Convergent Sequences

Part 3

Rafael Reno S. Cantuba, PhD
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Cauchy Sequences

Definition 1

A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ *is Cauchy* or is a *Cauchy sequence* in $\mathbb{R}$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $m, n \geq N$, we have $|a_m - a_n| < \varepsilon$.

Theorem 2

Every convergent sequence in $\mathbb{R}$ is Cauchy.

Proof.

We encounter in here another 'epsilon-over-two' technique. Suppose $(a_n)_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$, and let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies |a_n - a| < \frac{\varepsilon}{2}. \tag{1}$$

In particular, for any two indices $m, n \geq N$ that satisfy the hypothesis of (1), we have $|a_m - a| < \frac{\varepsilon}{2}$ and $|a_n - a| = |a - a_n| < \frac{\varepsilon}{2}$. By the triangle inequality,

$$|a_m - a_n| = |(a_m - a) + (a - a_n)| \leq |a_m - a| + |a - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $(a_n)_{n\in\mathbb{N}}$ is Cauchy. $\square$

Proving the converse of Theorem 2 shall take us into some longer argumentation. First, we need the sort of dual of the notion of limit superior. The *limit inferior* or *lower limit* of a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is defined as $\liminf_{n\to\infty} a_n := \sup_{n\in\mathbb{N}} \inf_{k\geq n} a_k$, which is analogously defined as how we defined $\inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k$ in the previous lecture.

Lemma 3

For any sequence $(a_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$,

(i) $\liminf\limits_{n\to\infty} a_n = -\limsup\limits_{n\to\infty}(-a_n)$;

(ii) $\liminf\limits_{n\to\infty} a_n \le \limsup\limits_{n\to\infty} a_n$;

(iii) *if* M *is a real number such that* $M \le a_n$ *for any* $n \in \mathbb{N}$, *then* $M \le \liminf\limits_{n\to\infty} a_n$;

(iv) *the condition* $\liminf\limits_{n\to\infty} a_n \ge \limsup\limits_{n\to\infty} a_n$ *holds if and only if* $(a_n)_{n\in\mathbb{N}}$ *is convergent;*

(v) *if* $(a_n)_{n\in\mathbb{N}}$ *is indeed convergent, then* $(a_n)_{n\in\mathbb{N}}$ *converges to the common value of* $\liminf\limits_{n\to\infty} a_n$ *and* $\limsup\limits_{n\to\infty} a_n$.

Proof of (i)

The proof starts with two ideas: first is that $\sup_{k\geq n}(-a_k)$ is an upper bound of $\{-a_k \;:\; k \geq n\}$, and second is that $\inf_{k\geq n} a_k$ is a lower bound of $\{a_k \;:\; k \geq n\}$. From these, we have

$$h \geq n \implies \sup_{k\geq n}(-a_k) \geq -a_h, \tag{2}$$

$$h \geq n \implies \inf_{k\geq n} a_k \leq a_h. \tag{3}$$

We do not want to mislead the student that the index used in coming up with the supremum $\sup_{k\geq n}(-a_k)$ is dependent to the rest of the statement (2), hence we used a second index h. We did the same for (3). Multiplying both sides of each inequality in (2),(3) by -1, we have

$$h \geq n \implies -\sup_{k\geq n}(-a_k) \leq a_h, \tag{4}$$

$$h \geq n \implies -\inf_{k\geq n} a_k \geq -a_h. \tag{5}$$

We find from (22) that $-\sup_{k\geq n}(-a_k)$ is a lower bound of $\{a_h \;:\; h \geq n\}$, and should be less than or equal to the infimum of $\{a_h \;:\; h \geq n\}$. Similarly, (5) tells us that $-\inf_{k\geq n} a_k$ is an upper bound of $\{-a_h \;:\; h \geq n\}$, and should be greater than or equal to the supremum of $\{-a_h \;:\; h \geq n\}$. That is,

$$-\sup_{k\geq n}(-a_k) \leq \inf_{h\geq n} a_h, \tag{6}$$

$$-\inf_{k\geq n} a_k \geq \sup_{h\geq n}(-a_h). \tag{7}$$

The right-hand side of (6) is less than or equal to an upper bound $\left\{\inf_{h\geq n} a_h \;:\; n \in \mathbb{N}\right\}$, in particular by the supremum. Similarly, the right-hand side of (6) is greater than or equal to any lower bound of $\left\{\sup_{h\geq n} -a_h \;:\; n \in \mathbb{N}\right\}$, such as the infimum. This gives us

