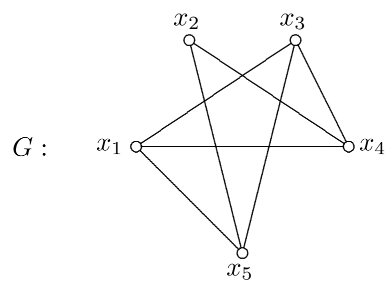
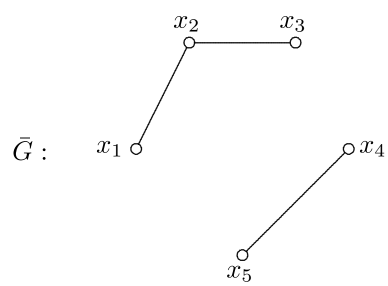
product of two graphs, the composition of two graphs and the conjunction of two graphs. These operations are formally defined as follows.

Definition 4.1. Let $G = (V(G), E(G)$ be a graph. The *complement* of G, denoted by $\bar{G} = (V(\bar{G}), E(\bar{G}))$, where $V(G) = V(\bar{G})$ and the edge $xy \in E(\bar{G})$ if and only if $xy \notin E(G)$.





The next operation is the rth power of a graph. This operation uses the notion of distance between vertices in a graph G. We define the distance between the vertices x, y in G, denoted by $d(x, y)$, to be the length of the shortest path between x and y, if any; otherwise $d(x, y) = \infty$. We note that in a connected graph, distance is a metric. This means that for any vertices x, y, z of G,

1. $d(x, y) \geq 0$, with $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) + d(y, z) \geq d(x, z)$.

Definition 4.2. Let G be a graph and r be a positive integer. The *rth power of* G, denoted by G^r, is obtained by adding the edge xy to the graph G whenever the distance between the vertices x and y of $V(G)$ is less than or equal to r.

Example 4.1. The $2nd$ power of C_6, C_6^2 is illustrated in Figure 15.

Remark 4.1. Observe that C_n^r, reduces to the complete graph K_n whenever $r \geq \frac{n-1}{2}$.

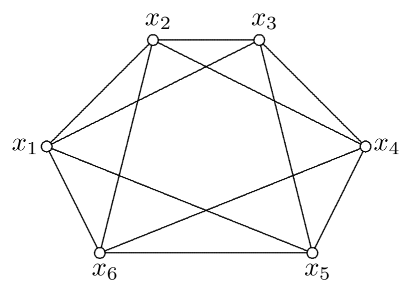


Figure 15: The graph C_6^2

Definition 4.3. Let G_1 and G_2 be graphs. The *sum* of G_1 and G_2, denoted by $G_1 + G_2$ is a graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{xy | x \in V(G_1), y \in V(G_2)\}.$$

Example 4.2. Consider the path, P_2 and the cycle, C_3. Suppose

$$V(P_2) = \{x_1, x_2\},\ E(P_2) = \{x_1x_2\}$$

and

$$V(C_3) = \{y_1, y_2, y_3\},\ E(C_3) = \{y_1y_2, y_2y_3, y_3y_1\}.$$

Then,

$$V(P_2 + C_3) = \{x_1, x_2, y_1, y_2, y_3\}$$

and

$$E(P_2 + C_3) = \{x_1x_2, y_1y_2, y_2y_3, y_3y_1, x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3\}.$$

A pictorial representation of $P_2 + C_3$ is given in Figure 16.

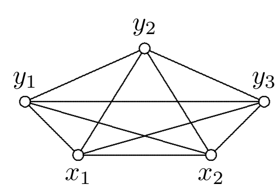


Figure 16: $P_2 + C_3$

Remark 4.2. Observe that $G_1 + G_2 \cong G_2 + G_1$.

For the next three operations, we will use the notation $[x, y]$ for the edge xy in the graph G.

