underset{i=1}{\boxed{a_i}}}, c))$ iff

¡Definition¿ —– $\mathfrak{N} \models \exists t(IE(y, S(y), t) \wedge$

$\exists y_0(IE(y_0, S(0), t) \wedge |\phi_g|^{\overset{n}{\underset{i=1}{\boxed{x_{g,i}}}}, y_g}_{\overset{n}{\underset{i=1}{\boxed{x_i}}}, y_0}) \wedge$

$(\forall i < z)(\exists u, v)(IE(u, S(i), t) \wedge IE(v, S(S(i)), t) \wedge |\phi_h|^{\overset{n+2}{\underset{i=1}{\boxed{x_{g,i}}}}, y_h}_{\overset{n}{\underset{i=1}{\boxed{x_i}}}, i, v, u})$) iff

¡Induction¿ —— $\mathfrak{N} \vDash |\phi|^{\overset{n}{\underset{i=1}{\boxed{x_i}}}, y}_{\overset{Arity(n)}{\underset{i=1}{\boxed{\overleftarrow{a_i}}}}, \overleftarrow{b}}$

– If $f(\overset{n}{\underset{i=1}{\boxed{x_i}}}) = \mu_{US}(g)(\overset{n}{\underset{i=1}{\boxed{x_i}}})$ and there exists $\phi_g(\overset{n+1}{\underset{i=1}{\boxed{x_{g,i}}}}, y_g) \in \Sigma_{Form}$, $g(\overset{n+1}{\underset{i=1}{\boxed{a_{g,i}}}}) = b_g$ iff $\mathfrak{N} \vDash |\phi_g|^{\overset{n+1}{\underset{i=1}{\boxed{x_{g,i}}}}, y_g}_{\overset{n+1}{\underset{i=1}{\boxed{\overleftarrow{a_{g,i}}}}}, \overleftarrow{b_g}}$), then

— There exists $\phi(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y) \in \Sigma_{Form}$, $\phi(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y) :\equiv (\forall i < y)(\phi_g(\overset{n}{\underset{i=1}{\boxed{x_i}}}, y, 0) \wedge \exists u(\phi_g(\overset{n}{\underset{i=1}{\boxed{x_i}}}, i, u) \wedge \neg(u \equiv 0))$

¡Definition¿ — $f(\overset{n}{\underset{i=1}{\boxed{a_i}}}) = b$ iff

—- $g(\overset{n}{\underset{i=1}{\boxed{a_i}}}, b) = 0$ and for any $b_< < b$, $g(\overset{n}{\underset{i=1}{\boxed{a_i}}}, b_<) > 0$ iff

¡Definition¿ —- $\mathfrak{N} \vDash |\phi|^{\overset{n}{\underset{i=1}{\boxed{x_i}}}, y}_{\overset{n}{\underset{i=1}{\boxed{\overleftarrow{a_i}}}}, \overleftarrow{b}}$

¡Misc. Semantics¿ – For any $f \in \mu$, there exists $\phi(\overset{Arity(f)}{\underset{i=1}{\boxed{x_i}}}, y) \in \Sigma_{Form}$, $f(\overset{Arity(f)}{\underset{i=1}{\boxed{a_i}}}) = b$ iff $\mathfrak{N} \vDash |\phi|^{\overset{Arity(f)}{\underset{i=1}{\boxed{x_i}}}, y}_{\overset{Arity(f)}{\underset{i=1}{\boxed{\overleftarrow{a_i}}}}, \overleftarrow{b}}$

¡Induction¿ ==

5.3.17 (Metatheorem) Sigma formulas can emulate SC sets

- For any $A \in SC$, there exists $\theta(x) \in \Sigma_{Form}$, $\mathfrak{N} \vDash |\theta|^{x}_{\overleftarrow{a}}$ iff $a \in A$
- Proof:
– For any $A \in SC$,
— There exists $f' \in \mu$, $dom(f') = A$
¡Definition¿ — There exists $g' \in \mu$, $g'(x) = 0 \dot{-} f'(x)$ ¡(I)¿
¡Misc. theorems¿ — $g'(x) = 0$ iff $x \in dom(f')$ iff $x \in A$
¡(I)¿ — There exists $\phi'(x, y) \in \Sigma_{Form}$, $g'(a) = 0$ iff $\mathfrak{N} \vDash |\phi'|^{x,y}_{\overleftarrow{a}, \overleftarrow{0}}$
¡Sigma formulas can emulate computable functions¿ — $\mathfrak{N} \vDash |\phi'|^{x,y}_{\overleftarrow{a}, \overleftarrow{0}}$ iff $g'(a) = 0$ iff $a \in A$ ¡(II)¿
¡(I)¿ — There exists $\theta(x)$, $\theta :\equiv |\phi'|^{y}_{\overleftarrow{0}}$
— There exists $\theta(x) \in \Sigma_{Form}$, $|\theta|^{x}_{\overleftarrow{a}}$ iff $a \in A$
¡(II)¿ - Basically, emulate the characteristic of the domain
==

