Contents

Chapter 1

Real Analysis

(1.5)

$OrderTrichotomy[<, S] := \forall_{x,y \in S}(x < y \lor x = y \lor y < x)$

$OrderTransitivity[<, S] := \forall_{x,y,z \in S}((x < y \land y < z) \implies x < z)$

$Order[<, S] := (OrderTrichotomy[<, S]) \land (OrderTransitivity[<, S])$

(1.7)

$BoundedAbove[E, S, <] := (Order[<, S]) \land (E \subset S) \land \left(\exists_{\beta \in S} \forall_{x \in E}(x \le \beta)\right)$

$BoundedBelow[E, S, <] := (Order[<, S]) \land (E \subset S) \land \left(\exists_{\beta \in S} \forall_{x \in E}(\beta \le x)\right)$

$UpperBound[\beta, E, S, <] := (Order[<, S]) \land (E \subset S) \land \left(\beta \in S \land \forall_{x \in E}(x \le \beta)\right)$

$LowerBound[\beta, E, S, <] := (Order[<, S]) \land (E \subset S) \land \left(\beta \in S \land \forall_{x \in E}(\beta \le x)\right)$

(1.8)

$LUB[\alpha, E, S, <] := (UpperBound[\alpha, E, S, <]) \land \left(\forall_{\gamma}(\gamma < \alpha \implies \neg UpperBound[\gamma, E, S, <])\right)$

$GLB[\alpha, E, S, <] := (LowerBound[\alpha, E, S, <]) \land \left(\forall_{\beta}(\alpha < \beta \implies \neg LowerBound[\beta, E, S, <])\right)$

(1.10)

$LUBProperty[S, <] := \forall_{E}\left(((\emptyset \ne E \subset S) \land (BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <]))\right)$

$GLBProperty[S, <] := \forall_{E}\left(((\emptyset \ne E \subset S) \land (BoundedBelow[E, S, <]) \implies \exists_{\alpha \in S}(GLB[\alpha, E, S, <]))\right)$

(1.11)

$LUBPropertyImpliesGLBProperty := LUBProperty[S, <] \implies GLBProperty[S, <]$

(1) $LUBProperty[S, <] \implies \dots$		wts: 2
(1.1) $(\emptyset \ne B \subset S \land BoundedBelow[B, S, <]) \implies \dots$		wts: 1.2
(1.1.1) $Order[<, S] \land \exists_{\delta' \in S}(LowerBound[\delta', B, S, <])$		from: *BoundedBelow*, 1.1
(1.1.2) $\|B\| = 1 \implies \dots$		wts: 1.1.3
(1.1.2.1) $\exists_{u'}(u' \in B) \; \blacksquare \; u := choice(\{u' : u' \in B\}) \; \blacksquare \; B = \{u\}$		from: 1.1.2
(1.1.2.2) $GLB[u, B, S, <] \; \blacksquare \; \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$		
(1.1.3) $\|B\| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$		
(1.1.4) $\|B\| \ne 1 \implies \dots$		wts: 1.1.5
(1.1.4.1) $\forall_{E}\left((\emptyset \ne E \subset S \land BoundedAbove[E, S, <] \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <]))\right)$		from: *LUBProperty*, 1
(1.1.4.2) $L := \{s \in S : LowerBound[s, B, S, <]\}$		
(1.1.4.3) $\|B\| > 1 \land OrderTrichotomy[<, S] \; \blacksquare \; \exists_{b_1' \in B} \exists_{b_0' \in B}(b_0' < b_1')$		from: *Order*, 1.1.1 / wts: 1.1.4.7
(1.1.4.4) $b_1 := choice\left(\{b_1' \in B : \exists_{b_0' \in B}(b_0' < b_1')\}\right) \; \blacksquare \; \neg LowerBound[b_1, B, S, <]$		from: 1.1.4.2
(1.1.4.5) $b_1 \notin L \; \blacksquare \; L \subset S$		
(1.1.4.6) $\delta := choice(\{\delta' \in S : LowerBound[\delta', B, S, <]\}) \; \blacksquare \; \delta \in L \; \blacksquare \; \emptyset \ne L$		from: 1.1.1
(1.1.4.7) $\emptyset \ne L \subset S$		from: 1.1.4.5, 1.1.4.6
(1.1.4.8) $\forall_{y \in L}(LowerBound[y_0, B, S, <]) \; \blacksquare \; \forall_{y \in L} \forall_{x \in B}(y_0 \le x)$		from: *LowerBound*, 1.1.4.2 / wts: 1.1.4.10

(1.1.4.9) $\forall_{x\in B}\left(x \in S \land \forall_{y\in L}(y_0 \leq x)\right)$ ▍ $\forall_{x\in B}(UpperBound[x, L, S, <])$

(1.1.4.10) $\exists_{x\in S}(UpperBound[x, L, S, <])$ ▍ $BoundedAbove[L, S, <]$

(1.1.4.11) $\emptyset \neq L \subset S \land BoundedAbove[L, S, <]$

(1.1.4.12) $\exists_{\alpha'\in S}(LUB[\alpha', L, S, <])$ ▍ $\alpha := choice(\{\alpha' \in S : (LUB[\alpha', L, S, <])\})$

(1.1.4.13) $\forall_x(x \in B \implies UpperBound[x, L, S, <])$

(1.1.4.14) $\forall_x(\neg UpperBound[x, L, S, <] \implies x \notin B)$

(1.1.4.15) $\gamma < \alpha \implies \ldots$

   (1.1.4.15.1) $\neg UpperBound[\gamma, L, S, <]$ ▍ $\gamma \notin B$

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B$ ▍ $\gamma \in B \implies \gamma \geq \alpha$

(1.1.4.17) $\forall_{\gamma\in B}(\alpha \leq \gamma)$ ▍ $LowerBound[\alpha, B, S, <]$

(1.1.4.18) $\alpha < \beta \implies \ldots$

   (1.1.4.18.1) $\forall_{y\in L}(y_0 \leq \alpha < \beta)$ ▍ $\forall_{y\in L}(y_0 \neq \beta)$

   (1.1.4.18.2) $\beta \notin L$ ▍ $\neg LowerBound[\beta, B, S, <]$

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound[\beta, B, S, <]$ ▍ $\forall_{\beta\in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.20) $LowerBound[\alpha, B, S, <] \land \forall_{\beta\in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.21) $GLB[\alpha, B, S, <]$ ▍ $\exists_{\epsilon_1\in S}(GLB[\epsilon_1, B, S, <])$

(1.1.5) $|B| \neq 1 \implies \exists_{\epsilon_1\in S}(GLB[\epsilon_1, B, S, <])$

(1.1.6) $\left(|B| = 1 \implies \exists_{\epsilon_0\in S}(GLB[\epsilon_0, B, S, <])\right) \land \left(|B| \neq 1 \implies \exists_{\epsilon_1\in S}(GLB[\epsilon_1, B, S, <])\right)$

(1.1.7) $(|B| = 1 \lor |B| \neq 1) \implies \exists_{\epsilon\in S}(GLB[\epsilon, B, S, <])$ ▍ $\exists_{\epsilon\in S}(GLB[\epsilon, B, S, <])$

(1.2) $(\emptyset \neq B \subset S \land BoundedBelow[B, S, <]) \implies \exists_{\epsilon\in S}(GLB[\epsilon, B, S, <])$

(1.3) $\forall_B\big((\emptyset \neq B \subset S \land BoundedBelow[B, S, <]) \implies \exists_{\epsilon\in S}(GLB[\epsilon, B, S, <])\big)$

(1.4) $GLBProperty[S, <]$

(2) $LUBProperty[S, <] \implies GLBProperty[S, <]$

(1.12)

$$Field[F, +, *] := \exists_{0,1\in F}\forall_{x,y,z\in F}\begin{cases} x + y \in F & \land & x * y \in F & \land \\ x + y = y + x & \land & x * y = y * x & \land \\ (x + y) + z = x + (y_0 + z) & \land & (x * y) * z = x * (y_0 * z) & \land \\ \mathbb{1} \neq \mathbb{0} & \land & x * (y_0 + z) = (x * y) + (x * z) & \land \\ \mathbb{0} + x = x & \land & \mathbb{1} * x = x & \land \\ \exists_{-x\in F}\big(x + (-x) = \mathbb{0}\big) \land \left(x \neq \mathbb{0} \implies \exists_{1/x\in F}\big(x * (1/x) = \mathbb{1}\big)\right) \end{cases}$$

*********************** $(Field[F, +, *] \land x, y, z \in F) \implies \ldots$ ***********************

(1.14)

$AdditiveCancellation := (x + y = x + z) \implies y = z$

(1) $y = \mathbb{0} + y = \big(x + (-x)\big) + y = \big((-x) + x\big) + y = (-x) + (x + y) = \ldots$

(2) $(-x) + (x + z) = \big((-x) + x\big) + z = \big(x + (-x)\big) + z = \mathbb{0} + z = z$

$AdditiveIdentityUniqueness := (x + y = x) \implies y = \mathbb{0}$

(1) $x + y = x = \mathbb{0} + x = x + \mathbb{0}$

(2) $y = 0$

$AdditiveInverseUniqueness := (x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$

(2) $y = -x$

$DoubleNegative := x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x$ ▍ $\mathbb{0} = (-x) + x$

(2) $x = -(-x)$

(1.15)

$MultiplicativeCancellation := (x \neq 0 \wedge x * y = x * z) \implies y = z$ —

$MultiplicativeIdentityUniqueness := (x \neq 0 \wedge x * y = x) \implies y = 1$ —

$MultiplicativeInverseUniqueness := (x \neq 0 \wedge x * y = 1) \implies y = 1/x$ —

$DoubleReciprocal := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)

$Domination := 0 * x = 0$

(1) $0 * x = (0 + 0) * x = 0 * x + 0 * x \quad\blacksquare\quad 0 * x = 0 * x + 0 * x$

(2) $0 * x = 0$

$NonDomination := (x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$

(1) $(x \neq 0 \wedge y \neq 0) \implies \ldots$

(1.1) $(x * y = 0) \implies \ldots$

(1.1.1) $1 = 1 * 1 = \left(x * (1/x)\right) * \left(y * (1/y)\right) = (x * y) * \left((1/x) * (1/y)\right) = 0 * \left((1/x) * (1/y)\right) = 0$

(1.1.2) $1 = 0 \wedge 1 \neq 0 \quad\blacksquare\quad \bot$

(1.2) $(x * y = 0) \implies \bot \quad\blacksquare\quad x * y \neq 0$

(2) $(x \neq 0 \wedge y \neq 0) \implies x * y \neq 0$

$NegationCommutativity := (-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = 0 * y = 0 \quad\blacksquare\quad x * y + (-x) * y = 0$

(2) $(-x) * y = -(x * y)$

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * 0 = 0 \quad\blacksquare\quad x * y + x * (-y) = 0$

(4) $x * (-y) = -(x * y)$

(5) $(-x) * y = -(x * y) = x * (-y)$

$NegativeMultiplication := (-x) * (-y) = x * y$

(1) $(-x) * (-y) = -\left(x * (-y)\right) = -\left(-(x * y)\right) = x * y$

(1.17)

$OrderedField[F, +, *, <] := \begin{pmatrix} Field[F, +, *] \quad\wedge\quad Order[<, F] \quad\wedge \\ \forall_{x,y,z \in F}(y_0 < z \implies x + y < x + z) \wedge \\ \forall_{x,y \in F}\left((x > 0 \wedge y > 0) \implies x * y > 0\right) \end{pmatrix}$

$\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast (OrderedField[F, +, *, <] \wedge x, y, z \in F) \implies \ldots \ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast$

(1.18)

$NegationOnOrder := x > 0 \iff -x < 0$

(1) $x > 0 \implies \ldots$

(1.1) $0 = (-x) + x > (-x) + 0 = -x \quad\blacksquare\quad 0 > -x \quad\blacksquare\quad -x < 0$

(2) $x > 0 \implies -x < 0$

(3) $-x < 0 \implies \ldots$

(3.1) $0 = x + (-x) < x + 0 = x \quad\blacksquare\quad 0 < x \quad\blacksquare\quad x > 0$

(4) $-x < 0 \implies x > 0$

(5) $x > 0 \implies -x < 0 \wedge -x < 0 \implies x > 0 \quad\blacksquare\quad x > 0 \iff -x < 0$

$PositiveFactorPreservesOrder := (x > 0 \wedge y < z) \implies x * y < x * z$

(1) $(x > 0 \wedge y < z) \implies \ldots$

(1.1) $(-y) + z > (-y) + y = 0 \quad\blacksquare\quad z + (-y) = 0$

(1.2) $x * \left(z + (-y)\right) > 0 \quad\blacksquare\quad x * z + x * (-y) > 0$

(1.3) $x * z = 0 + x * z = \big(x * y + -(x * y)\big) + x * z = \big(x * y + x * (-y)\big) + x * z = \ldots$ *from: Field, NegationCommutativity*

(1.4) $x * y + \big(x * z + x * (-y)\big) > x * y + 0 = x * y$ *from: Field, 1.2*

(1.5) $x * z > x * y$ *from: 1.3, 1.4*

(2) $(x > 0 \wedge y < z) \implies x * z > x * y$

$NegativeFactorFlipsOrder := (x < 0 \wedge y < z) \implies x * y > x * z$

(1) $(x < 0 \wedge y < z) \implies \ldots$

   (1.1) $-x > 0$ *from: NegationOnOrder*

   (1.2) $(-x) * y < (-x) * z$ ▮ $0 = x * y + (-x) * y < x * y + (-x) * z$ ▮ $0 < x * y + (-x) * z$ *from: PositiveFactorPreservesOrder*

   (1.3) $0 < (-x) * (-y + z)$ ▮ $0 > x * (-y + z)$ ▮ $0 > -(x * y) + x * z$ *from: NegationOnOrder*

   (1.4) $x * y > x * z$

(2) $(x < 0 \wedge y < z) \implies x * y > x * z$

$SquareIsPositive := (x \neq 0) \implies x * x > 0$

(1) $(x > 0) \implies x * x > 0$ *from: Ordered Field*

(2) $(x < 0) \implies \ldots$

   (2.1) $-x > 0$ ▮ $x * x = (-x) * (-x) > 0$ ▮ $x * x > 0$ *from: NegationOnOrder, Ordered Field, NegativeMultiplication*

(3) $(x < 0) \implies x * x > 0$

(4) $x \neq 0 \implies (x > 0 \vee x < 0) \implies x * x > 0$ ▮ $x \neq 0 \implies x * x > 0$ *from: OrderTrichotomy, 1, 3*

$OneIsPositive := 1 > 0$

(1) $1 \neq 0$ ▮ $1 = 1 * 1 > 0$ *from: Field, SquareIsPositive*

$ReciprocationOnOrder := (0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \ldots$

   (1.1) $x * (1/x) = 1 > 0$ ▮ $x * (1/x) > 0$ *from: Field, OneIsPositive*

   (1.2) $1/x < 0 \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \perp$ ▮ $1/x > 0$ *from: NegativeFactorFlipsOrder, 1*

   (1.3) $y * (1/y) = 1 > 0$ ▮ $y * (1/y) > 0$ *from: Field, OneIsPositive*

   (1.4) $1/y < 0 \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \perp$ ▮ $1/y > 0$ *from: NegativeFactorFlipsOrder, 1*

   (1.5) $(1/x) * (1/y) > 0$ *from: Ordered Field*

   (1.6) $0 < 1/y = \big((1/x) * (1/y)\big) * x < \big((1/x) * (1/y)\big) * y = 1/x$ *from: Ordered Field, 1, 1.4, 1.5*

(1.19)

$OrderedFieldQ := Ordered Field[\mathbb{Q}, +, *, <]$ —

$Subfield[K, F, +, *] := Field[F, +, *] \wedge K \subset F \wedge Field[K, +, *]$
$OrderedSubfield[K, F, +, *, <] := Ordered Field[F, +, *, <] \wedge K \subset F \wedge Ordered Field[K, +, *, <]$
$CutI[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$
$CutII[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
$CutIII[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} : CutI[\alpha] \wedge CutII[\alpha] \wedge CutIII[\alpha]\}$

$CutCorollaryI := (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \ldots$

   (1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ *from: CutII, 1*

   (1.2) $q < p \implies q \in \alpha$ ▮ $q \notin \alpha \implies q \geq p$ *from: 1*

   (1.3) $(q \notin \alpha) \implies \ldots$

     (1.3.1) $q \geq p$ *from: 1.2*

     (1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \perp$ ▮ $q \neq p$ *from: 1, 1.3*

$(1.3.3)$ $q \geq p \wedge q \neq p$ ▮ $p < q$

(1.4) $q \notin \alpha \implies p < q$ ▮ $p < q$ from: 1

(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

$CutCorollaryI1 := (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$

(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ from: $CutI1$, 1

(1.2) $s \in \alpha \implies \big(r \in \mathbb{Q} \implies (r < s \implies r \in \alpha) \big)$ ▮ $s \in \alpha \implies r \in \alpha$ from: 1, 1.1

(1.3) $r \notin \alpha \implies s \notin \alpha$ ▮ $s \notin \alpha$ from: 1, 1.2

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$<_{\mathbb{R}} [\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$OrderTrichotomyOfR := OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$

$(1.1.1)$ $\alpha \not\subset \beta \wedge \alpha \neq \beta$ from: $<_{\mathbb{R}}$, 1.1

$(1.1.2)$ $\exists_{p'} (p' \in \alpha \wedge p' \notin \beta)$ ▮ $p := choice(\{p' : p' \in \alpha \wedge p' \notin \beta\})$

$(1.1.3)$ $q \in \beta \implies \ldots$

$(1.1.3.1)$ $p, q \in \mathbb{Q}$

$(1.1.3.2)$ $q < p$ from: $CutCorollaryI$

$(1.1.3.3)$ $q \in \alpha$ from: $CutI1$

$(1.1.4)$ $q \in \beta \implies q \in \alpha$

$(1.1.5)$ $\forall_{q \in \beta}(q \in \alpha)$ ▮ $\beta \subseteq \alpha$

$(1.1.6)$ $\beta \subset \alpha$ ▮ $\beta <_{\mathbb{R}} \alpha$

(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$ ▮ $(\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

$OrderTransitivityOfR := OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

$(1.1.1)$ $\alpha \subset \beta \wedge \beta \subset \gamma$

$(1.1.2)$ $\forall_{a \in \alpha}(a \in \beta) \wedge \forall_{b \in \beta}(b \in \gamma)$

$(1.1.3)$ $\forall_{a \in \alpha}(\alpha \in \gamma)$ ▮ $\alpha \subset \gamma$ ▮ $\alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma \big)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}} \big((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma \big)$

(4) $OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

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from: $OrderTrichotomyR, OrderTransitivityR$
wts:

$OrderOfR := Order[<_{\mathbb{R}}, \mathbb{R}]$

$LUBPropertyOfR := LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A}(p \in \alpha)\}$

