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Choosability of Graphs with Bounded Order: Ohba's Conjecture and Beyond

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Abstract

We discuss some recent results and conjectures on bounding the choice number of a graph G under the condition that $|V(G)|$ is bounded above by a fixed function of $\chi(G)$.

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1 Results and Conjectures

Choosability is a natural generalization of the classical graph colouring problem. In choosability, the objective is to find a proper colouring of a graph G in which the colour of each vertex v belongs to a prescribed list $L(v)$ of *available* colours. Such a colouring is called an *acceptable* colouring for L. If there is an acceptable colouring for L whenever $|L(v)| \geq k$ for all $v \in V(G)$, then we say that G is *k-choosable.* The *choice number* of a graph G is defined as follows:

$$\mathrm{ch}(G) := \min\{k : G \text{ is } k\text{-choosable}\}.$$

Choosability was introduced independently by Vizing [18] and Erdős, Rubin and Taylor [4]. It has received a great deal of attention, and several interesting survey articles have been written on the topic [3,9,19].

A fundamental result of Erdős, Rubin and Taylor [4] is that there are bipartite graphs with arbitrarily large choice number. Therefore, for general graphs, there is no upper bound on the choice number in terms of the chromatic number. However, there are many results and conjectures concerning more specific graph classes. These include, for example, Thomassen's [16] celebrated proof that every planar graph is 5-choosable,[7] and the famous List Colouring Conjecture which asserts that ch $= \chi$ for every line graph (see [6]).

One way to observe a relationship between the choice number and the chromatic number is to consider graphs for which the number of vertices is bounded by a function of the chromatic number. In particular, we focus on a problem of this type due to Ohba [13].

Conjecture 1.1 (Ohba [13]) *If* $|V(G)| \leq 2\chi(G) + 1$, *then* $\mathrm{ch}(G) = \chi(G)$.

One of the motivating examples for Ohba's Conjecture comes from the original paper of Erdős, Rubin and Taylor [4]. For $m, k \geq 2$, let K_{m*k} denote the complete k-partite graph in which every part has size m. Erdős, Rubin and Taylor [4] calculated the choice number of K_{2*k} exactly. Their proof requires nothing more than a clever combination of induction and Hall's Theorem [5] on systems of distinct representatives.

Theorem 1.2 (Erdős, Rubin and Taylor [4]) $\mathrm{ch}\left(K_{2*k}\right) = k$.

Ohba's Conjecture has drawn interest from many different researchers, and numerous partial results have been proven. Most notably, Kostochka, Stiebitz and Woodall [8] proved it for graphs of stability number at most

[7] An exposition of Thomassen's proof can be found in Aigner and Ziegler's *Proofs from The Book* [1].

5, and Reed and Sudakov [15] applied the probabilistic method to prove the following asymptotic version: *If* $|V(G)| \leq (2-o(1))\chi(G)$*, then* $\mathrm{ch}(G) = \chi(G)$. In [11], we (Noel, Reed and Wu) proved Ohba's Conjecture.

Theorem 1.3 (Noel, Reed and Wu [11]) *If* $|V(G)| \leq 2\chi(G) + 1$*, then* $\mathrm{ch}(G) = \chi(G)$*.*

As in the proof of Theorem 1.2, the main elements of our proof are induction and Hall's Theorem, but with a novel approach. Our overarching philosophy is the following: instead of constructing an acceptable colouring *directly*, it is often easier to construct a special type of non-acceptable colouring, and then modify it via Hall's Theorem to obtain an acceptable colouring. Specifically, the proof is composed of three main lemmas, which we state next. To do so, we require some definitions. Throughout, we assume that G is a graph such that $|V(G)| \leq 2\chi(G) + 1$.

Definition 1.4 Say that a colour c is *frequent* if it is contained in at least $\chi(G) + 1$ lists.

Definition 1.5 Say that a proper colouring f is *near-acceptable* if for every vertex v, either

- $f(v) \in L(v)$, or
- $f^{-1}(f(v)) = \{v\}$ and $f(v)$ is frequent.

