3. Limits and Continuity of Functions on Intervals

Let $a, b \in \mathbb{R}$. We use the following notation for intervals in $\mathbb{R}$:

$$\begin{aligned}]a,b[&:= \{x \in \mathbb{R} : a < x < b\}, \\ [a,b] &:= \{x \in \mathbb{R} : a \leq x \leq b\}, \\ ]a,b] &:= \{x \in \mathbb{R} : a < x \leq b\}, \\ [a,b[&:= \{x \in \mathbb{R} : a \leq x < b\}. \end{aligned}$$

The interval $[a, b]$ is a singleton if $a = b$ and is empty if $a > b$, while all the other three intervals are empty if $a \geq b$. As for terminology, we bother only with calling $]a, b[$ as an *open interval* in $\mathbb{R}$, and $[a, b]$ as a *closed interval*.

Lemma 3.1. *If $a, b \in \mathbb{R}$ such that $]a, b[\neq \emptyset$, then there exist a sequence in $]a, b[$ that converges to a and a sequence that converges to b.*

Proof. First, we claim the existence of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$,

$$a < a_n < a + \frac{1}{n}, \tag{3.1}$$

$$b - \frac{1}{n} < b_n < b. \tag{3.2}$$

We construct the terms inductively. By the denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exist $a_1, b_1 \in \mathbb{Q}$ such that

$$a < a_1 < \min\{a + 1, b\}, \tag{3.3}$$

$$\max\{a, b - 1\} < b_1 < b. \tag{3.4}$$

We note here that $]a, b[\neq \emptyset$ implies $a < b$, and since $a < a+1$, we are guaranteed that a and $\min\{a+1, b\}$ are distinct. Similarly, since $b-1 < b$, we find that $\max\{a, b-1\}$ and b are distinct, so the use of the denseness of $\mathbb{Q}$ in $\mathbb{R}$ to produce (3.3),(3.4) is valid. Suppose that for some $n \in \mathbb{N}$, real numbers a_n, b_n have been determined such that

$$a < a_n < \min\left\{a + \frac{1}{n}, b\right\}, \tag{3.5}$$

$$\max\left\{a, b - \frac{1}{n}\right\} < b_n < b. \tag{3.6}$$

But then we use the denseness property of $\mathbb{Q}$ in $\mathbb{R}$ again to deduce the existence of $a_{n+1}, b_{n+1} \in \mathbb{Q}$ such that

$$\begin{aligned} a &< a_n < \min\left\{a + \frac{1}{n+1}, b\right\}, \\ \max\left\{a, b - \frac{1}{n+1}\right\} &< b_n < b. \end{aligned}$$

This completes the inductive construction of the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ such that (3.5),(3.6) hold for any $n \in \mathbb{N}$. The conditions (3.5),(3.6) imply (3.1),(3.2), which in turn imply that $a_n, b_n \in \,]a, b[$ *for any* $n \in \mathbb{N}$. This proves the claim. For the convergence, let $\varepsilon > 0$. We rewrite (3.1),(3.2) into

$$\begin{aligned} 0 &< a_n - a < \frac{1}{n}, \\ -\frac{1}{n} &< b_n - b < 0, \end{aligned}$$

and furthermore,

$$\begin{aligned} -\frac{1}{n} < 0 &< a_n - a < \frac{1}{n}, \\ -\frac{1}{n} &< b_n - b < 0 < \frac{1}{n}. \end{aligned}$$

Consequently,

$$\begin{aligned} |a_n - a| &< \frac{1}{n}, \\ |b_n - b| &< \frac{1}{n}, \end{aligned}$$

for any $n \in \mathbb{N}$. Choose a positive integer $N > \frac{1}{\varepsilon}$. If $n \geq N$, then

$$\begin{aligned} |a_n - a| &< \frac{1}{n} \leq \frac{1}{N} < \varepsilon, \\ |b_n - b| &< \frac{1}{n} \leq \frac{1}{N} < \varepsilon. \end{aligned}$$

Therefore, $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. □

Notation 3.2. Throughout Section 3, we fix $a, b \in \mathbb{R}$, and we use the symbol X to mean any of the intervals $]a, b[$, $[a, b]$, $]a, b]$, $[a, b[$. We call a, b the *endpoints* of X. We also assume henceforth that $|X| > 1$, i.e., $a < b$.

Lemma 3.3. *If c is an endpoint or an element of X, then there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in $X\backslash\{c\}$ such that $c = \lim_{n\to\infty} c_n$.*

Proof. For the case $c \in \{a, b\}$, we use Lemma 3.1 to deduce the existence of a sequence $(c_n)_{n\in\mathbb{N}}$ in $I := \,]a, b[\, \subseteq X$ such that $c = \lim_{n\to\infty} c_n$, whichever of the endpoints c may be. Since all the terms of $(c_n)_{n\in\mathbb{N}}$ are in $]a, b[$ and c is either a or b, we find that $(c_n)_{n\in\mathbb{N}}$ is a sequence in $X\backslash\{c\}$; i.e., each term of the sequence is not c. The proof for the case $c \in X\backslash\{a, b\}$ is similar, except that we use $I = \,]c, \frac{c+b}{2}[\, \subseteq X$ instead of $I = \,]a, b[$, and the sequence to be chosen according to Lemma 3.1 is, of course, that which converges to the endpoint c. $\square$

Definition 3.4. Let c be an endpoint or an element of X, let $L \in \mathbb{R}$, and consider a function $f : X\backslash\{c\} \to \mathbb{R}$. We say that L is the ***limit*** *of $f(x)$ as x approaches c*, or that *$f(x)$ approaches L as x approaches c*, and denote this in symbols by $\lim_{x\to c} f(x) = L$, if for any sequence $(c_n)_{n\in\mathbb{N}}$ in $X\backslash\{c\}$ such that $c = \lim_{n\to\infty} c_n$, we have $\lim_{n\to\infty} f(c_n) = L$. If there exists $L \in \mathbb{R}$ such that $\lim_{x\to c} f(x) = L$, then we say that *the limit* $\lim_{x\to c} f(x)$ *exists*.

The existence of sequences described in Lemma 3.3 at least assures us that Definition 3.4 makes sense. More precisely, the assertion "for any sequence $(c_n)_{n\in\mathbb{N}}$ in $X\backslash\{c\}$ such that $c = \lim_{n\to\infty} c_n$, we have $\lim_{n\to\infty} f(c_n) = L$," from Definition 3.4 would be true but useless if there were no sequences in $X\backslash\{c\}$ that converge to c, but Lemma 3.3 guarantees us that such is not the case.

Lemma 3.5. *If $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} f(x)$ is unique.*

Proof. Suppose $f(x)$ approaches some $L_1, L_2 \in \mathbb{R}$ as x approaches c. If $(c_n)_{n\in\mathbb{N}}$ is a sequence in $X\backslash\{c\}$ that converges to c, then by Definition 3.4, $(f(c_n))_{n\in\mathbb{N}}$ converges to L_1 and L_2. By the uniqueness of limits of sequences, Theorem 2.1, $L_1 = L_2$. $\square$

Theorem 3.6 (The Squeeze Theorem)**.** *Let $c \in X$, and consider functions $f, g, h : X\backslash\{c\} \to \mathbb{R}$ such that*

(i) $g(x) \leq f(x) \leq h(x)$ *for any* $x \in X\backslash\{c\}$; *and*

(ii) $L := \lim_{x\to c} g(x) = \lim_{x\to c} h(x)$.

Then $\lim_{x\to c} f(x)$ exists and is equal to L.

