Contents

Chapter 1

Real Analysis

(1.5)
$OrderTrichotomy[<, S] := \forall_{x,y \in S}(x < y \veebar x = y \veebar y < x)$
$OrderTransitivity[<, S] := \forall_{x,y,z \in S}((x < y \land y < z) \implies x < z)$
$Order[<, S] := (OrderTrichotomy[<, S]) \land (OrderTransitivity[<, S])$

(1.7)
$BoundedAbove[E, S, <] := (Order[<, S]) \land (E \subset S) \land (\exists_{\beta \in S}\forall_{x \in E}(x \le \beta))$
$BoundedBelow[E, S, <] := (Order[<, S]) \land (E \subset S) \land (\exists_{\beta \in S}\forall_{x \in E}(\beta \le x))$
$UpperBound[\beta, E, S, <] := (Order[<, S]) \land (E \subset S) \land (\beta \in S \land \forall_{x \in E}(x \le \beta))$
$LowerBound[\beta, E, S, <] := (Order[<, S]) \land (E \subset S) \land (\beta \in S \land \forall_{x \in E}(\beta \le x))$

(1.8)
$LUB[\alpha, E, S, <] := (UpperBound[\alpha, E, S, <]) \land (\forall_{\gamma}(\gamma < \alpha \implies \neg UpperBound[\gamma, E, S, <]))$
$GLB[\alpha, E, S, <] := (LowerBound[\alpha, E, S, <]) \land (\forall_{\beta}(\alpha < \beta \implies \neg LowerBound[\beta, E, S, <]))$

(1.10)
$LUBProperty[S, <] := \forall_{E}(((\emptyset \neq E \subset S) \land (BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <])))$
$GLBProperty[S, <] := \forall_{E}(((\emptyset \neq E \subset S) \land (BoundedBelow[E, S, <]) \implies \exists_{\alpha \in S}(GLB[\alpha, E, S, <])))$

(1.11)
$LUBPropertyImpliesGLBProperty := LUBProperty[S, <] \implies GLBProperty[S, <]$

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(1) $LUBProperty[S, <] \implies \ldots$ — wts: 2

(1.1) $(\emptyset \neq B \subset S \land BoundedBelow[B, S, <]) \implies \ldots$ — wts: 1.2

(1.1.1) $Order[<, S] \land \exists_{\delta' \in S}(LowerBound[\delta', B, S, <])$ — from: $BoundedBelow$, 1.1

(1.1.2) $|B| = 1 \implies \ldots$ — wts: 1.1.3

(1.1.2.1) $\exists_{u'}(u' \in B) \blacksquare u := choice(\{u' : u' \in B\}) \blacksquare B = \{u\}$ — from: 1.1.2

(1.1.2.2) $GLB[u, B, S, <] \blacksquare \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$

(1.1.3) $|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])$

(1.1.4) $|B| \neq 1 \implies \ldots$ — wts: 1.1.5

(1.1.4.1) $\forall_{E}((\emptyset \neq E \subset S \land BoundedAbove[E, S, <]) \implies \exists_{\alpha \in S}(LUB[\alpha, E, S, <]))$ — from: $LUBProperty$, 1

(1.1.4.2) $L := \{s \in S : LowerBound[s, B, S, <]\}$

(1.1.4.3) $|B| > 1 \land OrderTrichotomy[<, S] \blacksquare \exists_{b_1' \in B}\exists_{b_0' \in B}(b_0' < b_1')$ — from: $Order$, 1.1.1; wts: 1.1.4.7

(1.1.4.4) $b_1 := choice(\{b_1' \in B : \exists_{b_0' \in B}(b_0' < b_1')\}) \blacksquare \neg LowerBound[b_1, B, S, <]$ — from: 1.1.4.2

(1.1.4.5) $b_1 \notin L \blacksquare L \subset S$

(1.1.4.6) $\delta := choice(\{\delta' \in S : LowerBound[\delta', B, S, <]\}) \blacksquare \delta \in L \blacksquare \emptyset \neq L$ — from: 1.1.1

(1.1.4.7) $\emptyset \neq L \subset S$ — from: 1.1.4.5, 1.1.4.6

(1.1.4.8) $\forall_{y \in L}(LowerBound[y_0, B, S, <]) \blacksquare \forall_{y \in L}\forall_{x \in B}(y_0 \le x)$ — from: $LowerBound$, 1.1.4.2; wts: 1.1.4.10

(1.1.4.9) $\forall_{x \in B}(x \in S \land \forall_{y \in L}(y_0 \le x)) \blacksquare \forall_{x \in B}(UpperBound[x, L, S, <])$ — from: $UpperBound$

(1.1.4.10) $\exists_{x \in S}(UpperBound[x, L, S, <]) \blacksquare BoundedAbove[L, S, <]$

(1.1.4.11) $\emptyset \neq L \subset S \land BoundedAbove[L, S, <]$ from: 1.1.4.7, 1.1.4.10

(1.1.4.12) $\exists_{\alpha' \in S}(LUB[\alpha', L, S, <])$ ▌ $\alpha := choice(\{\alpha' \in S : (LUB[\alpha', L, S, <])\})$ from: 1.1.4.1 wts: 1.1.4.21

(1.1.4.13) $\forall_x(x \in B \implies UpperBound[x, L, S, <])$ from: 1.1.4.9 wts: 1.1.4.17

(1.1.4.14) $\forall_x(\neg UpperBound[x, L, S, <] \implies x \notin B)$

(1.1.4.15) $\gamma < \alpha \implies \dots$ wts: 1.1.4.16

(1.1.4.15.1) $\neg UpperBound[\gamma, L, S, <]$ ▌ $\gamma \notin B$ from: LUB, 1.1.4.12, 1.1.4.14

(1.1.4.16) $\gamma < \alpha \implies \gamma \notin B$ ▌ $\gamma \in B \implies \gamma \geq \alpha$

(1.1.4.17) $\forall_{\gamma \in B}(\alpha \leq \gamma)$ ▌ $LowerBound[\alpha, B, S, <]$ from: LowerBound

(1.1.4.18) $\alpha < \beta \implies \dots$ wts: 1.1.4.19

(1.1.4.18.1) $\forall_{y \in L}(y_0 \leq \alpha < \beta)$ ▌ $\forall_{y \in L}(y_0 \neq \beta)$ from: LUB, 1.1.4.12, 1.1.4.18

(1.1.4.18.2) $\beta \notin L$ ▌ $\neg LowerBound[\beta, B, S, <]$ from: 1.1.4.2

(1.1.4.19) $\alpha < \beta \implies \neg LowerBound[\beta, B, S, <]$ ▌ $\forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$

(1.1.4.20) $LowerBound[\alpha, B, S, <] \land \forall_{\beta \in S}(\alpha < \beta \implies \neg LowerBound[\beta, B, S, <])$ from: 1.1.4.17, 1.1.4.19

(1.1.4.21) $GLB[\alpha, B, S, <]$ ▌ $\exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.5) $|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <])$

(1.1.6) $(|B| = 1 \implies \exists_{\epsilon_0 \in S}(GLB[\epsilon_0, B, S, <])) \land (|B| \neq 1 \implies \exists_{\epsilon_1 \in S}(GLB[\epsilon_1, B, S, <]))$ from: 1.1.3, 1.1.5

(1.1.7) $(|B| = 1 \lor |B| \neq 1) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$ ▌ $\exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.2) $(\emptyset \neq B \subset S \land BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <])$

(1.3) $\forall_B((\emptyset \neq B \subset S \land BoundedBelow[B, S, <]) \implies \exists_{\epsilon \in S}(GLB[\epsilon, B, S, <]))$

(1.4) $GLBProperty[S, <]$

(2) $LUBProperty[S, <] \implies GLBProperty[S, <]$

(1.12)

$$Field[F, +, *] := \exists_{0,1 \in F}\forall_{x,y,z \in F}\begin{pmatrix} x + y \in F \quad \land \quad x * y \in F \quad \land \\ x + y = y + x \quad \land \quad x * y = y * x \quad \land \\ (x + y) + z = x + (y_0 + z) \;\land\; (x * y) * z = x * (y_0 * z) \;\land \\ \mathbb{1} \neq \mathbb{0} \quad \land \quad x * (y_0 + z) = (x * y) + (x * z) \quad \land \\ \mathbb{0} + x = x \quad \land \quad \mathbb{1} * x = x \quad \land \\ \exists_{-x \in F}(x + (-x) = \mathbb{0}) \land (x \neq \mathbb{0} \implies \exists_{1/x \in F}(x * (1/x) = \mathbb{1})) \end{pmatrix}$$

*********************** $(Field[F, +, *] \land x, y, z \in F) \implies \dots$ ***********************

(1.14)

$AdditiveCancellation := (x + y = x + z) \implies y = z$

(1) $y = \mathbb{0} + y = (x + (-x)) + y = ((-x) + x) + y = (-x) + (x + y) = \dots$ from: Field

(2) $(-x) + (x + z) = ((-x) + x) + z = (x + (-x)) + z = \mathbb{0} + z = z$ from: Field

$AdditiveIdentityUniqueness := (x + y = x) \implies y = \mathbb{0}$

(1) $x + y = x = \mathbb{0} + x = x + \mathbb{0}$ from: Field

(2) $y = 0$ from: AdditiveCancellation

$AdditiveInverseUniqueness := (x + y = \mathbb{0}) \implies y = -x$

(1) $x + y = \mathbb{0} = x + (-x)$ from: Field

(2) $y = -x$ from: AdditiveCancellation

$DoubleNegative := x = -(-x)$

(1) $\mathbb{0} = x + (-x) = (-x) + x$ ▌ $\mathbb{0} = (-x) + x$ from: Field

(2) $x = -(-x)$ from: AdditiveInverseUniqueness

(1.15)

$\textit{MultiplicativeCancellation} := (x \neq 0 \land x * y = x * z) \implies y = z$ —

$\textit{MultiplicativeIdentityUniqueness} := (x \neq 0 \land x * y = x) \implies y = \mathbb{1}$ —

$\textit{MultiplicativeInverseUniqueness} := (x \neq 0 \land x * y = \mathbb{1}) \implies y = 1/x$ —

$\textit{DoubleReciprocal} := (x \neq 0) \implies x = 1/(1/x)$ —

(1.16)

$\textit{Domination} := \mathbb{0} * x = \mathbb{0}$

(1) $\mathbb{0} * x = (\mathbb{0} + \mathbb{0}) * x = \mathbb{0} * x + \mathbb{0} * x \;\blacksquare\; \mathbb{0} * x = \mathbb{0} * x + \mathbb{0} * x$ — from: *Field*

(2) $\mathbb{0} * x = \mathbb{0}$ — from: *AdditiveIdentityUniqueness*

$\textit{NonDomination} := (x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq \mathbb{0}$

(1) $(x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies \ldots$

(1.1) $(x * y = 0) \implies \ldots$

(1.1.1) $\mathbb{1} = \mathbb{1} * \mathbb{1} = (x * (1/x)) * (y * (1/y)) = (x * y) * ((1/x) * (1/y)) = \mathbb{0} * ((1/x) * (1/y)) = \mathbb{0}$ — from: *Field*, *Domination*, 1, 1.1

(1.1.2) $\mathbb{1} = \mathbb{0} \land \mathbb{1} \neq \mathbb{0} \;\blacksquare\; \bot$ — from: *Field*

(1.2) $(x * y = 0) \implies \bot \;\blacksquare\; x * y \neq 0$

(2) $(x \neq \mathbb{0} \land y \neq \mathbb{0}) \implies x * y \neq 0$

$\textit{NegationCommutativity} := (-x) * y = -(x * y) = x * (-y)$

(1) $x * y + (-x) * y = (x + -x) * y = \mathbb{0} * y = \mathbb{0} \;\blacksquare\; x * y + (-x) * y = \mathbb{0}$ — from: *Field*, *Domination*; wts: 2

(2) $(-x) * y = -(x * y)$ — from: *AdditiveInverseUniqueness*

(3) $x * y + x * (-y) = x * (y_0 + -y) = x * \mathbb{0} = \mathbb{0} \;\blacksquare\; x * y + x * (-y) = \mathbb{0}$ — from: *Field*, *Domination*; wts: 4

(4) $x * (-y) = -(x * y)$ — from: *AdditiveInverseUniqueness*

(5) $(-x) * y = -(x * y) = x * (-y)$ — from: 2, 4

$\textit{NegativeMultiplication} := (-x) * (-y) = x * y$

(1) $(-x) * (-y) = -(x * (-y)) = -(-(x * y)) = x * y$ — from: *NegationCommutativity*, *DoubleNegative*

(1.17)

$$\textit{OrderedField}[F, +, *, <] := \begin{pmatrix} \textit{Field}[F, +, *] \quad \land \quad \textit{Order}[<, F] \quad \land \\ \forall_{x,y,z \in F}(y_0 < z \implies x + y < x + z) \land \\ \forall_{x,y \in F}((x > \mathbb{0} \land y > \mathbb{0}) \implies x * y > \mathbb{0}) \end{pmatrix}$$

*********************** $(\textit{OrderedField}[F, +, *, <] \land x, y, z \in F) \implies \ldots$ ***********************

(1.18)

$\textit{NegationOnOrder} := x > \mathbb{0} \iff -x < \mathbb{0}$

(1) $x > \mathbb{0} \implies \ldots$

(1.1) $\mathbb{0} = (-x) + x > (-x) + \mathbb{0} = -x \;\blacksquare\; \mathbb{0} > -x \;\blacksquare\; -x < \mathbb{0}$ — from: *OrderedField*

(2) $x > \mathbb{0} \implies -x < \mathbb{0}$

(3) $-x < \mathbb{0} \implies \ldots$

(3.1) $\mathbb{0} = x + (-x) < x + \mathbb{0} = x \;\blacksquare\; \mathbb{0} < x \;\blacksquare\; x > \mathbb{0}$ — from: *OrderedField*

(4) $-x < \mathbb{0} \implies x > \mathbb{0}$

(5) $x > \mathbb{0} \implies -x < \mathbb{0} \land -x < \mathbb{0} \implies x > \mathbb{0} \;\blacksquare\; x > \mathbb{0} \iff -x < \mathbb{0}$ — from: 2, 4

$\textit{PositiveFactorPreservesOrder} := (x > \mathbb{0} \land y < z) \implies x * y < x * z$

(1) $(x > \mathbb{0} \land y < z) \implies \ldots$

(1.1) $(-y) + z > (-y) + y = \mathbb{0} \;\blacksquare\; z + (-y) = \mathbb{0}$ — from: *OrderedField*

(1.2) $x * (z + (-y)) > \mathbb{0} \;\blacksquare\; x * z + x * (-y) > \mathbb{0}$ — from: *OrderedField*

(1.3) $x * z = \mathbb{0} + x * z = (x * y + -(x * y)) + x * z = (x * y + x * (-y)) + x * z = \ldots$ — from: *Field*, *NegationCommutativity*

(1.4) $x * y + (x * z + x * (-y)) > x * y + \mathbb{0} = x * y$ — from: *Field*, 1.2

(1.5) $x * z > x * y$ — from: 1.3, 1.4

(2) $(x > \mathbb{0} \wedge y < z) \implies x * z > x * y$

$NegativeFactorFlipsOrder := (x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

(1) $(x < \mathbb{0} \wedge y < z) \implies \ldots$

(1.1) $-x > \mathbb{0}$ — from: $NegationOnOrder$

(1.2) $(-x) * y < (-x) * z$ ▌ $\mathbb{0} = x * y + (-x) * y < x * y + (-x) * z$ ▌ $\mathbb{0} < x * y + (-x) * z$ — from: $PositiveFactorPreservesOrder$

(1.3) $\mathbb{0} < (-x) * (-y + z)$ ▌ $\mathbb{0} > x * (-y + z)$ ▌ $\mathbb{0} > -(x * y) + x * z$ — from: $NegationOnOrder$

(1.4) $x * y > x * z$

(2) $(x < \mathbb{0} \wedge y < z) \implies x * y > x * z$

$SquareIsPositive := (x \neq \mathbb{0}) \implies x * x > \mathbb{0}$

(1) $(x > \mathbb{0}) \implies x * x > 0$ — from: $OrderedField$

(2) $(x < \mathbb{0}) \implies \ldots$

(2.1) $-x > \mathbb{0}$ ▌ $x * x = (-x) * (-x) > \mathbb{0}$ ▌ $x * x > \mathbb{0}$ — from: $NegationOnOrder, OrderedField, NegativeMultiplication$

(3) $(x < \mathbb{0}) \implies x * x > \mathbb{0}$

(4) $x \neq \mathbb{0} \implies (x > \mathbb{0} \vee x < \mathbb{0}) \implies x * x > \mathbb{0}$ ▌ $x \neq \mathbb{0} \implies x * x > \mathbb{0}$ — from: $OrderTrichotomy$, 1, 3

