Contents

Chapter 1

Real Analysis

(1.5)
$\mathit{OrderTrichotomy}[<, S] := \forall_{x,y\in S}(x < y \veebar x = y \veebar y < x)$
$\mathit{OrderTransitivity}[<, S] := \forall_{x,y,z\in S}((x < y \wedge y < z) \implies x < z)$
$\mathit{Order}[<, S] := (\mathit{OrderTrichotomy}[<, S]) \wedge (\mathit{OrderTransitivity}[<, S])$

(1.7)
$\mathit{BoundedAbove}[E, S, <] := (\mathit{Order}[<, S]) \wedge (E \subset S) \wedge (\exists_{\beta\in S}\forall_{x\in E}(x \leq \beta))$
$\mathit{BoundedBelow}[E, S, <] := (\mathit{Order}[<, S]) \wedge (E \subset S) \wedge (\exists_{\beta\in S}\forall_{x\in E}(\beta \leq x))$
$\mathit{UpperBound}[\beta, E, S, <] := (\mathit{Order}[<, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x\in E}(x \leq \beta))$
$\mathit{LowerBound}[\beta, E, S, <] := (\mathit{Order}[<, S]) \wedge (E \subset S) \wedge (\beta \in S \wedge \forall_{x\in E}(\beta \leq x))$

(1.8)
$\mathit{LUB}[\alpha, E, S, <] := (\mathit{UpperBound}[\alpha, E, S, <]) \wedge (\forall_{\gamma}(\gamma < \alpha \implies \neg\mathit{UpperBound}[\gamma, E, S, <]))$
$\mathit{GLB}[\alpha, E, S, <] := (\mathit{LowerBound}[\alpha, E, S, <]) \wedge (\forall_{\beta}(\alpha < \beta \implies \neg\mathit{LowerBound}[\beta, E, S, <]))$

(1.10)
$\mathit{LUBProperty}[S, <] := \forall_{E}(((\emptyset \neq E \subset S) \wedge (\mathit{BoundedAbove}[E, S, <]) \implies \exists_{\alpha\in S}(\mathit{LUB}[\alpha, E, S, <])))$
$\mathit{GLBProperty}[S, <] := \forall_{E}(((\emptyset \neq E \subset S) \wedge (\mathit{BoundedBelow}[E, S, <]) \implies \exists_{\alpha\in S}(\mathit{GLB}[\alpha, E, S, <])))$

(1.11)
$\mathit{LUBPropertyImpliesGLBProperty} := \mathit{LUBProperty}[S, <] \implies \mathit{GLBProperty}[S, <]$

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(1) $\mathit{LUBProperty}[S, <] \implies \ldots$ — wts: 2

(1.1) $(\emptyset \neq B \subset S \wedge \mathit{BoundedBelow}[B, S, <]) \implies \ldots$ — wts: 1.2

(1.1.1) $\mathit{Order}[<, S] \wedge \exists_{\delta'\in S}(\mathit{LowerBound}[\delta', B, S, <])$ — from: $\mathit{BoundedBelow}$, 1.1

(1.1.2) $|B| = 1 \implies \ldots$ — wts: 1.1.3

(1.1.2.1) $\exists_{u'}(u' \in B)$ ▮ $u := \mathit{choice}(\{u' : u' \in B\})$ ▮ $B = \{u\}$ — from: 1.1.2

(1.1.2.2) $\mathit{GLB}[u, B, S, <]$ ▮ $\exists_{\epsilon_0\in S}(\mathit{GLB}[\epsilon_0, B, S, <])$

(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0\in S}(\mathit{GLB}[\epsilon_0, B, S, <])$

(1.1.4) $|B| \neq 1 \implies \ldots$ — wts: 1.1.5

(1.1.4.1) $\forall_{E}((\emptyset \neq E \subset S \wedge \mathit{BoundedAbove}[E, S, <]) \implies \exists_{\alpha\in S}(\mathit{LUB}[\alpha, E, S, <]))$ — from: $\mathit{LUBProperty}$, 1

(1.1.4.2) $L := \{s \in S : \mathit{LowerBound}[s, B, S, <]\}$

(1.1.4.3) $|B| > 1 \wedge \mathit{OrderTrichotomy}[<, S]$ ▮ $\exists_{b_1'\in B}\exists_{b_0'\in B}(b_0' < b_1')$ — from: Order, 1.1.1; wts: 1.1.4.7

(1.1.4.4) $b_1 := \mathit{choice}(\{b_1' \in B : \exists_{b_0'\in B}(b_0' < b_1')\})$ ▮ $\neg\mathit{LowerBound}[b_1, B, S, <]$ — from: 1.1.4.2

(1.1.4.5) $b_1 \notin L$ ▮ $L \subset S$

(1.1.4.6) $\delta := \mathit{choice}(\{\delta' \in S : \mathit{LowerBound}[\delta', B, S, <]\})$ ▮ $\delta \in L$ ▮ $\emptyset \neq L$ — from: 1.1.1

(1.1.4.7) $\emptyset \neq L \subset S$ — from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y\in L}(\mathit{LowerBound}[y_0, B, S, <])$ ▮ $\forall_{y\in L}\forall_{x\in B}(y_0 \leq x)$ — from: $\mathit{LowerBound}$, 1.1.4.2; wts: 1.1.4.10

(1.1.4.9) $\forall_{x\in B}(x \in S \wedge \forall_{y\in L}(y_0 \leq x))$ ▮ $\forall_{x\in B}(\mathit{UpperBound}[x, L, S, <])$ — from: $\mathit{UpperBound}$

(1.1.4.10) $\exists_{x\in S}(\mathit{UpperBound}[x, L, S, <])$ ▮ $\mathit{BoundedAbove}[L, S, <]$

(1.1.4.11) $\emptyset \neq L \subset S \wedge BoundedAbove[L, S, <]$ — from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha' \in S}(LUB[\alpha', L, S, <]) \blacksquare \alpha := choice(\{\alpha' \in S : (LUB[\alpha', L, S, <])\})$ — from: 1.1.4.1, wts: 1.1.4.21

(1.1.4.13) $\forall_x(x \in B \implies UpperBound[x, L, S, <])$ — from: 1.1.4.9, wts: 1.1.4.17

(1.1.4.14) $\forall_x(\neg UpperBound[x, L, S, <] \implies x \notin B)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$ — wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound[\gamma, L, S, <] \blacksquare \gamma \notin B$ — from: LUB, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B \blacksquare \gamma \in B \implies \gamma \geq \alpha$

(1.1.4.17) $\forall_{\gamma \in B}(\alpha \leq \gamma) \blacksquare LowerBound[\alpha, B, S, <]$ — from: LowerBound

(1.1.4.18) $\alpha < \beta \implies \ldots$ — wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y \in L}(y_0 \leq \alpha < \beta) \blacksquare \forall_{y \in L}(y_0 \neq \beta)$ — from: LUB, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta \notin L \blacksquare \neg LowerBound[\beta, B, S, <]$ — from: 1.1.4.2

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound[\beta, B, S, <] \blacksquare \forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.20) $LowerBound[\alpha, B, S, <] \wedge \forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$ — from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB[\alpha, B, S, <] \blacksquare \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.5) $|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.6) $(|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])) \wedge (|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <]))$ — from: 1.1.3, 1.1.5

(1.1.7) $(|B| = 1 \vee |B| \neq 1) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <]) \blacksquare \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.2) $(\emptyset \neq B \subset S \wedge BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.3) $\forall_B((\emptyset \neq B \subset S \wedge BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <]))$

(1.4) $GLBProperty[S, <]$

(2) $LUBProperty[S, <] \implies GLBProperty[S, <]$

(1.12)

$$Field[F,+,*] := \exists_{0,1 \in F}\forall_{x,y,z \in F}\left(\begin{array}{l} x + y \in F \quad \wedge \quad x * y \in F \quad \wedge \\ x + y = y + x \quad \wedge \quad x * y = y * x \quad \wedge \\ (x + y) + z = x + (y_0 + z) \;\wedge\; (x * y) * z = x * (y_0 * z) \;\wedge \\ \mathbb{1} \neq \mathbb{0} \quad \wedge \quad x * (y_0 + z) = (x * y) + (x * z) \quad \wedge \\ \mathbb{0} + x = x \quad \wedge \quad \mathbb{1} * x = x \quad \wedge \\ \exists_{-x \in F}(x + (-x) = \mathbb{0}) \wedge (x \neq \mathbb{0} \implies \exists_{1/x \in F}(x * (1/x) = \mathbb{1})) \end{array}\right)$$

*********************** $(Field[F,+,*] \wedge x, y, z \in F) \implies \ldots$ ***********************

(1.14)

$AdditiveCancellation := (x + y = x + z) \implies y = z$

(1) $y = \mathbb{0} + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \ldots$ — from: Field

(2) $(-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = \mathbb{0} + z = z$ — from: Field

$AdditiveIdentityUniqueness := (x + y = x) \implies y = \mathbb{0}$

(1) $x + y = x = \mathbb{0} + x = x + \mathbb{0}$ — from: Field

(2) $y = 0$ — from: AdditiveCancellation

$AdditiveInverseUniqueness := (x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$ — from: Field

(2) $y = -x$ — from: AdditiveCancellation

$DoubleNegative := x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x \blacksquare \mathbb{0} = (-x) + x$ — from: Field

(2) $x = -(-x)$ — from: AdditiveInverseUniqueness

(1.15)

$MultiplicativeCancellation := (x \neq 0 \land x * y = x * z) \implies y = z$ —
$MultiplicativeIdentityUniqueness := (x \neq 0 \land x * y = x) \implies y = \mathbb{1}$ —
$MultiplicativeInverseUniqueness := (x \neq 0 \land x * y = \mathbb{1}) \implies y = 1/x$ —
$DoubleReciprocal := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)
$Domination := \mathbb{0} * x = \mathbb{0}$

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x \;\blacksquare\; \mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$ — from: $Field$

(2) $\mathbb{0} * x = \mathbb{0}$ — from: $AdditiveIdentityUniqueness$

$NonDomination := (x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies \ldots$

(1.1) $(x * y = 0) \implies \ldots$

(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = \mathbb{0} * ((1/x) * (1/y)) = \mathbb{0}$ — from: $Field$, $Domination$, 1, 1.1

(1.1.2) $\mathbb{1} = \mathbb{0} \land \mathbb{1} \neq \mathbb{0} \;\blacksquare\; \bot$ — from: $Field$

(1.2) $(x * y = 0) \implies \bot \;\blacksquare\; x * y \neq 0$

(2) $(x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq 0$

$NegationCommutativity := (-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0} \;\blacksquare\; x * y + (-x) * y = \mathbb{0}$ — from: $Field$, $Domination$; wts: 2

(2) $(-x) * y = -(x * y)$ — from: $AdditiveInverseUniqueness$

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * \mathbb{0} = \mathbb{0} \;\blacksquare\; x * y + x * (-y) = \mathbb{0}$ — from: $Field$, $Domination$; wts: 4

(4) $x * (-y) = -(x * y)$ — from: $AdditiveInverseUniqueness$

(5) $(-x) * y = -(x * y) = x * (-y)$ — from: 2, 4

$NegativeMultiplication := (-x) * (-y) = x * y$

(1) $(-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$ — from: $NegationCommutativity$, $DoubleNegative$

(1.17)

$$OrderedField[F, +, *, <] := \left(\begin{array}{l} Field[F, +, *] \quad \land \quad Order[<, F] \quad \land \\ \forall_{x,y,z \in F}(y_0 < z \implies x + y < x + z) \land \\ \forall_{x,y \in F}((x > \mathbb{0} \land y > \mathbb{0}) \implies x * y > \mathbb{0}) \end{array} \right)$$

************************* $(OrderedField[F, +, *, <] \land x, y, z \in F) \implies \ldots$ *************************

(1.18)
$NegationOnOrder := x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \ldots$

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x \;\blacksquare\; \mathbb{0} > -x \;\blacksquare\; -x < \mathbb{0}$ — from: $OrderedField$

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \ldots$

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x \;\blacksquare\; \mathbb{0} < x \;\blacksquare\; x > \mathbb{0}$ — from: $OrderedField$

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \land -x < \mathbb{0} \implies x > \mathbb{0} \;\blacksquare\; x > \mathbb{0} \iff -x < \mathbb{0}$ — from: 2, 4

$PositiveFactorPreservesOrder := (x > \mathbb{0} \land y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \land y < z) \implies \ldots$

(1.1) $(-y) + z > (-y) + y = \mathbb{0} \;\blacksquare\; z + (-y) = \mathbb{0}$ — from: $OrderedField$

(1.2) $x * (z + (-y)) > \mathbb{0} \;\blacksquare\; x * z + x * (-y) > \mathbb{0}$ — from: $OrderedField$

(1.3) $x * z = \mathbb{0} + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \ldots$ — from: $Field$, $NegationCommutativity$

(1.4) $x * y + (x * z + x * (-y)) > x * y + \mathbb{0} = x * y$ — from: $Field$, 1.2

(1.5) $x * z > x * y$ — from: 1.3, 1.4

(2) $(x > \mathbb{0} \wedge y < z) \implies x * z > x * y$

$NegativeFactorFlipsOrder := (x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \wedge y < z) \implies \ldots$

(1.1) $-x > \mathbb{0}$ — from: $NegationOnOrder$

(1.2) $(-x) * y < (-x) * z \;\blacksquare\; \mathbb{0} = x * y + (-x) * y < x * y + (-x) * z \;\blacksquare\; \mathbb{0} < x * y + (-x) * z$ — from: $PositiveFactorPreservesOrder$

(1.3) $\mathbb{0} < (-x) * (-y + z) \;\blacksquare\; \mathbb{0} > x * (-y + z) \;\blacksquare\; \mathbb{0} > -(x * y) + x * z$ — from: $NegationOnOrder$

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

$SquareIsPositive := (x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ — from: $OrderedField$

(2) $(x < \mathbb{0}) \implies \ldots$

(2.1) $-x > \mathbb{0} \;\blacksquare\; x * x = (-x) * (-x) > \mathbb{0} \;\blacksquare\; x * x > \mathbb{0}$ — from: $NegationOnOrder, OrderedField, NegativeMultiplication$

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \vee x < \mathbb{0}) \implies x * x > \mathbb{0} \;\blacksquare\; x \neq \mathbb{0} \implies x * x > \mathbb{0}$ — from: $OrderTrichotomy, 1, 3$

$OneIsPositive := \mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0} \;\blacksquare\; \mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ — from: $Field, SquareIsPositive$

$ReciprocationOnOrder := (0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \ldots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0} \;\blacksquare\; x * (1/x) > 0$ — from: $Field, OneIsPositive$

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp \;\blacksquare\; 1/x > \mathbb{0}$ — from: $NegativeFactorFlipsOrder, 1$

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0} \;\blacksquare\; y * (1/y) > 0$ — from: $Field, OneIsPositive$

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp \;\blacksquare\; 1/y > \mathbb{0}$ — from: $NegativeFactorFlipsOrder, 1$

(1.5) $(1/x) * (1/y) > \mathbb{0}$ — from: $OrderedField$

(1.6) $\mathbb{0} < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$ — from: $OrderedField, 1, 1.4, 1.5$

(1.19)
$OrderedFieldQ := OrderedField[\mathbb{Q}, +, *, <]$ —

$Subfield[K, F, +, *] := Field[F, +, *] \wedge K \subset F \wedge Field[K, +, *]$
$OrderedSubfield[K, F, +, *, <] := OrderedField[F, +, *, <] \wedge K \subset F \wedge OrderedField[K, +, *, <]$
$CutI[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$
$CutII[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
$CutIII[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} : CutI[\alpha] \wedge CutII[\alpha] \wedge CutIII[\alpha]\}$

$CutCorollaryI := (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \ldots$

(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ — from: $CutII, 1$

(1.2) $q < p \implies q \in \alpha \;\blacksquare\; q \notin \alpha \implies q \geq p$ — from: 1

(1.3) $(q \notin \alpha) \implies \ldots$

(1.3.1) $q \geq p$ — from: 1.2

(1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp \;\blacksquare\; q \neq p$ — from: 1, 1.3

(1.3.3) $q \geq p \wedge q \neq p \;\blacksquare\; p < q$

(1.4) $q \notin \alpha \implies p < q \;\blacksquare\; p < q$ — from: 1

(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

$\textit{CutCorollaryII} := (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$

(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ from: *CutII*, 1

(1.2) $s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha))$ ▌ $s \in \alpha \implies r \in \alpha$ from: 1, 1.1