Proof. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence in $X\backslash\{c\}$ that converges to c, and let $\varepsilon > 0$. By *(ii)*, $\lim\limits_{n\to\infty} g(c_n) = L = \lim\limits_{n\to\infty} h(c_n)$, and so there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_1 \implies \quad & |g(c_n) - L| < \varepsilon, \\ & -\varepsilon < g(c_n) - L < \varepsilon, \\ & L - \varepsilon < g(c_n) < L + \varepsilon, \\ & L - \varepsilon < g(c_n), \qquad (3.7) \\ n \geq N_2 \implies \quad & |h(c_n) - L| < \varepsilon, \\ & -\varepsilon < h(c_n) - L < \varepsilon, \\ & L - \varepsilon < h(c_n) < L + \varepsilon, \\ & h(c_n) < L + \varepsilon. \qquad (3.8) \end{aligned}$$

Since $c_n \in X\backslash\{c\}$ for any $n \in \mathbb{N}$, by *(i)*, we have

$$g(c_n) \leq f(c_n) \leq h(c_n). \qquad (3.9)$$

If $n \geq N := \max\{N_1, N_2\}$, then both $n \geq N_1$ and $n \geq N_2$ are true. This means that both (3.7), (3.8) hold, and in conjunction with (3.9), we have

$$\begin{array}{ccccc} L - \varepsilon & < & f(c_n) & < & L + \varepsilon, \\ -\varepsilon & < & f(c_n) - L & < & \varepsilon, \end{array} \qquad (3.10)$$

so $|f(c_n) - L| < \varepsilon$. At this point, we have shown that $\lim\limits_{n\to\infty} f(c_n) = L$ for an arbitrary sequence $(c_n)_{n\in\mathbb{N}}$ in $X\backslash\{c\}$ that converges to c. Therefore, $\lim\limits_{x\to c} f(x) = L$. $\square$

Pointwise function operations. Let $k \in \mathbb{R}$, let $S \subseteq \mathbb{R}$, and consider functions $f_1, f_2, f_3 : S \to \mathbb{R}$. We further assume that $f_3(x) \neq 0$ for any $x \in S$. Recall the functions f_1+f_2, kf_1, f_1f_2, $\frac{1}{f_3}$, $|f_1|$, $f_1 \vee f_2$ and $f_1 \wedge f_2$ defined by the rules

$$\begin{aligned} (f_1 + f_2)(x) &:= f_1(x) + f_2(x), \\ (kf_1)(x) &:= k \cdot f_1(x), \\ (f_1f_2)(x) &:= f_1(x) \cdot f_2(x), \\ \left(\frac{1}{f_3}\right)(x) &:= \frac{1}{f_3(x)}, \\ |f_1|(x) &:= |f_1(x)|, \\ (f_1 \vee f_2)(x) &:= f_1(x) \vee f_2(x), \\ (f_1 \wedge f_2)(x) &:= f_1(x) \wedge f_2(x). \end{aligned}$$

We also have the function $-f_1 := (-1)\cdot f_1$, the function $f_1 - f_2 := f_1 + (-f_2)$, and $\frac{f_1}{f_3} := f_1 \cdot \frac{1}{f_3}$. For the *constant function* with value k, we use the notation $k : X \to \mathbb{R}$ with $k(x) = k$ for all $x \in \mathbb{R}$. That is, we are imposing an abuse of notation—the symbol k means both the real number k and the constant function with k as value. With reference to the *zero function* 0, we define the *positive part* f_1^+ and *negative part* f_1^- of f_1 by $f_1^+ := f_1 \vee 0$ and $f_1^- := (-f_1) \vee 0$, respectively. We have the important identities $f_1 = f_1^+ - f_1^-$ and $|f_1| = f_1^+ + f_1^-$. By the *identity function*, we mean the function $\mathrm{id} : X \to \mathbb{R}$ with $\mathrm{id}(x) = x$ for all $x \in \mathbb{R}$.

Lemma 3.7. *Let $k \in \mathbb{R}$, $c \in X$, and consider functions $f_1, f_2, f_3 : X\backslash\{c\} \to \mathbb{R}$. Suppose further that $f_3(x) \neq 0$ for any $x \in X\backslash\{c\}$, and $\lim\limits_{x\to c} f_3(x) \neq 0$. We have*

$$\begin{aligned}
\lim_{x\to c} x &= c, && (3.11)\\
\lim_{x\to c} k &= k, && (3.12)\\
\lim_{x\to c} [f_1(x) + f_2(x)] &= \lim_{x\to c} f_1(x) + \lim_{x\to c} f_2(x), && (3.13)\\
\lim_{x\to c} k\cdot f_1(x) &= k \lim_{x\to c} f_1(x), && (3.14)\\
\lim_{x\to c} [f_1(x) f_2(x)] &= \lim_{x\to c} f_1(x) \cdot \lim_{x\to c} f_2(x), && (3.15)\\
\lim_{x\to c} \frac{1}{f_3(x)} &= \frac{1}{\lim\limits_{x\to c} f_3(x)}, && (3.16)\\
\lim_{x\to c} |f_1(x)| &= \left|\lim_{x\to c} f_1(x)\right|, && (3.17)\\
\lim_{x\to c} (f_1(x) \vee f_2(x)) &= \left(\lim_{x\to c} f_1(x)\right) \vee \left(\lim_{x\to c} f_2(x)\right), && (3.18)\\
\lim_{x\to c} (f_1(x) \wedge f_2(x)) &= \left(\lim_{x\to c} f_1(x)\right) \wedge \left(\lim_{x\to c} f_2(x)\right). && (3.19)
\end{aligned}$$

Proof. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence in $X\backslash\{c\}$ that converges to c.

To prove (3.11), we need $f = \mathrm{id}|_{X\backslash\{c\}}$. That is, $f(x) = x$ for all $x \in X\backslash\{c\}$, and so $\lim\limits_{n\to\infty} f(c_n) = \lim\limits_{n\to\infty} c_n = c$. Therefore, $\lim\limits_{x\to c} x = \lim\limits_{x\to c} f(x) = c$.

Let $L_1 = \lim\limits_{x\to c} f_1(x)$, $L_2 = \lim\limits_{x\to c} f_2(x)$, $L_3 = \lim\limits_{x\to c} f_3(x)$. The left-hand sides of (3.12)–(3.19) are the limits, as x approaches c, of the functions $k|_{X\backslash\{c\}}$, $f_1 + f_2$, kf_1, $f_1 f_2$, $\frac{1}{f_3}$, $|f_1|$, $f_1 \vee f_2$ and $f_1 \wedge f_2$, respectively. We now take the sequence limits of the image of $(c_n)_{n\in\mathbb{N}}$ under these functions as in:

$$\lim_{n\to\infty} k|_{X\setminus\{c\}}(c_n) = \lim_{n\to\infty} k = k, \tag{3.20}$$
$$\lim_{n\to\infty}(f_1+f_2)(c_n) = \lim_{n\to\infty}\left[f_1(c_n)+f_2(c_n)\right], \tag{3.21}$$
$$= \lim_{n\to\infty} f_1(c_n) + \lim_{n\to\infty} f_2(c_n) = L_1+L_2, \tag{3.22}$$
$$\lim_{n\to\infty}(kf_1)(c_n) = \lim_{n\to\infty}\left[k\cdot f_1(c_n)\right], \tag{3.23}$$
$$= k\lim_{n\to\infty} f_1(c_n) = kL_1, \tag{3.24}$$
$$\lim_{n\to\infty}(f_1f_2)(c_n) = \lim_{n\to\infty}\left[f_1(c_n)f_2(c_n)\right], \tag{3.25}$$
$$= \lim_{n\to\infty} f_1(c_n)\cdot\lim_{n\to\infty} f_2(c_n) = L_1L_2, \tag{3.26}$$
$$\lim_{n\to\infty}\left(\frac{1}{f_3}\right)(c_n) = \lim_{n\to\infty}\frac{1}{f_3(c_n)}, \tag{3.27}$$
$$= \frac{1}{\lim_{n\to\infty} f_3(c_n)} = \frac{1}{L_3}, \tag{3.28}$$
$$\lim_{n\to\infty}|f_1|(c_n) = \lim_{n\to\infty}|f_1(c_n)|, \tag{3.29}$$
$$= \left|\lim_{n\to\infty} f_1(c_n)\right| = |L_1|, \tag{3.30}$$
$$\lim_{n\to\infty}(f_1\vee f_2)(c_n) = \lim_{n\to\infty}(f_1(c_n))\vee(f_2(c_n)), \tag{3.31}$$
$$= \left(\lim_{n\to\infty} f_1(c_n)\right)\vee\left(\lim_{n\to\infty} f_2(c_n)\right) = L_1\vee L_2, \tag{3.32}$$
$$\lim_{n\to\infty}(f_1\wedge f_2)(c_n) = \lim_{n\to\infty}(f_1(c_n))\wedge(f_2(c_n)), \tag{3.33}$$
$$= \left(\lim_{n\to\infty} f_1(c_n)\right)\wedge\left(\lim_{n\to\infty} f_2(c_n)\right) = L_1\wedge L_2. \tag{3.34}$$

The first equality in (3.20) is because of the meaning of the function $k|_{X\setminus\{c\}}$, while the second equality is because of the 'constant rule' for the limits of sequences, Corollary 2.8*(i)*, and so by Definition 3.4, $\lim_{x\to c} k = \lim_{x\to c} k|_{X\setminus\{c\}}(x) = k$. This proves the 'constant rule' for limits of functions, (3.12).