$OneIsPositive := \mathbb{1} > \mathbb{0}$

(1) $\mathbb{1} \neq \mathbb{0}$ ▌ $\mathbb{1} = \mathbb{1} * \mathbb{1} > \mathbb{0}$ — from: $Field, SquareIsPositive$

$ReciprocationOnOrder := (0 < x < y) \implies 0 < 1/y < 1/x$

(1) $(0 < x < y) \implies \ldots$

(1.1) $x * (1/x) = \mathbb{1} > \mathbb{0}$ ▌ $x * (1/x) > 0$ — from: $Field, OneIsPositive$

(1.2) $1/x < \mathbb{0} \implies x * (1/x) < 0 \wedge x * (1/x) > 0 \implies \bot$ ▌ $1/x > \mathbb{0}$ — from: $NegativeFactorFlipsOrder$, 1

(1.3) $y * (1/y) = \mathbb{1} > \mathbb{0}$ ▌ $y * (1/y) > 0$ — from: $Field, OneIsPositive$

(1.4) $1/y < \mathbb{0} \implies y * (1/y) < 0 \wedge y * (1/y) > 0 \implies \bot$ ▌ $1/y > \mathbb{0}$ — from: $NegativeFactorFlipsOrder$, 1

(1.5) $(1/x) * (1/y) > \mathbb{0}$ — from: $OrderedField$

(1.6) $\mathbb{0} < 1/y = ((1/x) * (1/y)) * x < ((1/x) * (1/y)) * y = 1/x$ — from: $OrderedField$, 1, 1.4, 1.5

(1.19)
$OrderedFieldQ := OrderedField[\mathbb{Q}, +, *, <]$ —

$Subfield[K, F, +, *] := Field[F, +, *] \wedge K \subset F \wedge Field[K, +, *]$
$OrderedSubfield[K, F, +, *, <] := OrderedField[F, +, *, <] \wedge K \subset F \wedge OrderedField[K, +, *, <]$
$CutI[\alpha] := \emptyset \neq \alpha \subset \mathbb{Q}$
$CutII[\alpha] := \forall_{p \in \alpha} \forall_{q \in \mathbb{Q}} (q < p \implies q \in \alpha)$
$CutIII[\alpha] := \forall_{p \in \alpha} \exists_{r \in \alpha} (p < r)$
$\mathbb{R} := \{\alpha \in \mathbb{Q} : CutI[\alpha] \wedge CutII[\alpha] \wedge CutIII[\alpha]\}$

$CutCorollaryI := (\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

(1) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies \ldots$

(1.1) $\forall_{p' \in \alpha} \forall_{q' \in \mathbb{Q}} (q' < p' \implies q' \in \alpha)$ — from: $CutII$, 1

(1.2) $q < p \implies q \in \alpha$ ▌ $q \notin \alpha \implies q \geq p$ — from: 1

(1.3) $(q \notin \alpha) \implies \ldots$

(1.3.1) $q \geq p$ — from: 1.2

(1.3.2) $(q = p) \implies (p \in \alpha \wedge p \notin \alpha) \implies \bot$ ▌ $q \neq p$ — from: 1, 1.3

(1.3.3) $q \geq p \wedge q \neq p$ ▌ $p < q$

(1.4) $q \notin \alpha \implies p < q$ ▌ $p < q$ — from: 1

(2) $(\alpha \in \mathbb{R} \wedge p \in \alpha \wedge q \in \mathbb{Q} \wedge q \notin \alpha) \implies p < q$

$\color{red}{CutCorollaryII} := (\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

(1) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies \ldots$

(1.1) $\forall_{s' \in \alpha} \forall_{r' \in \mathbb{Q}} (r' < s' \implies r' \in \alpha)$ — from: *CutII*, 1

(1.2) $s \in \alpha \implies (r \in \mathbb{Q} \implies (r < s \implies r \in \alpha)) \blacksquare\ s \in \alpha \implies r \in \alpha$ — from: 1, 1.1

(1.3) $r \notin \alpha \implies s \notin \alpha \blacksquare\ s \notin \alpha$ — from: 1, 1.2

(2) $(\alpha \in \mathbb{R} \wedge r, s \in \mathbb{Q} \wedge r < s \wedge r \notin \alpha) \implies s \notin \alpha$

$\color{red}{<_{\mathbb{R}}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge \alpha \subset \beta$

$\color{red}{OrderTrichotomyOfR} := OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \ldots$

(1.1.1) $\alpha \not\subset \beta \wedge \alpha \neq \beta$ — from: $<_{\mathbb{R}}$, 1.1

(1.1.2) $\exists_{p'}(p' \in \alpha \wedge p' \notin \beta) \blacksquare\ p := choice(\{p' : p' \in \alpha \wedge p' \notin \beta\})$

(1.1.3) $q \in \beta \implies \ldots$

(1.1.3.1) $p, q \in \mathbb{Q}$

(1.1.3.2) $q < p$ — from: *CutCorollaryI*

(1.1.3.3) $q \in \alpha$ — from: *CutII*

(1.1.4) $q \in \beta \implies q \in \alpha$

(1.1.5) $\forall_{q \in \beta}(q \in \alpha) \blacksquare\ \beta \subseteq \alpha$

(1.1.6) $\beta \subset \alpha \blacksquare\ \beta <_{\mathbb{R}} \alpha$

(1.2) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \implies \beta <_{\mathbb{R}} \alpha$

(1.3) $\neg(\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta) \blacksquare\ (\beta <_{\mathbb{R}} \alpha) \vee (\alpha <_{\mathbb{R}} \beta \vee \alpha = \beta)$

(1.4) $\alpha = \beta \implies \neg(\alpha <_{\mathbb{R}} \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.5) $\alpha <_{\mathbb{R}} \beta \implies \neg(\alpha = \beta \vee \beta <_{\mathbb{R}} \alpha)$

(1.6) $\beta <_{\mathbb{R}} \alpha \implies \neg(\alpha = \beta \vee \alpha <_{\mathbb{R}} \beta)$

(1.7) $\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta$

(2) $(\alpha, \beta \in \mathbb{R}) \implies (\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(3) $\forall_{\alpha, \beta \in \mathbb{R}}(\alpha <_{\mathbb{R}} \beta \veebar \alpha = \beta \veebar \alpha <_{\mathbb{R}} \beta)$

(4) $OrderTrichotomy[\mathbb{R}, <_{\mathbb{R}}]$

$\color{red}{OrderTransitivityOfR} := OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \ldots$

(1.1.1) $\alpha \subset \beta \wedge \beta \subset \gamma$

(1.1.2) $\forall_{a \in \alpha}(a \in \beta) \wedge \forall_{b \in \beta}(b \in \gamma)$

(1.1.3) $\forall_{a \in \alpha}(\alpha \in \gamma) \blacksquare\ \alpha \subset \gamma \blacksquare\ \alpha <_{\mathbb{R}} \gamma$

(1.2) $(\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(3) $\forall_{\alpha, \beta, \gamma \in \mathbb{R}}((\alpha <_{\mathbb{R}} \beta \wedge \beta <_{\mathbb{R}} \gamma) \implies \alpha <_{\mathbb{R}} \gamma)$

(4) $OrderTransitivity[\mathbb{R}, <_{\mathbb{R}}]$

$\color{red}{OrderOfR} := Order[<_{\mathbb{R}}, \mathbb{R}]$ — from: *OrderTrichotomyR*, *OrderTransitivityR*

wts:

$\color{red}{LUBPropertyOfR} := LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

(1) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \ldots$

(1.1) $\gamma := \{p \in \mathbb{Q} : \exists_{\alpha \in A}(p \in \alpha)\}$

(1.2) $A \neq \emptyset \blacksquare\ \exists_{\alpha}(\alpha \in A) \blacksquare\ \alpha_0 := choice(\{\alpha : \alpha \in A\})$

(1.3) $\alpha_0 \neq \emptyset \blacksquare\ \exists_a(a \in \alpha_0) \blacksquare\ a_0 := choice(\{a : a \in \alpha_0\}) \blacksquare\ a_0 \in \gamma \blacksquare\ \gamma \neq \emptyset$

(1.4) $BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}] \blacksquare\ \exists_{\beta}(UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.5) $\beta_0 := choice(\{\beta : UpperBound[\beta, A, \mathbb{R}, <_{\mathbb{R}}]\})$

(1.6) $UpperBound[\beta_0, A, \mathbb{R}, <_{\mathbb{R}}]$ ▮ $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \beta_0)$ ▮ $\forall_{\alpha \in A}(\alpha \subseteq \beta_0)$ ▮ $\forall_{\alpha \in A}\forall_{a \in \alpha}(a \in \beta_0)$

(1.7) $(\alpha \in A \wedge a \in \alpha) \iff a \in \gamma$ ▮ $\forall_{a \in \gamma}(a \in \beta_0)$ ▮ $\gamma \subseteq \beta_0$

(1.8) $\beta_0 \subset \mathbb{Q}$ ▮ $\gamma \subseteq \beta_0 \subset \mathbb{Q}$ ▮ $\gamma \subset \mathbb{Q}$

(1.9) $\emptyset \neq \gamma \subset \mathbb{Q}$ ▮ $CutI[\gamma]$

(1.10) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

  (1.10.1) $p \in \gamma$ ▮ $\exists_{\alpha \in A}(p \in \alpha)$ ▮ $\alpha_1 := choice(\{\alpha \in A : p \in \alpha\})$

  (1.10.2) $p \in \alpha_1 \wedge q \in \mathbb{Q} \wedge q < p$ ▮ $q \in \alpha_1$ ▮ $q \in \gamma$

(1.11) $(p \in \gamma \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \gamma$ ▮ $\forall_{p \in \gamma}\forall_{q \in \mathbb{Q}}(q < p \implies q \in \gamma)$ ▮ $CutII[\gamma]$

(1.12) $p \in \gamma \implies \ldots$

  (1.12.1) $\exists_{\alpha \in A}(p \in \alpha)$ ▮ $\alpha_2 := choice(\{\alpha \in A : p \in \alpha\})$

  (1.12.2) $\alpha_2 \in \mathbb{R}$ ▮ $CutII[\alpha_2]$ ▮ $\exists_{r \in \alpha_2}(p < r)$ ▮ $r_0 := choice(\{r \in \alpha_2 : p < r\})$

  (1.12.3) $r_0 \in \alpha_2$ ▮ $r_0 \in \gamma$

  (1.12.4) $p < r_0$ ▮ $p < r_0 \wedge r_0 \in \gamma$ ▮ $\exists_{r \in \gamma}(p < r)$

(1.13) $p \in \gamma \implies \exists_{r \in \gamma}(p < r)$ ▮ $\forall_{p \in \gamma}\exists_{r \in \gamma}(p < r)$ ▮ $CutIII[\gamma]$

(1.14) $CutI[\gamma] \wedge CutII[\gamma] \wedge CutIII[\gamma]$ ▮ $\gamma \in \mathbb{R}$

(1.15) $\forall_{\alpha \in A}(\alpha \subseteq \gamma)$ ▮ $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma)$

(1.16) $\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \gamma) \wedge \gamma \in \mathbb{R}$ ▮ $UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.17) $\delta <_{\mathbb{R}} \gamma \implies \ldots$

  (1.17.1) $\delta \subset \gamma$ ▮ $\exists_s(s \in \gamma \wedge s \notin \delta)$ ▮ $s_0 := choice(\{s \in \mathbb{Q} : s \in \gamma \wedge s \notin \delta\})$

  (1.17.2) $s_0 \in \gamma$ ▮ $\exists_{\alpha \in A}(s_0 \in \alpha)$ ▮ $\alpha_3 := choice(\{\alpha \in A : s_0 \in \alpha\})$

  (1.17.3) $s_0 \in \alpha_3 \wedge s_0 \notin \delta$ ▮ $\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

  (1.17.4) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \ldots$

    (1.17.4.1) $\alpha_3 \subseteq \delta$ ▮ $\forall_{s \in \mathbb{Q}}(s \in \alpha_3 \implies s \in \delta)$ ▮ $\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$

    (1.17.4.2) $\neg\exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta) \wedge \exists_{s \in \mathbb{Q}}(s \in \alpha_3 \wedge s \notin \delta)$ ▮ $\perp$

  (1.17.5) $\delta \geq_{\mathbb{R}} \alpha_3 \implies \perp$ ▮ $\delta <_{\mathbb{R}} \alpha_3$ ▮ $\exists_{\alpha \in A}(\delta <_{\mathbb{R}} \alpha)$ ▮ $\exists_{\alpha \in A}(\neg(\alpha \leq_{\mathbb{R}} \delta))$

  (1.17.6) $\neg\forall_{\alpha \in A}(\alpha \leq_{\mathbb{R}} \delta)$ ▮ $\neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}]$

(1.18) $\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$ ▮ $\forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.19) $UpperBound[\gamma, A, \mathbb{R}, <_{\mathbb{R}}] \wedge \forall_\delta(\delta <_{\mathbb{R}} \gamma \implies \neg UpperBound[\delta, A, \mathbb{R}, <_{\mathbb{R}}])$

(1.20) $LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]$ ▮ $\exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(2) $(\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}])$

(3) $\forall_A((\emptyset \neq A \subset \mathbb{R} \wedge BoundedAbove[A, \mathbb{R}, <_{\mathbb{R}}]) \implies \exists_{\gamma \in S}(LUB[\gamma, A, \mathbb{R}, <_{\mathbb{R}}]))$ ▮ $LUBProperty[\mathbb{R}, <_{\mathbb{R}}]$

$+_{\mathbb{R}}[\alpha, \beta] := \alpha, \beta \in \mathbb{R} \wedge (\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

$0_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 0\}$

$ZeroInR := 0_{\mathbb{R}} \in \mathbb{R}$

(1) $-1 \in 0_{\mathbb{R}} \wedge 1 \notin 0_{\mathbb{R}}$ ▮ $\emptyset \neq 0_{\mathbb{R}} \subseteq \mathbb{Q}$ ▮ $CutI[0_{\mathbb{R}}]$

(2) $(x \in 0_{\mathbb{R}} \wedge y \in \mathbb{Q} \wedge y < x) \implies y < x < 0 \implies y < 0 \implies y \in 0_{\mathbb{R}}$ ▮ $\forall_{x \in 0_{\mathbb{R}}}\forall_{y \in \mathbb{Q}}(y_0 < x \implies y \in 0_{\mathbb{R}})$ ▮ $CutII[0_{\mathbb{R}}]$

(3) $y := x/2$ ▮ $(x \in 0_{\mathbb{R}}) \implies (x < y < 0) \implies \exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▮ $\forall_{x \in 0_{\mathbb{R}}}\exists_{y \in 0_{\mathbb{R}}}(x < y)$ ▮ $CutIII[0_{\mathbb{R}}]$

(4) $CutI[0_{\mathbb{R}}] \wedge CutII[0_{\mathbb{R}}] \wedge CutIII[0_{\mathbb{R}}]$ ▮ $0_{\mathbb{R}} \in \mathbb{R}$

$FieldAdditionClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

(1) $(\alpha, \beta \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) = \{r + s : r \in \alpha \wedge s \in \beta\}$

(1.2) $\emptyset \neq \alpha \subset \mathbb{Q} \wedge \emptyset \neq \beta \subset \mathbb{Q}$

(1.3) $\exists_a(a \in \alpha)\,;\, \exists_b(b \in \beta)$ ▮ $a_0 := choice(\{a : a \in \alpha\})\,;\, b_0 := choice(\{b : b \in \beta\})$ ▮ $a_0 + b_0 \in \alpha +_{\mathbb{R}} \beta$

(1.4) $\exists_x(x \notin \alpha)\,;\, \exists_y(y_0 \notin \beta)$ ▮ $x_0 := choice(\{x : x \notin \alpha\})\,;\, y_0 := choice(\{y : y \notin \beta\})$

(1.5) $\forall_{r \in \alpha}(r < x_0)\,;\, \forall_{s \in \beta}(s < y_0)$ ▮ $\forall_{r \in \alpha}\forall_{s \in \beta}(r + s < x_0 + y_0)$ ▮ $x_0 + y_0 \notin \alpha +_{\mathbb{R}} \beta$

(1.6) $\emptyset \neq \alpha +_{\mathbb{R}} \beta \subset \mathbb{Q}$ ▮ $CutI[\alpha +_{\mathbb{R}} \beta]$

(1.7) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.7.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▌ $(r_0, s_0) := choice((r, s) \in \alpha \times \beta : p = r + s)$

(1.7.2) $q < p = r_0 + s_0$ ▌ $(q - s_0) < r_0$ ▌ $(q - s_0) \in \alpha$

(1.7.3) $s_0 \in \beta$ ▌ $q = (q - s_0) + s_0 \in \alpha +_{\mathbb{R}} \beta$ ▌ $q \in \alpha +_{\mathbb{R}} \beta$

