Contents

Chapter 1

Philosopherers

(1.5)
$OrderTrichotomy(<, S) := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$
$OrderTransitivity(<, S) := \forall_{x,y,z \in S}\big((x < y \wedge y < z) \implies x < z\big)$
$Order(<, S) := OrderTrichotomy(<, S) \wedge OrderTransitivity(<, S)$

(1.7)
$BoundedAbove(E, S, <) := Order(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(x \leq \beta)$
$BoundedBelow(E, S, <) := Order(<, S) \wedge E \subset S \wedge \exists_{\beta \in S}\forall_{x \in E}(\beta \leq x)$
$UpperBound(\beta, E, S, <) := Order(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(x \leq \beta)$
$LowerBound(\beta, E, S, <) := Order(<, S) \wedge E \subset S \wedge \beta \in S \wedge \forall_{x \in E}(\beta \leq x)$

(1.8)
$LUB(\alpha, E, S, <) := UpperBound(\alpha, E, S, <) \wedge \forall_{\gamma}\big(\gamma < \alpha \implies \neg UpperBound(\gamma, E, S, <)\big)$
$GLB(\alpha, E, S, <) := LowerBound(\alpha, E, S, <) \wedge \forall_{\beta}\big(\alpha < \beta \implies \neg LowerBound(\beta, E, S, <)\big)$

(1.10)
$LUBProperty(S, <) := \forall_{E}\Big(\big(\emptyset \neq E \subset S \wedge BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$
$GLBProperty(S, <) := \forall_{E}\Big(\big(\emptyset \neq E \subset S \wedge BoundedBelow(E, S, <)\big) \implies \exists_{\alpha \in S}\big(GLB(\alpha, E, S, <)\big)\Big)$

(1.11)
$\boxed{\text{LUBPropertyImpliesGLBProperty}}$ $LUBProperty(S, <) \implies GLBProperty(S, <)$

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(1) $LUBProperty(S, <) \implies \ldots$ wts: 2

(1.1) $\big(\emptyset \neq B \subset S \wedge BoundedBelow(B, S, <)\big) \implies \ldots$ wts: 1.2

(1.1.1) $Order(<, S) \wedge \exists_{\delta' \in S}\big(LowerBound(\delta', B, S, <)\big)$ from: BoundedBelow, 1.1

(1.1.2) $|B| = 1 \implies \ldots$ wts: 1.1.3

(1.1.2.1) $\exists_{u'}(u' \in B)$ ▌ $u := choice(\{u' | u' \in B\})$ ▌ $B = \{u\}$ from: 1.1.2

(1.1.2.2) $GLB(u, B, S, <)$ ▌ $\exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$

(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S}\big(GLB(\epsilon_0, B, S, <)\big)$

(1.1.4) $|B| \neq 1 \implies \ldots$ wts: 1.1.5

(1.1.4.1) $\forall_{E}\Big(\big(\emptyset \neq E \subset S \wedge BoundedAbove(E, S, <)\big) \implies \exists_{\alpha \in S}\big(LUB(\alpha, E, S, <)\big)\Big)$ from: LUBProperty, 1

(1.1.4.2) $L := \{s \in S | LowerBound(s, B, S, <)\}$

(1.1.4.3) $|B| > 1 \wedge OrderTrichotomy(<, S)$ ▌ $\exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$ from: Order, 1.1.1 wts: 1.1.4.7

(1.1.4.4) $b_1 := choice\Big(\{b_1' \in B | \exists_{b_0' \in B}(b_0' < b_1')\}\Big)$ ▌ $\neg LowerBound(b_1, B, S, <)$ from: 1.1.4.2

(1.1.4.5) $b_1 \notin L$ ▌ $L \subset S$

(1.1.4.6) $\delta := choice\big(\{\delta' \in S | LowerBound(\delta', B, S, <)\}\big)$ ▌ $\delta \in L$ ▌ $\emptyset \neq L$ from: 1.1.1

(1.1.4.7) $\emptyset \neq L \subset S$ from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y\in L}\big(LowerBound(y,B,S,<)\big) \blacksquare \forall_{y\in L}\forall_{x\in B}(y\leq x)$ — from: LowerBound, 1.1.4.2; wts: 1.1.4.10