The equation (3.21) is because of the meaning of f_1+f_2. The first equality in (3.22) is because of the 'addition rule' for limits of sequences, equation (2.8) from Theorem 2.3, while the second equality is because $L_1 = \lim_{x\to c} f_1(x)$, $L_2 = \lim_{x\to c} f_2(x)$. Now, we have $\lim_{n\to\infty}(f_1+f_2)(c_n) = L_1+L_2$ where $(c_n)_{n\in\mathbb{N}}$ is an arbitrary sequence in $X\setminus\{c\}$ that converges to c. By Definition 3.4, this means $\lim_{x\to c}(f_1+f_2)(x) = L_1+L_2$, which proves the 'addition rule' (3.13) for the limits of functions. Similar reasoning may be used on

(3.23)–(3.34), so that from the rules for limits of sequences from Section 2.1, we may be able to deduce the corresponding rules for limits of functions in (3.14)–(3.19). □

In summary, the proofs of Lemmas 3.5 and 3.7 are illustrations of how the rules for limits of sequences from Section 2.1 can be 'converted' into rules for limits of functions. This was made possible because we defined limits in terms of sequences. Now, we look at a notion closely related to that of the limit of a function.

Definition 3.8. A function $f : X \to \mathbb{R}$ is *continuous at* $c \in X$ if for any sequence $(c_n)_{n\in\mathbb{N}}$ in X that converges to c, we have $\lim_{n\to\infty} f(c_n) = f(c)$. Continuity of a function on some element of its domain is often called the *local definition of continuity* or *continuity at a point*. If f is continuous on every $c \in X$, we say that f is *continuous on* X.

Denote by $\mathbb{R}[x]$ the set of all polynomial functions on the real variable x, and more importantly:

Definition 3.9. Let $C(X)$ be the set of all functions continuous on the interval X.

Theorem 3.10. *Let $k \in \mathbb{R}$, let $f_1, f_2, f_3 \in C(X)$, and let $p \in \mathbb{R}[x]$. Suppose further that $f_3(x) \neq 0$ for all $x \in X$, and that for any convergent sequence $(c_n)_{n\in\mathbb{N}}$ in X, we have $\lim_{n\to\infty} f_3(c_n) \neq 0$. Then*

$$\mathrm{id},\ k,\ f_1 + f_2,\ kf_1,\ f_1f_2,\ \frac{1}{f_3},\ |f_1|,\ f_1 \vee f_2,\ f_1 \wedge f_2,\ p|_X \in C(X).$$

Proof. Let $c \in X$. Suppose $(c_n)_{n\in\mathbb{N}}$ is a sequence in X that converges to c. Since $f_1, f_2, f_3 \in C(X)$, we have $\lim_{n\to\infty} f_1(c_n) = f_1(c)$, $\lim_{n\to\infty} f_2(c_n) = f_2(c)$, $\lim_{n\to\infty} f_3(c_n) =$

$f_3(c)$, and using the rules for limits of sequences from Section 2.1, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{id}(c_n) &= \lim_{n\to\infty} c_n = c = \mathrm{id}(c),\\
\lim_{n\to\infty} k(c_n) &= \lim_{n\to\infty} k = k = k(c),\\
\lim_{n\to\infty} (f_1+f_2)(c_n) &= \lim_{n\to\infty} \left[f_1(c_n)+f_2(c_n)\right],\\
&= \lim_{n\to\infty} f_1(c_n) + \lim_{n\to\infty} f_2(c_n) = f_1(c)+f_2(c),\\
&= (f_1+f_2)(c),\\
\lim_{n\to\infty} (kf_1)(c_n) &= \lim_{n\to\infty} \left[k\cdot f_1(c_n)\right],\\
&= k\lim_{n\to\infty} f_1(c_n) = k\cdot f_1(c) = (kf_1)(c),\\
\lim_{n\to\infty} (f_1f_2)(c_n) &= \lim_{n\to\infty} \left[f_1(c_n)f_2(c_n)\right],\\
&= \lim_{n\to\infty} f_1(c_n)\cdot \lim_{n\to\infty} f_2(c_n) = f_1(c)f_2(c) = (f_1f_2)(c),\\
\lim_{n\to\infty} \left(\frac{1}{f_3}\right)(c_n) &= \lim_{n\to\infty} \frac{1}{f_3(c_n)} = \frac{1}{\lim\limits_{n\to\infty} f_3(c_n)} = \frac{1}{f_3(c)} = \left(\frac{1}{f_3}\right)(c),\\
\lim_{n\to\infty} |f_1|(c_n) &= \lim_{n\to\infty} |f_1(c_n)| = \left|\lim_{n\to\infty} f_1(c_n)\right| = |f_1(c)|,\\
\lim_{n\to\infty} (f_1\vee f_2)(c_n) &= \lim_{n\to\infty} (f_1(c_n)\vee f_2(c_n)) = \lim_{n\to\infty} f_1(c_n) \vee \lim_{n\to\infty} f_2(c_n),\\
&= f_1(c)\vee f_2(c) = (f_1\vee f_2)(c),\\
\lim_{n\to\infty} (f_1\wedge f_2)(c_n) &= \lim_{n\to\infty} (f_1(c_n)\wedge f_2(c_n)) = \lim_{n\to\infty} f_1(c_n) \wedge \lim_{n\to\infty} f_2(c_n),\\
&= f_1(c)\wedge f_2(c) = (f_1\wedge f_2)(c).
\end{aligned}$$

The above computations show us that

$$\mathrm{id},\ k,\ f_1+f_2,\ kf_1,\ f_1f_2,\ \frac{1}{f_3},\ |f_1|,\ f_1\vee f_2,\ f_1\wedge f_2\ \in\ C(X). \tag{3.35}$$

As for $p|_X$, we use induction on the polynomial degree of p. From (3.35), we find that $\mathrm{id}, k \in C(X)$, so $p|_X \in C(X)$ whenever the degree of p is one or zero, respectively. Suppose that if $q \in \mathbb{R}[x]$ with degree strictly less than that of p, then $q|_X \in C(X)$. However, $p|_X = \mathrm{id}\cdot r|_X$ for some $r \in \mathbb{R}[x]$ with degree stricly less than that of p. Earlier, we have shown that $\mathrm{id} \in C(X)$ while by the inductive hypothesis, we have $r|_X \in C(X)$, then the product function $p|_X$ must also be in $C(X)$. By induction, $p|_X \in C(X)$ with p of any degree. □

Theorem 3.11. *Consider the case $X=[a,b]$, and let $f \in C(X)$.*

(i) [The Intermediate Value Theorem.] *If* $f(a) \neq f(b)$ *and*

$$k \in [f(a) \wedge f(b), f(b) \vee f(a)], \tag{3.36}$$

then there exists $c \in X$ *such that* $f(c) = k$*. [i.e., The equation* $f(x)$=k *always has a solution in* X *for any* k *that satisfies* (3.36)*.]*

(ii) [The Extreme Value Theorem.] *There exist* $c_1, c_2 \in X$ *such that for any* $x \in X$*, we have* $f(x) \in [f(c_1), f(c_2)]$*. [i.e., The function* f *attains its maximum and minimum values—extreme values—in* X*.]*