(1.2)	$A \neq \emptyset$ ▎ $\exists_\alpha (\alpha \in A)$ ▎ $\alpha_0 := choice(\{\alpha \: : \: \alpha \in A\})$
(1.3)	$\alpha_0 \neq \emptyset$ ▎ $\exists_a (a \in \alpha_0)$ ▎ $a_0 := choice(\{a \: : \: a \in \alpha_0\})$ ▎ $a_0 \in \gamma$ ▎ $\gamma \neq \emptyset$
(1.4)	*Bounded Above*$[A, \mathbb{R}, <_\mathbb{R}]$ ▎ $\exists_\beta (UpperBound[\beta, A, \mathbb{R}, <_\mathbb{R}])$
(1.5)	$\beta_0 := choice(\{\beta \: : \: UpperBound[\beta, A, \mathbb{R}, <_\mathbb{R}]\})$
(1.6)	*UpperBound*$[\beta_0, A, \mathbb{R}, <_\mathbb{R}]$ ▎ $\forall_{\alpha \in A}(\alpha \leq_\mathbb{R} \beta_0)$ ▎ $\forall_{\alpha \in A}(\alpha \subseteq \beta_0)$ ▎ $\forall_{\alpha \in A}\forall_{a \in \alpha}(a \in \beta_0)$
(1.7)	$(\alpha \in A \land a \in \alpha) \iff a \in \gamma$ ▎ $\forall_{a \in \gamma}(a \in \beta_0)$ ▎ $\gamma \subseteq \beta_0$
(1.8)	$\beta_0 \subset \mathbb{Q}$ ▎ $\gamma \subseteq \beta_0 \subset \mathbb{Q}$ ▎ $\gamma \subset \mathbb{Q}$
(1.9)	$\emptyset \neq \gamma \subset \mathbb{Q}$ ▎ *CutI*$[\gamma]$
(1.10)	$(p \in \gamma \land q \in \mathbb{Q} \land q < p) \implies \dots$
(1.10.1)	$p \in \gamma$ ▎ $\exists_{\alpha \in A}(p \in \alpha)$ ▎ $\alpha_1 := choice(\{\alpha \in A \: : \: p \in \alpha\})$
(1.10.2)	$p \in \alpha_1 \land q \in \mathbb{Q} \land q < p$ ▎ $q \in \alpha_1$ ▎ $q \in \gamma$
(1.11)	$(p \in \gamma \land q \in \mathbb{Q} \land q < p) \implies q \in \gamma$ ▎ $\forall_{p \in \gamma}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma)$ ▎ *CutII*$[\gamma]$
(1.12)	$p \in \gamma \implies \dots$
(1.12.1)	$\exists_{\alpha \in A}(p \in \alpha)$ ▎ $\alpha_2 := choice(\{\alpha \in A \: : \: p \in \alpha\})$
(1.12.2)	$\alpha_2 \in \mathbb{R}$ ▎ *CutII*$[\alpha_2]$ ▎ $\exists_{r \in \alpha_2}(p < r)$ ▎ $r_0 := choice(\{r \in \alpha_2 \: : \: p < r\})$
(1.12.3)	$r_0 \in \alpha_2$ ▎ $r_0 \in \gamma$
(1.12.4)	$p < r_0$ ▎ $p < r_0 \land r_0 \in \gamma$ ▎ $\exists_{r \in \gamma}(p < r)$
(1.13)	$p \in \gamma \implies \exists_{r \in \gamma}(p < r)$ ▎ $\forall_{p \in \gamma}\exists_{r \in \gamma}(p < r)$ ▎ *CutIII*$[\gamma]$
(1.14)	*CutI*$[\gamma] \land CutII[\gamma] \land CutIII[\gamma]$ ▎ $\gamma \in \mathbb{R}$
(1.15)	$\forall_{\alpha \in A}(\alpha \subseteq \gamma)$ ▎ $\forall_{\alpha \in A}(\alpha \leq_\mathbb{R} \gamma)$
(1.16)	$\forall_{\alpha \in A}(\alpha \leq_\mathbb{R} \gamma) \land \gamma \in \mathbb{R}$ ▎ *UpperBound*$[\gamma, A, \mathbb{R}, <_\mathbb{R}]$
(1.17)	$\delta <_\mathbb{R} \gamma \implies \dots$
(1.17.1)	$\delta \subset \gamma$ ▎ $\exists_s (s \in \gamma \land s \notin \delta)$ ▎ $s_0 := choice(\{s \in \mathbb{Q} \: : \: s \in \gamma \land s \notin \delta\})$
(1.17.2)	$s_0 \in \gamma$ ▎ $\exists_{\alpha \in A}(s_0 \in \alpha)$ ▎ $\alpha_3 := choice(\{\alpha \in A \: : \: s_0 \in \alpha\})$
(1.17.3)	$s_0 \in \alpha_3 \land s_0 \notin \delta$ ▎ $\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \land s \notin \delta)$
(1.17.4)	$\delta \geq_\mathbb{R} \alpha_3 \implies \dots$
(1.17.4.1)	$\alpha_3 \subseteq \delta$ ▎ $\forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta)$ ▎ $\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \land s \notin \delta)$
(1.17.4.2)	$\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \land s \notin \delta) \land \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \land s \notin \delta)$ ▎ $\bot$
(1.17.5)	$\delta \geq_\mathbb{R} \alpha_3 \implies \bot$ ▎ $\delta <_\mathbb{R} \alpha_3$ ▎ $\exists_{\alpha \in A}(\delta <_\mathbb{R} \alpha)$ ▎ $\exists_{\alpha \in A}(\neg(\alpha \leq_\mathbb{R} \delta))$
(1.17.6)	$\neg\forall_{\alpha \in A}(\alpha \leq_\mathbb{R} \delta)$ ▎ $\neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}]$
(1.18)	$\delta <_\mathbb{R} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}])$ ▎ $\forall_\delta(\delta <_\mathbb{R} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}])$
(1.19)	*UpperBound*$[\gamma, A, \mathbb{R}, <_\mathbb{R}] \land \forall_\delta(\delta <_\mathbb{R} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_\mathbb{R}])$
(1.20)	*LUB*$[\gamma, A, \mathbb{R}, <_\mathbb{R}]$ ▎ $\exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_\mathbb{R}])$
(2)	$(\emptyset \neq A \subset \mathbb{R} \land BoundedAbove[A, \mathbb{R}, <_\mathbb{R}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_\mathbb{R}])$
(3)	$\forall_A \Big((\emptyset \neq A \subset \mathbb{R} \land BoundedAbove[A, \mathbb{R}, <_\mathbb{R}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_\mathbb{R}]) \Big)$ ▎ *LUBProperty*$[\mathbb{R}, <_\mathbb{R}]$

$+_\mathbb{R}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \land (\alpha +_\mathbb{R} \beta) = \{r + s \: : \: r \in \alpha \land s \in \beta\}$
$0_\mathbb{R} := \{x \in \mathbb{Q} \: : \: x < 0\}$

ZeroInR $:= 0_\mathbb{R} \in \mathbb{R}$

(1)	$-1 \in 0_\mathbb{R} \land 1 \notin 0_\mathbb{R}$ ▎ $\emptyset \neq 0_\mathbb{R} \subseteq \mathbb{Q}$ ▎ *CutI*$[0_\mathbb{R}]$
(2)	$(x \in 0_\mathbb{R} \land y \in \mathbb{Q} \land y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_\mathbb{R}$ ▎ $\forall_{x \in 0_\mathbb{R}}\forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_\mathbb{R})$ ▎ *CutII*$[0_\mathbb{R}]$
(3)	$y := x/2$ ▎ $(x \in 0_\mathbb{R}) \implies (x < y < 0) \implies \exists_{y \in 0_\mathbb{R}}(x < y)$ ▎ $\forall_{x \in 0_\mathbb{R}}\exists_{y \in 0_\mathbb{R}}(x < y)$ ▎ *CutIII*$[0_\mathbb{R}]$
(4)	*CutI*$[0_\mathbb{R}] \land CutII[0_\mathbb{R}] \land CutIII[0_\mathbb{R}]$ ▎ $0_\mathbb{R} \in \mathbb{R}$

FieldAdditionClosureOfR $:= (\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_\mathbb{R} \beta) \in \mathbb{R}\big)$

(1)	$(\alpha, \beta \in \mathbb{R}) \implies \dots$
(1.1)	$(\alpha +_\mathbb{R} \beta) = \{r + s \: : \: r \in \alpha \land s \in \beta\}$
(1.2)	$\emptyset \neq \alpha \subset \mathbb{Q} \land \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha)$; $\exists_b(b \in \beta)$ ▮ $a_0 := choice(\{a : a \in \alpha\})$; $b_0 := choice(\{b : b \in \beta\})$ ▮ $a_0 + b_0 \in \alpha +_\mathbb{R} \beta$

(1.4) $\exists_x(x \notin \alpha)$; $\exists_y(y_0 \notin \beta)$ ▮ $x_0 := choice(\{x : x \notin \alpha\})$; $y_0 := choice(\{y : y \notin \beta\})$

(1.5) $\forall_{r \in \alpha}(r < x_0)$; $\forall_{s \in \beta}(s < y_0)$ ▮ $\forall_{r \in \alpha}\forall_{s \in \beta}(r + s < x_0 + y_0)$ ▮ $x_0 + y_0 \notin \alpha +_\mathbb{R} \beta$

(1.6) $\emptyset \neq \alpha +_\mathbb{R} \beta \subset \mathbb{Q}$ ▮ $\textcolor{blue}{Cut\,I[\alpha +_\mathbb{R} \beta]}$

(1.7) $(p \in \alpha +_\mathbb{R} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

$\quad(1.7.1)$ $\exists_{r \in \alpha}\exists_{s \in \beta}(p = r + s)$ ▮ $(r_0, s_0) := choice\big((r, s) \in \alpha \times \beta : p = r + s\big)$

$\quad(1.7.2)$ $q < p = r_0 + s_0$ ▮ $(q - s_0) < r_0$ ▮ $(q - s_0) \in \alpha$

$\quad(1.7.3)$ $s_0 \in \beta$ ▮ $q = (q - s_0) + s_0 \in \alpha +_\mathbb{R} \beta$ ▮ $q \in \alpha +_\mathbb{R} \beta$

(1.8) $(p \in \alpha +_\mathbb{R} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_\mathbb{R} \beta$ ▮ $\forall_{p \in \alpha +_\mathbb{R} \beta}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_\mathbb{R} \beta)$ ▮ $\textcolor{blue}{Cut\,II[\alpha +_\mathbb{R} \beta]}$

(1.9) $p \in \alpha \implies \ldots$

$\quad(1.9.1)$ $\exists_{r \in \alpha}\exists_{s \in \beta}(p = r + s)$ ▮ $(r_1, s_1) := choice\big(\{(r, s) \in \alpha \times \beta : p = r + s\}\big)$

$\quad(1.9.2)$ $r_1 \in \alpha$ ▮ $\exists_{t \in \alpha}(r_1 < t)$ ▮ $t_0 := choice(\{t \in \alpha : r_1 < t\})$

$\quad(1.9.3)$ $s_1 \in \beta$ ▮ $t + s_1 \in \alpha +_\mathbb{R} \beta \wedge p = r_1 + s_1 < t + s_1$ ▮ $\exists_{r \in \alpha +_\mathbb{R} \beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r \in \alpha +_\mathbb{R} \beta}(p < r)$ ▮ $\forall_{p \in \alpha +_\mathbb{R} \beta}\exists_{r \in \alpha +_\mathbb{R} \beta}(p < r)$ ▮ $\textcolor{blue}{Cut\,III[\alpha +_\mathbb{R} \beta]}$

(1.11) $\textcolor{blue}{Cut\,I[\alpha +_\mathbb{R} \beta]} \wedge \textcolor{blue}{Cut\,II[\alpha +_\mathbb{R} \beta]} \wedge \textcolor{blue}{Cut\,III[\alpha +_\mathbb{R} \beta]}$ ▮ $\alpha +_\mathbb{R} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies \big((\alpha +_\mathbb{R} \beta) \in \mathbb{R}\big)$

$\textcolor{red}{Field\,Addition\,Commutativity\,Of\,R} := (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_\mathbb{R} \beta = \beta +_\mathbb{R} \alpha)$

(1) $\alpha +_\mathbb{R} \beta = \{r + s : r \in \alpha \wedge s \in \beta\} = \{s + r : s \in \beta \wedge r \in \alpha\} = \beta +_\mathbb{R} \alpha$

$\textcolor{red}{Field\,Addition\,Associativity\,Of\,R} := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma)\big)$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

$\quad(1.1)$ $(\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \{(a + b) + c : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

$\quad(1.2)$ $\{a + (b + c) : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_\mathbb{R} \beta) +_\mathbb{R} \gamma = \alpha +_\mathbb{R} (\beta +_\mathbb{R} \gamma)$

$\textcolor{red}{Field\,Addition\,Identity\,Of\,R} := (\alpha \in \mathbb{R}) \implies 0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

$\quad(1.1)$ $(r \in \alpha \wedge s \in 0_\mathbb{R}) \implies \ldots$

$\quad\quad(1.1.1)$ $s < 0$ ▮ $r + s < r + 0 = r$ ▮ $r + s < r$ ▮ $r + s \in \alpha$

$\quad(1.2)$ $(r \in \alpha \wedge s \in 0_\mathbb{R}) \implies r + s \in \alpha$ ▮ $\forall_{r \in \alpha}\forall_{s \in 0_\mathbb{R}}(r + s \in \alpha)$

$\quad(1.3)$ $(r \in \alpha \wedge s \in 0_\mathbb{R}) \iff (r + s \in \alpha +_\mathbb{R} 0_\mathbb{R})$ ▮ $\forall_{p \in \alpha +_\mathbb{R} 0_\mathbb{R}}(p \in \alpha)$ ▮ $\alpha +_\mathbb{R} 0_\mathbb{R} \subseteq \alpha$

$\quad(1.4)$ $p \in \alpha \implies \ldots$

$\quad\quad(1.4.1)$ $\exists_{r \in \alpha}(p < r)$ ▮ $r_2 := choice(\{r \in \alpha : p < r\})$

$\quad\quad(1.4.2)$ $p < r_2$ ▮ $p - r_2 < r_2 - r_2 = 0$ ▮ $(p - r_2) < 0$ ▮ $(p - r_2) \in 0_\mathbb{R}$

$\quad\quad(1.4.3)$ $r_2 \in \alpha$ ▮ $p = r_2 + (p - r_2) \in \alpha +_\mathbb{R} 0_\mathbb{R}$ ▮ $p \in \alpha +_\mathbb{R} 0_\mathbb{R}$

$\quad(1.5)$ $p \in \alpha \implies p \in \alpha +_\mathbb{R} 0_\mathbb{R}$ ▮ $\forall_{p \in \alpha}(p \in \alpha +_\mathbb{R} 0_\mathbb{R})$ ▮ $\alpha \subseteq \alpha +_\mathbb{R} 0_\mathbb{R}$

$\quad(1.6)$ $\alpha +_\mathbb{R} 0_\mathbb{R} \subseteq \alpha \wedge \alpha \subseteq \alpha +_\mathbb{R} 0_\mathbb{R}$ ▮ $0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_\mathbb{R} +_\mathbb{R} \alpha = \alpha$

$\textcolor{red}{Field\,Addition\,Inverse\,Of\,R} := (\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}}\big(\alpha +_\mathbb{R} (-\alpha) = 0_\mathbb{R}\big)$

(1) $\alpha \in \mathbb{R} \implies \ldots$

$\quad(1.1)$ $\beta := \{p \in \mathbb{Q} : \exists_{r>0}(-p - r \notin \alpha)\}$

$\quad(1.2)$ $\alpha \subset \mathbb{Q}$ ▮ $\exists_{s \in \mathbb{Q}}(s \notin \alpha)$ ▮ $s_0 := choice(\{s : s \notin \alpha\})$ ▮ $p_0 := -s_0 - 1$

$\quad(1.3)$ $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha$ ▮ $-p_0 - 1 \notin \alpha$ ▮ $\exists_{r>0}(-p_0 - r \notin \alpha)$ ▮ $p_0 \in \beta$

$\quad(1.4)$ $\emptyset \neq \alpha$ ▮ $\exists_{q \in \alpha}$ ▮ $q_0 := choice(\{q \in \mathbb{Q} : q \in \alpha\})$

$\quad(1.5)$ $r > 0 \implies \ldots$

$\quad\quad(1.5.1)$ $q_0 \in \alpha$ ▮ $-(-q_0) - r = q_0 - r < q_0$ ▮ $-(-q_0) - r < q_0$ ▮ $-(-q_0) - r \in \alpha$

$\quad(1.6)$ $\forall_{r>0}\big(-(-q_0) - r \in \alpha\big)$ ▮ $\neg\exists_{r>0}\big(-(-q_0) - r \notin \alpha\big)$ ▮ $-q_0 \notin \beta$

$\quad(1.7)$ $\emptyset \neq \beta \subset \mathbb{Q}$ ▮ $\textcolor{blue}{Cut\,I[\beta]}$

(1.8)	$(p \in \beta \land q \in \mathbb{Q} \land q < p) \implies \dots$
(1.8.1)	$p \in \beta$ ▮ $\exists_{r>0}(-p - r \notin \alpha)$ ▮ $r_0 := choice(\{r > 0 : -p - r \notin \alpha\})$
(1.8.2)	$q < p$ ▮ $-p - r < -q - r$
(1.8.3)	$-q - r \notin \alpha$ ▮ $q \in \beta$
(1.9)	$(p \in \beta \land q \in \mathbb{Q} \land q < p) \implies q \in \beta$ ▮ $\forall_{p\in\beta}\forall_{q\in\mathbb{Q}}(q < p \implies q \in \beta)$ ▮ $Cut\,II[\beta]$
(1.10)	$p \in \beta \implies \dots$
(1.10.1)	$p \in \beta$ ▮ $\exists_{r>0}(-p - r \notin \alpha)$ ▮ $r_1 := choice(\{r > 0 : -p - r \notin \alpha\})$
(1.10.2)	$t_0 := p + (r_1/2)$
(1.10.3)	$r_1 > 0$ ▮ $r_1/2 > 0$
(1.10.4)	$t_0 > t_0 - (r_1/2) = p$ ▮ $t_0 > p$
(1.10.5)	$-t_0 - (r_1/2) = -\big(p + (r_1/2)\big) - (r_1/2) = -p - r_1$
(1.10.6)	$-p - r_1 \notin \alpha$ ▮ $-t_0 - (r_1/2) \notin \alpha$ ▮ $\exists_{r>0}(-t_0 - r \notin \alpha)$ ▮ $t_0 \in \beta$
(1.10.7)	$t_0 > p \land t_0 \in \beta$ ▮ $\exists_{t\in\beta}(p < t)$
(1.11)	$p \in \beta \implies \exists_{t\in\beta}(p < t)$ ▮ $\forall_{p\in\beta}\exists_{t\in\beta}(p < t)$ ▮ $Cut\,III[\beta]$
(1.12)	$Cut\,I[\beta] \land Cut\,II[\beta] \land Cut\,III[\beta]$ ▮ $\beta \in \mathbb{R}$
(1.13)	$(r \in \alpha \land s \in \beta) \implies \dots$
(1.13.1)	$s \in \beta$ ▮ $\exists_{t>0}(-s - t \notin \alpha)$ ▮ $t_1 := choice(\{t > 0 : -s - t \notin \alpha\})$ ▮ $-s - t_1 < -s$
(1.13.2)	$\alpha \in \mathbb{R} \land s, t_1 \in \mathbb{Q} \land -s - t_1 < -s \land -s - t_1 \notin \alpha$ ▮ $-s \notin \alpha$
(1.13.3)	$\alpha \in \mathbb{R} \land r \in \alpha \land -s \notin \alpha$ ▮ $r < -s$ ▮ $r + s < 0$ ▮ $r + s \in 0_{\mathbb{R}}$
(1.14)	$(r \in \alpha \land s \in \beta) \implies r + s \in 0_{\mathbb{R}}$ ▮ $\forall_{(r,s)\in\alpha\times\beta}(r + s \in 0_{\mathbb{R}})$ ▮ $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$
(1.15)	$v \in 0_{\mathbb{R}} \implies \dots$
(1.15.1)	$v < 0$ ▮ $w_0 := -v/2$ ▮ $w > 0$
(1.15.2)	$\exists_{n\in\mathbb{Z}}\big(nw_0 \in \alpha \land (n+1)w_0 \notin \alpha\big)$ ▮ $n_0 := choice\big(\{n \in \mathbb{Z} : nw_0 \in \alpha \land (n+1)w_0 \notin \alpha\}\big)$
(1.15.3)	$p_0 := -(n_0 + 2)w_0$ ▮ $-p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha$ ▮ $-p_0 - w_0 \notin \alpha$ ▮ $p_0 \in \beta$
(1.15.4)	$n_0 w_0 \in \alpha \land p_0 \in \beta$ ▮ $n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$
(1.16)	$v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta$ ▮ $\forall_{v\in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta)$ ▮ $0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$
(1.17)	$\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \land 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$ ▮ $\alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$
(1.18)	$\beta \in \mathbb{R} \land \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$ ▮ $\exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$
(2)	$\alpha \in \mathbb{R} \implies \exists_{-\alpha\in\mathbb{R}}\big(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}}\big)$

from: $ARCHIMEDEAN\,PROPERTY\,OF\,Q + LUB???$

$*_{\mathbb{R}}[\alpha, \beta] :=$ ⸻
$1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$1IsNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ ⸻
$1InR := 1_{\mathbb{R}} \in \mathbb{R}$ ⸻
$FieldMultiplicationClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R}\big)$ ⸻
$FieldMultiplicationCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ ⸻
$FieldMultiplicationAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \big((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma)\big)$ ⸻
$FieldMultiplicationIdentityOfR := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ ⸻
$FieldMultiplicationInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha\in\mathbb{R}}\big(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}}\big)$ ⸻

$FieldDistributativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ ⸻

$FieldWithR := Field[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}]$ ⸻
$OrderedFieldWithR := OrderedField[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ ⸻

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$
$QROrderedSubfieldOfR := OrderedSubfield[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ ⸻
$QIsomorphicToQR := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ ⸻
$CompletenessOfR := \exists_{\mathbb{R}}(LUBProperty[\mathbb{R}, <_{\mathbb{R}}] \land OrderedSubfield[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}])$ ⸻

(1.20)
$ArchimedeanPropertyOfR := \forall_{x,y\in\mathbb{R}}\big(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y)\big)$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx : n \in \mathbb{N}^+\}$ ▋ $(\emptyset \neq A \subset \mathbb{R}) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$

(1.2) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n \in \mathbb{N}^+}(nx > y)$ ▋ $\forall_{n \in \mathbb{N}^+}(nx \leq y)$ ▋ $UpperBound[y_0, A, \mathbb{R}, <]$ ▋ $BoundedAbove[A, \mathbb{R}, <]$

(1.2.2) $CompletenessOfR$ ▋ $LUBProperty[\mathbb{R}, <]$

(1.2.3) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (BoundedAbove[A, \mathbb{R}, <])$ ▋ $\exists_{\alpha \in \mathbb{R}}(LUB[\alpha, A, \mathbb{R}, <]) \ldots$

(1.2.4) $\ldots \alpha_0 := choice(\{\alpha \in \mathbb{R} : LUB[\alpha, A, \mathbb{R}, <]\})$ ▋ $LUB[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0$ ▋ $\alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge (LUB[\alpha_0, A, \mathbb{R}, <])$ ▋ $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$ ▋ $\exists_{c \in A}(\alpha_0 - x < c) \ldots$

(1.2.8) $\ldots c_0 := choice(\{c \in A : \alpha_0 - x < c\})$ ▋ $(c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$ ▋ $\exists_{m \in \mathbb{N}^+}(mx = c_0) \ldots$

(1.2.10) $\ldots m_0 := choice(\{m \in \mathbb{N}^+ : mx = c_0\})$ ▋ $(m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0)$ ▋ $\alpha_0 - x < c_0 = m_0 x$ ▋ $\alpha_0 < m_0 x + x$ ▋ $\alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+$ ▋ $m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge (a \in A \iff \exists_{m \in \mathbb{N}^+}(mx = a))$ ▋ $(m_0 + 1)x \in A$

(1.2.14) $(\alpha_0 < (m_0 + 1)x) \wedge ((m_0 + 1)x \in A)$ ▋ $\exists_{c \in A}(\alpha_0 < c)$

(1.2.15) $LUB[\alpha_0, A, \mathbb{R}, <]$ ▋ $UpperBound[\alpha_0, A, \mathbb{R}, <]$ ▋ $\forall_{c \in A}(c \leq \alpha_0)$ ▋ $\neg\exists_{c \in A}(c > \alpha_0)$ ▋ $\neg\exists_{c \in A}(\alpha_0 < c)$

(1.2.16) $(\exists_{c \in A}(\alpha_0 < c)) \wedge (\neg\exists_{c \in A}(\alpha_0 < c))$ ▋ $\perp$

(1.3) $\neg\exists_{n \in \mathbb{N}^+}(nx > y) \implies \perp$ ▋ $\exists_{n \in \mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n \in \mathbb{N}^+}(nx > y)$ ▋ $\forall_{x,y \in \mathbb{R}}(x > 0 \implies \exists_{n \in \mathbb{N}^+}(nx > y))$