(1.1.4.9) $\forall_{x\in B}\Big(x\in S\wedge\forall_{y\in L}(y\leq x)\Big) \blacksquare \forall_{x\in B}\big(UpperBound(x,L,S,<)\big)$ — from: UpperBound

(1.1.4.10) $\exists_{x\in S}\big(UpperBound(x,L,S,<)\big) \blacksquare BoundedAbove(L,S,<)$

(1.1.4.11) $\emptyset\neq L\subset S\wedge BoundedAbove(L,S,<)$ — from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha'\in S}\big(LUB(\alpha',L,S,<)\big) \blacksquare \alpha := choice\Big(\{\alpha'\in S|\big(LUB(\alpha',L,S,<)\big)\}\Big)$ — from: 1.1.4.1; wts: 1.1.4.21

(1.1.4.13) $\forall_x\big(x\in B\implies UpperBound(x,L,S,<)\big)$ — from: 1.1.4.9; wts: 1.1.4.17

(1.1.4.14) $\forall_x\big(\neg UpperBound(x,L,S,<)\implies x\notin B\big)$

(1.1.4.15) $\gamma<\alpha\implies\dots$ — wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound(\gamma,L,S,<) \blacksquare \gamma\notin B$ — from: LUB, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma<\alpha\implies\gamma\notin B \blacksquare \gamma\in B\implies\gamma\geq\alpha$

(1.1.4.17) $\forall_{\gamma\in B}(\alpha\leq\gamma) \blacksquare LowerBound(\alpha,B,S,<)$ — from: LowerBound

(1.1.4.18) $\alpha<\beta\implies\dots$ — wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y\in L}(y\leq\alpha<\beta) \blacksquare \forall_{y\in L}(y\neq\beta)$ — from: LUB, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta\notin L \blacksquare \neg LowerBound(\beta,B,S,<)$ — from: 1.1.4.2

(1.1.4.19) $\alpha<\beta\implies\neg LowerBound(\beta,B,S,<) \blacksquare \forall_{\beta\in S}\big(\alpha<\beta\implies\neg LowerBound(\beta,B,S,<)\big)$

(1.1.4.20) $LowerBound(\alpha,B,S,<)\wedge\forall_{\beta\in S}\big(\alpha<\beta\implies\neg LowerBound(\beta,B,S,<)\big)$ — from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB(\alpha,B,S,<) \blacksquare \exists_{\epsilon_1\in S}\big(GLB(\epsilon_1,B,S,<)\big)$

(1.1.5) $|B|\neq 1\implies\exists_{\epsilon_1\in S}\big(GLB(\epsilon_1,B,S,<)\big)$

(1.1.6) $\Big(|B|=1\implies\exists_{\epsilon_0\in S}\big(GLB(\epsilon_0,B,S,<)\big)\Big)\wedge\Big(|B|\neq 1\implies\exists_{\epsilon_1\in S}\big(GLB(\epsilon_1,B,S,<)\big)\Big)$ — from: 1.1.3, 1.1.5

(1.1.7) $(|B|=1\vee|B|\neq 1)\implies\exists_{\epsilon\in S}\big(GLB(\epsilon,B,S,<)\big) \blacksquare \exists_{\epsilon\in S}\big(GLB(\epsilon,B,S,<)\big)$

(1.2) $\big(\emptyset\neq B\subset S\wedge BoundedBelow(B,S,<)\big)\implies\exists_{\epsilon\in S}\big(GLB(\epsilon,B,S,<)\big)$

(1.3) $\forall_B\Big(\big(\emptyset\neq B\subset S\wedge BoundedBelow(B,S,<)\big)\implies\exists_{\epsilon\in S}\big(GLB(\epsilon,B,S,<)\big)\Big)$

