LECTURE NOTES IN GRAPH THEORY

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1 Undirected graphs

1.1 Simple Graph, Multigraph and Pseudograph

Definition 1.1. A *graph* G is an ordered pair $(V(G), E(G))$. The set $V = V(G)$ called the *vertex set* of G is a set of vertices and the set $E = E(G)$, called the *edge set* of G is a set of unordered pairs of distinct elements of V. The cardinality of the $V(G)$ and of $E(G)$ are called the *order* and *size* of G, respectively.

Example 1.1. Let $G = (V, E)$, where $V = \{x_1, x_2, x_3, x_4, x_5\}$ and

$$E = \{[x_1, x_3], [x_1, x_5], [x_2, x_4], [x_3, x_5], [x_4, x_1], [x_4, x_3], [x_5, x_2]\}.$$

A graph can be shown pictorially. The vertices of a graph can be represented by points and edges can be represented by lines connecting the vertices. A pictorial representation of the graph G is given in Example 1.1 is given in Figure 1 below.

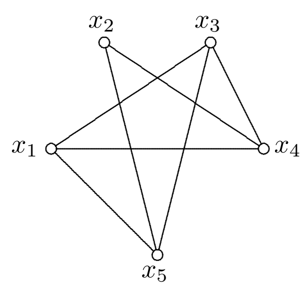


Figure 1: The graph G

We will use the notation $x_i x_j$ to denote the edge $[x_i, x_j]$. Furthermore, if $x_i x_j$ is an edge of the graph G, we then say that x_i and x_j are *adjacent* and the vertices x_i and x_j are *incident* with the edge $x_i x_j$.

Remark 1.1. We note that the definition of a graph in Definition 1.1 permits no edges of the form $x_i x_i$, that is edges joining points to itself. These edges are called *loops*.

There are several variations of a graph. Two of these are called multigraph and pseudographs. A *multigraph* do not allow loops but allows multiple edges connecting the same vertices. A *pseudograph* allows both loops and multiple edges.

A graph which has neither loops nor multiple edges is called a *simple graph*. For this notes, when we say graphs, we mean simple graphs.

Definition 1.2. A *subgraph* $H = (V(H), E(H))$ of the graph $G = (V(G), E(G))$ is the graph satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is called a *spanning subgraph* of G.

Example 1.2. Consider the graph in Exmp. 1.1. Let $H = (V(H), E(H))$, where $V = \{x_1, x_2, x_3, x_4, x_5\}$ and $E(H) = \{x_1x_5, x_2x_4, x_3x_5, x_4x_1, x_4x_3\}$. The graph H is a spanning subgraph of G.

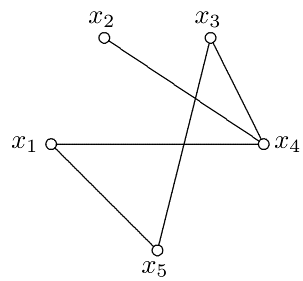


Figure 2: The spanning subgraph H of G

Definition 1.3. Let $S \subseteq V(G)$, where G is a graph. The *induced subgraph* $< S >$ is the maximal subgraph of G with point set S. In other words, two vertices of S are adjacent in $< S >$ if and only if they are adjacent is G.

Remark 1.2. A subgraph H of a graph G may be obtained by performing the following operations:

1. *Removal of a point.* Let G be a graph and v_i be a vertex of G. The removal of v_i results in a subgraph $G - v_i$ of G consisting of all the vertices of G except v_i and all edges incident to v_i. Thus, $G - v_i$ is a maximal subgraph of G not containing v_i;

2. *Removal of an edge e_j.* Let G be a graph and e_j be an edge of G. The removal of the edge e_j results in a subgraph $G - e_j$ of G consisting of all edges of G except e_j. Thus, $G - e_j$ is a spanning subgraph of G.

Example 1.3. Consider the graph in Exmp. 1.1. The graph $G - x_2$ is

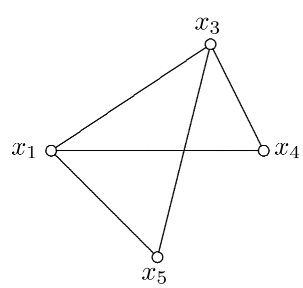


Figure 3: The graph $G - x_2$

Example 1.4. Consider the graph in Exmp. 1.1. The graph $G - x_1x_5$ is

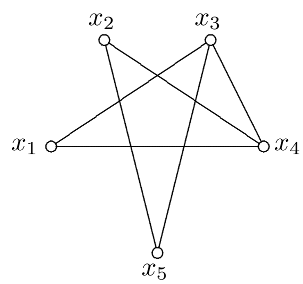


Figure 4: The graph $G - x_1x_5$

Example 1.5. Let $G = (V, E)$ be graph with

$$V = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5\}$$

and

$$E = \{x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5, x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5, y_1y_2, y_2y_3, y_3y_4, y_4y_5, y_5y_1\}.$$

Do the following:

1. Draw a pictorial representation of G;
2. Let $H = (V(H), E(H))$ be a subgraph of G, where $E(H) = \{x_iy_i | i = 1, 2, 3, 4, 5\}$.
3. Draw a pictorial representation of $< S_1 >$, where $S_1 = \{x_1, x_2, x_3, x_4, x_5\}$.
4. Draw a pictorial representation of $< S_2 >$, where $S_2 = \{y_1, y_2, y_3, y_4, y_5\}$.

