Solutions Manual to Walter Rudin's *Principles of Mathematical Analysis*

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Chapter 10

Integration of Differential Forms

Exercise 10.1 Let H be a compact convex set in R^k with nonempty interior. Let $f \in \mathcal{C}(H)$, put $f(\mathbf{x}) = 0$ in the complement of H and define $\int_H f$ as in Definition 10.3.

Prove that $\int_H f$ is independent of the order in which the integrations are carried out.

Hint: Approximate f by functions that are continuous on R^k and whose supports are in H, as was done in Example 10.4.

Solution. We first give the definition of $\int_H f$, namely $\int_I f$, where I is any k-cell containing H. This definition is unambiguous, since if I and J are both k-cells containing H, each of the single integrals carried out is an integral over the same line segment for both cells, namely the intersection of the path of integration with H.

There seems to be no way to avoid somehow proving that the boundary of H, denoted ∂H, has "measure zero." The definition of the integral as an iterated integral makes that problem slightly more difficult than it would be otherwise, although we can show how to avoid this approach in two dimensions. We shall reserve that discussion until after the proof, which is rather lengthy. The length of the proof is due to the fact that integrals are really defined only over parallelepipeds. The point of the exercise is to enlarge the class of sets over which one can integrate. Our challenge is to show that the boundary of H can be enclosed in a finite set of parallelepipeds whose total volume can be arbitrarily small.

Our first job is to show that the hypersphere in R^k has measure zero. As the proof of that fact involves some work with $(k-2)$-dimensional hyperspheres in k-dimensional space, we need to make several definitions in order to express these ideas properly.

First, for each real number z and each positive number r, $S_r^{k-2}(z)$ denotes the $(k-2)$-dimensional hypersphere in R^k having radius r and center at the point $(0, 0, \ldots, 0, z)$, that is,

$$S_r^{k-2}(z) = \{\mathbf{x} : x_1^2 + \cdots + x_{k-1}^2 = r^2,\ x_k = z\}.$$

When $z = 0$, we shall write simply S_r^{k-2} and identify this sphere with the same $(k-2)$-dimensional hypersphere in R^{k-1}. We observe that $S_r^0(z)$ consists of the two points (r, z) and $(-r, z)$ in R^2. Another way of defining the set $S_r^{k-2}(z)$ is as the intersection of the $(k-1)$-dimensional sphere of radius r centered at $(0, 0, \ldots, 0, z)$ in R^k with its equatorial hyperplane $P_z = \{(x_1, \ldots, x_k) : x_k = z\}$.

Second, for each $\mathbf{a} \in R^k$ and each $\delta > 0$, $I_{\mathbf{a}}^k(\delta)$ is the closed hypercube of side δ in R^k whose "lower left" corner is $\mathbf{a}$, that is

$$I_{\mathbf{a}}^k(\delta) = \{\mathbf{x} : a_j \le x_j \le a_j + \delta,\ j = 1, \ldots, k\}.$$

Third, the set of points $(m_1, \ldots, m_k) \in R^k$ having integer coordinates will be denoted $\mathcal{Z}^k$.

Fourth, for all real numbers r and δ such that $0 < \delta < r$, $C_{r,\delta}^k$ is the set of lattice points $\mathbf{m} \in \mathcal{Z}^k$ for which the closed hypercube of side δ with lower left corner $\delta\mathbf{m}$ intersects the hypersphere S_r^{k-1} in R^k. That is,

$$C_{r,\delta}^k = \{\mathbf{m} : I_{\delta\mathbf{m}}^k(\delta) \cap S_r^{k-1} \ne \emptyset\}.$$

Fifth, $N(r, \delta, k)$ is the number of points in $C_{r,\delta}^k(z)$, that is, the number of hypercubes of side δ with lower left hand corner at a point $\delta\mathbf{m}$, $\mathbf{m} \in \mathcal{Z}^k$, that intersect the hypersphere S_r^{k-1}. Our main goal in the first stage of the proof will be to prove the estimate $N(r, \delta, k) \le 6^{k^2}\left(\frac{r}{\delta}\right)^{k-1}$. (A smaller constant than 12^{k^2} could easily be attained, but we have no need of any improvement, and this constant seems to be the one that makes the argument simplest.)