Proof. *(i)* From the hypothesis $f(a) \neq f(b)$, the Trichotomy Law gives us two cases $f(a) < f(b)$ and $f(a) > f(b)$. We first consider $f(a) < f(b)$. Then (3.36) simplifies into $k \in [f(a), f(b)]$. That is,

$$f(a) \leq k \leq f(b). \tag{3.37}$$

Since $f(a) \leq k$, we find that a is an element of the set

$$S := \{x \in X \,:\, f(x) \leq k\} \subseteq X, \tag{3.38}$$

and so $S \neq \emptyset$. Since $S \subseteq X$, we find that b is an upper bound of S, and by the Completeness Axiom, $\sup S$ exists as a real number. Given any $n \in \mathbb{N}$, the number $-\frac{1}{n} + \sup S$ is not an upper bound of S, and so there exists $c_n \in S$ such that $c_n > -\frac{1}{n} + \sup S$. Since $c_n \in S$ and $\sup S$ is an upper bound of S, we have $c_n \leq \sup S$, and furthermore,

$$-\frac{1}{n} + \sup S < c_n \leq \sup S < \sup S + \frac{1}{n},$$

which, after some manipulation using properties of inequalities and absolute value, gives us

$$\forall n \in \mathbb{N} \left[|c_n - \sup S| < \frac{1}{n} \right].$$

If $\varepsilon > 0$, then by the Archimedean property of $\mathbb{R}$, we can choose a positive integer $N > \frac{1}{\varepsilon}$, and if $n \geq N$, then

$$|c_n - \sup S| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Therefore, $\lim_{n\to\infty} c_n = \sup S$. Every term $c_n \in S \subseteq X = [a,b]$ in the sequence $(c_n)_{n\in\mathbb{N}}$ satisfies $a \leq c_n \leq b$, which, by Lemmas 2.10 and 2.17*(iii)*,*(v)*, implies

$$a \leq \liminf_{n\to\infty} c_n = \lim_{n\to\infty} c_n = \limsup_{n\to\infty} c_n \leq b,$$

and so $\sup S = \lim_{n\to\infty} c_n \in [a,b] = X$. At least this assures us that $\sup S$ is in the domain of X, and $f(\sup S)$ is not undefined.

By the continuity of f on X, the condition $\lim_{n\to\infty} c_n = \sup S$ implies $\lim_{n\to\infty} f(c_n) = f(\sup S)$. Since every term c_n is in S, we have $f(c_n) \leq k$ for any $n \in \mathbb{N}$. By Lemmas 2.10 and 2.17*(v)*, this gives us $f(\sup S) = \lim_{n\to\infty} f(c_n) = \limsup_{n\to\infty} f(c_n) \leq k$. That is,

$$f(\sup S) \leq k. \tag{3.39}$$

Our next goal is to prove equality. For this, we consider two cases: since $\sup S \in [a,b]$, we have either $\sup S = b$ or $\sup S \neq b$. If $\sup S = b$, then since f is a function, we have $f(\sup S) = f(b)$, and in conjunction with (3.37) and (3.39),

$$f(\sup S) \leq k \leq f(b) = f(\sup S).$$

Therefore, $f(\sup S) = k$. Consider the case $\sup S \neq b$. Since $\sup S \in [a,b]$, we further have $\sup S < b$ in this case, and so the interval $]\sup S, b[$ is not empty. By Lemma 3.1, there exists a sequence $(d_n)_{n\in\mathbb{N}}$ in $]\sup S, b[$ such that $\lim_{n\to\infty} d_n = \sup S$. Since $d_n > \sup S$ for all $n \in \mathbb{N}$ with $\sup S$ an upper bound of S, we find from (3.38) that $d_n \notin S$, or equivalently $f(d_n) > k$, for all $n \in \mathbb{N}$. We can deduce from here the weaker condition

$$\forall n \in \mathbb{N}\ [f(d_n) \geq k]\,. \tag{3.40}$$

By the continuity of f on X, the condition $\lim_{n\to\infty} d_n = \sup S$ implies $\lim_{n\to\infty} f(d_n) = f(\sup S)$. By Lemma 2.17*(iii)*,*(v)* and (3.40), we have

$$f(\sup S) = \lim_{n\to\infty} f(d_n) = \liminf_{n\to\infty} f(d_n) \geq k.$$

That is, $f(\sup S) \geq k$, and by (3.39), $f(\sup S) = k$. i.e., The solution $c \in X$ we desire for the equation $f(c) = k$ is $c := \sup S$. In summary we have shown that

$$f \in C(X) \wedge f(a) < f(b) \wedge k \in [f(a), f(b)] \implies \exists c \in X\ [f(c) = k]\,. \tag{3.41}$$

For the case $f(a) > f(b)$, we have $(-f)(a) < (-f)(b)$, and (3.36) implies $f(b) \leq k \leq f(a)$ or equivalently $-k \in [(-f)(a), (-f)(b)]$. Also, $f \in C(X)$ implies $-f \in C(X)$. We see from here that all three hypotheses of (3.41) are true for $-f$ in place of f, and $-k$ in place of k. Thus, there exists $c' \in X$ [a different variable is required according to Existential Instantiation] such that $(-f)(c') = -k$, or equivalently, $f(c') = k$, as desired.

(ii) Since $X \neq \emptyset$ and f is a function, the set $f(X) = \{f(x) : x \in X\}$ is not empty. Let F be the set of all upper bounds of $f(X)$. The Completeness Axiom applied to $f(X)$ can be written symbolically as

$$\exists u \in \mathbb{R}\ [u \in F] \implies \exists s \in \mathbb{R}\ [s = \sup f(X)],$$

which by contraposition is equivalent to

$$\forall s \in \mathbb{R}\ [s \neq \sup f(X)] \implies \forall u \in \mathbb{R}\ [u \notin F]. \tag{3.42}$$

We claim that the set $f(X)$ has a supremum in $\mathbb{R}$. Suppose otherwise. Then by (3.42), any real number is not in F, that is, not an upper bound of $f(X)$. In particular, so is any $n \in \mathbb{N}$, which means that for any such n, there exists $k_n \in f(X)$ such that $k_n > n$. By the definition of the image of a function, we further have some $c_n \in X$ such that

$$f(c_n) = k_n > n. \tag{3.43}$$

That is, we have now constructed a sequence $(c_n)_{n\in\mathbb{N}}$ in $X = [a, b]$, and so this sequence must be bounded. By the Bolzano-Weierstrass Theorem, $(c_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(c_{N_i})_{i\in\mathbb{N}}$. Since every term of $(c_n)_{n\in\mathbb{N}}$ is in X, so is every term of $(c_{N_i})_{i\in\mathbb{N}}$. That is, for any i, we have

$$a \leq c_{N_i} \leq b, \tag{3.44}$$

which by Lemmas 2.10 and 2.17*(iii)*,*(v)*, further implies

$$a \leq \liminf_{i\to\infty} c_{N_i} = \lim_{i\to\infty} c_{N_i} = \limsup_{i\to\infty} c_{N_i} \leq b, \tag{3.45}$$

and so

$$c := \lim_{i\to\infty} c_{N_i} \in [a, b] = X. \tag{3.46}$$

By the continuity of f on X, the sequence $(f(c_{N_i}))_{i\in\mathbb{N}}$ is convergent [in particular, $\lim_{i\to\infty} f(c_{N_i}) = f(c)$], which by Theorem 2.15*(i)* is bounded. That is, there exists $M >$

0 such that for any $i \in \mathbb{N}$, we have $|f(c_{N_i})| \leq M$. Using the Archimedean Property, there exists a positive integer $I > M \geq f(c_{N_i})$ for any $i \in \mathbb{N}$. Using the property (2.47) of subsequences from Section 2.2, we have $I \leq N_I$, and in summary we have

$$\forall i \in \mathbb{N} \; [|f(c_{N_i})| < N_I]. \tag{3.47}$$

However, by construction, (3.43) holds for any term of the sequence $(c_n)_{n\in\mathbb{N}}$, and the same holds for any term of the subsequence $(c_{N_i})_{i\in\mathbb{N}}$. That is,

$$\forall i \in \mathbb{N} \; [f(c_{N_i}) > N_i]. \tag{3.48}$$

Instantiating (3.47) and (3.48) both at the value $i = I$, we obtain

$$f(c_{N_I}) \leq |f(c_{N_I})| < N_I < f(c_{N_I}), \tag{3.49}$$

from which we get $f(c_{N_I}) < f(c_{N_I})$, contradicting the Trichotomy Law. Therefore, $\sup f(X)$ exists as a real number, proving the claim. Let $n \in \mathbb{N}$. The real number $-\frac{1}{n} + \sup f(X)$ is not an upper bound of $f(X)$ and so there exists $h_n \in f(X)$ such that $h_n > -\frac{1}{n} + \sup f(X)$, and consequently, there is some $d_n \in X$ such that $f(d_n) = h_n$. That is, we have