The graph G in Example 1.5 is called the *Petersen graph*.

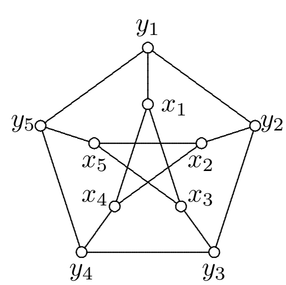


Figure 5: The Petersen graph

1.2 Walk, path, cycle of a graph

Definition 1.4. A *walk* of length n in a graph is a sequence of vertices $x_1, x_2, \ldots, x_n, x_{n+1}$ such that x_ix_{i+1} is an edge for each $i = 1, 2, \ldots, n$. A walk is said to be *closed* if $x_1 = x_{n+1}$.

Definition 1.5. A walk is called a *path* if $x_1, x_2, \ldots, x_n, x_{n+1}$ are distinct.

Definition 1.6. A closed walk is called a *cycle* if $(n+1) > 1$ and $x_1, x_2, \ldots, x_n$ is a path.

Remark 1.3. A cycle with three vertices is called a *triangle*. The length n of a walk is the number of edges in it.

Definition 1.7. The *girth* of a graph G, denoted by $g(G)$, is the length of the shortest cycle in G. The *circumference* of a graph G, denoted by $c(G)$ if the length of the longest cycle.

Remark 1.4. The terms girth and circumference of G are undefined if G has no cycles.

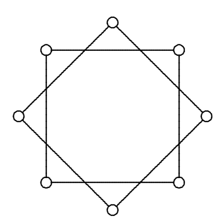
1.3 Connected graph, components of a graph and regular graphs

Definition 1.8. A graph is said to be *connected* if every pair of vertices are joined by a path. A maximal connected subgraph of G is called a *connected component* or simply a *component* of G.

Definition 1.9. Let G be a graph and v be a vertex of G. The *degree* of v, denoted by $deg(v)$, is the number edges incident with it. If for every vertex v_i of G, $deg(v_i) = r$, we then say that G is *r-regular*.

Remark 1.5. We can see that the Petersen graph is a connected graph and is 3-regular.

Example 1.6. Given the graph below. Is this graph connected? If not,



how many component does it have? Is this a regular graph?

Theorem 1.1. *Let $G = (V, E)$. The sum of the degrees of the vertices of a graph G is twice the number of edges,*

$$\sum_{v \in V} deg(v) = 2|E|.$$

Corollary 1.1.1. *In any graph, the number of vertices of odd degree is even.*

1.4 The adjacency matrix

For every graph G of order n, there is an associated $n \times n$ matrix called the adjacency matrix of G. We define the adjacency matrix below.

Definition 1.10. Let G be a graph of order n with $V(G) = \{x_1, x_2, \ldots, x_n\}$. The $n \times n$ matrix denoted by $\mathcal{A}(G) = [a_{ij}]$ and defined by

$$a_{ij} = \begin{cases} 1 \text{ if } x_i x_j \in E(G) \\ 0 \text{ if } x_i x_j \notin E(G) \end{cases}$$

is called the *adjacency matrix* of G.

Example 1.7. The adjacency matrix of the graph given in Example 1.1 is

$$\mathcal{A}(G) = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Example 1.8. Give the adjacency matrices for the Petersen graph and the graph given in Example 1.6.

2 Some special classes of graphs

This chapter introduces some special classes of graphs.

2.1 Path

Definition 2.1. The *path of order n* which is denoted by P_n, is defined to be the graph with $V(P_n) = \{x_1, x_2, \ldots, x_n\}$ and $E(P_n) = \{x_i x_{i+1} | i = 1, 2, \ldots, n-1\}$.

The path of order 6 is illustrated in Figure 6

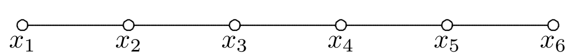


Figure 6: The path P_6

Example 2.1. The paths of orders 4 and 7 are given below. Give the adjacency matrices $\mathcal{A}(P_4)$ and $\mathcal{A}(P_7)$.

