

Convergent Sequences

Part 2

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Some notes on subsequences

Given a sequence $(a_n)_{n \in \mathbb{N}}$ in \mathbb{R} , and a function $\mathbb{N} \rightarrow \mathbb{N}$ denoted by $i \mapsto N_i$ such that

$$i < j \implies N_i < N_j, \quad (1)$$

we call $(a_{N_i})_{i \in \mathbb{N}}$ a **subsequence** of $(a_n)_{n \in \mathbb{N}}$.

Given a subsequence $(a_{N_i})_{i \in \mathbb{N}}$ of a sequence $(a_n)_{n \in \mathbb{N}}$ in \mathbb{R} , by the Trichotomy Law, the condition $i \neq j$ means that either $i < j$ or $i > j$. Then by (1), we have either $N_i < N_j$ or $N_i > N_j$, which implies $N_i \neq N_j$. We have thus shown that $i \neq j$ implies $N_i \neq N_j$, and by contraposition,

$$N_i = N_j \implies i = j. \quad (2)$$

Therefore, $i \mapsto N_i$ is injective. The converse

$$i = j \implies N_i = N_j, \quad (3)$$

of (2) is true because $i \mapsto N_i$ is a function. Also, if $N_i < N_j$, then $N_i \neq N_j$, and by the contrapositive of (3), we have $i \neq j$.

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If $i > j$, then we get, from (1), the contradiction $N_i > N_j$, and so the only possibility is $i < j$. That is,

$$N_i < N_j \implies i < j. \quad (4)$$

From (1)–(4), we obtain

$$i \leq j \iff N_i \leq N_j. \quad (5)$$

Using an elementary proof, the equivalence (5) can be used to prove that the conditions

$$\forall \varepsilon > 0 \quad \exists N_I \in \mathbb{N} \quad \forall N_i \geq N_I \quad |a_{N_i} - a| < \varepsilon, \quad (6)$$

$$\forall \varepsilon > 0 \quad \exists I \in \mathbb{N} \quad \forall i \geq I \quad |a_{N_i} - a| < \varepsilon, \quad (7)$$

are equivalent. Hence, if the subsequence $(a_{N_i})_{i \in \mathbb{N}}$ converges to some $a \in \mathbb{R}$, both notations $\lim_{N_i \rightarrow \infty} a_{N_i}$ and $\lim_{i \rightarrow \infty} a_{N_i}$ are valid, and

furthermore,

$$\lim_{N_i \rightarrow \infty} a_{N_i} = a \iff \lim_{i \rightarrow \infty} a_{N_i} = a.$$

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i.e., The limiting process for the convergent subsequence $(a_{N_i})_{i \in \mathbb{N}}$ is the same regardless of whether we view this limiting process in terms of the original indices, as in $N_i \rightarrow \infty$, or in terms of the 'secondary' indices, as in $i \rightarrow \infty$.

Another important property of a subsequence $(a_{N_i})_{i \in \mathbb{N}}$ of $(a_n)_{n \in \mathbb{N}}$ is that

$$\forall i \in \mathbb{N} \ [i \leq N_i]. \quad (8)$$

If $i = 1$, then by the fact that $N_i \in \mathbb{N}$, we have $N_i \geq 1 = i$.

Suppose $i \leq N_i$ for some $i \in \mathbb{N}$. Tending towards a contradiction, suppose $i + 1 > N_{i+1}$. Since both $i + 1$ and N_i are integers, we further have $i \geq N_{i+1}$. By the inductive hypothesis, $N_i \geq i \geq N_{i+1}$. But this contradicts $N_i < N_{i+1}$ because of (1) and $i < i + 1$. Therefore, $i + 1 \leq N_{i+1}$, and we have proven (8) by induction.

Proposition 1

If $(a_n)_{n \in \mathbb{N}}$ converges to $a \in \mathbb{R}$, then any convergent subsequence of $(a_n)_{n \in \mathbb{N}}$ also converges to a .