(1.3) $r \notin \alpha \implies s \notin \alpha$ ▌ $s \notin \alpha$ from: 1, 1.2

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$\textit{OrderTrichotomyOfR} := \textit{OrderTrichotomy}[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$

(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$ from: $<_{\mathbb{R}}$, 1.1

(1.1.2) $\exists_{p'}(p' \in \alpha \wedge p' \notin \beta)$ ▌ $p := \textit{choice}(\{p' : p' \in \alpha \wedge p' \notin \beta\})$

(1.1.3) $q \in \beta \implies \ldots$

(1.1.3.1) $p, q \in \mathbb{Q}$

(1.1.3.2) $q < p$ from: *CutCorollaryI*

(1.1.3.3) $q \in \alpha$ from: *CutII*

(1.1.4) $q \in \beta \implies q \in \alpha$

(1.1.5) $\forall_{q \in \beta}(q \in \alpha)$ ▌ $\beta \subseteq \alpha$

(1.1.6) $\beta \subset \alpha$ ▌ $\beta <_{\mathbb{R}} \alpha$

(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$ ▌ $(\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $\textit{OrderTrichotomy}[\mathbb{R}, <_{\mathbb{R}}]$

$\textit{OrderTransitivityOfR} := \textit{OrderTransitivity}[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a \in \alpha}(a \in \beta) \wedge \forall_{b \in \beta}(b \in \gamma)$

(1.1.3) $\forall_{a \in \alpha}(\alpha \in \gamma)$ ▌ $\alpha \subset \gamma$ ▌ $\alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}}((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(4) $\textit{OrderTransitivity}[\mathbb{R}, <_{\mathbb{R}}]$

$\textit{OrderOfR} := \textit{Order}[<_{\mathbb{R}}, \mathbb{R}]$ from: *OrderTrichotomyR*, *OrderTransitivityR*

$\textit{LUBPropertyOfR} := \textit{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}]$ wts:

(1) $(\emptyset \neq A \subset \mathbb{R} \wedge \textit{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset$ ▌ $\exists_{\alpha}(\alpha \in A)$ ▌ $\alpha_0 := \textit{choice}(\{\alpha : \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset$ ▌ $\exists_{a}(a \in \alpha_0)$ ▌ $a_0 := \textit{choice}(\{a : a \in \alpha_0\})$ ▌ $a_0 \in \gamma$ ▌ $\gamma \neq \emptyset$

(1.4) $\textit{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]$ ▌ $\exists_{\beta}(\textit{UpperBound}[\beta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.5) $\beta_0 := choice(\{\beta : \mathit{UpperBound}[\beta, A, \mathbb{R}, <_{\mathbb{R}}]\})$

(1.6) $\mathit{UpperBound}[\beta_0, A, \mathbb{R}, <_{\mathbb{R}}]$ ▌ $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \beta_0)$ ▌ $\forall_{\alpha \in A}(\alpha \subseteq \beta_0)$ ▌ $\forall_{\alpha \in A}\forall_{a \in \alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma$ ▌ $\forall_{a \in \gamma}(a \in \beta_0)$ ▌ $\gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q}$ ▌ $\gamma \subseteq \beta_0 \subset \mathbb{Q}$ ▌ $\gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q}$ ▌ $\mathit{CutI}[\gamma]$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.10.1) $p \in \gamma$ ▌ $\exists_{\alpha \in A}(p \in \alpha)$ ▌ $\alpha_1 := choice(\{\alpha \in A : p \in \alpha\})$

(1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p$ ▌ $q \in \alpha_1$ ▌ $q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma$ ▌ $\forall_{p \in \gamma}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma)$ ▌ $\mathit{CutII}[\gamma]$

(1.12) $p \in \gamma \implies \ldots$

(1.12.1) $\exists_{\alpha \in A}(p \in \alpha)$ ▌ $\alpha_2 := choice(\{\alpha \in A : p \in \alpha\})$

(1.12.2) $\alpha_2 \in \mathbb{R}$ ▌ $\mathit{CutII}[\alpha_2]$ ▌ $\exists_{r \in \alpha_2}(p < r)$ ▌ $r_0 := choice(\{r \in \alpha_2 : p < r\})$

(1.12.3) $r_0 \in \alpha_2$ ▌ $r_0 \in \gamma$

(1.12.4) $p < r_0$ ▌ $p < r_0 \wedge r_0 \in \gamma$ ▌ $\exists_{r \in \gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r \in \gamma}(p < r)$ ▌ $\forall_{p \in \gamma}\exists_{r \in \gamma}(p < r)$ ▌ $\mathit{CutIII}[\gamma]$

(1.14) $\mathit{CutI}[\gamma] \wedge \mathit{CutII}[\gamma] \wedge \mathit{CutIII}[\gamma]$ ▌ $\gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha \in A}(\alpha \subseteq \gamma)$ ▌ $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma)$

(1.16) $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R}$ ▌ $\mathit{UpperBound}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.17) $\delta <_{\mathbb{R}} \gamma \implies \ldots$

(1.17.1) $\delta \subset \gamma$ ▌ $\exists_s(s \in \gamma \wedge s \notin \delta)$ ▌ $s_0 := choice(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$

(1.17.2) $s_0 \in \gamma$ ▌ $\exists_{\alpha \in A}(s_0 \in \alpha)$ ▌ $\alpha_3 := choice(\{\alpha \in A : s_0 \in \alpha\})$

(1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta$ ▌ $\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \ldots$

(1.17.4.1) $\alpha_3 \subseteq \delta$ ▌ $\forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta)$ ▌ $\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

(1.17.4.2) $\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$ ▌ $\perp$

(1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \perp$ ▌ $\delta <_{\mathbb{R}} \alpha_3$ ▌ $\exists_{\alpha \in A}(\delta <_{\mathbb{R}} \alpha)$ ▌ $\exists_{\alpha \in A}(\neg(\alpha \leq_{\mathbb{R}} \delta))$

(1.17.6) $\neg\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \delta)$ ▌ $\neg\mathit{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg\mathit{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$ ▌ $\forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg\mathit{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.19) $\mathit{UpperBound}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \wedge \forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg\mathit{UpperBound}[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.20) $\mathit{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$ ▌ $\exists_{\gamma \in S}(\mathit{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge \mathit{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(\mathit{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(3) $\forall_A((\emptyset \neq A \subset \mathbb{R} \wedge \mathit{BoundedAbove}[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(\mathit{LUB}[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]))$ ▌ $\mathit{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}]$

$+_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

$0_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 0\}$

$\mathit{ZeroInR} := 0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}}$ ▌ $\emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q}$ ▌ $\mathit{CutI}[0_{\mathbb{R}}]$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}}$ ▌ $\forall_{x \in 0_{\mathbb{R}}}\forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}})$ ▌ $\mathit{CutII}[0_{\mathbb{R}}]$

(3) $y := x/2$ ▌ $(x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $\forall_{x \in 0_{\mathbb{R}}}\exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▌ $\mathit{CutIII}[0_{\mathbb{R}}]$

(4) $\mathit{CutI}[0_{\mathbb{R}}] \wedge \mathit{CutII}[0_{\mathbb{R}}] \wedge \mathit{CutIII}[0_{\mathbb{R}}]$ ▌ $0_{\mathbb{R}} \in \mathbb{R}$

$\mathit{FieldAdditionClosureOfR} := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha)\,;\, \exists_b(b \in \beta)$ ▌ $a_0 := choice(\{a : a \in \alpha\})\,;\, b_0 := choice(\{b : b \in \beta\})$ ▌ $a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha)\,;\, \exists_y(y_0 \notin \beta)$ ▌ $x_0 := choice(\{x : x \notin \alpha\})\,;\, y_0 := choice(\{y : y \notin \beta\})$

(1.5) $\forall_{r \in \alpha}(r < x_0)\,;\, \forall_{s \in \beta}(s < y_0)$ ▌ $\forall_{r \in \alpha}\forall_{s \in \beta}(r + s < x_0 + y_0)$ ▌ $x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q}$ ▌ $\mathit{CutI}[\alpha +_{\mathbb{R}} \beta]$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.7.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▌ $(r_0, s_0) := choice((r, s) \in \alpha \times \beta : p = r + s)$

(1.7.2) $q < p = r_0 + s_0$ ▌ $(q - s_0) < r_0$ ▌ $(q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta$ ▌ $q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta$ ▌ $q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta)$ ▌ $CutII[\alpha +_{\mathbb{R}} \beta]$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▌ $(r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta : p = r + s\})$

(1.9.2) $r_1 \in \alpha$ ▌ $\exists_{t \in \alpha}(r_1 < t)$ ▌ $t_0 := choice(\{t \in \alpha : r_1 < t\})$

(1.9.3) $s_1 \in \beta$ ▌ $t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1$ ▌ $\exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} \beta} \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▌ $CutIII[\alpha +_{\mathbb{R}} \beta]$

(1.11) $CutI[\alpha +_{\mathbb{R}} \beta] \wedge CutII[\alpha +_{\mathbb{R}} \beta] \wedge CutIII[\alpha +_{\mathbb{R}} \beta]$ ▌ $\alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

$FieldAdditionCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s : r \in \alpha \wedge s \in \beta\} = \{s + r : s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

$FieldAdditionAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma))$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

$FieldAdditionIdentityOfR := (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \ldots$

(1.1.1) $s < 0$ ▌ $r + s < r + 0 = r$ ▌ $r + s < r$ ▌ $r + s \in \alpha$

(1.2) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha$ ▌ $\forall_{r \in \alpha} \forall_{s \in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}}(p \in \alpha)$ ▌ $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r \in \alpha}(p < r)$ ▌ $r_2 := choice(\{r \in \alpha : p < r\})$

(1.4.2) $p < r_2$ ▌ $p - r_2 < r_2 - r_2 = 0$ ▌ $(p - r_2) < 0$ ▌ $(p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha$ ▌ $p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▌ $p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▌ $\forall_{p \in \alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▌ $\alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▌ $0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

$FieldAdditionInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} : \exists_{r > 0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q}$ ▌ $\exists_{s \in \mathbb{Q}}(s \notin \alpha)$ ▌ $s_0 := choice(\{s : s \notin \alpha\})$ ▌ $p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha$ ▌ $-p_0 - 1 \notin \alpha$ ▌ $\exists_{r > 0}(-p_0 - r \notin \alpha)$ ▌ $p_0 \in \beta$

(1.4) $\emptyset \neq \alpha$ ▌ $\exists_{q \in \alpha}$ ▌ $q_0 := choice(\{q \in \mathbb{Q} : q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha$ ▌ $-(-q_0) - r = q_0 - r < q_0$ ▌ $-(-q_0) - r < q_0$ ▌ $-(-q_0) - r \in \alpha$

(1.6) $\forall_{r > 0}(-(-q_0) - r \in \alpha)$ ▌ $\neg\exists_{r > 0}(-(-q_0) - r \notin \alpha)$ ▌ $-q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q}$ ▌ $CutI[\beta]$

(1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.8.1) $p \in \beta$ ▌ $\exists_{r > 0}(-p - r \notin \alpha)$ ▌ $r_0 := choice(\{r > 0 : -p - r \notin \alpha\})$

(1.8.2) $q < p$ ▌ $-p - r < -q - r$

(1.8.3) $-q - r \notin \alpha$ ▌ $q \in \beta$

(1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta$ ▌ $\forall_{p \in \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \beta)$ ▌ $CutII[\beta]$

(1.10) $p \in \beta \implies \ldots$

(1.10.1) $p \in \beta$ ▌ $\exists_{r>0}(-p-r \notin \alpha)$ ▌ $r_1 := choice(\{r > 0 : -p-r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0$ ▌ $r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p$ ▌ $t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha$ ▌ $-t_0 - (r_1/2) \notin \alpha$ ▌ $\exists_{r>0}(-t_0 - r \notin \alpha)$ ▌ $t_0 \in \beta$

(1.10.7) $t_0 > p \wedge t_0 \in \beta$ ▌ $\exists_{t\in\beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t\in\beta}(p < t)$ ▌ $\forall_{p\in\beta}\exists_{t\in\beta}(p < t)$ ▌ $\mathit{CutIII}[\beta]$

(1.12) $\mathit{CutI}[\beta] \wedge \mathit{CutII}[\beta] \wedge \mathit{CutIII}[\beta]$ ▌ $\beta \in \mathbb{R}$

(1.13) $(r \in \alpha \wedge s \in \beta) \implies \ldots$

(1.13.1) $s \in \beta$ ▌ $\exists_{t>0}(-s-t \notin \alpha)$ ▌ $t_1 := choice(\{t > 0 : -s-t \notin \alpha\})$ ▌ $-s - t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha$ ▌ $-s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha$ ▌ $r < -s$ ▌ $r + s < 0$ ▌ $r + s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}}$ ▌ $\forall_{(r,s)\in\alpha\times\beta}(r + s \in 0_{\mathbb{R}})$ ▌ $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \ldots$

(1.15.1) $v < 0$ ▌ $w_0 := -v/2$ ▌ $w > 0$

from: $ARCHIMEDEANPROPERTYOFQ + LUB$???

(1.15.2) $\exists_{n\in\mathbb{Z}}(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha)$ ▌ $n_0 := choice(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$

(1.15.3) $p_0 := -(n_0 + 2)w_0$ ▌ $-p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha$ ▌ $-p_0 - w_0 \notin \alpha$ ▌ $p_0 \in \beta$

(1.15.4) $n_0 w_0 \in \alpha \wedge p_0 \in \beta$ ▌ $n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{v\in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta)$ ▌ $0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$ ▌ $\alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$ ▌ $\exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

$*_{\mathbb{R}}[\alpha, \beta] :=$ —

$1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$\mathit{1IsNot0} := 0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ —

$\mathit{1InR} := 1_{\mathbb{R}} \in \mathbb{R}$ —

$\mathit{FieldMultiplicationClosureOfR} := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R})$ —

$\mathit{FieldMultiplicationCommutativityOfR} := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ —

$\mathit{FieldMultiplicationAssociativityOfR} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma))$ —

$\mathit{FieldMultiplicationIdentityOfR} := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ —

$\mathit{FieldMultiplicationInverseOfR} := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha\in\mathbb{R}}(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}})$ —

$\mathit{FieldDistributativityOfR} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ —

$\mathit{FieldWithR} := \mathit{Field}[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}]$ —

$\mathit{OrderedFieldWithR} := \mathit{OrderedField}[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$

$\mathit{QROrderedSubfieldOfR} := \mathit{OrderedSubfield}[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$\mathit{QIsomorphicToQR} := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ —

$\mathit{CompletenessOfR} := \exists_{\mathbb{R}}(\mathit{LUBProperty}[\mathbb{R}, <_{\mathbb{R}}] \wedge \mathit{OrderedSubfield}[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}])$ —

(1.20)

$\mathit{ArchimedeanPropertyOfR} := \forall_{x,y\in\mathbb{R}}(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y))$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx : n \in \mathbb{N}^+\}$ ▌ $(\emptyset \neq A \subset \mathbb{R}) \wedge (a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a))$

(1.2) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n\in\mathbb{N}^+}(nx > y)$ ▌ $\forall_{n\in\mathbb{N}^+}(nx \leq y)$ ▌ $\mathit{UpperBound}[y_0, A, \mathbb{R}, <]$ ▌ $\mathit{BoundedAbove}[A, \mathbb{R}, <]$

(1.2.2) $\mathit{CompletenessOfR}$ ▌ $\mathit{LUBProperty}[\mathbb{R}, <]$

(1.2.3) $(\mathit{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (\mathit{BoundedAbove}[A, \mathbb{R}, <])$ ▌ $\exists_{\alpha\in\mathbb{R}}(\mathit{LUB}[\alpha, A, \mathbb{R}, <])$ . . .