Definition 4.4. Let G_1 and G_2 be graphs. The *cartesian product of* G_1 *and* G_2, denoted by $G_1 \times G_2$ is graph with

$$V(G_1 \times G_2) = \{[x, y] | x \in V(G_1), y \in V(G_2)\}$$

and

$E(G_1 \times G_2) = \{[(a,b),(c,d)] |\, \text{either}\, \{a = b\, \text{and}\, [\text{b}, \text{d}] \in \text{E}(\text{G}_2)\}\, \text{or}\, \{\text{b} = \text{d}\, \text{and}\, [\text{a}, \text{c}] \in \text{E}(\text{G}_1)\}\}$

Example 4.3. Consider the path, P_2 and the cycle, C_3 in Example 4.2. Then the cartesian product of P_2 and C_3, $P_2 \times C_3$ has as its vertex set

$$V(P_2 \times C_3) = \{(x_1,y_1),(x_1,y_2),(x_1,y_3),(x_2,y_1),(x_2,y_2),(x_2,y_3)\}$$

and edge set

$$\begin{aligned} E(P_2 \times C_3) \quad = \quad & \{[(x_1,y_1),(x_1,y_2)],[(x_1,y_2),(x_1,y_3)],[(x_1,y_3),(x_1,y_1)], \\ & [(x_1,y_1),(x_2,y_1)],[(x_1,y_2),(x_2,y_2)],[(x_1,y_3),(x_2,y_3)], \\ & [(x_2,y_1),(x_2,y_2)],[(x_2,y_2),(x_2,y_3)],[(x_2,y_3),(x_2,y_1)]\} \end{aligned}$$

A graphical representation of $P_2 \times C_3$ is given in Figure 17

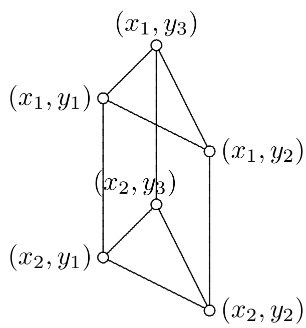


Figure 17: $P_2 \times C_3$

Definition 4.5. Let G and H be graphs. The *composition* of G and H, denoted by $G \circ H$, is the graph with

$$V(G \circ H) = V(G) \times V(H)$$

and where two vertices (a,b) and (c,d) are adjacent if and only if (1.) $a = c$ and $[b,d] \in E(H)$ or (2.) $[a,c] \in E(G)$.

Example 4.4. Consider the cycle C_4 with $V(C_4) = \{x_1,x_2,x_3,x_4\}$ and the path P_2 with $V(P_2) = \{y_1,y_2\}$. Then the composition $C_4 \circ P_2$ has

$V(C_4 \circ P_2) = \{(x_1,y_1),(x_1,y_2),(x_2,y_1),(x_2,y_2),(x_3,y_1),(x_3,y_2),(x_4,y_1),(x_4,y_2)\}$

and

$$\begin{aligned} E(C_4 \circ P_2) &= \{[(x_1,y_1),(x_1,y_2)],[(x_2,y_1),(x_2,y_2)],[(x_3,y_1),(x_3,y_2)],\\ &\quad [(x_4,y_1),(x_4,y_2)],[(x_1,y_1),(x_2,y_1)],[(x_1,y_1),(x_2,y_2)],\\ &\quad [(x_1,y_2),(x_2,y_1)],[(x_1,y_2),(x_2,y_2)],[(x_2,y_1),(x_3,y_1)],\\ &\quad [(x_2,y_1),(x_3,y_2)],[(x_2,y_2),(x_3,y_1)],[(x_2,y_2),(x_3,y_2)],\\ &\quad [(x_3,y_1),(x_4,y_1)],[(x_3,y_1),(x_4,y_2)],[(x_3,y_2),(x_4,y_1)],\\ &\quad [(x_3,y_2),(x_4,y_2)],[(x_4,y_1),(x_1,y_1)],[(x_4,y_1),(x_1,y_2)],\\ &\quad [(x_4,y_2),(x_1,y_1)],[(x_4,y_2),(x_1,y_2)]\}. \end{aligned}$$