5.3.18 (Metatheorem) Sigma formulas can emulate K

- There exists $\theta(x) \in \Sigma_{Form}$, $\mathfrak{N} \vDash |\theta|^{x}_{\overleftarrow{a}}$ iff $a \in \mathcal{K}$
- Proof:
¡Sigma formulas can emulate SC sets¿ ==

5.3.19 (Metatheorem) Pi formulas can emulate N-complement of K

- There exists $\psi(x) \in \Pi_{Form}$, $\mathfrak{N} \vDash |\psi|^{x}_{\overleftarrow{a}}$ iff $a \in \bar{\mathcal{K}}$ and
– There exists $\theta(x) \in \Sigma_{Form}$, $\mathfrak{N} \vDash |\theta|^{x}_{\overleftarrow{a}}$ iff $a \in \mathcal{K}$ and

– $|\psi|^x_{\overleftarrow{a}} \vDash \neg|\theta|^x_{\overleftarrow{a}}$ and $\neg|\theta|^x_{\overleftarrow{a}} \vDash |\psi|^x_{\overleftarrow{a}}$
- Proof:
¡Sigma formulas can emulate K, De Morgan's¿ ================================

5.3.20 (Definition) Weak number theory conjunction

- The formula $N_\wedge$ is defined as $N_\wedge = \wedge(\underset{\phi\in N}{\boxed{\phi}})$
================================

5.3.21 (Metatheorem) Undecidability of the Entscheidungsproblem

- The set of all valid formulas $\mathcal{E}$ is defined as $\mathcal{E} = \{GN(\phi) : \phi \in Form(\mathcal{L}_{NT})(\text{ and }) \vDash \phi\}$
- $\chi_\mathcal{E} \notin \mu$
- Proof:
– $a \in \mathcal{K}$ iff
— There exists $\phi(x) \in \Sigma_{Form}$, $\mathfrak{N} \vDash |\phi|^x_{\overleftarrow{a}}$ iff
— $N \vdash |\phi|^x_{\overleftarrow{a}}$ iff
¡TODO 5.3.13¿ — $\vdash N_\wedge \implies |\phi|^x_{\overleftarrow{a}}$ iff
¡Deduction theorem¿ — $\vDash N_\wedge \implies |\phi|^x_{\overleftarrow{a}}$
¡Completeness theorem¿ – There exists $\phi(x) \in \Sigma_{Form}$, $a \in \mathcal{K}$ iff $\vDash N_\wedge \implies |\phi|^x_{\overleftarrow{a}}$
¡Abbreviate¿ – There exists $g \in \mu$, $g(n) = GN(N_\wedge \implies |\phi|^x_{\overleftarrow{n}})$
¡Misc. theorems¿ – If $\chi_\mathcal{E} \in \mu$, then
— There exists $f \in \mu$, $f(x) = \chi_\mathcal{E}(g(x))$
¡Misc. theorems¿ — $f(n) = 0$ iff
—- $\chi_\mathcal{E}(g(n)) = 0$ iff
—- $\vDash N_\wedge \implies |\phi|^x_{\overleftarrow{n}}$ iff
—- $n \in \mathcal{K}$
— $f = \chi_\mathcal{K}$
— $\chi_\mathcal{K} \notin \mu$
¡K is not computable¿ — $\chi_\mathcal{K} \in \mu$ and $\chi_\mathcal{K} \notin \mu$
¡Conjunction¿ — CONTRADICTION !! – $\chi_\mathcal{E} \notin \mu$
¡Metaproof by contradiction¿ ================================