$\color{red}{QDenseInR} := \forall_{x,y \in \mathbb{R}}\left(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y)\right)$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y$ ▋ $(0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $ArchimedeanPropertyOfR \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R})$ ▋ $\exists_{n \in \mathbb{N}^+}(n(y - x) > 1) \ldots$

(1.3) $\ldots n_0 := choice(\{n \in \mathbb{N}^+ : n(y - x) > 1\})$ ▋ $(n_0 \in \mathbb{N}^+) \wedge (n_0(y - x) > 1)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R})$ ▋ $n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R})$ ▋ $\exists_{m \in \mathbb{N}^+}(m(1) > n_0 x) \ldots$

(1.6) $\ldots m_1 := choice(\{m \in \mathbb{N}^+ : m(1) > n_0 x\})$ ▋ $(m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R})$ ▋ $\exists_{m \in \mathbb{N}^+}(m(1) > -n_0 x) \ldots$

(1.8) $\ldots m_2 := choice(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\})$ ▋ $(m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x)$ ▋ $-m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+$ ▋ $|m_1 - (-m_2)| \geq 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge (|m_1 - (-m_2)| \geq 2)$ ▋ $\exists_{m \in \mathbb{Z}}((-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)) \ldots$

(1.12) $\ldots m_0 := choice(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \leq n_0 x < m)\})$ ▋ $(-m_2 < m_0 < m_1) \wedge (m_0 - 1 \leq n_0 x < m_0)$

(1.13) $(n_0(y - x) > 1) \wedge (m_0 - 1 \leq n_0 x < m_0)$ ▋ $n_0 x < m_0 \leq 1 + n_0 x < n_0 y$ ▋ $n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y)$ ▋ $x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z}$ ▋ $m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y)$ ▋ $\exists_{p \in \mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p \in \mathbb{Q}}(x < p < y)$ ▋ $\forall_{x,y \in \mathbb{R}}\left(x < y \implies \exists_{p \in \mathbb{Q}}(x < p < y)\right)$

(1.21)

$\color{red}{RootLemma} := (0 < a < b) \implies \left(b^n - a^n \leq (b - a)nb^{n-1}\right)$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b$ ▋ $b/a > 1$

(1.3) $b/a > 1$ ❙ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}\left(b^{n-i}a^{i-1}(b/a)^{i-1}\right) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$ ❙ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b-a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le (b-a)nb^{n-1}$ ❙ $b^n - a^n \le (b-a)nb^{n-1}$

(2) $(0 < a < b) \implies \left(b^n - a^n \le (b-a)nb^{n-1}\right)$

$RootExistenceInR := \forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}(y_0{}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\}$ ❙ $t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$

(1.2) $t_0 := x/(1+x)$ ❙ $\left(t_0 = x/(1+x)\right) \wedge (t_0 \in \mathbb{R})$

(1.3) $0 < x$ ❙ $0 < x < 1 + x$ ❙ $t_0 = x/(1+x) > 0$ ❙ $t_0 > 0$

(1.4) $1 = (1+x)/(1+x) > x/(1+x) = t_0$ ❙ $1 > t_0$

(1.5) $(t_0 > 0) \wedge (1 > t_0)$ ❙ $0 < t_0 < 1$

(1.6) $(0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1)$ ❙ $t_0{}^n \le t_0$

(1.7) $0 < x$ ❙ $x > x/(1+x) = t_0$ ❙ $x > t_0$

(1.8) $(t_0{}^n \le t_0) \wedge (x > t_0)$ ❙ $t_0{}^n < x$

(1.9) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge (t_0{}^n < x)$ ❙ $t_0 \in E$ ❙ $\emptyset \ne E$

(1.10) $t_1 := choice(\{t \in \mathbb{R} : t > 1 + x\})$ ❙ $(t_1 \in \mathbb{R}) \wedge (t_1 > 1 + x)$

(1.11) $x > 0$ ❙ $t_1 > 1 + x > 1$ ❙ $t_1 > 1$ ❙ $t_1{}^n \ge t_1$

(1.12) $(t_1{}^n \ge t_1) \wedge (t_1 > 1 + x) \wedge (1 > 0)$ ❙ $t_1{}^n \ge t_1 > 1 + x > x$ ❙ $t_1{}^n > x$

(1.13) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge (t_1{}^n > x)$ ❙ $t_1 \notin E$ ❙ $E \subset \mathbb{R}$

(1.14) $(\emptyset \ne E) \wedge (E \subset \mathbb{R})$ ❙ $\emptyset \ne E \subset \mathbb{R}$

(1.15) $t \in E \implies \ldots$

(1.15.1) $(t \in E) \wedge \left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right)$ ❙ $t^n < x$

(1.15.2) $(t_1{}^n > x) \wedge (t^n < x)$ ❙ $t^n < x < t_1{}^n$ ❙ $t < t_1$

(1.16) $t \in E \implies t < t_1$ ❙ $\forall_{t\in E}(t \le t_1)$ ❙ $UpperBound[t_1, E, \mathbb{R}, <]$ ❙ $BoundedAbove[E, \mathbb{R}, <]$

(1.17) $CompletenessOfR$ ❙ $LUBProperty[\mathbb{R}, <]$

(1.18) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \ne E \subset \mathbb{R}) \wedge (BoundedAbove[E, \mathbb{R}, <])$ ❙ $\exists_{y\in\mathbb{R}}(LUB[y, E, \mathbb{R}, <]) \ldots$

(1.19) $\ldots y_0 := choice(\{y \in \mathbb{R} : LUB[y, E, \mathbb{R}, <]\})$ ❙ $LUB[y_0, E, \mathbb{R}, <]$

(1.20) $(LUB[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0)$ ❙ $0 < t_0 \le y_0 \in \mathbb{R}$ ❙ $0 < y_0 \in \mathbb{R}$

(1.21) $y_0{}^n < x \implies \ldots$

(1.21.1) $k_0 := \frac{x - y_0{}^n}{n(y_0 + 1)^{n-1}}$ ❙ $k_0 \in \mathbb{R}$

(1.21.2) $y_0{}^n < x$ ❙ $0 < x - y_0{}^n$

(1.21.3) $(n > 0) \wedge (y_0 > 0)$ ❙ $0 < n(y_0 + 1)^{n-1}$

(1.21.4) $(0 < x - y_0{}^n) \wedge \left(0 < n(y_0 + 1)^{n-1}\right)$ ❙ $0 < \frac{x - y_0{}^n}{n(y_0 + 1)^{n-1}} = k_0$ ❙ $0 < k_0$

(1.21.5) $(0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R})$ ❙ $0 < min(1, k_0) \in \mathbb{R}$

(1.21.6) $QDenseInR \wedge \left(0, min(1, k_0) \in \mathbb{R}\right) \wedge \left(0 < min(1, k_0)\right)$ ❙ $\exists_{h\in\mathbb{Q}}\left(0 < h < min(1, k_0)\right) \ldots$

(1.21.7) $\ldots h_0 := choice\left(\{h \in \mathbb{Q} : 0 < h < min(1, k_0)\}\right)$ ❙ $(0 < h_0 < 1) \wedge \left(h_0 < k_0 = \frac{x - y_0{}^n}{n(y_0 + 1)^{n-1}}\right)$

(1.21.8) $(y_0 > 0) \wedge (h_0 > 0)$ ❙ $0 < y_0 < y_0 + h_0$

(1.21.9) $RootLemma \wedge (0 < y_0 < y_0 + h_0)$ ❙ $(y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + h_0)^{n-1}$

(1.21.10) $h_0 < 1$ ❙ $h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$

(1.21.11) $\left((y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + h_0)^{n-1}\right) \wedge \left(h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}\right)$ ❙ $(y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + 1)^{n-1}$

(1.21.12) $\left(0 < n(y_0 + 1)^{n-1}\right) \wedge \left(h_0 < k_0 = \frac{x - y_0{}^n}{n(y_0 + 1)^{n-1}}\right)$ ❙ $h_0 n(y_0 + 1)^{n-1} < x - y_0{}^n$

(1.21.13) $\left((y_0 + h_0)^n - y_0{}^n < h_0 n(y_0 + 1)^{n-1}\right) \wedge \left(h_0 n(y_0 + 1)^{n-1} < x - y_0{}^n\right)$ ❙ $(y_0 + h_0)^n - y_0{}^n < x - y_0{}^n$ ❙ $(y_0 + h_0)^n < x$

(1.21.14) $(y_0 + h_0)^n - y_0{}^n < x - y_0{}^n$ ❙ $(y_0 + h_0)^n < x$

(1.21.15) $(0 < y_0 \mathbb{R}) \wedge (0 < h_0 < \mathbb{R})$ ❙ $0 < y_0 < y_0 + h_0 \in \mathbb{R}$

(1.21.16) $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge \left((y_0 + h_0)^n < x\right) \wedge (0 < y_0 + h_0 \in \mathbb{R})$ ❙ $(y_0 + h_0)^n \in E$

$(1.21.17)$ $\left((y_0 + h_0)^n \in E\right) \wedge (y_0 < y_0 + h_0)$ ▌ $\exists_{e \in E}(y_0 < e)$

$(1.21.18)$ $LUB[y_0, E, \mathbb{R}, <]$ ▌ $UpperBound[y_0, E, \mathbb{R}, <]$ ▌ $\forall_{e \in E}(e \leq y_0)$ ▌ $\neg\exists_{e \in E}(e > y_0)$

$(1.21.19)$ $\left(\exists_{e \in E}(e > y_0)\right) \wedge \left(\neg\exists_{e \in E}(e > y_0)\right)$ ▌ $\bot$

(1.22) $y_0{}^n < x \implies \bot$ ▌ $y_0{}^n \geq x$

(1.23) $y_0{}^n > x \implies \ldots$

$(1.23.1)$ $k_1 := \frac{y_0{}^n - x}{n y_0{}^{n-1}}$ ▌ $(k_1 \in \mathbb{R}) \wedge (k_1 n y_0{}^{n-1} = y_0{}^n - x)$

$(1.23.2)$ $(0 < x) \wedge (0 < n \in \mathbb{Z})$ ▌ $y_0{}^n - x < y_0{}^n \leq n y_0{}^n$ ▌ $y_0{}^n - x < n y_0{}^n$

$(1.23.3)$ $y_0{}^n - x < n y_0{}^n$ ▌ $k_1 = \frac{y_0{}^n - x}{n y_0{}^{n-1}} < \frac{n y_0{}^n}{n y_0{}^{n-1}} = y_0$ ▌ $k_1 < y_0$

$(1.23.4)$ $y_0{}^n > x$ ▌ $0 < y_0{}^n - x$

$(1.23.5)$ $(n > 0) \wedge (y_0 > 0)$ ▌ $0 < n y_0{}^{n-1}$

$(1.23.6)$ $(0 < y_0{}^n - x) \wedge 0 < (n y_0{}^{n-1})$ ▌ $0 < \frac{y_0{}^n - x}{n y_0{}^{n-1}} = k_1$ ▌ $0 < k_1$

$(1.23.7)$ $(k_1 < y_0) \wedge (0 < k_1)$ ▌ $(0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$

$(1.23.8)$ $t \geq y_0 - k_1 \implies \ldots$

$(1.23.8.1)$ $t \geq y_0 - k_1$ ▌ $t^n \geq (y_0 - k_1)^n$ ▌ $-t^n \leq -(y_0 - k_1)^n$ ▌ $y_0{}^n - t^n \leq y_0{}^n - (y_0 - k_1)^n$

$(1.23.8.2)$ $RootLemma \wedge (0 < y_0 - k_1 < y_0)$ ▌ $y_0{}^n - (y_0 - k_1)^n < k_1 n y_0{}^{n-1}$

$(1.23.8.3)$ $\left(y_0{}^n - t^n \leq y_0{}^n - (y_0 - k_1)^n\right) \wedge \left(y_0{}^n - (y_0 - k_1)^n < k_1 n y_0{}^{n-1}\right)$ ▌ $y_0{}^n - t^n < k_1 n y_0{}^{n-1}$

$(1.23.8.4)$ $(k_1 n y_0{}^{n-1} = y_0{}^n - x) \wedge (y_0{}^n - t^n < k_1 n y_0{}^{n-1})$ ▌ $y_0{}^n - t^n < y_0{}^n - x$ ▌ $-t^n < -x$ ▌ $t^n > x$

$(1.23.8.5)$ $\left(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)\right) \wedge (t^n > x)$ ▌ $t \notin E$

$(1.23.9)$ $t \geq y_0 - k_1 \implies t \notin E$ ▌ $t \in E \implies t < y_0 - k_1$ ▌ $\forall_{t \in E}(t \leq y_0 - k_1)$ ▌ $UpperBound[y_0 - k_1, E, \mathbb{R}, <]$

$(1.23.10)$ $\left(LUB[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)\right)$ ▌ $\neg UpperBound[y_0 - k_1, E, \mathbb{R}, <]$

$(1.23.11)$ $(UpperBound[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg UpperBound[y_0 - k_1, E, \mathbb{R}, <])$ ▌ $\bot$

(1.24) $y_0{}^n > x \implies \bot$ ▌ $y_0{}^n \leq x$

(1.25) $Order[\mathbb{R}, <]$ ▌ $OrderTrichotomy[\mathbb{R}, <]$

(1.26) $(OrderTrichotomy[\mathbb{R}, <]) \wedge (y_0{}^n \geq x) \wedge (y_0{}^n \leq x)$ ▌ $y_0{}^n = x$

(1.27) $(y_0{}^n = x) \wedge (y_0 \in \mathbb{R})$ ▌ $\exists_{y \in \mathbb{R}}(y^n = x)$

(1.28) $y_1, y_2 := choice(\{y \in \mathbb{R} : y^n = x\})$

(1.29) $y_1 \neq y_2 \implies \ldots$

$(1.29.1)$ $(OrderTrichotomy[\mathbb{R}, <]) \wedge (y_1 \neq y_2)$ ▌ $(y_1 < y_2) \vee (y_2 < y_1)$ $\ldots$

$(1.29.2)$ $\ldots (x = y_1{}^n < y_2{}^n = x) \vee (x = y_2{}^n < y_1{}^n = x)$ ▌ $(x < x) \vee (x > x)$ ▌ $\bot \vee \bot$ ▌ $\bot$

(1.30) $y_1 \neq y_2 \implies \bot$ ▌ $y_1 = y_2$ ▌ $\forall_{a,b \in \mathbb{R}}\left((a^n = x \wedge b^n = x) \implies a = b\right)$

(1.31) $\left(\exists_{y \in \mathbb{R}}(y^n = x)\right) \wedge \left(\forall_{a,b \in \mathbb{R}}\left((a^n = x \wedge b^n = x) \implies a = b\right)\right)$ ▌ $\exists!_{y \in \mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}}(y^n = x)$ ▌ $\forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}}(y_0{}^n = x)$

$RootExistenceInRCorollary := \forall_{0 < a \in \mathbb{R}} \forall_{0 < b \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}}\left((ab)^{1/n} = a^{1/n}b^{1/n}\right)$ ———

$ExtendedRealSystem[\bar{\mathbb{R}}, +, *, <] := \begin{cases} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \wedge x - \infty = -\infty \wedge \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \wedge \\ (x > 0) \implies \left(x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty\right) \wedge \\ (x < 0) \implies \left(x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty\right) \end{cases}$

$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$

$+_{\mathbb{C}}[\langle a, b\rangle, \langle c, d\rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d\rangle$

$*_{\mathbb{C}}[\langle a, b\rangle, \langle c, d\rangle] := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c\rangle$

$FieldC := Field[\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}}]$ ———

$RSubfieldC := Subfield[\mathbb{R}, \mathbb{C}, +, *]$ ———

$i := \langle 0, 1 \rangle \in \mathbb{C}$

$iProperty := i^2 = -1$ ———

$CProperty := (a, b \in \mathbb{R}) \implies (\langle a, b\rangle = a + bi)$ ———

Conjugate$[\overline{a + bi}] := a - bi$

ConjugateProperties $:= (w, z \in \mathbb{C}) \implies \dots$ ——

(1)	$\overline{z + w} = \overline{z} + \overline{w}$
(2)	$\overline{z * w} = \overline{z} * \overline{w}$
(3)	$Re(z) = (1/2)(z + \overline{z}) \wedge Im(z) = (1/2)(z - \overline{z})$
(4)	$0 \leq z * \overline{z} \in \mathbb{R}$

AbsoluteValueC$[|z|] = (z * \overline{z})^{1/2}$
AbsoluteValueProperties $:= (z, w \in \mathbb{C}) \implies \dots$ ——

(1)	123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

2.1 Functions

$Rel[r, X] := (X \neq \emptyset) \land (r \subseteq X)$

$Func[f, X, Y] := (Rel[f, X \times Y]) \land \left(\forall_{x \in X} \exists!_{y \in Y} (\langle x, y \rangle \in f) \right)$

$Comp[g \circ f, f, g, X, Y, Z] := (Func[f, X, Y]) \land (Func[g, Y, Z]) \land (Func[g \circ f, X, Z]) \land \left(g \circ f = \{ \langle x, g(f(x)) \rangle \in X \times Z \, | \, x \in X \} \right)$

$CompAssoc := h \circ (g \circ f) = (h \circ g) \circ f$

(1)	TODO

$Domain[dom(f), f, X, Y] := (Func[f, X, Y]) \land \left(dom(f) = X \right)$

$Codomain[cod(f), f, X, Y] := (Func[f, X, Y]) \land \left(cod(f) = Y \right)$

$Image[im(A), A, f, X, Y] := (Func[f, X, Y]) \land (A \subseteq X) \land \left(im(A) = \{ f(a) \in Y \, | \, a \in A \} \right)$

$Preimage[pim(B), B, f, X, Y] := (Func[f, X, Y]) \land (B \subseteq Y) \land \left(pim(B) = \{ a \in X \, | \, f(a) \in B \} \right)$

$Range[rng(f), f, X, Y] := (Func[f, X, Y]) \land \left(Image[rng(f), dom(f), f, X, Y] \right)$

$Inj[f, X, Y] := (Func[f, X, Y]) \land \left(\forall_{x_1, x_2 \in X} \left((f(x_1) = f(x_2)) \implies (x_1 = x_2) \right) \right)$

$Surj[f, X, Y] := (Func[f, X, Y]) \land \left(\forall_{y \in Y} \exists_{x \in X} (y = f(x)) \right)$

$Bij[f, X, Y] := (Inj[f, X, Y]) \land (Surj[f, X, Y])$

$Inv[f^{-1}, f, X, Y] := (Func[f, X, Y]) \land (Func[f^{-1}, Y, X]) \land (f \circ f^{-1} = I_Y) \land (f^{-1} \circ f = I_X)$

$SurjEquiv := (Surj[f, X, Y]) \iff \left(rng(f) = cod(f) \right)$

(1)	TODO

$BijEquiv := (Bij[f, X, Y]) \iff \left(\exists_{f_{-1}} (Inv[f^{-1}, f, X, Y]) \right)$

(1)	TODO

$InjComp := \left((Inj[f]) \land (Inj[g]) \right) \implies (Inj[g \circ f])$

(1)	TODO

$SurjComp := \left((Surj[f]) \land (Surj[g]) \right) \implies (Surj[g \circ f])$

(1)	TODO

2.2 Divisibility, Equivalence Relations, Paritions

$DivisionAlgorithm := \forall_{b \in \mathbb{Z}} \forall_{a \in \mathbb{Z}^+} \exists!_{q, r \in \mathbb{Z}} \left((b = aq + r) \land (0 \leq r < q) \right)$

(1)	TODO

$Divides[a, b] := (a, b \in \mathbb{Z}) \wedge \left(\exists_{c \in \mathbb{Z}}(b = ac)\right)$

$ComDiv[a, b, c] := (Divides[a, b]) \wedge (Divides[a, c])$

$GCD[a, b, c] := (ComDiv[a, b, c]) \wedge \left(\forall_{d \in \mathbb{Z}}\left(\left((Divides[d, b]) \wedge (Divides[d, c])\right) \implies (Divides[d, a])\right)\right)$

$RelPrime[a, b] := GCD[1, a, b]$

$CongRel[a, b, n] := Divides[n, a - b]$

$Partition[\mathcal{P}, S] := \left(\forall_{P \in \mathcal{P}}(P \neq \emptyset)\right) \wedge \left(S = \cup_{P \in \mathcal{P}}(P)\right) \wedge \left(\forall_{P_1, P_2 \in \mathcal{P}}\left((P_1 \neq P_2) \implies (P_1 \cap P_2 = \emptyset)\right)\right)$

$EqRel[\sim, S] := (Rel[\sim, S]) \wedge \left(\forall_{a \in S}(a \sim a)\right) \wedge \left(\forall_{a,b \in S}\left((a \sim b) \implies (b \sim a)\right)\right) \wedge \left(\forall_{a,b,c \in S}\left(((a \sim b) \wedge (b \sim c)) \implies (a \sim c)\right)\right)$

$EqClass[[s], s, \sim, S] := (Rel[\sim, S]) \wedge (s \in S) \wedge ([s] = \{x \in S | x \sim s\})$

$PartitionInducesEqRel := (Partition[\mathcal{P}, S]) \implies \left(\exists_{\sim}(EqRel[\sim, S])\right)$

(1)   TODO : $\sim = \{\langle a, b \rangle \in S \times S | (P \in \mathcal{P}) \wedge (a, b \in P)\}$

$EqRelInducesPartition := (EqRel[\sim, S]) \implies \left(\exists_{\mathcal{P}}(Partition[\mathcal{P}, S])\right)$

(1)   TODO : $Partition[EqClass_1, EqClass_2, ...]$

$EqRelCong := \forall_{n \in \mathbb{Z}^+}(EqRel[CongRel, \mathbb{Z}])$

(1)   TODO

2.3   Groups

$Group[G, *] := \left(\begin{array}{l} (Function[*, G, G]) \qquad \wedge \\ \left(\forall_{a,b,c \in G}\left((a * b) * c = a * (b * c)\right)\right) \wedge \\ \left(\exists_{e \in G}\forall_{a \in G}(a * e = a = e * a)\right) \qquad \wedge \\ \left(\forall_{a \in G}\exists_{a^{-1} \in G}(a * a^{-1} = e = a^{-1} * a)\right) \end{array} \right)$