A pictorial representation of $C_4 \circ P_2$ is given in Figure 18

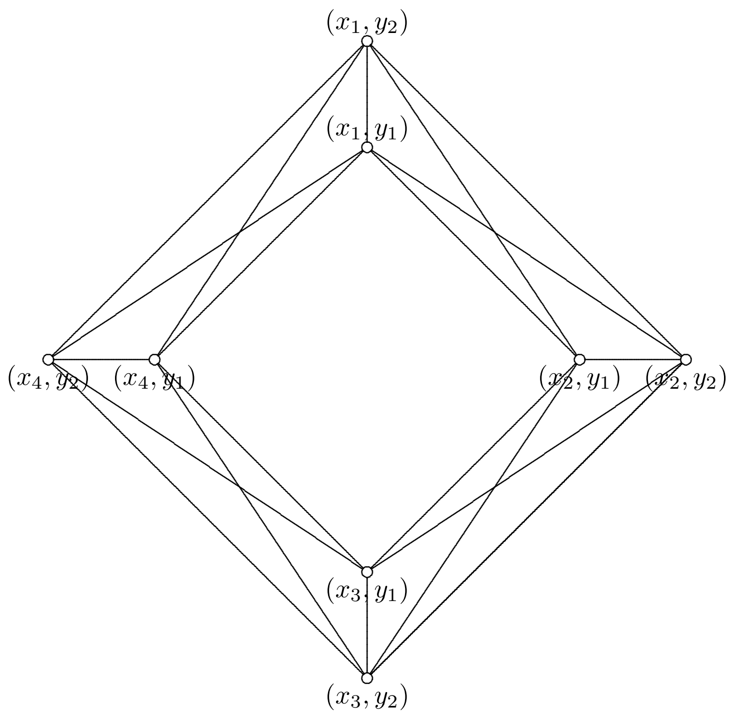


Figure 18: The composition $C_4 \circ P_2$

Definition 4.6. Let G and H be graphs. The *conjunction* of G and H, denoted by $G \wedge H$ is a graph with

$$V(G \wedge H) = V(G) \times V(H)$$

and

$$E(G \wedge H) = \{[(x_1,x_2),(y_1,y_2)] \mid [x_1,y_1] \in E(G) \,\text{and}\, [x_2,y_2] \in E(H)\}.$$

Example 4.5. Consider the cycle C_4 with $V(C_4) = \{x_1, x_2, x_3, x_4\}$ and the path P_2 with $V(P_2) = \{y_1, y_2\}$. Then the conjunction $C_4 \wedge P_2$ has

$$V(C_4 \wedge P_2) = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2), (x_4, y_1), (x_4, y_2)\}$$

and

$$\begin{aligned} E(C_4 \wedge P_2) &= \{[(x_1, y_1), (x_2, y_2)], [(x_2, y_2), (x_3, y_1)], [(x_3, y_1), (x_4, y_2)], [(x_4, y_2), (x_1, y_1)], \\ &\quad [(x_1, y_2), (x_2, y_1)], [(x_2, y_1), (x_3, y_2)], [(x_3, y_2), (x_4, y_1)], [(x_4, y_1), (x_1, y_2)]\} \end{aligned}$$