5.3.22 (Metatheorem) SC axioms yields SC theorems

- If $\{\phi(x)\} \cup A \subseteq Form(\mathcal{L}_{NT})$ and $\{GN(\eta) : A \vdash \eta\} \in SC$, then $\{a : A \vdash |\phi|^x_{\overleftarrow{a}}\} \in SC$
- Proof:
– If $\phi(x) \in Form(\mathcal{L}_{NT})$ and $A \subseteq Form(\mathcal{L}_{NT}$ and $\{GN(\eta) : A \vdash \eta\} \in SC$, then
— There exists $f \in \mu$, $dom(f) = \{GN(\eta) : A \vdash \eta\}$ ¡(I)¿
¡Definition¿ — There exists $g \in \mu$, $g(n) = GN(|\phi|^x_{\overleftarrow{n}})$
¡Misc. theorems¿ — There exists $h \in \mu$, $h(m) = f(g(m))$
¡Misc. theorems¿ — $a \in dom(h)$ iff
—- There exists $b \in \mathbb{N}$, $h(a) = b$ iff
¡Definition¿ —- There exists $b \in \mathbb{N}$, $f(GN(|\phi|^x_{\overleftarrow{a}})) = b$ iff
—- $A \vdash |\phi|^x_{\overleftarrow{a}}$
¡(I)¿ — There exists $h \in \mu$, $dom(h) = \{a : A \vdash |\phi|^x_{\overleftarrow{a}}\} \in SC$
— $\{a : A \vdash |\phi|^x_{\overleftarrow{a}}\} \in SC$
¡Definition¿ ================================

5.3.23 (Metatheorem) Incompleteness theorem version I

- If $A \subseteq Form(\mathcal{L}_{NT})$ and $\mathfrak{N} \vDash A$ and $\{GN(\eta) : A \vdash \eta\} \in SC$, then there exists $\theta \in \Pi_{Form}$, $\mathfrak{N} \vDash \theta$ and $A \nvdash \theta$
- Proof:
– If $A \subseteq Form(\mathcal{L}_{NT})$ and $\mathfrak{N} \vDash A$ and $\{GN(\eta) : A \vdash \eta\} \in SC$, then
— There exists $\psi(x) \in \Pi_{Form}$, $\mathfrak{N} \vDash |\psi|^x_{\overleftarrow{a}}$ iff $a \in \underline{r}\mathcal{K}$
¡Pi formulas can emulate N-complement of K¿ — $\bar{\mathcal{K}} = \{a : \mathfrak{N} \vDash |\psi|^x_{\overleftarrow{a}}\}$
— $\{a : A \vDash |\psi|^x_{\overleftarrow{a}}\} \subseteq \{a : \mathfrak{N} \vDash |\psi|^x_{\overleftarrow{a}}\}$
¡Hypothesis¿ — $\{a : A \vdash |\psi|^x_{\overleftarrow{a}}\} \subseteq \{a : A \vDash |\psi|^x_{\overleftarrow{a}}\}$
¡Soundness theorem¿ — $\{a : A \vdash |\psi|^x_{\overleftarrow{a}}\} \subseteq \bar{\mathcal{K}}$

¡Conjunction¿ — $\{a : A \vdash |\psi|^{x}_{\overleftarrow{a}}\} \in SC$
¡SC axioms yields SC theorems¿ — $\bar{\mathcal{K}} \notin SC$
¡N-complement of K is not semi-computable¿ — $\bar{\mathcal{K}} \neq \{a : A \vdash |\psi|^{x}_{\overleftarrow{a}}\}$
¡Conjunction¿ — There exists $\theta \in \bar{\mathcal{K}} \setminus \{a : A \vdash |\psi|^{x}_{\overleftarrow{a}}\}$,
—– $\theta \in \bar{\mathcal{K}}$ and $\theta \notin \{a : A \vdash |\psi|^{x}_{\overleftarrow{a}}\}$
—– $\mathfrak{N} \vDash |\theta|^{x}_{\overleftarrow{a}}$ and $A \nvdash |\theta|^{x}_{\overleftarrow{a}}$
==

5.3.24 (Definition) Theory extension

- The theory A extends the theory B ($extends(A, B)$) iff for any $\phi \in \mathcal{L}$, if $B \vdash \phi$, then $A \vdash \phi$
- Alternative definition: $extends(A, B)$ iff $A \vdash B$
==