$AbelianGroup[G, *] := (Group[G, *]) \wedge \left(\forall_{a,b \in G}(a * b = b * a)\right)$

$CancelLaws := \forall_G\left((Group[G, *]) \implies \left(\forall_{a,b,c \in G}\left(((a * b = a * c) \implies (b = c)) \wedge ((a * c = b * c) \implies (a = b))\right)\right)\right)$

(1) $(a * b = a * c) \implies ...$

  (1.1) $a \in G$ ▌ $\exists_{a^{-1} \in G}(a * a^{-1} = e = a^{-1} * a)$

  (1.2) $Function[*, G, G]$ ▌ $a^{-1} * a * b = a^{-1} * a * c$

  (1.3) $\left(\forall_{a,b,c \in G}\left((a * b) * c = a * (b * c)\right)\right) \wedge \left(\forall_{a \in G}\exists_{a^{-1} \in G}(a * a^{-1} = e = a^{-1} * a)\right)$ ▌ $b = c$

(2) $(a * b = a * c) \implies (b = c)$

(3) $(a * c = b * c) \implies ...$

  (3.1)   TODO

(4) $(a * c = b * c) \implies (a = b)$

(5) $\left((a * b = a * c) \implies (b = c)\right) \wedge \left((a * c = b * c) \implies (a = b)\right)$

$IdUniq := \forall_G\left((Group[G, *]) \implies \left(\forall_{e_1,e_2 \in G}\forall_{a \in G}\left(((a * e_1 = a = e_1 * a) \wedge (a * e_2 = a = e_2 * a)) \implies (e_1 = e_2)\right)\right)\right)$

(1) $(CancelLaws) \wedge \left(\forall_{a \in G}\exists_{a^{-1} \in G}(a * a^{-1} = e = a^{-1} * a)\right)$ ▌ $a * e_1 = a = a * e_2$ ▌ $e_1 = e_2$

$InvUniq := \forall_G\left((Group[G, *]) \implies \left(\forall_{a \in G}\forall_{a_1^{-1}, a_2^{-1} \in G}\left(((a * a_1^{-1} = e = a_1^{-1} * a) \wedge (a * a_2^{-1} = e = a_2^{-1} * a)) \implies (a_1^{-1} = a_2^{-1})\right)\right)\right)$

(1) $(CancelLaws) \wedge \left(\forall_{a \in G} \exists_{a^{-1} \in G} (a * a^{-1} = e = a^{-1} * a) \right)$ ▮ $a * a_1^{-1} = e = a * a_2^{-1}$ ▮ $a_1^{-1} = a_2^{-1}$

$InvProd := \forall_G \forall_{a,b \in G} \left((a * b)^{-1} = b^{-1} * a^{-1} \right)$

(1) $(a * b) * (a * b)^{-1} = e$

(2) $(a * b) * (b^{-1} * a^{-1}) = \left(a * (b * b^{-1}) * a^{-1} \right) = e$

(3) $InvUniq$ ▮ $(a * b)^{-1} = b^{-1} * a^{-1}$

$OrderG[o(G), G, *] := (Group[G, *]) \wedge \left(o(G) = |G| \right)$
$gWitness[n, g, G, *] := (Group[G, *]) \wedge (n \in \mathbb{Z}^+) \wedge (g^n = e) \wedge \left(\forall_{m \in \mathbb{Z}^+} (m < n) \implies (g^m \neq e) \right)$
$Orderg[o(g), g, G, *] := (Group[G, *]) \wedge \left(\left(\exists_n (gWitness[n, g, G, *]) \right) \implies \left(o(g) = n \right) \right) \wedge \left(\left(\neg \exists_n (gWitness[n, g, G, *]) \right) \implies \left(o(g) = \infty \right) \right)$

2.4 Subgroups

$Subgroup[H, G, *] := (Group[G, *]) \wedge (H \subseteq G) \wedge (Group[H, *])$
$TrivSubgroup[H, G, *] := (H = \{e\}) \vee (H = G)$
$PropSubgroup[H, G, *] := (Subgroup[H, G, *]) \wedge (\neg TrivSubgroup[H, G, *])$

$SubgroupEquiv := \forall_{H,G} \left(\begin{array}{l} (Subgroup[H, G, *]) \qquad\qquad\qquad \Longleftrightarrow \\ \left((Group[G, *]) \wedge (\emptyset \neq H \subseteq G) \wedge (Function[*, H, H]) \wedge \left(\forall_{a \in H} \exists_{a^{-1} \in H} (a * a^{-1} = e = a^{-1} * a) \right) \right) \end{array} \right)$

(1) $(Subgroup[H, G, *]) \implies \left((\emptyset \neq H \subseteq G) \wedge (Function[*, H, H]) \wedge \left(\forall_{a \in H} \exists_{a^{-1} \in H} (a * a^{-1} = e = a^{-1} * a) \right) \right)$

(2) $\left((\emptyset \neq H \subseteq G) \wedge (Function[*, H, H]) \wedge \left(\forall_{a \in H} \exists_{a^{-1} \in H} (a * a^{-1} = e = a^{-1} * a) \right) \right) \implies \dots$

  (2.1) $Group[G, *]$ ▮ $(a, b, c \in H) \implies (a, b, c \in G) \implies \left((a * b) * c = a * (b * c) \right)$ ▮ $\forall_{a,b,c \in H} \left((a * b) * c = a * (b * c) \right)$

  (2.2) $\emptyset \neq H$ ▮ $\exists_h (h \in H)$

  (2.3) $h \in H$ ▮ $\exists_{h^{-1} \in H} (h * h^{-1} = e = h^{-1} * h)$

  (2.4) $Function[*, H, H]$ ▮ $e = h * h^{-1} \in H$ ▮ $e \in H$ ▮ $\exists_{e \in H} \forall_{a \in H} (a * e = a = e * a)$

  (2.5) $(Function[*, H, H]) \wedge \left(\forall_{a,b,c \in H} \left((a * b) * c = a * (b * c) \right) \right) \wedge \left(\exists_{e \in H} \forall_{a \in H} (a * e = a = e * a) \wedge (\forall_{a \in H} \exists_{a^{-1} \in H} (a * a^{-1} = e = a^{-1} * a) \right)$

  (2.6) $Group[H, *]$

  (2.7) $(Group[G, *]) \wedge (H \subseteq G) \wedge (Group[H, *])$ ▮ $Subgroup[H, G, *]$

(3) $\left((\emptyset \neq H \subseteq G) \wedge (Function[*, H, H]) \wedge \left(\forall_{a \in H} \exists_{a^{-1} \in H} (a * a^{-1} = e = a^{-1} * a) \right) \right) \implies (Subgroup[H, G, *])$

(4) $(Subgroup[H, G, *]) \Longleftrightarrow \left((Group[G, *]) \wedge (\emptyset \neq H \subseteq G) \wedge (Function[*, H, H]) \wedge \left(\forall_{a \in H} \exists_{a^{-1} \in H} (a * a^{-1} = e = a^{-1} * a) \right) \right)$

$SubgroupEquivOST := \forall_{H,G} \left((Subgroup[H, G, *]) \Longleftrightarrow \left((Group[G, *]) \wedge (\emptyset \neq H \subseteq G) \wedge \left(\forall_{a,b \in H} (a * b^{-1} \in H) \right) \right) \right)$

(1) TODO

$SubgroupIntersection := \forall_{H_1, H_2, G} \left(\left((Subgroup[H_1, G, *]) \wedge (Subgroup[H_2, G, *]) \right) \implies (Subgroup[H_1 \cap H_2, G, *]) \right)$

(1) $Group[G, *]$

(2) $(e \in H_1) \wedge (e \in H_2)$ ▮ $e \in H_1 \cap H_2$ ▮ $\emptyset \neq H_1 \cap H_2$

(3) $(H_1 \subseteq G) \wedge (H_2 \subseteq G)$ ▮ $H_1 \cap H_2 \subseteq G$

(4) $\emptyset \neq H_1 \cap H_2 \subseteq G$

(5) $(a, b \in H_1 \cap H_2) \implies \dots$

  (5.1) $a, b \in H_1$ ▮ $a * b \in H_1$

  (5.2) $a, b \in H_2$ ▮ $a * b \in H_2$

(5.3) $a * b \in H_1 \cap H_2$	

(6) $(a, b \in H_1 \cap H_2) \implies (a * b \in H_1 \cap H_2)$ ▌ $Function[*, H_1 \cap H_2, H_1 \cap H_2]$

(7) $(a \in H_1 \cap H_2) \implies \ldots$

(7.1) $(a^{-1} \in H_1) \wedge (a^{-1} \in H_2)$ ▌ $a^{-1} \in H_1 \cap H_2$

(8) $(a \in H_1 \cap H_2) \implies (a^{-1} \in H_1 \cap H_2)$ ▌ $\forall_{a \in H_1 \cap H_2} \exists_{a^{-1} \in H_1 \cap H_2} (a * a^{-1} = e = a^{-1} * a)$

(9) $(SubgroupEquiv) \wedge (Group[G, *]) \wedge (\emptyset \neq H_1 \cap H_2 \subseteq G) \wedge (Function[*, H_1 \cap H_2, H_1 \cap H_2]) \wedge \ldots$

(10) $\ldots \left(\forall_{a \in H_1 \cap H_2} \exists_{a^{-1} \in H_1 \cap H_2} (a * a^{-1} = e = a^{-1} * a) \right)$ ▌ $Subgroup[H_1 \cap H_2, G, *]$

centers centralizers you're fucking retarded

Chapter 3

Linear Algebra

3.1 Matrix Operations and Special Matrices

$Matrix[A, m, n] := [a_{i,j}]_{m \times n} :=$ m rows, n columns of real numbers
$\mathcal{M}_{m,n} := \{A : Matrix[A, m, n]\}$
$O_{m,n} := (Matrix[O, m, n]) \wedge (a_{i,j} = 0)$
$Square[A, n] := Matrix[A, n, n]$
$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$
$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$
$Diagonal[A, n] := (Square[A, n]) \wedge (i \neq j \implies a_{i,j} = 0)$
$Scalar[A, n, k] := (Diagonal[A, n]) \wedge (a_{i,i} = k)$
$I_n := Scalar[I, n, 1]$

$+(A, B) := \big((Matrix[A, m, n]) \wedge (Matrix[B, m, n])\big) \implies (A + B = [a_{i,j} + b_{i,j}]_{m \times n})$
$*(r, A) := \big((r \in \mathbb{R}) \wedge (Matrix[A, m, n])\big) \implies (r * A = [ra_{i,j}]_{m \times n})$
$*(A, B) := \big((Matrix[A, m, p]) \wedge (Matrix[B, p, n])\big) \implies \left(A * B = \left[\sum_{k=1}^{p}(a_{i,k} b_{k,j})\right]_{m \times n}\right)$
$^T[A] := (Matrix[A, m, n]) \implies (A^T = [a_{j,i}]_{n \times m})$

$AddCom := \forall_{A,B \in \mathcal{M}}(A + B = B + A)$

(1)	$A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C \in \mathcal{M}}\big((A + B) + C = A + (B + C)\big)$

(1)	$(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A \in \mathcal{M}} \exists!_{O \in \mathcal{M}}(A + O = A = O + A)$

(1)	$A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$
(2)	$A + O_1 = A = A + O_2 \ \blacksquare \ O_1 = O_2$

$AddInv := \forall_{A \in \mathcal{M}} \exists!_{(-A) \in \mathcal{M}}\big(A + (-A) = O = (-A) + A\big)$

(1)	$A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$
(2)	$A + (-A_1) = O = A + (-A_2) \ \blacksquare \ -A_1 = -A_2 \ \blacksquare \ A_1 = A_2$

$MulAssoc := \forall_{A,B,C \in \mathcal{M}}\big((A * B) * C = A * (B * C)\big)$

(1)	$(A * B) * C = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,j})\right] * C = \left[\sum_{k_2=1}^{p_2}\left(\sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,k_2})c_{k_2,j}\right)\right] = \left[\sum_{k_2=1}^{p_2} \sum_{k_1=1}^{p_1}(a_{i,k_1} b_{k_1,k_2} c_{k_2,j})\right] = \ldots$
(2)	$\ldots \left[\sum_{k_1=1}^{p_1} \sum_{k_2=1}^{p_2}(a_{i,k_1} b_{k_1,k_2} c_{k_2,j})\right] = \left[\sum_{k_1=1}^{p_1}\left(a_{i,k_1} \sum_{k_2=1}^{p_2}(b_{k_1,k_2} c_{k_2,j})\right)\right] = \ldots = A * (B * C)$

$MulId := \forall_{A : Square[A,n]}(A * I_n = A = I_n * A)$

$$(1) \quad A * I_n = \left[\sum_{k=1}^{n} \left(a_{i,k} \left(\begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \right) \right) \right] = [a_{i,j}] = A$$

$(2) \quad TODO = A$

$ScalAssoc := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} \left(r(sA) = (rs)A = s(rA) \right)$

$(1) \quad r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$

$(2) \quad (rs)A = [rsa_{i,j}]$

$(3) \quad s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$

$TransCancel := \forall_{A \in \mathcal{M}} \left(A = (A^T)^T \right)$

$(1) \quad A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$

$ScalMulCom := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} \left((rA) * B = r(A * B) = A * (rB) \right)$

$(1) \quad (rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[\sum_{k=1}^{p} (ra_{i,k} b_{k,j}) \right] = r(A * B)$

$(2) \quad A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[\sum_{k=1}^{p} (a_{i,k} rb_{k,j}) \right] = \left[\sum_{k=1}^{p} (ra_{i,k} b_{k,j}) \right] = r(A * B)$

$ScalDistLeft := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} \left((r + s)A = rA + sA \right)$

$(1) \quad TODO$

$ScalDistRight := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} \left(r(A + B) = rA + rB \right)$

$(1) \quad TODO$

$MulDistRight := \forall_{A,B,C \in \mathcal{M}} \left((A + B) * C = A * C + B * C \right)$

$(1) \quad (A + B) * C = [a_{i,j} + b_{i,j}] * C = \left[\sum_{k=1}^{p} \left((a_{i,k} + b_{i,k}) c_{k,j} \right) \right] = \ \ldots$

$(2) \quad \ldots \left[\sum_{k=1}^{p} (a_{i,k} c_{k,j} + b_{i,k} c_{k,j}) \right] = \left[\sum_{k=1}^{p} (a_{i,k} c_{k,j}) \right] + \left[\sum_{k=1}^{p} (b_{i,k} c_{k,j}) \right] = A * C + B * C$

$MulDistLeft := \forall_{A,B,C \in \mathcal{M}} \left(C * (A + B) = C * A + C * B \right)$

$(1) \quad TODO$

$TransAddDist := \forall_{A,B \in \mathcal{M}} \left((A + B)^T = A^T + B^T \right)$

$(1) \quad TODO$

$TransMulDist := \forall_{A,B \in \mathcal{M}} \left((A * B)^T = B^T * A^T \right)$

$(1) \quad (A * B)^T = \left[\sum_{k=1}^{p} (a_{i,k} b_{k,j}) \right]^T = \left[\sum_{k=1}^{p} (a_{j,k} b_{k,i}) \right] = \left[\sum_{k=1}^{p} (b_{k,i} a_{j,k}) \right] = \left[\sum_{k=1}^{p} (b_{i,k}^T a_{k,j}^T) \right] = B^T * A^T$

$Sym[A] := A = A^T$
$SkewSym[A] := A = -A^T$
$Invertible[A] := (Square[A, n]) \wedge \left(\exists_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A) \right)$

$SymGen := \forall_{A \in \mathcal{M}} (Sym[A + A^T])$

$(1) \quad (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$SkewSymGen := \forall_{A \in \mathcal{M}} (SkewSym[A - A^T])$

$(1) \quad -(A - A^T)^T = -\left(A^T - (A^T)^T \right) = -(A^T - A) = (A - A^T)$

$SymDecomp := \forall_{A \in \mathcal{M}} \exists!_{B:Sym[B]} \exists!_{C:SkewSym[C]} (A = B + C)$

(1)	$B := (1/2) * (A + A^T)$; $C := (1/2) * (A - A^T)$
(2)	$SymGen[B] \wedge SkewSymGen[C]$
(3)	$A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$
(4)	$(1/2) * (A_1 + A_1{}^T) = (1/2) * (A_2 + A_2{}^T)$ ▉ $A_1 = A_2$
(5)	$(1/2) * (A_3 - A_3{}^T) = (1/2) * (A_4 - A_4{}^T)$ ▉ $A_3 = A_4$

$InvId := \forall_{A:Invertible[A]} \left(\exists!_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A) \right)$

(1)	$A^{-1}{}_1 = A^{-1}{}_1 * I_n = A^{-1}{}_1 * (A * A^{-1}{}_2) = (A^{-1}{}_1 * A) * A^{-1}{}_2 = I_n * A^{-1}{}_2 = A^{-1}{}_2$

$InvCancel := \forall_{A:Invertible[A]} \left((A^{-1})^{-1} = A \right)$

(1)	$(A * A^{-1})^{-1} = I_n{}^{-1} = I_n$
(2)	$(A^{-1})^{-1} * A^{-1} = I_n$ ▉ $A^{-1})^{-1} = I_n * A = A$

$InvDist := \forall_{A:Invertible[A]} \forall_{B:Invertible[B]} \left((A * B)^{-1} = B^{-1} * A^{-1} \right)$

(1)	$(A * B) * (A * B)^{-1} = I$ ▉ $B * (A * B)^{-1} = A^{-1}$ ▉ $(A * B)^{-1} = B^{-1} * A^{-1}$

$InvTrans := \forall_{A:Invertible[A]} \left((A^T)^{-1} = (A^{-1})^T \right)$ ▉ $\Leftarrow$

(1)	$A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I$ ▉ $(A^{-1})^T = (A^T)^{-1}$

3.2 Elementary Matrices on Invertibility and Systems of Linear Equations

$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$
$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$
$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X (Sol[X, A, B])$
$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$
$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$

$HomoSysProps := (Sys[A, O]) \implies \dots$

(1)	$u_0 := O$; $u_1 := choice(\{X \in \mathcal{M} | X \neq O\})$; $k := choice(\mathbb{R})$
(2)	$TrivSol[u_0, A]$
(3)	$(NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$
(4)	$(TrivSol[\overrightarrow{X}, A]) \implies \left(TrivSol[LC(\overrightarrow{X}), A] \right)$

$ElemMat[E] := (E = Swap[I_n, i, j]) \vee (Scale_*(I_n, i, c)) \vee (Combine_*(I_n, i, c, j))$
$ElemMatProd[E^*] := \exists_{\langle E \rangle} \left(\forall_{E_i \in E^*} (ElemMat[E_i]) \wedge \left(E^* = \Pi_{E_i \in E^*}(E_i) \right) \right)$
$RowEquiv[A, B] := \exists_{E^*} ((ElemMatProd[E^*]) \wedge (B = E^* * A))$

$ElemMatInv := \forall_{E \in \mathcal{M}} ((ElemMat[E]) \implies (Invertible[E]))$

(1)	$E - RowSwap[E] \implies TODO$; $E - RowScale_*(E) \implies TODO$; $E - RowCombine_*(E) \implies TODO$

$ElemMatProdInv := \forall_{E^*} ((ElemMatProd[E^*]) \implies (Invertible[E^*]))$

(1)	TODO

$RowEquivSys := \forall_{A,B,C,D,X \in \mathcal{M}} \left(((Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]])) \implies (Sol[X, A, B] \iff Sol[X, C, D]) \right)$

(1)	$\exists_{E^*:ElemMatProd[E^*]} ([CD] = E^* * [AB])$

| (2) | $(E^* * A = C) \wedge (E^* * B = D)$ |

| (3) | $Sol[Y, A, B] \implies \ldots$ |

(3.1) $A * Y = B$

(3.2) $C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D \;\blacksquare\; Sol[Y, C, D]$

(4) $Sol[Y, A, B] \implies Sol[Y, C, D]$

(5) $\left(A = (E^*)^{-1} * C \right) \wedge \left(B = (E^*)^{-1} * D \right)$

(6) $Sol[Z, C, D] \implies \ldots$

(6.1) $C * Z = D$

(6.2) $A * Z = \left((E^*)^{-1} * C \right) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$

(7) $Sol[Z, C, D] \implies Sol[Z, A, B]$

(8) $Sol[X, A, B] \iff Sol[X, C, D]$

$RowEquivHomoSysSol := \forall_{A,C,X \in \mathcal{M}} \Big((RowEquiv[A, C]) \implies \big((Sol[X, A, O]) \iff (Sol[X, C, O]) \big) \Big)$

(1) Set $B = D = O$

$RREF[A] := (A \in \mathcal{M}) \wedge \left\{ \begin{array}{ll} \text{All zero rows are at the bottom of the matrix.} & \wedge \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} \wedge \\ \text{The leading entry in any nonzero row is 1.} & \wedge \\ \text{All entries in the column above and below a leading 1 are zero.} & \wedge \end{array} \right\}$

$GaussJordanElim := \forall_{A \in \mathcal{M}} \exists!_{B \in \mathcal{M}} \big((RREF[B]) \wedge (RowEquiv[A, B]) \big)$

(1) Hit A with $ElemMat$'s until it becomes B

(2) $(B = E^* * A) \wedge (RREF[B])$

$HasZero[A] := \big(Matrix(A, m, n) \big) \wedge \big(\exists_{i \le m}(A_{i,:} = O) \big)$

$HasZeroNonInvertible := \forall_{A \in \mathcal{M}} \big((HasZero[A]) \implies (\neg Invertible[A]) \big)$

(1) $i := choice(\{i \le m | A_{i,:} = O\})$

(2) $(B \in \mathcal{M}) \implies \ldots$

(2.1) $(A * B)_{i,:} = O \ne I_{n i,:} \;\blacksquare\; A * B \ne I_n$

(3) $(B \in \mathcal{M}) \implies (A * B \ne I_n) \;\blacksquare\; \forall_{B \in \mathcal{M}}(A * B \ne I_n) \;\blacksquare\; \neg Invertible[A]$