Sixth, and finally, $A_{r,\delta}^k$ is the union of all the hypercubes $I_{\delta\mathbf{m})}^k(\delta)$ that intersect the hypersphere S_r^{k-1}, that is, for which $\mathbf{m} \in C_{r,\delta}^k$. This set is a finite union of compact sets, hence is compact. Obviously it contains the hypersphere S_r^{k-1}. What is slightly less obvious is that its interior contains this sphere. In fact no point of the sphere can be a limit point of points exterior to $A_{r,\delta}^k$, since if $\{\mathbf{x}_n\}$ is a sequence of points such that each $\mathbf{x}_n$ belongs to a hypercube $I_{\delta\mathbf{m}_n}^k(\delta)$ not contained in $A_{r,\delta}^k$, and $\mathbf{x}_n \to \mathbf{x}$, some set $I_{\delta\mathbf{m}_{n_0}}^k(\delta)$ must occur infinitely often. (Any bounded neighborhood of $\mathbf{x}$ intersects only finitely many of these hypercubes.) Since $I_{\delta\mathbf{m}_{n_0}}^k(\delta)$ is closed, this implies that $\mathbf{x}$ belongs to $I_{\delta\mathbf{m}_{n_0}}^k(\delta)$. Since $I_{\delta\mathbf{m}_{n_0}}^k(\delta)$ is not contained in $A_{r,\delta}^k$ it follows that $\mathbf{x} \notin S_r^{k-1}$. Thus no sequence of points exterior to $A_{r,\delta}^k$ can approach a point of S_r^{k-1}. It follows that S_r^{k-1} contains no points of the boundary of $A_{r,\delta}^k$ and is therefore contained in the interior of this set.

With these definitions out of the way we can proceed to the proof, which we break into several stages, each broken into several steps, in order to make navigating easier.

Stage 1. Establish that the sphere S_r^{k-1} in R^k has k-dimensional content 0.

Step 1. Establish that $A^k_{r,\delta} \cup A^k_{r+\delta}$ contains the closed k-dimensional annulus consisting of the region between S^{k-1}_r and $S^{k-1}_{r+\delta}$, that is, all the points $\mathbf{x} \in R^k$ such that $r \leq |\mathbf{x}| \leq r + \delta$.

To this end, let $\mathbf{x}$ belong to this annulus, so that $r \leq |\mathbf{x}| \leq r + \delta$. Since $S^{k-1}_s \subset A^k_{s,\delta}$ for each s, we can assume $r < |\mathbf{x}| < r + \delta$. Let $\mathbf{m} = (m_1, \ldots, m_k)$ be a lattice point in $\mathcal{Z}^k$ such that $\mathbf{x} \in I^k_{\delta\mathbf{m}}(\delta)$. Let $n_j = m_j$ if $m_j \geq 0$ and $n_j = m_j + 1$ if $m_j < 0$, so that $(\delta n_1, \ldots, \delta n_k)$ is a corner of $I_{\delta\mathbf{m}}(\delta)$. Since $|n_j| = \min(|m_j|, |m_j + 1|)$ for all j, $(\delta\mathbf{n})$ is the unique point of $I^k_{\delta\mathbf{m}}(\delta)$ closest to the origin. In particular $|\delta\mathbf{n}| \leq |\mathbf{x}|$. The lattice point $\mathbf{n}' = (n_1 + \varepsilon_1(n_1), \ldots, n_k + \varepsilon_k(n_k))$, where $\varepsilon_j(t)$ is 1 if $t \geq 0$ and -1 if $t < 0$, is such that $\delta\mathbf{n}'$ is the corner of $I_{\delta\mathbf{m}}(\delta)$ opposite to $\delta\mathbf{n}$ and is the unique point of $K_{\delta\mathbf{m}}(\delta)$ farthest from the origin. In particular $|\delta\mathbf{n}'| \geq |\mathbf{x}|$.

We claim first that $|\mathbf{n}'| - |\mathbf{n}| \geq 1$. Indeed, we have

$$\begin{aligned}|\mathbf{n}'|^2 &= (n_1^2 + \cdots + n_k^2) + 2(n_1\varepsilon_1(n_1) + \cdots + n_k\varepsilon_k(n_k) + ((\varepsilon_1(n_1))^2 + \cdots + (\varepsilon_k(n_k))^2) \\ &= (n_1^2 + \cdots + n_k^2) + 2(|n_1| + \cdots + |n_k|) + k,\end{aligned}$$

and

$$\left(|\mathbf{n}| + 1\right)^2 = (n_1^2 + \cdots + n_k^2) + 2\sqrt{n_1^2 + \cdots + n_k^2} + 1,$$

so that the desired inequality follows from the two inequalities $k \geq 1$ and $\sqrt{n_1^2 + \cdots + n_k^2} \leq |n_1| + \cdots + |n_k|$. This argument shows in general that, for any $\mathbf{m} \in \mathcal{Z}^k$, if $\mathbf{b}$ and $\mathbf{c}$ are the points in $I^k_{\delta\mathbf{m}}(\delta)$ of minimal and maximal absolute value respectively, then $|\mathbf{c}| - |\mathbf{b}| \geq \delta$.