(1.4) $GLBProperty(S,<)$

(2) $LUBProperty(S,<)\implies GLBProperty(S,<)$

(1.12)

$$Field(F,+,*) := \exists_{\mathbb{0},\mathbb{1}\in F}\forall_{x,y,z\in F}\left(\begin{array}{l} x+y\in F \quad\wedge\quad x*y\in F \quad\wedge\\ x+y=y+x \quad\wedge\quad x*y=y*x \quad\wedge\\ (x+y)+z=x+(y+z) \quad\wedge\quad (x*y)*z=x*(y*z) \quad\wedge\\ \mathbb{1}\neq\mathbb{0} \quad\wedge\quad x*(y+z)=(x*y)+(x*z) \quad\wedge\\ \mathbb{0}+x=x \quad\wedge\quad \mathbb{1}*x=x \quad\wedge\\ \exists_{-x\in F}\big(x+(-x)=\mathbb{0}\big)\wedge\Big(x\neq\mathbb{0}\implies\exists_{1/x\in F}\big(x*(1/x)=\mathbb{1}\big)\Big)\end{array}\right)$$

$*********************** \big(Field(F,+,*)\wedge x,y,z\in F\big)\implies\dots ***********************$

(1.14)

AdditiveCancellation $(x+y=x+z)\implies y=z$

(1) $y=\mathbb{0}+y=\big(x+(-x)\big)+y=\big((-x)+x\big)+y=(-x)+(x+y)=\dots$ — from: Field

(2) $(-x)+(x+z)=\big((-x)+x\big)+z=\big(x+(-x)\big)+z=\mathbb{0}+z=z$ — from: Field

AdditiveIdentityUniqueness $(x+y=x)\implies y=\mathbb{0}$

(1) $x+y=x=\mathbb{0}+x=x+\mathbb{0}$ — from: Field

(2) $y=0$ — from: AdditiveCancellation

$\boxed{\text{AdditiveInverseUniqueness}}$ $(x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$ from: Field

(2) $y = -x$ from: AdditiveCancellation

$\boxed{\text{DoubleNegative}}$ $x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x$ ▌ $\mathbb{0} = (-x) + x$ from: Field

(2) $x = -(-x)$ from: AdditiveInverseUniqueness

(1.15)
$\boxed{\text{MultiplicativeCancellation}}$ $(x \neq 0 \wedge x * y = x * z) \implies y = z$ —

$\boxed{\text{MultiplicativeIdentityUniqueness}}$ $(x \neq 0 \wedge x * y = x) \implies y = \mathbb{1}$ —

$\boxed{\text{MultiplicativeInverseUniqueness}}$ $(x \neq 0 \wedge x * y = \mathbb{1}) \implies y = 1/x$ —

$\boxed{\text{DoubleReciprocal}}$ $(x \neq 0) \implies x = 1/(1/x)$ —

(1.16)
$\boxed{\text{Domination}}$ $\mathbb{0} * x = \mathbb{0}$

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x$ ▌ $\mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$ from: Field

(2) $\mathbb{0} * x = \mathbb{0}$ from: AdditiveIdentityUniqueness

$\boxed{\text{NonDomination}}$ $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies \ldots$

(1.1) $(x * y = 0) \implies \ldots$

(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = \big(x * (1/x)\big) * \big(y * (1/y)\big) = (x * y) * \big((1/x) * (1/y)\big) = \mathbb{0} * \big((1/x) * (1/y)\big) = \mathbb{0}$ from: Field, Domination, 1, 1.1

(1.1.2) $\mathbb{1} = \mathbb{0} \wedge \mathbb{1} \neq \mathbb{0}$ ▌ $\perp$ from: Field

(1.2) $(x * y = 0) \implies \perp$ ▌ $x * y \neq 0$

(2) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq 0$

$\boxed{\text{NegationCommutativity}}$ $(-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0}$ ▌ $x * y + (-x) * y = \mathbb{0}$ from: Field, Domination wts: 2

(2) $(-x) * y = -(x * y)$ from: AdditiveInverseUniqueness

(3) $x * y + x * (-y) = x * (y + -y) = x * \mathbb{0} = \mathbb{0}$ ▌ $x * y + x * (-y) = \mathbb{0}$ from: Field, Domination wts: 4