$$-\frac{1}{n} + \sup f(X) < f(d_n) \leq \sup f(X) < \sup f(X) + \frac{1}{n},$$

for any $n \in \mathbb{N}$, which further gives us

$$\forall n \in \mathbb{N} \left[|f(d_n) - \sup f(X)| < \frac{1}{n} \right].$$

If $\varepsilon > 0$, we choose a positive integer $N > \frac{1}{\varepsilon}$, and if $n \geq N$, then

$$|f(d_n) - \sup f(X)| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Therefore, $\lim\limits_{n\to\infty} f(d_n) = \sup f(X)$, which also means that $\lim\limits_{n\to\infty} f(d_n)$ is an upper bound of $f(X)$. That is,

$$\forall x \in X \left[f(x) \leq \lim_{n\to\infty} f(d_n) \right]. \tag{3.50}$$

Since $d_n \in X$ for any $n \in \mathbb{N}$, $(d_n)_{n\in\mathbb{N}}$ is a sequence in X, and is hence bounded. By the Bolzano-Weierstrass Theorem, $(d_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(d_{N_i})_{i\in\mathbb{N}}$, which is also a sequence in X. By reasoning similar to that done in (3.44)–(3.46), we have

$$c_2 := \lim_{i\to\infty} d_{N_i} \in [a, b] = X.$$

By Proposition 2.9 and by the continuity of f on X, we have

$$\lim_{n\to\infty} f(d_n) = \lim_{i\to\infty} f(d_{N_i}) = f(c_2),$$

and by (3.50), $f(x) \leq f(c_2)$ for any $x \in X$. In summary, we have shown

$$\exists c_2 \in X \quad \forall x \in X \ [f(x) \leq f(c_2)]\,. \tag{3.51}$$

But since $f \in C(X)$ is arbitrary, (3.51) also holds for $-f \in C(X)$ in place of f. That is, there exists $c_1 \in X$ such that $(-f)(x) \leq (-f)(c_1)$, or equivalently, $f(c_1) \leq f(x)$ for any $x \in X$. This completes our proof that f assumes its minimum and maximum values for some $c_1, c_2 \in X$, respectively. □

Corollary 3.12. *(i) If $f \in C(X)$, $c_1, c_2 \in X$ and*

$$k \in [f(c_1) \wedge f(c_2), f(c_1) \vee f(c_2)]\,,$$

then there exists $c \in X$ such that $f(c) = k$.

(ii) If $X = [a, b]$, then for any $f \in C(X)$, the set $f(X)$ is a closed interval.

Proof. *(i)* If $c_1 = c_2$, since f is a function, $f(c_1) = k = f(c_2)$, and we simply choose $c = c_1$. Suppose $c_1 \neq c_2$. If $f(c_1) = f(c_2)$, then $k = f(c_1)$, and we again choose $c = c_1$. For the case $c_1 \neq c_2$ and $f(c_1) \neq f(c_2)$, the trick is to use the Intermediate Value Theorem on the restriction $f : [c_1, c_2] \to \mathbb{R}$.

(ii) By the Extreme Value Theorem, there exist $c_1, c_2 \in X$ such that for all $x \in [a, b]$, we have $f(c_1) \leq f(x) \leq f(c_2)$, which implies $f(X) \subseteq [f(c_1), f(c_2)]$. If $k \in [f(c_1), f(c_2)]$, then by part *(i)*, there exists $c \in X$ such that $k = f(c) \in f(X)$. Thus, $[f(c_1), f(c_2)] \subseteq f(X)$. □

If the limit of $f(x)$ is taken as x approaches an element of the domain of f, which in our case is X, then we have the following relationship between the notions of limit and continuity.

Theorem 3.13. *A function $f : X \to \mathbb{R}$ is continuous at $c \in X$ if and only if*

$$\lim_{x\to c} f|_{X\setminus\{c\}}(x) = f(c). \tag{3.52}$$

Proof. Our use in (3.52) of the function $f|_{X\setminus\{c\}}$, which is the restriction of f to $X\setminus\{c\}$, is so that the left-hand side of (3.52) is consistent with Definition 3.4—the function of which we are taking the limit should have $X\setminus\{c\}$ as domain. In view of (3.52) as having $L = f(c)$ in Definition 3.4, we find that the statement of this theorem is precisely the assertion that the two statements *(i)* for any sequence $(c_n)_{n\in\mathbb{N}}$ in X that converges to c, we have $\lim_{n\to\infty} f(c_n) = f(c)$,

(ii) for any sequence $(c_n)_{n\in\mathbb{N}}$ in $X\backslash\{c\}$ that converges to c, we have $\lim_{n\to\infty} f(c_n) = f(c)$,

are equivalent. The only difference in the phrasing of the said statements is whether the sequence $(c_n)_{n\in\mathbb{N}}$ is in X or in $X\backslash\{c\}$. If the condition $\lim_{n\to\infty} f(c_n) = f(c)$ holds for any sequence $(c_n)_{n\in\mathbb{N}}$ in X that converges to c, then the same holds, in particular, for any sequence in $X\backslash\{c\} \subseteq X$ that converges to c, and this takes care of *(i)*$\Longrightarrow$*(ii)*. The converse is not trivial, which we now prove by contraposition. Suppose there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in X that converges to c but $\lim_{n\to\infty} f(c_n) \neq f(c)$. Then there exists $\varepsilon > 0$ such that

$$\forall N \in \mathbb{N} \quad \exists n \geq N \; [|f(c_n) - f(c)| \geq \varepsilon]\,. \tag{3.53}$$

Given any $x \in X$, the condition $|f(x) - f(c)| \geq \varepsilon$ implies $|f(x) - f(c)| > 0$, which by the identity of indiscernibles in $\mathbb{R}$, means that $f(x) \neq f(c)$. Since f is a function, this further implies $x \neq c$. That is,

$$|f(x) - f(c)| \geq \varepsilon \implies x \neq c. \tag{3.54}$$

Let $i \in \mathbb{N}$. If $i = 1$, then we instantiate (3.53) with $N = 2$ [and n as N_1]: there exists $N_1 \geq 2$ [consequently $N_1 > 1$] such that

$$\begin{aligned} |f(c_{N_1}) - f(c)| &\geq \varepsilon, & (3.55)\\ c_{N_1} &\neq c, & (3.56) \end{aligned}$$

where (3.56) is because of (3.54),(3.55). We again instantiate (3.53) for $i = 2$ with $N = N_1 + 1$, n as N_2: there exists $N_2 \geq N_1 + 1 > N_1$ such that $|f(c_{N_2}) - f(c)| \geq \varepsilon$ but $c_{N_2} \neq c$. Suppose that for some $i \in \mathbb{N}$ with $i \geq 2$, we have produced c_{N_i} such that

$$\begin{aligned} N_i &> N_{i-1}, & (3.57)\\ |f(c_{N_i}) - f(c)| &\geq \varepsilon, & (3.58)\\ c_{N_i} &\neq c. & (3.59) \end{aligned}$$

Now, we instantiate (3.53) at $N = N_i + 1$ and n as N_{i+1}: there exists $N_{i+1} > N_i$ such that $|f(c_{N_{i+1}}) - f(c)| \geq \varepsilon$ and by (3.54), $c_{N_{i+1}} \neq c$. By induction, we have produced c_{N_i} such that (3.58)–(3.59) hold for any $i \in \mathbb{N}$, and (3.57) holds for any $i \geq 2$. Using

(3.57), it can be shown by induction that $i < j$ implies $N_i < N_j$, and so indeed $(c_{N_i})_{i\in\mathbb{N}}$ is a subsequence of $(c_n)_{n\in\mathbb{N}}$. Furthermore, (3.59) tells us that $(c_{N_i})_{i\in\mathbb{N}}$ is a sequence in $X\backslash\{c\}$. Since $(c_n)_{n\in\mathbb{N}}$ converges to c, by Proposition 2.9, $(c_{N_i})_{i\in\mathbb{N}}$ also converges to c. If N_I is an arbitrary index of the sequence $(c_{N_i})_{i\in\mathbb{N}}$, we take $N_i = N_{I+1}$, which by (3.57) satisfies $N_i \geq N_I$ and by (3.58), $|f(c_{N_i}) - f(c)| \geq \varepsilon$. At this point we have shown that

$$\exists\varepsilon > 0 \quad \forall N_I \quad \exists N_i \geq N_I \;\left[|f(c_{N_i}) - f(c)| \geq \varepsilon\right].$$