Proof of (i)

$$\begin{aligned} -\sup_{k\geq n}(-a_k) &\leq \inf_{h\geq n} a_h \leq \sup_{n\in\mathbb{N}} \inf_{h\geq n} a_h = \liminf_{n\to\infty} a_n, \\ -\inf_{k\geq n} a_k &\geq \sup_{h\geq n}(-a_h) \geq \inf_{n\in\mathbb{N}} \sup_{h\geq n}(-a_h) = \limsup_{n\to\infty}(-a_n), \end{aligned}$$

which simplify into

$$\begin{aligned} -\sup_{k\geq n}(-a_k) &\leq \liminf_{n\to\infty} a_n, \\ -\inf_{k\geq n} a_k &\geq \limsup_{n\to\infty}(-a_n). \end{aligned}$$

Multiplying both sides of each inequality by -1, we have

$$\begin{aligned} \sup_{k\geq n}(-a_k) &\geq -\liminf_{n\to\infty} a_n, \\ \inf_{k\geq n} a_k &\leq -\limsup_{n\to\infty}(-a_n), \end{aligned}$$

which imply that $-\liminf_{n\to\infty} a_n$ is a lower bound of

$\left\{ \sup_{k\geq n}(-a_k) \ : \ n \in \mathbb{N} \right\}$, and is less than or equal to the infimum.
Analogously, $-\limsup_{n\to\infty}(-a_n)$ is an upper bound of
$\left\{ \inf_{k\geq n} a_k \ : \ n \in \mathbb{N} \right\}$, and is greater than or equal to the supremum.
That is,

$$\begin{aligned} \inf_{n\in\mathbb{N}} \sup_{k\geq n}(-a_k) &\geq -\liminf_{n\to\infty} a_n, \\ \sup_{n\in\mathbb{N}} \inf_{k\geq n} a_k &\leq -\limsup_{n\to\infty}(-a_n), \end{aligned}$$

where the left-hand sides may be simplified so that

$$\begin{aligned} \limsup_{n\to\infty}(-a_n) &\geq -\liminf_{n\to\infty} a_n, \\ \liminf_{n\to\infty} a_n &\leq -\limsup_{n\to\infty}(-a_n), \end{aligned}$$

from which we get

Proof of (i)

$$\begin{aligned}\liminf_{n\to\infty} a_n &\geq -\limsup_{n\to\infty}(-a_n),\\ \liminf_{n\to\infty} a_n &\leq -\limsup_{n\to\infty}(-a_n),\end{aligned}$$

and finally we get (i).

Proof of (ii)

Let $n \in \mathbb{N}$. Since the set $\{a_h \ : \ h \geq n\}$ has $\sup\limits_{k \geq n} a_k$ as an upperbound and $\inf\limits_{k \geq n} a_h$ as a lower bound, we have, for any $h \geq n$,

$$\inf_{k \geq n} a_k \leq a_h \leq \sup_{k \geq n} a_k,$$

$$\inf_{k \geq n} a_k \leq \sup_{k \geq n} a_k,$$

which implies that the number $\sup\limits_{k \geq n} a_k$ is an upper bound of $\left\{\inf\limits_{k \geq n} a_k \ : \ n \in \mathbb{N}\right\}$, and so the supremum of the said set must be less than or equal to $\sup\limits_{k \geq n} a_k$, that is

$$\sup_{n \in \mathbb{N}} \inf_{k \geq n} a_k \leq \sup_{k \geq n} a_k,$$

which now tells us that the number $\sup\limits_{n \in \mathbb{N}} \inf\limits_{k \geq n} a_k$ is a lower bound of

Proof of (ii)

$\left\{ \sup\limits_{k\geq n} a_k \ : \ n \in \mathbb{N} \right\}$, and so this lower bound must be less than or equal to the infimum of the said set. Thus,

$$\sup_{n\in\mathbb{N}} \inf_{k\geq n} a_k \quad \leq \quad \inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k,$$

from which we get (ii).

Proof of (iii)

If $M \leq a_n$ for any $n \in \mathbb{N}$, then $-a_n \leq -M$ for any $n \in \mathbb{N}$, and by a lemma from the previous lecture, we have $\limsup_{n\to\infty}(-a_n) \leq -M$, or equivalently, $M \leq -\limsup_{n\to\infty}(-a_n)$. By (i), we have $M \leq \liminf_{n\to\infty} a_n$.

