Contents

Chapter 1

Real Analysis

(1.5)
$OrderTrichotomy[<, S] := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$
$OrderTransitivity[<, S] := \forall_{x,y,z \in S}((x < y \wedge y < z) \implies x < z)$
$Order[<, S] := (OrderTrichotomy[<, S]) \wedge (OrderTransitivity[<, S])$

(1.7)
$BoundedAbove[E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge (\exists_{\beta \in S} \forall_{x \in E}(x \leq \beta))$
$BoundedBelow[E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge (\exists_{\beta \in S} \forall_{x \in E}(\beta \leq x))$
$UpperBound[\beta, E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E}(x \leq \beta))$
$LowerBound[\beta, E, S, <] := (Order[<, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x \in E}(\beta \leq x))$

(1.8)
$LUB[\alpha, E, S, <] := (UpperBound[\alpha, E, S, <]) \wedge (\forall_{\gamma}(\gamma < \alpha \implies \neg UpperBound[\gamma, E, S, <]))$
$GLB[\alpha, E, S, <] := (LowerBound[\alpha, E, S, <]) \wedge (\forall_{\beta}(\alpha < \beta \implies \neg LowerBound[\beta, E, S, <]))$

(1.10)
$LUBProperty[S, <] := \forall_{E}(((\emptyset \neq E \subset S) \wedge (BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <])))$
$GLBProperty[S, <] := \forall_{E}(((\emptyset \neq E \subset S) \wedge (BoundedBelow[E, S, <]) \implies \exists_{\alpha \in S}(GLB[\alpha, E, S, <])))$

(1.11)
$LUBPropertyImpliesGLBProperty := LUBProperty[S, <] \implies GLBProperty[S, <]$

(1)	$LUBProperty[S, <] \implies \ldots$	wts: 2
(1.1)	$(\emptyset \neq B \subset S \wedge BoundedBelow[B, S, <]) \implies \ldots$	wts: 1.2
(1.1.1)	$Order[<, S] \wedge \exists_{\delta' \in S}(LowerBound[\delta', B, S, <])$	from: $BoundedBelow$, 1.1
(1.1.2)	$\lvert B \rvert = 1 \implies \ldots$	wts: 1.1.3
(1.1.2.1)	$\exists_{u'}(u' \in B)$ ▌ $u := choice(\{u' : u' \in B\})$ ▌ $B = \{u\}$	from: 1.1.2
(1.1.2.2)	$GLB[u, B, S, <]$ ▌ $\exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$	
(1.1.3)	$\lvert B \rvert = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$	
(1.1.4)	$\lvert B \rvert \neq 1 \implies \ldots$	wts: 1.1.5
(1.1.4.1)	$\forall_{E}((\emptyset \neq E \subset S \wedge BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <]))$	from: $LUBProperty$, 1
(1.1.4.2)	$L := \{s \in S : LowerBound[s, B, S, <]\}$	
(1.1.4.3)	$\lvert B \rvert > 1 \wedge OrderTrichotomy[<, S]$ ▌ $\exists_{b_1' \in B} \exists_{b_0' \in B}(b_0' < b_1')$	from: $Order$, 1.1.1 wts: 1.1.4.7
(1.1.4.4)	$b_1 := choice(\{b_1' \in B : \exists_{b_0' \in B}(b_0' < b_1')\})$ ▌ $\neg LowerBound[b_1, B, S, <]$	from: 1.1.4.2
(1.1.4.5)	$b_1 \notin L$ ▌ $L \subset S$	
(1.1.4.6)	$\delta := choice(\{\delta' \in S : LowerBound[\delta', B, S, <]\})$ ▌ $\delta \in L$ ▌ $\emptyset \neq L$	from: 1.1.1
(1.1.4.7)	$\emptyset \neq L \subset S$	from: 1.1.4.5, 1.1.4.6
(1.1.4.8)	$\forall_{y \in L}(LowerBound[y_0, B, S, <])$ ▌ $\forall_{y \in L} \forall_{x \in B}(y_0 \leq x)$	from: $LowerBound$, 1.1.4.2 wts: 1.1.4.10
(1.1.4.9)	$\forall_{x \in B}(x \in S \wedge \forall_{y \in L}(y_0 \leq x))$ ▌ $\forall_{x \in B}(UpperBound[x, L, S, <])$	from: $UpperBound$
(1.1.4.10)	$\exists_{x \in S}(UpperBound[x, L, S, <])$ ▌ $BoundedAbove[L, S, <]$	

(1.1.4.11) $\emptyset \neq L \subset S \wedge BoundedAbove[L, S, <]$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha' \in S}(LUB[\alpha', L, S, <])$ ▌ $\alpha := choice(\{\alpha' \in S : (LUB[\alpha', L, S, <])\})$ from: 1.1.4.1 wts: 1.1.4.21

(1.1.4.13) $\forall_x(x \in B \implies UpperBound[x, L, S, <])$ from: 1.1.4.9 wts: 1.1.4.17

(1.1.4.14) $\forall_x(\neg UpperBound[x, L, S, <] \implies x \notin B)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$ wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound[\gamma, L, S, <]$ ▌ $\gamma \notin B$ from: LUB, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B$ ▌ $\gamma \in B \implies \gamma \geq \alpha$

(1.1.4.17) $\forall_{\gamma \in B}(\alpha \leq \gamma)$ ▌ $LowerBound[\alpha, B, S, <]$ from: LowerBound

(1.1.4.18) $\alpha < \beta \implies \ldots$ wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y \in L}(y_0 \leq \alpha < \beta)$ ▌ $\forall_{y \in L}(y_0 \neq \beta)$ from: LUB, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta \notin L$ ▌ $\neg LowerBound[\beta, B, S, <]$ from: 1.1.4.2

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound[\beta, B, S, <]$ ▌ $\forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.20) $LowerBound[\alpha, B, S, <] \wedge \forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB[\alpha, B, S, <]$ ▌ $\exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.5) $|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.6) $(|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])) \wedge (|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <]))$ from: 1.1.3, 1.1.5

(1.1.7) $(|B| = 1 \vee |B| \neq 1) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$ ▌ $\exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.2) $(\emptyset \neq B \subset S \wedge BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.3) $\forall_B((\emptyset \neq B \subset S \wedge BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <]))$

(1.4) $GLBProperty[S, <]$

(2) $LUBProperty[S, <] \implies GLBProperty[S, <]$

(1.12)

$$Field[F, +, *] := \exists_{0,1 \in F}\forall_{x,y,z \in F}\begin{pmatrix} x + y \in F & \wedge & x * y \in F & \wedge \\ x + y = y + x & \wedge & x * y = y * x & \wedge \\ (x + y) + z = x + (y_0 + z) & \wedge & (x * y) * z = x * (y_0 * z) & \wedge \\ \mathbb{1} \neq \mathbb{0} & \wedge & x * (y_0 + z) = (x * y) + (x * z) & \wedge \\ \mathbb{0} + x = x & \wedge & \mathbb{1} * x = x & \wedge \\ \multicolumn{4}{l}{\exists_{-x \in F}(x + (-x) = \mathbb{0}) \wedge (x \neq \mathbb{0} \implies \exists_{1/x \in F}(x * (1/x) = \mathbb{1}))} \end{pmatrix}$$

*********************** $(Field[F, +, *] \wedge x, y, z \in F) \implies \ldots$ ***********************

(1.14)

$AdditiveCancellation := (x + y = x + z) \implies y = z$

(1) $y = \mathbb{0} + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \ldots$ from: Field

(2) $(-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = \mathbb{0} + z = z$ from: Field

$AdditiveIdentityUniqueness := (x + y = x) \implies y = \mathbb{0}$

(1) $x + y = x = \mathbb{0} + x = x + \mathbb{0}$ from: Field

(2) $y = 0$ from: AdditiveCancellation

$AdditiveInverseUniqueness := (x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$ from: Field

(2) $y = -x$ from: AdditiveCancellation

$DoubleNegative := x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x$ ▌ $\mathbb{0} = (-x) + x$ from: Field

(2) $x = -(-x)$ from: AdditiveInverseUniqueness

(1.15)

$MultiplicativeCancellation := (x \neq 0 \wedge x * y = x * z) \implies y = z$ —
$MultiplicativeIdentityUniqueness := (x \neq 0 \wedge x * y = x) \implies y = \mathbb{1}$ —
$MultiplicativeInverseUniqueness := (x \neq 0 \wedge x * y = \mathbb{1}) \implies y = 1/x$ —
$DoubleReciprocal := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)
$Domination := \mathbb{0} * x = \mathbb{0}$

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x$ █ $\mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$ — from: $Field$

(2) $\mathbb{0} * x = \mathbb{0}$ — from: $AdditiveIdentityUniqueness$

$NonDomination := (x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies \ldots$

(1.1) $(x * y = 0) \implies \ldots$

(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = \mathbb{0} * ((1/x) * (1/y)) = \mathbb{0}$ — from: $Field$, $Domination$, 1, 1.1

(1.1.2) $\mathbb{1} = \mathbb{0} \wedge \mathbb{1} \neq \mathbb{0}$ █ $\perp$ — from: $Field$

(1.2) $(x * y = 0) \implies \perp$ █ $x * y \neq 0$

(2) $(x \neq \mathbb{0} \wedge y \neq \mathbb{0}) \implies x * y \neq 0$

$NegationCommutativity := (-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0}$ █ $x * y + (-x) * y = \mathbb{0}$ — from: $Field$, $Domination$; wts: 2

(2) $(-x) * y = -(x * y)$ — from: $AdditiveInverseUniqueness$

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * \mathbb{0} = \mathbb{0}$ █ $x * y + x * (-y) = \mathbb{0}$ — from: $Field$, $Domination$; wts: 4

(4) $x * (-y) = -(x * y)$ — from: $AdditiveInverseUniqueness$

(5) $(-x) * y = -(x * y) = x * (-y)$ — from: 2, 4

$NegativeMultiplication := (-x) * (-y) = x * y$

(1) $(-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$ — from: $NegationCommutativity$, $DoubleNegative$

(1.17)

$$OrderedField[F, +, *, <] := \left(\begin{array}{l} Field[F, +, *] \quad \wedge \quad Order[<, F] \quad \wedge \\ \forall_{x,y,z \in F}(y_0 < z \implies x + y < x + z) \wedge \\ \forall_{x,y \in F}((x > \mathbb{0} \wedge y > \mathbb{0}) \implies x * y > \mathbb{0}) \end{array} \right)$$

************************ $(OrderedField[F, +, *, <] \wedge x, y, z \in F) \implies \ldots$ ************************

(1.18)
$NegationOnOrder := x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \ldots$

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x$ █ $\mathbb{0} > -x$ █ $-x < \mathbb{0}$ — from: $OrderedField$

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \ldots$

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x$ █ $\mathbb{0} < x$ █ $x > \mathbb{0}$ — from: $OrderedField$

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \wedge -x < \mathbb{0} \implies x > \mathbb{0}$ █ $x > \mathbb{0} \iff -x < \mathbb{0}$ — from: 2, 4

$PositiveFactorPreservesOrder := (x > \mathbb{0} \wedge y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \wedge y < z) \implies \ldots$

(1.1) $(-y) + z > (-y) + y = \mathbb{0}$ █ $z + (-y) = \mathbb{0}$ — from: $OrderedField$

(1.2) $x * (z + (-y)) > \mathbb{0}$ █ $x * z + x * (-y) > \mathbb{0}$ — from: $OrderedField$

(1.3) $x * z = \mathbb{0} + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \ldots$ — from: $Field$, $NegationCommutativity$

(1.4) $x * y + (x * z + x * (-y)) > x * y + \mathbb{0} = x * y$ — from: $Field$, 1.2

(1.5) $x * z > x * y$ — from: 1.3, 1.4

(2) $(x > \mathbb{0} \wedge y < z) \implies x * z > x * y$

$NegativeFactorFlipsOrder := (x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \wedge y < z) \implies \ldots$

(1.1) $-x > \mathbb{0}$ — from: $NegationOnOrder$

(1.2) $(-x) * y < (-x) * z$ ▮ $\mathbb{0} = x * y + (-x) * y < x * y + (-x) * z$ ▮ $\mathbb{0} < x * y + (-x) * z$ — from: $PositiveFactorPreservesOrder$

(1.3) $\mathbb{0} < (-x) * (-y + z)$ ▮ $\mathbb{0} > x * (-y + z)$ ▮ $\mathbb{0} > -(x * y) + x * z$ — from: $NegationOnOrder$

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

$SquareIsPositive := (x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ — from: $OrderedField$

(2) $(x < \mathbb{0}) \implies \ldots$

(2.1) $-x > \mathbb{0}$ ▮ $x * x = (-x) * (-x) > \mathbb{0}$ ▮ $x * x > \mathbb{0}$ — from: $NegationOnOrder, OrderedField, NegativeMultiplication$

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \vee x < \mathbb{0}) \implies x * x > \mathbb{0}$ ▮ $x \neq \mathbb{0} \implies x * x > \mathbb{0}$ — from: $OrderTrichotomy$, 1, 3

$OneIsPositive := \mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0}$ ▮ $\mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ — from: $Field, SquareIsPositive$