From this we deduce a corollary: *Let r be any positive real number larger than δ. If $r \leq s \leq r + \delta$ and $\mathbf{m} \in C^k_{s,\delta}$, then either $\mathbf{m} \in C^k_{r,\delta}$ or $\mathbf{m} \in C^k_{r+\delta,\delta}$.* In plain words, if $I^k_{\delta\mathbf{m}}(\delta)$ meets S^{k-1}_s for some $s \in [r, r + \delta]$, it must meet either S^{k-1}_r or $S^{k-1}_{r+\delta}$.

To prove this corollary, we note that the assumption $\mathbf{m} \in C^k_{s,\delta}$ says that there exists $\mathbf{x} \in R^k$ such that $\mathbf{x} \in S^{k-1}_s \cap I^k_{\delta\mathbf{m}}(\delta)$. Now suppose $\mathbf{m}$ belongs to neither of the sets $C^k_{r,\delta}$ and $C^k_{r+\delta,\delta}$. Then the set $I^k_{\delta\mathbf{m}}(\delta)$ contains no points of S^{k-1}_r. If $\mathbf{b}$ is the point of $I^k_{\delta\mathbf{m}}(\delta)$ of smallest norm, it follows that $|\mathbf{b}| > r$. (For $I^k_{\delta\mathbf{m}}(\delta)$ contains the point $\mathbf{x}$ of norm $s \geq r$. Since $I^k_{\delta\mathbf{m}}(\delta)$ is a connected set, if it contained a point of norm less than or equal to r it would also contain a point of S^{k-1}_r.) Similarly, if the set $I^k_{\delta\mathbf{m}}(\delta)$ contains no points of $S^{k-1}_{r+\delta}$, then $|\mathbf{c}| < r + \delta$. But then it follows that $|\mathbf{c}| - |\mathbf{b}| < r + \delta - r = \delta$, contrary to what has been proved.

Another way of stating what was just proved is that if $\mathbf{x} \in R^k$ is such that $r \leq |\mathbf{x}| \leq r + \delta$, then $\mathbf{x} \in A^k_{r,\delta} \cup A^k_{r+\delta,\delta}$. That is, the union $A^k_{r,\delta} \cup A^k_{r+\delta,\delta}$ contains the entire annulus of points $\mathbf{x}$ such that $r \leq |\mathbf{x}| \leq r + \delta$. Step 1 of the proof is now complete.

Step 2. Assuming $k > 1$, estimate the number of k-dimensional hypercubes $I^k_{\delta\mathbf{m}}(\delta)$ that intersect various zones on the $(k-1)$-sphere S^{k-1}_r in R^k.

We divide the upper hemisphere, consisting of $\mathbf{x}$ such that $|\mathbf{x}| = r$ and $x_k \geq 0$ into half-open zones

$$Z_p = \{(x_1, \ldots, x_{k-1}, x_k) : x_1^2 + \cdots + x_k^2 = r, p\delta \leq x_k < (p+1)\delta\},$$

for $0 \leq p \leq [\frac{r}{\delta}] - 1$. Here $[a]$ denotes the integer part of a, that is, the integer q such that $q \leq a < q+1$, and we assume $0 < \delta < r$. Between the top zone $Z_{[\frac{r}{\delta}]-1}$ and the "north pole" (the point $(0, 0, \ldots, r)$), there is a closed "cap" of height η for some $\eta \in [0, \delta)$. The hypercubes $I^k_{\delta\mathbf{m}}(\delta)$ (where $m_k = [\frac{r}{\delta}] - 1$ or $m_k = [\frac{r}{\delta}]$) intersecting this cap must be handled separately from those intersecting the other zones.