(4) $x * (-y) = -(x * y)$ from: AdditiveInverseUniqueness

(5) $(-x) * y = -(x * y) = x * (-y)$ from: 2, 4

$\boxed{\text{NegativeMultiplication}}$ $(-x) * (-y) = x * y$

(1) $(-x) * (-y) = -\big(x * (-y)\big) = -\big(-(x * y)\big) = x * y$ from: NegationCommutativity, DoubleNegative

(1.17)

$$OrderedField(F, +, *, <) := \begin{pmatrix} Field(F, +, *) \quad \wedge \quad Order(<, F) \quad \wedge \\ \forall_{x,y,z \in F}(y < z \implies x + y < x + z) \quad \wedge \\ \forall_{x,y \in F}\big((x > \mathbb{0} \wedge y > \mathbb{0}) \implies x * y > \mathbb{0}\big) \end{pmatrix}$$

$************************ \big(OrderedField(F, +, *, <) \wedge x, y, z \in F\big) \implies \ldots ************************$

(1.18)
$\boxed{\text{NegationOnOrder}}$ $x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \dots$

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x$ ▮ $\mathbb{0} > -x$ ▮ $-x < \mathbb{0}$ — from: OrderedField

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \dots$

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x$ ▮ $\mathbb{0} < x$ ▮ $x > \mathbb{0}$ — from: OrderedField

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \land -x < \mathbb{0} \implies x > \mathbb{0}$ ▮ $x > \mathbb{0} \iff -x < \mathbb{0}$ — from: 2, 4

PositiveFactorPreservesOrder $(x > \mathbb{0} \land y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \land y < z) \implies \dots$

(1.1) $(-y) + z > (-y) + y = \mathbb{0}$ ▮ $z + (-y) = \mathbb{0}$ — from: OrderedField

(1.2) $x * (z + (-y)) > \mathbb{0}$ ▮ $x * z + x * (-y) > \mathbb{0}$ — from: OrderedField

(1.3) $x * z = \mathbb{0} + x * z = \big(x * y + -(x * y)\big) + x * z = \big(x * y + x * (-y)\big) + x * z = \dots$ — from: Field, NegationCommutativity

(1.4) $x * y + \big(x * z + x * (-y)\big) > x * y + \mathbb{0} = x * y$ — from: Field, 1.2

(1.5) $x * z > x * y$ — from: 1.3, 1.4

(2) $(x > \mathbb{0} \land y < z) \implies x * z > x * y$

NegativeFactorFlipsOrder $(x < \mathbb{0} \land y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \land y < z) \implies \dots$

(1.1) $-x > \mathbb{0}$ — from: NegationOnOrder

(1.2) $(-x) * y < (-x) * z$ ▮ $\mathbb{0} = x * y + (-x) * y < x * y + (-x) * z$ ▮ $\mathbb{0} < x * y + (-x) * z$ — from: PositiveFactorPreservesOrder

(1.3) $\mathbb{0} < (-x) * (-y + z)$ ▮ $\mathbb{0} > x * (-y + z)$ ▮ $\mathbb{0} > -(x * y) + x * z$ — from: NegationOnOrder

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \land y < z) \implies x * y > x * z$

SquareIsPositive $(x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ — from: OrderedField

(2) $(x < \mathbb{0}) \implies \dots$

(2.1) $-x > \mathbb{0}$ ▮ $x * x = (-x) * (-x) > \mathbb{0}$ ▮ $x * x > \mathbb{0}$ — from: NegationOnOrder, OrderedField, NegativeMultiplication

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \lor x < \mathbb{0}) \implies x * x > \mathbb{0}$ ▮ $x \neq \mathbb{0} \implies x * x > \mathbb{0}$ — from: OrderTrichotomy, 1, 3

OneIsPositive $\mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0}$ ▮ $\mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ — from: Field, SquareIsPositive

ReciprocationOnOrder $(0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \dots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0}$ ▮ $x * (1/x) > 0$ — from: Field, OneIsPositive

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \land x * (1/x) > 0 \implies \bot$ ▮ $1/x > \mathbb{0}$ — from: NegativeFactorFlipsOrder, 1

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0}$ ▮ $y * (1/y) > 0$ — from: Field, OneIsPositive