$ReciprocationOnOrder := (0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \ldots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0}$ ▮ $x * (1/x) > 0$ — from: $Field, OneIsPositive$

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp$ ▮ $1/x > \mathbb{0}$ — from: $NegativeFactorFlipsOrder$, 1

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0}$ ▮ $y * (1/y) > 0$ — from: $Field, OneIsPositive$

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp$ ▮ $1/y > \mathbb{0}$ — from: $NegativeFactorFlipsOrder$, 1

(1.5) $(1/x) * (1/y) > \mathbb{0}$ — from: $OrderedField$

(1.6) $\mathbb{0} < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$ — from: $OrderedField$, 1, 1.4, 1.5

(1.19)
$OrderedFieldQ := OrderedField[\mathbb{Q}, +, *, <]$ —

$Subfield[K, F, +, *] := Field[F, +, *] \wedge K \subset F \wedge Field[K, +, *]$
$OrderedSubfield[K, F, +, *, <] := OrderedField[F, +, *, <] \wedge K \subset F \wedge OrderedField[K, +, *, <]$
$CutI[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$
$CutII[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
$CutIII[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} : CutI[\alpha] \wedge CutII[\alpha] \wedge CutIII[\alpha]\}$

$CutCorollaryI := (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \ldots$

(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ — from: $CutII$, 1

(1.2) $q < p \implies q \in \alpha$ ▮ $q \notin \alpha \implies q \geq p$ — from: 1

(1.3) $(q \notin \alpha) \implies \ldots$

(1.3.1) $q \geq p$ — from: 1.2

(1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp$ ▮ $q \neq p$ — from: 1, 1.3

(1.3.3) $q \geq p \wedge q \neq p$ ▮ $p < q$

(1.4) $q \notin \alpha \implies p < q$ ▮ $p < q$ — from: 1

(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

$\color{red}{CutCorollaryII} := (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$

(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ — from: *CutII*, 1

(1.2) $s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)) \blacksquare s \in \alpha \implies r \in \alpha$ — from: 1, 1.1

(1.3) $r \notin \alpha \implies s \notin \alpha \blacksquare s \notin \alpha$ — from: 1, 1.2

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$\color{red}{<_{\mathbb{R}}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$\color{red}{OrderTrichotomyOfR} := OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$

(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$ — from: $<_{\mathbb{R}}$, 1.1

(1.1.2) $\exists_{p'}(p' \in \alpha \wedge p' \notin \beta) \blacksquare p := choice(\{p' : p' \in \alpha \wedge p' \notin \beta\})$

(1.1.3) $q \in \beta \implies \ldots$

(1.1.3.1) $p, q \in \mathbb{Q}$

(1.1.3.2) $q < p$ — from: *CutCorollaryI*

(1.1.3.3) $q \in \alpha$ — from: *CutII*

(1.1.4) $q \in \beta \implies q \in \alpha$

(1.1.5) $\forall_{q \in \beta}(q \in \alpha) \blacksquare \beta \subseteq \alpha$

(1.1.6) $\beta \subset \alpha \blacksquare \beta <_{\mathbb{R}} \alpha$

(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \blacksquare (\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

$\color{red}{OrderTransitivityOfR} := OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a \in \alpha}(a \in \beta) \wedge \forall_{b \in \beta}(b \in \gamma)$

(1.1.3) $\forall_{a \in \alpha}(\alpha \in \gamma) \blacksquare \alpha \subset \gamma \blacksquare \alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}}((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(4) $OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

$\color{red}{OrderOfR} := Order[<_{\mathbb{R}}, \mathbb{R}]$ — from: *OrderTrichotomyR*, *OrderTransitivityR*

$\color{red}{LUBPropertyOfR} := LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$ — wts:

(1) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset \blacksquare \exists_{\alpha}(\alpha \in A) \blacksquare \alpha_0 := choice(\{\alpha : \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset \blacksquare \exists_{a}(a \in \alpha_0) \blacksquare a_0 := choice(\{a : a \in \alpha_0\}) \blacksquare a_0 \in \gamma \blacksquare \gamma \neq \emptyset$

(1.4) $BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare \exists_{\beta}(UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.5) $\beta_0 := choice(\{\beta : UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}]\})$

(1.6) $UpperBound[\beta_0, A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \beta_0) \blacksquare \forall_{\alpha \in A}(\alpha \subseteq \beta_0) \blacksquare \forall_{\alpha \in A}\forall_{a \in \alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma \blacksquare \forall_{a \in \gamma}(a \in \beta_0) \blacksquare \gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q} \blacksquare \gamma \subseteq \beta_0 \subset \mathbb{Q} \blacksquare \gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q} \blacksquare CutI[\gamma]$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.10.1) $p \in \gamma \blacksquare \exists_{\alpha \in A}(p \in \alpha) \blacksquare \alpha_1 := choice(\{\alpha \in A : p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p \blacksquare q \in \alpha_1 \blacksquare q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma \blacksquare \forall_{p \in \gamma}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma) \blacksquare CutII[\gamma]$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha \in A}(p \in \alpha) \blacksquare \alpha_2 := choice(\{\alpha \in A : p \in \alpha\})$

(1.12.2) $\alpha_2 \in \mathbb{R} \blacksquare CutII[\alpha_2] \blacksquare \exists_{r \in \alpha_2}(p < r) \blacksquare r_0 := choice(\{r \in \alpha_2 : p < r\})$

(1.12.3) $r_0 \in \alpha_2 \blacksquare r_0 \in \gamma$

(1.12.4) $p < r_0 \blacksquare p < r_0 \wedge r_0 \in \gamma \blacksquare \exists_{r \in \gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r \in \gamma}(p < r) \blacksquare \forall_{p \in \gamma}\exists_{r \in \gamma}(p < r) \blacksquare CutIII[\gamma]$

(1.14) $CutI[\gamma] \wedge CutII[\gamma] \wedge CutIII[\gamma] \blacksquare \gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha \in A}(\alpha \subseteq \gamma) \blacksquare \forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma)$

(1.16) $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R} \blacksquare UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.17) $\delta <_{\mathbb{R}} \gamma \implies \ldots$

(1.17.1) $\delta \subset \gamma \blacksquare \exists_s(s \in \gamma \wedge s \notin \delta) \blacksquare s_0 := choice(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$

(1.17.2) $s_0 \in \gamma \blacksquare \exists_{\alpha \in A}(s_0 \in \alpha) \blacksquare \alpha_3 := choice(\{\alpha \in A : s_0 \in \alpha\})$

(1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta \blacksquare \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \ldots$

(1.17.4.1) $\alpha_3 \subseteq \delta \blacksquare \forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta) \blacksquare \neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4.2) $\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \blacksquare \perp$

(1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \perp \blacksquare \delta <_{\mathbb{R}} \alpha_3 \blacksquare \exists_{\alpha \in A}(\delta <_{\mathbb{R}} \alpha) \blacksquare \exists_{\alpha \in A}(\neg(\alpha \leq_{\mathbb{R}} \delta))$

(1.17.6) $\neg\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \delta) \blacksquare \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}]) \blacksquare \forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.19) $UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \wedge \forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.20) $LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(3) $\forall_A((\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])) \blacksquare LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

$+_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

$0_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 0\}$

$ZeroInR := 0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}} \blacksquare \emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q} \blacksquare CutI[0_{\mathbb{R}}]$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}} \blacksquare \forall_{x \in 0_{\mathbb{R}}}\forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}}) \blacksquare CutII[0_{\mathbb{R}}]$

(3) $y := x/2 \blacksquare (x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y) \blacksquare \forall_{x \in 0_{\mathbb{R}}}\exists_{y \in 0_{\mathbb{R}}}(x < y) \blacksquare CutIII[0_{\mathbb{R}}]$

(4) $CutI[0_{\mathbb{R}}] \wedge CutII[0_{\mathbb{R}}] \wedge CutIII[0_{\mathbb{R}}] \blacksquare 0_{\mathbb{R}} \in \mathbb{R}$

$FieldAdditionClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha)\, ;\, \exists_b(b \in \beta) \blacksquare a_0 := choice(\{a : a \in \alpha\})\, ;\, b_0 := choice(\{b : b \in \beta\}) \blacksquare a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha)\, ;\, \exists_y(y_0 \notin \beta) \blacksquare x_0 := choice(\{x : x \notin \alpha\})\, ;\, y_0 := choice(\{y : y \notin \beta\})$

(1.5) $\forall_{r \in \alpha}(r < x_0)\, ;\, \forall_{s \in \beta}(s < y_0) \blacksquare \forall_{r \in \alpha}\forall_{s \in \beta}(r + s < x_0 + y_0) \blacksquare x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q} \blacksquare CutI[\alpha +_{\mathbb{R}} \beta]$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.7.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s) \;\blacksquare\; (r_0, s_0) := choice((r, s) \in \alpha \times \beta : p = r + s)$

(1.7.2) $q < p = r_0 + s_0 \;\blacksquare\; (q - s_0) < r_0 \;\blacksquare\; (q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta \;\blacksquare\; q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta \;\blacksquare\; q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta \;\blacksquare\; \forall_{p\in\alpha+_{\mathbb{R}}\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta) \;\blacksquare\; CutII[\alpha +_{\mathbb{R}} \beta]$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r\in\alpha}\exists_{s\in\beta}(p = r + s) \;\blacksquare\; (r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta : p = r + s\})$

(1.9.2) $r_1 \in \alpha \;\blacksquare\; \exists_{t\in\alpha}(r_1 < t) \;\blacksquare\; t_0 := choice(\{t \in \alpha : r_1 < t\})$

(1.9.3) $s_1 \in \beta \;\blacksquare\; t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1 \;\blacksquare\; \exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r) \;\blacksquare\; \forall_{p\in\alpha+_{\mathbb{R}}\beta}\exists_{r\in\alpha+_{\mathbb{R}}\beta}(p < r) \;\blacksquare\; CutIII[\alpha +_{\mathbb{R}} \beta]$

(1.11) $CutI[\alpha +_{\mathbb{R}} \beta] \wedge CutII[\alpha +_{\mathbb{R}} \beta] \wedge CutIII[\alpha +_{\mathbb{R}} \beta] \;\blacksquare\; \alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

$FieldAdditionCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s : r \in \alpha \wedge s \in \beta\} = \{s + r : s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

$FieldAdditionAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma))$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

$FieldAdditionIdentityOfR := (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \ldots$

(1.1.1) $s < 0 \;\blacksquare\; r + s < r + 0 = r \;\blacksquare\; r + s < r \;\blacksquare\; r + s \in \alpha$

(1.2) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha \;\blacksquare\; \forall_{r\in\alpha}\forall_{s\in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \;\blacksquare\; \forall_{p\in\alpha+_{\mathbb{R}}0_{\mathbb{R}}}(p \in \alpha) \;\blacksquare\; \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r\in\alpha}(p < r) \;\blacksquare\; r_2 := choice(\{r \in \alpha : p < r\})$

(1.4.2) $p < r_2 \;\blacksquare\; p - r_2 < r_2 - r_2 = 0 \;\blacksquare\; (p - r_2) < 0 \;\blacksquare\; (p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha \;\blacksquare\; p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; \forall_{p\in\alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}) \;\blacksquare\; \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}} \;\blacksquare\; 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

$FieldAdditionInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} : \exists_{r>0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q} \;\blacksquare\; \exists_{s\in\mathbb{Q}}(s \notin \alpha) \;\blacksquare\; s_0 := choice(\{s : s \notin \alpha\}) \;\blacksquare\; p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha \;\blacksquare\; -p_0 - 1 \notin \alpha \;\blacksquare\; \exists_{r>0}(-p_0 - r \notin \alpha) \;\blacksquare\; p_0 \in \beta$

(1.4) $\emptyset \neq \alpha \;\blacksquare\; \exists_{q\in\alpha} \;\blacksquare\; q_0 := choice(\{q \in \mathbb{Q} : q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha \;\blacksquare\; -(-q_0) - r = q_0 - r < q_0 \;\blacksquare\; -(-q_0) - r < q_0 \;\blacksquare\; -(-q_0) - r \in \alpha$

(1.6) $\forall_{r>0}(-(-q_0) - r \in \alpha) \;\blacksquare\; \neg\exists_{r>0}(-(-q_0) - r \notin \alpha) \;\blacksquare\; -q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q} \;\blacksquare\; CutI[\beta]$

(1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.8.1) $p \in \beta \;\blacksquare\; \exists_{r>0}(-p - r \notin \alpha) \;\blacksquare\; r_0 := choice(\{r > 0 : -p - r \notin \alpha\})$

(1.8.2) $q < p \;\blacksquare\; -p - r < -q - r$

(1.8.3) $-q - r \notin \alpha \;\blacksquare\; q \in \beta$

(1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta \;\blacksquare\; \forall_{p\in\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \beta) \;\blacksquare\; CutII[\beta]$

(1.10) $p \in \beta \implies \ldots$

(1.10.1) $p \in \beta$ ▌ $\exists_{r>0}(-p - r \notin \alpha)$ ▌ $r_1 := choice(\{r > 0 : -p - r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0$ ▌ $r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p$ ▌ $t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha$ ▌ $-t_0 - (r_1/2) \notin \alpha$ ▌ $\exists_{r>0}(-t_0 - r \notin \alpha)$ ▌ $t_0 \in \beta$

(1.10.7) $t_0 > p \wedge t_0 \in \beta$ ▌ $\exists_{t\in\beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t\in\beta}(p < t)$ ▌ $\forall_{p\in\beta}\exists_{t\in\beta}(p < t)$ ▌ $CutIII[\beta]$

(1.12) $CutI[\beta] \wedge CutII[\beta] \wedge CutIII[\beta]$ ▌ $\beta \in \mathbb{R}$

(1.13) $(r \in \alpha \wedge s \in \beta) \implies \ldots$

(1.13.1) $s \in \beta$ ▌ $\exists_{t>0}(-s - t \notin \alpha)$ ▌ $t_1 := choice(\{t > 0 : -s - t \notin \alpha\})$ ▌ $-s - t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha$ ▌ $-s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha$ ▌ $r < -s$ ▌ $r + s < 0$ ▌ $r + s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}}$ ▌ $\forall_{(r,s)\in\alpha\times\beta}(r + s \in 0_{\mathbb{R}})$ ▌ $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \ldots$

(1.15.1) $v < 0$ ▌ $w_0 := -v/2$ ▌ $w > 0$

(1.15.2) $\exists_{n\in\mathbb{Z}}(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha)$ ▌ $n_0 := choice(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$ from: $ARCHIMEDEANPROPERTYOFQ + LUB$???