(1.8) $(p \in \alpha +_{\mathbb{R}} \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \alpha +_{\mathbb{R}} \beta$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \alpha +_{\mathbb{R}} \beta)$ ▌ $CutII[\alpha +_{\mathbb{R}} \beta]$

(1.9) $p \in \alpha \implies \ldots$

(1.9.1) $\exists_{r \in \alpha} \exists_{s \in \beta}(p = r + s)$ ▌ $(r_1, s_1) := choice(\{(r, s) \in \alpha \times \beta : p = r + s\})$

(1.9.2) $r_1 \in \alpha$ ▌ $\exists_{t \in \alpha}(r_1 < t)$ ▌ $t_0 := choice(\{t \in \alpha : r_1 < t\})$

(1.9.3) $s_1 \in \beta$ ▌ $t + s_1 \in \alpha +_{\mathbb{R}} \beta \wedge p = r_1 + s_1 < t + s_1$ ▌ $\exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$

(1.10) $p \in \alpha \implies \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} \beta} \exists_{r \in \alpha +_{\mathbb{R}} \beta}(p < r)$ ▌ $CutIII[\alpha +_{\mathbb{R}} \beta]$

(1.11) $CutI[\alpha +_{\mathbb{R}} \beta] \wedge CutII[\alpha +_{\mathbb{R}} \beta] \wedge CutIII[\alpha +_{\mathbb{R}} \beta]$ ▌ $\alpha +_{\mathbb{R}} \beta \in \mathbb{R}$

(2) $(\alpha, \beta \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) \in \mathbb{R})$

$FieldAdditionCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta = \beta +_{\mathbb{R}} \alpha)$

(1) $\alpha +_{\mathbb{R}} \beta = \{r + s : r \in \alpha \wedge s \in \beta\} = \{s + r : s \in \beta \wedge r \in \alpha\} = \beta +_{\mathbb{R}} \alpha$

$FieldAdditionAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma))$

(1) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies \ldots$

(1.1) $(\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \{(a + b) + c : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \ldots$

(1.2) $\{a + (b + c) : a \in \alpha \wedge b \in \beta \wedge c \in \gamma\} = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

(2) $(\alpha, \beta, \gamma \in \mathbb{R}) \implies (\alpha +_{\mathbb{R}} \beta) +_{\mathbb{R}} \gamma = \alpha +_{\mathbb{R}} (\beta +_{\mathbb{R}} \gamma)$

$FieldAdditionIdentityOfR := (\alpha \in \mathbb{R}) \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies \ldots$

(1.1.1) $s < 0$ ▌ $r + s < r + 0 = r$ ▌ $r + s < r$ ▌ $r + s \in \alpha$

(1.2) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \implies r + s \in \alpha$ ▌ $\forall_{r \in \alpha} \forall_{s \in 0_{\mathbb{R}}}(r + s \in \alpha)$

(1.3) $(r \in \alpha \wedge s \in 0_{\mathbb{R}}) \iff (r + s \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▌ $\forall_{p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}}(p \in \alpha)$ ▌ $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha$

(1.4) $p \in \alpha \implies \ldots$

(1.4.1) $\exists_{r \in \alpha}(p < r)$ ▌ $r_2 := choice(\{r \in \alpha : p < r\})$

(1.4.2) $p < r_2$ ▌ $p - r_2 < r_2 - r_2 = 0$ ▌ $(p - r_2) < 0$ ▌ $(p - r_2) \in 0_{\mathbb{R}}$

(1.4.3) $r_2 \in \alpha$ ▌ $p = r_2 + (p - r_2) \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▌ $p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.5) $p \in \alpha \implies p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▌ $\forall_{p \in \alpha}(p \in \alpha +_{\mathbb{R}} 0_{\mathbb{R}})$ ▌ $\alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$

(1.6) $\alpha +_{\mathbb{R}} 0_{\mathbb{R}} \subseteq \alpha \wedge \alpha \subseteq \alpha +_{\mathbb{R}} 0_{\mathbb{R}}$ ▌ $0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

(2) $\alpha \in \mathbb{R} \implies 0_{\mathbb{R}} +_{\mathbb{R}} \alpha = \alpha$

$FieldAdditionInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{-\alpha \in \mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(1) $\alpha \in \mathbb{R} \implies \ldots$

(1.1) $\beta := \{p \in \mathbb{Q} : \exists_{r > 0}(-p - r \notin \alpha)\}$

(1.2) $\alpha \subset \mathbb{Q}$ ▌ $\exists_{s \in \mathbb{Q}}(s \notin \alpha)$ ▌ $s_0 := choice(\{s : s \notin \alpha\})$ ▌ $p_0 := -s_0 - 1$

(1.3) $-p_0 - 1 = -(-s_0 - 1) - 1 = s_0 \notin \alpha$ ▌ $-p_0 - 1 \notin \alpha$ ▌ $\exists_{r > 0}(-p_0 - r \notin \alpha)$ ▌ $p_0 \in \beta$

(1.4) $\emptyset \neq \alpha$ ▌ $\exists_{q \in \alpha}$ ▌ $q_0 := choice(\{q \in \mathbb{Q} : q \in \alpha\})$

(1.5) $r > 0 \implies \ldots$

(1.5.1) $q_0 \in \alpha$ ▌ $-(-q_0) - r = q_0 - r < q_0$ ▌ $-(-q_0) - r < q_0$ ▌ $-(-q_0) - r \in \alpha$

(1.6) $\forall_{r > 0}(-(-q_0) - r \in \alpha)$ ▌ $\neg\exists_{r > 0}(-(-q_0) - r \notin \alpha)$ ▌ $-q_0 \notin \beta$

(1.7) $\emptyset \neq \beta \subset \mathbb{Q}$ ▌ $CutI[\beta]$

(1.8) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies \ldots$

(1.8.1) $p \in \beta$ ▌ $\exists_{r > 0}(-p - r \notin \alpha)$ ▌ $r_0 := choice(\{r > 0 : -p - r \notin \alpha\})$

(1.8.2) $q < p$ ▌ $-p - r < -q - r$

(1.8.3) $-q - r \notin \alpha$ ▌ $q \in \beta$

(1.9) $(p \in \beta \wedge q \in \mathbb{Q} \wedge q < p) \implies q \in \beta$ ▌ $\forall_{p \in \beta} \forall_{q \in \mathbb{Q}}(q < p \implies q \in \beta)$ ▌ $CutII[\beta]$

(1.10) $p \in \beta \implies \ldots$

(1.10.1) $p \in \beta \;\blacksquare\; \exists_{r>0}(-p - r \notin \alpha) \;\blacksquare\; r_1 := choice(\{r > 0 : -p - r \notin \alpha\})$

(1.10.2) $t_0 := p + (r_1/2)$

(1.10.3) $r_1 > 0 \;\blacksquare\; r_1/2 > 0$

(1.10.4) $t_0 > t_0 - (r_1/2) = p \;\blacksquare\; t_0 > p$

(1.10.5) $-t_0 - (r_1/2) = -(p + (r_1/2)) - (r_1/2) = -p - r_1$

(1.10.6) $-p - r_1 \notin \alpha \;\blacksquare\; -t_0 - (r_1/2) \notin \alpha \;\blacksquare\; \exists_{r>0}(-t_0 - r \notin \alpha) \;\blacksquare\; t_0 \in \beta$

(1.10.7) $t_0 > p \wedge t_0 \in \beta \;\blacksquare\; \exists_{t\in\beta}(p < t)$

(1.11) $p \in \beta \implies \exists_{t\in\beta}(p < t) \;\blacksquare\; \forall_{p\in\beta}\exists_{t\in\beta}(p < t) \;\blacksquare\; CutIII[\beta]$

(1.12) $CutI[\beta] \wedge CutII[\beta] \wedge CutIII[\beta] \;\blacksquare\; \beta \in \mathbb{R}$

(1.13) $(r \in \alpha \wedge s \in \beta) \implies \ldots$

(1.13.1) $s \in \beta \;\blacksquare\; \exists_{t>0}(-s - t \notin \alpha) \;\blacksquare\; t_1 := choice(\{t > 0 : -s - t \notin \alpha\}) \;\blacksquare\; -s - t_1 < -s$

(1.13.2) $\alpha \in \mathbb{R} \wedge s, t_1 \in \mathbb{Q} \wedge -s - t_1 < -s \wedge -s - t_1 \notin \alpha \;\blacksquare\; -s \notin \alpha$

(1.13.3) $\alpha \in \mathbb{R} \wedge r \in \alpha \wedge -s \notin \alpha \;\blacksquare\; r < -s \;\blacksquare\; r + s < 0 \;\blacksquare\; r + s \in 0_{\mathbb{R}}$

(1.14) $(r \in \alpha \wedge s \in \beta) \implies r + s \in 0_{\mathbb{R}} \;\blacksquare\; \forall_{(r,s)\in\alpha\times\beta}(r + s \in 0_{\mathbb{R}}) \;\blacksquare\; \alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}}$

(1.15) $v \in 0_{\mathbb{R}} \implies \ldots$

(1.15.1) $v < 0 \;\blacksquare\; w_0 := -v/2 \;\blacksquare\; w > 0$

from: $ARCHIMEDEANPROPERTYOFQ + LUB$???

(1.15.2) $\exists_{n\in\mathbb{Z}}(nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha) \;\blacksquare\; n_0 := choice(\{n \in \mathbb{Z} : nw_0 \in \alpha \wedge (n+1)w_0 \notin \alpha\})$

(1.15.3) $p_0 := -(n_0 + 2)w_0 \;\blacksquare\; -p_0 - w_0 = (n_0 + 2)w_0 - w_0 = (n_0 + 1)w_0 \notin \alpha \;\blacksquare\; -p_0 - w_0 \notin \alpha \;\blacksquare\; p_0 \in \beta$

(1.15.4) $n_0 w_0 \in \alpha \wedge p_0 \in \beta \;\blacksquare\; n_0 w_0 + p_0 = n_0(-v/2) + -(n_0 + 2) - v/2 = v \in \alpha +_{\mathbb{R}} \beta$

(1.16) $v \in 0_{\mathbb{R}} \implies v \in \alpha +_{\mathbb{R}} \beta \;\blacksquare\; \forall_{v\in 0_{\mathbb{R}}}(v \in \alpha +_{\mathbb{R}} \beta) \;\blacksquare\; 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta$

(1.17) $\alpha +_{\mathbb{R}} \beta \subseteq 0_{\mathbb{R}} \wedge 0_{\mathbb{R}} \subseteq \alpha +_{\mathbb{R}} \beta \;\blacksquare\; \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}}$

(1.18) $\beta \in \mathbb{R} \wedge \alpha +_{\mathbb{R}} \beta = 0_{\mathbb{R}} \;\blacksquare\; \exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

(2) $\alpha \in \mathbb{R} \implies \exists_{-\alpha\in\mathbb{R}}(\alpha +_{\mathbb{R}} (-\alpha) = 0_{\mathbb{R}})$

$*_{\mathbb{R}}[\alpha, \beta] :=$ —

$1_{\mathbb{R}} := \{x \in \mathbb{Q} : x < 1\}$

$1IsNot0 := 0_{\mathbb{R}} \neq 1_{\mathbb{R}}$ —

$1InR := 1_{\mathbb{R}} \in \mathbb{R}$ —

$FieldMultiplicationClosureOfR := (\alpha, \beta \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) \in \mathbb{R})$ —

$FieldMultiplicationCommutativityOfR := (\alpha, \beta \in \mathbb{R}) \implies (\alpha *_{\mathbb{R}} \beta = \beta *_{\mathbb{R}} \alpha)$ —

$FieldMultiplicationAssociativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies ((\alpha *_{\mathbb{R}} \beta) *_{\mathbb{R}} \gamma = \alpha *_{\mathbb{R}} (\beta *_{\mathbb{R}} \gamma))$ —

$FieldMultiplicationIdentityOfR := (\alpha \in \mathbb{R}) \implies 1_{\mathbb{R}} *_{\mathbb{R}} \alpha = \alpha$ —

$FieldMultiplicationInverseOfR := (\alpha \in \mathbb{R}) \implies \exists_{1/\alpha\in\mathbb{R}}(\alpha *_{\mathbb{R}} (1/\alpha) = 1_{\mathbb{R}})$ —

$FieldDistributativityOfR := (\alpha, \beta, \gamma \in \mathbb{R}) \implies \gamma *_{\mathbb{R}} (\alpha +_{\mathbb{R}} \beta) = \gamma *_{\mathbb{R}} \alpha + \gamma *_{\mathbb{R}} \beta$ —

$FieldWithR := Field[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}]$ —

$OrderedFieldWithR := OrderedField[\mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$\mathbb{Q}_{\mathbb{R}} := \{\{r \in \mathbb{Q} : r < q\} : q \in \mathbb{Q}\}$

$QROrderedSubfieldOfR := OrderedSubfield[\mathbb{Q}_{\mathbb{R}}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}]$ —

$QIsomorphicToQR := \mathbb{Q}_{\mathbb{R}} \simeq \mathbb{Q}$ —

$CompletenessOfR := \exists_{\mathbb{R}}(LUBProperty[\mathbb{R}, <_{\mathbb{R}}] \wedge OrderedSubfield[\mathbb{Q}, \mathbb{R}, +_{\mathbb{R}}, *_{\mathbb{R}}, <_{\mathbb{R}}])$ —

(1.20)

$ArchimedeanPropertyOfR := \forall_{x,y\in\mathbb{R}}(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y))$

(1) $(x, y \in \mathbb{R} \wedge x > 0) \implies \ldots$

(1.1) $A := \{nx : n \in \mathbb{N}^+\} \;\blacksquare\; (\emptyset \neq A \subset \mathbb{R}) \wedge (a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a))$

(1.2) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \implies \ldots$

(1.2.1) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \;\blacksquare\; \forall_{n\in\mathbb{N}^+}(nx \leq y) \;\blacksquare\; UpperBound[y_0, A, \mathbb{R}, <] \;\blacksquare\; BoundedAbove[A, \mathbb{R}, <]$

(1.2.2) $CompletenessOfR \;\blacksquare\; LUBProperty[\mathbb{R}, <]$

(1.2.3) $(LUBProperty[\mathbb{R}, <]) \wedge (\emptyset \neq A \subset \mathbb{R}) \wedge (BoundedAbove[A, \mathbb{R}, <]) \;\blacksquare\; \exists_{\alpha\in\mathbb{R}}(LUB[\alpha, A, \mathbb{R}, <])$ . . .