Thus, $\lim\limits_{N_i\to\infty} f(c_{N_i}) \neq f(c)$. In summary, we have produced a sequence $(c_{N_i})_{i\in\mathbb{N}}$ in $X\backslash\{c\}$ that converges to c, but $\lim\limits_{N_i\to\infty} f(c_{N_i}) \neq f(c)$, as desired. □

Notation 3.14. With reference to Theorem 3.13, we drop the symbol of function restriction in the left-hand side of (3.52). Also, since $c = \lim\limits_{x\to c} x$ from the rules for limits of functions, we rewrite (3.52) as

$$\lim_{x\to c} f(x) = f\left(\lim_{x\to c} x\right),$$

which now serves as one characterization for the continuity of f at c. Also, the above equation suggests that continuous functions, in some sense, 'commute' with taking limits. A similar property holds for continuous functions in terms of limits of sequences, since from Definition 3.8, we have

$$\lim_{n\to\infty} f(c_n) = f\left(\lim_{n\to\infty} c_n\right).$$

3.1. The function algebra $C(X)$ for the interval X

In this section, we give a discussion analogous to that done in Section 2.1, but instead of showing how an algebra of sequences arose from the rules on limits of sequences, in this section, we show how an algebra of functions shall arise because of the concept of continuity. But first, we recall some set-theoretic notions. Suppose f, g are two functions both with codomain $\mathbb{R}$ and with domains D_f, D_g, respectively. By the *equality of functions* $f = g$ we mean

(i) $D_f = D_g$.

(ii) $\forall x\ [f(x) = g(x)]$.

Simply writing $f = g$ in place of *(i)*,*(ii)* above is indeed a very convenient simplification. Furthermore, we shall only be considering functions in $C(X)$, which by Definition 3.8, have X as domain, so *(i)* shall hold throughout for any two functions we shall consider. Thus, all equations, idenitities or properties involving functions that we shall show in this section are intended to mean *(ii)*. One example is the associativity of addition of functions

$$f_1 + (f_2 + f_3) = (f_1 + f_2) + f_3,$$

for all $f_1, f_2, f_3 \in C(X)$, which means that the function value $(f_1 + (f_2 + f_3))(x)$ is equal to $((f_1 + f_2) + f_3)(x)$ for any $x \in X$, and this can be routinely verified using the definition of addition of functions and the rules for the usual addition in $\mathbb{R}$. Recall our abuse of notation for constant functions: if $k \in \mathbb{R}$, then by $k \in C(X)$ we mean the function in $C(X)$ defined by the rule $x \mapsto k$. In particular, the *zero function* $0 \in C(X)$ has the properties

$$\begin{aligned} f_1 + 0 &= f_1 = 0 + f_1, \\ f_1 + (-f_1) &= 0 = (-f_1) + f_1, \end{aligned}$$

for all $f_1 \in C(X)$. That is, $0 \in C(X)$ is the identity for addition of functions, and the additive inverse of f_1 is $-f_1$. We also have the commutative property of addition of functions in $C(X)$, which is

$$f_1 + f_2 = f_2 + f_1,$$

for all $f_1, f_2 \in C(X)$. By Theorem 3.10, $C(X)$ is closed under addition of functions, a property which, together with the previously mentioned properties of associativity, existence of an identity, existence of inverses and commutativity, implies that $C(X)$ is an abelian group under addition of functions. Theorem 3.10 also asserts that $C(X)$ is closed under left-multiplication by a constant. Together with the properties

$$\begin{aligned} c_1(f_1 + f_2) &= c_1 f_1 + c_2 f_2, \\ (c_1 + c_2) f_1 &= c_1 f_1 + c_2 f_1, \\ c_1(c_2 f_1) &= (c_1 c_2) f_1, \\ 1 \cdot f_1 &= f_1, \qquad\qquad (1 \in \mathbb{R}), \end{aligned}$$

that hold for any $c_1, c_2 \in \mathbb{R}$ and any $f_1, f_2 \in C(X)$, we find that $C(X)$ is a vector space over $\mathbb{R}$ with addition of functions as vector addition, and left-multiplication by a constant as scalar multiplication. Thus, $C(X)$ is a vector space over $\mathbb{R}$.

The closure of the vector space $C(X)$ under the lattice operations $\vee$ and $\wedge$, according to Theorem 3.10, gives $C(X)$ the structure of a *vector lattice*. Furthermore, the positive and negative parts of every function in $C(X)$ are also in $C(X)$. These are important concepts in the theory of integration we shall develop later.

Also by Theorem 3.10, $C(X)$ is closed under multiplication of functions, which is bilinear:

$$(c_1 f_1 + c_2 f_2)(c_3 f_3 + c_4 f_4) = c_1 c_3 f_1 f_3 + c_1 c_4 f_1 f_4 + c_2 c_3 f_2 f_3 + c_2 c_4 f_2 f_4,$$

for any $c_1, c_2, c_3, c_4 \in \mathbb{R}$, $f_1, f_2, f_3, f_4 \in C(X)$, and associative:

$$f_1(f_2 f_3) = (f_1 f_2) f_3.$$

Therefore, $C(X)$ is an associative algebra over $\mathbb{R}$. Because the constant function 1 has the property

$$f_1 \cdot 1 = f_1 = 1 \cdot f_1, \qquad (1 \in C(X)),$$

the associative algebra $C(X)$ is unital, with multiplicative identity $1 \in C(X)$. Furthermore, because

$$f_1 f_2 = f_2 f_1,$$

for any $f_1, f_2 \in C(X)$, the associative algebra $C(X)$ is commutative. Also, Theorem 3.10 tells us that a function $f \in C(X)$ is a unit of the associative algebra $C(X)$ whenever $f(x) \neq 0$ for any $x \in X$, and $\lim_{n\to\infty} f(c_n) \neq 0$ for any convergent sequence $(c_n)_{n\in\mathbb{N}}$ in X.

There is additional structure on $C(X)$ when $X = [a, b]$. By the Extreme Value Theorem, for any $f \in C(X)$, there exist $c_1, c_2 \in X$ such that for any $x \in X$, we have the inequalities $f(c_1) \leq f(x) \leq f(c_2)$, or equivalently,

$$\begin{aligned} -f(x) &\leq -f(c_1), \\ f(x) &\leq f(c_2). \end{aligned}$$

Since $|f(x)|$ is either $-f(x)$ or $f(x)$, we find that, in any case,

$$|f(x)| \leq \max\{-f(c_1), f(c_2)\},$$

but since this is true for an arbitrary $x \in X$, we find that the set

$$\{|f(x)| \,:\, x \in X\}, \tag{3.60}$$

has an upper bound in $\mathbb{R}$, and so

$$\|f\|_\infty := \sup\{|f(x)| \, : \, x \in X\},$$

exists as a real number. By the uniqueness of the supremum, this gives us a function $C(X) \to \mathbb{R}$ given by $f \mapsto \|f\|_\infty$, which we call the ***uniform norm***, the ***sup norm***, the ***supremum norm***, the ***Chebyshev norm*** or the ***infinity norm on $C(X)$***.

Proposition 3.15. *For any $f, g \in C(X)$ and any $k \in \mathbb{R}$,*

(i) $\|f\|_\infty = 0 \iff f = 0$.

(ii) $\|kf\|_\infty \leq |k| \cdot \|f\|_\infty$.