(1.15.3) $p_0 := -(n_0 + 2)w_0$ ▌ $-p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha$ ▌ $-p_0 - w_0 \notin \alpha$ ▌ $p_0 \in \beta$

(1.15.4) $n_0w_0 \in \alpha \wedge p_0 \in \beta$ ▌ $n_0w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{v\in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta)$ ▌ $0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$ ▌ $\alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$ ▌ $\exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

$*_{\mathbb{R}}[\alpha, \beta] :=$ —

$1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$1IsNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ —

$1InR := 1_{\mathbb{R}} \in \mathbb{R}$ —

$FieldMultiplicationClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R})$ —

$FieldMultiplicationCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ —

$FieldMultiplicationAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma))$ —

$FieldMultiplicationIdentityOfR := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ —

$FieldMultiplicationInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha\in\mathbb{R}}(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}})$ —

$FieldDistributativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ —

$FieldWithR := Field[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}]$ —

$OrderedFieldWithR := OrderedField[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$

$QROrderedSubfieldOfR := OrderedSubfield[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$QIsomorphicToQR := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ —

$CompletenessOfR := \exists_{\mathbb{R}}(LUBProperty[\mathbb{R}, <_{\mathbb{R}}] \wedge OrderedSubfield[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}])$ —

(1.20)

$ArchimedeanPropertyOfR := \forall_{x,y\in\mathbb{R}}(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y))$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx : n \in \mathbb{N}^+\}$ ▌ $(\emptyset \neq A \subset \mathbb{R}) \wedge (a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a))$

(1.2) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n\in\mathbb{N}^+}(nx > y)$ ▌ $\forall_{n\in\mathbb{N}^+}(nx \leq y)$ ▌ $UpperBound[y_0, A, \mathbb{R}, <]$ ▌ $BoundedAbove[A, \mathbb{R}, <]$

(1.2.2) $CompletenessOfR$ ▌ $LUBProperty[\mathbb{R}, <]$

(1.2.3) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (BoundedAbove[A, \mathbb{R}, <])$ ▌ $\exists_{\alpha\in\mathbb{R}}(LUB[\alpha, A, \mathbb{R}, <])$ . . .

(1.2.4) $\ldots \alpha_0 := choice(\{\alpha \in \mathbb{R} : LUB[\alpha, A, \mathbb{R}, <]\})$ ▌ $LUB[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0$ ▌ $\alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge (LUB[\alpha_0, A, \mathbb{R}, <])$ ▌ $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$ ▌ $\exists_{c \in A}(\alpha_0 - x < c)\ \ldots$

(1.2.8) $\ldots c_0 := choice(\{c \in A : \alpha_0 - x < c\})$ ▌ $(c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$ ▌ $\exists_{m \in \mathbb{N}^+}(mx = c_0)\ \ldots$

(1.2.10) $\ldots m_0 := choice(\{m \in \mathbb{N}^+ : mx = c_0\})$ ▌ $(m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0)$ ▌ $\alpha_0 - x < c_0 = m_0 x$ ▌ $\alpha_0 < m_0 x + x$ ▌ $\alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+$ ▌ $m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$ ▌ $(m_0 + 1)x \in A$

(1.2.14) $(\alpha_0 < (m_0 + 1)x) \wedge ((m_0 + 1)x \in A)$ ▌ $\exists_{c \in A}(\alpha_0 < c)$

(1.2.15) $LUB[\alpha_0, A, \mathbb{R}, <]$ ▌ $UpperBound[\alpha_0, A, \mathbb{R}, <]$ ▌ $\forall_{c \in A}(c \le \alpha_0)$ ▌ $\neg\exists_{c \in A}(c > \alpha_0)$ ▌ $\neg\exists_{c \in A}(\alpha_0 < c)$

(1.2.16) $(\exists_{c \in A}(\alpha_0 < c)) \wedge (\neg\exists_{c \in A}(\alpha_0 < c))$ ▌ $\perp$

(1.3) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \perp$ ▌ $\exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y)$ ▌ $\forall_{x, y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

$QDenseInR := \forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y$ ▌ $(0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $ArchimedeanPropertyOfR \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R})$ ▌ $\exists_{n \in \mathbb{N}^+}(n(y - x) > 1)\ \ldots$

(1.3) $\ldots n_0 := choice(\{n \in \mathbb{N}^+ : n(y - x) > 1\})$ ▌ $(n_0 \in \mathbb{N}^+) \wedge (n_0(y - x) > 1)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R})$ ▌ $n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m \in \mathbb{N}^+}(m(1) > n_0 x)\ \ldots$

(1.6) $\ldots m_1 := choice(\{m \in \mathbb{N}^+ : m(1) > n_0 x\})$ ▌ $(m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m \in \mathbb{N}^+}(m(1) > -n_0 x)\ \ldots$

(1.8) $\ldots m_2 := choice(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\})$ ▌ $(m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x)$ ▌ $-m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+$ ▌ $|m_1 - (-m_2)| \ge 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge (|m_1 - (-m_2)| \ge 2)$ ▌ $\exists_{m \in \mathbb{Z}}((-m_2 < m < m_1) \wedge (m - 1 \le n_0 x < m))\ \ldots$

(1.12) $\ldots m_0 := choice(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \le n_0 x < m)\})$ ▌ $(-m_2 < m_0 < m_1) \wedge (m_0 - 1 \le n_0 x < m_0)$

(1.13) $(n_0(y - x) > 1) \wedge (m_0 - 1 \le n_0 x < m_0)$ ▌ $n_0 x < m_0 \le 1 + n_0 x < n_0 y$ ▌ $n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y)$ ▌ $x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z}$ ▌ $m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y)$ ▌ $\exists_{p \in \mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y)$ ▌ $\forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1.21)
$RootLemma := (0 < a < b) \implies (b^n - a^n \le (b - a)nb^{n-1})$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b$ ▌ $b/a > 1$

(1.3) $b/a > 1$ ▌ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}(b^{n-i}a^{i-1}(b/a)^{i-1}) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$ ▌ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le (b - a)nb^{n-1}$ ▌ $b^n - a^n \le (b - a)nb^{n-1}$

(2) $(0 < a < b) \implies (b^n - a^n \le (b - a)nb^{n-1})$

$RootExistenceInR := \forall_{0 < x \in \mathbb{R}}\forall_{0 < n \in \mathbb{Z}}\exists!_{0 < y \in \mathbb{R}}({y_0}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\}$ ▌ $t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$

(1.2) $t_0 := x/(1 + x)$ ▌ $(t_0 = x/(1 + x)) \wedge (t_0 \in \mathbb{R})$

(1.3) $0 < x$ ▌ $0 < x < 1 + x$ ▌ $t_0 = x/(1 + x) > 0$ ▌ $t_0 > 0$

(1.4) $1 = (1+x)/(1+x) > x/(1+x) = t_0 \;\blacksquare\; 1 > t_0$

(1.5) $(t_0 > 0) \wedge (1 > t_0) \;\blacksquare\; 0 < t_0 < 1$

(1.6) $(0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1) \;\blacksquare\; {t_0}^n \leq t_0$

(1.7) $0 < x \;\blacksquare\; x > x/(1+x) = t_0 \;\blacksquare\; x > t_0$

(1.8) $({t_0}^n \leq t_0) \wedge (x > t_0) \;\blacksquare\; {t_0}^n < x$

(1.9) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge ({t_0}^n < x) \;\blacksquare\; t_0 \in E \;\blacksquare\; \emptyset \neq E$

(1.10) $t_1 := choice(\{t \in \mathbb{R} : t > 1 + x\}) \;\blacksquare\; (t_1 \in \mathbb{R}) \wedge (t_1 > 1 + x)$

(1.11) $x > 0 \;\blacksquare\; t_1 > 1 + x > 1 \;\blacksquare\; t_1 > 1 \;\blacksquare\; {t_1}^n \geq t_1$

(1.12) $({t_1}^n \geq t_1) \wedge (t_1 > 1 + x) \wedge (1 > 0) \;\blacksquare\; {t_1}^n \geq t_1 > 1 + x > x \;\blacksquare\; {t_1}^n > x$

(1.13) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ({t_1}^n > x) \;\blacksquare\; t_1 \notin E \;\blacksquare\; E \subset \mathbb{R}$

(1.14) $(\emptyset \neq E) \wedge (E \subset \mathbb{R}) \;\blacksquare\; \emptyset \neq E \subset \mathbb{R}$

(1.15) $t \in E \implies \ldots$

(1.15.1) $(t \in E) \wedge (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \;\blacksquare\; t^n < x$

(1.15.2) $({t_1}^n > x) \wedge (t^n < x) \;\blacksquare\; t^n < x < {t_1}^n \;\blacksquare\; t < t_1$

(1.16) $t \in E \implies t < t_1 \;\blacksquare\; \forall_{t \in E}(t \leq t_1) \;\blacksquare\; \mathit{UpperBound}[t_1, E, \mathbb{R}, <] \;\blacksquare\; \mathit{BoundedAbove}[E, \mathbb{R}, <]$

(1.17) $\mathit{CompletenessOfR} \;\blacksquare\; \mathit{LUBProperty}[\mathbb{R}, <]$

(1.18) $(\mathit{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (\mathit{BoundedAbove}[E, \mathbb{R}, <]) \;\blacksquare\; \exists_{y \in \mathbb{R}}(\mathit{LUB}[y, E, \mathbb{R}, <]) \;\ldots$

(1.19) $\ldots y_0 := choice(\{y \in \mathbb{R} : \mathit{LUB}[y, E, \mathbb{R}, <]\}) \;\blacksquare\; \mathit{LUB}[y_0, E, \mathbb{R}, <]$

(1.20) $(\mathit{LUB}[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0) \;\blacksquare\; 0 < t_0 \leq y_0 \in \mathbb{R} \;\blacksquare\; 0 < y_0 \in \mathbb{R}$

(1.21) ${y_0}^n < x \implies \ldots$

(1.21.1) $k_0 := \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} \;\blacksquare\; k_0 \in \mathbb{R}$

(1.21.2) ${y_0}^n < x \;\blacksquare\; 0 < x - {y_0}^n$

(1.21.3) $(n > 0) \wedge (y_0 > 0) \;\blacksquare\; 0 < n(y_0 + 1)^{n-1}$

(1.21.4) $(0 < x - {y_0}^n) \wedge (0 < n(y_0 + 1)^{n-1}) \;\blacksquare\; 0 < \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} = k_0 \;\blacksquare\; 0 < k_0$

(1.21.5) $(0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R}) \;\blacksquare\; 0 < min(1, k_0) \in \mathbb{R}$

(1.21.6) $\mathit{QDenseInR} \wedge (0, min(1, k_0) \in \mathbb{R}) \wedge (0 < min(1, k_0)) \;\blacksquare\; \exists_{h \in \mathbb{Q}}(0 < h < min(1, k_0)) \;\ldots$

(1.21.7) $\ldots h_0 := choice(\{h \in \mathbb{Q} : 0 < h < min(1, k_0)\}) \;\blacksquare\; (0 < h_0 < 1) \wedge (h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}})$

(1.21.8) $(y_0 > 0) \wedge (h_0 > 0) \;\blacksquare\; 0 < y_0 < y_0 + h_0$

(1.21.9) $\mathit{RootLemma} \wedge (0 < y_0 < y_0 + h_0) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}$

(1.21.10) $h_0 < 1 \;\blacksquare\; h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$

(1.21.11) $((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}) \wedge (h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}$

(1.21.12) $(0 < n(y_0 + 1)^{n-1}) \wedge (h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}) \;\blacksquare\; h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n$

(1.21.13) $((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}) \wedge (h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n) \;\blacksquare\; (y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \;\blacksquare\; (y_0 + h_0)^n < x$

(1.21.14) $(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n \;\blacksquare\; (y_0 + h_0)^n < x$

