MTH223A LECTURE NOTES CHAPTER 5

Yvette Fajardo-Lim

Mathematics and Statistics Department
De La Salle University - Manila

Outline

Outline

1. Conjugacy
   - Conjugacy Classes
   - Normal Subgroups
   - Quotient Groups

Conjugacy Classes

Definition

If $a, b \in G$, we say that b is conjugate to a if there exists $x \in G$ with $b = x^{-1}ax$.

Example

1. *In D_4 we have a^3 conjugate to a, since with $x = b$ we have $b^{-1}ab = bab = a^3bb = a^3$.*
2. *In any group G, the only element conjugate to the identity is itself, since for all $x \in G$ we have $x^{-1}ex = x^{-1}x = e$.*

Conjugacy Classes

Definition

If $a, b \in G$, we say that b is conjugate to a if there exists $x \in G$ with $b = x^{-1}ax$.

Example

1. *In D_4 we have a^3 conjugate to a, since with $x = b$ we have $b^{-1}ab = bab = a^3bb = a^3$.*
2. *In any group G, the only element conjugate to the identity is itself, since for all $x \in G$ we have $x^{-1}ex = x^{-1}x = e$.*

Conjugacy Classes

Definition

If $a, b \in G$, we say that b is conjugate to a if there exists $x \in G$ with $b = x^{-1}ax$.

Example

1. *In D_4 we have a^3 conjugate to a, since with $x = b$ we have $b^{-1}ab = bab = a^3bb = a^3$.*
2. *In any group G, the only element conjugate to the identity is itself, since for all $x \in G$ we have $x^{-1}ex = x^{-1}x = e$.*

Conjugacy Classes

Theorem

Conjugacy is an equivalence relation on G.

Definition

The equivalence classes for the relation of conjugacy are called **conjugacy classes** *of G; we write $\mathbf{C}_g$ for the conjugacy class containing the element g.*

Conjugacy Classes

Theorem

Conjugacy is an equivalence relation on G.

Definition

The equivalence classes for the relation of conjugacy are called **conjugacy classes** *of G; we write $\mathbf{C}_g$ for the conjugacy class containing the element g.*

Conjugacy Classes

Theorem

If $g \in G$ then $\mathbf{C}_g = \{g\}$ if and only if $g \in Z(G)$.

Corollary

G is abelian if and only if $\mathbf{C}_g = \{g\}$ for all $g \in G$.

Conjugacy Classes

Theorem

If $g \in G$ then $\mathbf{C}_g = \{g\}$ if and only if $g \in Z(G)$.

Corollary

G is abelian if and only if $\mathbf{C}_g = \{g\}$ for all $g \in G$.

Conjugacy Classes

Theorem

If $g, h \in G$ and h is conjugate to g, then $o(h) = o(g)$.

Example

In D_4 the elements a and a^3 are conjugate, and each has order 4.

Conjugacy Classes

Theorem

If $g, h \in G$ and h is conjugate to g, then $o(h) = o(g)$.

Example

In D_4 the elements a and a^3 are conjugate, and each has order 4.

Conjugacy Classes

Theorem

If $g, h \in G$ and h is conjugate to g, then $o(h) = o(g)$.

Example

In D_4 the elements a and a^3 are conjugate, and each has order 4.

Remark

The converse need not be true, for example, in $\mathbb{Z}_4$ the elements $[1]$ and $[3]$ both have order 4, but they are not conjugate as the group is abelian.

Conjugacy Classes

Theorem

If $g, h \in G$ and h is conjugate to g, then $o(h) = o(g)$.

Example

In D_4 the elements a and a^3 are conjugate, and each has order 4.

Remark

The converse need not be true, for example, in $\mathbb{Z}_4$ the elements $[1]$ and $[3]$ both have order 4, but they are not conjugate as the group is abelian.

Conjugacy Classes

Theorem

If $g, h \in G$ and h is conjugate to g, then $o(h) = o(g)$.

Example

In D_4 the elements a and a^3 are conjugate, and each has order 4.

Remark

The converse need not be true, for example, in $\mathbb{Z}_4$ the elements $[1]$ and $[3]$ both have order 4, but they are not conjugate as the group is abelian.