We shall prove that *the lattice points $\mathbf{m} \in \mathcal{Z}^k$ for which $m_k = p$ and $I^k_{\delta\mathbf{m}}(\delta)$ intersects Z_p are precisely those whose bottom face $x_k = p\delta$ intersects the half-closed $(k-1)$-dimensional annulus between $S^{k-2}_s(p\delta)$ and $S^{k-2}_t(p\delta)$ in the hyperplane $x_k = p\delta$.* (This annulus is closed at t and open at s, where $s = \sqrt{r^2 - ((p+1)\delta)^2}$ and $t = \sqrt{r^2 - (p\delta)^2}$.)

Indeed, this fact is nearly obvious, as the zone Z_p is the union of the $(k-2)$-dimensional spheres $S^{k-2}_{\sqrt{r^2-u^2}}(u)$ for $p\delta \leq u < (p+1)\delta$. If $(x_1, \ldots, x_{k-1}, u) \in I^k_{\delta\mathbf{m}}(\delta) \cap Z_p$ and $m_k = p$, then $p\delta \leq u < (p+1)\delta$ and $x_1^2 + \cdots + x_{k-1}^2 = r^2 - u^2$, so that $s < \sqrt{x_1^2 + \cdots + x_{k-1}^2} \leq t$. Thus the point $(x_1, \ldots, x_{k-1}, \delta p)$ belongs to both $I^k_{\delta\mathbf{m}}(\delta)$ and to the annulus. Conversely, if $I^k_{\delta\mathbf{m}}(\delta)$ with $m_k = p$ intersects the annulus, then this hypercube contains a point $(x_1, \ldots, x_{k-1}, p\delta)$ with $s^2 < x_1^2 + \cdots + x_{k-1}^2 \leq t^2$. Setting $u = \sqrt{r^2 - (x_1^2 + \cdots + x_{k-1}^2)}$, we have $p\delta \leq u < (p+1)\delta$, and therefore $(x_1, \ldots, x_{k-1}, u) \in Z_p \cap I^k_{\delta\mathbf{m}}(\delta)$.

To estimate the total number of hypercubes $I^k_{\delta\mathbf{m}}(\delta)$ that intersect the hypersphere S^{k-1}_r, we need an estimate of the number that intersect each zone Z_p. If $I^k_{\delta\mathbf{m}}(\delta)$ intersects Z_p, then $m_k = p$ or $m_k = p - 1$. If $I^k_{(\delta m_1, \ldots, \delta m_{k-1}, \delta(p-1))}(\delta)$ intersects Z_p, the intersection must be in the hyperplane $x_k = p\delta$, and hence $I^k_{(\delta m_1, \ldots, \delta m_{k-1}, \delta p)}(\delta)$ also intersects Z_p. Hence we can get a (loose, but safe) upper bound on the number of hypercubes $I^k_{\delta\mathbf{m}}(\delta)$ that intersect Z_p by counting those for which $m_k = p$ and doubling. (The case of the bottom layer Z_0 is special, and the "northern cap" mentioned above will be handled separately.)

The fact that a hypercube $I^k_{\delta\mathbf{m}}(\delta)$ intersecting Z_p must intersect the annulus shows that we need only estimate of the width of the annulus, that is, the number $t - s = \sqrt{r^2 - (p\delta)^2} - \sqrt{r^2 - ((p+1)\delta)^2}$. For that width we have the following simple result:

$$\sqrt{r^2 - (p\delta)^2} - \sqrt{r^2 - ((p+1)\delta)^2} \leq \frac{(2p+1)\delta^2}{\sqrt{r^2 - (p\delta)^2}}.$$

The proof of this inequality is straightforward:

$$\begin{aligned}\sqrt{r^2 - (p\delta)^2} - \sqrt{r^2 - ((p+1)\delta)^2} &= \frac{(r^2 - (p\delta)^2) - (r^2 - ((p+1)\delta)^2)}{\sqrt{r^2 - (p\delta)^2} + \sqrt{r^2 - ((p+1)\delta)^2}} \\ &\leq \frac{(2p+1)\delta^2}{\sqrt{r^2 - (p\delta)^2}}.\end{aligned}$$

Now let j be the largest integer not larger than $\sqrt{\left(\frac{r}{\delta}\right)^2-(p+1)^2}$ and l the smallest integer not smaller than $\sqrt{\left(\frac{r}{\delta}\right)^2-p^2}$. We have just shown that

$$0 < l - j < 2 + \frac{(2p+1)\delta}{\sqrt{r^2-(p\delta)^2}} = 2 + \frac{2p+1}{\sqrt{\left(\frac{r}{\delta}\right)^2-p^2}}.$$

It is this inequality that provides the required estimate of the width of the annulus corresponding to the zone Z_p.