(1.2.4) $\ldots \alpha_0 := choice(\{\alpha \in \mathbb{R} : \mathit{LUB}[\alpha, A, \mathbb{R}, <]\}) \;\blacksquare\; \mathit{LUB}[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0 \;\blacksquare\; \alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge (\mathit{LUB}[\alpha_0, A, \mathbb{R}, <]) \;\blacksquare\; \neg \mathit{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg \mathit{UpperBound}[\alpha_0 - x, A, \mathbb{R}, <] \;\blacksquare\; \exists_{c \in A}(\alpha_0 - x < c) \;\ldots$

(1.2.8) $\ldots c_0 := choice(\{c \in A : \alpha_0 - x < c\}) \;\blacksquare\; (c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)) \;\blacksquare\; \exists_{m \in \mathbb{N}^+}(mx = c_0) \;\ldots$

(1.2.10) $\ldots m_0 := choice(\{m \in \mathbb{N}^+ : mx = c_0\}) \;\blacksquare\; (m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0) \;\blacksquare\; \alpha_0 - x < c_0 = m_0 x \;\blacksquare\; \alpha_0 < m_0 x + x \;\blacksquare\; \alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+ \;\blacksquare\; m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a)) \;\blacksquare\; (m_0 + 1)x \in A$

(1.2.14) $(\alpha_0 < (m_0 + 1)x) \wedge ((m_0 + 1)x \in A) \;\blacksquare\; \exists_{c \in A}(\alpha_0 < c)$

(1.2.15) $\mathit{LUB}[\alpha_0, A, \mathbb{R}, <] \;\blacksquare\; \mathit{UpperBound}[\alpha_0, A, \mathbb{R}, <] \;\blacksquare\; \forall_{c \in A}(c \le \alpha_0) \;\blacksquare\; \neg \exists_{c \in A}(c > \alpha_0) \;\blacksquare\; \neg \exists_{c \in A}(\alpha_0 < c)$

(1.2.16) $(\exists_{c \in A}(\alpha_0 < c)) \wedge (\neg \exists_{c \in A}(\alpha_0 < c)) \;\blacksquare\; \bot$

(1.3) $\neg \exists_{n \in \mathbb{N}^+}(nx > y) \implies \bot \;\blacksquare\; \exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y) \;\blacksquare\; \forall_{x, y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

$\mathit{QDenseInR} := \forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y \;\blacksquare\; (0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $\mathit{ArchimedeanPropertyOfR} \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R}) \;\blacksquare\; \exists_{n \in \mathbb{N}^+}(n(y - x) > 1) \;\ldots$

(1.3) $\ldots n_0 := choice(\{n \in \mathbb{N}^+ : n(y - x) > 1\}) \;\blacksquare\; (n_0 \in \mathbb{N}^+) \wedge (n_0(y - x) > 1)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R}) \;\blacksquare\; n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $\mathit{ArchimedeanPropertyOfR} \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R}) \;\blacksquare\; \exists_{m \in \mathbb{N}^+}(m(1) > n_0 x) \;\ldots$

(1.6) $\ldots m_1 := choice(\{m \in \mathbb{N}^+ : m(1) > n_0 x\}) \;\blacksquare\; (m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $\mathit{ArchimedeanPropertyOfR} \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R}) \;\blacksquare\; \exists_{m \in \mathbb{N}^+}(m(1) > -n_0 x) \;\ldots$

(1.8) $\ldots m_2 := choice(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\}) \;\blacksquare\; (m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x) \;\blacksquare\; -m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+ \;\blacksquare\; |m_1 - (-m_2)| \ge 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge (|m_1 - (-m_2)| \ge 2) \;\blacksquare\; \exists_{m \in \mathbb{Z}}((-m_2 < m < m_1) \wedge (m - 1 \le n_0 x < m)) \;\ldots$

(1.12) $\ldots m_0 := choice(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \le n_0 x < m)\}) \;\blacksquare\; (-m_2 < m_0 < m_1) \wedge (m_0 - 1 \le n_0 x < m_0)$

(1.13) $(n_0(y - x) > 1) \wedge (m_0 - 1 \le n_0 x < m_0) \;\blacksquare\; n_0 x < m_0 \le 1 + n_0 x < n_0 y \;\blacksquare\; n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y) \;\blacksquare\; x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z} \;\blacksquare\; m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y) \;\blacksquare\; \exists_{p \in \mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y) \;\blacksquare\; \forall_{x, y \in \mathbb{R}}(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y))$

(1.21)

$\mathit{RootLemma} := (0 < a < b) \implies (b^n - a^n \le (b - a)nb^{n-1})$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b \;\blacksquare\; b/a > 1$

(1.3) $b/a > 1 \;\blacksquare\; \sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}(b^{n-i}a^{i-1}(b/a)^{i-1}) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1} \;\blacksquare\; \sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le (b - a)nb^{n-1} \;\blacksquare\; b^n - a^n \le (b - a)nb^{n-1}$

(2) $(0 < a < b) \implies (b^n - a^n \le (b - a)nb^{n-1})$

$\mathit{RootExistenceInR} := \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}}({y_0}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\} \;\blacksquare\; t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$

(1.2) $t_0 := x/(1 + x) \;\blacksquare\; (t_0 = x/(1 + x)) \wedge (t_0 \in \mathbb{R})$

(1.3) $0 < x \;\blacksquare\; 0 < x < 1 + x \;\blacksquare\; t_0 = x/(1 + x) > 0 \;\blacksquare\; t_0 > 0$

(1.4)	$1 = (1+x)/(1+x) > x/(1+x) = t_0$ ▮ $1 > t_0$
(1.5)	$(t_0 > 0) \wedge (1 > t_0)$ ▮ $0 < t_0 < 1$
(1.6)	$(0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1)$ ▮ ${t_0}^n \leq t_0$
(1.7)	$0 < x$ ▮ $x > x/(1+x) = t_0$ ▮ $x > t_0$
(1.8)	$({t_0}^n \leq t_0) \wedge (x > t_0)$ ▮ ${t_0}^n < x$
(1.9)	$(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge ({t_0}^n < x)$ ▮ $t_0 \in E$ ▮ $\emptyset \neq E$
(1.10)	$t_1 := choice(\{t \in \mathbb{R} : t > 1 + x\})$ ▮ $(t_1 \in \mathbb{R}) \wedge (t_1 > 1 + x)$
(1.11)	$x > 0$ ▮ $t_1 > 1 + x > 1$ ▮ $t_1 > 1$ ▮ ${t_1}^n \geq t_1$
(1.12)	$({t_1}^n \geq t_1) \wedge (t_1 > 1 + x) \wedge (1 > 0)$ ▮ ${t_1}^n \geq t_1 > 1 + x > x$ ▮ ${t_1}^n > x$
(1.13)	$(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ({t_1}^n > x)$ ▮ $t_1 \notin E$ ▮ $E \subset \mathbb{R}$
(1.14)	$(\emptyset \neq E) \wedge (E \subset \mathbb{R})$ ▮ $\emptyset \neq E \subset \mathbb{R}$
(1.15)	$t \in E \implies \ldots$
(1.15.1)	$(t \in E) \wedge (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x))$ ▮ $t^n < x$
(1.15.2)	$({t_1}^n > x) \wedge (t^n < x)$ ▮ $t^n < x < {t_1}^n$ ▮ $t < t_1$
(1.16)	$t \in E \implies t < t_1$ ▮ $\forall_{t \in E}(t \leq t_1)$ ▮ $UpperBound[t_1, E, \mathbb{R}, <]$ ▮ $BoundedAbove[E, \mathbb{R}, <]$
(1.17)	$CompletenessOfR$ ▮ $LUBProperty[\mathbb{R}, <]$
(1.18)	$(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq E \subset \mathbb{R}) \wedge (BoundedAbove[E, \mathbb{R}, <])$ ▮ $\exists_{y \in \mathbb{R}}(LUB[y, E, \mathbb{R}, <])$ ...
(1.19)	$\ldots y_0 := choice(\{y \in \mathbb{R} : LUB[y, E, \mathbb{R}, <]\})$ ▮ $LUB[y_0, E, \mathbb{R}, <]$
(1.20)	$(LUB[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0)$ ▮ $0 < t_0 \leq y_0 \in \mathbb{R}$ ▮ $0 < y_0 \in \mathbb{R}$
(1.21)	${y_0}^n < x \implies \ldots$
(1.21.1)	$k_0 := \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}$ ▮ $k_0 \in \mathbb{R}$
(1.21.2)	${y_0}^n < x$ ▮ $0 < x - {y_0}^n$
(1.21.3)	$(n > 0) \wedge (y_0 > 0)$ ▮ $0 < n(y_0 + 1)^{n-1}$
(1.21.4)	$(0 < x - {y_0}^n) \wedge (0 < n(y_0+1)^{n-1})$ ▮ $0 < \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} = k_0$ ▮ $0 < k_0$
(1.21.5)	$(0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R})$ ▮ $0 < min(1, k_0) \in \mathbb{R}$
(1.21.6)	$QDenseInR \wedge (0, min(1, k_0) \in \mathbb{R}) \wedge (0 < min(1, k_0))$ ▮ $\exists_{h \in \mathbb{Q}}(0 < h < min(1, k_0))$ ...
(1.21.7)	$\ldots h_0 := choice(\{h \in \mathbb{Q} : 0 < h < min(1, k_0)\})$ ▮ $(0 < h_0 < 1) \wedge (h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}})$
(1.21.8)	$(y_0 > 0) \wedge (h_0 > 0)$ ▮ $0 < y_0 < y_0 + h_0$
(1.21.9)	$RootLemma \wedge (0 < y_0 < y_0 + h_0)$ ▮ $(y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}$
(1.21.10)	$h_0 < 1$ ▮ $h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$
(1.21.11)	$((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}) \wedge (h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1})$ ▮ $(y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}$
(1.21.12)	$(0 < n(y_0 + 1)^{n-1}) \wedge (h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}})$ ▮ $h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n$
(1.21.13)	$((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}) \wedge (h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n)$ ▮ $(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n$ ▮ $(y_0 + h_0)^n < x$
(1.21.14)	$(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n$ ▮ $(y_0 + h_0)^n < x$
(1.21.15)	$(0 < y_0 \mathbb{R}) \wedge (0 < h_0 < \mathbb{R})$ ▮ $0 < y_0 < y_0 + h_0 \in \mathbb{R}$
(1.21.16)	$(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ((y_0 + h_0)^n < x) \wedge (0 < y_0 + h_0 \in \mathbb{R})$ ▮ $(y_0 + h_0)^n \in E$
(1.21.17)	$((y_0 + h_0)^n \in E) \wedge (y_0 < y_0 + h_0)$ ▮ $\exists_{e \in E}(y_0 < e)$
(1.21.18)	$LUB[y_0, E, \mathbb{R}, <]$ ▮ $UpperBound[y_0, E, \mathbb{R}, <]$ ▮ $\forall_{e \in E}(e \leq y_0)$ ▮ $\neg\exists_{e \in E}(e > y_0)$
(1.21.19)	$(\exists_{e \in E}(e > y_0)) \wedge (\neg\exists_{e \in E}(e > y_0))$ ▮ $\perp$
(1.22)	${y_0}^n < x \implies \perp$ ▮ ${y_0}^n \geq x$
(1.23)	${y_0}^n > x \implies \ldots$
(1.23.1)	$k_1 := \frac{{y_0}^n - x}{n{y_0}^{n-1}}$ ▮ $(k_1 \in \mathbb{R}) \wedge (k_1 n{y_0}^{n-1} = {y_0}^n - x)$
(1.23.2)	$(0 < x) \wedge (0 < n \in \mathbb{Z})$ ▮ ${y_0}^n - x < {y_0}^n \leq n{y_0}^n$ ▮ ${y_0}^n - x < n{y_0}^n$
(1.23.3)	${y_0}^n - x < n{y_0}^n$ ▮ $k_1 = \frac{{y_0}^n - x}{n{y_0}^{n-1}} < \frac{n{y_0}^n}{n{y_0}^{n-1}} = y_0$ ▮ $k_1 < y_0$
(1.23.4)	${y_0}^n > x$ ▮ $0 < {y_0}^n - x$
(1.23.5)	$(n > 0) \wedge (y_0 > 0)$ ▮ $0 < n{y_0}^{n-1}$
(1.23.6)	$(0 < {y_0}^n - x) \wedge 0 < (n{y_0}^{n-1})$ ▮ $0 < \frac{{y_0}^n - x}{n{y_0}^{n-1}} = k_1$ ▮ $0 < k_1$

(1.23.7) $(k_1 < y_0) \wedge (0 < k_1) \;\blacksquare\; (0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$

(1.23.8) $t \geq y_0 - k_1 \implies \ldots$

(1.23.8.1) $t \geq y_0 - k_1 \;\blacksquare\; t^n \geq (y_0 - k_1)^n \;\blacksquare\; -t^n \leq -(y_0 - k_1)^n \;\blacksquare\; {y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n$

(1.23.8.2) $\mathit{RootLemma} \wedge (0 < y_0 - k_1 < y_0) \;\blacksquare\; {y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}$

(1.23.8.3) $({y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n) \wedge ({y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}) \;\blacksquare\; {y_0}^n - t^n < k_1 n {y_0}^{n-1}$

(1.23.8.4) $(k_1 n {y_0}^{n-1} = {y_0}^n - x) \wedge ({y_0}^n - t^n < k_1 n {y_0}^{n-1}) \;\blacksquare\; {y_0}^n - t^n < {y_0}^n - x \;\blacksquare\; -t^n < -x \;\blacksquare\; t^n > x$

(1.23.8.5) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t^n > x) \;\blacksquare\; t \notin E$

(1.23.9) $t \geq y_0 - k_1 \implies t \notin E \;\blacksquare\; t \in E \implies t < y_0 - k_1 \;\blacksquare\; \forall_{t \in E}(t \leq y_0 - k_1) \;\blacksquare\; \mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.10) $(\mathit{LUB}[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)) \;\blacksquare\; \neg \mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.11) $(\mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg \mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \;\blacksquare\; \bot$

(1.24) ${y_0}^n > x \implies \bot \;\blacksquare\; {y_0}^n \leq x$

(1.25) $\mathit{Order}[\mathbb{R}, <] \;\blacksquare\; \mathit{OrderTrichotomy}[\mathbb{R}, <]$

(1.26) $(\mathit{OrderTrichotomy}[\mathbb{R}, <]) \wedge ({y_0}^n \geq x) \wedge ({y_0}^n \leq x) \;\blacksquare\; {y_0}^n = x$

(1.27) $({y_0}^n = x) \wedge (y_0 \in \mathbb{R}) \;\blacksquare\; \exists_{y \in \mathbb{R}}(y^n = x)$

(1.28) $y_1, y_2 := choice(\{y \in \mathbb{R} : y^n = x\})$

(1.29) $y_1 \neq y_2 \implies \ldots$

(1.29.1) $(\mathit{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_1 \neq y_2) \;\blacksquare\; (y_1 < y_2) \veebar (y_2 < y_1) \;\ldots$

(1.29.2) $\ldots (x = {y_1}^n < {y_2}^n = x) \veebar (x = {y_2}^n < {y_1}^n = x) \;\blacksquare\; (x < x) \veebar (x > x) \;\blacksquare\; \bot \veebar \bot \;\blacksquare\; \bot$

(1.30) $y_1 \neq y_2 \implies \bot \;\blacksquare\; y_1 = y_2 \;\blacksquare\; \forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b)$

(1.31) $(\exists_{y \in \mathbb{R}}(y^n = x)) \wedge (\forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b)) \;\blacksquare\; \exists!_{y \in \mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}}(y^n = x) \;\blacksquare\; \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}}({y_0}^n = x)$

$\mathit{RootExistenceInRCorollary} := \forall_{0 < a \in \mathbb{R}} \forall_{0 < b \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}}((ab)^{1/n} = a^{1/n} b^{1/n})$ —

$$\mathit{ExtendedRealSystem}[\bar{\mathbb{R}}, +, *, <] := \left(\begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \;\wedge\; x - \infty = -\infty \;\wedge\; \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \;\wedge \\ (x > 0) \implies (x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty) \wedge \\ (x < 0) \implies (x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty) \end{array} \right)$$

$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$
$+_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d \rangle$
$*_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c \rangle$

$\mathit{FieldC} := \mathit{Field}[\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}}]$ —
$\mathit{RSubfieldC} := \mathit{Subfield}[\mathbb{R}, \mathbb{C}, +, *]$ —

$i := \langle 0, 1 \rangle \in \mathbb{C}$

$\mathit{iProperty} := i^2 = -1$ —
$\mathit{CProperty} := (a, b \in \mathbb{R}) \implies (\langle a, b \rangle = a + bi)$ —