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \land y * (1/y) > 0 \implies \bot$ ▮ $1/y > \mathbb{0}$ — from: NegativeFactorFlipsOrder, 1

(1.5) $(1/x) * (1/y) > \mathbb{0}$ — from: OrderedField

(1.6) $\mathbb{0} < 1/y = \big((1/x) * (1/y)\big) * x < \big((1/x) * (1/y)\big) * y = 1/x$ — from: OrderedField, 1, 1.4, 1.5

(1.19)
FieldQ $OrderedField(\mathbb{Q},+,*,<)$ —

$Subfield(K,F,+,*) := Field(F,+,*) \wedge K \subset F \wedge Field(K,+,*)$
$OrderedSubfield(K,F,+,*,<) := OrderedField(F,+,*,<) \wedge K \subset F \wedge OrderedField(K,+,*,<)$
$CutI(\alpha) := \emptyset \neq \alpha \subset \mathbb{Q}$
$CutII(\alpha) := \forall_{p\in\alpha}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \alpha)$
$CutIII(\alpha) := \forall_{p\in\alpha}\exists_{r\in\alpha}(p < r)$
$SetR := \mathbb{R} := \{\alpha \in \mathbb{Q} | CutI(\alpha) \wedge CutII(\alpha) \wedge CutIII(\alpha)\}$

CutCorollaryI $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \ldots$
(1.1) $\forall_{p'\in\alpha}\forall_{q'\in\mathbb{Q}}(q' < p' \implies q' \in \alpha)$ from: CutII, 1
(1.2) $q < p \implies q \in \alpha$ ▌ $q \notin \alpha \implies q \geq p$ from: 1
(1.3) $(q \notin \alpha) \implies \ldots$
(1.3.1) $q \geq p$ from: 1.2
(1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \bot$ ▌ $q \neq p$ from: 1, 1.3
(1.3.3) $q \geq p \wedge q \neq p$ ▌ $p < q$
(1.4) $q \notin \alpha \implies p < q$ ▌ $p < q$ from: 1
(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

CutCorollaryII $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$
(1.1) $\forall_{s'\in\alpha}\forall_{r'\in\mathbb{Q}}(r' < s' \implies r' \in \alpha)$ from: CutII, 1
(1.2) $s \in \alpha \implies \big(r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)\big)$ ▌ $s \in \alpha \implies r \in \alpha$ from: 1, 1.1
(1.3) $r \notin \alpha \implies s \notin \alpha$ ▌ $s \notin \alpha$ from: 1, 1.2
(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<(\alpha,\beta) := \alpha,\beta \in \mathbb{R} \wedge \alpha \subset \beta$

OrderTrichotomyR $OrderTrichotomy(\mathbb{R},<)$

(1) $(\alpha,\beta \in \mathbb{R}) \implies \ldots$
(1.1) $\neg(\alpha < \beta \vee \alpha = \beta) \implies \ldots$
(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$ from: <, 1.1
(1.1.2) $\exists_{p'}(p' \in \alpha \wedge p' \notin \beta)$ ▌ $p := choice(\{p' | p' \in \alpha \wedge p' \notin \beta\})$
(1.1.3) $q \in \beta \implies \ldots$
(1.1.3.1) $p, q \in \mathbb{Q}$
(1.1.3.2) $q < p$ from: CutCorollaryI
(1.1.3.3) $q \in \alpha$ from: CutII
(1.1.4) $q \in \beta \implies q \in \alpha$
(1.1.5) $\forall_{q\in\beta}(q \in \alpha)$ ▌ $\beta \subseteq \alpha$
(1.1.6) $\beta \subset \alpha$ ▌ $\beta < \alpha$
(1.2) $\neg(\alpha < \beta \vee \alpha = \beta) \implies \beta < \alpha$
(1.3) $\neg(\alpha < \beta \vee \alpha = \beta) \vee (\alpha < \beta \vee \alpha = \beta)$ ▌ $(\beta < \alpha) \vee (\alpha < \beta \vee \alpha = \beta)$
(1.4) $\alpha = \beta \implies \neg(\alpha < \beta \vee \beta < \alpha)$
(1.5) $\alpha < \beta \implies \neg(\alpha = \beta \vee \beta < \alpha)$
(1.6) $\beta < \alpha \implies \neg(\alpha = \beta \vee \alpha < \beta)$
(1.7) $\alpha < \beta \veebar \alpha = \beta \veebar \alpha < \beta$
(2) $(\alpha,\beta \in \mathbb{R}) \implies (\alpha < \beta \veebar \alpha = \beta \veebar \alpha < \beta)$
(3) $\forall_{\alpha,\beta\in\mathbb{R}}(\alpha < \beta \veebar \alpha = \beta \veebar \alpha < \beta)$