Step 3. Prove the estimate $N^k_{r,\delta} \le 6^{k^2}\left(\frac{r}{\delta}\right)^{k-1}$.

We first do the case $k=1$. This case is very straightforward. If $I^1_{(\delta m,z)}(\delta) \cap S^0_r$, then either $m\delta \le r \le (m+1)\delta$ or $m\delta \le -r \le (m+1)\delta$. If $r = k\delta$ for some integer k, there are four values of m for which one of these two sets of inequalities hold, namely $-k-1$, $-k$, $k-1$, and k. Otherwise there are only two such integers m, namely $\left[\frac{r}{\delta}\right]$ and $\left[\frac{-r}{\delta}\right]$. Thus we actually have $N^0_{r,\delta} \le 4 < 6^1\left(\frac{r}{\delta}\right)^0$.

We now proceed by induction, supposing the theorem proved for dimensions less than k, and we assume $k \ge 2$. We consider all the lattice points $\mathbf{m} \in R^k$ such that $m_k = p$ and $I^k_{\delta\mathbf{m}}(\delta)$ intersects Z_p. From what we have shown above in Step 1 and Step 2, if $\mathbf{m}$ has this property, then the bottom face of $I^k_{\delta\mathbf{m}}(\delta)$ intersects one of the spheres $S^{k-2}_{s\delta}(p\delta)$, where s is an integer such that $j \le s \le l$. The number of such $\mathbf{m}$ is at most $N^{k-1}_{s\delta,\delta}$ and hence is at most $6^{(k-1)^2}s^{k-2}$, which is certainly no larger than

$$6^{(k-1)^2}\left(1+\sqrt{\left(\frac{r}{\delta}\right)^2-p^2}\right)^{k-2}.$$

Because of our estimate of $l-j$, we see that the total number of $\mathbf{m}$ for which $m_k = p$ and $I^k_{\delta\mathbf{m}}(\delta)$ intersects Z_p is at most

$$6^{(k-1)^2}\left(2+\frac{2p+1}{\sqrt{\left(\frac{r}{\delta}\right)^2-p^2}}\right)\left(2^{k-2}+2^{k-2}\left(\left(\frac{r}{\delta}\right)^2-p^2\right)^{\frac{k-2}{2}}\right),$$

where we have used the inequality $(1+t)^q \le 2^q + (2t)^q$ with $q = k-2$. We expand this last product into a sum of six terms:

$$\begin{aligned}
&6^{(k-1)^2}\Bigg(2^{k-1}+2^{k-1}\left(\left(\frac{r}{\delta}\right)^2-p^2\right)^{\frac{k-2}{2}}+2^{k-1}\frac{p}{\sqrt{\left(\frac{r}{\delta}\right)^2-p^2}}\\
&\quad+\frac{2^{k-2}}{\sqrt{\left(\frac{r}{\delta}\right)^2-p^2}}+2^{k-1}p\left(\left(\frac{r}{\delta}\right)^2-p^2\right)^{\frac{k-3}{2}}+2^{k-2}\left(\left(\frac{r}{\delta}\right)^2-p^2\right)^{\frac{k-3}{2}}\Bigg)\\
&=6^{(k-1)^2}\big(I_1(p)+I_2(p)+I_3(p)+I_4(p)+I_5(p)+I_6(p)\big).
\end{aligned}$$

We need to estimate the sum of each of these terms over p from 0 to $\left[\frac{r}{\delta}\right]-1$.

For $I_1(p)$ we have the simple estimate

$$\sum_{p=0}^{[\frac{r}{\delta}]-1} I_1(p) = 2^{k-1}\Big[\frac{r}{\delta}\Big] \le 2^{k-1}\Big(\frac{r}{\delta}\Big) < 2^{k-1}\Big(\frac{r}{\delta}\Big)^{k-1},$$

since $k > 1$ and $r > \delta$.