Lemma 1.6 *If there is a near-acceptable colouring, then there is an acceptable colouring.*

Lemma 1.7 *If there are at least* $\chi(G)$ *frequent colours, then there is a near-acceptable colouring.*

Lemma 1.8 *If Ohba's Conjecture is false, then there is a counterexample with at least* $\chi(G)$ *frequent colours.*

To give the reader a taste of our techniques, we provide the proof of Lemma 1.6 from [11].

Proof of Lemma 1.6 Suppose that $|V(G)| \leq 2\chi(G) + 1$ and that $|L(v)| \geq \chi(G)$ for every vertex $v \in V(G)$. Let f be a near-acceptable colouring for L, and let $V_f := \{f^{-1}(c) : c \in f(V(G))\}$ be the colour classes for f.

Our goal is to use Hall's Theorem to obtain an acceptable colouring for L with the same colour classes as f. To this end, we let B_f be the bipartite graph with bipartition $\left(V_f, \cup_{v\in V(G)} L(v)\right)$ where each $f^{-1}(c) \in V_f$ is joined to the colours of $\cap_{v\in f^{-1}(c)} L(v)$. That is, $f^{-1}(c)$ is joined to the colours which are available for *every* vertex of $f^{-1}(c)$. Clearly there is an acceptable colouring for L with the same colour classes as f if and only if there is a matching in B_f which saturates V_f.

If no such matching exists, then by Hall's Theorem there is a set $S \subseteq V_f$ such that $|N_{B_f}(S)| < |S|$. In particular, there must be a colour c such that $f^{-1}(c) \in S$ but $c \notin N_{B_f}(S)$. However, since $\cap_{v \in f^{-1}(c)} L(v) \subseteq N_{B_f}(S)$, this implies that there is a vertex v such that $f(v) = c$ and $c \notin L(v)$. Since f is near-acceptable, we have that

- c is frequent, and
- $f^{-1}(c) = \{v\}$.

Since $f^{-1}(c) = \{v\} \in S$, it must be the case that $L(v) = \cap_{v \in f^{-1}(c)} L(v) \subseteq N_{B_f}(S)$. Therefore, by our choice of S we have

$$|S| > |N_{B_f}(S)| \geq |L(v)| \geq \chi(G)$$

and so $|S| \geq \chi(G) + 1$. However, since $c \notin N_{B_f}(S)$, each colour class in S must contain a vertex for which c is not available. Therefore, $|S|$ is at most the number of vertices of G for which c is not available. Since c is frequent and $|V(G)| \leq 2\chi(G) + 1$, this implies that

$$|S| \leq |V(G)| - (\chi(G) + 1) \leq \chi(G).$$

However, we have already proved that $|S| \geq \chi(G) + 1$, and so this is a contradiction. It follows that no such S can exist, and so the proof is complete. □

To prove Lemma 1.7, we exploit the relaxed properties of a near-acceptable colouring to show that, if there are at least $\chi(G)$ frequent colours, then a near-acceptable colouring can be constructed via a greedy colouring procedure. We then prove Lemma 1.8 by considering a theoretical extremal counterexample to Ohba's Conjecture and applying some clever counting arguments.

As it turns out, Ohba's Conjecture is best possible since $|V(K_{3,3})| = 2\chi(K_{3,3}) + 2$ and $\mathrm{ch}(K_{3,3}) = 3$ (see Fig. 1). However, it is still reasonable to obtain an upper bound on the choice number of a graph on more than $2\chi + 1$ vertices in terms of its chromatic number and order. In [12], we (Noel, West, Wu and Zhu) proved a result of this type, which strengthens Theorem 1.3.