(4) $OrderTrichotomy(\mathbb{R}, <)$

OrderTransitivityR $OrderTransitivity(\mathbb{R}, <)$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha < \beta \land \beta < \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \land \beta \subset \gamma$

(1.1.2) $\forall_{a \in \alpha}(a \in \beta) \land \forall_{b \in \beta}(b \in \gamma)$

(1.1.3) $\forall_{a \in \alpha}(\alpha \in \gamma) \;\blacksquare\; \alpha \subset \gamma \;\blacksquare\; \alpha < \gamma$

(1.2) $(\alpha < \beta \land \beta < \gamma) \implies \alpha < \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha < \beta \land \beta < \gamma) \implies \alpha < \gamma\big)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}}\big((\alpha < \beta \land \beta < \gamma) \implies \alpha < \gamma\big)$

(4) $OrderTransitivity(\mathbb{R}, <)$

OrderR $Order(<, \mathbb{R})$

from: OrderTrichotomyR, OrderTransitivityR
wts:

LUBPropertyR $LUBProperty(\mathbb{R}, <)$

(1) $\big(\emptyset \neq A \subset \mathbb{R} \land BoundedAbove(A, \mathbb{R}, <)\big) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} | \exists_{\alpha \in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset \;\blacksquare\; \exists_{\alpha}(\alpha \in A) \;\blacksquare\; \alpha_0 := choice(\{\alpha | \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset \;\blacksquare\; \exists_{a}(a \in \alpha_0) \;\blacksquare\; a_0 := choice(\{a | a \in \alpha_0\}) \;\blacksquare\; a_0 \in \gamma \;\blacksquare\; \gamma \neq \emptyset$

(1.4) $BoundedAbove(A, \mathbb{R}, <) \;\blacksquare\; \exists_{\beta}\big(UpperBound(\beta, A, \mathbb{R}, <)\big)$

(1.5) $\beta_0 := choice(\{\beta | UpperBound(\beta, A, \mathbb{R}, <)\})$

(1.6) $UpperBound(\beta_0, A, \mathbb{R}, <) \;\blacksquare\; \forall_{\alpha \in A}(\alpha \leq \beta_0) \;\blacksquare\; \forall_{\alpha \in A}(\alpha \subseteq \beta_0)$

(1.7) $\forall_{\alpha \in A}\forall_{a \in \alpha}(a \in \beta_0) \;\blacksquare\; \forall_{a \in \gamma}(a \in \beta_0) \;\blacksquare\; \gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q} \;\blacksquare\; \gamma \subseteq \beta_0 \subset \mathbb{Q} \;\blacksquare\; \gamma \subseteq \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q} \;\blacksquare\; CutI(\gamma)$

(1.10) $(p \in \gamma \land q \in \mathbb{Q} \land q < p) \implies \ldots$

(1.10.1) $p \in \gamma \;\blacksquare\; \exists_{\alpha \in A}(p \in \alpha) \;\blacksquare\; \alpha_1 := choice(\{\alpha \in A | p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \land q \in \mathbb{Q} \land q < p \;\blacksquare\; q \in \alpha_1 \;\blacksquare\; q \in \gamma$

(1.11) $(p \in \gamma \land q \in \mathbb{Q} \land q < p) \implies q \in \gamma \;\blacksquare\; \forall_{p \in \gamma}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma) \;\blacksquare\; CutII(\gamma)$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha \in A}(p \in \alpha) \;\blacksquare\; \alpha_2 := choice(\{\alpha \in A | p \in \alpha\})$