Proof of (iv)

We first prove necessity. Let $\varepsilon > 0$. The condition $\liminf_{n\to\infty} a_n \geq \limsup_{n\to\infty} a_n$ can be written in two equivalent ways

$$\sup_{n\in\mathbb{N}} \inf_{k\geq n} a_k \geq \limsup_{n\to\infty} a_n, \tag{8}$$

$$\liminf_{n\to\infty} a_n \geq \inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k. \tag{9}$$

To the right-hand side of (8), we subtract ε, and to the left-hand side of (9), we add ε to obtain the strict inequalities

$$\sup_{n\in\mathbb{N}} \inf_{k\geq n} a_k > \limsup_{n\to\infty} a_n - \varepsilon, \tag{10}$$

$$\varepsilon + \liminf_{n\to\infty} a_n > \inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k. \tag{11}$$

The inequality (10) tells us that the number $\limsup_{n\to\infty} a_n - \varepsilon$ is already lower than the supremum of the set $\left\{\inf_{k\geq n} a_k : n \in \mathbb{N}\right\}$, so

Proof of (iv)

$\limsup_{n\to\infty} a_n - \varepsilon$ is not a lower bound of $\left\{\inf_{k\geq n} a_k \ : \ n \in \mathbb{N}\right\}$. This means that $\left\{\inf_{k\geq n} a_k \ : \ n \in \mathbb{N}\right\}$ has an element not bounded above by $\limsup_{n\to\infty} a_n - \varepsilon$. Similarly, (11) means that the set $\left\{\sup_{k\geq n} a_k \ : \ n \in \mathbb{N}\right\}$ has an element not bounded below by $\varepsilon + \liminf_{n\to\infty} a_n$. In terms of indices, we find that there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\inf_{k\geq N_1} a_k > \limsup_{n\to\infty} a_n - \varepsilon, \tag{12}$$

$$\varepsilon + \liminf_{n\to\infty} a_n > \sup_{k\geq N_2} a_k. \tag{13}$$

Since $\inf_{k\geq N_1} a_k$ is a lower bound of $\{a_k \ : \ k \geq N_1\}$, the inequality (12) means that every element of $\{a_k \ : \ k \geq N_1\}$ is strictly greater than $\limsup_{n\to\infty} a_n - \varepsilon$. Similarly, (13) tells us that every element of

Proof of (iv)

$\{a_k \ : \ k \geq N_2\}$ is strictly less than $\varepsilon + \liminf_{n\to\infty} a_n$. That is, we have the conditions

$$k \geq N_1 \implies a_k > \limsup_{n\to\infty} a_n - \varepsilon, \tag{14}$$

$$k \geq N_2 \implies a_k < \liminf_{n\to\infty} a_n + \varepsilon. \tag{15}$$

Thus, if a term of the sequence $(a_n)_{n\in\mathbb{N}}$ has an index $n \geq N := \max\{N_1, N_2\}$, then both hypotheses of (14),(15) are true for $k = n$, and we further have

$$a_n - \limsup_{n\to\infty} a_n > -\varepsilon, \tag{16}$$

$$a_n - \liminf_{n\to\infty} a_n < \varepsilon. \tag{17}$$

However, the assumption $\liminf_{n\to\infty} a_n \geq \limsup_{n\to\infty} a_n$ combined with 2 gives us

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n, \tag{18}$$

Proof of (iv)

and so (16),(17) may be simplified into

$$\begin{aligned} -\varepsilon \quad &< \quad a_n - \limsup_{n\to\infty} a_n \quad < \quad \varepsilon, \\ &\left| a_n - \limsup_{n\to\infty} a_n \right| \quad < \quad \varepsilon. \end{aligned}$$

In summary, we have shown

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \;\left[\left| a_n - \limsup_{n\to\infty} a_n \right| < \varepsilon \right].$$

Therefore,

$$\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n. \tag{19}$$

Proof of (iv)