For $I_2(p)$ we have

$$\begin{aligned}\sum_{p=0}^{[\frac{r}{\delta}]-1} I_2(p) = 2^{k-1}\Big(\frac{r}{\delta}\Big)^{k-2} \sum_{p=0}^{[\frac{r}{\delta}]-1} \Big(1-\frac{p\delta}{r}\Big)^{\frac{k-2}{2}} \\ < 2^{k-1}\Big(\frac{r}{\delta}\Big)^{k-2}\Big(\frac{r}{\delta}\Big) = 2^{k-1}\Big(\frac{r}{\delta}\Big)^{k-1}.\end{aligned}$$

Here we have used the fact that $0 < 1 - \big(\frac{p\delta}{r}\big)^2 \le 1$ for the values of p in the range of summation, as we shall do twice more below.

For $I_3(p)$ we have

$$\begin{aligned}\sum_{p=0}^{[\frac{r}{\delta}]-1} I_3(p) &= 2^{k-1}\Big(\frac{\delta}{r}\Big) \sum_{p=0}^{[\frac{r}{\delta}]-1} \frac{p}{\sqrt{1-\big(\frac{p\delta}{r}\big)^2}} \\ &\le 2^{k-1}\Big(\frac{\delta}{r}\Big)\int_1^{\frac{r}{\delta}} \frac{x}{\sqrt{1-\big(\frac{\delta x}{r}\big)^2}}\,dx \\ &= 2^{k-1}\Big(\frac{r}{\delta}\Big)\int_{\frac{\delta}{r}}^1 \frac{y}{\sqrt{1-y^2}}\,dy \\ &< 2^{k-1}\Big(\frac{r}{\delta}\Big)\int_0^1 \frac{y}{\sqrt{1-y^2}}\,dy \\ &= 2^{k-1}\frac{r}{\delta} \le 2^{k-1}\Big(\frac{r}{\delta}\Big)^{k-1}.\end{aligned}$$

Since $I_4(p) \le \frac{1}{2}I_3(p)$ for $p = 1, 2, \ldots, [\frac{r}{\delta}] - 1$, it is clear that

$$\sum_{p=0}^{[\frac{r}{\delta}]-1} I_4(p) < 2^{k-2} + 2^{k-2}\Big(\frac{r}{\delta}\Big)^{k-1} < 2^{k-1}\Big(\frac{r}{\delta}\Big)^{k-1}.$$

For $I_5(p)$ we have

$$\begin{aligned}\sum_{p=0}^{[\frac{r}{\delta}]-1} I_5(p) = 2^{k-1}\Big(\frac{r}{\delta}\Big)^{k-3} \sum_{p=0}^{[\frac{r}{\delta}]-1} p\Big(1-\frac{p\delta}{r}\Big)^{\frac{k-3}{2}} \\ < 2^{k-2}\Big(\frac{r}{\delta}\Big)^{k-3}\Big(\frac{r}{\delta}\Big)^2 = 2^{k-2}\Big(\frac{r}{\delta}\Big)^{k-1}.\end{aligned}$$

For $I_6(p)$ we have

$$\sum_{p=0}^{[\frac{r}{\delta}]-1} I_6(p) = 2^{k-2}\Big(\frac{r}{\delta}\Big)^{k-3} \sum_{p=0}^{[\frac{r}{\delta}]-1} \Big(1 - \frac{p\delta}{r}\Big)^{\frac{k-3}{2}}$$

$$< 2^{k-2}\Big(\frac{r}{\delta}\Big)^{k-3}\Big(\frac{r}{\delta}\Big) < 2^{k-2}\Big(\frac{r}{\delta}\Big)^{k-1}.$$

Adding all these estimates, we find a sum that is at most

$$6^{(k-1)^2}(2^{k-1}+2^{k-1}+2^{k-1}+2^{k-2}+2^{k-2}+2^{k-2})\Big(\frac{r}{\delta}\Big)^{k-1} \le 6^{(k-1)^2+1}2^{k-1}\Big(\frac{r}{\delta}\Big)^{k-1}.$$

If we wish to count the total number of hypercubes $I^k_{\delta\mathbf{m}}(\delta)$ that intersect the zones $Z_1, \ldots, Z_{[\frac{r}{\delta}]-1}$, we recall our previous observation that such a hypercube can intersect Z_p only if $m_k = p$ or $m_k = p-1$, and if a hypercube $I^k_{\delta\mathbf{m}}(\delta)$ with $m_k = p-1$ intersects Z_p, then so does the hypercube $I^k_{\delta(\mathbf{m}+\mathbf{e}_k)}(\delta)$, and the latter has already been counted. Hence in estimating the total number of hypercubes that intersect one of these zones we are more than safe in simply doubling the estimate we have already obtained.