$\mathit{Conjugate}[\overline{a + bi}] := a - bi$

$\mathit{ConjugateProperties} := (w, z \in \mathbb{C}) \implies \ldots$ —

(1) $\overline{z + w} = \overline{z} + \overline{w}$

(2) $\overline{z * w} = \overline{z} * \overline{w}$

(3) $Re(z) = (1/2)(z + \overline{z}) \wedge Im(z) = (1/2)(z - \overline{z})$

(4) $0 \leq z * \overline{z} \in \mathbb{R}$

$\mathit{AbsoluteValueC}[|z|] = (z * \overline{z})^{1/2}$
$\mathit{AbsoluteValueProperties} := (z, w \in \mathbb{C}) \implies \ldots$ —

(1) 123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y}((x, y) \in f)$

$(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \ldots$

(1)	$Domain(f) := X; Codomain(f) := Y$
(2)	$Image(f, A) := \{f(a) : a \in A\}; Preimage(f, B) := \{a : f(a) \in B\}$
(3)	$Range(f) := Image(Domain(f))$

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X}(x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$
$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X}(y_0 = f(x))$
$Bijective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$
$SurjectiveEquivalent := (Range(f) = Codomain(f)) \implies Surjective(f)$

$(Function(f, X, Y) \wedge Function(g, Y, Z)) \implies (f \circ g)(x) := f(g(x)); Function(f \circ g, X, Z)$

$Propertiesof Functions := (Function(f, A, B) \wedge Function(g, B, C) \wedge Function(h, C, D)) \implies \ldots$

(1)	$h \circ (g \circ f) = (h \circ g) \circ f$
(2)	$(Injective(f) \wedge Injective(g)) \implies Injective(g \circ f)$
(3)	$(Surjective(f) \wedge Surjective(g)) \implies Surjective(g \circ f)$
(4)	$(Bijective(f, A, B)) \implies \exists_{f^{-1}}(Function(f^{-}1, B, A) \wedge \forall_{a \in A}(f^{-1}(f(a)) = a) \wedge \forall_{b \in B}(f(f^{-1}(b)) = b))$

$|(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}}(b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \ldots$

(1)	$(a \vert b) \implies a \vert bc$
(2)	$(a \vert b \wedge b \vert c) \implies a \vert c$
(3)	$(a \vert b \wedge b \vert c) \implies a \vert (bx + cy)$
(4)	$(a \vert b \wedge b \vert a) \implies a = \pm b$
(5)	$(a \vert b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$
(6)	$(a \vert b) \iff (m \neq 0 \wedge ma \vert mb)$

$DivisionAlgorithm := (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}}(b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \wedge a : b \wedge a : c$
$GCD(a, b, c) := CD(a, b, c) \wedge \forall_d((d : b \wedge d : c) \implies d : a)$
$GCDEquivalent := 123123$

Chapter 3

Linear Algebra

3.1 Matrix Operations and Special Matrices

$Matrix[A, m, n] := [a_{i,j}]_{m\times n}$:= m rows, n columns of real numbers
$\mathcal{M}_{m,n} := \{A : Matrix[A, m, n]\}$
$O_{m,n} := (Matrix[O, m, n]) \wedge (a_{i,j} = 0)$
$Square[A, n] := Matrix[A, n, n]$
$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$
$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$
$Diagonal[A, n] := (Square[A, n]) \wedge (i \neq j \implies a_{i,j} = 0)$
$Scalar[A, n, k] := (Diagonal[A, n]) \wedge (a_{i,i} = k)$
$I_n := Scalar[I, n, 1]$

$+(A, B) := ((Matrix[A, m, n]) \wedge (Matrix[B, m, n])) \implies (A + B = [a_{i,j} + b_{i,j}]_{m\times n})$
$*\,(r, A) := ((r \in \mathbb{R}) \wedge (Matrix[A, m, n])) \implies (r * A = [ra_{i,j}]_{m\times n})$
$*\,(A, B) := ((Matrix[A, m, p]) \wedge (Matrix[B, p, n])) \implies (A * B = \left[\sum_{k=1}^{p}(a_{i,k} b_{k,j})\right]_{m\times n})$
$^{T}[A] := (Matrix[A, m, n]) \implies (A^T = [a_{j,i}]_{n\times m})$

$AddCom := \forall_{A,B\in\mathcal{M}}(A + B = B + A)$

(1) $A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C\in\mathcal{M}}((A + B) + C = A + (B + C))$

(1) $(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A\in\mathcal{M}}\exists!_{O\in\mathcal{M}}(A + O = A = O + A)$

(1) $A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$
(2) $A + O_1 = A = A + O_2$ ▮ $O_1 = O_2$

$AddInv := \forall_{A\in\mathcal{M}}\exists!_{(-A)\in\mathcal{M}}(A + (-A) = O = (-A) + A)$

(1) $A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$
(2) $A + (-A_1) = O = A + (-A_2)$ ▮ $-A_1 = -A_2$ ▮ $A_1 = A_2$

$MulAssoc := \forall_{A,B,C\in\mathcal{M}}((A * B) * C = A * (B * C))$

(1) $(A * B) * C = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,j})\right] * C = \left[\sum_{k_2=1}^{p_2}(\sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,k_2})c_{k_2,j})\right] = \left[\sum_{k_2=1}^{p_2}\sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,k_2} c_{k_2,j})\right] = \ldots$
(2) $\ldots\left[\sum_{k_1=1}^{p_1}\sum_{k_2=1}^{p_2}(a_{i,k_1} b_{k_1,k_2} c_{k_2,j})\right] = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1}\sum_{k_2=1}^{p_2}(b_{k_1,k_2} c_{k_2,j}))\right] = \ldots = A * (B * C)$

$MulId := \forall_{A:Square[A,n]}(A * I_n = A = I_n * A)$

(1) $A * I_n = \left[\sum_{k=1}^{n}\left(a_{i,k}\left(\begin{cases}1 & k=j\\ 0 & k\neq j\end{cases}\right)\right)\right] = [a_{i,j}] = A$

(2) $TODO = A$

$ScalAssoc := \forall_{r,s\in\mathbb{R}}\forall_{A\in\mathcal{M}}(r(sA) = (rs)A = s(rA))$

(1) $r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$

(2) $(rs)A = [rsa_{i,j}]$

(3) $s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$

$TransCancel := \forall_{A\in\mathcal{M}}(A = (A^T)^T)$

(1) $A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$

$ScalMulCom := \forall_{r\in\mathbb{R}}\forall_{A,B\in\mathcal{M}}((rA) * B = r(A * B) = A * (rB))$

(1) $(rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[\sum_{k=1}^{p}(ra_{i,k}b_{k,j})\right] = r(A * B)$

(2) $A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[\sum_{k=1}^{p}(a_{i,k}rb_{k,j})\right] = \left[\sum_{k=1}^{p}(ra_{i,k}b_{k,j})\right] = r(A * B)$

$ScalDistLeft := \forall_{r,s\in\mathbb{R}}\forall_{A\in\mathcal{M}}((r+s)A = rA + sA)$

(1) TODO

$ScalDistRight := \forall_{r\in\mathbb{R}}\forall_{A,B\in\mathcal{M}}(r(A+B) = rA + rB)$

(1) TODO

$MulDistRight := \forall_{A,B,C\in\mathcal{M}}((A+B) * C = A * C + B * C)$

(1) $(A+B) * C = [a_{i,j} + b_{i,j}] * C = \left[\sum_{k=1}^{p}((a_{i,k} + b_{i,k})c_{k,j})\right] = \ ...$

(2) $...\left[\sum_{k=1}^{p}(a_{i,k}c_{k,j} + b_{i,k}c_{k,j})\right] = \left[\sum_{k=1}^{p}(a_{i,k}c_{k,j})\right] + \left[\sum_{k=1}^{p}(b_{i,k}c_{k,j})\right] = A * C + B * C$

$MulDistLeft := \forall_{A,B,C\in\mathcal{M}}(C * (A+B) = C * A + C * B)$

(1) TODO

$TransAddDist := \forall_{A,B\in\mathcal{M}}((A+B)^T = A^T + B^T)$

(1) TODO

$TransMulDist := \forall_{A,B\in\mathcal{M}}((A * B)^T = B^T * A^T)$

(1) $(A * B)^T = \left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]^T = \left[\sum_{k=1}^{p}(a_{j,k}b_{k,i})\right] = \left[\sum_{k=1}^{p}(b_{k,i}a_{j,k})\right] = \left[\sum_{k=1}^{p}(b^T_{i,k}a^T_{k,j})\right] = B^T * A^T$

$Sym[A] := A = A^T$
$SkewSym[A] := A = -A^T$
$Invertible[A] := (Square[A,n]) \wedge (\exists_{A^{-1}\in\mathcal{M}}(A * A^{-1} = I_n = A^{-1} * A))$

$SymGen := \forall_{A\in\mathcal{M}}(Sym[A + A^T])$

(1) $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$SkewSymGen := \forall_{A\in\mathcal{M}}(SkewSym[A - A^T])$

(1) $-(A - A^T)^T = -(A^T - (A^T)^T) = -(A^T - A) = (A - A^T)$

$SymDecomp := \forall_{A \in \mathcal{M}} \exists!_{B : Sym[B]} \exists!_{C : SkewSym[C]} (A = B + C)$

(1) $B := (1/2) * (A + A^T)\ ;\ C := (1/2) * (A - A^T)$

(2) $SymGen[B] \wedge SkewSymGen[C]$

(3) $A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$

(4) $(1/2) * (A_1 + {A_1}^T) = (1/2) * (A_2 + {A_2}^T)\ \blacksquare\ A_1 = A_2$

(5) $(1/2) * (A_3 - {A_3}^T) = (1/2) * (A_4 - {A_4}^T)\ \blacksquare\ A_3 = A_4$

$InvId := \forall_{A : Invertible[A]} (\exists!_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A))$

(1) ${A^{-1}}_1 = {A^{-1}}_1 * I_n = {A^{-1}}_1 * (A * {A^{-1}}_2) = ({A^{-1}}_1 * A) * {A^{-1}}_2 = I_n * {A^{-1}}_2 = {A^{-1}}_2$

$InvCancel := \forall_{A : Invertible[A]} ((A^{-1})^{-1} = A)$

(1) $(A * A^{-1})^{-1} = {I_n}^{-1} = I_n$

(2) $(A^{-1})^{-1} * A^{-1} = I_n\ \blacksquare\ A^{-1})^{-1} = I_n * A = A$

$InvDist := \forall_{A : Invertible[A]} \forall_{B : Invertible[B]} ((A * B)^{-1} = B^{-1} * A^{-1})$

(1) $(A * B) * (A * B)^{-1} = I\ \blacksquare\ B * (A * B)^{-1} = A^{-1}\ \blacksquare\ (A * B)^{-1} = B^{-1} * A^{-1}$

$InvTrans := \forall_{A : Invertible[A]} ((A^T)^{-1} = (A^{-1})^T)\ \blacksquare \Leftarrow$

(1) $A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I\ \blacksquare\ (A^{-1})^T = (A^T)^{-1}$

3.2 Elementary Matrices on Invertibility and Systems of Linear Equations

$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$
$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$
$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X (Sol[X, A, B])$
$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$
$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$

$HomoSysProps := (Sys[A, O]) \implies \ldots$

(1) $u_0 := O\ ;\ u_1 := choice(\{X \in \mathcal{M} | X \neq O\})\ ;\ k := choice(\mathbb{R})$

(2) $TrivSol[u_0, A]$

(3) $(NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$

(4) $(TrivSol[\overrightarrow{X}, A]) \implies (TrivSol[LC(\overrightarrow{X}), A])$

$ElemMat[E] := (E = Swap[I_n, i, j]) \vee (Scale_*(I_n, i, c)) \vee (Combine_*(I_n, i, c, j))$
$ElemMatProd[E^*] := \exists_{\langle E \rangle} (\forall_{E_i \in E^*} (ElemMat[E_i]) \wedge (E^* = \Pi_{E_i \in E^*} (E_i)))$
$RowEquiv[A, B] := \exists_{E^*} ((ElemMatProd[E^*]) \wedge (B = E^* * A))$

$ElemMatInv := \forall_{E \in \mathcal{M}} ((ElemMat[E]) \implies (Invertible[E]))$

(1) $E - RowSwap[E] \implies TODO\ ;\ E - RowScale_*(E) \implies TODO\ ;\ E - RowCombine_*(E) \implies TODO$

$ElemMatProdInv := \forall_{E^*} ((ElemMatProd[E^*]) \implies (Invertible[E^*]))$

(1) TODO

$RowEquivSys := \forall_{A,B,C,D,X \in \mathcal{M}} (((Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]])) \implies (Sol[X, A, B] \iff Sol[X, C, D]))$

(1) $\exists_{E^* : ElemMatProd[E^*]} ([CD] = E^* * [AB])$

(2) $(E^* * A = C) \wedge (E^* * B = D)$

(3) $Sol[Y, A, B] \implies \ldots$

(3.1) $A * Y = B$

(3.2) $C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D \blacksquare Sol[Y, C, D]$

(4) $Sol[Y, A, B] \implies Sol[Y, C, D]$

(5) $(A = (E^*)^{-1} * C) \wedge (B = (E^*)^{-1} * D)$

(6) $Sol[Z, C, D] \implies \ldots$

(6.1) $C * Z = D$

(6.2) $A * Z = ((E^*)^{-1} * C) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$

(7) $Sol[Z, C, D] \implies Sol[Z, A, B]$

(8) $Sol[X, A, B] \iff Sol[X, C, D]$

$RowEquivHomoSysSol := \forall_{A,C,X \in \mathcal{M}}((RowEquiv[A, C]) \implies ((Sol[X, A, O]) \iff (Sol[X, C, O])))$

(1) Set $B = D = O$

$RREF[A] := (A \in \mathcal{M}) \wedge \left(\begin{array}{l} \text{All zero rows are at the bottom of the matrix.} \qquad \wedge \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} \wedge \\ \text{The leading entry in any nonzero row is 1.} \qquad \wedge \\ \text{All entries in the column above and below a leading 1 are zero.} \qquad \wedge \end{array} \right)$

$GaussJordanElim := \forall_{A \in \mathcal{M}} \exists!_{B \in \mathcal{M}}((RREF[B]) \wedge (RowEquiv[A, B]))$

(1) Hit A with $ElemMat$'s until it becomes B

(2) $(B = E^* * A) \wedge (RREF[B])$

$HasZero[A] := (Matrix(A, m, n)) \wedge (\exists_{i \leq m}(A_{i,:} = O))$

$HasZeroNonInvertible := \forall_{A \in \mathcal{M}}((HasZero[A]) \implies (\neg Invertible[A]))$

(1) $i := choice(\{i \leq m | A_{i,:} = O\})$

(2) $(B \in \mathcal{M}) \implies \ldots$

(2.1) $(A * B)_{i,:} = O \neq I_{ni,:} \blacksquare A * B \neq I_n$

(3) $(B \in \mathcal{M}) \implies (A * B \neq I_n) \blacksquare \forall_{B \in \mathcal{M}}(A * B \neq I_n) \blacksquare \neg Invertible[A]$

$InvIffRowEquivI := \forall_{A \in \mathcal{M}}((Invertible[A]) \iff (RowEquiv[A, I_n]))$

(1) $(Invertible[A]) \implies \ldots$

(1.1) $(RREF[B]) \wedge (RowEquiv[A, B])$

(1.2) $B = E^* * A$

(1.3) $(Invertible[E^*]) \wedge (Invertible[A]) \blacksquare Invertible[B]$

(1.4) $Invertible[B] \blacksquare \neg HasZero[B]$

(1.5) $(RREF[B]) \wedge (\neg HasZero[B]) \blacksquare B = I_n$

(1.6) $RowEquiv[A, I_n]$

(2) $(Invertible[A]) \implies (RowEquiv[A, I_n])$

(3) $(RowEquiv[A, I_n]) \implies \ldots$

(3.1) $I_n = E^* * A \blacksquare (E^*)^{-1} = A$

(3.2) $A^{-1} = E^*_{DescSort} \blacksquare Invertible[A]$

(4) $(RowEquiv[A, I_n]) \implies (Invertible[A])$

(5) $(Invertible[A]) \iff (RowEquiv[A, I_n])$

$RowEquivIIffTrivSol := \forall_{A \in \mathcal{M}}((RowEquiv[A, I_n]) \iff (\forall_X((X = O) \iff (Sol[X, A, O]))))$