5.3.25 (Metatheorem) Incompleteness theorem version II

- If $A \subseteq Form(\mathcal{L}_{NT})$ and $A \nvdash \overleftarrow{\perp}$ and $\{GN(\eta) : A \vdash \eta\} \in SC$, then there exists $\theta \in \Pi_{Form}$, $\mathfrak{N} \vDash \theta$ and $A \nvdash \theta$
- Proof:
– If $extends(A, N)$, then
— $A \nvdash N$
¡Definition¿ — There exists $\alpha \in N$, $A \nvdash \alpha$
¡Definition¿ — There exists $\theta \in \Pi_{Form}$, $\theta :\equiv N_{\wedge}$
¡Definition¿ — $\mathfrak{N} \vDash \theta$
— $A \nvdash \theta$
¡PC¿ — There exists $\theta \in \Pi_{Form}$, $\mathfrak{N} \vDash \theta$ and $A \nvdash \theta$
– If $extends(A, N)$, then
— $A \vdash N$ ¡(I)¿
¡Definition¿ — There exists $\phi(x) \in \Sigma_{Form}$, $\mathfrak{N} \vDash |\phi|^{x}_{\overleftarrow{a}}$ iff $a \in \mathcal{K}$
¡Sigma formulas can emulate K¿ — $a \in \mathcal{K}$ iff
—– $\mathfrak{N} \vDash |\phi|^{x}_{\overleftarrow{a}}$ iff
—– $N \vdash |\phi|^{x}_{\overleftarrow{a}}$
¡TODO 5.3.13¿ — $a \in \mathcal{K}$ iff $N \vdash |\phi|^{x}_{\overleftarrow{a}}$
¡Abbreviate¿ — If $N \vdash |\phi|^{x}_{\overleftarrow{a}}$, then $A \vdash |\phi|^{x}_{\overleftarrow{a}}$
¡(I)¿ — If $a \in \mathcal{K}$, then $A \vdash |\phi|^{x}_{\overleftarrow{a}}$
¡Conjunction¿ — If $A \nvdash |\phi|^{x}_{\overleftarrow{a}}$, then $a \notin \mathcal{K}$ ¡(II)¿
¡Contrapositive¿ — If $A \vdash \neg|\phi|^{x}_{\overleftarrow{a}}$, then
—– $A \nvdash |\phi|^{x}_{\overleftarrow{a}}$
¡Hypothesis¿ —– $a \notin \mathcal{K}$
¡(II)¿ —– $a \in \bar{\mathcal{K}}$
¡Definition¿ — If $A \vdash \neg|\phi|^{x}_{\overleftarrow{a}}$, then $a \in \bar{\mathcal{K}}$ ¡(III)¿
¡Abbreviate¿ — There exists $\psi(x) \in \Pi_{Form}$, ¡(IV)¿
¡Pi formulas can emulate N-complement of K¿ —– $|\psi|^{x}_{\overleftarrow{a}} \vDash |\neg\phi|^{x}_{\overleftarrow{a}}$ and
—– $|\neg\phi|^{x}_{\overleftarrow{a}} \vDash |\psi|^{x}_{\overleftarrow{a}} \vDash$ and
—– $\mathfrak{N} \vDash |\psi|^{x}_{\overleftarrow{a}}$ iff $a \in \bar{\mathcal{K}}$
— $A \vdash |\psi|^{x}_{\overleftarrow{a}}$ iff
—– $A \vDash |\psi|^{x}_{\overleftarrow{a}}$ iff
¡Soundness theorem¿ —– $A \vDash |\neg\phi|^{x}_{\overleftarrow{a}}$ iff
¡(IV)¿ —– $A \vdash |\neg\phi|^{x}_{\overleftarrow{a}}$
¡Completeness theorem¿ — $A \vdash |\psi|^{x}_{\overleftarrow{a}}$ iff $A \vdash |\neg\phi|^{x}_{\overleftarrow{a}}$ ¡(V)¿
¡Abbreviate¿ — If $A \vdash |\psi|^{x}_{\overleftarrow{a}}$, then
—– $A \vdash |\neg\phi|^{x}_{\overleftarrow{a}}$
¡(V)¿ —– $a \in \bar{\mathcal{K}}$
¡(III)¿ —– $\mathfrak{N} \vDash |\psi|^{x}_{\overleftarrow{a}}$
¡(IV)¿ — If $A \vdash |\psi|^{x}_{\overleftarrow{a}}$, then $\mathfrak{N} \vDash |\psi|^{x}_{\overleftarrow{a}}$
¡Abbreviate¿ — $\{a : A \vdash |\psi|^{x}_{\overleftarrow{a}}\} \subseteq \{a : \mathfrak{N} \vDash |\psi|^{x}_{\overleftarrow{a}}\}$
— $\bar{\mathcal{K}} = \{a : \mathfrak{N} \vDash |\psi|^{x}_{\overleftarrow{a}}\}$
¡(IV)¿ — $\{a : A \vdash |\psi|^{x}_{\overleftarrow{a}}\} \subseteq \{a : A \vDash |\psi|^{x}_{\overleftarrow{a}}\}$
¡Soundness theorem¿ — $\{a : A \vdash |\psi|^{x}_{\overleftarrow{a}}\} \subseteq \bar{\mathcal{K}}$
¡Conjunction¿ — $\{a : A \vdash |\psi|^{x}_{\overleftarrow{a}}\} \in SC$
¡SC axioms yields SC theorems¿ — $\bar{\mathcal{K}} \notin SC$