Proof of Proposition 1

Suppose $(a_{N_i})_{i \in \mathbb{N}}$ is a subsequence of $(a_n)_{n \in \mathbb{N}}$ that converges to $b \in \mathbb{R}$, and let $\varepsilon > 0$. The conditions $a = \lim_{n \rightarrow \infty} a_n$ and $b = \lim_{N_i \rightarrow \infty} a_{N_i}$ imply that there exist $N, N_I \in \mathbb{N}$ such that

$$n \geq N \implies |a - a_n| = |a_n - a| < \frac{\varepsilon}{2}, \quad (9)$$

$$N_i \geq N_I \implies |a_{N_i} - b| < \frac{\varepsilon}{2}. \quad (10)$$

Let us consider those indices N_i such that $i > \max\{N, N_I\}$. Using (8), we have $N_i \geq i > N$, so the conclusion of (9) is true for $n = N_i$. Also using (8), we have $N_i \geq i > N_I$, so the conclusion of (10) is also true. By the triangle inequality,

$$|a - b| = |(a - a_{N_i}) + (a_{N_i} - b)| \leq |a - a_{N_i}| + |a_{N_i} - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

That is $|a - b| < \varepsilon$ for an arbitrary $\varepsilon > 0$. Therefore, $a = b$. \square

The limit superior of a sequence

Let us return our attention to an arbitrary sequence $(a_n)_{n \in \mathbb{N}}$ in \mathbb{R} . Given $n \in \mathbb{N}$, let us collect the terms of the sequence “at index n and beyond” in the following set:

$$\{a_k : k \geq n\} = \{a_n, a_{n+1}, a_{n+2}, \dots\}. \quad (11)$$

If the set (11) has an upper bound $M \in \mathbb{R}$, then its supremum

$$\sup_{k \geq n} a_k := \sup\{a_k : k \geq n\} = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}, \quad (12)$$

exists as an element of \mathbb{R} . Otherwise, we define $\sup_{k \geq n} a_k$ as ∞ . Note that the number $\sup_{k \geq n} a_k$ depends on n , and so we now have a new sequence

$$\sup_{k \geq 1} a_k, \quad \sup_{k \geq 2} a_k, \quad \sup_{k \geq 3} a_k, \quad \dots, \quad \sup_{k \geq n} a_k, \quad \dots \quad (13)$$

of extended real numbers, where in the subscripts after the “ $k \geq$ ” we find the indices of the terms of the sequence (13).

The limit superior of a sequence

Observe that the supremum (12) of (11) need not be one of the terms in (11), and so it is important to note here that (13) is not necessarily a subsequence of $(a_n)_{n \in \mathbb{N}}$. If the set of all terms in the sequence (13) has a lower bound $M' \in \mathbb{R}$, then the infimum

$$\begin{aligned}\limsup_{n \rightarrow \infty} a_n &= \inf_{n \in \mathbb{N}} \sup_{k \geq n} a_k := \inf \left\{ \sup_{k \geq n} a_k : n \in \mathbb{N} \right\}, \\ &= \inf \left\{ \sup_{k \geq 1} a_k, \sup_{k \geq 2} a_k, \dots \right\},\end{aligned}$$

of the set of all terms of (13) exists as an element of \mathbb{R} .

Otherwise, we define $\limsup_{n \rightarrow \infty} a_n$ as $-\infty$. We call the number

$\limsup_{n \rightarrow \infty} a_n$ the *limit superior or upper limit* of the sequence $(a_n)_{n \in \mathbb{N}}$.

Lemma 2

Let $M \in \mathbb{R}$, and consider a sequence $(a_n)_{n \in \mathbb{N}}$ in \mathbb{R} . If $a_n \leq M$ for any $n \in \mathbb{N}$, then $\limsup_{n \rightarrow \infty} a_n \leq M$.