(1.21.15) $(0 < y_0 \mathbb{R}) \wedge (0 < h_0 < \mathbb{R}) \;\blacksquare\; 0 < y_0 < y_0 + h_0 \in \mathbb{R}$

(1.21.16) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ((y_0 + h_0)^n < x) \wedge (0 < y_0 + h_0 \in \mathbb{R}) \;\blacksquare\; (y_0 + h_0)^n \in E$

(1.21.17) $((y_0 + h_0)^n \in E) \wedge (y_0 < y_0 + h_0) \;\blacksquare\; \exists_{e \in E}(y_0 < e)$

(1.21.18) $\mathit{LUB}[y_0, E, \mathbb{R}, <] \;\blacksquare\; \mathit{UpperBound}[y_0, E, \mathbb{R}, <] \;\blacksquare\; \forall_{e \in E}(e \leq y_0) \;\blacksquare\; \neg\exists_{e \in E}(e > y_0)$

(1.21.19) $(\exists_{e \in E}(e > y_0)) \wedge (\neg\exists_{e \in E}(e > y_0)) \;\blacksquare\; \bot$

(1.22) ${y_0}^n < x \implies \bot \;\blacksquare\; {y_0}^n \geq x$

(1.23) ${y_0}^n > x \implies \ldots$

(1.23.1) $k_1 := \frac{{y_0}^n - x}{n{y_0}^{n-1}} \;\blacksquare\; (k_1 \in \mathbb{R}) \wedge (k_1 n{y_0}^{n-1} = {y_0}^n - x)$

(1.23.2) $(0 < x) \wedge (0 < n \in \mathbb{Z}) \;\blacksquare\; {y_0}^n - x < {y_0}^n \leq n{y_0}^n \;\blacksquare\; {y_0}^n - x < n{y_0}^n$

(1.23.3) ${y_0}^n - x < n{y_0}^n \;\blacksquare\; k_1 = \frac{{y_0}^n - x}{n{y_0}^{n-1}} < \frac{n{y_0}^n}{n{y_0}^{n-1}} = y_0 \;\blacksquare\; k_1 < y_0$

(1.23.4) ${y_0}^n > x \;\blacksquare\; 0 < {y_0}^n - x$

(1.23.5) $(n > 0) \wedge (y_0 > 0) \;\blacksquare\; 0 < n{y_0}^{n-1}$

(1.23.6) $(0 < {y_0}^n - x) \wedge 0 < (n{y_0}^{n-1}) \;\blacksquare\; 0 < \frac{{y_0}^n - x}{n{y_0}^{n-1}} = k_1 \;\blacksquare\; 0 < k_1$

(1.23.7) $(k_1 < y_0) \wedge (0 < k_1)$ ▌ $(0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$

(1.23.8) $t \geq y_0 - k_1 \implies \ldots$

(1.23.8.1) $t \geq y_0 - k_1$ ▌ $t^n \geq (y_0 - k_1)^n$ ▌ $-t^n \leq -(y_0 - k_1)^n$ ▌ ${y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n$

(1.23.8.2) $RootLemma \wedge (0 < y_0 - k_1 < y_0)$ ▌ ${y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}$

(1.23.8.3) $({y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n) \wedge ({y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1})$ ▌ ${y_0}^n - t^n < k_1 n {y_0}^{n-1}$

(1.23.8.4) $(k_1 n {y_0}^{n-1} = {y_0}^n - x) \wedge ({y_0}^n - t^n < k_1 n {y_0}^{n-1})$ ▌ ${y_0}^n - t^n < {y_0}^n - x$ ▌ $-t^n < -x$ ▌ $t^n > x$

(1.23.8.5) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t^n > x)$ ▌ $t \notin E$

(1.23.9) $t \geq y_0 - k_1 \implies t \notin E$ ▌ $t \in E \implies t < y_0 - k_1$ ▌ $\forall_{t \in E}(t \leq y_0 - k_1)$ ▌ $UpperBound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.10) $(LUB[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0))$ ▌ $\neg UpperBound[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.11) $(UpperBound[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg UpperBound[y_0 - k_1, E, \mathbb{R}, <])$ ▌ $\perp$

(1.24) ${y_0}^n > x \implies \perp$ ▌ ${y_0}^n \leq x$

(1.25) $Order[\mathbb{R}, <]$ ▌ $OrderTrichotomy[\mathbb{R}, <]$

(1.26) $(OrderTrichotomy[\mathbb{R}, <]) \wedge ({y_0}^n \geq x) \wedge ({y_0}^n \leq x)$ ▌ ${y_0}^n = x$

(1.27) $({y_0}^n = x) \wedge (y_0 \in \mathbb{R})$ ▌ $\exists_{y \in \mathbb{R}}(y^n = x)$

(1.28) $y_1, y_2 := choice(\{y \in \mathbb{R} : y^n = x\})$

(1.29) $y_1 \neq y_2 \implies \ldots$

(1.29.1) $(OrderTrichotomy[\mathbb{R}, <]) \wedge (y_1 \neq y_2)$ ▌ $(y_1 < y_2) \veebar (y_2 < y_1)$ . . .

(1.29.2) . . . $(x = {y_1}^n < {y_2}^n = x) \veebar (x = {y_2}^n < {y_1}^n = x)$ ▌ $(x < x) \veebar (x > x)$ ▌ $\perp \veebar \perp$ ▌ $\perp$

(1.30) $y_1 \neq y_2 \implies \perp$ ▌ $y_1 = y_2$ ▌ $\forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b)$

(1.31) $(\exists_{y \in \mathbb{R}}(y^n = x)) \wedge (\forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b))$ ▌ $\exists!_{y \in \mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}}(y^n = x)$ ▌ $\forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}}({y_0}^n = x)$

$RootExistenceInRCorollary := \forall_{0<a\in\mathbb{R}} \forall_{0<b\in\mathbb{R}} \forall_{0<n\in\mathbb{Z}}((ab)^{1/n} = a^{1/n} b^{1/n})$ —

$$ExtendedRealSystem[\bar{\mathbb{R}}, +, *, <] := \left(\begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \ \wedge \ x - \infty = -\infty \ \wedge \ \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \ \wedge \\ (x > 0) \implies (x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty) \wedge \\ (x < 0) \implies (x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty) \end{array} \right)$$

$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$
$+_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d \rangle$
$*_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c \rangle$

$FieldC := Field[\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}}]$ —
$RSubfieldC := Subfield[\mathbb{R}, \mathbb{C}, +, *]$ —

$i := \langle 0, 1 \rangle \in \mathbb{C}$

$iProperty := i^2 = -1$ —
$CProperty := (a, b \in \mathbb{R}) \implies (\langle a, b \rangle = a + bi)$ —

$Conjugate[\overline{a + bi}] := a - bi$

$ConjugateProperties := (w, z \in \mathbb{C}) \implies \ldots$ —

(1) $\overline{z + w} = \overline{z} + \overline{w}$

(2) $\overline{z * w} = \overline{z} * \overline{w}$

(3) $Re(z) = (1/2)(z + \overline{z}) \wedge Im(z) = (1/2)(z - \overline{z})$

(4) $0 \leq z * \overline{z} \in \mathbb{R}$

$AbsoluteValueC[|z|] = (z * \overline{z})^{1/2}$
$AbsoluteValueProperties := (z, w \in \mathbb{C}) \implies \ldots$ —

(1) 123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y}((x, y) \in f)$

$(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \ldots$

(1)	$Domain(f) := X; Codomain(f) := Y$
(2)	$Image(f, A) := \{f(a) : a \in A\}; Preimage(f, B) := \{a : f(a) \in B\}$
(3)	$Range(f) := Image(Domain(f))$

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X}(x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$
$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X}(y_0 = f(x))$
$Bijective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$
$SurjectiveEquivalent := (Range(f) = Codomain(f)) \implies Surjective(f)$

$(Function(f, X, Y) \wedge Function(g, Y, Z)) \implies (f \circ g)(x) := f(g(x)); Function(f \circ g, X, Z)$

$PropertiesofFunctions := (Function(f, A, B) \wedge Function(g, B, C) \wedge Function(h, C, D)) \implies \ldots$

(1)	$h \circ (g \circ f) = (h \circ g) \circ f$
(2)	$(Injective(f) \wedge Injective(g)) \implies Injective(g \circ f)$
(3)	$(Surjective(f) \wedge Surjective(g)) \implies Surjective(g \circ f)$
(4)	$(Bijective(f, A, B)) \implies \exists_{f^{-1}}(Function(f^-1, B, A) \wedge \forall_{a \in A}(f^{-1}(f(a)) = a) \wedge \forall_{b \in B}(f(f^{-1}(b)) = b))$

$|(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}}(b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \ldots$

(1)	$(a \mid b) \implies a \mid bc$
(2)	$(a \mid b \wedge b \mid c) \implies a \mid c$
(3)	$(a \mid b \wedge b \mid c) \implies a \mid (bx + cy)$
(4)	$(a \mid b \wedge b \mid a) \implies a = \pm b$
(5)	$(a \mid b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$
(6)	$(a \mid b) \iff (m \neq 0 \wedge ma \mid mb)$

$DivisionAlgorithm := (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}}(b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \wedge a : b \wedge a : c$
$GCD(a, b, c) := CD(a, b, c) \wedge \forall_d((d : b \wedge d : c) \implies d : a)$
$GCDEquivalent := 123123$

Chapter 3

Linear Algebra

$Matrix[A,m,n] := [a_{i,j}]_{m\times n}$:= m rows, n columns of real numbers
$\mathcal{M}_{m,n} := \{A : Matrix[A,m,n]\}$
$O_{m,n} := (Matrix[O,m,n]) \wedge (a_{i,j} = 0)$
$Square[A,n] := Matrix[A,n,n]$
$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$
$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$
$Diagonal[A,n] := (Square[A,n]) \wedge (i \neq j \implies a_{i,j} = 0)$
$Scalar[A,n,k] := (Diagonal[A,n]) \wedge (a_{i,i} = k)$
$I_n := Scalar[I,n,1]$

$+(A,B) := ((Matrix[A,m,n]) \wedge (Matrix[B,m,n])) \implies (A + B = [a_{i,j} + b_{i,j}]_{m\times n})$
$*(r,A) := ((r \in \mathbb{R}) \wedge (Matrix[A,m,n])) \implies (r * A = [ra_{i,j}]_{m\times n})$
$*(A,B) := ((Matrix[A,m,p]) \wedge (Matrix[B,p,n])) \implies \left(A * B = \left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]_{m\times n}\right)$
$^{T}[A] := (Matrix[A,m,n]) \implies (A^T = [a_{j,i}]_{n\times m})$

$AddCom := \forall_{A,B\in\mathcal{M}}(A + B = B + A)$

(1) $A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C\in\mathcal{M}}((A + B) + C = A + (B + C))$

(1) $(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A\in\mathcal{M}}\exists!_{O\in\mathcal{M}}(A + O = A = O + A)$

(1) $A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$

(2) $A + O_1 = A = A + O_2 \blacksquare O_1 = O_2$

$AddInv := \forall_{A\in\mathcal{M}}\exists!_{(-A)\in\mathcal{M}}(A + (-A) = O = (-A) + A)$

(1) $A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$

(2) $A + (-A_1) = O = A + (-A_2) \blacksquare -A_1 = -A_2 \blacksquare A_1 = A_2$

$MulAssoc := \forall_{A,B,C\in\mathcal{M}}((A * B) * C = A * (B * C))$

(1) $(A * B) * C = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,j})\right] * C = \left[\sum_{k_2=1}^{p_2}(\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2})c_{k_2,j})\right] = \left[\sum_{k_2=1}^{p_2}\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right] = \ldots$

(2) $\ldots \left[\sum_{k_1=1}^{p_1}\sum_{k_2=1}^{p_2}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right] = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1}\sum_{k_2=1}^{p_2}(b_{k_1,k_2}c_{k_2,j}))\right] = \ldots = A * (B * C)$

$MulId := \forall_{A:Square[A,n]}(A * I_n = A = I_n * A)$

(1) $A * I_n = \left[\sum_{k=1}^{n}\left(a_{i,k}\left(\begin{cases}1 & k = j\\ 0 & k \neq j\end{cases}\right)\right)\right] = [a_{i,j}] = A$

(2) $TODO = A$

$ScalAssoc := \forall_{r,s\in\mathbb{R}}\forall_{A\in\mathcal{M}}(r(sA) = (rs)A = s(rA))$

(1) $r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$

(2) $(rs)A = [rsa_{i,j}]$

(3) $s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$

$TransCancel := \forall_{A\in\mathcal{M}}(A = (A^T)^T)$

(1) $A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$

$ScalMulCom := \forall_{r\in\mathbb{R}}\forall_{A,B\in\mathcal{M}}((rA) * B = r(A * B) = A * (rB))$

(1) $(rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[\sum_{k=1}^{p}(ra_{i,k}b_{k,j})\right] = r(A * B)$

(2) $A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[\sum_{k=1}^{p}(a_{i,k}rb_{k,j})\right] = \left[\sum_{k=1}^{p}(ra_{i,k}b_{k,j})\right] = r(A * B)$

$ScalDistLeft := \forall_{r,s\in\mathbb{R}}\forall_{A\in\mathcal{M}}((r + s)A = rA + sA)$