(1.2.4) $\ldots \alpha_0 := choice(\{\alpha \in \mathbb{R} : LUB[\alpha, A, \mathbb{R}, <]\})$ ▌ $LUB[\alpha_0, A, \mathbb{R}, <]$

(1.2.5) $x > 0$ ▌ $\alpha_0 - x < \alpha_0$

(1.2.6) $(\alpha_0 - x < \alpha_0) \wedge (LUB[\alpha_0, A, \mathbb{R}, <])$ ▌ $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$

(1.2.7) $\neg UpperBound[\alpha_0 - x, A, \mathbb{R}, <]$ ▌ $\exists_{c\in A}(\alpha_0 - x < c) \ \ldots$

(1.2.8) $\ldots c_0 := choice(\{c \in A : \alpha_0 - x < c\})$ ▌ $(c_0 \in A) \wedge (\alpha_0 - x < c_0)$

(1.2.9) $(c_0 \in A) \wedge (a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a))$ ▌ $\exists_{m\in\mathbb{N}^+}(mx = c_0) \ \ldots$

(1.2.10) $\ldots m_0 := choice(\{m \in \mathbb{N}^+ : mx = c_0\})$ ▌ $(m_0 \in \mathbb{N}^+) \wedge (m_0 x = c_0)$

(1.2.11) $(\alpha_0 - x < c_0) \wedge (m_0 x = c_0)$ ▌ $\alpha_0 - x < c_0 = m_0 x$ ▌ $\alpha_0 < m_0 x + x$ ▌ $\alpha_0 < (m_0 + 1)x$

(1.2.12) $m_0 \in \mathbb{N}^+$ ▌ $m_0 + 1 \in \mathbb{N}^+$

(1.2.13) $(m_0 + 1 \in \mathbb{N}^+) \wedge (a \in A \iff \exists_{m\in\mathbb{N}^+}(mx = a))$ ▌ $(m_0 + 1)x \in A$

(1.2.14) $(\alpha_0 < (m_0 + 1)x) \wedge ((m_0 + 1)x \in A)$ ▌ $\exists_{c\in A}(\alpha_0 < c)$

(1.2.15) $LUB[\alpha_0, A, \mathbb{R}, <]$ ▌ $UpperBound[\alpha_0, A, \mathbb{R}, <]$ ▌ $\forall_{c\in A}(c \le \alpha_0)$ ▌ $\neg\exists_{c\in A}(c > \alpha_0)$ ▌ $\neg\exists_{c\in A}(\alpha_0 < c)$

(1.2.16) $(\exists_{c\in A}(\alpha_0 < c)) \wedge (\neg\exists_{c\in A}(\alpha_0 < c))$ ▌ $\bot$

(1.3) $\neg\exists_{n\in\mathbb{N}^+}(nx > y) \implies \bot$ ▌ $\exists_{n\in\mathbb{N}^+}(nx > y)$

(2) $(x, y \in \mathbb{R} \wedge x > 0) \implies \exists_{n\in\mathbb{N}^+}(nx > y)$ ▌ $\forall_{x,y\in\mathbb{R}}(x > 0 \implies \exists_{n\in\mathbb{N}^+}(nx > y))$

$QDenseInR := \forall_{x,y\in\mathbb{R}}(x < y \implies \exists_{p\in\mathbb{Q}}(x < p < y))$

(1) $(x, y \in \mathbb{R} \wedge x < y) \implies \ldots$

(1.1) $x < y$ ▌ $(0 < y - x) \wedge (y - x \in \mathbb{R})$

(1.2) $ArchimedeanPropertyOfR \wedge (0 < y - x) \wedge (y - x, 1 \in \mathbb{R})$ ▌ $\exists_{n\in\mathbb{N}^+}(n(y - x) > 1) \ \ldots$

(1.3) $\ldots n_0 := choice(\{n \in \mathbb{N}^+ : n(y - x) > 1\})$ ▌ $(n_0 \in \mathbb{N}^+) \wedge (n_0(y - x) > 1)$

(1.4) $(n_0 \in \mathbb{N}^+) \wedge (x \in \mathbb{R})$ ▌ $n_0 x, -n_0 x \in \mathbb{R}$

(1.5) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m\in\mathbb{N}^+}(m(1) > n_0 x) \ \ldots$

(1.6) $\ldots m_1 := choice(\{m \in \mathbb{N}^+ : m(1) > n_0 x\})$ ▌ $(m_1 \in \mathbb{N}^+) \wedge (m_1 > n_0 x)$

(1.7) $ArchimedeanPropertyOfR \wedge (1 > 0) \wedge (-n_0 x, 1 \in \mathbb{R})$ ▌ $\exists_{m\in\mathbb{N}^+}(m(1) > -n_0 x) \ \ldots$

(1.8) $\ldots m_2 := choice(\{m \in \mathbb{N}^+ : m(1) > -n_0 x\})$ ▌ $(m_2 \in \mathbb{N}^+) \wedge (m_2 > -n_0 x)$

(1.9) $(m_1 > n_0 x) \wedge (m_2 > -n_0 x)$ ▌ $-m_2 < n_0 x < m_1$

(1.10) $m_1, m_2 \in \mathbb{N}^+$ ▌ $|m_1 - (-m_2)| \ge 2$

(1.11) $(-m_2 < n_0 x < m_1) \wedge (|m_1 - (-m_2)| \ge 2)$ ▌ $\exists_{m\in\mathbb{Z}}((-m_2 < m < m_1) \wedge (m - 1 \le n_0 x < m)) \ \ldots$

(1.12) $\ldots m_0 := choice(\{m \in \mathbb{Z} : (-m_2 < m < m_1) \wedge (m - 1 \le n_0 x < m)\})$ ▌ $(-m_2 < m_0 < m_1) \wedge (m_0 - 1 \le n_0 x < m_0)$

(1.13) $(n_0(y - x) > 1) \wedge (m_0 - 1 \le n_0 x < m_0)$ ▌ $n_0 x < m_0 \le 1 + n_0 x < n_0 y$ ▌ $n_0 x < m_0 < n_0 y$

(1.14) $(n_0 \in \mathbb{N}^+) \wedge (n_0 x < m_0 < n_0 y)$ ▌ $x < m_0/n_0 < y$

(1.15) $m_0, n_0 \in \mathbb{Z}$ ▌ $m_0/n_0 \in \mathbb{Q}$

(1.16) $(m_0/n_0 \in \mathbb{Q}) \wedge (x < m_0/n_0 < y)$ ▌ $\exists_{p\in\mathbb{Q}}(x < p < y)$

(2) $(x, y \in \mathbb{R} \wedge x < y) \implies \exists_{p\in\mathbb{Q}}(x < p < y)$ ▌ $\forall_{x,y\in\mathbb{R}}(x < y \implies \exists_{p\in\mathbb{Q}}(x < p < y))$

(1.21)

$RootLemma := (0 < a < b) \implies (b^n - a^n \le (b - a)nb^{n-1})$

(1) $(0 < a < b) \implies \ldots$

(1.1) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1})$

(1.2) $0 < a < b$ ▌ $b/a > 1$

(1.3) $b/a > 1$ ▌ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}(b^{n-i}a^{i-1}(b/a)^{i-1}) = \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$ ▌ $\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le \sum_{i=1}^{n}(b^{n-1}) = nb^{n-1}$

(1.4) $b^n - a^n = (b - a)\sum_{i=1}^{n}(b^{n-i}a^{i-1}) \le (b - a)nb^{n-1}$ ▌ $b^n - a^n \le (b - a)nb^{n-1}$

(2) $(0 < a < b) \implies (b^n - a^n \le (b - a)nb^{n-1})$

$RootExistenceInR := \forall_{0<x\in\mathbb{R}}\forall_{0<n\in\mathbb{Z}}\exists!_{0<y\in\mathbb{R}}({y_0}^n = x)$

(1) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \ldots$

(1.1) $E := \{t \in \mathbb{R} : t > 0 \wedge t^n < x\}$ ▌ $t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)$

(1.2) $t_0 := x/(1 + x)$ ▌ $(t_0 = x/(1 + x)) \wedge (t_0 \in \mathbb{R})$

(1.3) $0 < x$ ▌ $0 < x < 1 + x$ ▌ $t_0 = x/(1 + x) > 0$ ▌ $t_0 > 0$

(1.4) $1 = (1+x)/(1+x) > x/(1+x) = t_0$ ▮ $1 > t_0$

(1.5) $(t_0 > 0) \wedge (1 > t_0)$ ▮ $0 < t_0 < 1$

(1.6) $(0 < n \in \mathbb{Z}) \wedge (0 < t_0 < 1)$ ▮ ${t_0}^n \le t_0$

(1.7) $0 < x$ ▮ $x > x/(1+x) = t_0$ ▮ $x > t_0$

(1.8) $({t_0}^n \le t_0) \wedge (x > t_0)$ ▮ ${t_0}^n < x$

(1.9) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t_0 \in \mathbb{R}) \wedge (t_0 > 0) \wedge ({t_0}^n < x)$ ▮ $t_0 \in E$ ▮ $\emptyset \ne E$

(1.10) $t_1 := choice(\{t \in \mathbb{R} : t > 1 + x\})$ ▮ $(t_1 \in \mathbb{R}) \wedge (t_1 > 1 + x)$

(1.11) $x > 0$ ▮ $t_1 > 1 + x > 1$ ▮ $t_1 > 1$ ▮ ${t_1}^n \ge t_1$

(1.12) $({t_1}^n \ge t_1) \wedge (t_1 > 1 + x) \wedge (1 > 0)$ ▮ ${t_1}^n \ge t_1 > 1 + x > x$ ▮ ${t_1}^n > x$

(1.13) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ({t_1}^n > x)$ ▮ $t_1 \notin E$ ▮ $E \subset \mathbb{R}$

(1.14) $(\emptyset \ne E) \wedge (E \subset \mathbb{R})$ ▮ $\emptyset \ne E \subset \mathbb{R}$

(1.15) $t \in E \implies \ldots$

(1.15.1) $(t \in E) \wedge (t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x))$ ▮ $t^n < x$

(1.15.2) $({t_1}^n > x) \wedge (t^n < x)$ ▮ $t^n < x < {t_1}^n$ ▮ $t < t_1$

(1.16) $t \in E \implies t < t_1$ ▮ $\forall_{t \in E}(t \le t_1)$ ▮ $\mathit{UpperBound}[t_1, E, \mathbb{R}, <]$ ▮ $\mathit{BoundedAbove}[E, \mathbb{R}, <]$

(1.17) $\mathit{CompletenessOfR}$ ▮ $\mathit{LUBProperty}[\mathbb{R}, <]$

(1.18) $(\mathit{LUBProperty}[\mathbb{R}, <]) \wedge (\emptyset \ne E \subset \mathbb{R}) \wedge (\mathit{BoundedAbove}[E, \mathbb{R}, <])$ ▮ $\exists_{y \in \mathbb{R}}(\mathit{LUB}[y, E, \mathbb{R}, <])$ $\ldots$

(1.19) $\ldots y_0 := choice(\{y \in \mathbb{R} : \mathit{LUB}[y, E, \mathbb{R}, <]\})$ ▮ $\mathit{LUB}[y_0, E, \mathbb{R}, <]$

(1.20) $(\mathit{LUB}[y_0, E, \mathbb{R}, <]) \wedge (t_0 \in E) \wedge (t_0 > 0)$ ▮ $0 < t_0 \le y_0 \in \mathbb{R}$ ▮ $0 < y_0 \in \mathbb{R}$

(1.21) ${y_0}^n < x \implies \ldots$

(1.21.1) $k_0 := \frac{x - {y_0}^n}{n(y_0+1)^{n-1}}$ ▮ $k_0 \in \mathbb{R}$

(1.21.2) ${y_0}^n < x$ ▮ $0 < x - {y_0}^n$

(1.21.3) $(n > 0) \wedge (y_0 > 0)$ ▮ $0 < n(y_0 + 1)^{n-1}$

(1.21.4) $(0 < x - {y_0}^n) \wedge (0 < n(y_0 + 1)^{n-1})$ ▮ $0 < \frac{x - {y_0}^n}{n(y_0+1)^{n-1}} = k_0$ ▮ $0 < k_0$

(1.21.5) $(0 < 1 \in \mathbb{R}) \wedge (0 < k_0 \in \mathbb{R})$ ▮ $0 < min(1, k_0) \in \mathbb{R}$

(1.21.6) $\mathit{QDenseInR} \wedge (0, min(1, k_0) \in \mathbb{R}) \wedge (0 < min(1, k_0))$ ▮ $\exists_{h \in \mathbb{Q}}(0 < h < min(1, k_0))$ $\ldots$

(1.21.7) $\ldots h_0 := choice(\{h \in \mathbb{Q} : 0 < h < min(1, k_0)\})$ ▮ $(0 < h_0 < 1) \wedge (h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}})$

(1.21.8) $(y_0 > 0) \wedge (h_0 > 0)$ ▮ $0 < y_0 < y_0 + h_0$

(1.21.9) $\mathit{RootLemma} \wedge (0 < y_0 < y_0 + h_0)$ ▮ $(y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}$

(1.21.10) $h_0 < 1$ ▮ $h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1}$

(1.21.11) $((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + h_0)^{n-1}) \wedge (h_0 n(y_0 + h_0)^{n-1} < h_0 n(y_0 + 1)^{n-1})$ ▮ $(y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}$

(1.21.12) $(0 < n(y_0 + 1)^{n-1}) \wedge (h_0 < k_0 = \frac{x - {y_0}^n}{n(y_0+1)^{n-1}})$ ▮ $h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n$

(1.21.13) $((y_0 + h_0)^n - {y_0}^n < h_0 n(y_0 + 1)^{n-1}) \wedge (h_0 n(y_0 + 1)^{n-1} < x - {y_0}^n)$ ▮ $(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n$ ▮ $(y_0 + h_0)^n < x$

(1.21.14) $(y_0 + h_0)^n - {y_0}^n < x - {y_0}^n$ ▮ $(y_0 + h_0)^n < x$

(1.21.15) $(0 < y_0 \mathbb{R}) \wedge (0 < h_0 < \mathbb{R})$ ▮ $0 < y_0 < y_0 + h_0 \in \mathbb{R}$

(1.21.16) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge ((y_0 + h_0)^n < x) \wedge (0 < y_0 + h_0 \in \mathbb{R})$ ▮ $(y_0 + h_0)^n \in E$

(1.21.17) $((y_0 + h_0)^n \in E) \wedge (y_0 < y_0 + h_0)$ ▮ $\exists_{e \in E}(y_0 < e)$

(1.21.18) $\mathit{LUB}[y_0, E, \mathbb{R}, <]$ ▮ $\mathit{UpperBound}[y_0, E, \mathbb{R}, <]$ ▮ $\forall_{e \in E}(e \le y_0)$ ▮ $\neg\exists_{e \in E}(e > y_0)$

(1.21.19) $(\exists_{e \in E}(e > y_0)) \wedge (\neg\exists_{e \in E}(e > y_0))$ ▮ $\bot$

(1.22) ${y_0}^n < x \implies \bot$ ▮ ${y_0}^n \ge x$

(1.23) ${y_0}^n > x \implies \ldots$

(1.23.1) $k_1 := \frac{{y_0}^n - x}{n{y_0}^{n-1}}$ ▮ $(k_1 \in \mathbb{R}) \wedge (k_1 n{y_0}^{n-1} = {y_0}^n - x)$

(1.23.2) $(0 < x) \wedge (0 < n \in \mathbb{Z})$ ▮ ${y_0}^n - x < {y_0}^n \le n{y_0}^n$ ▮ ${y_0}^n - x < n{y_0}^n$

(1.23.3) ${y_0}^n - x < n{y_0}^n$ ▮ $k_1 = \frac{{y_0}^n - x}{n{y_0}^{n-1}} < \frac{n{y_0}^n}{n{y_0}^{n-1}} = y_0$ ▮ $k_1 < y_0$

(1.23.4) ${y_0}^n > x$ ▮ $0 < {y_0}^n - x$

(1.23.5) $(n > 0) \wedge (y_0 > 0)$ ▮ $0 < n{y_0}^{n-1}$

(1.23.6) $(0 < {y_0}^n - x) \wedge 0 < (n{y_0}^{n-1})$ ▮ $0 < \frac{{y_0}^n - x}{n{y_0}^{n-1}} = k_1$ ▮ $0 < k_1$

(1.23.7) $(k_1 < y_0) \wedge (0 < k_1) \blacksquare (0 < k_1 < y_0) \wedge (0 < y_0 - k_1 < y_0)$

(1.23.8) $t \geq y_0 - k_1 \implies \ldots$

(1.23.8.1) $t \geq y_0 - k_1 \blacksquare t^n \geq (y_0 - k_1)^n \blacksquare -t^n \leq -(y_0 - k_1)^n \blacksquare {y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n$

(1.23.8.2) $\mathit{RootLemma} \wedge (0 < y_0 - k_1 < y_0) \blacksquare {y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}$

(1.23.8.3) $({y_0}^n - t^n \leq {y_0}^n - (y_0 - k_1)^n) \wedge ({y_0}^n - (y_0 - k_1)^n < k_1 n {y_0}^{n-1}) \blacksquare {y_0}^n - t^n < k_1 n {y_0}^{n-1}$

(1.23.8.4) $(k_1 n {y_0}^{n-1} = {y_0}^n - x) \wedge ({y_0}^n - t^n < k_1 n {y_0}^{n-1}) \blacksquare {y_0}^n - t^n < {y_0}^n - x \blacksquare -t^n < -x \blacksquare t^n > x$

(1.23.8.5) $(t \in E \iff (t \in \mathbb{R} \wedge t > 0 \wedge t^n < x)) \wedge (t^n > x) \blacksquare t \notin E$

(1.23.9) $t \geq y_0 - k_1 \implies t \notin E \blacksquare t \in E \implies t < y_0 - k_1 \blacksquare \forall_{t \in E}(t \leq y_0 - k_1) \blacksquare \mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.10) $(\mathit{LUB}[y_0, E, \mathbb{R}, <] \wedge (y_0 - k_1 < y_0)) \blacksquare \neg \mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]$

(1.23.11) $(\mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \wedge (\neg \mathit{UpperBound}[y_0 - k_1, E, \mathbb{R}, <]) \blacksquare \perp$

(1.24) ${y_0}^n > x \implies \perp \blacksquare {y_0}^n \leq x$

(1.25) $\mathit{Order}[\mathbb{R}, <] \blacksquare \mathit{OrderTrichotomy}[\mathbb{R}, <]$

(1.26) $(\mathit{OrderTrichotomy}[\mathbb{R}, <]) \wedge ({y_0}^n \geq x) \wedge ({y_0}^n \leq x) \blacksquare {y_0}^n = x$

(1.27) $({y_0}^n = x) \wedge (y_0 \in \mathbb{R}) \blacksquare \exists_{y \in \mathbb{R}}(y^n = x)$

