0.1 Problem Set 2

Suppose $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$ and $\lim\limits_{n\to\infty}(c_n) \neq 0$ and for any $n \in \mathbb{N}$, $c_n \neq 0$.

1.) Prove $\lim\limits_{n\to\infty}(c) = c$:
Observe that $(c)_{n\in\mathbb{N}} = c(1)_{n\in\mathbb{N}}$. By Theorem 4, $\lim\limits_{n\to\infty}(c) = \lim\limits_{n\to\infty}(c * 1) = c \lim\limits_{n\to\infty}(1)$. Since $\lim\limits_{n\to\infty}(1) = 1$, $\lim\limits_{n\to\infty}(c) = c * 1 = c$. ■

2.) Prove $\lim\limits_{n\to\infty}(|a_n|) = |\lim\limits_{n\to\infty}(a_n)|$:
Let $\epsilon > 0$. By the hypotheses, $(a_n)_{n\in\mathbb{N}}$ converges to some $a \in \mathbb{R}$. Furthermore, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$, $|a_n - a| < \epsilon$. This with the reverse triangle inequality yields $||a_n| - |a|| \leq |a_n - a| < \epsilon$, for any $n \geq N$. By transitivity of the order relation on $\mathbb{R}$, $||a_n| - |a|| < \epsilon$, for any $n \geq N$. Therefore, $\lim\limits_{n\to\infty}(|a_n|) = |a| = |\lim\limits_{n\to\infty}(a_n)|$. ■

3.) Prove $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty}(a_n) - \lim\limits_{n\to\infty}(b_n)$:
Observe that $(a_n - b_n)_{n\in\mathbb{N}} = (a_n)_{n\in\mathbb{N}} + (-1)(b_n)_{n\in\mathbb{N}}$. By Theorem 4, $(-1)(b_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$ and $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty}\left(a_n + (-1)b_n\right) = \lim\limits_{n\to\infty}(a_n) + (-1)\lim\limits_{n\to\infty}(b_n) = \lim\limits_{n\to\infty}(a_n) - \lim\limits_{n\to\infty}(b_n)$ ■

4.) Prove $\lim\limits_{n\to\infty}\left(\frac{a_n}{c_n}\right) = \frac{\lim\limits_{n\to\infty}(a_n)}{\lim\limits_{n\to\infty}(c_n)}$:
Observe that $\left(\frac{a_n}{c_n}\right)_{n\in\mathbb{N}} = (a_n)_{n\in\mathbb{N}} * \left(\frac{1}{c_n}\right)_{n\in\mathbb{N}}$. By the hypotheses, $(a_n)_{n\in\mathbb{N}}$ converges to some $a \in \mathbb{R}$ and $(c_n)_{n\in\mathbb{N}}$ converges to some $c \in \mathbb{R}$. We can apply Theorem 8 on $(c_n)_{n\in\mathbb{N}}$ since it satisfies the necessary assumptions. Thus, $\left(\frac{1}{c_n}\right)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$ and $\lim\limits_{n\to\infty}\left(\frac{1}{c_n}\right) = \frac{1}{c}$. We can now apply Theorem 6 to get $\lim\limits_{n\to\infty}\left(\frac{a_n}{c_n}\right) = \lim\limits_{n\to\infty}\left(a_n * \frac{1}{c_n}\right) = \lim\limits_{n\to\infty}(a_n) * \lim\limits_{n\to\infty}\left(\frac{1}{c_n}\right) = a * \frac{1}{c} = \frac{\lim\limits_{n\to\infty}(a_n)}{\lim\limits_{n\to\infty}(c_n)}$. ■

5.) Prove $\lim\limits_{n\to\infty}(a_n \vee b_n) = \lim\limits_{n\to\infty}(a_n) \vee \lim\limits_{n\to\infty}(b_n)$:
By the definition of the lattice operations,

$$(a_n \vee b_n)_{n\in\mathbb{N}} = \frac{1}{2}(a_n)_{n\in\mathbb{N}} + \frac{1}{2}(b_n)_{n\in\mathbb{N}} + \frac{1}{2}(|a_n - b_n|)_{n\in\mathbb{N}} \tag{1}$$

By the hypotheses, $(a_n)_{n\in\mathbb{N}}$ converges to some $a \in \mathbb{R}$ and $(b_n)_{n\in\mathbb{N}}$ converges to some $b \in \mathbb{R}$. Using Item 3, Item 2, and Theorem 4, we can say that the RHS of (1) must also be convergent, and we apply them on (1)

$$\lim_{n\to\infty}(a_n \vee b_n) = \frac{1}{2}\lim_{n\to\infty}(a_n) + \frac{1}{2}\lim_{n\to\infty}(b_n) + \frac{1}{2}\lim_{n\to\infty}(|a_n - b_n|) = \frac{1}{2}(a + b + |a - b|) = a \vee b = \lim_{n\to\infty}(a_n) \vee \lim_{n\to\infty}(b_n) \tag{2}$$

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6.) Prove $\lim\limits_{n\to\infty}(a_n \wedge b_n) = \lim\limits_{n\to\infty}(a_n) \wedge \lim\limits_{n\to\infty}(b_n)$:
By the definition of the lattice operations,

$$(a_n \wedge b_n)_{n\in\mathbb{N}} = \frac{1}{2}(a_n)_{n\in\mathbb{N}} + \frac{1}{2}(b_n)_{n\in\mathbb{N}} + \frac{-1}{2}(|a_n - b_n|)_{n\in\mathbb{N}} \tag{3}$$

By the hypotheses, $(a_n)_{n\in\mathbb{N}}$ converges to some $a \in \mathbb{R}$ and $(b_n)_{n\in\mathbb{N}}$ converges to some $b \in \mathbb{R}$. Using Item 3, Item 2, and Theorem 4, we can say that the RHS of (3) must also be convergent, and we apply them on (3)

$$\lim_{n\to\infty}(a_n \wedge b_n) = \frac{1}{2}\lim_{n\to\infty}(a_n) + \frac{1}{2}\lim_{n\to\infty}(b_n) + \frac{-1}{2}\lim_{n\to\infty}(|a_n - b_n|) = \frac{1}{2}(a + b - |a - b|) = a \wedge b = \lim_{n\to\infty}(a_n) \wedge \lim_{n\to\infty}(b_n) \tag{4}$$

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