(1) TODO

$ScalDistRight := \forall_{r\in\mathbb{R}}\forall_{A,B\in\mathcal{M}}(r(A + B) = rA + rB)$

(1) TODO

$MulDistRight := \forall_{A,B,C\in\mathcal{M}}((A + B) * C = A * C + B * C)$

(1) $(A + B) * C = [a_{i,j} + b_{i,j}] * C = \left[\sum_{k=1}^{p}((a_{i,k} + b_{i,k})c_{k,j})\right] = \ldots$

(2) $\ldots \left[\sum_{k=1}^{p}(a_{i,k}c_{k,j} + b_{i,k}c_{k,j})\right] = \left[\sum_{k=1}^{p}(a_{i,k}c_{k,j})\right] + \left[\sum_{k=1}^{p}(b_{i,k}c_{k,j})\right] = A * C + B * C$

$MulDistLeft := \forall_{A,B,C\in\mathcal{M}}(C * (A + B) = C * A + C * B)$

(1) TODO

$TransAddDist := \forall_{A,B\in\mathcal{M}}((A + B)^T = A^T + B^T)$

(1) TODO

$TransMulDist := \forall_{A,B\in\mathcal{M}}((A * B)^T = B^T * A^T)$

(1) $(A * B)^T = \left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]^T = \left[\sum_{k=1}^{p}(a_{j,k}b_{k,i})\right] = \left[\sum_{k=1}^{p}(b_{k,i}a_{j,k})\right] = \left[\sum_{k=1}^{p}(b^T_{i,k}a^T_{k,j})\right] = B^T * A^T$

$Sym[A] := A = A^T$
$SkewSym[A] := A = -A^T$
$Invertible[A] := (Square[A, n]) \wedge (\exists_{A^{-1}\in\mathcal{M}}(A * A^{-1} = I_n = A^{-1} * A))$

$SymGen := \forall_{A\in\mathcal{M}}(Sym[A + A^T])$

(1) $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$SkewSymGen := \forall_{A\in\mathcal{M}}(SkewSym[A - A^T])$

(1) $-(A - A^T)^T = -(A^T - (A^T)^T) = -(A^T - A) = (A - A^T)$

$SymDecomp := \forall_{A\in\mathcal{M}}\exists!_{B:Sym[B]}\exists!_{C:SkewSym[C]}(A = B + C)$

(1) $B := (1/2) * (A + A^T)\,;\ C := (1/2) * (A - A^T)$

(2) $SymGen[B] \wedge SkewSymGen[C]$

(3) $A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$

(4) $(1/2) * (A_1 + {A_1}^T) = (1/2) * (A_2 + {A_2}^T) \; \blacksquare \; A_1 = A_2$

(5) $(1/2) * (A_3 - {A_3}^T) = (1/2) * (A_4 - {A_4}^T) \; \blacksquare \; A_3 = A_4$

$InvId := \forall_{A:Invertible[A]}(\exists!_{A^{-1} \in \mathcal{M}}(A * A^{-1} = I_n = A^{-1} * A))$

(1) $A^{-1}{}_1 = A^{-1}{}_1 * I_n = A^{-1}{}_1 * (A * A^{-1}{}_2) = (A^{-1}{}_1 * A) * A^{-1}{}_2 = I_n * A^{-1}{}_2 = A^{-1}{}_2$

$InvCancel := \forall_{A:Invertible[A]}((A^{-1})^{-1} = A)$

(1) $(A * A^{-1})^{-1} = {I_n}^{-1} = I_n$

(2) $(A^{-1})^{-1} * A^{-1} = I_n \; \blacksquare \; A^{-1})^{-1} = I_n * A = A$

$InvDist := \forall_{A:Invertible[A]}\forall_{B:Invertible[B]}((A * B)^{-1} = B^{-1} * A^{-1})$

(1) $(A * B) * (A * B)^{-1} = I \; \blacksquare \; B * (A * B)^{-1} = A^{-1} \; \blacksquare \; (A * B)^{-1} = B^{-1} * A^{-1}$

$InvTrans := \forall_{A:Invertible[A]}((A^T)^{-1} = (A^{-1})^T) \; \blacksquare \Leftarrow$

(1) $A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I \; \blacksquare \; (A^{-1})^T = (A^T)^{-1}$

$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$
$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$
$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X(Sol[X, A, B])$
$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$
$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$

$HomoSysProps := (Sys[A, O]) \implies \ldots$

(1) $u_0 := O \; ; u_1 := choice(\{X \in \mathcal{M} | X \neq O\}) \; ; k := choice(\mathbb{R})$

(2) $TrivSol[u_0, A]$

(3) $(NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$

(4) $(TrivSol[\overrightarrow{X}, A]) \implies (TrivSol[LC(\overrightarrow{X}), A])$

$ElemMat[E] := (E = Swap[I_n, i, j]) \vee (Scale_*(I_n, i, c)) \vee (Combine_*(I_n, i, c, j))$
$ElemMatProd[E^*] := \exists_{\langle E \rangle}(\forall_{E_i \in E^*}(ElemMat[E_i]) \wedge (E^* = \Pi_{E_i \in E^*}(E_i)))$
$RowEquiv[A, B] := \exists_{E^*}((ElemMatProd[E^*]) \wedge (B = E^* * A))$

$ElemMatInv := \forall_{E \in \mathcal{M}}((ElemMat[E]) \implies (Invertible[E]))$

(1) $E - RowSwap[E] \implies TODO \; ; E - RowScale_*(E) \implies TODO \; ; E - RowCombine_*(E) \implies TODO$

$ElemMatProdInv := \forall_{E^*}((ElemMatProd[E^*]) \implies (Invertible[E^*]))$

(1) TODO

$RowEquivSys := \forall_{A,B,C,D,X \in \mathcal{M}}(((Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]])) \implies (Sol[X, A, B] \iff Sol[X, C, D]))$

(1) $\exists_{E^*:ElemMatProd[E^*]}([CD] = E^* * [AB])$

(2) $(E^* * A = C) \wedge (E^* * B = D)$

(3) $Sol[Y, A, B] \implies \ldots$

(3.1) $A * Y = B$

(3.2) $C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D \; \blacksquare \; Sol[Y, C, D]$

(4) $Sol[Y, A, B] \implies Sol[Y, C, D]$

(5) $(A = (E^*)^{-1} * C) \wedge (B = (E^*)^{-1} * D)$

(6) $Sol[Z, C, D] \implies \ldots$

(6.1) $C * Z = D$

(6.2) $A * Z = ((E^*)^{-1} * C) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$

(7) $Sol[Z, C, D] \implies Sol[Z, A, B]$

(8) $Sol[X, A, B] \iff Sol[X, C, D]$

$RowEquivHomoSysSol := \forall_{A,C,X\in\mathcal{M}}((RowEquiv[A,C]) \implies ((Sol[X,A,O]) \iff (Sol[X,C,O])))$

(1) Set $B = D = O$

$$RREF[A] := (A \in \mathcal{M}) \wedge \left(\begin{array}{l} \text{All zero rows are at the bottom of the matrix.} \wedge \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} \wedge \\ \text{The leading entry in any nonzero row is 1.} \wedge \\ \text{All entries in the column above and below a leading 1 are zero.} \wedge \end{array}\right)$$

$GaussJordanElim := \forall_{A\in\mathcal{M}}\exists!_{B\in\mathcal{M}}((RREF[B]) \wedge (RowEquiv[A,B]))$

(1) Hit A with $ElemMat$'s until it becomes B

(2) $(B = E^* * A) \wedge (RREF[B])$

$HasZero[A] := (Matrix(A,m,n)) \wedge (\exists_{i\le m}(A_{i,:} = O))$

$HasZeroNonInvertible := \forall_{A\in\mathcal{M}}((HasZero[A]) \implies (\neg Invertible[A]))$

(1) $i := choice(\{i \le m | A_{i,:} = O\})$

(2) $(B \in \mathcal{M}) \implies \ldots$

  (2.1) $(A * B)_{i,:} = O \neq I_{ni,:}$ $\blacksquare$ $A * B \neq I_n$

(3) $(B \in \mathcal{M}) \implies (A * B \neq I_n)$ $\blacksquare$ $\forall_{B\in\mathcal{M}}(A * B \neq I_n)$ $\blacksquare$ $\neg Invertible[A]$

$InvIffRowEquivI := \forall_{A\in\mathcal{M}}((Invertible[A]) \iff (RowEquiv[A,I_n]))$

(1) $(Invertible[A]) \implies \ldots$

  (1.1) $(RREF[B]) \wedge (RowEquiv[A,B])$

  (1.2) $B = E^* * A$

  (1.3) $(Invertible[E^*]) \wedge (Invertible[A])$ $\blacksquare$ $Invertible[B]$

  (1.4) $Invertible[B]$ $\blacksquare$ $\neg HasZero[B]$

  (1.5) $(RREF[B]) \wedge (\neg HasZero[B])$ $\blacksquare$ $B = I_n$

  (1.6) $RowEquiv[A,I_n]$

(2) $(Invertible[A]) \implies (RowEquiv[A,I_n])$

(3) $(RowEquiv[A,I_n]) \implies \ldots$

  (3.1) $I_n = E^* * A$ $\blacksquare$ $(E^*)^{-1} = A$

  (3.2) $A^{-1} = E^*_{DescSort}$ $\blacksquare$ $Invertible[A]$

(4) $(RowEquiv[A,I_n]) \implies (Invertible[A])$

(5) $(Invertible[A]) \iff (RowEquiv[A,I_n])$

$RowEquivIIffTrivSol := \forall_{A\in\mathcal{M}}((RowEquiv[A,I_n]) \iff (\forall_X((X = O) \iff (Sol[X,A,O]))))$

(1) $(RowEquiv[A,I_n]) \implies \ldots$

  (1.1) $RowEquiv[A,I_n]$ $\blacksquare$ $Invertible[A]$

  (1.2) $(Sol[X,A,O]) \implies \ldots$

    (1.2.1) $A * X = O$ $\blacksquare$ $X = A^{-1} * O = O$ $\blacksquare$ $X = O$

  (1.3) $(Sol[X,A,O]) \implies (X = O)$

  (1.4) $(X = O) \implies (Sol[X,A,O])$

  (1.5) $(X = O) \iff (Sol[X,A,O])$ $\blacksquare$ $\forall_X((X = O) \iff (Sol[X,A,O]))$

(2) $(RowEquiv[A,I_n]) \implies (\forall_X((X = O) \iff (Sol[X,A,O])))$

(3) $(\forall_X((X = O) \iff (Sol[X,A,O]))) \implies \ldots$

  (3.1) $(RREF[B]) \wedge (RowEquiv[A,B])$

  (3.2) $Sol[X,B,O]$

  (3.3) $(B \neq I_n) \implies \ldots$

    (3.3.1) $(\exists_{Y\neq X}(Sol[Y,B,O]))$

    (3.3.2) $Sol[Y,A,O]$ $\blacksquare$ $Y = X$

    (3.3.3) $(Y \neq X) \wedge (Y = X)$ $\blacksquare$ $\perp$

(3.4)	$(B \neq I_n) \implies \bot$ ▌ $B = I_n$
(3.5)	$(RowEquiv[A, B]) \wedge (B = I_n)$ ▌ $RowEquiv[A, I_n]$
(4)	$(\forall_X((X = O) \iff (Sol[X, A, O]))) \implies (RowEquiv[A, I_n])$
(5)	$(RowEquiv[A, I_n]) \iff (\forall_X((X = O) \iff (Sol[X, A, O])))$

$InvIffUniqSol := \forall_{A \in \mathcal{M}}((Invertible[A]) \iff (\forall_{B \in \mathcal{M}} \exists !_{X \in \mathcal{M}}(Sol[X, A, B])))$

(1)	$(Invertible[A] \wedge B \in \mathcal{M}) \implies \ldots$
(1.1)	$(Invertible[A]) \wedge (Sys[A, B])$
(1.2)	$(X = A^{-1} * B) \iff (Sol[X, A, B])$ ▌ $\exists !_{X \in \mathcal{M}}(Sol[X, A, B])$
(2)	$(\forall_{B \in \mathcal{M}} \exists !_{X \in \mathcal{M}}(Sol[X, A, B])) \implies \ldots$
(2.1)	$X_i := choice(\{X_i \vert Sol[X_i, A, I_{n:,i}]\})$
(2.2)	$A * [X_1 \ldots X_n] = [(A * X_1) \ldots (A * X_n)] = [I_{n:,1} \ldots I_{n:,n}] = I_n$
(2.3)	$A^{-1} = [X_1 \ldots X_n]$
(3)	$(\forall_{B \in \mathcal{M}} \exists !_{X \in \mathcal{M}}(Sol[X, A, B])) \implies (Invertible[A])$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}} \begin{pmatrix} (Invertible[A]) \iff \\ (RowEquiv[A, I_n]) \iff \\ (\forall_X((X = O) \iff (Sol[X, A, O]))) \iff \\ (\forall_{B \in \mathcal{M}} \exists !_{X \in \mathcal{M}}(Sol[X, A, B])) \end{pmatrix}$$

$$VectorSpace[V, +, *] := \exists_{O \in V} \forall_{\alpha, \beta \in \mathbb{R}} \forall_{u,v,w \in V} \begin{pmatrix} (u + v \in V) \wedge (u + v = v + u) \wedge ((u + v) + w = u + (v + w)) \wedge \\ (u + O = u) \wedge (\exists_{-u \in V}(u + (-u) = O)) \wedge \\ (\alpha * u \in V) \wedge (\alpha * (\beta * u) = (\alpha\beta) * u) \wedge (1 * u = u) \wedge \\ (\alpha * (u + v) = (\alpha * u) + (\alpha * v)) \wedge ((\alpha + \beta) * u = (\alpha * u) + (\beta * u)) \end{pmatrix}$$