$InvIffRowEquivI := \forall_{A \in \mathcal{M}} \big((Invertible[A]) \iff (RowEquiv[A, I_n]) \big)$

(1) $(Invertible[A]) \implies \ldots$

(1.1) $(RREF[B]) \wedge (RowEquiv[A, B])$

(1.2) $B = E^* * A$

(1.3) $(Invertible[E^*]) \wedge (Invertible[A]) \;\blacksquare\; Invertible[B]$

(1.4) $Invertible[B] \;\blacksquare\; \neg HasZero[B]$

(1.5) $(RREF[B]) \wedge (\neg HasZero[B]) \;\blacksquare\; B = I_n$

(1.6) $RowEquiv[A, I_n]$

(2) $(Invertible[A]) \implies (RowEquiv[A, I_n])$

(3) $(RowEquiv[A, I_n]) \implies \ldots$

(3.1) $I_n = E^* * A \;\blacksquare\; (E^*)^{-1} = A$

(3.2) $A^{-1} = E^*_{DescSort} \;\blacksquare\; Invertible[A]$

(4) $(RowEquiv[A, I_n]) \implies (Invertible[A])$

(5) $(Invertible[A]) \iff (RowEquiv[A, I_n])$

$RowEquivIIffTrivSol := \forall_{A \in \mathcal{M}} \Big((RowEquiv[A, I_n]) \iff \big(\forall_X ((X = O) \iff (Sol[X, A, O])) \big) \Big)$

(1) $(RowEquiv[A, I_n]) \implies \ldots$

(1.1) $RowEquiv[A, I_n] \;\blacksquare\; Invertible[A]$

(1.2) $(Sol[X, A, O]) \implies \ldots$

   (1.2.1) $A * X = O$ ▌ $X = A^{-1} * O = O$ ▌ $X = O$

(1.3) $(Sol[X, A, O]) \implies (X = O)$

(1.4) $(X = O) \implies (Sol[X, A, O])$

(1.5) $(X = O) \iff (Sol[X, A, O])$ ▌ $\forall_X((X = O) \iff (Sol[X, A, O]))$

(2) $(RowEquiv[A, I_n]) \implies \left(\forall_X((X = O) \iff (Sol[X, A, O]))\right)$

(3) $\left(\forall_X((X = O) \iff (Sol[X, A, O]))\right) \implies \ldots$

  (3.1) $(RREF[B]) \land (RowEquiv[A, B])$

  (3.2) $Sol[X, B, O]$

  (3.3) $(B \neq I_n) \implies \ldots$

   (3.3.1) $\left(\exists_{Y \neq X}(Sol[Y, B, O])\right)$

   (3.3.2) $Sol[Y, A, O]$ ▌ $Y = X$

   (3.3.3) $(Y \neq X) \land (Y = X)$ ▌ $\perp$

  (3.4) $(B \neq I_n) \implies \perp$ ▌ $B = I_n$

  (3.5) $(RowEquiv[A, B]) \land (B = I_n)$ ▌ $RowEquiv[A, I_n]$

(4) $\left(\forall_X((X = O) \iff (Sol[X, A, O]))\right) \implies (RowEquiv[A, I_n])$

(5) $(RowEquiv[A, I_n]) \iff \left(\forall_X((X = O) \iff (Sol[X, A, O]))\right)$

$InvIffUniqSol := \forall_{A \in \mathcal{M}}\left((Invertible[A]) \iff \left(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\right)\right)$

(1) $(Invertible[A] \land B \in \mathcal{M}) \implies \ldots$

  (1.1) $(Invertible[A]) \land (Sys[A, B])$

  (1.2) $(X = A^{-1} * B) \iff (Sol[X, A, B])$ ▌ $\exists!_{X \in \mathcal{M}}(Sol[X, A, B])$

(2) $\left(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\right) \implies \ldots$

  (2.1) $X_i := choice(\{X_i | Sol[X_i, A, I_{n:,i}]\})$

  (2.2) $A * [X_1 \ldots X_n] = [(A * X_1) \ldots (A * X_n)] = [I_{n:,1} \ldots I_{n:,n}] = I_n$

  (2.3) $A^{-1} = [X_1 \ldots X_n]$

(3) $\left(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\right) \implies (Invertible[A])$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}}\left(\begin{array}{ll} (Invertible[A]) & \iff \\ (RowEquiv[A, I_n]) & \iff \\ \left(\forall_X((X = O) \iff (Sol[X, A, O]))\right) & \iff \\ \left(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])\right) & \end{array}\right)$$

3.3 Vector Spaces

$$VectorSpace[V, +, *] := \exists_{O \in V} \forall_{\alpha, \beta \in \mathbb{R}} \forall_{u,v,w \in V}\left\{\begin{array}{l} (u + v \in V) \ \land \ (u + v = v + u) \ \land \ ((u + v) + w = u + (v + w)) \ \land \\ (u + O = u) \qquad \land \qquad \left(\exists_{-u \in V}(u + (-u) = O)\right) \qquad \land \\ (\alpha * u \in V) \ \land \ (\alpha * (\beta * u) = (\alpha\beta) * u) \ \land \ (1 * u = u) \ \land \\ (\alpha * (u + v) = (\alpha * u) + (\alpha * v)) \land ((\alpha + \beta) * u = (\alpha * u) + (\beta * u)) \end{array}\right\}$$

$ZeroVectorUniq := \forall_{O',v \in V}\left((v + O' = v) \implies (O' = O)\right)$

(1) $O' = O' + O = O + O' = O$ ▌ $O' = O$

$AddInvUniq := \forall_{-v',v \in V}\left((v + -v' = O) \implies (-v' = -v)\right)$

(1) $-v' = -v' + O = -v' + (v + -v) = (-v' + v) + -v = (v + -v') + -v = O + -v = -v$ ▌ $-v' = -v$

$AddInvGen := \forall_{v \in V}\left((-1) * v = -v\right)$

(1) $v + (-1) * v = (1 - 1) * v = 0 * v = O$ ▮ $(-1) * v = -v$

$ZeroVectorGenLeft := \forall_{v \in V}(0 * v = O)$

(1) $0 * v = (0 + 0) * v = (0 * v) + (0 * v)$ ▮ $O = 0 * v$

$ZeroVectorGenRight := \forall_{r \in \mathbb{R}}(r * O = O)$

(1) $r * O = r * (O + O) = (r * O) + (r * O)$ ▮ $O = r * O$

$ZeroVectorEquiv := \forall_{r \in \mathbb{R}} \forall_{v \in V}\Big((r * v = O) \iff \big((v = O) \vee (r = 0)\big)\Big)$

(1) $(ZeroVectorGenLeft) \wedge (ZeroVectorGenRight)$ ▮ $\big((v = O) \vee (r = 0)\big) \implies (r * v = O))$

(2) $(r * v = O) \implies \dots$

  (2.1) $(r \neq 0) \implies \dots$

    (2.1.1) $r \neq 0$ ▮ $r^{-1} \in \mathbb{R}$

    (2.1.2) $ZeroVectorGenRight$ ▮ $O = r^{-1} * O = r^{-1} * (r * v) = (r^{-1}r) * v = 1 * v = v$ ▮ $O = v$

  (2.2) $(r \neq 0) \implies (v = O)$ ▮ $(r = 0) \vee (v = O)$

(3) $(r * v = O) \implies \big((r = 0) \vee (v = O)\big)$

(4) $(r * v = O) \iff \big((r = 0) \vee (v = O)\big)$

3.4   Subspaces and Special Subspaces

$Subspace[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (S \subseteq V) \wedge (VectorSpace[S, +, *])$

$SubspaceEquiv := \forall_{V,S}\left(\begin{array}{l} (VectorSpace[V, +, *]) \hspace{3.5cm} \implies \\ \Big((Subspace[S, V, +, *]) \iff \big((\emptyset \neq S \subseteq V) \wedge (\forall_{r,s \in S}(r + s \in S)) \wedge (\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S))\big)\Big) \end{array} \right)$

(1) $(Subspace[S, V, +, *]) \implies \dots$

  (1.1) $Subspace[S, V, +, *]$ ▮ $S \subseteq V$

  (1.2) $VectorSpace[S, V, +, *]$ ▮ $\exists_{O \in V} \forall_{v \in V}(v + O = v)$ ▮ $O \in S$ ▮ $\emptyset \neq S$

  (1.3) $(\emptyset \neq S) \wedge (S \subseteq V)$ ▮ $\emptyset \neq S \subseteq V$

  (1.4) $VectorSpace[S, V, +, *]$ ▮ $\big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)$

  (1.5) $(\emptyset \neq S \subseteq V) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)$

(2) $(Subspace[S, V, +, *]) \implies \Big((\emptyset \neq S \subseteq V) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)\Big)$

(3) $\Big((\emptyset \neq S \subseteq V) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)\Big) \implies \dots$

  (3.1) $\big((\emptyset \neq S) \wedge (\alpha, \beta \in \mathbb{R}) \wedge (u, v, w \in S)\big) \implies \dots$

    (3.1.1) $\emptyset \neq S$ ▮ $\exists_x(x \in V)$

    (3.1.2) $(ZeroVectorGenLeft) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big) \wedge (x \in V)$ ▮ $O = 0 * x \in S$ ▮ $O \in S$

    (3.1.3) $u, v \in V$ ▮ $u + v = v + u$

    (3.1.4) $u, v, w \in V$ ▮ $(u + v) + w = u + (v + w)$

    (3.1.5) $u \in V$ ▮ $u + O = u$

    (3.1.6) $(AddInvGen) \wedge (u \in S)$ ▮ $(-1) * u = -u \in S$

    (3.1.7) $u \in V$ ▮ $\alpha * (\beta * u) = (\alpha\beta) * u$

    (3.1.8) $u \in V$ ▮ $1 * u = u$

    (3.1.9) $u, v \in V$ ▮ $\alpha * (u + v) = (\alpha * u) + (\alpha * v)$

    (3.1.10) $u \in V$ ▮ $(\alpha + \beta) * u = (\alpha * u) + (\beta * u)$

(4) $\Big((\emptyset \neq S) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)\Big) \implies (Subspace[S, V, +, *])$

(5) $(Subspace[S, V, +, *]) \iff \Big((\emptyset \neq S) \wedge \big(\forall_{r,s \in S}(r + s \in S)\big) \wedge \big(\forall_{\alpha \in \mathbb{R}} \forall_{s \in S}(\alpha * s \in S)\big)\Big)$

$SetSum[A + B, A, B, V, +, *] := (VectorSpace[V, +, *]) \wedge (A, B \subseteq V) \wedge \left(A + B = \{a + b | (a \in A) \wedge (b \in B)\} \right)$

$SumSubContains := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])) \implies \\ ((Subspace[A + B, V, +, *]) \wedge (A, B \subseteq A + B)) \end{array} \right)$

(1) $(Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *])$ ▍ $(O \in A) \wedge (O \in B)$

(2) $(SetSum[A + B, A, B, V, +, *]) \wedge (O \in A) \wedge (O \in B)$ ▍ $O = O + O \in A + B$ ▍ $\emptyset \neq A + B$

(3) $(v \in A + B) \implies \ldots$

　(3.1) $\exists_{a \in A} \exists_{b \in B} (v = a + b)$

　(3.2) $(A \subseteq V) \wedge (B \subseteq V)$ ▍ $a, b \in V$

　(3.3) $VectorSpace[V, +, *]$ ▍ $v = a + b \in V$

(4) $(v \in A + B) \implies (v \in V)$ ▍ $A + B \subseteq V$

(5) $(\emptyset \neq A + B) \wedge (A + B \subseteq V)$ ▍ $\emptyset \neq A + B \subseteq V$

(6) $(u, v \in A + B) \implies \ldots$

　(6.1) $\left(\exists_{a_1 \in A} \exists_{b_1 \in B} (u = a_1 + b_1) \right) \wedge \left(\exists_{a_2 \in A} \exists_{b_2 \in B} (v = a_2 + b_2) \right)$

　(6.2) $u + v = (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2)$

　(6.3) $(a_1 + a_2 \in A) \wedge (b_1 + b_2 \in B)$ ▍ $u + v \in A + B$

(7) $(u, v \in A + B) \implies (u + v \in A + B)$ ▍ $\forall_{u,v \in A+B} (u + v \in A + B)$

(8) $\left((r \in \mathbb{R}) \wedge (v \in A + B) \right) \implies \ldots$

　(8.1) $\exists_{a \in A} \exists_{b \in B} (v = a + b)$

　(8.2) $r * v = r * (a + b) = r * a + r * b$

　(8.3) $(r * a \in A) \wedge (r * b) \in B$ ▍ $r * v \in A + B$

(9) $\left((r \in \mathbb{R}) \wedge (v \in A + B) \right) \implies (r * v \in A + B)$ ▍ $\forall_{r \in \mathbb{R}} \forall_{v \in A+B} (r * v \in A + B)$

(10) $(SubspaceEquiv) \wedge (\emptyset \neq A + B \subseteq V) \wedge \left(\forall_{u,v \in A+B} (u + v \in A + B) \right) \wedge \left(\forall_{r \in \mathbb{R}} \forall_{v \in A+B} (r * v \in A + B) \right)$ ▍ $Subspace[A + B, V, +, *]$

(11) $(O \in B) \wedge \left(\forall_{a \in A} (a + O = a) \right)$ ▍ $A \subseteq A + B$

(12) $(O \in A) \wedge \left(\forall_{b \in B} (b + O = b) \right)$ ▍ $B \subseteq A + B$

(13) $(A \subseteq A + B) \wedge (B \subseteq A + B)$ ▍ $A, B \subseteq A + B$

(14) $(Subspace[A + B, V, +, *]) \wedge (A, B \subseteq A + B)$

$SumSubMinContains := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])) \implies \\ \forall_C ((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies (A + B \subseteq C) \end{array} \right)$

(1) $SumSub$ ▍ $(A, B \subseteq A + B) \wedge (Subspace[A + B, V, +, *])$

(2) $\left((Subspace[C, V, +, *]) \wedge (A, B \subseteq C) \right) \implies \ldots$

　(2.1) $(s \in A + B) \implies \ldots$

　　(2.1.1) $\exists_{a \in A} \exists_{b \in B} (s = a + b)$

　　(2.1.2) $(A, B \subseteq C)$ ▍ $a, b \in C$

　　(2.1.3) $(VectorSpace[C, V, +, *]) \wedge (a, b \in C)$ ▍ $s = a + b \in C$

　(2.2) $(s \in A + B) \implies (s \in C)$ ▍ $A + B \subseteq C$

(3) $\left((Subspace[C, V, +, *]) \wedge (A, B \subseteq C) \right) \implies (A + B \subseteq C)$

$DirSum[A \oplus B, A, B, V, +, *] := \left(\begin{array}{ll} (Subspace[A, V, +, *]) & \wedge \quad (Subspace[B, V, +, *]) \quad \wedge \\ (SetSum[A + B, A, B, V, +, *]) \wedge \left(\forall_{s \in A+B} \exists!_{\langle a,b \rangle \in A \times B} (s = a + b) \right) \end{array} \right)$

$DirSumEquiv := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])) \implies \\ (DirSum[A \oplus B, A, B, V, +, *]) \iff \left(\exists!_{\langle a,b \rangle \in A \times B} (O = a + b) \right) \end{array} \right)$

(1) $(DirSum[A \oplus B, A, B, V, +, *]) \implies \ldots$

　(1.1) $(Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *])$ ▍ $(O \in A) \wedge (O \in B)$

(1.2) $(SubSum[A \oplus B, A, B, V, +, *]) \wedge (O \in A) \wedge (O \in B)$ ▌ $O = O + O \in A \oplus B$

(1.3) $(DirSum[A \oplus B, A, B, V, +, *]) \wedge (O \in A \oplus B)$ ▌ $\exists!_{\langle a,b \rangle \in A \times B}(O = a + b)$

(2) $(DirSum[A \oplus B, A, B, V, +, *]) \implies \left(\exists!_{\langle a,b \rangle \in A \times B}(O = a + b) \right)$

(3) $\left(\exists!_{\langle a,b \rangle \in A \times B}(O = a + b) \right) \implies \ldots$

(3.1) $(s \in A \oplus B) \implies \ldots$

(3.1.1) $\left(\exists_{\langle a,b \rangle \in A \times B}(s = a + b) \right)$

(3.1.2) $\left((s = a_1 + b_1) \wedge (s = a_2 + b_2) \right) \implies \ldots$

(3.1.2.1) $O = s - s = (a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2)$

(3.1.2.2) $(Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *])$ ▌ $(a_1 - a_2 \in A) \wedge (b_1 - b_2 \in B)$

(3.1.2.3) $\left((a_1 - a_2 \neq O) \vee (b_1 - b_2 \neq O) \right) \implies \left(\neg \exists!_{\langle a,b \rangle \in A \times B}(O = a + b) \right) \implies \bot$

(3.1.2.4) $(a_1 - a_2 = O) \wedge (b_1 - b_2 = O)$ ▌ $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$

(3.1.3) $\left((s = a_1 + b_1) \wedge (s = a_2 + b_2) \right) \implies \langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$

(3.1.4) $\forall_{\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B} \left(\left((s = a_1 + b_1) \wedge (s = a_2 + b_2) \right) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle) \right)$

(3.1.5) $\exists_{\langle a,b \rangle \in A \times B}(s = a + b) \wedge \forall_{\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B} \left(\left((s = a_1 + b_1) \wedge (s = a_2 + b_2) \right) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle) \right)$ ▌ $\exists!_{\langle a,b \rangle \in A \times B}(s = a + b)$

(3.2) $(s \in A + B) \implies \exists!_{\langle a,b \rangle \in A \times B}(s = a + b)$ ▌ $\forall_{s \in A + B} \exists!_{\langle a,b \rangle \in A \times B}(s = a + b)$ ▌ $DirSum[A \oplus B, A, B, V, +, *]$

(4) $\left(\exists!_{\langle a,b \rangle \in A \times B}(O = a + b) \right) \implies (DirSum[A \oplus B, A, B, V, +, *])$

(5) $(DirSum[A \oplus B, A, B, V, +, *]) \iff \left(\exists!_{\langle a,b \rangle \in A \times B}(O = a + b) \right)$

$DirSumSubspace := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *]) \wedge (SetSum[A + B, A, B, V, +, *])) \implies \\ ((DirSum[A \oplus B, A, B, V, +, *]) \iff (A \cap B = \{O\})) \end{array} \right)$

(1) $(DirSum[A \oplus B, A, B, V, +, *]) \implies \ldots$

(1.1) $(v \in A \cap B) \implies \ldots$

(1.1.1) $(v \in A \cap B) \wedge (VectorSpace[B, +, *])$ ▌ $(v \in A) \wedge (v \in B)$ ▌ $(v \in A) \wedge (-v \in B)$

(1.1.2) $(v \in A) \wedge (-v \in B)$ ▌ $v + (-v) = O \in A + B$

(1.1.3) $DirSum[A \oplus B, A, B, V, +, *]$ ▌ $\exists!_{\langle a,b \rangle \in A \times B}(O = a + b)$

(1.1.4) $(v \neq O) \implies \left(\neg \exists!_{\langle a,b \rangle \in A \times B}(O = a + b) \right) \implies \bot$ ▌ $v = O$

(1.2) $(v \in A \cap B) \implies (v = O)$ ▌ $A + B \subseteq \{O\}$

(1.3) $(v = O) \implies \ldots$

(1.3.1) $(Subspace[A, V, +, *]) \wedge (Subspace[B, V, +, *])$ ▌ $(O \in A) \wedge (O \in B)$ ▌ $v = O \in A \cup B$

(1.4) $(v = O) \implies (v \in A \cap B)$ ▌ $\{O\} \subseteq A \cap B$

(1.5) $(A + B \subseteq \{O\}) \wedge (\{O\} \subseteq A \cap B)$ ▌ $A \cap B = \{O\}$

(2) $(DirSum[A \oplus B, A, B, V, +, *]) \implies (A \cap B = \{O\})$

(3) $(A \cap B = \{O\}) \implies \ldots$

(3.1) $(O \in A) \wedge (O \in B) \wedge (O = O + O \in A + B)$ ▌ $\exists_{\langle a,b \rangle \in A \times B}(O = a + b)$

(3.2) $\left((\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B) \wedge (O = a_1 + b_1) \wedge (O = a_2 + b_2) \right) \implies \ldots$

(3.2.1) $(O = a_1 + b_1) \wedge (O = a_2 + b_2)$ ▌ $(a_1 = -b_1) \wedge (a_2 = -b_2)$

(3.2.2) $VectorSpace[B, +, *]$ ▌ $-b_1, -b_2 \in B$

(3.2.3) $(a_1 \in A) \wedge (a_1 = -b_1 \in B)$ ▌ $a_1 \in A \cap B$ ▌ $a_1 = O$ ▌ $a_1 = b_1 = O$

(3.2.4) $(a_2 \in A) \wedge (a_2 = -b_2 \in B)$ ▌ $a_2 \in A \cap B$ ▌ $a_2 = O$ ▌ $a_2 = b_2 = O$

(3.2.5) $\langle a_1, b_1 \rangle = \langle O, O \rangle = \langle a_2, b_2 \rangle$

(3.3) $\left((\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B) \wedge (O = a_1 + b_1) \wedge (O = a_2 + b_2) \right) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle)$

(3.4) $\forall_{\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B} \left(\left((O = a_1 + b_1) \wedge (O = a_2 + b_2) \right) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle) \right)$

(3.5) $\left(\exists_{\langle a,b \rangle \in A \times B}(O = a + b) \right) \wedge \left(\forall_{\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \in A \times B} \left(\left((O = a_1 + b_1) \wedge (O = a_2 + b_2) \right) \implies (\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle) \right) \right)$

(3.6)	$\left(\exists!_{\langle a,b\rangle\in A\times B}(O = a + b)\right) \wedge (DirSumEquiv)\ \blacksquare\ DirSum[A \oplus B, A, B, V, +, *]$