(1) $(RowEquiv[A, I_n]) \implies \ldots$

(1.1) $RowEquiv[A, I_n] \blacksquare Invertible[A]$

(1.2) $(Sol[X, A, O]) \implies \ldots$

(1.2.1) $A * X = O \blacksquare X = A^{-1} * O = O \blacksquare X = O$

(1.3) $(Sol[X, A, O]) \implies (X = O)$

(1.4) $(X = O) \implies (Sol[X, A, O])$

(1.5) $(X = O) \iff (Sol[X, A, O])$ ▌ $\forall_X((X = O) \iff (Sol[X, A, O]))$

(2) $(RowEquiv[A, I_n]) \implies (\forall_X((X = O) \iff (Sol[X, A, O])))$

(3) $(\forall_X((X = O) \iff (Sol[X, A, O]))) \implies \dots$

(3.1) $(RREF[B]) \wedge (RowEquiv[A, B])$

(3.2) $Sol[X, B, O]$

(3.3) $(B \neq I_n) \implies \dots$

(3.3.1) $(\exists_{Y \neq X}(Sol[Y, B, O]))$

(3.3.2) $Sol[Y, A, O]$ ▌ $Y = X$

(3.3.3) $(Y \neq X) \wedge (Y = X)$ ▌ $\perp$

(3.4) $(B \neq I_n) \implies \perp$ ▌ $B = I_n$

(3.5) $(RowEquiv[A, B]) \wedge (B = I_n)$ ▌ $RowEquiv[A, I_n]$

(4) $(\forall_X((X = O) \iff (Sol[X, A, O]))) \implies (RowEquiv[A, I_n])$

(5) $(RowEquiv[A, I_n]) \iff (\forall_X((X = O) \iff (Sol[X, A, O])))$

$InvIffUniqSol := \forall_{A \in \mathcal{M}}((Invertible[A]) \iff (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])))$

(1) $(Invertible[A] \wedge B \in \mathcal{M}) \implies \dots$

(1.1) $(Invertible[A]) \wedge (Sys[A, B])$

(1.2) $(X = A^{-1} * B) \iff (Sol[X, A, B])$ ▌ $\exists!_{X \in \mathcal{M}}(Sol[X, A, B])$

(2) $(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies \dots$

(2.1) $X_i := choice(\{X_i | Sol[X_i, A, I_{n:,i}]\})$

(2.2) $A * [X_1 \dots X_n] = [(A * X_1) \dots (A * X_n)] = [I_{n:,1} \dots I_{n:,n}] = I_n$

(2.3) $A^{-1} = [X_1 \dots X_n]$

(3) $(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies (Invertible[A])$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}} \begin{pmatrix} (Invertible[A]) \iff \\ (RowEquiv[A, I_n]) \iff \\ (\forall_X((X = O) \iff (Sol[X, A, O]))) \iff \\ (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \end{pmatrix}$$

3.3 Vector Spaces

$$VectorSpace[V, +, *] := \exists_{O \in V} \forall_{\alpha, \beta \in \mathbb{R}} \forall_{u,v,w \in V} \begin{pmatrix} (u + v \in V) \wedge (u + v = v + u) \wedge ((u + v) + w = u + (v + w)) \wedge \\ (u + O = u) \wedge (\exists_{-u \in V}(u + (-u) = O)) \wedge \\ (\alpha * u \in V) \wedge (\alpha * (\beta * u) = (\alpha\beta) * u) \wedge (1 * u = u) \wedge \\ (\alpha * (u + v) = (\alpha * u) + (\alpha * v)) \wedge ((\alpha + \beta) * u = (\alpha * u) + (\beta * u)) \end{pmatrix}$$

$ZeroVectorUniq := \forall_{O', v \in V}((v + O' = v) \implies (O' = O))$

(1) $O' = O' + O = O + O' = O$ ▌ $O' = O$

$AddInvUniq := \forall_{-v', v \in V}((v + -v' = O) \implies (-v' = -v))$

(1) $-v' = -v' + O = -v' + (v + -v) = (-v' + v) + -v = (v + -v') + -v = O + -v = -v$ ▌ $-v' = -v$

$AddInvGen := \forall_{v \in V}((-1) * v = -v)$

(1) $v + (-1) * v = (1 - 1) * v = 0 * v = O$ ▌ $(-1) * v = -v$

$ZeroVectorGenLeft := \forall_{v \in V}(0 * v = O)$

(1) $0 * v = (0 + 0) * v = (0 * v) + (0 * v)$ ▌ $O = 0 * v$

$ZeroVectorGenRight := \forall_{r \in \mathbb{R}}(r * O = O)$

(1) $r * O = r * (O + O) = (r * O) + (r * O)$ ▌ $O = r * O$

$ZeroVectorEquiv := \forall_{r\in\mathbb{R}}\forall_{v\in V}((r*v=O) \iff ((v=O)\vee(r=0)))$

(1) $(ZeroVectorGenLeft)\wedge(ZeroVectorGenRight)$ $\blacksquare$ $((v=O)\vee(r=0)) \implies (r*v=O))$

(2) $(r*v=O) \implies \ldots$

(2.1) $(r\neq 0) \implies \ldots$

(2.1.1) $r\neq 0$ $\blacksquare$ $r^{-1}\in\mathbb{R}$

(2.1.2) $ZeroVectorGenRight$ $\blacksquare$ $O=r^{-1}*O=r^{-1}*(r*v)=(r^{-1}r)*v=1*v=v$ $\blacksquare$ $O=v$

(2.2) $(r\neq 0) \implies (v=O)$ $\blacksquare$ $(r=0)\vee(v=O)$

(3) $(r*v=O) \implies ((r=0)\vee(v=O))$

(4) $(r*v=O) \iff ((r=0)\vee(v=O))$

3.4 Subspaces and Special Subspaces

$Subspace[S,V,+,*] := (VectorSpace[V,+,*])\wedge(S\subseteq V)\wedge(VectorSpace[S,+,*])$

$$SubspaceEquiv := \forall_{V,S}\left(\begin{array}{ll}(VectorSpace[V,+,*]) & \implies \\ ((Subspace[S,V,+,*]) \iff ((\emptyset\neq S\subseteq V)\wedge(\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S)))) & \end{array}\right)$$

(1) $(Subspace[S,V,+,*]) \implies \ldots$

(1.1) $Subspace[S,V,+,*]$ $\blacksquare$ $S\subseteq V$

(1.2) $VectorSpace[S,V,+,*]$ $\blacksquare$ $\exists_{O\in V}\forall_{v\in V}(v+O=v)$ $\blacksquare$ $O\in S$ $\blacksquare$ $\emptyset\neq S$

(1.3) $(\emptyset\neq S)\wedge(S\subseteq V)$ $\blacksquare$ $\emptyset\neq S\subseteq V$

(1.4) $VectorSpace[S,V,+,*]$ $\blacksquare$ $(\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))$

(1.5) $(\emptyset\neq S\subseteq V)\wedge(\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))$

(2) $(Subspace[S,V,+,*]) \implies ((\emptyset\neq S\subseteq V)\wedge(\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S)))$

(3) $((\emptyset\neq S\subseteq V)\wedge(\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))) \implies \ldots$

(3.1) $((\emptyset\neq S)\wedge(\alpha,\beta\in\mathbb{R})\wedge(u,v,w\in S)) \implies \ldots$

(3.1.1) $\emptyset\neq S$ $\blacksquare$ $\exists_x(x\in V)$

(3.1.2) $(ZeroVectorGenLeft)\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))\wedge(x\in V)$ $\blacksquare$ $O=0*x\in S$ $\blacksquare$ $O\in S$

(3.1.3) $u,v\in V$ $\blacksquare$ $u+v=v+u$

(3.1.4) $u,v,w\in V$ $\blacksquare$ $(u+v)+w=u+(v+w)$

(3.1.5) $u\in V$ $\blacksquare$ $u+O=u$

(3.1.6) $(AddInvGen)\wedge(u\in S)$ $\blacksquare$ $(-1)*u=-u\in S$

(3.1.7) $u\in V$ $\blacksquare$ $\alpha*(\beta*u)=(\alpha\beta)*u$

(3.1.8) $u\in V$ $\blacksquare$ $1*u=u$

(3.1.9) $u,v\in V$ $\blacksquare$ $\alpha*(u+v)=(\alpha*u)+(\alpha*v)$

(3.1.10) $u\in V$ $\blacksquare$ $(\alpha+\beta)*u=(\alpha*u)+(\beta*u)$

(4) $((\emptyset\neq S)\wedge(\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))) \implies (Subspace[S,V,+,*])$

(5) $(Subspace[S,V,+,*]) \iff ((\emptyset\neq S)\wedge(\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S)))$

$SetSum[A+B,A,B,V,+,*] := (VectorSpace[V,+,*])\wedge(A,B\subseteq V)\wedge(A+B=\{a+b|(a\in A)\wedge(b\in B)\})$

$$SumSubContains := \forall_{A,B,V}\left(\begin{array}{l}((Subspace[A,V,+,*])\wedge(Subspace[B,V,+,*])\wedge(SetSum[A+B,A,B,V,+,*])) \implies \\ ((Subspace[A+B,V,+,*])\wedge(A,B\subseteq A+B))\end{array}\right)$$

(1) $(Subspace[A,V,+,*])\wedge(Subspace[B,V,+,*])$ $\blacksquare$ $(O\in A)\wedge(O\in B)$

(2) $(SetSum[A+B,A,B,V,+,*])\wedge(O\in A)\wedge(O\in B)$ $\blacksquare$ $O=O+O\in A+B$ $\blacksquare$ $\emptyset\neq A+B$

(3) $(v\in A+B) \implies \ldots$

(3.1) $\exists_{a\in A}\exists_{b\in B}(v=a+b)$

(3.2) $(A\subseteq V)\wedge(B\subseteq V)$ $\blacksquare$ $a,b\in V$

(3.3) $VectorSpace[V,+,*]$ $\blacksquare$ $v=a+b\in V$

(4) $(v \in A+B) \implies (v \in V) \blacksquare A+B \subseteq V$

(5) $(\emptyset \neq A+B) \wedge (A+B \subseteq V) \blacksquare \emptyset \neq A+B \subseteq V$

(6) $(u, v \in A+B) \implies \ldots$

(6.1) $(\exists_{a_1 \in A} \exists_{b_1 \in B}(u = a_1 + b_1)) \wedge (\exists_{a_2 \in A} \exists_{b_2 \in B}(v = a_2 + b_2))$

(6.2) $u + v = (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$

(6.3) $(a_1 + a_2 \in A) \wedge (b_1 + b_2 \in B) \blacksquare u + v \in A + B$

(7) $(u, v \in A+B) \implies (u + v \in A+B) \blacksquare \forall_{u,v \in A+B}(u + v \in A+B)$

(8) $((r \in \mathbb{R}) \wedge (v \in A+B)) \implies \ldots$

(8.1) $\exists_{a \in A} \exists_{b \in B}(v = a + b)$

(8.2) $r * v = r * (a + b) = r * a + r * b$

(8.3) $(r * a \in A) \wedge (r * b) \in B \blacksquare r * v \in A + B$

(9) $((r \in \mathbb{R}) \wedge (v \in A+B)) \implies (r * v \in A+B) \blacksquare \forall_{r \in \mathbb{R}} \forall_{v \in A+B}(r * v \in A+B)$

(10) $(SubspaceEquiv) \wedge (\emptyset \neq A+B \subseteq V) \wedge (\forall_{u,v \in A+B}(u + v \in A+B)) \wedge (\forall_{r \in \mathbb{R}} \forall_{v \in A+B}(r * v \in A+B)) \blacksquare Subspace[A+B, V, +, *]$

(11) $(O \in B) \wedge (\forall_{a \in A}(a + O) = a) \blacksquare A \subseteq A+B$

(12) $(O \in A) \wedge (\forall_{b \in B}(b + O) = b) \blacksquare B \subseteq A+B$

(13) $(A \subseteq A+B) \wedge (B \subseteq A+B) \blacksquare A, B \subseteq A+B$

(14) $(Subspace[A+B, V, +, *]) \wedge (A, B \subseteq A+B)$

$$SumSubMinContains := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \wedge (SetSum[A+B,A,B,V,+,*])) \implies \\ (\forall_C((Subspace[C,V,+,*]) \wedge (A,B \subseteq C)) \implies (A+B \subseteq C)) \end{array} \right)$$

(1) $SumSub \blacksquare (A, B \subseteq A+B) \wedge (Subspace[A+B, V, +, *])$

(2) $((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies \ldots$

(2.1) $(s \in A+B) \implies \ldots$

(2.1.1) $\exists_{a \in A} \exists_{b \in B}(s = a + b)$

(2.1.2) $(A, B \subseteq C) \blacksquare a, b \in C$

(2.1.3) $(VectorSpace[C, V, +, *]) \wedge (a, b \in C) \blacksquare s = a + b \in C$

(2.2) $(s \in A+B) \implies (s \in C) \blacksquare A+B \subseteq C$

(3) $((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies (A+B \subseteq C)$

$$DirSum[A \oplus B, A, B, V, +, *] := \left(\begin{array}{l} (Subspace[A,V,+,*]) \quad \wedge \quad (Subspace[B,V,+,*]) \quad \wedge \\ (SetSum[A+B,A,B,V,+,*]) \wedge (\forall_{s \in A+B} \exists!_{\langle a,b \rangle \in A \times B}(s = a + b)) \end{array} \right)$$

$$DirSumEquiv := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \wedge (SetSum[A+B,A,B,V,+,*])) \implies \\ ((DirSum[A \oplus B,A,B,V,+,*]) \iff (\exists!_{\langle a,b \rangle \in A \times B}(O = a + b))) \end{array} \right)$$

(1) $(DirSum[A \oplus B, A, B, V, +, *]) \implies \ldots$

(1.1) $(Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \blacksquare (O \in A) \wedge (O \in B)$

(1.2) $(SubSum[A \oplus B, A, B, V, +, *]) \wedge (O \in A) \wedge (O \in B) \blacksquare O = O + O \in A \oplus B$

(1.3) $(DirSum[A \oplus B, A, B, V, +, *]) \wedge (O \in A \oplus B) \blacksquare \exists!_{\langle a,b \rangle \in A \times B}(O = a + b)$

(2) $(DirSum[A \oplus B, A, B, V, +, *]) \implies (\exists!_{\langle a,b \rangle \in A \times B}(O = a + b))$

(3) $(\exists!_{\langle a,b \rangle \in A \times B}(O = a + b)) \implies \ldots$

(3.1) $(s \in A \oplus B) \implies \ldots$

(3.1.1) $(\exists_{\langle a,b \rangle \in A \times B}(s = a + b))$

(3.1.2) $((s = a_1 + b_1) \wedge (s = a_2 + b_2)) \implies \ldots$

(3.1.2.1) $O = s - s = (a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2)$

(3.1.2.2) $(Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \blacksquare (a_1 - a_2 \in A) \wedge (b_1 - b_2 \in B)$

(3.1.2.3) $((a_1 - a_2 \neq O) \vee (b_1 - b_2 \neq O)) \implies (\neg \exists!_{\langle a,b \rangle \in A \times B}(O = a + b)) \implies \bot$

(3.1.2.4) $(a_1 - a_2 = O) \wedge (b_1 - b_2 = O) \blacksquare \langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$

(3.1.3) $((s = a_1 + b_1) \wedge (s = a_2 + b_2)) \implies \langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$

(3.1.4) $\forall_{\langle a_1,b_1 \rangle, \langle a_2,b_2 \rangle \in A \times B}(((s = a_1 + b_1) \wedge (s = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle))$

(3.1.5) $\exists_{\langle a,b \rangle \in A \times B}(s = a + b) \wedge \forall_{\langle a_1,b_1 \rangle, \langle a_2,b_2 \rangle \in A \times B}(((s = a_1 + b_1) \wedge (s = a_2 + b_2)) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle)) \blacksquare \exists!_{\langle a,b \rangle \in A \times B}(s = a + b)$

(3.2) $(s \in A+B) \implies \exists!_{\langle a,b \rangle \in A \times B}(s = a + b) \blacksquare \forall_{s \in A+B} \exists!_{\langle a,b \rangle \in A \times B}(s = a + b) \blacksquare DirSum[A \oplus B, A, B, V, +, *]$

(4) $(\exists!_{\langle a,b\rangle \in A\times B}(O = a+b)) \implies (DirSum[A \oplus B, A, B, V, +, *])$

(5) $(DirSum[A \oplus B, A, B, V, +, *]) \iff (\exists!_{\langle a,b\rangle \in A\times B}(O = a+b))$

$$DirSumSubspace := \forall_{A,B,V}\left(\begin{array}{l} ((Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \wedge (SetSum[A+B,A,B,V,+,*])) \implies \\ ((DirSum[A \oplus B, A, B, V, +, *]) \iff (A \cap B = \{O\})) \end{array}\right)$$