(1.28) $y_1, y_2 := \mathit{choice}(\{y \in \mathbb{R} : y^n = x\})$

(1.29) $y_1 \neq y_2 \implies \ldots$

(1.29.1) $(\mathit{OrderTrichotomy}[\mathbb{R}, <]) \wedge (y_1 \neq y_2) \blacksquare (y_1 < y_2) \veebar (y_2 < y_1) \;...$

(1.29.2) $...\,(x = {y_1}^n < {y_2}^n = x) \veebar (x = {y_2}^n < {y_1}^n = x) \blacksquare (x < x) \veebar (x > x) \blacksquare \perp \veebar \perp \blacksquare \perp$

(1.30) $y_1 \neq y_2 \implies \perp \blacksquare y_1 = y_2 \blacksquare \forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b)$

(1.31) $(\exists_{y \in \mathbb{R}}(y^n = x)) \wedge (\forall_{a,b \in \mathbb{R}}((a^n = x \wedge b^n = x) \implies a = b)) \blacksquare \exists!_{y \in \mathbb{R}}(y^n = x)$

(2) $(0 < x \in \mathbb{R} \wedge 0 < n \in \mathbb{Z}) \implies \exists!_{y \in \mathbb{R}}(y^n = x) \blacksquare \forall_{0 < x \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}} \exists!_{0 < y \in \mathbb{R}}({y_0}^n = x)$

$\mathit{RootExistenceInRCorollary} := \forall_{0 < a \in \mathbb{R}} \forall_{0 < b \in \mathbb{R}} \forall_{0 < n \in \mathbb{Z}}((ab)^{1/n} = a^{1/n} b^{1/n})$ —

$$\mathit{ExtendedRealSystem}[\bar{\mathbb{R}}, +, *, <] := \left(\begin{array}{l} \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\} \quad \wedge \quad -\infty < x < \infty \quad \wedge \\ x + \infty = +\infty \ \wedge \ x - \infty = -\infty \ \wedge \ \frac{x}{+\infty} = \frac{x}{-\infty} = 0 \wedge \\ (x > 0) \implies (x * (+\infty) = +\infty \wedge x * (-\infty) = -\infty) \wedge \\ (x < 0) \implies (x * (+\infty) = -\infty \wedge x * (-\infty) = +\infty) \end{array} \right)$$

$\mathbb{C} := \{\langle a, b \rangle \in \mathbb{R} \times \mathbb{R}\}$
$+_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a +_{\mathbb{R}} c, b +_{\mathbb{R}} d \rangle$
$*_{\mathbb{C}}[\langle a, b \rangle, \langle c, d \rangle] := \langle a *_{\mathbb{R}} c - b *_{\mathbb{R}} d, a *_{\mathbb{R}} d + b *_{\mathbb{R}} c \rangle$

$\mathit{FieldC} := \mathit{Field}[\mathbb{C}, +_{\mathbb{C}}, *_{\mathbb{C}}]$ —
$\mathit{RSubfieldC} := \mathit{Subfield}[\mathbb{R}, \mathbb{C}, +, *]$ —

$i := \langle 0, 1 \rangle \in \mathbb{C}$

$\mathit{iProperty} := i^2 = -1$ —
$\mathit{CProperty} := (a, b \in \mathbb{R}) \implies (\langle a, b \rangle = a + bi)$ —

$\mathit{Conjugate}[\overline{a + bi}] := a - bi$

$\mathit{ConjugateProperties} := (w, z \in \mathbb{C}) \implies \ldots$ —

(1) $\overline{z + w} = \overline{z} + \overline{w}$

(2) $\overline{z * w} = \overline{z} * \overline{w}$

(3) $Re(z) = (1/2)(z + \overline{z}) \wedge Im(z) = (1/2)(z - \overline{z})$

(4) $0 \leq z * \overline{z} \in \mathbb{R}$

$\mathit{AbsoluteValueC}[|z|] = (z * \overline{z})^{1/2}$
$\mathit{AbsoluteValueProperties} := (z, w \in \mathbb{C}) \implies \ldots$ —

(1) 123123

TODO: - MORE EXPLICIT MODUS PONENS ON OrderTrichotomyR ??? - name all properties - hyperlink all definitions ???

Chapter 2

Abstract Algebra

$Relation(f, X) := f \subseteq X$
$Function(f, X, Y) := X \neq \emptyset \neq Y \wedge Relation(f, X \times Y) \wedge \forall_{x \in X} \exists!_{y \in Y}((x, y) \in f)$

$(Function(f, X, Y) \wedge A \subseteq X \wedge B \subseteq Y) \implies \dots$

(1)	$Domain(f) := X; Codomain(f) := Y$
(2)	$Image(f, A) := \{f(a) : a \in A\}; Preimage(f, B) := \{a : f(a) \in B\}$
(3)	$Range(f) := Image(Domain(f))$

$Injective(f, X, Y) := Function(f, X, Y) \wedge \forall_{x_1, x_2 \in X}(x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$
$Surjective(f, X, Y) := Function(f, X, Y) \wedge \forall_{y \in Y} \exists_{x \in X}(y_0 = f(x))$
$Bijective(f, X, Y) := Injective(f, X, Y) \wedge Surjective(f, X, Y)$
$SurjectiveEquivalent := (Range(f) = Codomain(f)) \implies Surjective(f)$

$(Function(f, X, Y) \wedge Function(g, Y, Z)) \implies (f \circ g)(x) := f(g(x)); Function(f \circ g, X, Z)$

$Propertiesof Functions := (Function(f, A, B) \wedge Function(g, B, C) \wedge Function(h, C, D)) \implies \dots$

(1)	$h \circ (g \circ f) = (h \circ g) \circ f$
(2)	$(Injective(f) \wedge Injective(g)) \implies Injective(g \circ f)$
(3)	$(Surjective(f) \wedge Surjective(g)) \implies Surjective(g \circ f)$
(4)	$(Bijective(f, A, B)) \implies \exists_{f^{-1}}(Function(f^{-}1, B, A) \wedge \forall_{a \in A}(f^{-1}(f(a)) = a) \wedge \forall_{b \in B}(f(f^{-1}(b)) = b))$

$|(a, b) := a, b \in \mathbb{Z} \wedge a \neq 0 \wedge \exists_{c \in \mathbb{Z}}(b = ac)$

$DivisibilityTheorems := (a, b, c, m, x, y \in \mathbb{Z}) \implies \dots$

(1)	$(a \vert b) \implies a \vert bc$
(2)	$(a \vert b \wedge b \vert c) \implies a \vert c$
(3)	$(a \vert b \wedge b \vert c) \implies a \vert (bx + cy)$
(4)	$(a \vert b \wedge b \vert a) \implies a = \pm b$
(5)	$(a \vert b \wedge a > 0 \wedge b > 0) \implies (a \leq b)$
(6)	$(a \vert b) \iff (m \neq 0 \wedge ma \vert mb)$

$DivisionAlgorithm := (a, b \in \mathbb{Z} \wedge a > 0) \implies \exists!_{q, r \in \mathbb{Z}}(b = aq + r)$

$CD(a, b, c) := a, b, c \in \mathbb{Z} \wedge a : b \wedge a : c$
$GCD(a, b, c) := CD(a, b, c) \wedge \forall_d((d : b \wedge d : c) \implies d : a)$
$GCDEquivalent := 123123$

Chapter 3

Linear Algebra

3.1 Matrix Operations and Special Matrices

$Matrix[A, m, n] := [a_{i,j}]_{m\times n}$:= m rows, n columns of real numbers
$\mathcal{M}_{m,n} := \{A : Matrix[A, m, n]\}$
$O_{m,n} := (Matrix[O, m, n]) \wedge (a_{i,j} = 0)$
$Square[A, n] := Matrix[A, n, n]$
$UpperTriangular[A] := (Square[A]) \wedge (i > j \implies a_{i,j} = 0)$
$LowerTriangular[A] := (Square[A]) \wedge (i < j \implies a_{i,j} = 0)$
$Diagonal[A, n] := (Square[A, n]) \wedge (i \neq j \implies a_{i,j} = 0)$
$Scalar[A, n, k] := (Diagonal[A, n]) \wedge (a_{i,i} = k)$
$I_n := Scalar[I, n, 1]$

$+(A, B) := ((Matrix[A, m, n]) \wedge (Matrix[B, m, n])) \implies (A + B = [a_{i,j} + b_{i,j}]_{m\times n})$
$*\,(r, A) := ((r \in \mathbb{R}) \wedge (Matrix[A, m, n])) \implies (r * A = [ra_{i,j}]_{m\times n})$
$*\,(A, B) := ((Matrix[A, m, p]) \wedge (Matrix[B, p, n])) \implies (A * B = \left[\sum_{k=1}^{p}(a_{i,k}b_{k,j})\right]_{m\times n})$
${}^{T}[A] := (Matrix[A, m, n]) \implies (A^T = [a_{j,i}]_{n\times m})$

$AddCom := \forall_{A,B\in\mathcal{M}}(A + B = B + A)$

(1) $A + B = [a_{i,j} + b_{i,j}] = [b_{i,j} + a_{i,j}] = B + A$

$AddAssoc := \forall_{A,B,C\in\mathcal{M}}((A + B) + C = A + (B + C))$

(1) $(A + B) + C = [(a_{i,j} + b_{i,j}) + c_{i,j}] = [a_{i,j} + (b_{i,j} + c_{i,j})] = A + (B + C)$

$AddId := \forall_{A\in\mathcal{M}}\exists!_{O\in\mathcal{M}}(A + O = A = O + A)$

(1) $A + O = [a_{i,j} + 0] = A = [0 + a_{i,j}] = O + A$
(2) $A + O_1 = A = A + O_2 \;\blacksquare\; O_1 = O_2$

$AddInv := \forall_{A\in\mathcal{M}}\exists!_{(-A)\in\mathcal{M}}(A + (-A) = O = (-A) + A)$

(1) $A + (-A) = [a_{i,j} - a_{i,j}] = O = [-a_{i,j} + a_{i,j}] = (-A) + A$
(2) $A + (-A_1) = O = A + (-A_2) \;\blacksquare\; -A_1 = -A_2 \;\blacksquare\; A_1 = A_2$

$MulAssoc := \forall_{A,B,C\in\mathcal{M}}((A * B) * C = A * (B * C))$

(1) $(A * B) * C = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,j})\right] * C = \left[\sum_{k_2=1}^{p_2}(\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2})c_{k_2,j})\right] = \left[\sum_{k_2=1}^{p_2}\sum_{k_1=1}^{p_1}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right] = \ldots$
(2) $\ldots\left[\sum_{k_1=1}^{p_1}\sum_{k_2=1}^{p_2}(a_{i,k_1}b_{k_1,k_2}c_{k_2,j})\right] = \left[\sum_{k_1=1}^{p_1}(a_{i,k_1}\sum_{k_2=1}^{p_2}(b_{k_1,k_2}c_{k_2,j}))\right] = \ldots = A * (B * C)$

$MulId := \forall_{A:Square[A,n]}(A * I_n = A = I_n * A)$

(1) $A * I_n = \left[\sum_{k=1}^{n} \left(a_{i,k} \left(\begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \right) \right) \right] = [a_{i,j}] = A$

(2) $TODO = A$

$ScalAssoc := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} (r(sA) = (rs)A = s(rA))$

(1) $r(sA) = r[sa_{i,j}] = [rsa_{i,j}]$

(2) $(rs)A = [rsa_{i,j}]$

(3) $s(rA) = s[ra_{i,j}] = [sra_{i,j}] = [rsa_{i,j}]$

$TransCancel := \forall_{A \in \mathcal{M}} (A = (A^T)^T)$

(1) $A = [a_{i,j}] = [a_{j,i}]^T = ([a_{i,j}]^T)^T = (A^T)^T$

$ScalMulCom := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} ((rA) * B = r(A * B) = A * (rB))$

(1) $(rA) * B = [ra_{i,l}] * [b_{l,j}] = \left[\sum_{k=1}^{p} (ra_{i,k} b_{k,j}) \right] = r(A * B)$

(2) $A * (rB) = [a_{i,l}] * [rb_{l,j}] = \left[\sum_{k=1}^{p} (a_{i,k} rb_{k,j}) \right] = \left[\sum_{k=1}^{p} (ra_{i,k} b_{k,j}) \right] = r(A * B)$

$ScalDistLeft := \forall_{r,s \in \mathbb{R}} \forall_{A \in \mathcal{M}} ((r + s)A = rA + sA)$

(1) TODO

$ScalDistRight := \forall_{r \in \mathbb{R}} \forall_{A,B \in \mathcal{M}} (r(A + B) = rA + rB)$

(1) TODO

$MulDistRight := \forall_{A,B,C \in \mathcal{M}} ((A + B) * C = A * C + B * C)$

(1) $(A + B) * C = [a_{i,j} + b_{i,j}] * C = \left[\sum_{k=1}^{p} ((a_{i,k} + b_{i,k}) c_{k,j}) \right] = \ \ldots$

(2) $\ldots \left[\sum_{k=1}^{p} (a_{i,k} c_{k,j} + b_{i,k} c_{k,j}) \right] = \left[\sum_{k=1}^{p} (a_{i,k} c_{k,j}) \right] + \left[\sum_{k=1}^{p} (b_{i,k} c_{k,j}) \right] = A * C + B * C$

$MulDistLeft := \forall_{A,B,C \in \mathcal{M}} (C * (A + B) = C * A + C * B)$

(1) TODO

$TransAddDist := \forall_{A,B \in \mathcal{M}} ((A + B)^T = A^T + B^T)$

(1) TODO

$TransMulDist := \forall_{A,B \in \mathcal{M}} ((A * B)^T = B^T * A^T)$

(1) $(A * B)^T = \left[\sum_{k=1}^{p} (a_{i,k} b_{k,j}) \right]^T = \left[\sum_{k=1}^{p} (a_{j,k} b_{k,i}) \right] = \left[\sum_{k=1}^{p} (b_{k,i} a_{j,k}) \right] = \left[\sum_{k=1}^{p} (b_{i,k}^T a_{k,j}^T) \right] = B^T * A^T$

$Sym[A] := A = A^T$
$SkewSym[A] := A = -A^T$
$Invertible[A] := (Square[A, n]) \wedge (\exists_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A))$

$SymGen := \forall_{A \in \mathcal{M}} (Sym[A + A^T])$

(1) $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$

$SkewSymGen := \forall_{A \in \mathcal{M}} (SkewSym[A - A^T])$

(1) $-(A - A^T)^T = -(A^T - (A^T)^T) = -(A^T - A) = (A - A^T)$

$SymDecomp := \forall_{A \in \mathcal{M}} \exists!_{B:Sym[B]} \exists!_{C:SkewSym[C]} (A = B + C)$

(1) $B := (1/2) * (A + A^T)$; $C := (1/2) * (A - A^T)$

(2) $SymGen[B] \wedge SkewSymGen[C]$

(3) $A = (1/2) * (A + A^T) + (1/2) * (A - A^T) = B + C$

(4) $(1/2) * (A_1 + {A_1}^T) = (1/2) * (A_2 + {A_2}^T)$ ▮ $A_1 = A_2$

(5) $(1/2) * (A_3 - {A_3}^T) = (1/2) * (A_4 - {A_4}^T)$ ▮ $A_3 = A_4$

$InvId := \forall_{A:Invertible[A]} (\exists!_{A^{-1} \in \mathcal{M}} (A * A^{-1} = I_n = A^{-1} * A))$

(1) ${A^{-1}}_1 = {A^{-1}}_1 * I_n = {A^{-1}}_1 * (A * {A^{-1}}_2) = ({A^{-1}}_1 * A) * {A^{-1}}_2 = I_n * {A^{-1}}_2 = {A^{-1}}_2$

$InvCancel := \forall_{A:Invertible[A]} ((A^{-1})^{-1} = A)$

(1) $(A * A^{-1})^{-1} = {I_n}^{-1} = I_n$

(2) $(A^{-1})^{-1} * A^{-1} = I_n$ ▮ $A^{-1})^{-1} = I_n * A = A$

$InvDist := \forall_{A:Invertible[A]} \forall_{B:Invertible[B]} ((A * B)^{-1} = B^{-1} * A^{-1})$