(4)	$(A \cap B = \{O\}) \implies (DirSum[A \oplus B, A, B, V, +, *])$
(5)	$(DirSum[A \oplus B, A, B, V, +, *]) \iff (A \cap B = \{O\})$

$NullSpace[N, A, m, n] := (Matrix[A, m, n]) \wedge (N = \{x \in \mathbb{R}^n | A * x = O\})$
$RowSpace[R, A, m, n] := (Matrix[A, m, n]) \wedge (R = \{x^T * A \in \mathbb{R}^n | x \in \mathbb{R}^m\})$
$ColSpace[C, A, m, n] := (Matrix[A, m, n]) \wedge (C = \{A * x \in \mathbb{R}^m | x \in \mathbb{R}^n\})$

$NullSubspace := (NullSpace[N, A, m, n]) \implies (Subspace[N, \mathbb{R}^n, +, *])$

(1)	TODO

$RowSubspace := (RowSpace[R, A, m, n]) \implies (Subspace[R, \mathbb{R}^n, +, *])$

(1)	TODO

$ColSubspace := (ColSpace[C, A, m, n]) \implies (Subspace[C, \mathbb{R}^m, +, *])$

(1)	TODO

3.5 Linear Combination, Linear Span, Linear Independence

$LinComb[c, U, K, V, +, *] := (VectorSpace[V, +, *]) \wedge (n \in \mathbb{N}) \wedge (U \in V^n) \wedge (K \in \mathbb{R}^n) \wedge \left(c = \sum_{i=1}^{n}(k_i * u_i)\right)$

$LinSpan[S', S, V, +, *] := \left(\begin{array}{c} (VectorSpace[V, +, *]) \wedge (S \in V^n) \wedge ((S = \emptyset) \implies (S' = \{O\})) \quad \wedge \\ ((S \neq \emptyset) \implies (S' = \{c \in V | (K \in \mathbb{R}^n) \wedge (LinComb[c, S, K, V, +, *])\})) \end{array} \right)$

$LinSpanSubContains := \forall_{S',S,V}\left((LinSpan[S', S, V, +, *]) \implies ((Subspace[S', V, +, *]) \wedge (S \subseteq S'))\right)$

(1)	$(S = \emptyset) \implies$ ...
(1.1)	$LinSpan[S', S, V, +, *]\ \blacksquare\ S' = \{O\}$
(1.2)	$Subspace[\{O\}, V, +, *]\ \blacksquare\ Subspace[S', V, +, *]$
(1.3)	$S = \emptyset \subseteq \{O\} = S'\ \blacksquare\ S \subseteq S'$
(1.4)	$(Subspace[S', V, +, *]) \wedge (S \subseteq S')$
(2)	$(S = \emptyset) \implies ((Subspace[S', V, +, *]) \wedge (S \subseteq S'))$
(3)	$(S \neq \emptyset) \implies$ ...
(3.1)	$LinSpan[S', S, V, +, *]\ \blacksquare\ S' = \{c \in V | (K \in \mathbb{R}^n) \wedge (LinComb[c, S, K, V, +, *])\}\ \blacksquare\ S' \subseteq V$
(3.2)	$(\{0\}^n \subseteq \mathbb{R}^n) \wedge (LinComb[O, S, \{0\}^n, V, +, *])\ \blacksquare\ O \in S'\ \blacksquare\ \emptyset \neq S'$
(3.3)	$(S' \subseteq V) \wedge (\emptyset \neq S')\ \blacksquare\ \emptyset \neq S' \subseteq V$
(3.4)	$(a, b \in S') \implies$ ...
(3.4.1)	$\left(\exists_{K_a \in \mathbb{R}^n}(LinComb[a, S, K_a, V, +, *])\right) \wedge \left(\exists_{K_b \in \mathbb{R}^n}(LinComb[b, S, K_b, V, +, *])\right)\ \blacksquare\ \left(a = \sum_{i=1}^{n}(k_{ai} * s_i)\right) \wedge \left(b = \sum_{i=1}^{n}(k_{bi} * s_i)\right)$
(3.4.2)	$a + b = \sum_{i=1}^{n}(k_{ai} * s_i) + \sum_{i=1}^{n}(k_{bi} * s_i) = \sum_{i=1}^{n}((k_{ai} + k_{bi}) * s_i)\ \blacksquare\ a + b = \sum_{i=1}^{n}((k_{ai} + k_{bi}) * s_i)$
(3.4.3)	$\langle k_{ai} + k_{bi} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n$
(3.4.4)	$\left(a + b = \sum_{i=1}^{n}((k_{ai} + k_{bi}) * s_i)\right) \wedge (\langle k_{ai} + k_{bi} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n)$ ...
(3.4.5)	... $\exists_{M \in \mathbb{N}^n}\left(a + b = \sum_{i=1}^{n}(m_i * s_i)\right)\ \blacksquare\ \exists_{M \in \mathbb{N}^n}(LinComb[a + b, S, M, V, +, *])\ \blacksquare\ a + b \in S'$
(3.5)	$(a, b \in S') \implies (a + b \in S')\ \blacksquare\ \forall_{a,b \in S'}(a + b \in S')$
(3.6)	$((r \in \mathbb{R}) \wedge (u \in S')) \implies$ ...
(3.6.1)	$\exists_{K \in \mathbb{R}^n}(LinComb[u, S, K, V, +, *])\ \blacksquare\ u = \sum_{i=1}^{n}(k_i * s_i)$
(3.6.2)	$r * u = r * \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{n}(r * (k_i * s_i)) = \sum_{i=1}^{n}(rk_i) * s_i\ \blacksquare\ r * u = \sum_{i=1}^{n}(rk_i) * s_i$

(3.6.3) $\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n$

(3.6.4) $\left(r * u = \sum_{i=1}^{n} (rk_i) * s_i) \right) \wedge (\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n)$ ▌ $\exists_{M \in \mathbb{R}^n} \left(r * u = \sum_{i=1}^{n} (m_i * s_i) \right)$

(3.6.5) $\exists_{M \in \mathbb{R}^n} (LinComb[r * u, S, M, V, +, *])$ ▌ $r * u \in S'$

(3.7) $\left((r \in \mathbb{R}) \wedge (u \in S') \right) \implies (r * u \in S')$ ▌ $\forall_{r \in \mathbb{R}} \forall_{u \in S'} (r * u \in S')$

(3.8) $(SubspaceEquiv) \wedge (\emptyset \neq S' \subseteq V) \wedge \left(\forall_{a,b \in S'}(a + b \in S') \right) \wedge \left(\forall_{r \in \mathbb{R}} \forall_{u \in S'}(r * u \in S') \right)$ ▌ $Subspace[S', V, +, *]$

(3.9) $(s_i \in S) \implies \ldots$

(3.9.1) $K_s := \left\langle \left\{ \begin{matrix} 1 & j = i \\ 0 & j \neq i \end{matrix} \middle| j \in \mathbb{N}_{1,n} \right\rangle \right.$ ▌ $(K_s \in \mathbb{R}^n) \wedge \left(\sum_{j=1}^{n} (k_{s_j} * s_j) = s_i \right)$

(3.9.2) $\exists_{K \in \mathbb{R}^n} (LinComb[s_j, S, K, V, +, *])$ ▌ $s_j \in S'$

(3.10) $(s_i \in S) \implies (s_i \in S')$ ▌ $S \subseteq S'$

(3.11) $(Subspace[S', V, +, *]) \wedge (S \subseteq S')$

(4) $(S \neq \emptyset) \implies \left((Subspace[S', V, +, *]) \wedge (S \subseteq S') \right)$

(5) $\left((S = \emptyset) \implies \left((Subspace[S', V, +, *]) \wedge (S \subseteq S') \right) \right) \wedge \left((S \neq \emptyset) \implies \left((Subspace[S', V, +, *]) \wedge (S \subseteq S') \right) \right) \ldots$

(6) $\ldots (Subspace[S', V, +, *]) \wedge (S \subseteq S')$

$LinSpanSubMinContains := \forall_{S', S, V, +, *} \left((LinSpan[S', S, V, +, *]) \implies \left(\forall_W \left(((Subspace[W, V, +, *]) \wedge (S \subseteq W)) \implies (S' \subseteq W) \right) \right) \right)$

(1) $(s' \in S') \implies \ldots$

(1.1) $\exists_{K \in \mathbb{R}^n} (LinComb[s', S, K, V, +, *])$ ▌ $s' = \sum_{i=1}^{n} (k_i * s_i)$

(1.2) $(S \subseteq W) \wedge (VectorSpace[W, V, +, *])$ ▌ $s' = \sum_{i=1}^{n} (k_i * s_i) \in W$ ▌ $s' \in W$

(2) $(s' \in S') \implies (s' \in W)$ ▌ $S' \subseteq W$

$Spans[S, V, +, *] := LinSpan[V, S, V, +, *]$
$FinDim[V, +, *] := \exists_{S \in V^n} (Spans[S, V, +, *])$

$LinInd[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (S \in V^n) \wedge \left((S \neq \emptyset) \implies \left(\forall_{K \in \mathbb{R}^n} ((LinComb[O, S, K, V, +, *]) \implies (K = \{0\}^n)) \right) \right)$

$ZeroDependent := (O \in S) \implies (\neg LinInd[S, V, +, *])$

(1) $O \in S$ ▌ $\exists_{u_i \in S} (u_i = O)$ ▌ $K := \left\langle \left\{ \begin{matrix} 1 & u_i = O \\ 0 & u_i \neq O \end{matrix} \middle| i \in \mathbb{N}_{1,n} \right\rangle \right.$ ▌ $\{O\}^n \neq K \in \mathbb{R}^n$

(2) $O = \sum_{i=1}^{n} (k_i * s_i)$ ▌ $LinComb[O, S, K, V, +, *]$

(3) $(LinComb[O, S, K, V, +, *]) \wedge (\{O\}^n \neq K \in \mathbb{R}^n)$ ▌ $\exists_{K \in \mathbb{R}^n} \left((LinComb[O, S, K, V, +, *]) \wedge (K \neq \{0\}^n) \right)$ ▌ $\neg LinInd[S, V, +, *]$

$SingletonNonZeroIndependent := (v \neq O) \implies (LinInd[\langle v \rangle, V, +, *])$

(1) $\left((\langle r \rangle \in \mathbb{R}^1) \wedge (LinComb[O, \langle v \rangle, \langle r \rangle, V, +, *]) \right) \implies \ldots$

(1.1) $(ZeroVectorEquiv) \wedge (r * v = O)$ ▌ $(r * v = O) \iff ((r = 0) \vee (v \neq O))$

(1.2) $v \neq O$ ▌ $r = 0$

(2) $\left((\langle r \rangle \in \mathbb{R}^1) \wedge (LinComb[O, \langle v \rangle, \langle r \rangle, V, +, *]) \right) \implies (r = 0)$ ▌ $\forall_{r \in \mathbb{R}} ((LinComb[O, \langle v \rangle, \langle r \rangle, V, +, *]) \implies (r = 0))$

(3) $LinInd[\langle v \rangle, V, +, *]$

$SubIndependent := \forall_{V, A, B} \left(\begin{matrix} ((VectorSpace[V, +, *]) \wedge (A \subseteq B) \wedge (A \in V^n) \wedge (B \in V^m)) \implies \\ ((LinInd[B, V, +, *]) \implies (LinInd[A, V, +, *])) \end{matrix} \right)$

(1) $\left((K \in \mathbb{R}^n) \wedge (LinComb[O, A, K, V, +, *]) \right) \implies \ldots$

(1.1) $n \le m$ ▮ $L := \langle \left\{ \begin{matrix} k_j & j \le n \\ 0 & j > n \end{matrix} \right. | j \in \mathbb{N}_{1,m} \rangle$ ▮ $L \in \mathbb{R}^m$

(1.2) $A \subseteq B$ ▮ $\forall_{j \in \mathbb{N}_{1,n}}(a_j = b_j)$ ▮ $\sum_{i=1}^{n}(k_i * a_i)) = \sum_{j=1}^{m}(l_j * b_j))$

(1.3) $LinComb[O, A, K, V, +, *]$ ▮ $O = \sum_{i=1}^{n}(k_i * a_i) = \sum_{j=1}^{m}(l_j * b_j))$ ▮ $LinComb[O, B, L, V, +, *]$

(1.4) $(LinInd[B, V, +, *]) \wedge (LinComb[O, B, L, V, +, *])$ ▮ $L = \{0\}^m$ ▮ $K = \{0\}^n$

(2) $\big((K \in \mathbb{R}^n) \wedge (LinComb[O, A, K, V, +, *])\big) \implies (K = \{0\}^n)$ ▮ $LinInd[A, V, +, *]$

$SuperDependent := \forall_{V,A,B}\Big(\big((VectorSpace[V, +, *]) \wedge (A \subseteq B \subseteq V)\big) \implies \big((\neg LinInd[A, V, +, *]) \implies (\neg LinInd[B, V, +, *])\big)\Big)$

(1) $\neg LinInd[A, V, +, *]$ ▮ $\exists_K\big((LinComb[O, A, K, V, +, *]) \wedge (K \ne \{0\}^n)\big)$

(2) $n \le m$ ▮ $L := \langle \left\{ \begin{matrix} k_j & j \le n \\ 0 & j > n \end{matrix} \right. | j \in \mathbb{N}_{1,m} \rangle$ ▮ $L \in \mathbb{R}^m$

(3) $A \subseteq B$ ▮ $\forall_{j \in \mathbb{N}_{1,n}}(a_j = b_j)$ ▮ $\sum_{i=1}^{n}(k_i * a_i)) = \sum_{j=1}^{m}(l_j * b_j))$

(4) $LinComb[O, A, K, V, +, *]$ ▮ $LinComb[O, B, L, V, +, *]$

(5) $K \ne \{0\}^n$ ▮ $L \ne \{0\}^m$

(6) $\exists_L\big((LinComb[O, B, L, V, +, *]) \wedge (L \ne \{0\}^m)\big)$ ▮ $\neg LinInd[B, V, +, *]$

$LinDepProp := \forall_{S,V}\left((\neg LinInd[S, V, +, *]) \implies \left(\exists_{s_j \in S}\exists_{K \in \mathbb{R}^{n-1}}(LinComb[s_j, S \setminus \{s_j\}, K, V, +, *])\right)\right)$

(1) $\neg LinInd[S, V, +, *]$ ▮ $\exists_{K \in \mathbb{R}^n}\big((LinComb[O, S, K, V, +, *]) \wedge (K \ne \{0\}^n)\big)$

(2) $K \ne \{0\}^n$ ▮ $\exists_{j \in \mathbb{N}_{1,n}}\left((k_j \ne 0) \wedge \left(\forall_{i \in \mathbb{N}_{j+1,n}}(k_i = 0)\right)\right)$ $\ldots$

(3) $\ldots \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{j}(k_i * s_i) = \sum_{i=1}^{j-1}(k_i * s_i) + k_j * s_j$ ▮ $\sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{j-1}(k_i * s_i) + k_j * s_j$

(4) $(LinComb[O, S, K, V, +, *]) \wedge \left(\sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{j-1}(k_i * s_i) + k_j * s_j\right)$ ▮ $O = \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{j-1}(k_i * s_i) + k_j * s_j$

(5) $s_j = (-1/k_j)\sum_{i=1}^{j-1}(k_i * s_i) = \sum_{i=1}^{j-1}\left((-k_i/k_j) * s_i\right)$ ▮ $s_j = \sum_{i=1}^{j-1}\left((-k_i/k_j) * s_i\right)$

(6) $\exists_{K \in \mathbb{R}^{n-1}}(LinComb[s_j, S \setminus \{s_j\}, K, V, +, *])$

$LinDepPropCorollary := \forall_{P,S,V}\left(\big((\neg LinInd[S, V, +, *]) \wedge (LinSpan[P, S, V, +, *])\big) \implies \left(\exists_{s_j \in S}(LinSpan[P, S \setminus \{s_j\}, V, +, *])\right)\right)$

(1) $LinDepProp$ ▮ $\exists_{s_j \in S}\exists_{K \in \mathbb{R}^{n-1}}(LinComb[s_j, S \setminus \{s_j\}, K, V, +, *])$

(2) $\forall_{u \in P}\left(\left(\exists_{K_1}(LinComb[u, S, K_1, V, +, *])\right) \implies \left(\exists_{K_2}(LinComb[u, S \setminus \{s_j\}, K_2, V, +, *])\right)\right)$ ▮ $LinSpan[P, S \setminus \{s_j\}, V, +, *]$

$LinIndEquiv := \forall_{S,V}\left((LinInd[S, V, +, *]) \iff \left(\forall_{s_j \in S}\forall_{K \in \mathbb{R}^{n-1}}(\neg LinComb[s_j, S \setminus \{s_j\}, K, V, +, *])\right)\right)$

(1) $LinDepProp$ ▮ $(\neg LinInd[S, V, +, *]) \implies \left(\exists_{s_j \in S}\exists_{K \in \mathbb{R}^{n-1}}(LinComb[s_j, S \setminus \{s_j\}, K, V, +, *])\right)$ $\ldots$

(2) $\ldots \left(\forall_{s_j \in S}\forall_{K \in \mathbb{R}^{n-1}}(\neg LinComb[s_j, S \setminus \{s_j\}, K, V, +, *])\right) \implies (LinInd[S, V, +, *])$

(3) $\left(\exists_{s_j \in S}\exists_{K \in \mathbb{R}^{n-1}}(LinComb[s_j, S \setminus \{s_j\}, K, V, +, *])\right) \implies \ldots$

(3.1) $L := \langle \left\{ \begin{matrix} k_i & i \ne j \\ -1 & i = j \end{matrix} \right. | i \in \mathbb{N}_{1,n} \rangle$ ▮ $(L \in \mathbb{R}^n) \wedge (L \ne \{0\}^n)$

(3.2) $LinComb[s_j, S \setminus \{s_j\}, K, V, +, *]$ ▮ $\ldots$ ▮ $\sum_{i=1}^{j-1}(k_i * s_i) + k_j * s_j = \sum_{i=1}^{j-1}(k_i * s_i) + -\sum_{i=1}^{j-1}(k_i * s_i) = O$ $\ldots$

(3.3) $\ldots LinComb[O, S, L, V, +, *]$

(3.4) $(LinComb[O, S, L, V, +, *]) \wedge (L \ne \{0\}^n)$ ▮ $\exists_{L \in \mathbb{R}^n}\big((LinComb[O, S, L, V, +, *]) \wedge (L \ne \{0\}^n)\big)$ ▮ $(\neg LinInd[S, V, +, *])$

(4) $\left(\exists_{s_j \in S}\exists_{K \in \mathbb{R}^{n-1}}(LinComb[s_j, S \setminus \{s_j\}, K, V, +, *])\right) \implies (\neg LinInd[S, V, +, *])$

(5)	$(LinInd[S,V,+,*]) \implies \left(\forall_{s_j \in S} \forall_{K \in \mathbb{R}^{n-1}} (\neg LinComb[s_j, S \setminus \{s_j\}, K, V, +, *]) \right)$
(6)	$(LinInd[S,V,+,*]) \iff \left(\forall_{s_j \in S} \forall_{K \in \mathbb{R}^{n-1}} (\neg LinComb[s_j, S \setminus \{s_j\}, K, V, +, *]) \right)$

$LinIndSuperspace := \forall_{U,V} \left((Subspace[U,V]) \implies \left(\forall_W ((LinInd[W,U,+,*]) \implies (LinInd[W,V,+,*])) \right) \right)$

(1)	$(\neg LinInd[W,V,+,*]) \implies \dots$
(1.1)	$\exists_{j \in W}(LinComb[j, W \setminus \{j\}, +, *])$ ▌ $\neg LinInd[W,U,+,*]$
(1.2)	$(LinInd[W,U,+,*]) \land (\neg LinInd[W,U,+,*])$ ▌ $\bot$
(2)	$(\neg LinInd[W,V,+,*]) \implies \bot$ ▌ $LinInd[W,V,+,*]$

3.6   Bases and Dimensions

$Basis[S,V,+,*] := (Spans[S,V,+,*]) \land (LinInd[S,V,+,*])$

$BasisEquiv := \forall_{S,V} \left((Basis[S,V,+,*]) \iff (\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*])) \right)$

(1)	$(Basis[S,V,+,*]) \implies \dots$
(1.1)	$(v \in V) \implies \dots$
(1.1.1)	$Basis[S,V,+,*]$ ▌ $Spans[V,S,+,*]$ ▌ $\exists_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*])$
(1.1.2)	$\left((K_1, K_2 \in \mathbb{R}^n) \land (LinComb[v,S,K_1,V,+,*]) \land (LinComb[v,S,K_2,V,+,*]) \right) \implies \dots$
(1.1.2.1)	$\left(v = \sum(k_{1_i} * s_i) \right) \land \left(v = \sum(k_{2_i} * s_i) \right)$
(1.1.2.2)	$O = v - v = \sum(k_{1_i} * s_i) - \sum(k_{2_i} * s_i) = \sum\left((k_{1_i} - k_{2_i}) * s_i \right)$
(1.1.2.3)	$L := \langle k_{1_i} - k_{2_i} \mid i \in \mathbb{N}_{i=1}^n \rangle \in \mathbb{R}^n$
(1.1.2.4)	$(LinInd[S,V,+,*]) \land (LinComb[O,S,L,V,+,*])$ ▌ $L = \{0\}^n$ ▌ $K_2 = K_1$
(1.1.3)	$\left((K_1, K_2 \in \mathbb{R}^n) \land (LinComb[v,S,K_1,V,+,*]) \land (LinComb[v,S,K_2,V,+,*]) \right) \implies (K_1 = K_2)$
(1.1.4)	$\forall_{K_1,K_2 \in \mathbb{R}^n} \left((LinComb[v,S,K_1,V,+,*]) \land (LinComb[v,S,K_2,V,+,*]) \implies (K_1 = K_2) \right)$
(1.1.5)	$\exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*])$
(1.2)	$(v \in V) \implies \left(\exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*]) \right)$
(2)	$(Basis[S,V,+,*]) \implies \left(\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*]) \right)$
(3)	$\left(\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*]) \right) \implies \dots$
(3.1)	$\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*])$ ▌ $\forall_{v \in V} \exists_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*])$ ▌ $Spans[S,V,+,*]$
(3.2)	$O \in V$ ▌ $\exists!_{K \in \mathbb{R}^n}(LinComb[O,S,K,V,+,*])$
(3.3)	$(K \neq \{0\}^n) \implies \left(\neg \exists!_{K \in \mathbb{R}^n}(LinComb[O,S,K,V,+,*]) \right) \implies \bot$ ▌ $K = \{0\}^n$
(3.4)	$\left(\exists!_{K \in \mathbb{R}^n}(LinComb[O,S,K,V,+,*]) \right) \land (K = \{0\}^n)$ ▌ $LinInd[S,V,+,*]$
(3.5)	$(Spans[S,V,+,*]) \land (LinInd[S,V,+,*])$ ▌ $Basis[S,V,+,*]$
(4)	$\left(\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v,S,K,V,+,*]) \right) \implies (Basis[S,V,+,*])$