Figure 7: The path P_4

Figure 8: The path P_7

2.2 Cycle

Definition 2.2. The *cycle of order* n, denoted by C_n, is the graph with $V(C_n) = \{x_1, x_2, \ldots, x_n\}$ and $E(C_n) = \{x_i x_{i+1} | i = 1, 2, \ldots, n-1\} \cup \{x_n x_1\}$.

The cycle of order 6, C_6 is given in Figure 9.

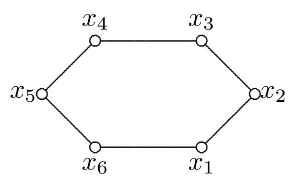
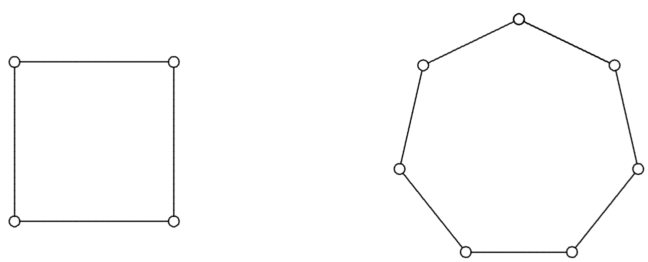


Figure 9: The cycle C_6

Example 2.2. The cycles of orders 4 and 7 are given below. Give the adjacency matrices $\mathcal{A}(C_4)$ and $\mathcal{A}(C_7)$.



C_4 $\qquad$ C_7

2.3 Fan

Definition 2.3. The *fan of order* $n+1$, denoted by F_n, is defined to be the graph with $V(F_n) = \{x_0, x_1, x_2, \ldots, x_n\}$ and $E(F_n) = \{x_i x_{i+1} | i = 1, 2, \ldots, n-1\} \cup \{x_0 x_i | i = 1, 2, \ldots, n\}$.

The fan of order 7, F_6 is illustated in Figure 10

Example 2.3. The fans of orders 4 and 8 are given below. Give the adjacency matrices $\mathcal{A}(F_3)$ and $\mathcal{A}(F_7)$.

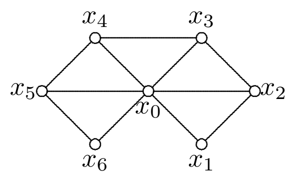
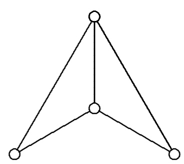
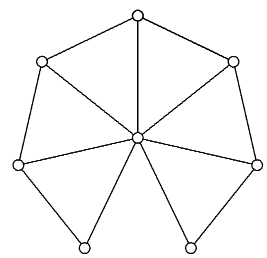


Figure 10: The fan F_6



F_3



F_7

2.4 Wheel

Definition 2.4. The *wheel of order* $n+1$, denoted by W_n, is defined to be the graph with $V(W_n) = \{x_0, x_1, x_2, \ldots, x_n\}$ and $E(W_n) = \{x_i x_{i+1} | i = 1, 2, \ldots, n-1\} \cup \{x_n x_1\} \cup \{x_0 x_i | i = 1, 2, \ldots, n\}$.

We can see that if the edge $x_n x_1$ is added to the edge set of the fan, F_n, we obtain the wheel of order $n+1$. The wheel of order 7, W_6 is given in Figure 11

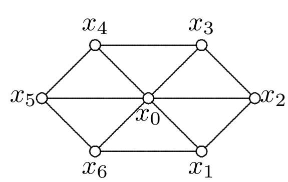
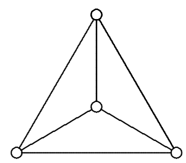
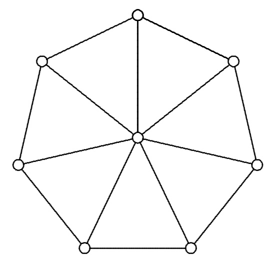


Figure 11: The wheel W_6

Example 2.4. The wheels of orders 4 and 8 are given below. Give the adjacency matrices $\mathcal{A}(W_3)$ and $\mathcal{A}(W_7)$.



W_3



W_7

2.5 Star

Definition 2.5. The *star of order* $n+1$, denoted by S_n, is the graph with vertex set and edge set as $V(S_n) = \{x_0, x_1, x_2, \ldots, x_n\}$ and $E(S_n) = \{x_0x_i | i = 1, 2, \ldots, n\}$.

The star of order 7, S_6 is illustrated in Figure 12.