Proof of Lemma 2

Since $a_n \leq M$ for any index n , in particular, given $k \in \mathbb{N}$, we have $a_n \leq M$ 'at index k and beyond.' That is,

$$k \geq n \implies a_k \leq M,$$

which means that M is an upper bound of $\{a_k : k \geq n\}$, and the relationship of this upper bound to the supremum is

$$\sup_{k \geq n} a_k \leq M.$$

But since $\inf_{n \in \mathbb{N}} \sup_{k \geq n} a_k$ is a lower bound of $\{\sup_{k \geq n} a_k : n \in \mathbb{N}\}$, we further have

$$\inf_{n \in \mathbb{N}} \sup_{k \geq n} a_k \leq \sup_{k \geq n} a_k \leq M.$$

Therefore, $\limsup_{n \rightarrow \infty} a_n \leq M$. \square

Lemma 3

If $-\infty < \limsup_{n \rightarrow \infty} a_n < \infty$, then there exists a subsequence $(a_{N_i})_{i \in \mathbb{N}}$ of $(a_n)_{n \in \mathbb{N}}$ such that for any $i \in \mathbb{N}$,

$$\left| a_{N_i} - \limsup_{n \rightarrow \infty} a_n \right| < \frac{1}{i}. \quad (14)$$

Proof of Lemma 3

Suppose $-\infty < \limsup_{n \rightarrow \infty} a_n < \infty$, and let $i \in \mathbb{N}$. Since $\frac{1}{i} > 0$, the number

$$\frac{1}{i} + \limsup_{n \rightarrow \infty} a_n \tag{15}$$

exceeds the infimum of

$$\left\{ \sup_{k \geq 1} a_k, \sup_{k \geq 2} a_k, \sup_{k \geq 3} a_k, \dots, \sup_{k \geq n} a_k, \dots \right\} \tag{16}$$

and is hence not a lower bound of (16). That is, the set (16) has an element not bounded below by (\neq) the number (15). This element has an index M_i that appears after the " $k \geq$ " and so we have

Proof of Lemma 3

$$\sup_{k \geq M_i} a_k < \frac{1}{i} + \limsup_{n \rightarrow \infty} a_n. \quad (17)$$

We note here that (17) cannot be possible if $\limsup_{n \rightarrow \infty} a_n = -\infty$, in

which case there shall be no number below

$\frac{1}{i} + \limsup_{n \rightarrow \infty} a_n = \frac{1}{i} - \infty = -\infty$. Hence, the assumption

$\limsup_{n \rightarrow \infty} a_n > -\infty$ is important. Since $-\frac{1}{i} < 0$, the number

$$-\frac{1}{i} + \sup_{k \geq M_i} a_k \quad (18)$$

is less than the supremum of

$$\{a_{M_i}, a_{M_i+1}, a_{M_i+2}, \dots\} \quad (19)$$

which means that (18) is not an upper bound of (19), and so (19) has an element not bounded above by (\nless) the number (18).

Proof of Lemma 3

This element has an index N_i which is one of the indices $M_i, M_i + 1, \dots$, which means $N_i \geq M_i$. We now have the inequality

$$a_{N_i} > -\frac{1}{i} + \sup_{k \geq M_i} a_k. \quad (20)$$

Since $\sup_{k \geq M_i} a_k$ is in (16) and $\limsup_{n \rightarrow \infty} a_n$ is a lower bound of (16), we can further extend the inequality (20) as

$$a_{N_i} > -\frac{1}{i} + \sup_{k \geq M_i} a_k \geq -\frac{1}{i} + \limsup_{n \rightarrow \infty} a_n. \quad (21)$$

The strict inequality in (21) would not be possible for the case $\limsup_{n \rightarrow \infty} a_n = \infty$, because in such a case, there would be no number above $-\frac{1}{i} + \limsup_{n \rightarrow \infty} a_n = -\frac{1}{i} + \infty = \infty$, and this tells us that the assumption $\limsup_{n \rightarrow \infty} a_n < \infty$ is important.

Proof of Lemma 3

Recall earlier that $N_i \geq M_i$, so a_{N_i} is in (19), and since $\sup_{k \geq M_i} a_k$ is an upper bound of (19), the inequality (17) can be extended as

$$a_{N_i} \leq \sup_{k \geq M_i} a_k < \frac{1}{i} + \limsup_{n \rightarrow \infty} a_n. \quad (22)$$

From (21) and (22), we get

$$\begin{aligned} -\frac{1}{i} + \limsup_{n \rightarrow \infty} a_n &< a_{N_i} < \frac{1}{i} + \limsup_{n \rightarrow \infty} a_n, \\ -\frac{1}{i} &< a_{N_i} - \limsup_{n \rightarrow \infty} a_n < \frac{1}{i}, \end{aligned} \quad (23)$$

from which we obtain (14). \square

Bounded sequences and the Bolzano-Weierstrass Theorem

Given a real number $M > 0$, we say that a sequence $(a_n)_{n \in \mathbb{N}}$ in \mathbb{R} is *bounded by* M if $|x_n| \leq M$ for all $n \in \mathbb{N}$. Any sequence bounded by some positive real number is a *bounded sequence*.