As for the zone Z_0, any hypercube $I^k_{\delta\mathbf{m}}(\delta)$ with $m_k = -1$ that intersects its bottom edge (the hypersphere S^{k-2}_r) also meets the reflection of Z_0 through the plane $x_k = 0$, and hence the reflection of that hypercube has already been counted among those that meet Z_0. When we double our count to include the hypercubes meeting the "southern" hemisphere, all these hypercubes will automatically be counted. Thus it remains only to estimate the hypercubes that meet the "northern arctic zone," then double the count.

Hence we now consider the cap at the top of the hemisphere, whose boundary is the $(k-2)$-dimensional hypersphere

$$S^{k-2}_s\Big(\Big[\frac{r}{\delta}\Big]\delta\Big),$$

where

$$s = \delta\sqrt{\Big(\frac{r}{\delta}\Big)^2 - \Big[\frac{r}{\delta}\Big]^2}.$$

If $\mathbf{m}$ is such that $I^k_{\delta\mathbf{m}}(\delta)$ meets this set, then we must have $[\frac{r}{\delta}] - 1 \le m_k \le [\frac{r}{\delta}]$, so that there are only two possible values for m_k. As for m_j, $j < k$, we certainly have $-\frac{s}{\delta} - 1 \le m_j \le \frac{s}{\delta}$, so that there are at most $2^k\Big(\frac{s}{\delta}+1\Big)^{k-1}$ such hypercubes not already counted.

Since $[\frac{r}{\delta}] \le \frac{r}{\delta} < [\frac{r}{\delta}] + 1$, we easily find that

$$s \le 2\sqrt{r\delta},$$

so that $\frac{s}{\delta}+1 \le 2\sqrt{\frac{r}{\delta}}+1$. Once more using the inequality $(1+t)^q \le 2^q(1+t^q)$ for positive t, with $q = k-1$, we find that the number of hypercubes meeting the northern polar cap is at most $2^{2k-1}(1 + (\frac{r}{\delta})^{\frac{k-1}{2}})$, which is less than $2^{2k}(\frac{r}{\delta})^{k-1}$

Adding the numbers up and then doubling to count the hypercubes that meet the southern hemisphere, we find that the hypersphere S_r^{k-1} meets at most

$$\left[6^{(k-1)^2+1}2^k + 2^{2k+1}\right)\left(\frac{r}{\delta}\right)^{k-1}.$$

The constant coefficient here is less than 6^{k^2}. This is directly computable for $k = 2$ and $k = 3$. Indeed, the quantity 2^{2k+1} is less than 6^k, while $2^k < 6^{\frac{k}{2}}$. The extremely weak inequality $a + b < ab$, which is valid for positive integers a and b both larger than 1, then implies that for $k \geq 4$ the coefficient is at most

$$6^{(k-1)^2+1+(k/2)+k} = 6^{k^2-k/2+2} \leq 6^{k^2}.$$

Stage 1 in the proof is now complete. We have shown that the $(k-1)$-sphere S_r^{k-1} intersects at most $6^{k^2}\left(\frac{r}{\delta}\right)^{k-1}$ hypercubes from the family $I^k_{\delta\mathbf{m}}(\delta)$. As the volume of each hypercube is δ^k, it follows that the $(k-1)$-sphere is contained in a finite union of cubes of total k-dimensional volume $6^{k^2}r^{k-1}\delta$. Since δ is an arbitrary positive number, the k-dimensional volume of S_r^{k-1} is zero.

We now move on to the second stage of the proof.

Stage 2. Given any convex set H in R^k with non-empty interior, construct a homeomorphism $\mathbf{T}$ of R^k onto itself that maps $S^{k-1} = S_1^{k-1}$ to ∂H, the inside of the unit ball to the interior of H, and the outside to the exterior of H, and satisfies a Lipschitz condition on a neighborhood of S^{k-1}. To get this result we need some more background work on general convex sets in R^k.

Let C be a bounded convex set in R^k, and let $\mathbf{z}$ be an interior point of C. For each point $\mathbf{x}$ on the unit sphere S^{k-1} in R^k, let $\psi(\mathbf{x})$ be the distance from $\mathbf{z}$ to the complement of C in the direction of $\mathbf{x}$, that is,

$$\psi(\mathbf{x}) = \inf\{t > 0 : \mathbf{z} + t\mathbf{x} \notin C\} = \sup\{t > 0 : \mathbf{z} + t\mathbf{x} \in C\}.$$

Step 1. Prove that the function $\psi : S^{k-1} \to (0, +\infty)$ is continuous, in fact, that it satisfies a Lipschitz condition: for some constant K, $|\psi(\mathbf{x}) - \psi(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|$.