Conjugacy Classes

Theorem

If $g, h \in G$ and $h = x^{-1}gx$, then $C_G(h) = x^{-1}C_G(g)x$.

Example

If $G = D_4$ we have $b^{-1}ab = a3$, and $C_G(a) = \{e, a, a^2, a^3\} = C_G(a^3)$; thus $b^{-1}C_G(a)b = \{b^{-1}eb, b^{-1}ab, b^{-1}a^2b, b^{-1}a^3b\} = \{e, a^3, a^2, a\} = b^{-1}C_G(a^3)b$.

Conjugacy Classes

Theorem

If $g, h \in G$ and $h = x^{-1}gx$, then $C_G(h) = x^{-1}C_G(g)x$.

Example

If $G = D_4$ we have $b^{-1}ab = a3$, and $C_G(a) = \{e, a, a^2, a^3\} = C_G(a^3)$; thus $b^{-1}C_G(a)b = \{b^{-1}eb, b^{-1}ab, b^{-1}a^2b, b^{-1}a^3b\} = \{e, a^3, a^2, a\} = b^{-1}C_G(a^3)b$.

Conjugacy Classes

Example

We shall determine the conjugacy classes in $G = D_3$. We know that $\{e\}$ is one conjugacy class; from a previous exercise $Z(G) = \{e\}$, so this is the only conjugacy class of order 1. Of the remaining elements, a and a^2 have order 3 while the rest have order 2; thus $\{a, a^2\}$ must be a conjugacy class, as must $\{b, ab, a^2b\}$ (otherwise there would be a class of order 1). To check this, note that $b^{-1}ab = bab = a2$ so that $a^2 \in \mathbf{C}_a$, while $(a2)^{-1}ba^2 = aba^2 = ba = a^2b$ and $a^{-1}ba = a^2ba = ab$ so that $a^2b, ab \in \mathbf{C}_b$. Thus we do indeed have three conjugacy classes $\mathbf{C}_e, \mathbf{C}_a$ and $\mathbf{C}_b$, of sizes 1, 2 and 3. Note that the centralizer sizes of representative elements are $|C_G(e)| = |G| = 6, |C_G(a)| = |\{e, a, a^2\}| = 3$ and $|C_G(b)| = |\{e, b\}| = 2$.

Conjugacy Classes

Example

We shall determine the conjugacy classes in $G = D_3$. We know that $\{e\}$ is one conjugacy class; from a previous exercise $Z(G) = \{e\}$, so this is the only conjugacy class of order 1. Of the remaining elements, a and a^2 have order 3 while the rest have order 2; thus $\{a, a^2\}$ must be a conjugacy class, as must $\{b, ab, a^2b\}$ (otherwise there would be a class of order 1). To check this, note that $b^{-1}ab = bab = a2$ so that $a^2 \in \mathbf{C}_a$, while $(a2)^{-1}ba^2 = aba^2 = ba = a^2b$ and $a^{-1}ba = a^2ba = ab$ so that $a^2b, ab \in \mathbf{C}_b$. Thus we do indeed have three conjugacy classes $\mathbf{C}_e, \mathbf{C}_a$ and $\mathbf{C}_b$, of sizes 1, 2 and 3. Note that the centralizer sizes of representative elements are $|C_G(e)| = |G| = 6, |C_G(a)| = |\{e, a, a^2\}| = 3$ and $|C_G(b)| = |\{e, b\}| = 2$.

Conjugacy Classes

Example

Take $G = D_3$ and $g = b$, so that $C_G(g) = \{e, b\}$. We let x run through the elements of G, and compute the conjugate $x^{-1}gx$ and the right coset $C_G(g)x$. We obtain the following.

x	$x^{-1}gx$	$C_G(g)x$
e	b	$\{e, b\}$
b	b	$\{e, b\}$
a	ab	$\{a, a^2b\}$
a^2b	ab	$\{a, a^2b\}$
a^2	a^2b	$\{a^2, ab\}$
ab	a^2b	$\{a^2, ab\}$

Thus each distinct conjugate $x^{-1}gx$ corresponds to a distinct right coset $C_G(g)x$.