(1.12.2) $\exists_{r \in \alpha_2}(p < r) \;\blacksquare\; r_0 := choice(\{r \in \alpha_2 | p < r\})$

(1.12.3) $r_0 \in \alpha_2 \;\blacksquare\; r_0 \in \gamma$

(1.12.4) $p < r_0 \;\blacksquare\; p < r_0 \land r_0 \in \gamma \;\blacksquare\; \exists_{r \in \gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r \in \gamma}(p < r) \;\blacksquare\; \forall_{p \in \gamma}\exists_{r \in \gamma}(p < r) \;\blacksquare\; CutIII(\gamma)$

(1.14) $CutI(\gamma) \land CutII(\gamma) \land CutIII(\gamma) \;\blacksquare\; \gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha \in A}(\alpha \subseteq \gamma) \;\blacksquare\; \forall_{\alpha \in A}(\alpha \leq \gamma) \;\blacksquare\; UpperBound(\gamma, A, \mathbb{R}, <)$

(1.16) $\delta < \gamma \implies \ldots$

(1.16.1) $\delta \subset \gamma \;\blacksquare\; \exists_{s}(s \in \gamma \land s \notin \delta) \;\blacksquare\; s_0 := choice(\{s \in \mathbb{Q} | s \in \gamma \land s \notin \delta\})$

(1.16.2) $s_0 \in \gamma \;\blacksquare\; \exists_{\alpha \in A}(s_0 \in \alpha) \;\blacksquare\; \alpha_3 := choice(\{\alpha \in A | s_0 \in \alpha\})$

(1.16.3) $s_0 \in \alpha_3 \land s_0 \notin \delta \;\blacksquare\; \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \land s \notin \delta)$

(1.16.4) $\delta \geq \alpha_3 \implies \ldots$

(1.16.4.1) $\alpha_3 \subseteq \delta \;\blacksquare\; \forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \;\blacksquare\; \neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \land s \notin \delta)$

(1.16.4.2) $\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \land s \notin \delta) \land \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \land s \notin \delta) \;\blacksquare\; \bot$

(1.16.5) $\delta \geq \alpha_3 \implies \bot \;\blacksquare\; \delta < \alpha_3 \;\blacksquare\; \exists_{\alpha \in A}(\delta < \alpha) \;\blacksquare\; \exists_{\alpha \in A}\big(\neg(\alpha \leq \delta)\big)$

(1.16.6) $\neg\forall_{\alpha \in A}(\alpha \leq \delta) \;\blacksquare\; \neg UpperBound(\delta, A, \mathbb{R}, <)$

(1.17) $\delta < \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <)) \;\blacksquare\; \forall_{\delta}\big(\delta < \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <)\big)$

(1.18) $UpperBound(\gamma, A, \mathbb{R}, <) \land \forall_{\delta}\big(\delta < \gamma \implies \neg UpperBound(\delta, A, \mathbb{R}, <)\big)$ ▮ $LUB(\delta, A, \mathbb{R}, <)$

(2) $\big(\emptyset \neq A \subset \mathbb{R} \land BoundedAbove(A, \mathbb{R}, <)\big) \implies \exists_{\gamma \in S}\big(LUB(\gamma, A, \mathbb{R}, <)\big)$

(3) $UNIVERSALCLOSURE$

(4) $LUBProperty(\mathbb{R}, <)$

$\boxed{\text{ExistenceOfR}}$ $\exists_{\mathbb{R}}\big(LUBProperty(\mathbb{R}, <) \land OrderedSubfield(\mathbb{Q}, \mathbb{R}, +, *, <)\big)$

(1) 123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

First Chapter

(1) First

(1.1) Second

(1.2) Third

(2) Fourth

This will be an empty chapter and I will put some text here

$$\sum_{i=0}^{\infty} a_i x^i \tag{2.1}$$

The equation 2.1 shows a sum that is divergent. This formula will later be used in the page **??**.
For further references see Something Linky or go to the next url: http://www.sharelatex.com or open the next file File.txt
It's also possible to link directly any word or any sentence in your document. supwithitSup With It Theorem
If you read this text, you will get no information. Really? Is there no information?

For instance this sentence. supwithit