(1) $(A * B) * (A * B)^{-1} = I$ ▮ $B * (A * B)^{-1} = A^{-1}$ ▮ $(A * B)^{-1} = B^{-1} * A^{-1}$

$InvTrans := \forall_{A:Invertible[A]} ((A^T)^{-1} = (A^{-1})^T)$ ▮ $\Leftarrow$

(1) $A^T * (A^{-1})^T = (A^{-1} * A)^T = I^T = I$ ▮ $(A^{-1})^T = (A^T)^{-1}$

3.2 Elementary Matrices on Invertibility and Systems of Linear Equations

$Sys[A, B] := (Matrix[A, m, n]) \wedge (Matrix[B, m, 1])$
$Sol[X, A, B] := (Sys[A, B]) \wedge (Matrix[X, n, 1]) \wedge (A * X = B)$
$ConsistentSys[A, B] := (Sys[A, B]) \wedge \exists_X (Sol[X, A, B])$
$TrivSol[X, A] := (Sol[X, A, O]) \wedge (X = O)$
$NonTrivSol[X, A] := (Sol[X, A, O]) \wedge (X \neq O)$

$HomoSysProps := (Sys[A, O]) \implies \ldots$

(1) $u_0 := O$; $u_1 := choice(\{X \in \mathcal{M} | X \neq O\})$; $k := choice(\mathbb{R})$

(2) $TrivSol[u_0, A]$

(3) $(NonTrivSol[u_1, A]) \implies (Sol[u_1 + ku_0])$

(4) $(TrivSol[\overrightarrow{X}, A]) \implies (TrivSol[LC(\overrightarrow{X}), A])$

$ElemMat[E] := (E = Swap[I_n, i, j]) \vee (Scale_*(I_n, i, c)) \vee (Combine_*(I_n, i, c, j))$
$ElemMatProd[E^*] := \exists_{\langle E \rangle} (\forall_{E_i \in E^*} (ElemMat[E_i]) \wedge (E^* = \Pi_{E_i \in E^*} (E_i)))$
$RowEquiv[A, B] := \exists_{E^*} ((ElemMatProd[E^*]) \wedge (B = E^* * A))$

$ElemMatInv := \forall_{E \in \mathcal{M}} ((ElemMat[E]) \implies (Invertible[E]))$

(1) $E - RowSwap[E] \implies TODO$; $E - RowScale_*(E) \implies TODO$; $E - RowCombine_*(E) \implies TODO$

$ElemMatProdInv := \forall_{E^*} ((ElemMatProd[E^*]) \implies (Invertible[E^*]))$

(1) TODO

$RowEquivSys := \forall_{A,B,C,D,X \in \mathcal{M}} (((Sys[A, B]) \wedge (Sys[C, D]) \wedge (RowEquiv[[AB], [CD]])) \implies (Sol[X, A, B] \iff Sol[X, C, D]))$

(1) $\exists_{E^*:ElemMatProd[E^*]} ([CD] = E^* * [AB])$

(2) $(E^* * A = C) \wedge (E^* * B = D)$

(3) $Sol[Y, A, B] \implies \ldots$

(3.1) $A * Y = B$

(3.2) $C * Y = (E^* * A) * Y = E^* * (A * Y) = E^* * B = D \blacksquare Sol[Y, C, D]$

(4) $Sol[Y, A, B] \implies Sol[Y, C, D]$

(5) $(A = (E^*)^{-1} * C) \wedge (B = (E^*)^{-1} * D)$

(6) $Sol[Z, C, D] \implies \ldots$

(6.1) $C * Z = D$

(6.2) $A * Z = ((E^*)^{-1} * C) * Z = (E^*)^{-1} * (C * Z) = (E^*)^{-1} * D = B$

(7) $Sol[Z, C, D] \implies Sol[Z, A, B]$

(8) $Sol[X, A, B] \iff Sol[X, C, D]$

$RowEquivHomoSysSol := \forall_{A,C,X \in \mathcal{M}}((RowEquiv[A, C]) \implies ((Sol[X, A, O]) \iff (Sol[X, C, O])))$

(1) Set $B = D = O$

$$RREF[A] := (A \in \mathcal{M}) \wedge \left(\begin{array}{l} \text{All zero rows are at the bottom of the matrix.} \wedge \\ \text{The leading entry after the first occurs to the right of the leading entry of the previous row.} \wedge \\ \text{The leading entry in any nonzero row is 1.} \wedge \\ \text{All entries in the column above and below a leading 1 are zero.} \wedge \end{array} \right)$$

$GaussJordanElim := \forall_{A \in \mathcal{M}} \exists!_{B \in \mathcal{M}}((RREF[B]) \wedge (RowEquiv[A, B]))$

(1) Hit A with $ElemMat$'s until it becomes B

(2) $(B = E^* * A) \wedge (RREF[B])$

$HasZero[A] := (Matrix(A, m, n)) \wedge (\exists_{i \leq m}(A_{i,:} = O))$

$HasZeroNonInvertible := \forall_{A \in \mathcal{M}}((HasZero[A]) \implies (\neg Invertible[A]))$

(1) $i := choice(\{i \leq m | A_{i,:} = O\})$

(2) $(B \in \mathcal{M}) \implies \ldots$

(2.1) $(A * B)_{i,:} = O \neq I_{ni,:} \blacksquare A * B \neq I_n$

(3) $(B \in \mathcal{M}) \implies (A * B \neq I_n) \blacksquare \forall_{B \in \mathcal{M}}(A * B \neq I_n) \blacksquare \neg Invertible[A]$

$InvIffRowEquivI := \forall_{A \in \mathcal{M}}((Invertible[A]) \iff (RowEquiv[A, I_n]))$

(1) $(Invertible[A]) \implies \ldots$

(1.1) $(RREF[B]) \wedge (RowEquiv[A, B])$

(1.2) $B = E^* * A$

(1.3) $(Invertible[E^*]) \wedge (Invertible[A]) \blacksquare Invertible[B]$

(1.4) $Invertible[B] \blacksquare \neg HasZero[B]$

(1.5) $(RREF[B]) \wedge (\neg HasZero[B]) \blacksquare B = I_n$

(1.6) $RowEquiv[A, I_n]$

(2) $(Invertible[A]) \implies (RowEquiv[A, I_n])$

(3) $(RowEquiv[A, I_n]) \implies \ldots$

(3.1) $I_n = E^* * A \blacksquare (E^*)^{-1} = A$

(3.2) $A^{-1} = E^*_{DescSort} \blacksquare Invertible[A]$

(4) $(RowEquiv[A, I_n]) \implies (Invertible[A])$

(5) $(Invertible[A]) \iff (RowEquiv[A, I_n])$

$RowEquivIIffTrivSol := \forall_{A \in \mathcal{M}}((RowEquiv[A, I_n]) \iff (\forall_X((X = O) \iff (Sol[X, A, O]))))$

(1) $(RowEquiv[A, I_n]) \implies \ldots$

(1.1) $RowEquiv[A, I_n] \blacksquare Invertible[A]$

(1.2) $(Sol[X, A, O]) \implies \ldots$

(1.2.1) $A * X = O \blacksquare X = A^{-1} * O = O \blacksquare X = O$

(1.3) $(Sol[X, A, O]) \implies (X = O)$

(1.4) $(X = O) \implies (Sol[X, A, O])$

(1.5) $(X = O) \iff (Sol[X, A, O]) \blacksquare \forall_X((X = O) \iff (Sol[X, A, O]))$

(2) $(RowEquiv[A, I_n]) \implies (\forall_X((X = O) \iff (Sol[X, A, O])))$

(3) $(\forall_X((X = O) \iff (Sol[X, A, O]))) \implies \dots$

(3.1) $(RREF[B]) \wedge (RowEquiv[A, B])$

(3.2) $Sol[X, B, O]$

(3.3) $(B \neq I_n) \implies \dots$

(3.3.1) $(\exists_{Y \neq X}(Sol[Y, B, O]))$

(3.3.2) $Sol[Y, A, O] \blacksquare Y = X$

(3.3.3) $(Y \neq X) \wedge (Y = X) \blacksquare \perp$

(3.4) $(B \neq I_n) \implies \perp \blacksquare B = I_n$

(3.5) $(RowEquiv[A, B]) \wedge (B = I_n) \blacksquare RowEquiv[A, I_n]$

(4) $(\forall_X((X = O) \iff (Sol[X, A, O]))) \implies (RowEquiv[A, I_n])$

(5) $(RowEquiv[A, I_n]) \iff (\forall_X((X = O) \iff (Sol[X, A, O])))$

$InvIffUniqSol := \forall_{A \in \mathcal{M}}((Invertible[A]) \iff (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])))$

(1) $(Invertible[A] \wedge B \in \mathcal{M}) \implies \dots$

(1.1) $(Invertible[A]) \wedge (Sys[A, B])$

(1.2) $(X = A^{-1} * B) \iff (Sol[X, A, B]) \blacksquare \exists!_{X \in \mathcal{M}}(Sol[X, A, B])$

(2) $(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies \dots$

(2.1) $X_i := choice(\{X_i | Sol[X_i, A, I_{n:,i}]\})$

(2.2) $A * [X_1 \dots X_n] = [(A * X_1) \dots (A * X_n)] = [I_{n:,1} \dots I_{n:,n}] = I_n$

(2.3) $A^{-1} = [X_1 \dots X_n]$

(3) $(\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \implies (Invertible[A])$

$$SquareTheorems_4 := \forall_{A \in \mathcal{M}} \begin{pmatrix} (Invertible[A]) \iff \\ (RowEquiv[A, I_n]) \iff \\ (\forall_X((X = O) \iff (Sol[X, A, O]))) \iff \\ (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \end{pmatrix}$$

3.3 Vector Spaces

$$VectorSpace[V, +, *] := \exists_{O \in V} \forall_{\alpha, \beta \in \mathbb{R}} \forall_{u, v, w \in V} \begin{pmatrix} (u + v \in V) \wedge (u + v = v + u) \wedge ((u + v) + w = u + (v + w)) \wedge \\ (u + O = u) \wedge (\exists_{-u \in V}(u + (-u) = O)) \wedge \\ (\alpha * u \in V) \wedge (\alpha * (\beta * u) = (\alpha\beta) * u) \wedge (1 * u = u) \wedge \\ (\alpha * (u + v) = (\alpha * u) + (\alpha * v)) \wedge ((\alpha + \beta) * u = (\alpha * u) + (\beta * u)) \end{pmatrix}$$

$ZeroVectorUniq := \forall_{O', v \in V}((v + O' = v) \implies (O' = O))$

(1) $O' = O' + O = O + O' = O \blacksquare O' = O$

$AddInvUnique := \forall_{-v', v \in V}((v + -v' = O) \implies (-v' = -v))$

(1) $-v' = -v' + O = -v' + (v + -v) = (-v' + v) + -v = (v + -v') + -v = O + -v = -v \blacksquare -v' = -v$

$AddInvGen := \forall_{v \in V}((-1) * v = -v)$

(1) $v + (-1) * v = (1 - 1) * v = 0 * v = O \blacksquare (-1) * v = -v$

$ZeroVectorGenLeft := \forall_{v \in V}(0 * v = O)$

(1) $0 * v = (0 + 0) * v = (0 * v) + (0 * v) \blacksquare O = 0 * v$

$ZeroVectorGenRight := \forall_{r \in \mathbb{R}}(r * O = O)$

(1) $r * O = r * (O + O) = (r * O) + (r * O) \blacksquare O = r * O$

$ZeroVectorEquiv := \forall_{r\in\mathbb{R}}\forall_{v\in V}((r*v=O) \iff ((v=O)\vee(r=0)))$

(1) $(ZeroVectorGenLeft)\wedge(ZeroVectorGenRight) \blacksquare ((v=O)\vee(r=0)) \implies (r*v=O))$

(2) $(r*v=O) \implies \ldots$

(2.1) $(r\neq 0) \implies \ldots$

(2.1.1) $r\neq 0 \blacksquare r^{-1}\in\mathbb{R}$

(2.1.2) $ZeroVectorGenRight \blacksquare O=r^{-1}*O=r^{-1}*(r*v)=(r^{-1}r)*v=1*v=v \blacksquare O=v$

(2.2) $(r\neq 0) \implies (v=O) \blacksquare (r=0)\vee(r\neq 0) \blacksquare (r=0)\vee(v=O)$

(3) $(r*v=O) \implies ((r=0)\vee(v=O))$

(4) $(r*v=O) \iff ((r=0)\vee(v=O))$

3.4 Subspaces and Special Subspaces

$Subspace[S,V,+,*] := (VectorSpace[V,+,*])\wedge(\emptyset\neq S\subseteq V)\wedge(VectorSpace[S,+,*])$

$$SubspaceEquiv := \forall_{V,S}\left(\begin{array}{ll}(VectorSpace[V,+,*]) & \implies \\ ((Subspace[S,V,+,*]) \iff ((\emptyset\neq S\subseteq V)\wedge(\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S)))) & \end{array}\right)$$

(1) $(Subspace[S,V,+,*]) \implies \ldots$

(1.1) $Subspace[S,V,+,*] \blacksquare \emptyset\neq S\subseteq V$

(1.2) $VectorSpace[S,+,*] \blacksquare (\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))$

(1.3) $(\emptyset\neq S\subseteq V)\wedge(\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))$

(2) $(Subspace[S,V,+,*]) \implies ((\emptyset\neq S\subseteq V)\wedge(\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S)))$

(3) $((\emptyset\neq S\subseteq V)\wedge(\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))) \implies \ldots$

(3.1) $((\alpha,\beta\in\mathbb{R})\wedge(\emptyset\neq S)\wedge(u,v,w\in S)) \implies \ldots$

(3.1.1) $u,v\in V \blacksquare u+v=v+u$

(3.1.2) $u,v,w\in V \blacksquare (u+v)+w=u+(v+w)$

(3.1.3) $(ZeroVectorGenLeft)\wedge(u\in S) \blacksquare 0*u=O\in S$

(3.1.4) $u\in V \blacksquare u+O=u$

(3.1.5) $(AddInvGen)\wedge(u\in S) \blacksquare (-1)*u=-u\in S$

(3.1.6) $u\in V \blacksquare \alpha*(\beta*u)=(\alpha\beta)*u$

(3.1.7) $u\in V \blacksquare 1*u=u$

(3.1.8) $u,v\in V \blacksquare \alpha*(u+v)=(\alpha*u)+(\alpha*v)$

(3.1.9) $u\in V \blacksquare (\alpha+\beta)*u=(\alpha*u)+(\beta*u)$

(4) $((\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S))) \implies (Subspace[S,V,+,*])$

(5) $(Subspace[S,V,+,*]) \iff ((\forall_{r,s\in S}(r+s\in S))\wedge(\forall_{\alpha\in\mathbb{R}}\forall_{s\in S}(\alpha*s\in S)))$

$$SumSubContains := \forall_{A,B,V}\left(\begin{array}{l}((Subspace[A,V,+,*])\wedge(Subspace[B,V,+,*])\wedge(SetSum[A+B,A,B,V,+,*])) \implies \\ ((Subspace[A+B,V,+,*])\wedge(A,B\subseteq A+B))\end{array}\right)$$

(1) $(Subspace[A,V,+,*])\wedge(Subspace[B,V,+,*]) \blacksquare (O\in A)\wedge(O\in B)$

(2) $(SetSum[A+B,A,B,V,+,*])\wedge(O\in A)\wedge(O\in B) \blacksquare O\in A+B \blacksquare \emptyset\neq A+B$

(3) $(Subspace[A,V,+,*])\wedge(Subspace[B,V,+,*]) \blacksquare A+B\subseteq V \blacksquare \emptyset\neq A+B\subseteq V$

(4) $(u,v\in A+B) \implies \ldots$

(4.1) $(\exists_{a_1\in A}\exists_{b_1\in B}(u=a_1+b_1))\wedge(\exists_{a_2\in A}\exists_{b_2\in B}(v=a_2+b_2))$

(4.2) $u+v=(a_1+b_1)+(a_2+b_2)=(a_1+a_2)+(b_1+b_2)$

(4.3) $(a_1+a_2\in A)\wedge(b_1+b_2\in B) \blacksquare u+v\in A+B$

(5) $(u,v\in A+B) \implies (u+v\in A+B) \blacksquare \forall_{u,v\in A+B}(u+v\in A+B)$

(6) $((r\in\mathbb{R})\wedge(v\in A+B)) \implies \ldots$

(6.1) $\exists_{a\in A}\exists_{b\in B}(v=a+b)$

(6.2) $r*v=r*(a+b)=r*a+r*b$

(6.3) $(r*a\in A)\wedge(r*b)\in B \blacksquare r*v\in A+B$

(7) $((r\in\mathbb{R})\wedge(v\in A+B)) \implies (r*v\in A+B) \blacksquare \forall_{r\in\mathbb{R}}\forall_{v\in A+B}(r*v\in A+B)$

(8) $(\emptyset \neq A+B \subseteq V) \wedge (\forall_{u,v \in A+B}(u+v \in A+B)) \wedge (\forall_{r \in \mathbb{R}} \forall_{v \in A+B}(r * v \in A+B))$ ▮ $Subspace[A+B, V, +, *]$