Outline

1. Conjugacy
   - Conjugacy Classes
   - Normal Subgroups
   - Quotient Groups

Normal Subgroups

Definition

The subgroup H of a group G is called a **normal subgroup** *if $g^{-1}hg$ for all $h \in H$ and $g \in G$; we write $H \lhd G$ if this condition is satisfied.*

Example

1. *If $G = D_4$, then $H = \{e, b\}$ is not a normal subgroup, since $a^{-1}ba = a^3ba = a^2b \notin H$.*
2. *If $G = D_4$, then $H = \{e, a^2\}$ is a normal subgroup, since for all $g \in G$ we have $g^{-1}eg = e$ and $g^{-1}a^2g = a^2$, as $a^2 \in Z(G)$.*

Normal Subgroups

Definition

The subgroup H of a group G is called a **normal subgroup** *if $g^{-1}hg$ for all $h \in H$ and $g \in G$; we write $H \triangleleft G$ if this condition is satisfied.*

Example

1. *If $G = D_4$, then $H = \{e, b\}$ is not a normal subgroup, since $a^{-1}ba = a^3ba = a^2b \notin H$.*
2. *If $G = D_4$, then $H = \{e, a^2\}$ is a normal subgroup, since for all $g \in G$ we have $g^{-1}eg = e$ and $g^{-1}a^2g = a^2$, as $a^2 \in Z(G)$.*

Normal Subgroups

Definition

The subgroup H of a group G is called a **normal subgroup** *if $g^{-1}hg$ for all $h \in H$ and $g \in G$; we write $H \triangleleft G$ if this condition is satisfied.*

Example

1. *If $G = D_4$, then $H = \{e, b\}$ is not a normal subgroup, since $a^{-1}ba = a^3ba = a^2b \notin H$.*
2. *If $G = D_4$, then $H = \{e, a^2\}$ is a normal subgroup, since for all $g \in G$ we have $g^{-1}eg = e$ and $g^{-1}a^2g = a^2$, as $a^2 \in Z(G)$.*

Remark

1. *Every subgroup of an abelian group is normal because if $gh = hg$ for any g and h then $g^{-1}hg = h$.*
2. *$\{e\}$ and G are normal subgroups of G.*
3. *$Z(G) \lhd G$ because if $h \in Z(G)$ then $g^{-1}hg = h$ for all $g \in G$.*
4. *A normal subgroup is a union of conjugacy classes of G.*

Normal Subgroups

Remark

1. *Every subgroup of an abelian group is normal because if* $gh = hg$ *for any* g *and* h *then* $g^{-1}hg = h$.
2. $\{e\}$ *and* G *are normal subgroups of* G.
3. $Z(G) \triangleleft G$ *because if* $h \in Z(G)$ *then* $g^{-1}hg = h$ *for all* $g \in G$.
4. *A normal subgroup is a union of conjugacy classes of* G.

Normal Subgroups

Remark

1. *Every subgroup of an abelian group is normal because if $gh = hg$ for any g and h then $g^{-1}hg = h$.*
2. $\{e\}$ *and* G *are normal subgroups of* G.
3. $Z(G) \triangleleft G$ *because if* $h \in Z(G)$ *then* $g^{-1}hg = h$ *for all* $g \in G$.
4. *A normal subgroup is a union of conjugacy classes of* G.

Normal Subgroups

Remark

1. *Every subgroup of an abelian group is normal because if $gh = hg$ for any g and h then $g^{-1}hg = h$.*
2. $\{e\}$ *and G are normal subgroups of G.*
3. $Z(G) \triangleleft G$ *because if $h \in Z(G)$ then $g^{-1}hg = h$ for all $g \in G$.*
4. *A normal subgroup is a union of conjugacy classes of G.*

Normal Subgroups

Example

The conjugacy classes of D_3 are $\{e\}, \{a, a^2\}$, and $\{b, ab, a^2b\}$. If $H \lhd G$, then H must be a union of some of these, including $\mathbf{C}_e$; there are four possibilities:

$$\mathbf{C}_e, \quad \mathbf{C}_e \cup \mathbf{C}_a, \quad \mathbf{C}_e \cup \mathbf{C}_b, \quad \mathbf{C}_e \cup C_a \cup \mathbf{C}_b$$

However, $|\mathbf{C}_e \cup \mathbf{C}_b| = 1 + 3 = 4$, which does not divide $|D6| = 6$, so this cannot be a subgroup; the remaining three are in fact all subgroups: we have $\mathbf{C}_e = \{e\}, \mathbf{C}_e \cup \mathbf{C}_a = \{e, a, a^2\}$ and $\mathbf{C}_e \cup C_a \cup \mathbf{C}_b$.