We now prove sufficiency. Suppose there exists $a \in \mathbb{R}$ such that $a = \lim_{n\to\infty} a_n$. Let $\varepsilon > 0$. [Our trick here is a change of notation: instead of $N \in \mathbb{N}$ and $n \geq N$ in the usual instantiations for the symbolic form of $a = \lim_{n\to\infty} a_n$, this time we use $n \in \mathbb{N}$ and $k \geq n$.]Then there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} k \geq n \quad \Longrightarrow \quad & |a_k - a| < \frac{\varepsilon}{2}, \\ & -\frac{\varepsilon}{2} < a_k - a < \frac{\varepsilon}{2}, \\ & a - \frac{\varepsilon}{2} < a_k < a + \frac{\varepsilon}{2}. \end{aligned} \tag{20}$$

The inequalities in (20) tell us that $a - \frac{\varepsilon}{2}$ is a lower bound of $\{a_k \ : \ k \geq n\}$, and so $a - \frac{\varepsilon}{2}$ must be at most the infimum of $\{a_k \ : \ k \geq n\}$. Similarly, $a + \frac{\varepsilon}{2}$ is at least the supremum of $\{a_k \ : \ k \geq n\}$. That is,

Proof of (iv)

$$a - \frac{\varepsilon}{2} \leq \inf_{k \geq n} a_k, \qquad (21)$$

$$a + \frac{\varepsilon}{2} \geq \sup_{k \geq n} a_k. \qquad (22)$$

The right-hand side of (21) must be less than or equal to any upper bound of the set $\left\{ \inf_{k \geq n} a_k \ : \ n \in \mathbb{N} \right\}$, while the right-hand side of (22) must be greater than or equal to any lower bound of $\left\{ \sup_{k \geq n} a_k \ : \ n \in \mathbb{N} \right\}$. In particular,

$$a - \frac{\varepsilon}{2} \leq \inf_{k \geq n} a_k \leq \sup_{n \in \mathbb{N}} \inf_{k \geq n} a_k = \liminf_{n \to \infty} a_n, \qquad (23)$$

$$a + \frac{\varepsilon}{2} \geq \sup_{k \geq n} a_k \geq \inf_{n \in \mathbb{N}} \sup_{k \geq n} a_k = \limsup_{n \to \infty} a_n, \qquad (24)$$

which can be simplified into

Proof of (iv)

$$a \leq \liminf_{n\to\infty} a_n + \frac{\varepsilon}{2}, \tag{25}$$

$$\limsup_{n\to\infty} a_n - a \leq \frac{\varepsilon}{2}. \tag{26}$$

Adding the left-hand sides and adding the right-hand sides of (25),(26), we obtain the inequality $\limsup_{n\to\infty} a_n \leq \liminf_{n\to\infty} a_n + \varepsilon$ where $\varepsilon > 0$ is arbitrary. By a property of inequalities, we get $\limsup_{n\to\infty} a_n \leq \liminf_{n\to\infty} a_n$ as desired.

Proof of (v)

This follows from (18),(19) from the proof of (iv). $\square$

Theorem 4 (Cauchy convegence criterion)

Every Cauchy sequence in $\mathbb{R}$ *is convergent.*

Proof of the Cauchy Convergence Theorem

Our proof bears much resemblance to the proof of sufficiency for Lemma 3(iv). The few differences lie in the instantiation of quantifiers. If $(a_n)_{n\in\mathbb{N}}$ is Cauchy, then there exists $n \in \mathbb{N}$ such that

$$
\begin{aligned}
k, h \geq n \quad \Longrightarrow \quad & |a_k - a_h| < \frac{\varepsilon}{2}, \\
& a_h - \frac{\varepsilon}{2} < a_k < a_h + \frac{\varepsilon}{2}.
\end{aligned}
$$

Taking infima and suprema on all terms a_k with $k \geq n$, similar to the argumentation from (20) to (24) [with a_h instead of a], we obtain

$$
\begin{aligned}
a_h - \frac{\varepsilon}{2} \quad & \leq \quad \inf_{k\geq n} a_k \leq \sup_{n\in\mathbb{N}} \inf_{k\geq n} a_k = \liminf_{n\to\infty} a_n, \\
a_h + \frac{\varepsilon}{2} \quad & \geq \quad \sup_{k\geq n} a_k \geq \inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k = \limsup_{n\to\infty} a_n,
\end{aligned}
$$

from which we get the inequality $\limsup_{n\to\infty} a_n \leq \liminf_{n\to\infty} a_n + \varepsilon$ where $\varepsilon > 0$ is arbitrary. By a property of inequalities, we get $\limsup_{n\to\infty} a_n \leq \liminf_{n\to\infty} a_n$, and by Lemma 3(iv), the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent. $\square$