$SpanReduceBasis := \forall_{S,V} \left((Spans[S,V,+,*]) \implies \left(\exists_B ((B \subseteq S) \land (Basis[B,V,+,*])) \right) \right)$

(1)	$LinDepPropCorollary$ ▌ $\exists_B ((B \subseteq S) \land (LinInd[B,V,+,*]) \land (Spans[B,V,+,*]))$ ▌ $\exists_B ((B \subseteq S) \land (Basis[B,V,+,*]))$
(2)	TODO - formalize removing latter entries first

$FinDimBasis := \forall_V \left((FinDim[V,+,*]) \implies \left(\exists_B (Basis[B,V,+,*]) \right) \right)$

(1)	$FinDim[V,+,*]$ ▌ $\exists_{S \in V^n}(Spans[S,V,+,*])$
(2)	$(SpanReduceBasis) \land (Spans[S,V,+,*])$ ▌ $\exists_B(Basis[B,V,+,*])$

$LinIndExpandBasis := \forall_{L,V} \left((LinInd[L,V,+,*]) \implies \left(\exists_B ((L \subseteq B) \land (Basis[B,V,+,*])) \right) \right)$

(1)	$FinDimBasis \ \vdash \ \exists_C(Basis[C, V, +, *])$
(2)	$S := L \cup C$
(3)	$Basis[C, V, +, *] \ \vdash \ Spans[C, V, +, *] \ \vdash \ Spans[S, V, +, *]$
(4)	$SpanReduceBasis \ \vdash \ \Big(\exists_B\big((B \subseteq S) \wedge (Basis[B, V, +, *])\big)\Big) \wedge (L \subseteq B)$

$$SpanLinIndLength := \forall_{S,T,V}\Big(\big((Span[S, V, +, *]) \wedge (LinInd[T, V, +, *])\big) \implies (|T| \le |S|)\Big)$$

(1)	$\big((Span[S, V, +, *]) \wedge (|T| > |S|)\big) \implies \ldots$
(1.1)	$Span[S, V, +, *] \ \vdash \ \forall_{i \in \mathbb{N}_{1,|H|}} \exists_{K_i \mathbb{R}^{|S|}}(LinComb[t_i, S, K_i V, +, *])$
(1.2)	$|H| > |S| \ \vdash \ \exists_{L \in \mathbb{R}^{|H|-1}}(LinComb[t_{|H|}, T \setminus \{t_{|H|}\}, L, V, +, *])$
(1.3)	$L = -1 * K \ \vdash \ \big(\sum(K + L) = O\big) \wedge (K + L \ne \{0\}^{|T|}) \ \vdash \ \neg LinInd[T, V, +, *]$
(1.4)	TODO - tidy up
(2)	$\big((Span[S, V, +, *]) \wedge (|T| > |S|)\big) \implies (\neg LinInd[T, V, +, *]) \ \vdash \ \big((Span[S, V, +, *]) \wedge (LinInd[T, V, +, *])\big) \implies (|T| \le |S|)$

$$BasisLength := \forall_{S,T,V}\Big(\big((Basis[S, V, +, *]) \wedge (Basis[T, V, +, *])\big) \implies (|T| = |S|)\Big)$$

(1)	$(Span[T, V, +, *]) \wedge (LinInd[S, V, +, *]) \ \vdash \ |S| \le |T|$
(2)	$(Span[S, V, +, *]) \wedge (LinInd[T, V, +, *]) \ \vdash \ |T| \le |S|$
(3)	$(|S| \le |T|) \wedge (|T| \le |S|) \ \vdash \ |T| = |S|$

$$Dim[d, V, +, *] := \big((V = \{O\}) \implies (d = 0)\big) \wedge \Big((V \ne \{O\}) \implies \big((\exists_B(Basis[B, V, +, *])) \wedge (d = |B|)\big)\Big)$$

$$LinIndLengthDim := \forall_{U,V}\Big(\big((LinInd[U, V, +, *]) \wedge (Dim[|U|, V, +, *])\big) \implies (Basis[U, V, +, *])\Big)$$

(1)	$(LinIndExpandBasis) \wedge (LinInd[U, V, +, *]) \ \vdash \ \exists_B\big((U \subseteq B) \wedge (Basis[B, V, +, *])\big)$
(2)	$(BasisLength) \wedge (Dim[|U|, V, +, *]) \wedge (Basis[B, V, +, *]) \ \vdash \ |B| = |U| \ \vdash \ B = U \ \vdash \ Basis[U, V, +, *]$

$$SpanLengthDim := \forall_{U,V}\Big(\big((Spans[U, V, +, *]) \wedge (Dim[|U|, V, +, *])\big) \implies (Basis[U, V, +, *])\Big)$$

(1)	$(SpanReduceBasis) \wedge (Spans[U, V, +, *]) \ \vdash \ \exists_B\big((B \subseteq U) \wedge (Basis[B, V, +, *])\big)$
(2)	$(BasisLength) \wedge (Dim[|U|, V, +, *]) \wedge (Basis[B, V, +, *]) \ \vdash \ |B| = |U| \ \vdash \ B = U \ \vdash \ Basis[U, V, +, *]$

$$LinDepLengthDim := \forall_{U,V}\Big(\big((U \subseteq V) \wedge (|U| > Dim[V])\big) \implies (\neg LinInd[U, V, +, *])\Big)$$

(1)	Contrapositive of $BasisLinearIndCard$
(2)	TODO - cleanup

$$NonSpanLengthDim := \forall_{U,V}\Big(\big((U \subseteq V) \wedge (|U| < Dim[V])\big) \implies (\neg Spans[U, V, +, *])\Big)$$

(1)	Suppose $Spans[U, V, +, *]$, $B = SpanReduceBasis[U]$ to form a basis, $(|B| \le |U| < Dim[V]) \wedge |B| = Dim[V] \ \vdash \ \bot$
(2)	$\neg Spans[U, V, +, *]$
(3)	TODO - cleanup

3.7 Rank

$Nullity[n, A] := (NullSpace[N, A]) \wedge (Dim[n, N, +, *])$
$Rank[r, A, m, n] := (Matrix[A, m, n]) \wedge (RowSpace[R, A, m, n]) \wedge (Dim[r, R, A, +, *])$

$$RowRankEqColRank := \forall_A(TODO)$$

(1)	TODO

$$RankNullity := \forall_A\big((Matrix[A, m, n]) \implies (Rank[A] + Nullity[A] = n)\big)$$

(1) TODO

$RankInv := \forall_A\Big((Matrix[A, m, n]) \implies ((Rank[A] = n) \iff (Inv[A]))\Big)$

(1) TODO

$RankNonTrivialSol := \Big(\exists_X\big((A * X = O) \wedge (X \neq O)\big)\Big) \iff (Rank[A] < n)$

(1) TODO

$RankUniqueSol := \big(\forall_{B \in \mathcal{M}}\exists!_{X \in \mathcal{M}}(Sol[X, A, B])\big) \iff (Rank[A] = n)$

(1) TODO

$$SquareTheorems_8 := \forall_{A \in \mathcal{M}}\begin{cases} (Invertible[A]) & \iff \\ (RowEquiv[A, I_n]) & \iff \\ \Big(\forall_X\big((X = O) \iff (Sol[X, A, O])\big)\Big) & \iff \\ (\forall_{B \in \mathcal{M}}\exists!_{X \in \mathcal{M}}(Sol[X, A, B])) & \iff \\ (Rank[A] = n) & \iff \\ (Nullity[A] = 0) & \iff \\ \big(\text{The rows form a linearly independent set of vectors (to get full rank)}\big) & \iff \\ \big(\text{The columns form a linearly independent set of vectors (to get full rank)}\big) & \iff \end{cases}$$

3.8 Linear Transformations

$LinTrans[L, V, +_v, *_v, W, +_w, *_w] := \left(\begin{array}{l} (Function[f, V, W]) \wedge (VectorSpace[V, +_v, *_v]) \wedge (VectorSpace[W, +_w, *_w]) \wedge \\ \Big(\forall_{\alpha, \beta \in V}\big(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta)\big)\Big) \wedge \Big(\forall_{r \in \mathbb{R}}\forall_{\alpha \in V}\big(L(r *_v \alpha) = r *_w L(\alpha)\big)\Big) \end{array}\right)$

$LinOp[L, V, +_v, *_v] := LinTrans[L, V, +_v, *_v, V, +_v, *_v]$

$\mathcal{L}[V, W] := \{L \mid LinTrans[L, V, +_v, *_v, W, +_w, *_w]\}$

$ZeroMapsToZero := \forall_{L,V,W}\Big((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies \big(L(O_v) = O_w\big)\Big)$

(1) $L(O_v) = L(O_v +_v O_v) = L(O_v) +_w L(O_v)$

(2) $O_w = L(O_v) - L(O_v) = L(O_v)$

$SplitAddInv := \forall_{L,V,W}\Big((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \implies \big(\forall_{\alpha, \beta \in V}\big(L(\alpha -_v \beta) = L(\alpha) -_w L(\beta)\big)\big)\Big)$

(1) $L(\alpha - \beta) = L\big(\alpha + (-\beta)\big) = L(\alpha) + L(-\beta) = L(\alpha) + (-1) * L(\beta) = L(\alpha) - L(\beta)$

$UniqBasisLT := \forall_{V,W}\left(\begin{array}{l} ((VectorSpace[V, +_v, *_v]) \wedge (VectorSpace[W, +_w, *_w]) \wedge (Basis[A, V, +_v, *_v]) \wedge (Basis[B, W, +_w, *_w])) \implies \\ \Big(\exists!_T\big((LinTrans[T, V, +_v, *_v, W, +_w, *_w]) \wedge \big(\forall_{i \in \mathbb{N}_{1,n}}\big(T(a_i) = b_i\big)\big)\big)\Big) \end{array}\right)$

(1) $T\big(\sum_{i=1}^n(k_i * a_i)\big) := \sum_{i=1}^n(k_i * b_i)$

(2) $(i \in \mathbb{N}_{1,n}) \implies \ldots$

   (2.1) $L := \Big\langle\Big\{\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}\Big| j \in \mathbb{N}_{1,n}\Big\rangle$ ▮ $L \in \mathbb{R}^n$

   (2.2) $T(a_i) = T\big(\sum_{i=1}^n(l_i * a_i)\big) = \sum_{i=1}^n(l_i * b_i) = b_i$ ▮ $T(a_i) = b_i$

(3) $(i \in \mathbb{N}_{1,n}) \implies \big(T(a_i) = b_i\big)$ ▮ $\forall_{i \in \mathbb{N}_{1,n}}\big(T(a_i) = b_i\big)$

(4) $(BasisEquiv) \wedge (Basis[A, V, +_v, *_v])$ ▮ $\forall_{v \in V}\exists!_{K \in \mathbb{R}^n}(LinComb[v, A, K, V, +, *])\ \ldots$

(5) $\ldots \forall_{v_1, v_2 \in V}\Big((v_1 = v_2) \implies \big(T(v_1) = T(v_2)\big)\Big)$ ▮ $Function[T, V, W]$

(6) $(\alpha, \beta \in V) \implies \ldots$

   (6.1) $\Big(\exists_{K_\alpha}(LinComb[\alpha, A, K_\alpha, V, +_v, *_v])\Big) \wedge \Big(\exists_{K_\beta}(LinComb[\beta, A, K_\beta, V, +_v, *_v])\Big)$ ▮ $\big(\alpha = \sum_{i=1}^n(k_{\alpha i} * a_i)\big) \wedge \big(\beta = \sum_{i=1}^n(k_{\beta i} * a_i)\big)$

(6.2) $T(\alpha + \beta) = T\left(\sum_{i=1}^{n}(k_{\alpha i} * a_i) + \sum_{i=1}^{n}(k_{\beta i} * a_i)\right) = T\left(\sum_{i=1}^{n}\left((k_{\alpha i} + k_{\beta i}) * a_i\right)\right)) = \sum_{i=1}^{n}\left((k_{\alpha i} + k_{\beta i}) * b_i\right) = \ldots$

(6.3) $\ldots \sum_{i=1}^{n}(k_{\alpha i} * b_i) + \sum_{i=1}^{n}(k_{\beta i} * b_i) = T\left(\sum_{i=1}^{n}(k_{\alpha i} * a_i)\right) + T\left(\sum_{i=1}^{n}(k_{\beta i} * a_i)\right) = T(\alpha) + T(\beta)$

(7) $(\alpha, \beta \in V) \implies \left(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta)\right)$ ▮ $\forall_{\alpha,\beta \in V}\left(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta)\right)$

(8) $\left((r \in \mathbb{R}) \wedge (\alpha \in V)\right) \implies \ldots$

(8.1) $\exists_K(LinComb[\alpha, A, K, V, +_v, *_v])$ ▮ $\alpha = \sum_{i=1}^{n}(k_i * a_i)$

(8.2) $L(r *_v \alpha) = L\left(r *_v \sum_{i=1}^{n}(k_i *_v a_i)\right) = L\left(\sum_{i=1}^{n}\left((rk_i) *_v a_i\right)\right) = \ldots$

(8.3) $\ldots \sum_{i=1}^{n}\left((rk_i) *_w b_i\right) = r *_w \sum_{i=1}^{n}(k_i *_w b_i) = r *_w L\left(\sum_{i=1}^{n}(k_i *_v a_i)\right) = r *_w L(\alpha)$

(9) $\left((r \in \mathbb{R}) \wedge (\alpha \in V)\right) \implies \left(L(r *_v \alpha) = r *_w L(\alpha)\right)$ ▮ $\forall_{r \in \mathbb{R}}\forall_{\alpha \in V}\left(L(r *_v \alpha) = r *_w L(\alpha)\right)$

(10) $\left(\forall_{i \in \mathbb{N}_{1,n}}\left(T(a_i) = b_i\right)\right) \wedge (Function[T, V, W]) \wedge \left(\forall_{\alpha,\beta \in V}\left(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta)\right)\right) \wedge \left(\forall_{r \in \mathbb{R}}\forall_{\alpha \in V}\left(L(r *_v \alpha) = r *_w L(\alpha)\right)\right) \wedge \ldots$

(11) $\ldots (VectorSpace[V, +_v, *_v]) \wedge (VectorSpace[W, +_w, *_w])$ ▮ $\left(\forall_{i \in \mathbb{N}_{1,n}}\left(T(a_i) = b_i\right)\right) \wedge (LinTrans[T, V, +_v, *_v, W, +_w, *_w])$

(12) $\left(\left(\forall_{i \in \mathbb{N}_{1,n}}\left(T_2(a_i) = b_i\right)\right) \wedge (LinTrans[T_2, V, +_v, *_v, W, +_w, *_w])\right) \implies \ldots$

(12.1) $\forall_{i \in \mathbb{N}_{1,n}}\left(T_2(a_i) = b_i\right)$ ▮ $\forall_{i \in \mathbb{N}_{1,n}}\left(T_2(c_i * a_i) = c_i * b_i\right)$ ▮ $T_2\left(\sum_{i=1}^{n}(c_i * a_i)\right) = \sum_{i=1}^{n}(c_i * b_i)$ ▮ $T_2 = T$

(13) $\left(\left(\forall_{i \in \mathbb{N}_{1,n}}\left(T_2(a_i) = b_i\right)\right) \wedge (LinTrans[T_2, V, +_v, *_v, W, +_w, *_w])\right) \implies (T_2 = T)$

$+_{\mathcal{L}}[S + T, S, T] := (S + T)(v) = S(v) + T(v)$
$*_{\mathcal{L}}[r * T, r, T] := (r * T)(v) = r * \left(T(v)\right)$
$LTVectorSpace := \forall_{V,W}(VectorSpace[\mathcal{L}[V, W], +_{\mathcal{L}}, *_{\mathcal{L}}])$

(1) TODO

$*_{\mathcal{L}}[S * T, S, T] := (S * T)(v) = S\left(T(v)\right)$
$LTProdProperties := (associativity) \wedge (identity) \wedge (distributive)$

(1) TODO

$Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge \left(ker_L = \{\alpha \in V | L(\alpha) = O_w\}\right)$

$KerSubspace := \forall_{L,V,W}\left((Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w]) \implies (Subspace[ker_L, V, +_v, *_v])\right)$

(1) $ZeroMapsToZero$ ▮ $L(O_v) = O_w$ ▮ $O_v \in ker_L$ ▮ $\emptyset \neq ker_L$ ▮ $\emptyset \neq ker_L \subseteq V$

(2) $(\alpha, \beta \in ker_L) \implies \ldots$

(2.1) $\left(L(\alpha) = O_w\right) \wedge \left(L(\beta) = O_w\right)$

(2.2) $L(\alpha + \beta) = L(\alpha) + L(\beta) = O_w + O_w = O_w$ ▮ $L(\alpha + \beta) \in ker_L$

(3) $(\alpha, \beta \in ker_L) \implies (\alpha + \beta \in ker_L)$ ▮ $\forall_{\alpha,\beta \in ker_L}(\alpha + \beta \in ker_L)$

(4) $\left((r \in \mathbb{R}) \wedge (\alpha \in ker_L)\right) \implies \ldots$

(4.1) $L(\alpha) = O_w$ ▮ $L(r * \alpha) = r * L(\alpha) = r * O_w = O_w$ ▮ $r * \alpha \in ker_L$

(5) $\left((r \in \mathbb{R}) \wedge (\alpha \in ker_L)\right) \implies (r * \alpha \in ker_L)$ ▮ $\forall_{r \in \mathbb{R}}\forall_{\alpha \in ker_L}(r * \alpha \in ker_L)$

(6) $(SubspaceEquiv) \wedge (\emptyset \neq ker_L \subseteq V) \wedge \left(\forall_{\alpha,\beta \in ker_L}(\alpha + \beta \in ker_L)\right) \wedge \left(\forall_{r \in \mathbb{R}}\forall_{\alpha \in ker_L}(r * \alpha \in ker_L)\right)$ ▮ $Subspace[ker_L, V, +_v, *_v]$

$Rng[rng_L, L, V, +_v, *_v, W, +_w, *_w] := (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge \left(rng_L = \{\beta \in W | \exists_{\alpha \in V}\left(\beta = L(\alpha)\right)\}\right)$

$RangeSubspace := \forall_{L,V,W}\left((Ran[rng_L, L, V, +_v, *_v, W, +_w, *_w]) \implies (Subspace[rng_L, W, +_w, *_w])\right)$

(1) $ZeroMapsToZero$ ▮ $O_w = L(O_v)$ ▮ $\exists_{\alpha \in V}\left(O_w = L(\alpha)\right)$ ▮ $O_w \in rng_L$ ▮ $\emptyset \neq rng_L$ ▮ $\emptyset \neq rng_L \subseteq W$

(2) $(\alpha, \beta \in rng_L) \implies \ldots$

(2.1) $\left(\exists_{u \in V}\big(\alpha = L(u)\big)\right) \wedge \left(\exists_{v \in V}\big(\beta = L(v)\big)\right)$

(2.2) $\alpha + \beta = L(u) + L(v) = L(u+v)$ ■ $\exists_{w \in V}\big(\alpha + \beta = L(w)\big)$ ■ $\alpha + \beta \in rng_L$

(3) $(\alpha, \beta \in rng_L) \implies (\alpha + \beta \in rng_L)$ ■ $\forall_{\alpha, \beta \in rng_L}(\alpha + \beta \in rng_L)$

(4) $\big((r \in \mathbb{R}) \wedge (\alpha \in rng_L)\big) \implies \ldots$

(4.1) $\exists_{v \in V}\big(\alpha = L(v)\big)$ ■ $L(r * v) = r * L(v) = r * \alpha$ ■ $\exists_{w \in V}\big(r * \alpha = L(w)\big)$ ■ $r * \alpha \in rng_L$

(5) $\big((r \in \mathbb{R}) \wedge (\alpha \in rng_L)\big) \implies (r * \alpha \in rng_L)$ ■ $\forall_{r \in \mathbb{R}}\forall_{\alpha \in rng_L}(r * \alpha \in rng_L)$

(6) $(SubspaceEquiv) \wedge (\emptyset \neq rng_L \subseteq W) \wedge \left(\forall_{\alpha, \beta \in rng_L}(\alpha + \beta \in rng_L)\right) \wedge \left(\forall_{r \in \mathbb{R}}\forall_{\alpha \in rng_L}(r * \alpha \in rng_L)\right)$ ■ $Subspace[rng_L, W, +_w, *_w]$

$KerInjective := \forall_{L,V,W}\Big((Ker[ker_L, L, V, +_v, *_v, W, +_w, *_w]) \implies \big((Injective[L,V,W]) \iff (ker_L = \{O_v\})\big)\Big)$

(1) $(Injective[L,V,W]) \implies \ldots$

(1.1) $ZeroMapsToZero$ ■ $L(O_v) = O_w$

(1.2) $O_v \in ker_L$ ■ $\{O_v\} \subseteq ker_L$

(1.3) $(v \in ker_L) \implies \ldots$

(1.3.1) $L(v) = O_w$

(1.3.2) $(Injective[L,V,W]) \wedge \big(L(O_v) = O_w\big)$ ■ $O_v = v$

(1.4) $(v \in ker_L) \implies (v = O_v)$ ■ $ker_L \subseteq \{O_v\}$

(1.5) $(\{O_v\} \subseteq ker_L) \wedge (ker_L \subseteq \{O_v\})$ ■ $ker_L = \{O_v\}$