Lemma 4

If $c \in \mathbb{R}$ and if $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are bounded sequences in \mathbb{R} , then the sequences

$$(a_n)_{n \in \mathbb{N}} + (b_n)_{n \in \mathbb{N}}, \quad c(a_n)_{n \in \mathbb{N}}, \quad (a_n)_{n \in \mathbb{N}}(b_n)_{n \in \mathbb{N}},$$

are also bounded.

Proof of Lemma 4

Suppose $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are bounded by M and N , respectively. By routine computations using the properties of inequalities in \mathbb{R} , we find that the sequences $(a_n)_{n \in \mathbb{N}} + (b_n)_{n \in \mathbb{N}}$, $c(a_n)_{n \in \mathbb{N}}$ and $(a_n)_{n \in \mathbb{N}}(b_n)_{n \in \mathbb{N}}$ are bounded by $M + N$, $|c| \cdot M$ and MN , respectively. \square

Corollary 5

The set $\ell^\infty(\mathbb{R})$ of all bounded sequences in \mathbb{R} is an associative algebra over \mathbb{R} that is unital and commutative.

Proof of Corollary 5

All the algebraic properties, except closure, of the three operations—addition of sequences as vector addition, left-multiplication by a constant as scalar multiplication, and multiplication of sequences as vector multiplication—that were discussed in the previous lecture are valid for all sequences, and, in particular, for all the sequences in $\ell^\infty(\mathbb{R})$. The closure of $\ell^\infty(\mathbb{R})$ under the said three operations is asserted in Lemma 4.

Lemma 6

If $(a_n)_{n \in \mathbb{N}}$ is bounded, then $-\infty < \limsup_{n \rightarrow \infty} a_n < \infty$.

Proof of Lemma 6

If for any $n \in \mathbb{N}$, we have $|a_n| \leq M$, or equivalently

$$-M \leq a_n \leq M, \quad (24)$$

then, in particular, $a_n \leq M$, and by Lemma 2, we have

$$\limsup_{n \rightarrow \infty} a_n \leq M < \infty. \quad (25)$$

By (24), we have $-M \leq a_n$ for any index n , and in particular for any index $k \geq n$. Thus means that $-M$ is a lower bound of $\{a_k : k \in \mathbb{N}\}$, but since $\sup_{k \geq n} a_k$ is an upper bound of $\{a_k : k \in \mathbb{N}\}$, we have

$$-M \leq \sup_{k \geq n} a_k. \quad (26)$$

Since (26) holds for any $n \in \mathbb{N}$, we find that $-M$ is a lower bound of $\left\{ \sup_{k \geq n} a_k : n \in \mathbb{N} \right\}$, and is thus less than or equal to the infimum of $\left\{ \sup_{k \geq n} a_k : n \in \mathbb{N} \right\}$. That is,

$$-M \leq \inf_{n \in \mathbb{N}} \sup_{k \geq n} a_k = \limsup_{n \rightarrow \infty} a_n,$$

which, in conjunction with (25), gives us $-\infty < M \leq \limsup_{n \rightarrow \infty} a_n < \infty$. \square

We summarize in the following the logical relationship between the notions of **boundedness and convergence of a sequence in \mathbb{R} .**

Theorem 7

- ① *A convergent sequence in \mathbb{R} is bounded.*
- ② *A bounded sequence in \mathbb{R} is not necessarily convergent.*
- ③ *[The Bolzano-Weierstrass Theorem.] A bounded sequence in \mathbb{R} has a convergent subsequence.*

Proof of Theorem 7(i)