(1) $(DirSum[A \oplus B, A, B, V, +, *]) \implies \ldots$

(1.1) $(v \in A \cap B) \implies \ldots$

(1.1.1) $(v \in A \cap B) \wedge (VectorSpace[B,+,*]) \blacksquare (v \in A) \wedge (v \in B) \blacksquare (v \in A) \wedge (-v \in B)$

(1.1.2) $(v \in A) \wedge (-v \in B) \blacksquare v + (-v) = O \in A + B$

(1.1.3) $DirSum[A \oplus B, A, B, V, +, *] \blacksquare \exists!_{\langle a,b\rangle \in A\times B}(O = a+b)$

(1.1.4) $(v \neq O) \implies (\neg\exists!_{\langle a,b\rangle \in A\times B}(O = a+b)) \implies \bot \blacksquare v = O$

(1.2) $(v \in A \cap B) \implies (v = O) \blacksquare A + B \subseteq \{O\}$

(1.3) $(v = O) \implies \ldots$

(1.3.1) $(Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \blacksquare (O \in A) \wedge (O \in B) \blacksquare v = O \in A \cup B$

(1.4) $(v = O) \implies (v \in A \cap B) \blacksquare \{O\} \subseteq A \cap B$

(1.5) $(A + B \subseteq \{O\}) \wedge (\{O\} \subseteq A \cap B) \blacksquare A \cap B = \{O\}$

(2) $(DirSum[A \oplus B, A, B, V, +, *]) \implies (A \cap B = \{O\})$

(3) $(A \cap B = \{O\}) \implies \ldots$

(3.1) $(O \in A) \wedge (O \in B) \wedge (O = O + O \in A + B) \blacksquare \exists_{\langle a,b\rangle \in A\times B}(O = a+b)$

(3.2) $((\langle a_1,b_1\rangle, \langle a_2,b_2\rangle \in A \times B) \wedge (O = a_1 + b_1) \wedge (O = a_2 + b_2)) \implies \ldots$

(3.2.1) $(O = a_1 + b_1) \wedge (O = a_2 + b_2) \blacksquare (a_1 = -b_1) \wedge (a_2 = -b_2)$

(3.2.2) $VectorSpace[B,+,*] \blacksquare -b_1, -b_2 \in B$

(3.2.3) $(a_1 \in A) \wedge (a_1 = -b_1 \in B) \blacksquare a_1 \in A \cap B \blacksquare a_1 = O \blacksquare a_1 = b_1 = O$

(3.2.4) $(a_2 \in A) \wedge (a_2 = -b_2 \in B) \blacksquare a_2 \in A \cap B \blacksquare a_2 = O \blacksquare a_2 = b_2 = O$

(3.2.5) $\langle a_1, b_1\rangle = \langle O, O\rangle = \langle a_2, b_2\rangle$

(3.3) $((\langle a_1,b_1\rangle, \langle a_2,b_2\rangle \in A \times B) \wedge (O = a_1 + b_1) \wedge (O = a_2 + b_2)) \implies (\langle a_1,b_1\rangle = \langle a_2,b_2\rangle)$

(3.4) $\forall_{\langle a_1,b_1\rangle, \langle a_2,b_2\rangle \in A\times B}(((O = a_1 + b_1) \wedge (O = a_2 + b_2)) \implies (\langle a_1,b_1\rangle = \langle a_2,b_2\rangle))$

(3.5) $(\exists_{\langle a,b\rangle \in A\times B}(O = a+b)) \wedge (\forall_{\langle a_1,b_1\rangle, \langle a_2,b_2\rangle \in A\times B}(((O = a_1 + b_1) \wedge (O = a_2 + b_2)) \implies (\langle a_1,b_1\rangle = \langle a_2,b_2\rangle)))$

(3.6) $(\exists!_{\langle a,b\rangle \in A\times B}(O = a+b)) \wedge (DirSumEquiv) \blacksquare DirSum[A \oplus B, A, B, V, +, *]$

(4) $(A \cap B = \{O\}) \implies (DirSum[A \oplus B, A, B, V, +, *])$

(5) $(DirSum[A \oplus B, A, B, V, +, *]) \iff (A \cap B = \{O\})$

$NullSpace[N, A, m, n] := (Matrix[A, m, n]) \wedge (N = \{x \in \mathbb{R}^n | A * x = O\})$
$RowSpace[R, A, m, n] := (Matrix[A, m, n]) \wedge (R = \{x^T * A \in \mathbb{R}^n | x \in \mathbb{R}^m\})$
$ColSpace[C, A, m, n] := (Matrix[A, m, n]) \wedge (C = \{A * x \in \mathbb{R}^m | x \in \mathbb{R}^n\})$

$NullSubspace := (NullSpace[N, A, m, n]) \implies (Subspace[N, \mathbb{R}^n, +, *])$

(1) TODO

$RowSubspace := (RowSpace[R, A, m, n]) \implies (Subspace[R, \mathbb{R}^n, +, *])$

(1) TODO

$ColSubspace := (ColSpace[C, A, m, n]) \implies (Subspace[C, \mathbb{R}^m, +, *])$

(1) TODO

3.5 Linear Combination, Linear Span, Linear Independence

$LinComb[c, U, K, V, +, *] := (VectorSpace[V, +, *]) \wedge (n \in \mathbb{N}) \wedge (U \in V^n) \wedge (K \in \mathbb{R}^n) \wedge (c = \sum_{i=1}^n (k_i * u_i))$

$$LinSpan[S', S, V, +, *] := \left(\begin{array}{l} (VectorSpace[V,+,*]) \wedge (S \in V^n) \wedge ((S = \emptyset) \implies (S' = \{O\})) \quad \wedge \\ ((S \neq \emptyset) \implies (S' = \{c \in V | (K \in \mathbb{R}^n) \wedge (LinComb[c, S, K, V, +, *])\})) \end{array}\right)$$

$LinSpanSubContains := \forall_{S',S,V}((LinSpan[S',S,V,+,*]) \implies ((Subspace[S',V,+,*]) \wedge (S \subseteq S')))$

(1) $(S = \emptyset) \implies \ldots$

(1.1) $LinSpan[S',S,V,+,*]$ ▌ $S' = \{O\}$

(1.2) $Subspace[\{O\},V,+,*]$ ▌ $Subspace[S',V,+,*]$

(1.3) $S = \emptyset \subseteq \{O\} = S'$ ▌ $S \subseteq S'$

(1.4) $(Subspace[S',V,+,*]) \wedge (S \subseteq S')$

(2) $(S = \emptyset) \implies ((Subspace[S',V,+,*]) \wedge (S \subseteq S'))$

(3) $(S \neq \emptyset) \implies \ldots$

(3.1) $LinSpan[S',S,V,+,*]$ ▌ $S' = \{c \in V|(K \in \mathbb{R}^n) \wedge (LinComb[c,S,K,V,+,*])\}$ ▌ $S' \subseteq V$

(3.2) $(\{0\}^n \subseteq \mathbb{R}^n) \wedge (LinComb[O,S,\{0\}^n,V,+,*])$ ▌ $O \in S'$ ▌ $\emptyset \neq S'$

(3.3) $(S' \subseteq V) \wedge (\emptyset \neq S')$ ▌ $\emptyset \neq S' \subseteq V$

(3.4) $(a,b \in S') \implies \ldots$

(3.4.1) $(\exists_{K_a \in \mathbb{R}^n}(LinComb[a,S,K_a,V,+,*])) \wedge (\exists_{K_b \in \mathbb{R}^n}(LinComb[b,S,K_b,V,+,*]))$ ▌ $(a = \sum_{i=1}^n (k_{ai} * s_i)) \wedge (b = \sum_{i=1}^n (k_{bi} * s_i))$

(3.4.2) $a + b = \sum_{i=1}^n (k_{ai} * s_i) + \sum_{i=1}^n (k_{bi} * s_i) = \sum_{i=1}^n ((k_{ai} + k_{bi}) * s_i)$ ▌ $a + b = \sum_{i=1}^n ((k_{ai} + k_{bi}) * s_i)$

(3.4.3) $\langle k_{ai} + k_{bi} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n$

(3.4.4) $(a + b = \sum_{i=1}^n ((k_{ai} + k_{bi}) * s_i)) \wedge (\langle k_{ai} + k_{bi} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n)$ . . .

(3.4.5) . . . $\exists_{M \in \mathbb{N}^n}(a + b = \sum_{i=1}^n (m_i * s_i))$ ▌ $\exists_{M \in \mathbb{N}^n}(LinComb[a+b,S,M,V,+,*])$ ▌ $a + b \in S'$

(3.5) $(a,b \in S') \implies (a + b \in S')$ ▌ $\forall_{a,b \in S'}(a + b \in S')$

(3.6) $((r \in \mathbb{R}) \wedge (u \in S')) \implies \ldots$

(3.6.1) $\exists_{K \in \mathbb{R}^n}(LinComb[u,S,K,V,+,*])$ ▌ $u = \sum_{i=1}^n (k_i * s_i)$

(3.6.2) $r * u = r * \sum_{i=1}^n (k_i * s_i) = \sum_{i=1}^n (r * (k_i * s_i)) = \sum_{i=1}^n (rk_i) * s_i$ ▌ $r * u = \sum_{i=1}^n (rk_i) * s_i$

(3.6.3) $\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n$

(3.6.4) $(r * u = \sum_{i=1}^n (rk_i) * s_i)) \wedge (\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n)$ ▌ $\exists_{M \in \mathbb{R}^n}(r * u = \sum_{i=1}^n (m_i * s_i))$

(3.6.5) $\exists_{M \in \mathbb{R}^n}(LinComb[r * u,S,M,V,+,*])$ ▌ $r * u \in S'$

(3.7) $((r \in \mathbb{R}) \wedge (u \in S')) \implies (r * u \in S')$ ▌ $\forall_{r \in \mathbb{R}}\forall_{u \in S'}(r * u \in S')$

(3.8) $(SubspaceEquiv) \wedge (\emptyset \neq S' \subseteq V) \wedge (\forall_{a,b \in S'}(a + b \in S')) \wedge (\forall_{r \in \mathbb{R}}\forall_{u \in S'}(r * u \in S'))$ ▌ $Subspace[S',V,+,*]$

(3.9) $(s_i \in S) \implies \ldots$

(3.9.1) $K_s := \langle \left(\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \right) | j \in \mathbb{N}_{1,n}\rangle$ ▌ $(K_s \in \mathbb{R}^n) \wedge (\sum_{j=1}^n (k_{sj} * s_j) = s_i)$

(3.9.2) $\exists_{K \in \mathbb{R}^n}(LinComb[s_j,S,K,V,+,*])$ ▌ $s_j \in S'$

(3.10) $(s_i \in S) \implies (s_i \in S')$ ▌ $S \subseteq S'$

(3.11) $(Subspace[S',V,+,*]) \wedge (S \subseteq S')$

(4) $(S \neq \emptyset) \implies ((Subspace[S',V,+,*]) \wedge (S \subseteq S'))$

(5) $((S = \emptyset) \implies ((Subspace[S',V,+,*]) \wedge (S \subseteq S'))) \wedge ((S \neq \emptyset) \implies ((Subspace[S',V,+,*]) \wedge (S \subseteq S')))$ . . .

(6) . . . $(Subspace[S',V,+,*]) \wedge (S \subseteq S')$

$LinSpanSubMinContains := \forall_{S',S,V,+,*}((LinSpan[S',S,V,+,*]) \implies (\forall_W(((Subspace[W,V,+,*]) \wedge (S \subseteq W)) \implies (S' \subseteq W)))$

(1) $(s' \in S') \implies \ldots$

(1.1) $\exists_{K \in \mathbb{R}^n}(LinComb[s',S,K,V,+,*])$ ▌ $s' = \sum_{i=1}^n (k_i * s_i)$

(1.2) $(S \subseteq W) \wedge (VectorSpace[W,V,+,*])$ ▌ $s' = \sum_{i=1}^n (k_i * s_i) \in W$ ▌ $s' \in W$

(2) $(s' \in S') \implies (s' \in W)$ ▌ $S' \subseteq W$

$Spans[S,V,+,*] := LinSpan[V,S,V,+,*]$
$FinDim[V,+,*] := \exists_{S \in V^n}(Spans[S,V,+,*])$

$LinInd[S,V,+,*] := (VectorSpace[V,+,*]) \wedge (S \in V^n) \wedge ((S \neq \emptyset) \implies (\forall_{K \in \mathbb{R}^n}((LinComb[O,S,K,V,+,*]) \implies (K = \{0\}^n))))$

$ZeroDependent := (O \in S) \implies (\neg LinInd[S,V,+,*])$

(1) $O \in S \;\blacksquare\; \exists_{u_i \in S}(u_i = O) \;\blacksquare\; K := \langle \left(\begin{cases} 1 & u_i = O \\ 0 & u_i \neq O \end{cases} \right) | i \in \mathbb{N}_{1,n} \rangle \;\blacksquare\; \{O\}^n \neq K \in \mathbb{R}^n$

(2) $O = \sum_{i=1}^{n}(k_i * s_i) \;\blacksquare\; LinComb[O,S,K,V,+,*]$

(3) $(LinComb[O,S,K,V,+,*]) \wedge (\{O\}^n \neq K \in \mathbb{R}^n) \;\blacksquare\; \exists_{K \in \mathbb{R}^n}((LinComb[O,S,K,V,+,*]) \wedge (K \neq \{0\}^n)) \;\blacksquare\; \neg LinInd[S,V,+,*]$

$SingletonNonZeroIndependent := (v \neq O) \implies (LinInd[\langle v \rangle, V,+,*])$

(1) $((\langle r \rangle \in \mathbb{R}^1) \wedge (LinComb[O,\langle v \rangle,\langle r \rangle,V,+,*])) \implies \ldots$

(1.1) $(ZeroVectorEquiv) \wedge (r * v = O) \;\blacksquare\; (r * v = O) \iff ((r = 0) \vee (v \neq O))$

(1.2) $v \neq O \;\blacksquare\; r = 0$

(2) $((\langle r \rangle \in \mathbb{R}^1) \wedge (LinComb[O,\langle v \rangle,\langle r \rangle,V,+,*])) \implies (r = 0) \;\blacksquare\; \forall_{r \in \mathbb{R}}((LinComb[O,\langle v \rangle,\langle r \rangle,V,+,*]) \implies (r = 0))$

(3) $LinInd[\langle v \rangle,V,+,*]$

$$SubIndependent := \forall_{V,A,B}\left(\begin{array}{l} ((VectorSpace[V,+,*]) \wedge (A \subseteq B) \wedge (A \in V^n) \wedge (B \in V^m)) \implies \\ ((LinInd[B,V,+,*]) \implies (LinInd[A,V,+,*])) \end{array} \right)$$

(1) $((K \in \mathbb{R}^n) \wedge (LinComb[O,A,K,V,+,*])) \implies \ldots$

(1.1) $n \leq m \;\blacksquare\; L := \langle \left(\begin{cases} k_j & j \leq n \\ 0 & j > n \end{cases} \right) | j \in \mathbb{N}_{1,m}) \rangle \;\blacksquare\; L \in \mathbb{R}^m$

(1.2) $A \subseteq B \;\blacksquare\; \forall_{j \in \mathbb{N}_{1,n}}(a_j = b_j) \;\blacksquare\; \sum_{i=1}^{n}(k_i * a_i)) = \sum_{j=1}^{m}(l_j * b_j))$

(1.3) $LinComb[O,A,K,V,+,*] \;\blacksquare\; O = \sum_{i=1}^{n}(k_i * a_i) = \sum_{j=1}^{m}(l_j * b_j)) \;\blacksquare\; LinComb[O,B,L,V,+,*]$

(1.4) $(LinInd[B,V,+,*]) \wedge (LinComb[O,B,L,V,+,*]) \;\blacksquare\; L = \{0\}^m \;\blacksquare\; K = \{0\}^n$

(2) $((K \in \mathbb{R}^n) \wedge (LinComb[O,A,K,V,+,*])) \implies (K = \{0\}^n) \;\blacksquare\; LinInd[A,V,+,*]$

$SuperDependent := \forall_{V,A,B}(((VectorSpace[V,+,*]) \wedge (A \subseteq B \subseteq V)) \implies ((\neg LinInd[A,V,+,*]) \implies (\neg LinInd[B,V,+,*])))$