(2) $(Injective[L,V,W]) \implies (ker_L = \{O_v\})$

(3) $(ker_L = \{O_v\}) \implies \ldots$

(3.1) $\left((u,v \in V) \wedge \big(L(u) = L(v)\big)\right) \implies \ldots$

(3.1.1) $O_w = L(u) - L(v) = L(u-v)$ ■ $u - v \in ker_L$

(3.1.2) $ker_L = \{O_v\}$ ■ $u - v = O_v$ ■ $u = v$

(3.2) $\left((u,v \in V) \wedge \big(L(u) = L(v)\big)\right) \implies (u = v)$ ■ $\forall_{u,v \in V}\left(\big(L(u) = L(v)\big) \implies (u = v)\right)$ ■ $Injective[L,V,W]$

(4) $(ker_L = \{O_v\}) \implies (Injective[L,V,W])$

(5) $(Injective[L,V,W]) \iff (ker_L = \{O_v\})$

$RankNullityLT := \forall_{L,V,W}\big((LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \implies (Dim[V] = Dim[ker_L] + Dim[rng_L])\big)$

(1) $KerSubspace$ ■ $\big(\exists_U(Basis[U, ker_L, +_v, *_v])\big) \wedge (Dim[ker_L] = |U|)$

(2) $(LinIndSuperspace) \wedge (LinInd[U, ker_L, +_v, *_v])$ ■ $LinInd[U, V, +_v, *_v]$

(3) $(LinIndExpandBasis) \wedge (LinInd[U, V, +_v, *_v])$ ■ $\left(\exists_B\big((U \subseteq B) \wedge (Basis[B, V, +_v, *_v])\big)\right) \wedge (Dim[V] = |B|)$

(4) $U \subseteq B$ ■ $\exists_T(B = U \cup T)$

(5) $(w \in rng_L) \implies \ldots$

(5.1) $\exists_{v \in V}\big(w = L(v)\big)$

(5.2) $(Basis[B, V, +_v, *_v]) \wedge (B = U \cup T)$ ■ $\exists_{K \in \mathbb{R}^{|B|}}\left(v = \sum_{i=1}^{|B|}(k_i * b_i) = \sum_{i=1}^{|U|}(k_i * u_i) + \sum_{i=1}^{|T|}(k_{|U|+i} * t_i)\right)$

(5.3) $w = L(v) = L\left(\sum_{i=1}^{|U|}(k_i * u_i) + \sum_{i=1}^{|T|}(k_{|U|+i} * t_i)\right) = L\left(\sum_{i=1}^{|U|}(k_i * u_i)\right) + L\left(\sum_{i=1}^{|T|}(k_{|U|+i} * t_i)\right) = \ldots$

(5.4) $O + L\left(\sum_{i=1}^{|T|}(k_{|U|+i} * t_i)\right) = \sum_{i=1}^{|T|}\left(L(k_{|U|+i} * t_i)\right) = \sum_{i=1}^{|T|}\left(k_{|U|+i} * L(t_i)\right)$ ■ $\exists_K\big(LinComb[w, L(T), K, W, +, *]\big)$

(6) $(w \in rng_L) \implies \left(\exists_L\big(LinComb[w, L(T), L, W, +, *]\big)\right)$ ■ $Spans[L(T), rng_L, W, +, *]$

(7) $\left((K \in \mathbb{R}^n) \wedge \big(LinComb[O_w, L(T), K, W, +_w, *_w]\big)\right) \implies \ldots$

(7.1) $O_w = \sum_{i=1}^n\big(k_i * L(t_i)\big) = L\big(\sum_{i=1}^n(k_i * t_i)\big)$ ■ $\sum_{i=1}^n(k_i * t_i) \in ker_L$

(7.2) $(Basis[U, ker_L, +_v, *_v]) \wedge \big(\sum_{i=1}^n(k_i * t_i) \in ker_L\big)$ ■ $\exists_{D \in \mathbb{R}^m}\big(\sum_{i=1}^n(k_i * t_i) = \sum_{i=1}^m(d_i * u_i)\big)$

(7.3) $Basis[B]$ ■ $LinInd[B]$ ■ $LinInd[U \cup T]$ ■ $\forall_{s_j \in U \cup T}\forall_{K \in \mathbb{R}^{n-1}}(\neg LinComb[s_j, U \cup T \setminus \{s_j\}, K, V, +, *])$

(7.4) $\big(\sum_{i=1}^n(k_i * t_i) = \sum_{i=1}^m(d_i * u_i)\big) \wedge \left(\forall_{s_j \in U \cup T}\forall_{K \in \mathbb{R}^{n-1}}(\neg LinComb[s_j, U \cup T \setminus \{s_j\}, K, V, +, *])\right)$ ■ $(D = \{O\}) \wedge (K = \{O\})$

(8) $\Big((K \in \mathbb{R}^n) \wedge \big(LinComb[O_w, L(T), K, W, +_w, *_w]\big)\Big) \implies (K = \{O\})$ ▐ $LinInd[L(T), W, +_w, *_w]$

(9) $(SubIndependent) \wedge \big(LinInd[L(T), W, +_w, *_w]\big)$ ▐ $LinInd[L(T), rng_L, +_w, *_w]$

(10) $\big(Spans[L(T), rng_L, W, +, *]\big) \wedge \big(LinInd[L(T), rng_L, +_w, *_w]\big)$ ▐ $Basis[L(T), rng_L, +_w, *_w]$ ▐ $Dim[rng_L] = |L(T)| = |T|$

(11) $B = U \cup T$ ▐ $|B| = |U| + |T|$ ▐ $Dim[V] = Dim[ker_L] + Dim[rng_L]$

$InjectiveSurjectiveEqualDim := \forall_{T,V,W} \Big(\begin{array}{l} ((LinTrans[T, V, +_v, *_v, W, +_w, *_w]) \wedge (Dim[V] = Dim[W]) \wedge (Injective[T, V, W])) \implies \\ (Surjective[T, V, W]) \end{array} \Big)$

(1) $(KerInjective) \wedge (Injective[T, V, W])$ ▐ $ker_T = \{O\}$ ▐ $Dim[ker_T] = 0$

(2) $(RankNullityLT) \wedge (Dim[ker_T] = 0)$ ▐ $Dim[V] = Dim[ker_T] + Dim[rng_T] = Dim[rng_T]$ ▐ $Dim[V] = Dim[rng_T]$

(3) $(Dim[V] = Dim[W]) \wedge (Dim[V] = Dim[rng_T])$ ▐ $Dim[W] = Dim[rng_T]$

(4) $RangeSubspace$ ▐ $Subspace[rng_T, W, +_w, *_w]$

(5) $(Subspace[rng_T, W, +_w, *_w]) \wedge (Dim[W] = Dim[rng_T])$ ▐ $\exists_B\big((Basis[B, W, +_w, *_w]) \wedge (Basis[B, rng_T, +_w, *_w])\big)$

(6) $(Spans[W] = Spans[rng_T])$ ▐ $W = rng_T$ ▐ $Surjective[T, V, W]$

$SurjectiveInjectiveEqualDim := \forall_{T,V,W} \Big(\begin{array}{l} ((LinTrans[T, V, +_v, *_v, W, +_w, *_w]) \wedge (Dim[V] = Dim[W]) \wedge (Surjective[T, V, W])) \implies \\ (Injective[T, V, W]) \end{array} \Big)$

(1) $RankNullityLT$ ▐ $Dim[V] = Dim[ker_T] + Dim[rng_T]$

(2) $Surjective[T, V, W]$ ▐ $rng_T = W$ ▐ $Dim[rng_T] = Dim[W]$

(3) $(Dim[V] = Dim[W]) \wedge (Dim[V] = Dim[ker_T] + Dim[rng_T]) \wedge (Dim[rng_T] = Dim[W])$ ▐ $Dim[ker_T] + Dim[rng_T] = Dim[rng_T]$ ▐ $Dim[ker_T] = 0$ ▐ $ker_T = \{O\}$

(4) $(KerInjective) \wedge (er_T = \{O\})$ ▐ $Injective[T, V, W]$

$SmallerMapNotInjective := \forall_{T,V,W} \Big(\big((LinTrans[T, V, +_v, *_v, W, +_w, *_w]) \wedge (Dim[V] > Dim[W]) \implies (\neg Injective[T, V, W])\big)\Big)$

(1) $(RankNullityLT) \wedge (Dim[W] \geq Dim[rng_T])$ ▐ $Dim[ker_T] = Dim[V] - Dim[rng_T] \geq Dim[V] - Dim[W] > 0$ ▐ $Dim[ker_T] \neq 0$

(2) $(KerInjective) \wedge (Dim[ker_T] \neq 0)$ ▐ $\neg Injective[T, V, W]$

$LargerMapNotSurjective := \forall_{T,V,W} \Big(\big((LinTrans[T, V, +_v, *_v, W, +_w, *_w]) \wedge (Dim[V] < Dim[W]) \implies (\neg Surjective[T, V, W])\big)\Big)$

(1) $RankNullityLT$ ▐ $Dim[rng_T] = Dim[V] - Dim[ker_T] \leq Dim[V] < Dim[W]$

(2) $Dim[rng_T] < Dim[W]$ ▐ $Dim[rng_T] \neq Dim[W]$ ▐ $\neg Surjective[T, V, W]$

A linear transformation L : V -> W is one-to-one if and only if the image of every linearly independent set of vectors in V is linearly independent set of vectors in W .

(1) TODO

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

(1) TODO

An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

(1) TODO

$LTInv[L^{-1}, L, V, +_v, *_v, W, +_w, *_w] := \Big(\begin{array}{lll} (LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \wedge (LinTrans[L^{-1}, W, +_w, *_w, V, +_v, *_v]) \wedge \\ (L^{-1} \circ L = 1_v) & \wedge & (L \circ L^{-1} = 1_w) \end{array} \Big)$

$LTInvUniq := \forall_{L_1^{-1}, L_2^{-1}} \Big(\big((LTInv[L_1^{-1}, L, V, +_v, *_v, W, +_w, *_w]) \wedge (LTInv[L_2^{-1}, L, V, +_v, *_v, W, +_w, *_w])\big) \implies (L_1^{-1} = L_2^{-1}) \Big)$

(1) $L_1^{-1} = L_1^{-1} \circ 1_w = L_1^{-1} \circ (L \circ L_2^{-1}) = (L_1^{-1} \circ L) \circ L_2^{-1} = 1_v \circ L_2^{-1} = L_2^{-1}$ ▐ $L_1^{-1} = L_2^{-1}$

$LTInvertible[L, V, +_v, *_v, W, +_w, *_w] := \exists_{L^{-1}}(LTInv[L^{-1}, L, V, +_v, *_v, W, +_w, *_w])$

$InvertibleBijectiveEquiv := \forall_L\Big((LTInvertible[L,V,+_v,*_v,W,+_w,*_w]) \iff \big((Injective[L,V,W]) \wedge (Surjective[L,V,W])\big)\Big)$

(1) $(LTInvertible[L,V,+_v,*_v,W,+_w,*_w]) \implies \ldots$

(1.1) $\exists_{L^{-1}}(LTInv[L^{-1},L,V,+_v,*_v,W,+_w,*_w])$

(1.2) $\big(L(u)=L(w)\big) \implies \ldots$

(1.2.1) $u = L^{-1}\big(L(u)\big) = L^{-1}\big(L(v)\big) = v \ \blacksquare \ u = v$

(1.3) $\big(L(u)=L(w)\big) \implies (u=w) \ \blacksquare \ \forall_{u,w}\Big(\big(L(u)=L(w)\big) \implies (u=w)\Big) \ \blacksquare \ Injective[L,V,W]$

(1.4) $(w \in W) \implies \ldots$

(1.4.1) $L^{-1}(w) \in V$

(1.4.2) $L \circ L^{-1} = 1_w \ \blacksquare \ L\Big(L^{-1}(w)=w\Big)$

(1.4.3) $\Big(L^{-1}(w) \in V\Big) \wedge \Big(L\big(L^{-1}(w)=w\big)\Big) \ \blacksquare \ \exists_{v \in V}\Big(w = \big(L(v)\big)\Big)$

(1.5) $(w \in W) \implies \Big(\exists_{v \in V}\big(w = (L(v))\big)\Big) \ \blacksquare \ \forall_{w \in W}\exists_{v \in V}\big(w = L(v)\big) \ \blacksquare \ Surjective[L,V,W]$

(1.6) $(Injective[L,V,W]) \wedge (Surjective[L,V,W])$

(2) $(LTInvertible[L,V,+_v,*_v,W,+_w,*_w]) \implies \big((Injective[L,V,W]) \wedge (Surjective[L,V,W])\big)$

(3) $\big((Injective[L,V,W]) \wedge (Surjective[L,V,W])\big) \implies \ldots$

(3.1) $(Injective[L,V,W]) \wedge (Surjective[L,V,W]) \ \blacksquare \ \forall_{w \in W}\exists!_{v \in V}\big(w = L(v)\big)$

(3.2) $S := \{(w,v) \in W \times V \mid w = L(v)\}$

(3.3) $\Big(\forall_{w \in W}\exists!_{v \in V}\big(w = L(v)\big)\Big) \wedge \big(S = \{(w,v) \in W \times V \mid w = L(v)\}\big) \ \blacksquare \ Function[S,W,V]$

(3.4) $\Big(\forall_{v \in V}\big(S(L(v))=v\big)\Big) \wedge \Big(\forall_{w \in W}\big(L(S(w))=w\big)\Big)$

(3.5) $(w_1,w_2 \implies W) \implies \ldots$

(3.5.1) $(LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \wedge \Big(\forall_{w \in W}\big(L(S(w))=w\big)\Big) \ \blacksquare \ L\big(S(w_1)+S(w_2)\big) = L\big(S(w_1)\big)+L\big(S(w_2)\big) = w_1 + w_2$

(3.5.2) $\Big(\forall_{w \in W}\big(L(S(w))=w\big)\Big) \wedge (w_1 + w_2 \in W) \ \blacksquare \ L\big(S(w_1+w_2)\big) = w_1 + w_2$

(3.5.3) $L\big(S(w_1)+S(w_2)\big) = w_1 + w_2 = L\big(S(w_1+w_2)\big) \ \blacksquare \ L\big(S(w_1)+S(w_2)\big) = L\big(S(w_1+w_2)\big)$

(3.5.4) $(Injective[L,V,W]) \wedge \Big(L\big(S(w_1)+S(w_2)\big) = L\big(S(w_1+w_2)\big)\Big) \ \blacksquare \ S(w_1)+S(w_2) = S(w_1+w_2)$

(3.6) $(w_1,w_2 \implies W) \implies \big(S(w_1+w_2)=S(w_1)+S(w_2)\big) \ \blacksquare \ \forall_{w_1,w_2 \in W}\big(S(w_1+w_2)=S(w_1)+S(w_2)\big)$

(3.7) $\big((r \in \mathbb{R}) \wedge (w \in W)\big) \implies \ldots$

(3.7.1) $(LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \wedge \Big(\forall_{w \in W}\big(L(S(w))=w\big)\Big) \ \blacksquare \ L\big(r*S(w)\big) = r*L\big(S(w)\big) = r*w$

(3.7.2) $\Big(\forall_{w \in W}\big(L(S(w))=w\big)\Big) \wedge (r*w \in W) \ \blacksquare \ L\big(S(r*w)\big) = r*w$

(3.7.3) $L\big(r*S(w)\big) = r*w = L\big(S(r*w)\big) \ \blacksquare \ L\big(r*S(w)\big) = L\big(S(r*w)\big)$

(3.7.4) $(Injective[L,V,W]) \wedge \Big(L\big(r*S(w)\big) = L\big(S(r*w)\big)\Big) \ \blacksquare \ r*S(w) = S(r*w)$

(3.8) $\big((r \in \mathbb{R}) \wedge (w \in W)\big) \implies \big(r*S(w) = S(r*w)\big) \ \blacksquare \ \forall_{r \in \mathbb{R}}\forall_{w \in W}\big(S(r*w) = r*S(w)\big)$

(3.9) $(Function[S,W,V]) \wedge \Big(\forall_{w_1,w_2 \in W}\big(S(w_1+w_2)=S(w_1)+S(w_2)\big)\Big) \wedge \Big(\forall_{r \in \mathbb{R}}\forall_{w \in W}\big(S(r*w) = r*S(w)\big)\Big)$

(3.10) $LinTrans[S,W,+_w,*_w,V,+_v,*_v]$

(3.11) $\forall_{v \in V}\Big(\big(S(L(v))=v\big)\Big) \ \blacksquare \ S \circ L = 1_v$

(3.12) $\forall_{w \in W}\big(L(S(w))=w\big) \ \blacksquare \ L \circ S = 1_w$

(3.13) $(LinTrans[S,W,+_w,*_w,V,+_v,*_v]) \wedge (S \circ L = 1_v) \wedge (L \circ S = 1_w) \ \blacksquare \ LTInv[S,L,V,+_v,*_v,W,+_w,*_w]$

(3.14) $\exists_{L^{-1}}(LTInv[L^{-1},L,V,+_v,*_v,W,+_w,*_w]) \ \blacksquare \ LTInvertible[L,V,+_v,*_v,W,+_w,*_w]$

(4) $\big((Injective[L,V,W]) \wedge (Surjective[L,V,W])\big) \implies (LTInvertible[L,V,+_v,*_v,W,+_w,*_w])$

(5) $(LTInvertible[L, V, +_v, *_v, W, +_w, *_w]) \iff ((Injective[L, V, W]) \land (Surjective[L, V, W]))$

TODO: some corollary of InjectiveSurjectiveEqualDim + SurjectiveInjectiveEqualDim + InvertibleBijectiveEquiv

$Isomorphism[L, V, +_v, *_v, W, +_w, *_w] := LTInvertible[L, V, +_v, *_v, W, +_w, *_w]$
$Isomorphic[V, +_v, *_v, W, +_w, *_w] := \exists_L (Isomorphism[L, V, +_v, *_v, W, +_w, *_w])$

3.9   Matrix of a Linear Transform

$CoordVec[[\alpha]_S, \alpha, S, V, +, *] := (Basis[S, V, +, *]) \land (S * [\alpha]_S = \alpha \in V)$

$$LTMatrix := \forall_{L, V, W} \left(\begin{array}{l} ((LinTrans[L, V, +_v, *_v, W, +_w, *_w]) \land (Basis[A, V, +_v, *_v]) \land (Basis[B, W, +_w, *_w])) \implies \\ (\forall_{v \in V} (CoordVec[[L(v)]_B, L(v), B, W, +_w, *_w] = \langle [L(a_i)]_B | a_i \in A \rangle * CoordVec[[v]_A, v, A, V, +_v, *_v])) \end{array} \right)$$

(1) $Basis[A, V, +_v, *_v]$ ▌ $\exists_{K \in \mathbb{R}^n} (v = \sum_{i=1}^n (k_i * a_i))$ ▌ $K^T = CoordVec[[v]_A, , v, A, V, +, *]$

(2) $[L(v)]_B = [L(\sum_{i=1}^n (k_i * a_i))]_B = [\sum_{i=1}^n (L(k_i * a_i))]_B = \sum_{i=1}^n ([L(k_i * a_i)]_B) = \sum_{i=1}^n ([k_i * L(a_i)]_B) = \sum_{i=1}^n (k_i * [L(a_i)]_B) = \dots$

(3) $\dots \langle [L(a)]_B | a \in A \rangle * K^T = \langle [L(a)]_B | a \in A \rangle * [v]_A$ ▌ $[L(v)]_B = \langle [L(a)]_B | a \in A \rangle * [v]_A$

Note: Shorthand is to RREF the augmented matrix [Columns of B | Columns of A] into [I | M], thus M is the transition matrix

$$TransitionMatrix := \forall_{L, V} \left(\begin{array}{l} ((Basis[A, V, +, *]) \land (Basis[B, V, +, *])) \implies \\ (\forall_{v \in V} (CoordVec[[v]_B, v, B, W, +_w, *_w] = \langle [a]_B | a \in A \rangle * CoordVec[[v]_A, v, A, V, +_v, *_v])) \end{array} \right)$$

(1) $(LTMatrix) \land (LinTrans[I, V, +, *, V, +, *])$ ▌ $[I(v)]_B = \langle [I(a)]_B | a \in A \rangle * [v]_A$ ▌ $[v]_B = \langle [a]_B | a \in A \rangle * [v]_A$

$$LTOverTransition := \left(([L(a)]_T = A * [a]_S) \land (P * [a]_{S'} = [a]_S) \land (Q * [L(a)]_{T'} = [L(a)]_T) \right) \implies \left([L(a)]_{T'} = (Q^{-1} * A * P) * [a]_{S'} \right)$$

(1) $[L(a)]_{T'} = Q^{-1} * [L(a)]_T = Q^{-1} * A * [a]_S = Q^{-1} * A * P * [a]_{S'}$ ▌ $[L(a)]_{T'} = (Q^{-1} * A * P) * [a]_{S'}$

$$LOOverTransition := \left(([L(a)]_S = A * [a]_S) \land (P * [a]_{S'} = [a]_S) \right) \implies \left([L(a)]_{S'} = (P^{-1} * A * P) * [a]_{S'} \right)$$

(1) $P * [a]_{S'} = [a]_S$ ▌ $P * [L(a)]_{S'} = [L(a)]_S$

(2) $LTOverTransition$ ▌ $[L(a)]_{S'} = P^{-1} * [L(a)]_S = P^{-1} * A * [a]_S = P^{-1} * A * P * [a]_{S'}$ ▌ $[L(a)]_{S'} = (P^{-1} * A * P) * [a]_{S'}$

$RankNullityRelation := (Rank[A] \equiv Dim[rng_L]) \land (Nullity[A] \equiv Dim[ker_L]) \land (RankNullity \equiv RankNullityLT)$

(1) TODO

$SimMatrix[A, B] := \exists_P (B = P * A * P^{-1})$

$SimMatrixEquiv := (SimMatrix[A, B]) \iff \left(([L(a)]_T = A * [a]_S) \land ([L(a)]_{T'} = B * [a]_{S'}) \right)$???