$ZeroVectorUniq := \forall_{O', v \in V}((v + O' = v) \implies (O' = O))$

(1)	$O' = O' + O = O + O' = O$ ▌ $O' = O$

$AddInvUnique := \forall_{-v', v \in V}((v + -v' = O) \implies (-v' = -v))$

(1)	$-v' = -v' + O = -v' + (v + -v) = (-v' + v) + -v = (v + -v') + -v = O + -v = -v$ ▌ $-v' = -v$

$AddInvGen := \forall_{v \in V}((-1) * v = -v)$

(1)	$v + (-1) * v = (1 - 1) * v = 0 * v = O$ ▌ $(-1) * v = -v$

$ZeroVectorGenLeft := \forall_{v \in V}(0 * v = O)$

(1)	$0 * v = (0 + 0) * v = (0 * v) + (0 * v)$ ▌ $O = 0 * v$

$ZeroVectorGenRight := \forall_{r \in \mathbb{R}}(r * O = O)$

(1)	$r * O = r * (O + O) = (r * O) + (r * O)$ ▌ $O = r * O$

$ZeroVectorEquiv := \forall_{r \in \mathbb{R}} \forall_{v \in V}((r * v = O) \iff ((v = O) \vee (r = 0)))$

(1)	$(ZeroVectorGenLeft) \wedge (ZeroVectorGenRight)$ ▌ $((v = O) \vee (r = 0)) \implies (r * v = O))$
(2)	$(r * v = O) \implies \ldots$
(2.1)	$(r \neq 0) \implies \ldots$
(2.1.1)	$r \neq 0$ ▌ $r^{-1} \in \mathbb{R}$
(2.1.2)	$ZeroVectorGenRight$ ▌ $O = r^{-1} * O = r^{-1} * (r * v) = (r^{-1}r) * v = 1 * v = v$ ▌ $O = v$
(2.2)	$(r \neq 0) \implies (v = O)$ ▌ $(r = 0) \vee (r \neq 0)$ ▌ $(r = 0) \vee (v = O)$
(3)	$(r * v = O) \implies ((r = 0) \vee (v = O))$
(4)	$(r * v = O) \iff ((r = 0) \vee (v = O))$

$Subspace[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (\emptyset \neq S \subseteq V) \wedge (VectorSpace[S, +, *])$

$$SubspaceEquiv := \forall_{V,S}\left(\begin{array}{l}(VectorSpace[V,+,*]) \implies \\ ((Subspace[S,V,+,*]) \iff ((\emptyset \neq S \subseteq V) \wedge (\forall_{r,s\in S}(r+s \in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha * s \in S))))\end{array}\right)$$

(1) $(Subspace[S,V,+,*]) \implies \ldots$

(1.1) $Subspace[S,V,+,*]$ ▌ $\emptyset \neq S \subseteq V$

(1.2) $VectorSpace[S,+,*]$ ▌ $(\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha * s \in S))$

(1.3) $(\emptyset \neq S \subseteq V) \wedge (\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha * s \in S))$

(2) $(Subspace[S,V,+,*]) \implies ((\emptyset \neq S \subseteq V) \wedge (\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha * s \in S)))$

(3) $((\emptyset \neq S \subseteq V) \wedge (\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha * s \in S))) \implies \ldots$

(3.1) $((\alpha,\beta \in \mathbb{R}) \wedge (\emptyset \neq S) \wedge (u,v,w \in S)) \implies \ldots$

(3.1.1) $u,v \in V$ ▌ $u+v = v+u$

(3.1.2) $u,v,w \in V$ ▌ $(u+v)+w = u+(v+w)$

(3.1.3) $(ZeroVectorGenLeft) \wedge (u \in S)$ ▌ $0 * u = O \in S$

(3.1.4) $u \in V$ ▌ $u + O = u$

(3.1.5) $(AddInvGen) \wedge (u \in S)$ ▌ $(-1) * u = -u \in S$

(3.1.6) $u \in V$ ▌ $\alpha * (\beta * u) = (\alpha\beta) * u$

(3.1.7) $u \in V$ ▌ $1 * u = u$

(3.1.8) $u,v \in V$ ▌ $\alpha * (u+v) = (\alpha * u) + (\alpha * v)$

(3.1.9) $u \in V$ ▌ $(\alpha+\beta) * u = (\alpha * u) + (\beta * u)$

(4) $((\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha * s \in S))) \implies (Subspace[S,V,+,*])$

(5) $(Subspace[S,V,+,*]) \iff ((\forall_{r,s\in S}(r+s\in S)) \wedge (\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha * s \in S)))$

$$SetSum[A+B,A,B,V,+,*] := (VectorSpace[V,+,*]) \wedge (A,B \subseteq V) \wedge (A+B = \{a+b|(a\in A) \wedge (b \in B)\})$$

$$SumSubContains := \forall_{A,B,V}\left(\begin{array}{l}((Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \wedge (SetSum[A+B,A,B,V,+,*])) \implies \\ ((Subspace[A+B,V,+,*]) \wedge (A,B \subseteq A+B))\end{array}\right)$$

(1) $(Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*])$ ▌ $(O \in A) \wedge (O \in B)$

(2) $(SetSum[A+B,A,B,V,+,*]) \wedge (O \in A) \wedge (O \in B)$ ▌ $O \in A+B$ ▌ $\emptyset \neq A+B$

(3) $(Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*])$ ▌ $A+B \subseteq V$ ▌ $\emptyset \neq A+B \subseteq V$

(4) $(u,v \in A+B) \implies \ldots$

(4.1) $(\exists_{a_1\in A}\exists_{b_1\in B}(u = a_1+b_1)) \wedge (\exists_{a_2\in A}\exists_{b_2\in B}(v = a_2+b_2))$

(4.2) $u+v = (a_1+b_1)+(a_2+b_2) = (a_1+a_2)+(b_1+b_2)$

(4.3) $(a_1+a_2 \in A) \wedge (b_1+b_2 \in B)$ ▌ $u+v \in A+B$

(5) $(u,v \in A+B) \implies (u+v \in A+B)$ ▌ $\forall_{u,v\in A+B}(u+v \in A+B)$

(6) $((r \in \mathbb{R}) \wedge (v \in A+B)) \implies \ldots$

(6.1) $\exists_{a\in A}\exists_{b\in B}(v = a+b)$

(6.2) $r * v = r * (a+b) = r * a + r * b$

(6.3) $(r * a \in A) \wedge (r * b) \in B$ ▌ $r * v \in A+B$

(7) $((r \in \mathbb{R}) \wedge (v \in A+B)) \implies (r * v \in A+B)$ ▌ $\forall_{r\in\mathbb{R}}\forall_{v\in A+B}(r * v \in A+B)$

(8) $(\emptyset \neq A+B \subseteq V) \wedge (\forall_{u,v\in A+B}(u+v \in A+B)) \wedge (\forall_{r\in\mathbb{R}}\forall_{v\in A+B}(r * v \in A+B))$ ▌ $Subspace[A+B,V,+,*]$

(9) $(\forall_{a\in A}(a+O) = a) \wedge (O \in B)$ ▌ $A \subseteq A+B$

(10) $(\forall_{b\in B}(b+O) = b) \wedge (O \in A)$ ▌ $B \subseteq A+B$

(11) $(A,B \subseteq A+B) \wedge (Subspace[A+B,V,+,*])$

$$SumSubMinContains := \forall_{A,B,V}\left(\begin{array}{l}((Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \wedge (SetSum[A+B,A,B,V,+,*])) \implies \\ (\forall_C((Subspace[C,V,+,*]) \wedge (A,B \subseteq C)) \implies (A+B \subseteq C))\end{array}\right)$$

(1) $SumSub$ ▌ $(A,B \subseteq A+B) \wedge (Subspace[A+B,V,+,*])$

(2) $((Subspace[C,V,+,*]) \wedge (A,B \subseteq C)) \implies \ldots$

(2.1) $(s \in A+B) \implies \ldots$

(2.1.1) $\exists_{a\in A}\exists_{b\in B}(s = a+b)$

(2.1.2) $(A,B \subseteq C)$ ▌ $a,b \in C$

(2.1.3) $Subspace[C,V,+,*]$ ▌ $s = a+b \in C$

(2.2) $(s \in A+B) \implies (s \in C)$ ▌ $A+B \subseteq C$

(3) $((Subspace[C, V, +, *]) \land (A, B \subseteq C)) \implies (A + B \subseteq C)$

$$DirSum[A \oplus B, A, B, V, +, *] := \left(\begin{array}{l} (Subspace[A, V, +, *]) \quad \land \quad (Subspace[B, V, +, *]) \quad \land \\ (SetSum[A + B, A, B, V, +, *]) \land (\forall_{s \in A+B} \exists!_{\langle a,b \rangle \in A \times B}(s = a + b)) \end{array} \right)$$

$$DirSumEquiv := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A, V, +, *]) \land (Subspace[B, V, +, *]) \land (SetSum[A + B, A, B, V, +, *])) \implies \\ ((DirSum[A \oplus B, A, B, V, +, *]) \iff (\exists!_{\langle a,b \rangle \in A \times B}(O = a + b))) \end{array} \right)$$

(1) $(DirSum[A \oplus B, A, B, V, +, *]) \implies \ldots$

(1.1) $(Subspace[A, V, +, *]) \land (Subspace[B, V, +, *]) \blacksquare (O \in A) \land (O \in B) \blacksquare O \in A \oplus B$

(1.2) $(DirSum[A \oplus B, A, B, V, +, *]) \land (O \in A \oplus B) \blacksquare \exists!_{\langle a,b \rangle \in A \times B}(O = a + b)$

(2) $(DirSum[A \oplus B, A, B, V, +, *]) \implies (\exists!_{\langle a,b \rangle \in A \times B}(O = a + b))$

(3) $(\exists!_{\langle a,b \rangle \in A \times B}(O = a + b)) \implies \ldots$

(3.1) $(s \in A + B) \implies \ldots$

(3.1.1) $s \in A + B \blacksquare (\exists_{\langle a,b \rangle \in A \times B}(s = a + b))$

(3.1.2) $((s = a_1 + b_1) \land (s = a_2 + b_2)) \implies \ldots$

(3.1.2.1) $O = s - s = (a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2)$

(3.1.2.2) $(a_1 - a_2 \in A) \land (b_1 - b_2 \in B)$

(3.1.2.3) $((a_1 - a_2 \neq O) \lor (b_1 - b_2 \neq O)) \implies (\neg \exists!_{\langle a,b \rangle \in A \times B}(O = a + b)) \implies \bot$

(3.1.2.4) $(a_1 - a_2 = O) \land (b_1 - b_2 = O) \blacksquare \langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$

(3.1.3) $((s = a_1 + b_1) \land (s = a_2 + b_2)) \implies \langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$

(3.1.4) $\forall_{\langle a_1,b_1 \rangle, \langle a_2,b_2 \rangle \in A \times B}(((s = a_1 + b_1) \land (s = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle))$

(3.1.5) $\exists_{\langle a,b \rangle \in A \times B}(s = a + b) \land \forall_{\langle a_1,b_1 \rangle, \langle a_2,b_2 \rangle \in A \times B}(((s = a_1 + b_1) \land (s = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle)) \blacksquare \exists!_{\langle a,b \rangle \in A \times B}(s = a + b)$

(3.2) $(s \in A + B) \implies \exists!_{\langle a,b \rangle \in A \times B}(s = a + b) \blacksquare \forall_{s \in A+B} \exists!_{\langle a,b \rangle \in A \times B}(s = a + b) \blacksquare DirSum[A \oplus B, A, B, V, +, *]$

(4) $(\exists!_{\langle a,b \rangle \in A \times B}(O = a + b)) \implies (DirSum[A \oplus B, A, B, V, +, *])$

(5) $(DirSum[A \oplus B, A, B, V, +, *]) \iff (\exists!_{\langle a,b \rangle \in A \times B}(O = a + b))$

$$DirSumProp := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A, V, +, *]) \land (Subspace[B, V, +, *]) \land (SetSum[A + B, A, B, V, +, *])) \implies \\ ((DirSum[A \oplus B, A, B, V, +, *]) \iff (A \cap B = \{O\})) \end{array} \right)$$

(1) $(DirSum[A \oplus B, A, B, V, +, *]) \implies \ldots$

(1.1) $(v \in A \cap B) \implies \ldots$

(1.1.1) $v \in A \cap B \blacksquare (v \in A) \land (v \in B) \blacksquare (v \in A) \land (-v \in B)$

(1.1.2) $(v \in A) \land (-v \in B) \blacksquare O = v + (-v) \in A + B$

(1.1.3) $(DirSum[A \oplus B, A, B, V, +, *]) \land (O \in A + B) \blacksquare \exists!_{\langle a,b \rangle \in A \times B}(O = a + b) \blacksquare v = O$