A pictorial representation of $C_4 \wedge P_2$ is given in Figure 19

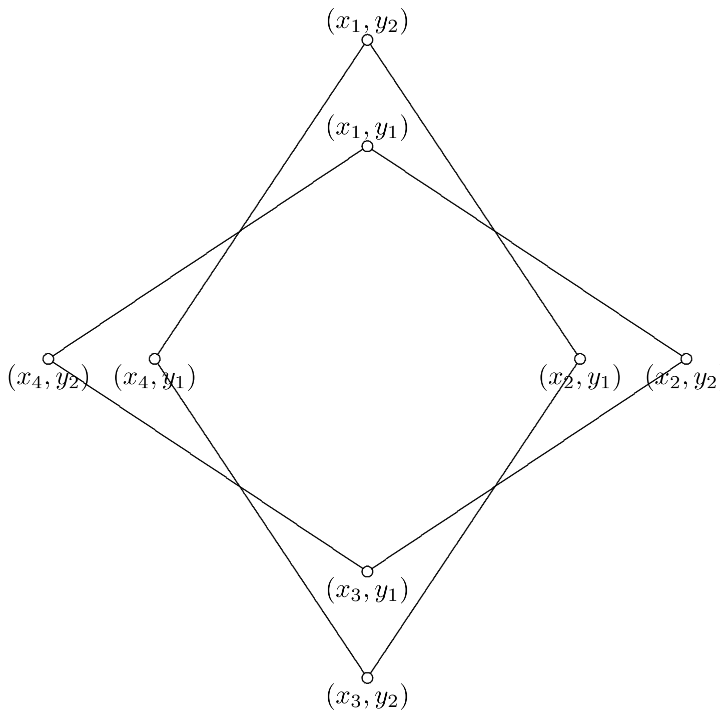


Figure 19: The conjunction $C_4 \wedge P_2$

The adjacency matrix of the conjunction of two graphs can be obtained by performing an operation on matrices called the Kronecker product. Below is its definition and an example.

Definition 4.7. Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix and $\mathbf{B} = [b_{ij}]$ be a $p \times q$ matrix. The *Kronecker product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is an $mp \times nq$ matrix defined by

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \dots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{bmatrix}.$$

Example 4.6. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix}.$$

Then,

$$\mathbf{A}\otimes\mathbf{B} = \begin{bmatrix} 1\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} & 2\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} & 3\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} \\ 4\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} & 5\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} & 6\begin{bmatrix} 7 & 8 \\ 9 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 7 & 8 & 14 & 16 & 21 & 24 \\ 9 & 0 & 18 & 0 & 27 & 0 \\ 28 & 32 & 35 & 40 & 42 & 48 \\ 36 & 0 & 45 & 0 & 56 & 0 \end{bmatrix}.$$

Remark 4.3. It can be shown that the adjacency matrix of the conjunction given in Example 4.5 can be expressed as

$$\mathcal{A}(C_4 \wedge P_2) = \mathcal{A}(P_2) \otimes \mathcal{A}(C_4).$$

In general if G and H are graphs, then

$$\mathcal{A}(G \wedge H) = \mathcal{A}(H) \otimes \mathcal{A}(G).$$

5 More on special classes of graphs

5.1 Trees

Definition 5.1. A *tree* is a connected graph with no cycles.

Example 5.1. Give all distinct trees with orders 1,2,3,4,5,6,7.

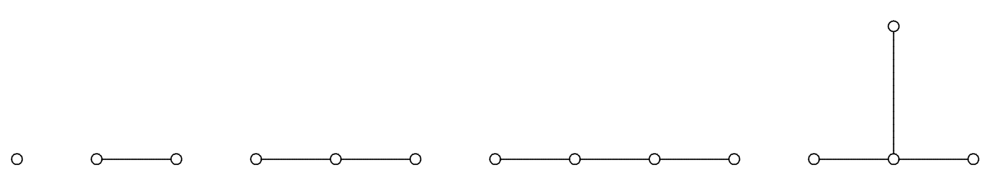


Figure 20: Trees of orders 1, 2, 3, 4

Remark 5.1. A graph with no cycles is called an *acyclic* graph. Thus, we can say that a tree is a connected acyclic graph. Furthermore, a graph (not necessarily connected) with no cycles is called a *forest*. This implies that the components of a forest are trees.

Theorem 5.1. *Let $G = (V, E)$ be a graph. The following statements are equivalent:*

1. *G is a tree.*