Theorem 1.9 (Noel, West, Wu and Zhu [12]) *For every graph G,*

$$\mathrm{ch}(G) \leq \max\left\{\chi(G), \left\lceil \frac{|V(G)| + \chi(G) - 1}{3} \right\rceil\right\}.$$

To prove Theorem 1.9, we use Theorem 1.3 as the 'base case' of an inductive argument. We then reduce Theorem 1.9 to the case of complete multipartite

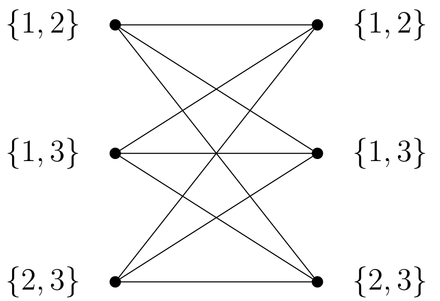

Fig. 1. A list assignment which demonstrates that $K_{3,3}$ is not 2-choosable.

graphs with stability number at most 4. Once this reduction is made, we complete the proof using a greedy procedure and, yet again, Hall's Theorem.

A result of Ohba [14] shows that Theorem 1.9 is tight for an infinite class of graphs (which generalizes the example of $K_{3,3}$ already mentioned).

Theorem 1.10 (Ohba [14]) *Let k_1 and k_3 be positive integers, and define $k := k_1 + k_3$ and $n := k_1 + 3k_3$. If G is the complete k-partite graph with k_1 parts of size 1 and k_3 parts of size 3, then*

$$\mathrm{ch}(G) = \max\left\{k, \left\lceil \frac{n+k-1}{3} \right\rceil\right\}.$$

A problem suggested by Erdős, Rubin and Taylor [4] is to determine good bounds on the choice number of K_{m*k}.[8] In general, Alon [2] proved that there are constants c_1 and c_2 such that $c_1 \log(m)k \leq \mathrm{ch}(K_{m*k}) \leq c_2 \log(m)k$. However, for fixes $m \geq 4$, the value of $\mathrm{ch}(K_{m*k})$ is not known for general k. Theorems 1.9 and 1.10 strengthen a result of Kierstead [7], who calculated $\mathrm{ch}(K_{3*k})$.

Theorem 1.11 (Kierstead [7]) $\mathrm{ch}(K_{3*k}) = \left\lceil \frac{4k-1}{3} \right\rceil$.

For $m = 4$ the bounds $\left\lfloor \frac{3k}{2} \right\rfloor \leq \mathrm{ch}(K_{4*k}) \leq \left\lceil \frac{7k}{4} \right\rceil$ are due to Yang [20]. As a corollary of Theorem 1.9, we obtain an improvement of the upper bound.

Corollary 1.12 (Noel, West, Wu and Zhu [12]) $\mathrm{ch}(K_{4*k}) \leq \left\lceil \frac{5k-1}{3} \right\rceil$.

By combining Theorem 1.9 with Theorem 1.11, we see that every graph G with $|V(G)| \leq 3\chi(G)$ satisfies $\mathrm{ch}(G) \leq \mathrm{ch}\left(K_{3*\chi(G)}\right)$. We conjecture that a similar property holds for graphs on at most $m\chi$ vertices for all m.

8 The original motivation for this problem was to estimate the choice number of the random graph $G(n, 1/2)$.

Conjecture 1.13 (Noel [10]) *For $m, k \geq 2$, every k-chromatic graph G on at most mk vertices satisfies* $\mathrm{ch}(G) \leq \mathrm{ch}(K_{m*k})$.

We also make the following, more general, conjecture.

Conjecture 1.14 (Noel [10]) *For $n \geq k \geq 2$, there exists a graph $G_{n,k}$ such that*

- *$G_{n,k}$ is a complete k-partite graph on n vertices,*
- *$\alpha(G_{n,k}) = \lceil \frac{n}{k} \rceil$, and*
- *every k-chromatic graph G on at most n vertices satisfies* $\mathrm{ch}(G) \leq \mathrm{ch}(G_{n,k})$.

Note that, by Theorems 1.3, 1.9 and 1.10, we see that Conjecture 1.14 is true in the case that $n \leq 2k + 1$ or $n \leq 3k$ and $n - k$ is even.

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