(iii) $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Proof. *(i)* Sufficiency is immediate from the fact that (3.60) is the singleton $\{0\}$ if $f = 0$. For necessity, assume $\|f\|_\infty = 0$, and tending towards a contradiction, suppose $f \neq 0$. From the definition of the constant functions [on X], we have

$$\begin{aligned} f = 0 &\iff \forall x \in X \; [f(x) = 0], \\ f \neq 0 &\iff \exists x \in X \; [f(x) \neq 0], \end{aligned}$$

and so $f \neq 0$ implies that indeed there is some $x_0 \in X$ such that $f(x_0) \neq 0$. Then $|f(x_0)| > 0 = \|f\|_\infty$. i.e., We have produced an element $|f(x_0)|$ of (3.60) that is greater than the upper bound $\|f\|_\infty$ of (3.60), a contradiction. Therefore, $f = 0$.
(ii)–(iii) Let $x \in X$. In addition to the set (3.60), we shall also be considering

$$\{|g(x)| \, : \, x \in X\}. \tag{3.61}$$

By properties of absolute value,

$$\begin{aligned} |kf(x)| &= |k| \cdot |f(x)|, \qquad (3.62) \\ |f(x)+g(x)| &\leq |f(x)| + |g(x)|. \end{aligned}$$

The equality asserted by (3.62) implies one of the weaker inequalities, of which we want $\leq$, so we further have

$$|kf(x)| \leq |k| \cdot |f(x)|, \tag{3.63}$$
$$|f(x)+g(x)| \leq |f(x)| + |g(x)|. \tag{3.64}$$

Since $|f(x)|$ and $|g(x)|$ are elements of (3.60),(3.61), respectively, we can replace $|f(x)|$ and $|g(x)|$ in the right-hand sides of (3.63)–(3.64) by upper bounds of the respective sets (3.60),(3.61) and still maintain the inequalities. In particular, we want the upper bound $\|f\|_\infty$ of (3.60), and the upper bound $\|g\|_\infty$ of (3.61). That is,

$$|kf(x)| \leq |k| \cdot \|f\|_\infty, \tag{3.65}$$
$$|f(x)+g(x)| \leq \|f\|_\infty + \|g\|_\infty. \tag{3.66}$$

The inequality (3.65) tells us that the number $|k| \cdot \|f\|_\infty$ is an upper bound of the set

$$\{|kf(x)| \, : \, x \in X\}, \tag{3.67}$$

and (3.66) asserts that the number $\|f\|_\infty + \|g\|_\infty$ is an upper bound of the set

$$\{|f(x)+g(x)| \, : \, x \in X\}, \tag{3.68}$$

and so the supremum of each of the sets (3.67)–(3.68) is less than or equal to the corresponding upper bound that was mentioned. This gives us *(ii),(iii)*. □

Corollary 3.16. *For any $f, g, h \in C(X)$,*

(i) $\|f-g\|_\infty = 0 \iff f = g$.

(ii) $\|f-g\|_\infty = \|g-f\|_\infty$.

(iii) $\|f-g\|_\infty \leq \|f-h\|_\infty + \|h-g\|_\infty$.

Proof. *(i)* Use $f-g \in C(X)$ in place of f in Proposition 3.15*(i)*.
(ii) Use $k=-1$ and $f-g \in C(X)$ in place of f in Proposition 3.15*(ii)*, and then use Proposition 3.15*(ii)* a second time also with $k=-1$ but this time with $g-f \in C(X)$ in place of f.
(iii) Use $f-h \in C(X)$ in place of f in Proposition 3.15*(iii)*, and $h-g \in C(X)$ in place of g. □

The properties of the sup norm on $C(X)$ outlined in Corollary 3.16 are analogous to the properties of the distance function in $\mathbb{R}$ from Corollary 1.5*(iv),(v),(vii)*. Thus, we have, for $C(X)$, a function $C(X) \times C(X) \to \mathbb{R}$ given by $(f,g) \mapsto \|f-g\|_\infty$, which satisfies an identity of indiscernibles, a property of symmetry, and a triangle inequality, if X is a closed interval.

3.2. Epsilon-delta characterizations

In the previous parts of the section, we showed how the characterizations, involving convergent sequences, of limits and continuity are helpful in algebraic results such as the laws of limits and the algebraic structure of $C(X)$. In the succeeding theorems, we give the classical 'epsilon-delta' characterizations of limits and continuity, which shall be useful later when we generalize the notions of continuity in metric and topological spaces.

Theorem 3.17. *Let $L \in \mathbb{R}$, let c be an element or an endpoint of X, and consider a function $f : X \backslash \{c\} \to \mathbb{R}$. The condition $\lim\limits_{x \to c} f(x) = L$ holds if and only if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X \; [0 < |x - c| < \delta \Longrightarrow |f(x) - L| < \varepsilon] . \tag{3.69}$$

Proof. ($\Longrightarrow$) We prove necessity by contraposition. Suppose the negation of (3.69) is true. That is, there exists $\varepsilon > 0$ such that

$$\forall \delta > 0 \quad \exists x \in X \; [0 < |x - c| < \delta \wedge |f(x) - L| \geq \varepsilon] . \tag{3.70}$$

Let $n \in \mathbb{N}$. Since $\frac{1}{n} > 0$, we can instantiate (3.70) at $\delta = \frac{1}{n}$. Thus, there exists $c_n \in X$ such that

$$\begin{aligned} 0 \quad < \quad |c_n - c| \quad &< \quad \frac{1}{n}, & (3.71) \\ |f(c_n) - L| \quad &\geq \quad \varepsilon. & (3.72) \end{aligned}$$

By the identity of indiscernibles in $\mathbb{R}$, the condition $0 < |c_n - c|$ from (3.71) implies $c_n \neq c$, and so we have produced $(c_n)_{n \in \mathbb{N}}$, which is a sequence in $X \backslash \{c\}$. To show $(c_n)_{n \in \mathbb{N}}$ converges to c, let $\eta > 0$. There exists a positive integer $N > \frac{1}{\eta}$, and if $n \geq N$, then by (3.71), we have

$$|c_n - c| < \frac{1}{n} \leq \frac{1}{N} < \eta.$$

Therefore, $\lim\limits_{n \to \infty} c_n = c$. To prove $(f(c_n))_{n \in \mathbb{N}}$ does not converge to L, let $M \in \mathbb{N}$, and take $m = M$, for which the inequality (3.72) holds at $n = m$. That is, we have shown,

$$\exists \varepsilon > 0 \quad \forall M \in \mathbb{N} \quad \exists m \geq M \; [|f(c_m) - L| \geq \varepsilon] .$$

Therefore, $\lim\limits_{m \to \infty} f(c_m) \neq L$, where $(c_n)_{n \in \mathbb{N}}$ is some sequence in $X \backslash \{c\}$ that converges to c. By Definition 3.4, $\lim\limits_{x \to c} f(x) \neq L$.

($\Longleftarrow$) Let $(c_n)_{n\in\mathbb{N}}$ be a sequence in $X\backslash\{c\}$ that converges to c. Let $\varepsilon > 0$. By (3.69), there exists $\delta > 0$ such that

$$0 < |x-c| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon. \tag{3.73}$$

Since $\lim\limits_{n\to\infty} c_n = c$ and $\delta > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Longrightarrow \quad |c_n - c| < \delta. \tag{3.74}$$

If indeed $n \geq N$, we find that $c_n \neq c$ since $(c_n)_{n\in\mathbb{N}}$ is a sequence in $X\backslash\{c\}$. By (3.74), we further have $0 < |c_n - c| < \delta$, which by (3.73), at $x = c_n$, further implies $|f(c_n) - L| < \varepsilon$. In summary,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \;\left[|f(c_n) - L| < \varepsilon\right].$$

Therefore, $\lim\limits_{n\to\infty} f(c_n) = L$, where $(c_n)_{n\in\mathbb{N}}$ is an arbitrary sequence in $X\backslash\{c\}$ that converges to c. By Definition 3.4, $\lim\limits_{x\to c} f(x) = L$. ☐

Theorem 3.18. *A function $f : X \to \mathbb{R}$ is continuous at $c \in X$ if and only if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X \;\left[|x-c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon\right]. \tag{3.75}$$

Proof. First, we consider the statement

$$\forall x \in X \;\left[0 < |x-c| < \delta \Longrightarrow \left| f|_{X\backslash\{c\}}(x) - f(c)\right| < \varepsilon\right]. \tag{3.76}$$

If indeed $x \in X$ and $0 < |x-c| < \delta$, then $x \in X\backslash\{c\}$, and we simply have $f|_{X\backslash\{c\}}(x) = f(x)$, so we can simplify (3.76) as

$$\forall x \in X \;\left[0 < |x-c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon\right],$$

and so, by Theorems 3.13 and 3.17, we find that $f : X \to \mathbb{R}$ is continuous at $c \in X$ if and only if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X \;\left[0 < |x-c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon\right]. \tag{3.77}$$

Our task then boils down to proving the equivalence of (3.75) and (3.77).