(1.2) $(v \in A \cap B) \implies (v = O) \blacksquare A + B \subseteq \{O\}$

(1.3) $(v = O) \implies (v \in A \cap B) \blacksquare \{O\} \subseteq A \cap B$

(1.4) $A \cap B = \{O\}$

(2) $(DirSum[A \oplus B, A, B, V, +, *]) \implies (A \cap B = \{O\})$

(3) $(A \cap B = \{O\}) \implies \ldots$

(3.1) $(O \in A) \land (O \in B) \land (O = O + O \in A + B) \blacksquare \exists_{\langle a,b \rangle \in A \times B}(O = a + b)$

(3.2) $(((\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B) \land (O = a_1 + b_1) \land (O = a_2 + b_2)) \implies \ldots$

(3.2.1) $(a_1 \in A) \land (a_1 = -b_1 \in B) \blacksquare a_1 \in A \cap B \blacksquare a_1 = O = b_1$

(3.2.2) $(a_2 \in A) \land (a_2 = -b_2 \in B) \blacksquare a_2 \in A \cap B \blacksquare a_2 = O = b_2$

(3.2.3) $\langle a_1, b_1 \rangle = \langle O, O \rangle = \langle a_2, b_2 \rangle$

(3.3) $(((\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B) \land (O = a_1 + b_1) \land (O = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle)$

(3.4) $\forall_{\langle a_1,b_1 \rangle, \langle a_2,b_2 \rangle \in A \times B}(((O = a_1 + b_1) \land (O = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle))$

(3.5) $(\exists_{\langle a,b \rangle \in A \times B}(O = a + b)) \land (\forall_{\langle a_1,b_1 \rangle, \langle a_2,b_2 \rangle \in A \times B}(((O = a_1 + b_1) \land (O = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle)))$

(3.6) $(\exists!_{\langle a,b \rangle \in A \times B}(O = a + b)) \land (DirSumEquiv) \blacksquare DirSum[A \oplus B, A, B, V, +, *]$

(4) $(A \cap B = \{O\}) \implies (DirSum[A \oplus B, A, B, V, +, *])$

(5) $(DirSum[A \oplus B, A, B, V, +, *]) \iff (A \cap B = \{O\})$

$$LinComb[c, U, K, V, +, *] := (VectorSpace[V, +, *]) \land (n \in \mathbb{N}) \land (U \in V^n) \land (K \in \mathbb{R}^n) \land (c = \textstyle\sum_{i=1}^{n}(k_i * u_i))$$

$$LinSpan[S', S, V, +, *] := \left(\begin{array}{l} (VectorSpace[V, +, *]) \wedge (S \in V^n) \wedge ((S = \emptyset) \implies (S' = \{O\}))\wedge \\ ((S \neq \emptyset) \implies (S' = \{c \in V | \exists_{K \in \mathbb{R}^n}(LinComb[c, S, K, V, +, *])\})) \end{array} \right)$$

$LinSpan(LinSpan[S', S, V, +, *]) \implies ((Subspace[S', V, +, *]) \wedge (S \subseteq S'))$

(1) $(S = \emptyset) \implies (S' = \{O\}) \implies (\emptyset \neq S')$

(2) $(S \neq \emptyset) \implies (LinComb[O, S, \{0\}^n, V, +, *]) \implies (O \in S') \implies (\emptyset \neq S')$

(3) $((S = \emptyset) \vee (S \neq \emptyset)) \implies (\emptyset \neq S') \blacksquare \emptyset \neq S'$

(4) $LinSpan[S', S, V, +, *] \blacksquare S' \subseteq V \blacksquare \emptyset \neq S' \subseteq V$

(5) $(a, b \in S') \implies \ldots$

(5.1) $(\exists_{K \in \mathbb{R}^n}(LinComb[a, S, K, V, +, *])) \wedge (\exists_{L \in \mathbb{R}^n}(LinComb[b, S, L, V, +, *]))$

(5.2) $a + b = \sum_{i=1}^{n}(k_i * s_i) + \sum_{i=1}^{n}(l_i * s_i) = \sum_{i=1}^{n}((k_i + l_i) * s_i) \blacksquare a + b = \sum_{i=1}^{n}((k_i + l_i) * s_i)$

(5.3) $\langle k_i + l_i | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{N}^n$

(5.4) $(a + b = \sum_{i=1}^{n}((k_i + l_i) * s_i)) \wedge (\langle k_i + l_i | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{N}^n) \blacksquare \exists_{M \in \mathbb{N}^n}(a + b = \sum_{i=1}^{n}(m_i * s_i))$

(5.5) $\exists_{M \in \mathbb{N}^n}(LinComb[a + b, S, M, V, +, *]) \blacksquare a + b \in S'$

(6) $(a, b \in S') \implies (a + b \in S') \blacksquare \forall_{a,b \in S'}(a + b \in S')$

(7) $((r \in \mathbb{R}) \wedge (u \in S')) \implies \ldots$

(7.1) $\exists_{K \in \mathbb{R}^n}(LinComb[u, S, K, V, +, *]) \blacksquare \exists_{K \in \mathbb{R}^n}(u = \sum_{i=1}^{n}(k_i * s_i))$

(7.2) $r * u = r * \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{n}(r * (k_i * s_i)) = \sum_{i=1}^{n}(rk_i) * s_i) \blacksquare r * u = \sum_{i=1}^{n}(rk_i) * s_i)$

(7.3) $\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n$

(7.4) $(\sum_{i=1}^{n}(rk_i) * s_i)) \wedge (\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n)$

(7.5) $\exists_{M \in \mathbb{R}^n}(r * u = \sum_{i=1}^{n}(m_i * s_i)) \blacksquare \exists_{M \in \mathbb{R}^n}(LinComb[r * u, S, M, V, +, *]) \blacksquare r * u \in S'$

(8) $((r \in \mathbb{R}) \wedge (u \in S')) \implies (r * u \in S') \blacksquare \forall_{r \in \mathbb{R}}\forall_{u \in S'}(r * u \in S')$

(9) $(SubspaceEquiv) \wedge (\emptyset \neq S' \subseteq V) \wedge (\forall_{a,b \in S'}(a + b \in S')) \wedge (\forall_{r \in \mathbb{R}}\forall_{u \in S'}(r * u \in S')) \blacksquare Subspace[S', V, +, *]$

(10) $(s_j \in S) \implies \ldots$

(10.1) $K := \langle \left(\begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \right) | \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n \blacksquare \sum_{i=1}^{n}(k_i * s_i) = s_j$

(10.2) $\ldots \blacksquare \exists_{K \in \mathbb{R}^n}(LinComb[s_j, S, K, V, +, *]) \blacksquare s_j \in S'$

(11) $(s_j \in S) \implies (s_j \in S') \blacksquare \forall_{x \in S}(x \in S') \blacksquare S \subseteq S'$

(12) $(Subspace[S', V, +, *]) \wedge (S \subseteq S')$

$LinSpanSubContains := \forall_{S', S, V, +, *}((LinSpan[S', S, V, +, *]) \implies ((Subspace[S', V, +, *]) \wedge (S \subseteq S')))$

(1) $(S = \emptyset) \implies (S' = \{O\}) \implies (\emptyset \neq S')$

(2) $(S \neq \emptyset) \implies (LinComb[O, S, \{0\}^n, V, +, *]) \implies (O \in S') \implies (\emptyset \neq S')$

(3) $((S = \emptyset) \vee (S \neq \emptyset)) \implies (\emptyset \neq S') \blacksquare \emptyset \neq S'$

(4) $LinSpan[S', S, V, +, *] \blacksquare S' \subseteq V \blacksquare \emptyset \neq S' \subseteq V$

(5) $(a, b \in S') \implies \ldots$

(5.1) $(\exists_{K \in \mathbb{R}^n}(LinComb[a, S, K, V, +, *])) \wedge (\exists_{L \in \mathbb{R}^n}(LinComb[b, S, L, V, +, *]))$

(5.2) $a + b = \sum_{i=1}^{n}(k_i * s_i) + \sum_{i=1}^{n}(l_i * s_i) = \sum_{i=1}^{n}((k_i + l_i) * s_i) \blacksquare a + b = \sum_{i=1}^{n}((k_i + l_i) * s_i)$

(5.3) $\langle k_i + l_i | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{N}^n$

(5.4) $(a + b = \sum_{i=1}^{n}((k_i + l_i) * s_i)) \wedge (\langle k_i + l_i | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{N}^n) \blacksquare \exists_{M \in \mathbb{N}^n}(a + b = \sum_{i=1}^{n}(m_i * s_i))$

(5.5) $\exists_{M \in \mathbb{N}^n}(LinComb[a + b, S, M, V, +, *]) \blacksquare a + b \in S'$

(6) $(a, b \in S') \implies (a + b \in S') \blacksquare \forall_{a,b \in S'}(a + b \in S')$

(7) $((r \in \mathbb{R}) \wedge (u \in S')) \implies \ldots$

(7.1) $\exists_{K \in \mathbb{R}^n}(LinComb[u, S, K, V, +, *]) \blacksquare \exists_{K \in \mathbb{R}^n}(u = \sum_{i=1}^{n}(k_i * s_i))$

(7.2) $r * u = r * \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{n}(r * (k_i * s_i)) = \sum_{i=1}^{n}(rk_i) * s_i) \blacksquare r * u = \sum_{i=1}^{n}(rk_i) * s_i)$

(7.3) $\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n$

(7.4) $(\sum_{i=1}^{n}(rk_i) * s_i)) \wedge (\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n)$

(7.5) $\exists_{M \in \mathbb{R}^n}(r * u = \sum_{i=1}^{n}(m_i * s_i)) \blacksquare \exists_{M \in \mathbb{R}^n}(LinComb[r * u, S, M, V, +, *]) \blacksquare r * u \in S'$

(8) $((r \in \mathbb{R}) \wedge (u \in S')) \implies (r * u \in S') \blacksquare \forall_{r \in \mathbb{R}}\forall_{u \in S'}(r * u \in S')$

(9) $(SubspaceEquiv) \wedge (\emptyset \neq S' \subseteq V) \wedge (\forall_{a,b \in S'}(a+b \in S')) \wedge (\forall_{r \in \mathbb{R}} \forall_{u \in S'}(r * u \in S'))$ ▮ $Subspace[S', V, +, *]$

(10) $(s_j \in S) \implies \ldots$

(10.1) $K := \langle \left(\begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \right) | \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n$ ▮ $\sum_{i=1}^{n}(k_i * s_i) = s_j$

(10.2) $\ldots$ ▮ $\exists_{K \in \mathbb{R}^n}(LinComb[s_j, S, K, V, +, *])$ ▮ $s_j \in S'$

(11) $(s_j \in S) \implies (s_j \in S')$ ▮ $\forall_{x \in S}(x \in S')$ ▮ $S \subseteq S'$

(12) $(Subspace[S', V, +, *]) \wedge (S \subseteq S')$

$LinSpanSubMinContains := \forall_{S',S,V,+,*}((LinSpan[S', S, V, +, *]) \implies (\forall_W(((Subspace[W, V, +, *]) \wedge (S \subseteq W)) \implies (S' \subseteq W)))$

(1) $(s' \in S') \implies \ldots$

(1.1) $\exists_{K \in \mathbb{R}^n}(LinComb[s', S, K, V, +, *])$ ▮ $s' = \sum_{i=1}^{n}(k_i * s_i)$

(1.2) $(S \subseteq W) \wedge (Subspace[W, V, +, *])$ ▮ $s' = \sum_{i=1}^{n}(k_i * s_i) \in W$

(2) $(s' \in S') \implies (s' \in W)$ ▮ $S' \subseteq W$

$Spans[S, V, +, *] := LinSpan[V, S, V, +, *]$
$FiniteDim[V, +, *] := \exists_{S \in V^n}(Spans[S, V, +, *])$
$LinInd[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (S \in V^n) \wedge ((S \neq \emptyset) \implies (\forall_{K \in \mathbb{R}^n}((LinComb[O, S, K, V, +, *]) \implies (K = \{0\}^n))))$

$$LinDepLemma := \forall_{S,V} \left(\begin{array}{l} (\neg LinInd[S, V, +, *]) \qquad \implies \\ \exists_{j \in \mathbb{N}_{1,n}}((s_j \in LinSpan[P_1, S_{1,j-1}, V, +, *]) \wedge (LinSpan[P_2, S, V, +, *] = LinSpan[P_3, S \setminus \{s_j\}, V, +, *])) \end{array} \right)$$

(1) $\neg LinInd[S, V, +, *]$ ▮ $\exists_{K \in \mathbb{R}^n}((LinComb[O, S, K, V, +, *]) \wedge (K \neq \{0\}^n))$

(2) $\exists_{j \in \mathbb{N}_{1,n}}((k_j \neq 0) \wedge (\forall_{i \in \mathbb{N}_{1,n}}((i > j) \implies (k_i = 0))))$

(3) $s_j = (-1/k_j)\sum_{i=1}^{j-1}(k_i * s_i) = \sum_{i=1}^{j-1}((-k_i/k_j) * s_i)$