¡N-complement of K is not semi-computable¿ — $\bar{\mathcal{K}} \neq \{a : A \vdash |\psi|^{x}_{\overleftarrow{a}}\}$
¡Conjunction¿ — There exists $\theta \in \bar{\mathcal{K}} \setminus \{a : A \vdash |\psi|^{x}_{\overleftarrow{a}}\}$,
—- $\theta \in \bar{\mathcal{K}}$ and $\theta \notin \{a : A \vdash |\psi|^{x}_{\overleftarrow{a}}\}$
—- $\mathfrak{N} \vDash |\theta|^{x}_{\overleftarrow{a}}$ and $A \nvdash |\theta|^{x}_{\overleftarrow{a}}$
— There exists $\theta \in \Pi_{Form}$, $\mathfrak{N} \vDash \theta$ and $A \nvdash \theta$
¡Abbreviate¿ – There exists $\theta \in \Pi_{Form}$, $\mathfrak{N} \vDash \theta$ and $A \nvdash \theta$
¡Conjunction¿ ==

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5.3.26 (Remarks) Incompleteness theorem intuition

- From an intuitive computability-theoretic point of view, the first Incompleteness Theorem is an inevitable consequence of the fact that we can define an undecidable set in the $\mathcal{L}_{NT}$-structure $\mathfrak{N}$. In other words, there exists $\phi(x) \in \mathcal{L}_{NT}$, $\mathfrak{N} \vDash |\phi|^{x}_{\overleftarrow{a}}$ iff $a \in \mathcal{K}$.
- Since we can define an undecidable set in $\mathfrak{N}$, no semi-decidable set of axioms of $\mathcal{L}_{NT}$ will be complete for $\mathfrak{N}$.
- If there were such a set of axioms, we could decide membership in an undecidable set. Otherwise, we could decide if a is a member of $\mathcal{K}$ by enumerating deductions until we encountered a proof or a refutation of $|\phi|^{x}_{\overleftarrow{a}}$.
- The expressive power (the standard interpretation) of the language $\mathcal{L}_{NT}$ is essential. To define an undecidable set like $\mathcal{K}$, we need an expressive language.

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TODO: Lowenhiem Skolem + model theory Rice's theorem Lindström's theorem

FORMAT: - out of scope lemma: (I, II, III, ...) - inside of scope lemma: (1, 2, 3, ...) - annotations: ¡NEW REF¿ ¡CAUSE REF¿

TODO: add Incomleteness theorem III, Rice's theorem, others ??? TODO: OVERLEFT ARROW ABBREVIATES vdcS... TODO: One liner theorems on comment header ??

? TODO: assumption contexts TODO: DEFINITIONS WITH: SATISFIES ANY OF THE FOLLOWING: ¡CONJUNCTIONS¿ IS MUCH CLEARER THAN IF X, THEN Y DEFINITIONS TODO: RECURSION BY STAGE + RECURSION BY STRUCTURE TODO: max largest biggest symbolic qualifier for sets like (set of all free variables contained in phi or something) TODO: do decidable metatheorems: 1.8.1.7 / 2.4.3.1-2

TODO check mistakes: - FIX BAD SMELL: IMPLICIT ASSUMPTIONS - re-write IF with IFF appropriate definitions like inferences - PC ONLY AFFECTS PROPOSITIONAL VAR, NOT ALPHABET VAR

5.3.27 (Notation) Retarded notation - free occurrence

- $\phi(x)$ means x is free in ϕ
- $\phi(t)$ means substitute x by t in ϕ

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