Normal Subgroups

Example

Suppose that we are given a group G of order 12 with four conjugacy classes, $\mathbf{C}_e, \mathbf{C}_a, \mathbf{C}_b,$ and $\mathbf{C}_c$; we are told that $|C_G(a)| = 4$ and $C_G(b) = C_G(c)$, and are asked to show that G can only have one possible proper non-trivial normal subgroup. We identify the class sizes: $|\mathbf{C}_e| = 1$ since $\mathbf{C}_e = \{e\}$; $|\mathbf{C}_a| = \frac{12}{4} = 3$; and as $|\mathbf{C}_b| = |\mathbf{C}_c|$ and the sum of the class sizes is $|G| = 12$, we must have $|\mathbf{C}_b| = |\mathbf{C}_c| = 4$. The only possible combination of 1,3,4,4 including 1 which gives a factor of 12 other than 1 or 12 is 1+3=4; so the only possible proper non-trivial normal subgroup is $\mathbf{C}_e \cup \mathbf{C}_a$.

Theorem

If H is a subgroup of G, then $H \triangleleft G$ if and only if the left and right cosets of H in G are the same.

Example

We saw that in D_4 the left and right cosets of the subgroup $H = \{e, b\}$ were different, so this gives another way of seeing that H is not a normal subgroup.

Normal Subgroups

Theorem

If H is a subgroup of G, then $H \triangleleft G$ if and only if the left and right cosets of H in G are the same.

Example

We saw that in D_4 the left and right cosets of the subgroup $H = \{e, b\}$ were different, so this gives another way of seeing that H is not a normal subgroup.

Normal Subgroups

Corollary

If G is a finite group and H is a subgroup of index 2 in G so that $|H| = \frac{1}{2}|G|$, then $H \triangleleft G$.

Example

1. *If $G = D_3$ then we have seen that $H = \{e, a, a^2\}$ is a normal subgroup of G, and its index in G is 2.*
2. *If $G = D_4$ then the subgroup $H = \{e, a, a^2, a^3\}$ is a normal subgroup, since its index in G is 2. We may see this directly as follows: clearly $g^{-1}hg \in H$ for all $g, h \in H$, so it suffices to consider $g \notin H$; then $g = a^i b, h = a^j$, and $g^{-1}hg = (a^i b)^{-1} a^j a^i b = b^{-1}(a^i)^{-1} a^j a^i b = b^{-1}(a^- i a^j a^i) b = b^{-1} a^j b = a^j \in H$*

Normal Subgroups

Corollary

If G is a finite group and H is a subgroup of index 2 in G so that $|H| = \frac{1}{2}|G|$, then $H \triangleleft G$.

Example

1. *If $G = D_3$ then we have seen that $H = \{e, a, a^2\}$ is a normal subgroup of G, and its index in G is 2.*
2. *If $G = D_4$ then the subgroup $H = \{e, a, a^2, a^3\}$ is a normal subgroup, since its index in G is 2. We may see this directly as follows: clearly $g^{-1}hg \in H$ for all $g, h \in H$, so it suffices to consider $g \notin H$; then $g = a^i b, h = a^j$, and $g^{-1}hg = (a^i b)^{-1} a^j a^i b = b^{-1}(a^i)^{-1} a^j a^i b = b^{-1}(a^{-}ia^j a^i)b = b^{-1} a^j b = a^j \in H$*

Normal Subgroups

Corollary

If G is a finite group and H is a subgroup of index 2 in G so that $|H| = \frac{1}{2}|G|$, then $H \triangleleft G$.