(4) $\langle -k_i/k_j | i \in \mathbb{N}_{1,j-1} \rangle \in \mathbb{R}^{j-1}$

(5) $\exists_{M \in \mathbb{R}^{j-1}}(LinComb[s_j, S_{1,j-1}, M, V, +, *])$ ▮ $s_j \in LinSpan[P_1, S_{1,j-1}, V, +, *]$

(6) $(v \in P_2) \iff (v \in LinSpan[P_2, S, V, +, *]) \iff \ldots$

(7) $\ldots(v = \sum_{i=1}^{n}(k_i * s_i)) = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^{n}(k_i * s_i)) + k_j * s_j = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^{n}(k_i * s_i)) + k_j * \sum_{i=1}^{j-1}((-k_i/k_j) * s_i) \iff \ldots$

(8) $(v \in LinSpan[P_3, S \setminus \{s_j\}, V, +, *]) \iff (v \in P_3)$ ▮ $(v \in P_2) \iff (v \in P_3)$ ▮ $P_2 = P_3$

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$LSSubspaceIdentity := (LinSpan[W', W, V, +, *]) \implies ((W' = W) \iff (Subspace[W, V, +, *]))$

(1) $(W' = W) \implies \ldots$

(1.1) $LSSubspaceContains$ ▮ $Subspace[W', V, +, *]$ ▮ $Subspace[W, V, +, *]$

(2) $(W' = W) \implies (Subspace[W, V, +, *])$

(3) $(Subspace[W, V, +, *]) \implies \ldots$

(3.1) $SubspaceEquiv$ ▮ $(\forall_{r,s \in W}(r + s \in W)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in W}(\alpha * s \in W))$ ▮ $\forall_{w \in W}(LinComb[w, W, K, V, +, *])$

(3.2) $(w \in W) \iff (LinComb[w, W, K, V, +, *]) \iff (w \in W')$ ▮ $W = W'$ ▮ $W' = W$

(4) $(Subspace[W, V, +, *]) \implies (W' = W)$

(5) $(W' = W) \iff (Subspace[W, V, +, *])$

$LSSubspaceSubset := ((LinSpan[S', S, V, +, *]) \wedge (Subspace[W, V, +, *]) \wedge (S \subseteq W)) \implies (Subspace[S', W, +, *])$

(1) $(LinSpan[S', S, V, +, *]) \wedge (S \subseteq W)$ ▮ $(LinSpan[S', S, W, +, *])$

(2) $(LSSubspaceContains) \wedge (LinSpan[S', S, W, +, *])$ ▮ $Subspace[S', W, +, *]$

$NullSpace[N, A, m, n] := (Matrix[A, m, n]) \wedge (N = \{x \in \mathbb{R}^n | A * x = O\})$
$RowSpace[R, A, m, n] := (Matrix[A, m, n]) \wedge (R = \{x^T * A \in \mathbb{R}^n | x \in \mathbb{R}^m\})$
$ColSpace[C, A, m, n] := (Matrix[A, m, n]) \wedge (C = \{A * x \in \mathbb{R}^m | x \in \mathbb{R}^n\})$

$NullSubspace := (NullSpace[N, A, m, n]) \implies (Subspace[N, \mathbb{R}^n, +, *])$

(1) TODO

$RowSubspace := (RowSpace[R, A, m, n]) \implies (Subspace[R, \mathbb{R}^n, +, *])$

(1) TODO

$ColSubspace := (ColSpace[C, A, m, n]) \implies (Subspace[C, \mathbb{R}^m, +, *])$

(1) TODO

$LinInd[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (\emptyset \neq S \in V^n) \wedge (\forall_{K \in \mathbb{R}^n}((LinComb[O, S, K, V, +, *]) \implies (K = \{0\}^n)))$

$ZeroDependent := (O \in S) \implies (\neg LinInd[S, V, +, *])$

(1) $K := \langle \left(\begin{cases} 1 & u_i = O \\ 0 & u_i \neq O \end{cases} \right) | (1 \leq i \leq n) \wedge (i \in \mathbb{N}) \rangle \blacksquare K \in \mathbb{R}^n$

(2) $(LinComb[O, S, K, V, +, *]) \wedge (K \neq \{O\}^n) \blacksquare \neg LinInd[S, V, +, *]$

$SingletonNonZeroIndependent := (v \neq O) \implies (LinInd[\langle v \rangle, V, +, *])$

(1) $(r * v = O) \iff ((r = 0) \vee (v \neq O))$

(2) $v \neq O \blacksquare r = 0$

(3) $\forall_{r \in \mathbb{R}}((r * v = O) \implies (r = 0))$

$SubIndependent := \forall_{V,A,B}(((VectorSpace[V, +, *]) \wedge (A \subseteq B \in V^m)) \implies ((LinInd[B, V, +, *]) \implies (LinInd[A, V, +, *])))$

(1) $(LinComb[O, A, K, V, +, *]) \implies \ldots$

(1.1) $L := \langle \left(\begin{cases} 1 & j \leq n \\ 0 & j > n \end{cases} \right) | (1 \leq j \leq m \wedge (j \in \mathbb{N}) \rangle \blacksquare L \in \mathbb{R}^m$

(1.2) $A \subseteq B \blacksquare \forall_{n \geq j \in \mathbb{N}}(a_j = b_j)$

(1.3) $\forall_{n \geq j \in \mathbb{N}}(a_j = b_j) \blacksquare \sum_{i=1}^{n}(k_i * a_i)) = \sum_{i=1}^{n}(k_i * a_i)) + O = \sum_{j=1}^{m}(l_j * b_j))$

(1.4) $LinComb[O, A, K, V, +, *] \blacksquare O = \sum_{i=1}^{n}(k_i * a_i)$

(1.5) $O = \sum_{i=1}^{n}(k_i * a_i) = \sum_{j=1}^{m}(l_j * b_j) \blacksquare LinComb[O, B, L, V, +, *]$

(1.6) $(LinInd[B, V, +, *]) \wedge (LinComb[O, B, L, V, +, *]) \blacksquare L = \{0\}^m$

(1.7) $(\forall_{n \geq j \in \mathbb{N}}(a_j = b_j)) \wedge (L = \{0\}^m) \blacksquare \forall_{n \geq j \in \mathbb{N}}(k_j * a_j = l_j * b * j = l_j * a_j) \blacksquare K = \{0\}^n$

(2) $(LinComb[O, A, K, V, +, *]) \implies (K = \{0\}^n) \blacksquare \forall_{K \in \subseteq \mathbb{R}^n}((LinComb[O, A, K, V, +, *]) \implies (K = \{0\}^n)) \blacksquare LinInd[A, V, +, *]$

$SuperDependent := \forall_{V,A,B}(((VectorSpace[V, +, *]) \wedge (A \subseteq B \subseteq V)) \implies ((\neg LinInd[A, V, +, *]) \implies (\neg LinInd[B, V, +, *])))$

(1) TODO

(2) $L := \langle \left(\begin{cases} 1 & j \leq n \\ 0 & j > n \end{cases} \right) | (1 \leq j \leq m \wedge (j \in \mathbb{N}) \rangle \blacksquare L \in \mathbb{R}^m$

(3) $\neg LinInd[A, V, +, *]$ $\blacksquare$ A has a non trivial solution $\blacksquare$ use the same non trivial solution in combination with B and L

$LinIndEquiv := \forall_{U,V}((LinInd[U, V, +, *]) \iff (\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])))$

(1) $\Gamma' = \Gamma \setminus \{j\}$

(2) $(\neg LinInd[U, V, +, *]) \implies \ldots$

(2.1) $(\exists_{\Gamma \in \mathbb{R}^{|U|}}((\sum(\gamma_i * u_i) = O) \wedge (\Gamma \neq \{0\}^{|U|})))$

(2.2) $\exists_{\gamma_k \in \Gamma}(\gamma_k \neq 0)$

(2.3) $\sum(\gamma_i' * u_i) = \sum(\gamma_i * u_i) - \gamma_k * u_k = -\gamma_k * u_j$

(2.4) $u_k = (-1/\gamma_k)(\sum(\gamma_i' * u_i)) = \sum((-\gamma_i'/\gamma_k) * u_i) \blacksquare \exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])$

(3) $(\neg LinInd[U, V, +, *]) \implies (\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *]))$

(4) $(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])) \implies (LinInd[U, V, +, *])$

(5) $(\exists_{j\in U}(LinComb[j, U \setminus \{j\}, +, *]) \implies \ldots$

(5.1) $\exists_{j\in U}(j = \sum(\gamma'_i * u_i))$

(5.2) $\Gamma := \Gamma' \cup \{-1\}$

(5.3) $(\sum(\gamma_i * u_i) = \sum(\gamma'_i * u_i) + (-1) * \gamma_j = O) \wedge (\Gamma \neq \{0\}^n)$ $\blacksquare$ $\neg LinInd[U, V, +, *]$

(6) $(\exists_{j\in U}(LinComb[j, U \setminus \{j\}, +, *]) \implies (\neg LinInd[U, V, +, *])$

(7) $(LinInd[U, V, +, *]) \implies (\forall_{j\in U}(\neg LinComb[j, U \setminus \{j\}, +, *]))$

(8) $(LinInd[U, V, +, *]) \iff (\forall_{j\in U}(\neg LinComb[j, U \setminus \{j\}, +, *]))$

CONT page 74, DEFER PROOFS OR JUST USE WORDS, THEN GET BACK ON TRACK

$Basis[S, V, +, *] := (Spans[S, V, +, *]) \wedge (LinInd[S, V, +, *])$

$UniqueLinComb := \forall_{S,V}((Basis[S, V]) \implies (\forall_{v,\Gamma,\Delta}(((v = \sum(\gamma_i * u_i)) \wedge (v = \sum(\delta_i * u_i))) \implies (\Gamma = \Delta))))$

(1) $(v \in V) \implies \ldots$

(1.1) $Spans[V, S, +, *]$ $\blacksquare$ $(\exists_{\Gamma\in\mathbb{R}^n}(v = \sum(\gamma_i * s_i))) \wedge (\exists_{\Gamma\in\mathbb{R}^n}(v = \sum(\gamma_i * s_i)))$

(1.2) $O = v - v = \sum(\gamma_i * s_i) - \sum(\delta_i * s_i) = \sum((\gamma_i - \delta_i) * s_i)$ $\blacksquare$ $\sum((\gamma_i - \delta_i) * s_i) = O$

(1.3) $(LinInd[S, V, +, *]) \wedge (\sum((\gamma_i - \delta_i) * s_i) = O)$ $\blacksquare$ $\{\gamma_i - \delta_i\} = \{0\}^n$ $\blacksquare$ $\{\gamma_i\} = \{\delta_i\}$ $\blacksquare$ $\Gamma = \Delta$

(2) $\Gamma = \Delta$

$BasisSubSpan := \forall_{S,V}((Spans[S, V, +, *]) \implies (\exists_{B\subseteq S}(Basis[B, V, +, *])))$

(1) $A = B$

(2) While $\neg LinInd(A, V, +, *)$, $\exists_{j\in A}(LinearCombination[j, A \setminus \{j\}, +, *])$, $A' = A \setminus \{j\}$

(3) $Spans[A', S, +, *]$, until $(LinInd[A', V, +, *]) \wedge (Spans[A', V, +, *])$

$BasisLinearIndCard := \forall_{S,T,V}(((Basis[S, V, +, *]) \wedge (LinInd[T, V, +, *])) \implies (|T| \leq |S|))$

(1) $(Basis[S, V, +, *]) \implies \ldots$

(1.1) $(|T| > |S|) \implies \ldots$

(1.1.1) $(Spans[S, V, +, *]) \wedge (T \subseteq V)$ $\blacksquare$ $t_{1\ldots t_j} = \sum(\gamma_i * s * i)\ \ldots$

(1.1.2) $\ldots t_j = \sum(\gamma'_i * t_i)$ $\blacksquare$ $\neg LinInd[T, V, +, *]$

(1.2) $(|T| > |S|) \implies (\neg LinInd[T, V, +, *])$ $\blacksquare$ $(LinInd[T, V, +, *]) \implies (|T| \leq |S|)$

(2) $((Basis[S, V, +, *]) \wedge (LinInd[T, V, +, *])) \implies (|T| \leq |S|)$

$BasisCard := \forall_{S,T,V}(((Basis[S, V, +, *]) \wedge (Basis[T, V, +, *])) \implies (|T| = |S|))$

(1) $Basis[S, V, +, *]$ $\blacksquare$ $LinInd[S, V, +, *]$

(2) $(Basis[T, V, +, *]) \wedge (LinInd[S, V, +, *])$ $\blacksquare$ $|S| \leq |T|$

(3) $Basis[T, V, +, *]$ $\blacksquare$ $LinInd[T, V, +, *]$

(4) $(Basis[S, V, +, *]) \wedge (LinInd[T, V, +, *])$ $\blacksquare$ $|T| \leq |S|$

(5) $(|S| \leq |T|) \wedge (|T| \leq |S|)$ $\blacksquare$ $|T| = |S|$

$Dim[d, V, +, *] := (\exists_B(Basis[B, V, +, *])) \wedge ((V = \{O\}) \implies (d = 0)) \wedge ((V \neq \{O\}) \implies (d = |B|))$

$Nullity[n, A] := (NullSpace[N, A]) \wedge (Dim[n, N, +, *])$

??????????????????????????????