(1) $\neg LinInd[A,V,+,*] \;\blacksquare\; \exists_K((LinComb[O,A,K,V,+,*]) \wedge (K \neq \{0\}^n))$

(2) $n \leq m \;\blacksquare\; L := \langle \left(\begin{cases} k_j & j \leq n \\ 0 & j > n \end{cases} \right) | j \in \mathbb{N}_{1,m}) \rangle \;\blacksquare\; L \in \mathbb{R}^m$

(3) $A \subseteq B \;\blacksquare\; \forall_{j \in \mathbb{N}_{1,n}}(a_j = b_j) \;\blacksquare\; \sum_{i=1}^{n}(k_i * a_i)) = \sum_{j=1}^{m}(l_j * b_j))$

(4) $LinComb[O,A,K,V,+,*] \;\blacksquare\; LinComb[O,B,L,V,+,*]$

(5) $K \neq \{0\}^n \;\blacksquare\; L \neq \{0\}^m$

(6) $\exists_L((LinComb[O,B,L,V,+,*]) \wedge (L \neq \{0\}^m)) \;\blacksquare\; \neg LinInd[B,V,+,*]$

$LinDepProp := \forall_{S,V}((\neg LinInd[S,V,+,*]) \implies (\exists_{s_j \in S}\exists_{K \in \mathbb{R}^{n-1}}(LinComb[s_j, S \setminus \{s_j\}, K,V,+,*])))$

(1) $\neg LinInd[S,V,+,*] \;\blacksquare\; \exists_{K \in \mathbb{R}^n}((LinComb[O,S,K,V,+,*]) \wedge (K \neq \{0\}^n))$

(2) $K \neq \{0\}^n \;\blacksquare\; \exists_{j \in \mathbb{N}_{1,n}}((k_j \neq 0) \wedge (\forall_{i \in \mathbb{N}_{j+1,n}}(k_i = 0))) \;\; . . .$

(3) $. . . \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{j}(k_i * s_i) = \sum_{i=1}^{j-1}(k_i * s_i) + k_j * s_j \;\blacksquare\; \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{j-1}(k_i * s_i) + k_j * s_j$

(4) $(LinComb[O,S,K,V,+,*]) \wedge (\sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{j-1}(k_i * s_i) + k_j * s_j) \;\blacksquare\; O = \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{j-1}(k_i * s_i) + k_j * s_j$

(5) $s_j = (-1/k_j)\sum_{i=1}^{j-1}(k_i * s_i) = \sum_{i=1}^{j-1}((-k_i/k_j) * s_i) \;\blacksquare\; s_j = \sum_{i=1}^{j-1}((-k_i/k_j) * s_i)$

(6) $\exists_{K \in \mathbb{R}^{n-1}}(LinComb[s_j, S \setminus \{s_j\}, K,V,+,*])$

$LinDepPropCorollary := \forall_{P,S,V}(((\neg LinInd[S,V,+,*]) \wedge (LinSpan[P,S,V,+,*])) \implies (\exists_{s_j \in S}(LinSpan[P, S \setminus \{s_j\},V,+,*])))$

(1) $LinDepProp \;\blacksquare\; \exists_{s_j \in S}\exists_{K \in \mathbb{R}^{n-1}}(LinComb[s_j, S \setminus \{s_j\}, K,V,+,*])$

(2) $\forall_{u \in P}((\exists_{K_1}(LinComb[u,S,K_1,V,+,*])) \implies (\exists_{K_2}(LinComb[u, S \setminus \{s_j\}, K_2,V,+,*]))) \;\blacksquare\; LinSpan[P, S \setminus \{s_j\},V,+,*]$

$LinIndEquiv := \forall_{S,V}((LinInd[S,V,+,*]) \iff (\forall_{s_j \in S}\forall_{K \in \mathbb{R}^{n-1}}(\neg LinComb[s_j, S \setminus \{s_j\}, K,V,+,*])))$

(1) $LinDepProp \;\blacksquare\; (\neg LinInd[S,V,+,*]) \implies (\exists_{s_j \in S} \exists_{K \in \mathbb{R}^{n-1}} (LinComb[s_j, S \setminus \{s_j\}, K, V, +, *]))\ \ldots$

(2) $\ldots (\forall_{s_j \in S} \forall_{K \in \mathbb{R}^{n-1}} (\neg LinComb[s_j, S \setminus \{s_j\}, K, V, +, *])) \implies (LinInd[S,V,+,*])$

(3) $(\exists_{s_j \in S} \exists_{K \in \mathbb{R}^{n-1}} (LinComb[s_j, S \setminus \{s_j\}, K, V, +, *])) \implies \ldots$

(3.1) $L := \langle \left(\begin{cases} k_i & i \neq j \\ -1 & i = j \end{cases} \right) | i \in \mathbb{N}_{1,n}) \rangle \;\blacksquare\; (L \in \mathbb{R}^n) \wedge (L \neq \{0\}^n)$

(3.2) $LinComb[s_j, S \setminus \{s_j\}, K, V, +, *] \;\blacksquare\; \ldots \;\blacksquare\; \sum_{i=1}^{j-1}(k_i * s_i) + k_j * s_j = \sum_{i=1}^{j-1}(k_i * s_i) + -\sum_{i=1}^{j-1}(k_i * s_i) = O\ \ldots$

(3.3) $\ldots LinComb[O, S, L, V, +, *]$

(3.4) $(LinComb[O,S,L,V,+,*]) \wedge (L \neq \{0\}^n) \;\blacksquare\; \exists_{L \in \mathbb{R}^n}((LinComb[O,S,L,V,+,*]) \wedge (L \neq \{0\}^n)) \;\blacksquare\; (\neg LinInd[S,V,+,*])$

(4) $(\exists_{s_j \in S} \exists_{K \in \mathbb{R}^{n-1}} (LinComb[s_j, S \setminus \{s_j\}, K, V, +, *])) \implies (\neg LinInd[S,V,+,*])$

(5) $(LinInd[S,V,+,*]) \implies (\forall_{s_j \in S} \forall_{K \in \mathbb{R}^{n-1}} (\neg LinComb[s_j, S \setminus \{s_j\}, K, V, +, *]))$

(6) $(LinInd[S,V,+,*]) \iff (\forall_{s_j \in S} \forall_{K \in \mathbb{R}^{n-1}} (\neg LinComb[s_j, S \setminus \{s_j\}, K, V, +, *]))$

$LinIndSuperspace := \forall_{U,V}((Subspace[U,V]) \implies (\forall_W((LinInd[W,U,+,*]) \implies (LinInd[W,V,+,*]))))$

(1) $(\neg LinInd[W,V,+,*]) \implies \ldots$

(1.1) $\exists_{j \in W}(LinComb[j, W \setminus \{j\}, +, *]) \;\blacksquare\; \neg LinInd[W,U,+,*]$

(1.2) $(LinInd[W,U,+,*]) \wedge (\neg LinInd[W,U,+,*]) \;\blacksquare\; \bot$

(2) $(\neg LinInd[W,V,+,*]) \implies \bot \;\blacksquare\; LinInd[W,V,+,*]$

3.6 Bases and Dimensions

$Basis[S,V,+,*] := (Spans[S,V,+,*]) \wedge (LinInd[S,V,+,*])$

$BasisEquiv := \forall_{S,V}((Basis[S,V,+,*]) \iff (\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*]))$

(1) $(Basis[S,V,+,*]) \implies \ldots$

(1.1) $(v \in V) \implies \ldots$

(1.1.1) $Basis[S,V,+,*] \;\blacksquare\; Spans[V,S,+,*] \;\blacksquare\; \exists_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*])$

(1.1.2) $((K_1, K_2 \in \mathbb{R}^n) \wedge (LinComb[v,S,K_1,V,+,*]) \wedge (LinComb[v,S,K_2,V,+,*])) \implies \ldots$

(1.1.2.1) $(v = \sum(k_{1i} * s_i)) \wedge (v = \sum(k_{2i} * s_i))$

(1.1.2.2) $O = v - v = \sum(k_{1i} * s_i) - \sum(k_{2i} * s_i) = \sum((k_{1i} - k_{2i}) * s_i)$

(1.1.2.3) $L := \langle k_{1i} - k_{2i} | i \in \mathbb{N}_{i=1}^n \rangle \in \mathbb{R}^n$

(1.1.2.4) $(LinInd[S,V,+,*]) \wedge (LinComb[O,S,L,V,+,*]) \;\blacksquare\; L = \{0\}^n \;\blacksquare\; K_2 = K_1$

(1.1.3) $((K_1, K_2 \in \mathbb{R}^n) \wedge (LinComb[v,S,K_1,V,+,*]) \wedge (LinComb[v,S,K_2,V,+,*])) \implies (K_1 = K_2)$

(1.1.4) $\forall_{K_1,K_2 \in \mathbb{R}^n}((LinComb[v,S,K_1,V,+,*]) \wedge (LinComb[v,S,K_2,V,+,*]) \implies (K_1 = K_2))$

(1.1.5) $\exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*])$

(1.2) $(v \in V) \implies (\exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*]))$

(2) $(Basis[S,V,+,*]) \implies (\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*]))$

(3) $(\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*])) \implies \ldots$

(3.1) $\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*]) \;\blacksquare\; \forall_{v \in V} \exists_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*]) \;\blacksquare\; Spans[S,V,+,*]$

(3.2) $O \in V \;\blacksquare\; \exists!_{K \in \mathbb{R}^n}(LinComb[O,S,K,V,+,*])$

(3.3) $(K \neq \{0\}^n) \implies (\neg \exists!_{K \in \mathbb{R}^n}(LinComb[O,S,K,V,+,*])) \implies \bot \;\blacksquare\; K = \{0\}^n$

(3.4) $(\exists!_{K \in \mathbb{R}^n}(LinComb[O,S,K,V,+,*])) \wedge (K = \{0\}^n) \;\blacksquare\; LinInd[S,V,+,*]$

(3.5) $(Spans[S,V,+,*]) \wedge (LinInd[S,V,+,*]) \;\blacksquare\; Basis[S,V,+,*]$

(4) $(\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*])) \implies (Basis[S,V,+,*])$

$SpanReduceBasis := \forall_{S,V}((Spans[S,V,+,*]) \implies (\exists_B((B \subseteq S) \wedge (Basis[B,V,+,*]))))$

(1) $LinDepPropCorollary \;\blacksquare\; \exists_B((B \subseteq S) \wedge (LinInd[B,V,+,*]) \wedge (Spans[B,V,+,*])) \;\blacksquare\; \exists_B((B \subseteq S) \wedge (Basis[B,V,+,*]))$

TODO - formalize removing latter entries first

$FinDimBasis := \forall_V((FinDim[V,+,*]) \implies (\exists_B(Basis[B,V,+,*])))$

(1) $FinDim[V,+,*] \blacksquare \exists_{S\in V^n}(Spans[S,V,+,*])$

(2) $(SpanReduceBasis) \wedge (Spans[S,V,+,*]) \blacksquare \exists_B(Basis[B,V,+,*])$

$LinIndExpandBasis := \forall_{L,V}((LinInd[L,V,+,*]) \implies (\exists_B((L \subseteq B) \wedge (Basis[B,V,+,*]))))$

(1) $FinDimBasis \blacksquare \exists_C(Basis[C,V,+,*])$

(2) $S := L \cup C$

(3) $Basis[C,V,+,*] \blacksquare Spans[C,V,+,*] \blacksquare Spans[S,V,+,*]$

(4) $SpanReduceBasis \blacksquare (\exists_B((B \subseteq S) \wedge (Basis[B,V,+,*]))) \wedge (L \subseteq B)$

CONTHERE $BasisLinearIndCard := \forall_{S,T,V}(((Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*])) \implies (|T| \leq |S|))$

(1) $(Basis[S,V,+,*]) \implies \ldots$

(1.1) $(|T| > |S|) \implies \ldots$

(1.1.1) $(Spans[S,V,+,*]) \wedge (T \subseteq V) \blacksquare t_{1\ldots t_j} = \sum(\gamma_i * s * i) \ \ldots$

(1.1.2) $\ldots t_j = \sum(\gamma'_i * t_i) \blacksquare \neg LinInd[T,V,+,*]$

(1.2) $(|T| > |S|) \implies (\neg LinInd[T,V,+,*]) \blacksquare (LinInd[T,V,+,*]) \implies (|T| \leq |S|)$

(2) $((Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*])) \implies (|T| \leq |S|)$

$BasisCard := \forall_{S,T,V}(((Basis[S,V,+,*]) \wedge (Basis[T,V,+,*])) \implies (|T| = |S|))$

(1) $Basis[S,V,+,*] \blacksquare LinInd[S,V,+,*]$

(2) $(Basis[T,V,+,*]) \wedge (LinInd[S,V,+,*]) \blacksquare |S| \leq |T|$

(3) $Basis[T,V,+,*] \blacksquare LinInd[T,V,+,*]$

(4) $(Basis[S,V,+,*]) \wedge (LinInd[T,V,+,*]) \blacksquare |T| \leq |S|$

(5) $(|S| \leq |T|) \wedge (|T| \leq |S|) \blacksquare |T| = |S|$

$Dim[d,V,+,*] := (\exists_B(Basis[B,V,+,*])) \wedge ((V = \{O\}) \implies (d = 0)) \wedge ((V \neq \{O\}) \implies (d = |B|))$

$LinIndLengthDim := \forall_{U,V}(((LinInd[U,V,+,*]) \wedge (Dim[|U|,V,+,*])) \implies (Basis[U,V,+,*]))$

(1) $(LinIndExpandBasis) \wedge (LinInd[U,V,+,*]) \blacksquare \exists_B((U \subseteq B) \wedge (Basis[B,V,+,*]))$

(2) $(BasisCard) \wedge (Dim[|U|,V,+,*]) \wedge (Basis[B,V,+,*]) \blacksquare |B| = |U| \blacksquare B = U \blacksquare Basis[U,V,+,*]$

$SpanLengthDim := \forall_{U,V}(((Spans[U,V,+,*]) \wedge (Dim[|U|,V,+,*])) \implies (Basis[U,V,+,*]))$

(1) $(SpanReduceBasis) \wedge (Spans[U,V,+,*]) \blacksquare \exists_B((B \subseteq U) \wedge (Basis[B,V,+,*]))$

(2) $(BasisCard) \wedge (Dim[|U|,V,+,*]) \wedge (Basis[B,V,+,*]) \blacksquare |B| = |U| \blacksquare B = U \blacksquare Basis[U,V,+,*]$

$LinDepLengthDim := \forall_{U,V}(((U \subseteq V) \wedge (|U| > Dim[V])) \implies (\neg LinInd[U,V,+,*]))$

(1) Contrapositive of $BasisLinearIndCard$

$LinDepLengthDim := \forall_{U,V}(((U \subseteq V) \wedge (|U| < Dim[V])) \implies (\neg Spans[U,V,+,*]))$

(1) Suppose $Spans[U,V,+,*]$, $B = SpanReduceBasis[U]$ to form a basis, $(|B| \leq |U| < Dim[V]) \wedge |B| = Dim[V] \blacksquare \perp$

(2) $\neg Spans[U,V,+,*]$

3.7 Rank

$Nullity[n,A] := (NullSpace[N,A]) \wedge (Dim[n,N,+,*])$
$Rank[r,A,m,n] := (Matrix[A,m,n]) \wedge (RowSpace[R,A,m,n]) \wedge (Dim[r,R,A,+,*])$

$RowRankEqColRank := \forall_A(TODO)$

(1) TODO

$RankNullity := \forall_A((Matrix[A,m,n]) \implies (Rank[A] + Nullity[A] = n))$

(1) TODO

$RankInv := \forall_A((Matrix[A,m,n]) \implies ((Rank[A] = n) \iff (Inv[A])))$

(1) TODO

$RankNonTrivialSol := (\exists_X((A * X = O) \wedge (X \neq O))) \iff (Rank[A] < n)$

(1) TODO

$RankUniqueSol := (\forall_{B\in\mathcal{M}}\exists!_{X\in\mathcal{M}}(Sol[X,A,B])) \iff (Rank[A] = n)$

(1) TODO

$$SquareTheorems_{()} := \forall_{A\in\mathcal{M}}\left(\begin{array}{ll} (Invertible[A]) & \iff \\ (RowEquiv[A, I_n]) & \iff \\ (\forall_X((X = O) \iff (Sol[X,A,O]))) & \iff \\ (\forall_{B\in\mathcal{M}}\exists!_{X\in\mathcal{M}}(Sol[X,A,B])) & \iff \\ (Rank[A] = n) & \iff \\ (Nullity[A] = 0) & \iff \\ \text{(The rows form a linearly independent set of vectors (to get full rank))} & \iff \\ \text{(The columns form a linearly independent set of vectors (to get full rank))} & \iff \end{array}\right)$$