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R} , and suppose $a = \lim_{n \rightarrow \infty} a_n$ for some $a \in \mathbb{R}$. In symbols,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad [|a_n - a| < \varepsilon]. \quad (27)$$

The trick is to instantiate (27) at the value $\varepsilon = 1$. That is, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - a| < 1. \quad (28)$$

The next trick is to use the reverse triangle inequality in the conclusion of (28). If $n \geq N$, then

$$\begin{aligned} |a_n - a| &< 1, \\ ||a_n| - |a|| &\leq |a_n - a| < 1, \\ |a_n| - |a| &< 1, \\ -1 &< |a_n| - |a| < 1, \\ |a_n| - |a| &< 1, \\ |a_n| &< 1 + |a|. \end{aligned}$$

Proof of Theorem 7(i)

and so (28) becomes

$$n \geq N \implies |a_n| < 1 + |a|. \quad (29)$$

Recall that our goal here is to find some $M \in \mathbb{R}$ such that every term of $(a_n)_{n \in \mathbb{N}}$ has absolute value less than or equal to M . The inequality in (29) tells us that all terms 'at index N and beyond' already have an absolute value less than $1 + |a|$. The only terms not covered are those with index $N - 1$ and below. Thus, we let

$$M := \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1 + |a|\}. \quad (30)$$

If $n \geq N$, then by (29), $|a_n| < 1 + |a| \leq M$, and if $n < N$, then by (30), $|a_n| \leq M$. Combining these two cases, we have $|a_n| \leq M$ for all $n \in \mathbb{N}$. Therefore, $(a_n)_{n \in \mathbb{N}}$ is bounded.

Proof of Theorem 7(ii)

Our goal here is to exhibit a sequence in \mathbb{R} that is both bounded and not convergent. For any $n \in \mathbb{N}$, let $a_n := (-1)^n$. That is $a_n = 1$ if n is even, and $a_n = -1$ if n is odd. Hence, $|a_n| = 1$, and consequently, $|a_n| \leq 1$ for any $n \in \mathbb{N}$, which means that $(a_n)_{n \in \mathbb{N}}$ is bounded. To show $(a_n)_{n \in \mathbb{N}}$ is not convergent, let $a \in \mathbb{R}$. We produce a value of ε by cases depending on the value of $|a - 1| \geq 0$. If $|a - 1| = 0$, then we set $\varepsilon = \frac{1}{2} > 0$, and if $|a - 1| > 0$, we set $\varepsilon = |a - 1| > 0$. Let $N \in \mathbb{N}$. If $|a - 1| = 0$, then $a = 1$, and we choose any odd $n \geq N$, for which $|a_n - a| = |-1 - 1| = 2 \geq \varepsilon$. If $|a - 1| > 0$, then we choose any even $n \geq N$, for which $|a_n - 1| = |1 - 1| = 0$, and by the triangle inequality,

$$\varepsilon = |a - 1| \leq |a - a_n| + |a_n - 1| = |a - a_n| + 0,$$

and we still have $|a_n - a| \geq \varepsilon$. We have thus shown

$$\exists \varepsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geq N \quad [|a_n - a| \geq \varepsilon],$$

with $a \in \mathbb{R}$ arbitrary. Therefore, $(a_n)_{n \in \mathbb{N}}$ does not converge to any element of \mathbb{R} .

Proof of Theorem 7(iii)

If $(a_n)_{n \in \mathbb{N}}$ is bounded, then by Lemma 6, we have $-\infty < \limsup_{n \rightarrow \infty} a_n < \infty$, which by Lemma 3 implies that there exists a subsequence $(a_{N_i})_{i \in \mathbb{N}}$ such that

$$\left| a_{N_i} - \limsup_{n \rightarrow \infty} a_n \right| < \frac{1}{i},$$

for any $i \in \mathbb{N}$. If $\varepsilon > 0$, then there exists [an integer] $I > \frac{1}{\varepsilon}$, and for any $i \geq I$,

$$\left| a_{N_i} - \limsup_{n \rightarrow \infty} a_n \right| < \frac{1}{i} \leq \frac{1}{I} < \varepsilon.$$

Therefore, $(a_{N_i})_{i \in \mathbb{N}}$ is convergent.