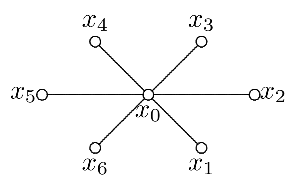
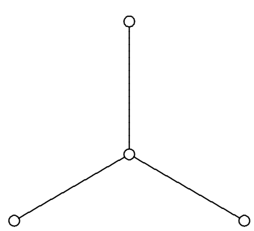
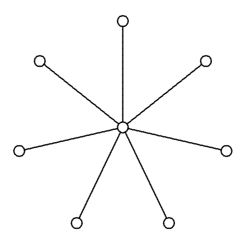


Figure 12: The star S_6

Example 2.5. The stars of orders 4 and 8 are given below. Give the adjacency matrices $\mathcal{A}(S_3)$ and $\mathcal{A}(S_7)$.



S_3



S_7

2.6 Complete Graph

Definition 2.6. The *complete graph* K_n is a graph of order n in which every pair of vertices are adjacent to each other.

The complete graph K_6 is illustated in Figure 13.

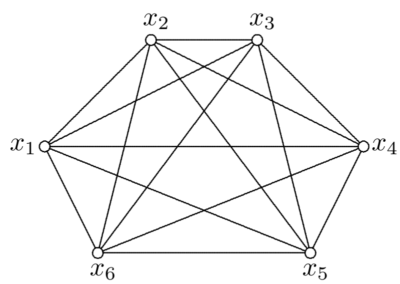
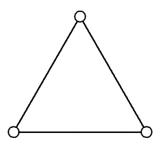
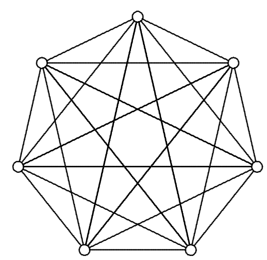


Figure 13: The graph K_6

Example 2.6. The complete graphs of orders 3 and 7 are given below. Give the adjacency matrices $\mathcal{A}(K_3)$ and $\mathcal{A}(K_7)$.



K_3



K_7

Remark 2.1. We note that K_3 and C_3 are the same graphs. We say that K_3 and C_3 are *isomorphic graphs* and write $K_3 \cong C_3$. We will have more on isomorphic graphs later.

Remark 2.2. Any graph G of order n is a spanning subgraph of K_n. Furthermore, any graph G of order n is a subgraph of K_m, where $m \geq n$.

2.7 Complete bipartite graph

Definition 2.7. The *complete bipartite graph* $K_{m,n}$ is a graph whose vertex set can be partitioned into two sets say X and Y containing m and n elements respectively, with the property that every vertex in X is joined by an edge with each of the vertices of Y and no pair of vertices both in X or both in Y are joined by an edge.

The graph is $K_{4,5}$ illustrated in Figure 14.

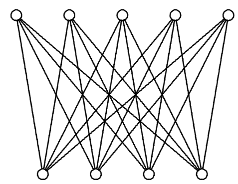


Figure 14: The complete bipartite graph $K_{4,5}$

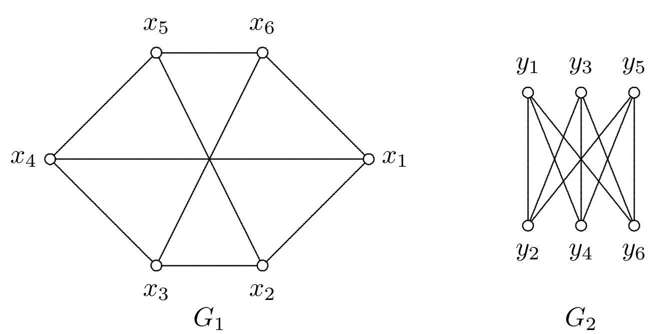
Remark 2.3. We note that $S_n \cong K_{1,n} \cong K_{n,1}$

Remark 2.4. If the vertex set of a graph G can be partitioned into two sets, X and Y with edges of G joining only points in X with points in Y, then G is said to be a *bipartite graph*.

3 Isomorphism of graphs

Definition 3.1. Two graphs $G_1 = (V(G_1), E(G_1)$ and $G_2 = (V(G_2), E(G_2))$ are *isomorphic* if there exists a bijective mapping $\phi : V(G_1) \to V(G_2)$ such that $ab \in E(G_1)$ if and only if $\phi(a)\phi(b) \in E(G_2)$. The mapping ϕ is called an *isomorphism*. If G_1 is isomorphic to G_2, we shall write $G_1 \cong G_2$, otherwise, we shall write $G_1 \ncong G_2$.

Example 3.1. The graphs G_1 and G_2 below are isomorphic. Give the bijective mapping ϕ.



The bijective mapping is

$$\phi(x_i) = y_i, \quad i = 1, 2, \ldots, 6.$$

We note that

$$x_i x_j \in E(G_1) \iff y_i y_j = \phi(x_i)\phi(x_j) \in E(G_2).$$

4 Some operations on graphs

We consider here some operations on graphs namely: the complement of a graph,the *rth* power of a graph, the sum of two graphs, the cartesian