3.8 Linear Transformations

$$LinTrans[L,V,+_v,*_v,W,+_w,*_w] := \left(\begin{array}{l} (Function[f,V,W]) \wedge (VectorSpace[V,+_v,*_v]) \wedge (VectorSpace[W,+_w,*_w]) \wedge \\ (\forall_{\alpha,\beta\in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))) \quad \wedge \quad (\forall_{r\in\mathbb{R}}\forall_{\alpha\in V}(L(r *_v \alpha) = r *_w L(\alpha))) \end{array}\right)$$

$LinOp[L,V,+_v,*_v] := LinTrans[L,V,+_v,*_v,V,+_v,*_v]$

$\mathcal{L}[V,W] := \{L | LinTrans[L,V,+_v,*_v,W,+_w,*_w]\}$

$ZeroMapsToZero := \forall_{L,V,W}((LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \implies (L(O_v) = O_w))$

(1) $L(O_v) = L(O_v +_v O_v) = L(O_v) +_w L(O_v)$

(2) $O_w = L(O_v) - L(O_v) = L(O_v)$

$SplitAddInv := \forall_{L,V,W}((LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \implies (\forall_{\alpha,\beta\in V}(L(\alpha -_v \beta) = L(\alpha) -_w L(\beta))))$

(1) $L(\alpha - \beta) = L(\alpha + (-\beta)) = L(\alpha) + L(-\beta) = L(\alpha) + (-1) * L(\beta) = L(\alpha) - L(\beta)$

$$BasisDomainInduceLinTrans := \forall_{V,W}\left(\begin{array}{l} ((Basis[A,V,+_v,*_v]) \wedge (B \subseteq W) \wedge (n = |B| = |A|) \wedge (VectorSpace[W,+_w,*_w])) \implies \\ (\exists!_T((LinTrans[T,V,+_v,*_v,W,+_w,*_w]) \wedge (\forall_{i\in\mathbb{N}_{1,n}}(T(a_i) = b_i)))) \end{array}\right)$$

(1) $T(\sum_{i=1}^n(k_i * a_i)) := \sum_{i=1}^n(k_i * b_i)$

(2) $(i \in \mathbb{N}_{1,n}) \implies \ldots$

(2.1) $L := \left\langle \left(\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \right) | j \in \mathbb{N}_{1,n} \right\rangle \;\blacksquare\; L \in \mathbb{R}^n$

(2.2) $T(a_i) = T(\sum_{i=1}^n(l_i * a_i)) = \sum_{i=1}^n(l_i * b_i) = b_i \;\blacksquare\; T(a_i) = b_i$

(3) $(i \in \mathbb{N}_{1,n}) \implies (T(a_i) = b_i) \;\blacksquare\; \forall_{i\in\mathbb{N}_{1,n}}(T(a_i) = b_i)$

(4) $(BasisEquiv) \wedge (Basis[A,V,+_v,*_v]) \;\blacksquare\; \forall_{v\in V}\exists!_{K\in\mathbb{R}^n}(LinComb[v,A,K,V,+,*]) \;\blacksquare\; T : Span[A] \to Span[B] \;\blacksquare\; Function[T,V,W]$

(5) $(\alpha, \beta \in V) \implies \ldots$

(5.1) $(\exists_{K_\alpha}(LinComb[\alpha,A,K_\alpha,V,+_v,*_v])) \wedge (\exists_{K_\beta}(LinComb[\beta,A,K_\beta,V,+_v,*_v])) \;\blacksquare\; (\alpha = \sum_{i=1}^n(k_{\alpha i} * a_i)) \wedge (\beta = \sum_{i=1}^n(k_{\beta_i} * a_i))$

(5.2) $T(\alpha + \beta) = T(\sum_{i=1}^n(k_{\alpha i} * a_i) + \sum_{i=1}^n(k_{\beta_i} * a_i)) = T(\sum_{i=1}^n((k_{\alpha i} + k_{\beta_i}) * a_i))) = \sum_{i=1}^n((k_{\alpha i} + k_{\beta_i}) * b_i) = \ldots$

(5.3) $\ldots \sum_{i=1}^n(k_{\alpha i} * b_i) + \sum_{i=1}^n(k_{\beta_i} * b_i) = T(\sum_{i=1}^n(k_{\alpha i} * a_i)) + T(\sum_{i=1}^n(k_{\beta_i} * a_i)) = T(\alpha) + T(\beta)$

(6) $(\alpha, \beta \in V) \implies (L(\alpha +_v \beta) = L(\alpha) +_w L(\beta)) \blacksquare \forall_{\alpha,\beta \in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))$

(7) $((r \in \mathbb{R}) \wedge (\alpha \in V)) \implies \ldots$

(7.1) $\exists_K(LinComb[\alpha, A, K, V, +_v, *_v]) \blacksquare \alpha = \sum_{i=1}^{n}(k_i * a_i)$

(7.2) $L(r *_v \alpha) = L(r *_v \sum_{i=1}^{n}(k_i *_v a_i)) = L(\sum_{i=1}^{n}((rk_i) *_v a_i)) = \ \ldots$

(7.3) $\ldots \sum_{i=1}^{n}((rk_i) *_w b_i) = r *_w \sum_{i=1}^{n}(k_i *_w b_i) = r *_w L(\sum_{i=1}^{n}(k_i *_v a_i)) = r *_w L(\alpha)$

(8) $((r \in \mathbb{R}) \wedge (\alpha \in V)) \implies (L(r *_v \alpha) = r *_w L(\alpha)) \blacksquare \forall_{r\in\mathbb{R}}\forall_{\alpha\in V}(L(r *_v \alpha) = r *_w L(\alpha))$

(9) $(\forall_{i\in\mathbb{N}_{1,n}}(T(a_i) = b_i)) \wedge (Function[T, V, W]) \wedge (\forall_{\alpha,\beta\in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))) \wedge (\forall_{r\in\mathbb{R}}\forall_{\alpha\in V}(L(r *_v \alpha) = r *_w L(\alpha))) \wedge \ \ldots$

(10) $\ldots (VectorSpace[V, +_v, *_v]) \wedge (VectorSpace[W, +_w, *_w]) \blacksquare LinTrans[T, V, +_v, *_v, W, +_w, *_w]$

$Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (ker_L = \{\alpha \in V | L(\alpha) = O_w\})$

$KerSub := \forall_{L,V,W}((Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w]) \implies (Subspace[ker_L, V, +_v, *_v]))$

(1) $ZeroMapsToZero \blacksquare L(O_v) = O_w \blacksquare O_v \in ker_L \blacksquare \emptyset \neq ker_L \blacksquare \emptyset \neq ker_L \subseteq V$

(2) $(\alpha, \beta \in ker_L) \implies \ldots$

(2.1) $(L(\alpha) = O_w) \wedge (L(\beta) = O_w)$

(2.2) $L(\alpha + \beta) = L(\alpha) + L(\beta) = O_w + O_w = O_w \blacksquare L(\alpha + \beta) \in ker_L$

(3) $(\alpha, \beta \in ker_L) \implies (\alpha + \beta \in ker_L) \blacksquare \forall_{\alpha,\beta\in ker_L}(\alpha + \beta \in ker_L)$

(4) $((r \in \mathbb{R}) \wedge (\alpha \in ker_L)) \implies \ldots$

(4.1) $L(\alpha) = O_w \blacksquare L(r * \alpha) = r * L(\alpha) = r * O_w = O_w \blacksquare r * \alpha \in ker_L$

(5) $((r \in \mathbb{R}) \wedge (\alpha \in ker_L)) \implies (r * \alpha \in ker_L) \blacksquare \forall_{r\in\mathbb{R}}\forall_{\alpha\in ker_L}(r * \alpha \in ker_L)$

(6) $(SubspaceEquiv) \wedge (\emptyset \neq ker_L \subseteq V) \wedge (\forall_{\alpha,\beta\in ker_L}(\alpha + \beta \in ker_L)) \wedge (\forall_{r\in\mathbb{R}}\forall_{\alpha\in ker_L}(r * \alpha \in ker_L)) \blacksquare Subspace[ker_L, V, +_v, *_v]$

$KerInjective := \forall_{L,V,W}((Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w]) \implies ((Injective[L, V, W]) \iff (ker_L = \{O_v\})))$

(1) $(Injective[L, V, W]) \implies \ldots$

(1.1) $ZeroMapsToZero \blacksquare L(O_v) = O_w$

(1.2) $O_v \in ker_L \blacksquare \{O_v\} \subseteq ker_L$

(1.3) $(v \in ker_L) \implies \ldots$

(1.3.1) $L(v) = O_w$

(1.3.2) $(Injective[L, V, W]) \wedge (L(O_v) = O_w) \blacksquare O_v = v$

(1.4) $(v \in ker_L) \implies (v = O_v) \blacksquare ker_L \subseteq \{O_v\}$

(1.5) $(\{O_v\} \subseteq ker_L) \wedge (ker_L \subseteq \{O_v\}) \blacksquare ker_L = \{O_v\}$

(2) $(Injective[L, V, W]) \implies (ker_L = \{O_v\})$

(3) $(ker_L = \{O_v\}) \implies \ldots$

(3.1) $((u, v \in V) \wedge (L(u) = L(v))) \implies \ldots$

(3.1.1) $O_w = L(u) - L(v) = L(u - v) \blacksquare u - v \in ker_L$

(3.1.2) $ker_L = \{O_v\} \blacksquare u - v = O_v \blacksquare u = v$

(3.2) $((u, v \in V) \wedge (L(u) = L(v))) \implies (u = v) \blacksquare \forall_{u,v\in V}((L(u) = L(v)) \implies (u = v)) \blacksquare Injective[L, V, W]$

(4) $(ker_L = \{O_v\}) \implies (Injective[L, V, W])$

(5) $(Injective[L, V, W]) \iff (ker_L = \{O_v\})$

$Rng[rng_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (rng_L = \{\beta \in W | \exists_{\alpha\in V}(\beta = L(\alpha))\})$

$RangeSub := \forall_{L,V,W}((Ran[rng_L, L, V, +_v, *_v, W, +_w, *_w]) \implies (Subspace[rng_L, W, +_w, *_w]))$

(1) $ZeroMapsToZero \blacksquare O_w = L(O_v) \blacksquare \exists_{\alpha\in V}(O_w = L(\alpha)) \blacksquare O_w \in rng_L \blacksquare \emptyset \neq rng_L \blacksquare \emptyset \neq rng_L \subseteq W$

(2) $(\alpha, \beta \in rng_L) \implies \ldots$

(2.1) $(\exists_{u\in V}(\alpha = L(u))) \wedge (\exists_{v\in V}(\beta = L(v)))$

(2.2) $\alpha + \beta = L(u) + L(v) = L(u + v) \blacksquare \exists_{w\in V}(\alpha + \beta = L(w)) \blacksquare \alpha + \beta \in rng_L$

(3) $(\alpha, \beta \in rng_L) \implies (\alpha + \beta \in rng_L) \blacksquare \forall_{\alpha,\beta\in rng_L}(\alpha + \beta \in rng_L)$

(4) $((r \in \mathbb{R}) \wedge (\alpha \in rng_L)) \implies \ldots$

(4.1) $\exists_{v\in V}(\alpha = L(v)) \blacksquare L(r * v) = r * L(v) = r * \alpha \blacksquare \exists_{w\in V}(r * \alpha = L(w)) \blacksquare r * \alpha \in rng_L$

(5) $((r \in \mathbb{R}) \wedge (\alpha \in rng_L)) \implies (r * \alpha \in rng_L)$ ▮ $\forall_{r\in\mathbb{R}}\forall_{\alpha\in rng_L}(r * \alpha \in rng_L)$

(6) $(SubspaceEquiv) \wedge (\emptyset \neq rng_L \subseteq W) \wedge (\forall_{\alpha,\beta\in rng_L}(\alpha + \beta \in rng_L)) \wedge (\forall_{r\in\mathbb{R}}\forall_{\alpha\in rng_L}(r * \alpha \in rng_L))$ ▮ $Subspace[rng_L, W, +_w, *_w]$

$RankKer := \forall_{L,V,W}((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies (Dim[V] = Dim[ker_L] + Dim[rng_L]))$

(1) $\exists_U(Basis[U, ker_L, +_v, *_v])$ ▮ $Dim[ker_L] = |U|$

(2) $(LinIndSuperspace) \wedge (LinInd[U, ker_L, +_v, *_v])$ ▮ $LinInd[U, V, +_v, *_v]$

(3) $LinIndExpandBasis$ ▮ $\exists_B((U \subseteq B) \wedge (Basis[B, V, +_v, *_v]))$ ▮ $Dim[V] = |B|$

(4) $T := B \setminus U$ ▮ $B = U \cup T$

(5) $m := |U|\ ;\ n := |T|\ ;\ p := |B|$

(6) $L(T) := \langle L(t_i) | i \in \mathbb{N}_{1,n}\rangle \subseteq W^n$

(7) $(w \in W) \implies \ldots$

(7.1) $\exists_{v\in V}(w = L(v))$

(7.2) $Basis[B, V, +_v, *_v]$ ▮ $\exists_{K\in\mathbb{R}^p}(v = \sum_{i=1}^{p}(k_i * b_i))$

(7.3) $v = \sum_{i=1}^{p}(k_i * b_i) = \sum_{i=1}^{m}(k_i * u_i) + \sum_{i=1}^{n}(\kappa_i * t_i)$

(7.4) $w = L(v) = L(\sum_{i=1}^{m}(k_i * u_i) + \sum_{i=1}^{n}(k_i * t_i)) = \sum_{i=1}^{m}(k_i * L(u_i)) + \sum_{i=1}^{n}(k_i * L(t_i)) = O_w + \sum_{i=1}^{n}(k_i * L(t_i)) = \sum_{i=1}^{n}(k_i * L(t_i))$

(7.5) $\exists_K(LinComb[w, L(T), K, W, +_w, *_w])$

(8) $(w \in W) \implies (\exists_K(LinComb[w, L(T), K, W, +_w, *_w]))$ ▮ $\forall_{w\in W}(\exists_K(LinComb[w, L(T), K, W, +_w, *_w]))$

(9) $Spans[L(T), W, +_w, *_w]$

(10) $((K \in \mathbb{R}^n) \wedge (LinComb[O_w, L(T), K, W, +_w, *_w])) \implies \ldots$

(10.1) $O_w = \sum_{i=1}^{n}(k_i * L(t_i)) = L(\sum_{i=1}^{n}(k_i * t_i))$ ▮ $\sum_{i=1}^{n}(k_i * t_i) \in ker_L$

(10.2) $Basis[U, ker_L, +_v, *_v]$ ▮ $\exists_{D\in\mathbb{R}^m}(\sum_{i=1}^{n}(k_i * t_i) = \sum_{i=1}^{m}(d_i * u_i))$

(10.3) $(LinInd[T \cup U]) \wedge (\sum_{i=1}^{n}(k_i * t_i) = \sum_{i=1}^{m}(d_i * u_i))$ ▮ $K = D = \{O\}$ ▮ $K = \{O\}$

(11) $((K \in \mathbb{R}^n) \wedge (LinComb[O_w, L(T), K, W, +_w, *_w])) \implies (K = \{O\})$

(12) $\forall_{K\in\mathbb{R}^n}((LinComb[O_w, L(T), K, W, +_w, *_w]) \implies (K = \{O\}))$

(13) $\forall_{K\in\mathbb{R}^n}((LinComb[O_w, L(T), K, W, +_w, *_w]) \iff (K = \{O\}))$ ▮ $LinInd[L(T), W, +_w, *_w]$

(14) $Basis[L(T), W, +_w, *_w]$ ▮ $Dim[V] = |B| = |U| + |L(T)| = Dim[ker_L] + Dim[rng_L]$

$LTSurInj := \forall_{L,V,W}(((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (Surjective[L, V, W])) \implies (Injective[L, V, W]))$

(1) TODO

$LTInjSur := \forall_{L,V,W}(((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (Injective[L, V, W])) \implies (Surjective[L, V, W]))$

(1) TODO

$LTInjLinInd := \forall_{L,V,W}((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies ((Injective[L, V, W]) \iff$ (LI In W are LI IN V)))

(1) TODO

3.9 Matrix of a Linear Transform