(9) $(\forall_{a \in A}(a+O) = a) \wedge (O \in B)$ ▮ $A \subseteq A+B$

(10) $(\forall_{b \in B}(b+O) = b) \wedge (O \in A)$ ▮ $B \subseteq A+B$

(11) $(A, B \subseteq A+B) \wedge (Subspace[A+B, V, +, *])$

$$SumSubMinContains := \forall_{A,B,V} \left(\begin{array}{l} ((Subspace[A,V,+,*]) \wedge (Subspace[B,V,+,*]) \wedge (SetSum[A+B,A,B,V,+,*])) \implies \\ (\forall_C((Subspace[C,V,+,*]) \wedge (A,B \subseteq C)) \implies (A+B \subseteq C)) \end{array} \right)$$

(1) $SumSub$ ▮ $(A, B \subseteq A+B) \wedge (Subspace[A+B, V, +, *])$

(2) $((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies \ldots$

(2.1) $(s \in A+B) \implies \ldots$

(2.1.1) $\exists_{a \in A} \exists_{b \in B}(s = a+b)$

(2.1.2) $(A, B \subseteq C)$ ▮ $a, b \in C$

(2.1.3) $Subspace[C, V, +, *]$ ▮ $s = a+b \in C$

(2.2) $(s \in A+B) \implies (s \in C)$ ▮ $A+B \subseteq C$

(3) $((Subspace[C, V, +, *]) \wedge (A, B \subseteq C)) \implies (A+B \subseteq C)$

$NullSpace[N, A, m, n] := (Matrix[A, m, n]) \wedge (N = \{x \in \mathbb{R}^n | A * x = O\})$
$RowSpace[R, A, m, n] := (Matrix[A, m, n]) \wedge (R = \{x^T * A \in \mathbb{R}^n | x \in \mathbb{R}^m\})$
$ColSpace[C, A, m, n] := (Matrix[A, m, n]) \wedge (C = \{A * x \in \mathbb{R}^m | x \in \mathbb{R}^n\})$

$NullSubspace := (NullSpace[N, A, m, n]) \implies (Subspace[N, \mathbb{R}^n, +, *])$

(1) TODO

$RowSubspace := (RowSpace[R, A, m, n]) \implies (Subspace[R, \mathbb{R}^n, +, *])$

(1) TODO

$ColSubspace := (ColSpace[C, A, m, n]) \implies (Subspace[C, \mathbb{R}^m, +, *])$

(1) TODO

3.5 Linear Combination, Linear Span, Linear Independence

$LinComb[c, U, K, V, +, *] := (VectorSpace[V, +, *]) \wedge (n \in \mathbb{N}) \wedge (U \in V^n) \wedge (K \in \mathbb{R}^n) \wedge (c = \sum_{i=1}^{n}(k_i * u_i))$

$$LinSpan[S', S, V, +, *] := \left(\begin{array}{l} (VectorSpace[V,+,*]) \wedge (S \in V^n) \wedge ((S = \emptyset) \implies (S' = \{O\})) \wedge \\ ((S \neq \emptyset) \implies (S' = \{c \in V | \exists_{K \in \mathbb{R}^n}(LinComb[c, S, K, V, +, *])\})) \end{array} \right)$$

$LinSpanSubContains := \forall_{S',S,V,+,*}((LinSpan[S', S, V, +, *]) \implies ((Subspace[S', V, +, *]) \wedge (S \subseteq S')))$

(1) $(S = \emptyset) \implies (S' = \{O\}) \implies (\emptyset \neq S')$

(2) $(S \neq \emptyset) \implies (LinComb[O, S, \{0\}^n, V, +, *]) \implies (O \in S') \implies (\emptyset \neq S')$

(3) $((S = \emptyset) \vee (S \neq \emptyset)) \implies (\emptyset \neq S')$ ▮ $\emptyset \neq S'$

(4) $LinSpan[S', S, V, +, *]$ ▮ $S' \subseteq V$ ▮ $\emptyset \neq S' \subseteq V$

(5) $(a, b \in S') \implies \ldots$

(5.1) $(\exists_{K \in \mathbb{R}^n}(LinComb[a, S, K, V, +, *])) \wedge (\exists_{L \in \mathbb{R}^n}(LinComb[b, S, L, V, +, *]))$

(5.2) $a + b = \sum_{i=1}^{n}(k_i * s_i) + \sum_{i=1}^{n}(l_i * s_i) = \sum_{i=1}^{n}((k_i + l_i) * s_i)$ ▮ $a + b = \sum_{i=1}^{n}((k_i + l_i) * s_i)$

(5.3) $\langle k_i + l_i | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{N}^n$

(5.4) $(a + b = \sum_{i=1}^{n}((k_i + l_i) * s_i)) \wedge (\langle k_i + l_i | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{N}^n)$ ▮ $\exists_{M \in \mathbb{N}^n}(a + b = \sum_{i=1}^{n}(m_i * s_i))$

(5.5) $\exists_{M \in \mathbb{N}^n}(LinComb[a+b, S, M, V, +, *])$ ▮ $a + b \in S'$

(6) $(a, b \in S') \implies (a + b \in S')$ ▮ $\forall_{a,b \in S'}(a + b \in S')$

(7) $((r \in \mathbb{R}) \wedge (u \in S')) \implies \ldots$

(7.1) $\exists_{K \in \mathbb{R}^n}(LinComb[u, S, K, V, +, *])$ ▮ $\exists_{K \in \mathbb{R}^n}(u = \sum_{i=1}^{n}(k_i * s_i))$

(7.2) $r * u = r * \sum_{i=1}^{n}(k_i * s_i) = \sum_{i=1}^{n}(r * (k_i * s_i)) = \sum_{i=1}^{n}(rk_i) * s_i)$ ▮ $r * u = \sum_{i=1}^{n}(rk_i) * s_i)$

(7.3) $\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n} \rangle \in \mathbb{R}^n$

(7.4) $(\sum_{i=1}^{n}(rk_i) * s_i)) \land (\langle rk_i \in \mathbb{R} | i \in \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n)$

(7.5) $\exists_{M\in\mathbb{R}^n}(r * u = \sum_{i=1}^{n}(m_i * s_i))$ ▌ $\exists_{M\in\mathbb{R}^n}(LinComb[r * u, S, M, V, +, *])$ ▌ $r * u \in S'$

(8) $((r \in \mathbb{R}) \land (u \in S')) \implies (r * u \in S')$ ▌ $\forall_{r\in\mathbb{R}}\forall_{u\in S'}(r * u \in S')$

(9) $(SubspaceEquiv) \land (\emptyset \neq S' \subseteq V) \land (\forall_{a,b\in S'}(a + b \in S')) \land (\forall_{r\in\mathbb{R}}\forall_{u\in S'}(r * u \in S'))$ ▌ $Subspace[S', V, +, *]$

(10) $(s_j \in S) \implies \ldots$

(10.1) $K := \langle\left(\begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}\right) | \mathbb{N}_{1,n}\rangle \in \mathbb{R}^n$ ▌ $\sum_{i=1}^{n}(k_i * s_i) = s_j$

(10.2) $\ldots$ ▌ $\exists_{K\in\mathbb{R}^n}(LinComb[s_j, S, K, V, +, *])$ ▌ $s_j \in S'$

(11) $(s_j \in S) \implies (s_j \in S')$ ▌ $\forall_{x\in S}(x \in S')$ ▌ $S \subseteq S'$

(12) $(Subspace[S', V, +, *]) \land (S \subseteq S')$

$LinSpanSubMinContains := \forall_{S',S,V,+,*}((LinSpan[S', S, V, +, *]) \implies (\forall_W(((Subspace[W, V, +, *]) \land (S \subseteq W)) \implies (S' \subseteq W)))$

(1) $(s' \in S') \implies \ldots$

(1.1) $\exists_{K\in\mathbb{R}^n}(LinComb[s', S, K, V, +, *])$ ▌ $s' = \sum_{i=1}^{n}(k_i * s_i)$

(1.2) $(S \subseteq W) \land (Subspace[W, V, +, *])$ ▌ $s' = \sum_{i=1}^{n}(k_i * s_i) \in W$

(2) $(s' \in S') \implies (s' \in W)$ ▌ $S' \subseteq W$

$ZeroDependent := (O \in S) \implies (\neg LinInd[S, V, +, *])$

(1) $K := \langle\left(\begin{cases} 1 & u_i = O \\ 0 & u_i \neq O \end{cases}\right) | (1 \leq i \leq n) \land (i \in \mathbb{N})\rangle$ ▌ $K \in \mathbb{R}^n$

(2) $(LinComb[O, S, K, V, +, *]) \land (K \neq \{O\}^n)$ ▌ $\neg LinInd[S, V, +, *]$

$SingletonNonZeroIndependent := (v \neq O) \implies (LinInd[\langle v\rangle, V, +, *])$

(1) $(r * v = O) \iff ((r = 0) \lor (v \neq O))$

(2) $v \neq O$ ▌ $r = 0$

(3) $\forall_{r\in\mathbb{R}}((r * v = O) \implies (r = 0))$

$SubIndependent := \forall_{V,A,B}(((VectorSpace[V, +, *]) \land (A \subseteq B \in V^m)) \implies ((LinInd[B, V, +, *]) \implies (LinInd[A, V, +, *])))$

(1) $(LinComb[O, A, K, V, +, *]) \implies \ldots$

(1.1) $L := \langle\left(\begin{cases} 1 & j \leq n \\ 0 & j > n \end{cases}\right) | (1 \leq j \leq m \land (j \in \mathbb{N})\rangle$ ▌ $L \in \mathbb{R}^m$

(1.2) $A \subseteq B$ ▌ $\forall_{n\geq j\in\mathbb{N}}(a_j = b_j)$

(1.3) $\forall_{n\geq j\in\mathbb{N}}(a_j = b_j)$ ▌ $\sum_{i=1}^{n}(k_i * a_i)) = \sum_{i=1}^{n}(k_i * a_i)) + O = \sum_{j=1}^{m}(l_j * b_j))$

(1.4) $LinComb[O, A, K, V, +, *]$ ▌ $O = \sum_{i=1}^{n}(k_i * a_i)$

(1.5) $O = \sum_{i=1}^{n}(k_i * a_i) = \sum_{j=1}^{m}(l_j * b_j)$ ▌ $LinComb[O, B, L, V, +, *]$

(1.6) $(LinInd[B, V, +, *]) \land (LinComb[O, B, L, V, +, *])$ ▌ $L = \{0\}^m$

(1.7) $(\forall_{n\geq j\in\mathbb{N}}(a_j = b_j)) \land (L = \{0\}^m)$ ▌ $\forall_{n\geq j\in\mathbb{N}}(k_j * a_j = l_j * b * j = l_j * a_j)$ ▌ $K = \{0\}^n$

(2) $(LinComb[O, A, K, V, +, *]) \implies (K = \{0\}^n)$ ▌ $\forall_{K\in\subseteq\mathbb{R}^n}((LinComb[O, A, K, V, +, *]) \implies (K = \{0\}^n))$ ▌ $LinInd[A, V, +, *]$

$SuperDependent := \forall_{V,A,B}(((VectorSpace[V, +, *]) \land (A \subseteq B \subseteq V)) \implies ((\neg LinInd[A, V, +, *]) \implies (\neg LinInd[B, V, +, *])))$

(1) TODO : A has a non trivial solution ▌ use the same non trivial solution in combination with B and L

$LinIndEquiv := \forall_{U,V}((LinInd[U, V, +, *]) \iff (\forall_{j\in U}(\neg LinComb[j, U \setminus \{j\}, +, *])))$

(1) $\Gamma' = \Gamma \setminus \{j\}$

(2) $(\neg LinInd[U, V, +, *]) \implies \ldots$

(2.1) $(\exists_{\Gamma\in\mathbb{R}^{|U|}}((\sum(\gamma_i * u_i) = O) \land (\Gamma \neq \{0\}^{|U|})))$

(2.2) $\exists_{\gamma_k \in \Gamma}(\gamma_k \neq 0)$

(2.3) $\sum(\gamma_i' * u_i) = \sum(\gamma_i * u_i) - \gamma_k * u_k = -\gamma_k * u_j$

(2.4) $u_k = (-1/\gamma_k)(\sum(\gamma_i' * u_i)) = \sum((-\gamma_i'/\gamma_k) * u_i) \ \blacksquare\ \exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])$

(3) $(\neg LinInd[U, V, +, *]) \implies (\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *]))$

(4) $(\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *])) \implies (LinInd[U, V, +, *])$

(5) $(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])) \implies \ldots$

(5.1) $\exists_{j \in U}(j = \sum(\gamma_i' * u_i))$

(5.2) $\Gamma := \Gamma' \cup \{-1\}$

(5.3) $(\sum(\gamma_i * u_i) = \sum(\gamma_i' * u_i) + (-1) * \gamma_j = O) \wedge (\Gamma \neq \{0\}^n) \ \blacksquare\ \neg LinInd[U, V, +, *]$

(6) $(\exists_{j \in U}(LinComb[j, U \setminus \{j\}, +, *])) \implies (\neg LinInd[U, V, +, *])$

(7) $(LinInd[U, V, +, *]) \implies (\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *]))$

(8) $(LinInd[U, V, +, *]) \iff (\forall_{j \in U}(\neg LinComb[j, U \setminus \{j\}, +, *]))$

$Spans[S, V, +, *] := LinSpan[V, S, V, +, *]$
$FinDim[V, +, *] := \exists_{S \in V^n}(Spans[S, V, +, *])$
$LinInd[S, V, +, *] := (VectorSpace[V, +, *]) \wedge (S \in V^n) \wedge ((S \neq \emptyset) \implies (\forall_{K \in \mathbb{R}^n}((LinComb[O, S, K, V, +, *]) \implies (K = \{0\}^n))))$

$$LinDepLemma := \forall_{S,V}\left(\begin{array}{ll} (\neg LinInd[S, V, +, *]) & \implies \\ \exists_{j \in \mathbb{N}_{1,n}}((s_j \in LinSpan[P_1, S_{1,j-1}, V, +, *]) \wedge (LinSpan[P_2, S, V, +, *] = LinSpan[P_3, S \setminus \{s_j\}, V, +, *])) & \end{array}\right)$$

(1) $\neg LinInd[S, V, +, *] \ \blacksquare\ \exists_{K \in \mathbb{R}^n}((LinComb[O, S, K, V, +, *]) \wedge (K \neq \{0\}^n))$

(2) $\exists_{j \in \mathbb{N}_{1,n}}((k_j \neq 0) \wedge (\forall_{i \in \mathbb{N}_{1,n}}((i > j) \implies (k_i = 0))))$

(3) $s_j = (-1/k_j)\sum_{i=1}^{j-1}(k_i * s_i) = \sum_{i=1}^{j-1}((-k_i/k_j) * s_i)$

(4) $\langle -k_i/k_j | i \in \mathbb{N}_{1,j-1}\rangle \in \mathbb{R}^{j-1}$

(5) $\exists_{M \in \mathbb{R}^{j-1}}(LinComb[s_j, S_{1,j-1}, M, V, +, *]) \ \blacksquare\ s_j \in LinSpan[P_1, S_{1,j-1}, V, +, *]$

(6) $(v \in P_2) \iff (v \in LinSpan[P_2, S, V, +, *]) \iff \ldots$

(7) $\ldots(v = \sum_{i=1}^{n}(k_i * s_i)) = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^{n}(k_i * s_i)) + k_j * s_j = \sum_{i=1}^{j-1}(k_i * s_i)) + \sum_{i=j+1}^{n}(k_i * s_i)) + k_j * \sum_{i=1}^{j-1}((-k_i/k_j) * s_i) \iff \ldots$

(8) $(v \in LinSpan[P_3, S \setminus \{s_j\}, V, +, *]) \iff (v \in P_3) \ \blacksquare\ (v \in P_2) \iff (v \in P_3) \ \blacksquare\ P_2 = P_3$

$LinIndLengthLeqSpan := \forall_{L,S}(((LinInd[L, V, +, *]) \wedge (Spans[S, V, +, *])) \implies (|L| \leq |S|))$

(1) TODO : form $B = L \cup S$, remove dependent elements in S such that $(Spans[B, V, +, *]) \wedge (|B| = |S|)$ by $LinDepLemma$, $|L| \leq |B| = |S|$

(2) $\forall_{l_i \in L} \cdots$

(2.1) $l_i \in V \ \blacksquare\ LinComb[l_i, S, K, V, +, *] \ \blacksquare\ \neg LinInd[\langle l_i \rangle \cup S, V, +, *]$

(2.2) $LinDepLemma \ \blacksquare\ \exists_{j \in \mathbb{N}_{1,n}}(LinSpan[V, S, V, +, *] = LinSpan[V, S \setminus \{s_j\}, V, +, *])$