(3.75) $\Longrightarrow$ (3.77). Let $\varepsilon > 0$. By (3.75), there exists $\delta > 0$ such that for any $x \in X$, we have

$$|x-c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon. \tag{3.78}$$

If $0 < |x - c| < \delta$, then in particular, $|x - c| < \delta$, and by (3.78), we have the inequality $|f(x) - f(c)| < \varepsilon$. Therefore, (3.77) holds.

(3.77) $\Longrightarrow$ (3.75). Let $\varepsilon > 0$. By (3.77), there exists $\delta > 0$ such that for any $x \in X$, we have

$$0 < |x - c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon. \tag{3.79}$$

Suppose $|x - c| < \delta$. We consider cases: $x = c$ or $x \neq c$.

If $x \neq c$, then $0 < |x - c|$. In conjunction with the assumed $|x - c| < \delta$, and by (3.79), we have $|f(x) - f(c)| < \varepsilon$ as desired.

If $x = c$, then $f(x) = f(c)$ because f is a function, and so $|f(x) - f(c)| = 0 < \varepsilon$. In any case, (3.75) is true. □

If (3.75) is the epsilon-delta characterization for continuity at a point, then continuity on an interval [the interval X] has the epsilon-delta characterization:

$$\forall c \in X \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in X \; [|x - c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon], \tag{3.80}$$

in which the number δ depends on the choice of $c \in X$ and $\varepsilon > 0$. A modification in the quantification in (3.80), one in which δ is made independent of $c \in X$, gives us the following notion related to continuity, which has far-reaching applications throughout the scope of the book and beyond.

Definition 3.19. A function $f : X \to \mathbb{R}$ is *uniformly continuous on* X if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall c \in X \quad \forall x \in X \; [|x - c| < \delta \Longrightarrow |f(x) - f(c)| < \varepsilon]. \tag{3.81}$$

If indeed f is uniformly continuous then the number $\delta = \delta(\varepsilon)$ in (3.81) is called a *modulus of continuity* of f.

The logical relationship between the notions of continuity and uniform continuity are summarized in the following.

Theorem 3.20. *(i) A uniformly continous function on X is continuous on X.*

(ii) A continuous function on X is not necessarily uniformly continuous on X.

(iii) If X is a closed interval, then any continuous function on X is uniformly continuous on X.

Proof. *(i)* We prove that (3.81) implies (3.80). Let $c \in X$. If $\varepsilon > 0$, then by (3.81), there exists $\delta > 0$ such that

$$\forall d \in X \quad \forall x \in X \ [|x - d| < \delta \Longrightarrow |f(x) - f(d)| < \varepsilon] \, ,$$

which we instantiate at $d = c$. Therefore, (3.80) holds.

(ii) Our goal is to exhibit a function continuous on some interval X but not uniformly continuous on X. We consider $X =]0, 1[$, and $f : X \to \mathbb{R}$ given by $f(x) = x^2$. Given $c \in X$, if $(c_n)_{n \in \mathbb{N}}$ is a sequence in X that converges to c, by the rules for limits of sequences, we have $\lim\limits_{n\to\infty} f(c_n) = \lim\limits_{n\to\infty} c_n^2 = \left(\lim\limits_{n\to\infty} c_n\right)^2 = c^2 = f(c)$. Therefore, $f \in C(X)$. To show f is not uniformly continuous on X, we show that the negation

$$\exists \varepsilon > 0 \quad \forall \delta > 0 \quad \exists c \in X \quad \exists x \in X \ [|x - c| < \delta \wedge |f(x) - f(c)| \geq \varepsilon] \, , \tag{3.82}$$

of (3.81) is true. To this end, we consider $\varepsilon = 1$, we assume an arbitrary $\delta > 0$, and we let

$$\begin{aligned} x &:= \min\left\{\frac{\delta}{2}, \frac{1}{2}\right\} > 0, && (3.83) \\ c &:= \frac{x}{1+x} > 0. && (3.84) \end{aligned}$$

Our goal here is to prove that

$$\begin{aligned} c, x &\in]0, 1[\, , && (3.85) \\ |x - c| &< \delta, && (3.86) \\ |f(x) - f(c)| &\geq 1. && (3.87) \end{aligned}$$

By some routine computations involving properties of inequalities in $\mathbb{R}$, the condition $0 < x$ from (3.83) implies $\frac{x}{1+x} < x$, and so $c < x$. From (3.83),(3.84), we further have

$$0 < c < x \leq \frac{1}{2} < 1, \tag{3.88}$$

which proves (3.83). We also have the inequalities

$$\begin{aligned} x - c &< x, && (3.89) \\ x - c &> 0, && (3.90) \end{aligned}$$

where (3.89) can be routinely obtained from $0 < c$, which is from (3.84), and (3.90) is from $c < x$, which is from (3.88). Using (3.89),(3.90) and then (3.83), we have

$$|x - c| = x - c < x \leq \frac{\delta}{2} < \delta,$$

and so (3.86) holds. Finally,

$$|f(x) - f(c)| = \left| \frac{1}{x} - \frac{1+x}{x} \right| = 1,$$

from which (3.87) follows. This completes the proof of (3.82). Therefore, f is not uniformly continuous on X.

(iii) We prove this by contradiction. If f is not uniformly continuous on X, then there exists $\varepsilon > 0$ such that

$$\forall \delta > 0 \quad \exists c \in X \quad \exists x \in X \ [|x - c| < \delta \wedge |f(x) - f(c)| \geq \varepsilon]. \tag{3.91}$$

Let $n \in \mathbb{N}$. Since $\frac{1}{n} > 0$, by (3.91), there exist $c_n, x_n \in X$ such that

$$|x_n - c_n| < \frac{1}{n}, \tag{3.92}$$

$$|f(x_n) - f(c_n)| \geq \varepsilon. \tag{3.93}$$

Thus, $(c_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are sequences in X, and are hence bounded. By the Bolzano-Weierstrass Theorem, these sequences have convergent subsequences $(c_{N_i})_{i\in\mathbb{N}}$, $(x_{N_i})_{i\in\mathbb{N}}$, respectively. Also, the terms in the said subsequences satisfy (3.92),(3.93). That is,

$$|x_{N_i} - c_{N_i}| < \frac{1}{N_i}, \tag{3.94}$$

$$|f(x_{N_i}) - f(c_{N_i})| \geq \varepsilon, \tag{3.95}$$

for all $i \in \mathbb{N}$. Let $s = \lim_{i\to\infty} c_{N_i}$. By Lemmas 2.10,2.17, we have $s \in X$. Given $\eta > 0$, we choose a positive integer $I > \frac{2}{\eta}$. By the property (2.47) of subsequences, we have $N_I \geq I > \frac{2}{\eta}$, and consequently,

$$N_i \geq N_I \Longrightarrow \frac{1}{N_i} \leq \frac{1}{N_I} < \frac{\eta}{2}. \tag{3.96}$$

Since $s = \lim_{N_i\to\infty} c_{N_i}$, there exists an index N_J such that

$$N_i \geq N_j \Longrightarrow |c_{N_i} - s| < \frac{\eta}{2}. \tag{3.97}$$

Let $N_k := \max\{N_I, N_J\}$. If $N_i \geq N_k$, then by (3.94),(3.96),(3.97),

$$|x_{N_i} - s| \leq |x_{N_i} - c_{N_i}| + |c_{N_i} - s| < \frac{1}{N_i} + \frac{\eta}{2} < \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

Therefore, $(x_{N_i})_{i\in\mathbb{N}}$ also converges to $s \in X$. By the continuity of f, we have

$$\lim_{N_i\to\infty} f(x_{N_i}) = f(s) = \lim_{N_i\to\infty} f(c_{N_i}),$$

and by the rules for operations on sequences, we further have

$$\lim_{n\to\infty} (f(x_{N_i}) - f(c_{N_i})) = \lim_{N_i\to\infty} f(x_{N_i}) - \lim_{N_i\to\infty} f(c_{N_i}) = 0,$$

which means that there exists an index N_h such that if $N_i \geq N_h$, then

$$|f(x_{N_i}) - f(c_{N_i})| = |(f(x_{N_i}) - f(c_{N_i})) - 0| < \varepsilon,$$

which contradicts (3.95) and the Trichotomy Law. Therefore, f is uniformly continuous on X. □