Example

1. *If $G = D_3$ then we have seen that $H = \{e, a, a^2\}$ is a normal subgroup of G, and its index in G is 2.*
2. *If $G = D_4$ then the subgroup $H = \{e, a, a^2, a^3\}$ is a normal subgroup, since its index in G is 2. We may see this directly as follows: clearly $g^{-1}hg \in H$ for all $g, h \in H$, so it suffices to consider $g \notin H$; then $g = a^i b, h = a^j$, and $g^{-1}hg = (a^i b)^{-1} a^j a^i b = b^{-1}(a^i)^{-1} a^j a^i b = b^{-1}(a^- i a^j a^i)b = b^{-1} a^j b = a^j \in H$*

Outline

Quotient Groups

Theorem

If $\phi : G \to H$ is a homomorphism, then $Ker\phi \lhd G$.

Theorem

If H is a normal subgroup of G, the set G/H of cosets of H in G is a group under the binary operation above; there is a homomorphism $\phi : G \to H$ defined by $\phi(g) = Hg$, and $ker\phi = H$.

Example

If $G = D_4$ and $H = \{e, b\}$, we know that H is not a normal subgroup of G; we cannot define a binary operation on the set of right cosets of H as above, as $Ha = \{a, a^3b\}$, $Ha^2 = \{a^2, a^2b\}$, but $aa^2 = a^3$ and $a^3ba^2b = a$ do not lie in the same right coset.

Quotient Groups

Theorem

If $\phi : G \to H$ is a homomorphism, then Ker$\phi \lhd G$.

Theorem

If H is a normal subgroup of G, the set G/H of cosets of H in G is a group under the binary operation above; there is a homomorphism $\phi : G \to H$ defined by $\phi(g) = Hg$, and ker$\phi = H$.

Example

If $G = D_4$ and $H = \{e, b\}$, we know that H is not a normal subgroup of G; we cannot define a binary operation on the set of right cosets of H as above, as $Ha = \{a, a^3b\}, Ha^2 = \{a^2, a^2b\}$, but $aa^2 = a^3$ and $a^3ba^2b = a$ do not lie in the same right coset.

Quotient Groups

Theorem

If $\phi : G \to H$ is a homomorphism, then Ker$\phi \lhd G$.

Theorem

If H is a normal subgroup of G, the set G/H of cosets of H in G is a group under the binary operation above; there is a homomorphism $\phi : G \to H$ defined by $\phi(g) = Hg$, and ker$\phi = H$.

Example

If $G = D_4$ and $H = \{e, b\}$, we know that H is not a normal subgroup of G; we cannot define a binary operation on the set of right cosets of H as above, as $Ha = \{a, a^3b\}, Ha^2 = \{a^2, a^2b\}$, but $aa^2 = a^3$ and $a^3ba^2b = a$ do not lie in the same right coset.

Quotient Groups

Example

If $G = D_4$ and $H = \{e, a^2\}$, then we have seen that $H \lhd G$; thus we do have a group G/H. If we order the elements of G according to the cosets of H, the Cayley table of G naturally gives that of G/H; here we have $G/H \cong$ Klein 4-group.

G	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	e	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	e	ab	a^2b	a^3b	b
a^2	a^2	a^3	e	a	a^2b	a^3b	b	ab
a^3	a^3	e	a	a^2	a^3b	b	ab	ab^2
b	b	a^3b	a^2b	ab	e	a^3	a^2	a
ab	ab	b	a^3b	a^2b	a	e	a^3	a^2
a^2b	a^2b	ab	b	a^3b	a^2	a	e	a^3
a^3b	a^3b	a^2b	ab	b	a^3	a^2	a	e

Quotient Groups

Example

G/H	He	Ha	Hb	Hab
He	He	Ha	Hb	Hab
Ha	Ha	He	Hab	Hb
Hb	Hb	Hab	He	Ha
Hab	Hab	Hb	Ha	He

Quotient Groups

Theorem

The Fundamental Theorem of Homomorphism *If $\phi : G \to G_1$ is a homomorphism, then $G/Ker\phi \cong \phi(G)$.*

Example

Let $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}_4, +)$ be the homomorphism defined by

$$\phi(x) = \phi(4q + r) = r$$

Then $N = Ker\ \phi = \{4k + k \in Z\}$ and $G/N = \{N, N+1, N+2, N+3\}$. The mapping ψ consists of the following: $N \mapsto 0,\ N+1 \mapsto 1,\ N+2 \mapsto 2,\ N+3 \mapsto 3.$ Clearly, ψ is an isomorphism. Moreover, ϕ is onto, so that $\phi(\mathbb{Z}) = \mathbb{Z}_4$. The group tables for $G/N = \mathbb{Z}/N$ and $G_1 = \mathbb{Z}_4$ are shown on the next slide.