(2.3) $B := \langle l_i \rangle \cup S \setminus \{s_j\} \ \blacksquare\ |B| = 1 + |S| - 1 = |S|$

(3) $|L| \leq |B| = |S| \ \blacksquare\ |L| \leq |S|$

$FinSubSpace := \forall_{U,V}(((Subspace[U, V, +, *]) \wedge (FinDim[V, +, *])) \implies (FinDim[U, +, *)])$

(1) TODO : take $Spans[S, V, +, *]$, remove all $s_j \in S$ such that $U \subseteq LinSpan[S \setminus \{s_j\}] \ \blacksquare\ S' = S \setminus \{s_j\}$ $(LinSpan[U, S', V, +, *]) \wedge (|S'| \leq |S|) \ \blacksquare\ FinDim[U, +, *]$

(2) $FinDim[V, +, *] \ \blacksquare\ \exists_{S \in V^n}(Spans[S, V, +, *])$

(3) $\forall_{(u_j \in U) \wedge (\neg LinSpan[U, S, V, +, *])} \cdots$

(3.1) $\neg LinSpan[U, S, V, +, *] \ \blacksquare\ \exists_{u_j \in U}(\neg LinComb[u_j, S_{1,j-1}, K_{1,j-1}, V, +, *])$

(3.2) $B := S_{1,j-1} \ \blacksquare\ |B| = |S| - 1 < |S|$

(4) $LinSpan[U, B, V, +, *] \ \blacksquare\ \exists_{B \in V^M}(Spans[B, U, +, *]) \ \blacksquare\ FinDim[U, +, *$

3.6 Bases and Dimensions

$Basis[S, V, +, *] := (Spans[S, V, +, *]) \wedge (LinInd[S, V, +, *])$

$BasisEquiv := \forall_{S,V}((Basis[S, V, +, *]) \iff (\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *]))$

(1) $(Basis[S, V, +, *]) \implies \dots$

(1.1) $(v \in V) \implies \dots$

(1.1.1) $Basis[S, V, +, *] \blacksquare Spans[V, S, +, *] \blacksquare \exists_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *])$

(1.1.2) $((K_1, K_2 \in \mathbb{R}^n) \wedge (LinComb[v, S, K_1, V, +, *]) \wedge (LinComb[v, S, K_2, V, +, *])) \implies \dots$

(1.1.2.1) $(v = \sum(k_{1i} * s_i)) \wedge (v = \sum(k_{2i} * s_i))$

(1.1.2.2) $O = v - v = \sum(k_{1i} * s_i) - \sum(k_{2i} * s_i) = \sum((k_{1i} - k_{2i}) * s_i)$

(1.1.2.3) $L := \langle k_{1i} - k_{2i} | i \in \mathbb{N}_{i=1}^n \rangle \in \mathbb{R}^n$

(1.1.2.4) $(LinInd[S, V, +, *]) \wedge (LinComb[O, S, L, V, +, *]) \blacksquare L = \{0\}^n \blacksquare K_2 = K_1$

(1.1.3) $((K_1, K_2 \in \mathbb{R}^n) \wedge (LinComb[v, S, K_1, V, +, *]) \wedge (LinComb[v, S, K_2, V, +, *])) \implies (K_1 = K_2)$

(1.1.4) $\forall_{K_1, K_2 \in \mathbb{R}^n}((LinComb[v, S, K_1, V, +, *]) \wedge (LinComb[v, S, K_2, V, +, *]) \implies (K_1 = K_2))$

(1.1.5) $\exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *])$

(1.2) $(v \in V) \implies (\exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *]))$

(2) $(Basis[S, V, +, *]) \implies (\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *]))$

(3) $(\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *])) \implies \dots$

(3.1) $\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *]) \blacksquare \forall_{v \in V} \exists_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *]) \blacksquare Spans[S, V, +, *]$

(3.2) $O \in V \blacksquare \exists!_{K \in \mathbb{R}^n}(LinComb[O, S, K, V, +, *])$

(3.3) $(K \neq \{0\}^n) \implies (\neg \exists!_{K \in \mathbb{R}^n}(LinComb[O, S, K, V, +, *])) \implies \perp \blacksquare K = \{0\}^n$

(3.4) $(\exists!_{K \in \mathbb{R}^n}(LinComb[O, S, K, V, +, *])) \wedge (K = \{0\}^n) \blacksquare LinInd[S, V, +, *]$

(3.5) $(Spans[S, V, +, *]) \wedge (LinInd[S, V, +, *]) \blacksquare Basis[S, V, +, *]$

(4) $(\forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v, S, K, V, +, *])) \implies (Basis[S, V, +, *])$

$SpanReduceBasis := \forall_{S,V}(Spans[S, V, +, *] \implies \exists_B((B \subseteq S) \wedge (Basis[B, V, +, *])))$

(1) TODO : Remove all dependent $s_j \in S \blacksquare (S' = S \setminus \{s_j\}) \wedge (LinSpan[S'] = LinSpan[S])$ until $LinInd[S'] \blacksquare Basis[S']$

$LinIndExpandBasis := \forall_{L,V}(LinInd[L, V, +, *] \implies \exists_B((L \subseteq B) \wedge (Basis[B, V, +, *])))$

(1) TODO : $FinDimBasis \blacksquare \exists_A(Basis[A, V, +, *])$, form $B = L \cup A \blacksquare Span[B]$,
use $SpanReduceBasis$ call it B', $(L \subseteq B') \wedge (Basis[B'])$

$FinDimBasis := \forall_V((FinDim[V, +, *]) \implies (\exists_B(Basis[B, V, +, *])))$

(1) $FinDim[V, +, *] \blacksquare \exists_{S \in V^n}(Spans[S, V, +, *])$

(2) $(SpanReduceBasis) \wedge (Spans[S, V, +, *]) \blacksquare \exists_B(Basis[B, V, +, *])$

$BasisLinearIndCard := \forall_{S,T,V}(((Basis[S, V, +, *]) \wedge (LinInd[T, V, +, *])) \implies (|T| \leq |S|))$

(1) $(Basis[S, V, +, *]) \implies \dots$

(1.1) $(|T| > |S|) \implies \dots$

(1.1.1) $(Spans[S, V, +, *]) \wedge (T \subseteq V) \blacksquare t_{1 \dots t_j} = \sum(\gamma_i * s * i)$ . . .

(1.1.2) . . . $t_j = \sum(\gamma_i' * t_i) \blacksquare \neg LinInd[T, V, +, *]$

(1.2) $(|T| > |S|) \implies (\neg LinInd[T, V, +, *]) \blacksquare (LinInd[T, V, +, *]) \implies (|T| \leq |S|)$

(2) $((Basis[S, V, +, *]) \wedge (LinInd[T, V, +, *])) \implies (|T| \leq |S|)$

$BasisCard := \forall_{S,T,V}(((Basis[S, V, +, *]) \wedge (Basis[T, V, +, *])) \implies (|T| = |S|))$

(1) $Basis[S, V, +, *] \blacksquare LinInd[S, V, +, *]$

(2) $(Basis[T, V, +, *]) \wedge (LinInd[S, V, +, *]) \blacksquare |S| \leq |T|$

(3) $Basis[T, V, +, *] \blacksquare LinInd[T, V, +, *]$

(4) $(Basis[S, V, +, *]) \wedge (LinInd[T, V, +, *]) \blacksquare |T| \leq |S|$

(5) $(|S| \leq |T|) \wedge (|T| \leq |S|) \blacksquare |T| = |S|$

$Dim[d, V, +, *] := (\exists_B(Basis[B, V, +, *])) \wedge ((V = \{O\}) \implies (d = 0)) \wedge ((V \neq \{O\}) \implies (d = |B|))$

$LinIndLengthDim := \forall_{U,V}(((LinInd[U, V, +, *]) \wedge (Dim[|U|, V, +, *])) \implies (Basis[U, V, +, *]))$

(1)	$(LinIndExpandBasis) \wedge (LinInd[U, V, +, *])$ ▮ $\exists_B((U \subseteq B) \wedge (Basis[B, V, +, *]))$
(2)	$(BasisCard) \wedge (Dim[|U|, V, +, *]) \wedge (Basis[B, V, +, *])$ ▮ $|B| = |U|$ ▮ $B = U$ ▮ $Basis[U, V, +, *]$

$SpanLengthDim := \forall_{U,V}(((Spans[U, V, +, *]) \wedge (Dim[|U|, V, +, *])) \implies (Basis[U, V, +, *]))$

(1)	$(SpanReduceBasis) \wedge (Spans[U, V, +, *])$ ▮ $\exists_B((B \subseteq U) \wedge (Basis[B, V, +, *]))$
(2)	$(BasisCard) \wedge (Dim[|U|, V, +, *]) \wedge (Basis[B, V, +, *])$ ▮ $|B| = |U|$ ▮ $B = U$ ▮ $Basis[U, V, +, *]$

$LinDepLengthDim := \forall_{U,V}(((U \subseteq V) \wedge (|U| > Dim[V])) \implies (\neg LinInd[U, V, +, *]))$

(1)	Contrapositive of $BasisLinearIndCard$

$LinDepLengthDim := \forall_{U,V}(((U \subseteq V) \wedge (|U| < Dim[V])) \implies (\neg Spans[U, V, +, *]))$

(1)	Suppose $Spans[U, V, +, *]$, $B = SpanReduceBasis[U]$ to form a basis, $(|B| \leq |U| < Dim[V]) \wedge |B| = Dim[V]$ ▮ $\perp$
(2)	$\neg Spans[U, V, +, *]$

3.7 Rank

$Nullity[n, A] := (NullSpace[N, A]) \wedge (Dim[n, N, +, *])$
$Rank[r, A, m, n] := (Matrix[A, m, n]) \wedge (RowSpace[R, A, m, n]) \wedge (Dim[r, R, A, +, *])$

$RowRankEqColRank := \forall_A(TODO)$

(1)	TODO

$RankNullity := \forall_A((Matrix[A, m, n]) \implies (Rank[A] + Nullity[A] = n))$

(1)	TODO

$RankInv := \forall_A((Matrix[A, m, n]) \implies ((Rank[A] = n) \iff (Inv[A])))$

(1)	TODO

$RankNonTrivialSol := (\exists_X((A * X = O) \wedge (X \neq O))) \iff (Rank[A] < n)$

(1)	TODO

$RankUniqueSol := (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) \iff (Rank[A] = n)$

(1)	TODO

$$SquareTheorems_{()} := \forall_{A \in \mathcal{M}} \left(\begin{array}{ll} (Invertible[A]) & \iff \\ (RowEquiv[A, I_n]) & \iff \\ (\forall_X((X = O) \iff (Sol[X, A, O]))) & \iff \\ (\forall_{B \in \mathcal{M}} \exists!_{X \in \mathcal{M}}(Sol[X, A, B])) & \iff \\ (Rank[A] = n) & \iff \\ (Nullity[A] = 0) & \iff \\ (\text{The rows form a linearly independent set of vectors (to get full rank)}) & \iff \\ (\text{The columns form a linearly independent set of vectors (to get full rank)}) & \iff \end{array} \right)$$

3.8 Linear Transformations

$$LinTrans[L,V,+_v,*_v,W,+_w,*_w] := \begin{pmatrix} (Function[f,V,W]) \wedge (VectorSpace[V,+_v,*_v]) \wedge (VectorSpace[W,+_w,*_w]) \wedge \\ (\forall_{\alpha,\beta \in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))) \quad \wedge \quad (\forall_{r \in \mathbb{R}} \forall_{\alpha \in V}(L(r *_v \alpha) = r *_w L(\alpha))) \end{pmatrix}$$

$LinOp[L,V,+_v,*_v] := LinTrans[L,V,+_v,*_v,V,+_v,*_v]$

$\mathcal{L}[V,W] := \{L | LinTrans[L,V,+_v,*_v,W,+_w,*_w]\}$

$ZeroMapsToZero := \forall_{L,V,W}((LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \implies (L(O_v) = O_w))$

(1) $L(O_v) = L(O_v +_v O_v) = L(O_v) +_w L(O_v)$

(2) $O_w = L(O_v) - L(O_v) = L(O_v)$

$SplitAddInv := \forall_{L,V,W}((LinTrans[L,V,+_v,*_v,W,+_w,*_w]) \implies (\forall_{\alpha,\beta \in V}(L(\alpha -_v \beta) = L(\alpha) -_w L(\beta))))$

(1) $L(\alpha - \beta) = L(\alpha + (-\beta)) = L(\alpha) + L(-\beta) = L(\alpha) + (-1) * L(\beta) = L(\alpha) - L(\beta)$

$$BasisDomainInduceLinTrans := \forall_{V,W} \begin{pmatrix} ((Basis[A,V,+_v,*_v]) \wedge (B \subseteq W) \wedge (n = |B| = |A|) \wedge (VectorSpace[W,+_w,*_w])) \implies \\ (\exists!_T((LinTrans[T,V,+_v,*_v,W,+_w,*_w]) \wedge (\forall_{i \in \mathbb{N}_{1,n}}(T(a_i) = b_i)))) \end{pmatrix}$$

(1) $T(\sum_{i=1}^{n}(k_i * a_i)) := \sum_{i=1}^{n}(k_i * b_i)$

(2) $(i \in \mathbb{N}_{1,n}) \implies \ldots$

(2.1) $L := \langle \left(\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \right) | j \in \mathbb{N}_{1,n} \rangle \;\blacksquare\; L \in \mathbb{R}^n$

(2.2) $T(a_i) = T(\sum_{i=1}^{n}(l_i * a_i)) = \sum_{i=1}^{n}(l_i * b_i) = b_i \;\blacksquare\; T(a_i) = b_i$

(3) $(i \in \mathbb{N}_{1,n}) \implies (T(a_i) = b_i) \;\blacksquare\; \forall_{i \in \mathbb{N}_{1,n}}(T(a_i) = b_i)$

(4) $(BasisEquiv) \wedge (Basis[A,V,+_v,*_v]) \;\blacksquare\; \forall_{v \in V} \exists!_{K \in \mathbb{R}^n}(LinComb[v,A,K,V,+,*]) \;\blacksquare\; T : Span[A] \to Span[B] \;\blacksquare\; Function[T,V,W]$

(5) $(\alpha, \beta \in V) \implies \ldots$

(5.1) $(LinComb[\alpha,A,K_\alpha,V,+_v,*_v]) \wedge (LinComb[\beta,A,K_\beta,V,+_v,*_v])$

(5.2) $T(\alpha + \beta) = T(\sum_{i=1}^{n}(k_{\alpha i} * a_i) + \sum_{i=1}^{n}(k_{\beta_i} * a_i)) = T(\sum_{i=1}^{n}((k_{\alpha i} + k_{\beta_i}) * a_i))) = \sum_{i=1}^{n}((k_{\alpha i} + k_{\beta_i}) * b_i) = \ldots$

(5.3) $\ldots \sum_{i=1}^{n}(k_{\alpha i} * b_i) + \sum_{i=1}^{n}(k_{\beta_i} * b_i) = T(\sum_{i=1}^{n}(k_{\alpha i} * a_i)) + T(\sum_{i=1}^{n}(k_{\beta_i} * a_i)) = T(\alpha) + T(\beta)$

(6) $(\alpha, \beta \in V) \implies (L(\alpha +_v \beta) = L(\alpha) +_w L(\beta)) \;\blacksquare\; \forall_{\alpha,\beta \in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))$

(7) $((r \in \mathbb{R}) \wedge (\alpha \in V)) \implies \ldots$

(7.1) $LinComb[\alpha,A,K,V,+_v,*_v]$

(7.2) $L(r *_v \alpha) = L(r *_v \sum_{i=1}^{n}(k_i *_v a_i)) = L(\sum_{i=1}^{n}((rk_i) *_v a_i)) = \sum_{i=1}^{n}((rk_i) *_w b_i) = r *_w$

(8) $((r \in \mathbb{R}) \wedge (\alpha \in V)) \implies (L(r *_v \alpha) = r *_w L(\alpha)) \;\blacksquare\; \forall_{r \in \mathbb{R}} \forall_{\alpha \in V}(L(r *_v \alpha) = r *_w L(\alpha))$

(9) $(\forall_{i \in \mathbb{N}_{1,n}}(T(a_i) = b_i)) \wedge (Function[T,V,W]) \wedge (\forall_{\alpha,\beta \in V}(L(\alpha +_v \beta) = L(\alpha) +_w L(\beta))) \wedge (\forall_{r \in \mathbb{R}} \forall_{\alpha \in V}(L(r *_v \alpha) = r *_w L(\alpha))) \wedge \ldots$

(10) $\ldots (VectorSpace[V,+_v,*_v]) \wedge (VectorSpace[W,+_w,*_w]) \;\blacksquare\; LinTrans[T,V,+_v,*_v,W,+_w,*_w]$