Quotient Groups

Theorem

The Fundamental Theorem of Homomorphism *If $\phi : G \to G_1$ is a homomorphism, then $G/Ker\phi \cong \phi(G)$.*

Example

Let $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}_4, +)$ be the homomorphism defined by

$$\phi(x) = \phi(4q + r) = r$$

Then $N = Ker\ \phi = \{4k + k \in Z\}$ and $G/N = \{N, N+1, N+2, N+3\}$. The mapping ψ consists of the following: $N \mapsto 0,\ N+1 \mapsto 1,\ N+2 \mapsto 2,\ N+3 \mapsto 3.$ Clearly, ψ is an isomorphism. Moreover, ϕ is onto, so that $\phi(\mathbb{Z}) = \mathbb{Z}_4$. The group tables for $G/N = \mathbb{Z}/N$ and $G_1 = \mathbb{Z}_4$ are shown on the next slide.

Quotient Groups

Example

$*$	N	$N+1$	$N+2$	$N+3$
N	N	$N+1$	$N+2$	$N+3$
$N+1$	$N+1$	$N+2$	$N+3$	N
$N+2$	$N+2$	$N+3$	N	$N+1$
$N+3$	$N+3$	N	$N+1$	$N+2$

$*$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Observe that each group table can be obtained from the other by replacing the entries in a table by the corresponding images or pre-images under the mapping ψ.

Quotient Groups

The fundamental homomorphism theorem is also known as the **First Isomorphism Theorem**. We also have the following isomorphism theorems.

Theorem

Second Isomorphism Theorem *Let H be a subgroup of a group G and N be a normal subgroup of G. Then $HN/N \cong H/H \cap N$.*

Theorem

Third Isomorphism Theorem *Let H and K be normal subgroups of a group G, with $K \leq H$. Then $G/H \cong (G/K)/(H/K)$.*

Quotient Groups

The fundamental homomorphism theorem is also known as the **First Isomorphism Theorem**. We also have the following isomorphism theorems.

Theorem

Second Isomorphism Theorem *Let H be a subgroup of a group G and N be a normal subgroup of G. Then $HN/N \cong H/H \cap N$.*

Theorem

Third Isomorphism Theorem *Let H and K be normal subgroups of a group G, with $K \leq H$. Then $G/H \cong (G/K)/(H/K)$.*

Quotient Groups

The fundamental homomorphism theorem is also known as the **First Isomorphism Theorem**. We also have the following isomorphism theorems.

Theorem

Second Isomorphism Theorem *Let H be a subgroup of a group G and N be a normal subgroup of G. Then* $HN/N \cong H/H \cap N$.

Theorem

Third Isomorphism Theorem *Let H and K be normal subgroups of a group G, with $K \leq H$. Then* $G/H \cong (G/K)/(H/K)$.

Quotient Groups

Example

Let $G = \mathbb{Z},\ H = 5\mathbb{Z},\ K = 10\mathbb{Z}$*, with respect to addition. Since* G *is abelian, every subgroup is normal, so* $H,\ K$ *are normal subgroups of* G*. We have*

$$\begin{aligned} G/H &= \mathbb{Z}/5\mathbb{Z} = \{\ H,\ H+1,\ H+2,\ H+3,\ H+4\ \} \cong \mathbb{Z}_5 \\ G/K &= \mathbb{Z}/10\mathbb{Z} = \{\ K,\ K+1, K+2,\ \ldots, K+9\ \} \cong \mathbb{Z}_{10} \\ H/K &= 5\mathbb{Z}/10\mathbb{Z} = \{K,\ K+5\ \} \cong \mathbb{Z}_2 \end{aligned}$$

On the other hand,
$(G/K)/(H/K) = \{H/K,\ H/K + (K+1),\ H/K + (K+2),\ H/K + (K+3),\ H/K + (K+4)\ \} \cong \mathbb{Z}_5$ *which shows that* $G/H \cong (G/K)/(H/K)$