There exist positive numbers a and b such that $a \leq \psi(\mathbf{x}) \leq b$ for all $\mathbf{x} \in S^{k-1}$. Indeed we can let a be the radius of the largest open ball about $\mathbf{z}$ that is contained in C and b the radius of the smallest closed ball about $\mathbf{z}$ containing C.

The result now follows from a lemma.

Let $0 \leq s \leq \psi(\mathbf{x})$. Then C contains the open ball of radius $\left(1 - \dfrac{s}{\psi(\mathbf{x})}\right)a$ about $\mathbf{z} + s\mathbf{x}$.

Proof: If $s = \psi(\mathbf{x})$, this ball is empty, and if $s = 0$ the assertion is merely the definition of a. Hence assume $0 < s < \psi(\mathbf{x})$. Now suppose $|\mathbf{y} - (\mathbf{z} + s\mathbf{x})| < \left(1 - \dfrac{s}{\psi(\mathbf{x})}\right)a$. Let $t = 1 - \dfrac{s}{\psi(\mathbf{x})}$, so that $0 < t < 1$, and let $r = \dfrac{1}{2}\min\left(a - \dfrac{|\mathbf{y} - (\mathbf{z} + s\mathbf{x})|}{t}, \dfrac{s}{t}\right)$, so that $r > 0$. Let $\mathbf{w} = \dfrac{\mathbf{y} - (\mathbf{z} + s\mathbf{x})}{t} + r\mathbf{x}$, and let $u = \dfrac{s-tr}{1-t}$. We claim that $\mathbf{z}+\mathbf{w} \in C$ and that $\mathbf{z}+u\mathbf{x} \in C$, so that $\mathbf{y} = t(\mathbf{z}+\mathbf{w}) + (1-t)(\mathbf{z}+u\mathbf{x}) \in C$.

The first claim will follow if we show that $|\mathbf{w}| < a$. In fact

$$\begin{aligned}
|\mathbf{w}| &\le \frac{|\mathbf{y}-(\mathbf{z}+s\mathbf{x})|}{t} + r \quad (\text{since } |\mathbf{x}| = 1),\\
&\le \frac{|\mathbf{y}-(\mathbf{z}+s\mathbf{x})|}{t} + \frac{1}{2}\Big(a - \frac{|\mathbf{y}-(\mathbf{z}+s\mathbf{x})|}{t}\Big)\\
&< \frac{|\mathbf{y}-(\mathbf{z}+s\mathbf{x})|}{t} + a - \frac{|\mathbf{y}-(\mathbf{z}+s\mathbf{x})|}{t} = a.
\end{aligned}$$

The second claim will follow if we prove $0 < u < \psi(\mathbf{x})$. In fact $u = (s - tr)\cdot\dfrac{1}{1-t} = (s-tr)\cdot\dfrac{\psi(\mathbf{x})}{s} = \Big(1 - \dfrac{tr}{s}\Big)\psi(\mathbf{x}) < \psi(\mathbf{x})$. Since $r \le \dfrac{s}{2t}$, we have $u \ge \dfrac{1}{2}\psi(\mathbf{x}) > 0$.

Finally, the last claim is a routine computation:

$$\begin{aligned}
t(\mathbf{z}+\mathbf{w}) + (1-t)(\mathbf{z}+u\mathbf{x}) &= \mathbf{z} + t\mathbf{w} + (1-t)u\mathbf{x}\\
&= \mathbf{z} + \mathbf{y} - (\mathbf{z}+s\mathbf{x}) + tr\mathbf{x} + (1-t)u\mathbf{x}\\
&= \mathbf{y} + \big(tr - s + (1-t)u\big)\mathbf{x}\\
&= \mathbf{y} \quad (\text{since } (1-t)u = s - tr).
\end{aligned}$$

The lemma is now proved.

Taking $\mathbf{y} = \mathbf{z} + s\mathbf{v}$ in this lemma (where $|\mathbf{v}| = 1$), we see that $\mathbf{y} \in C$ (and hence $\psi(\mathbf{v}) \ge s$) if

$$|\mathbf{v}-\mathbf{x}| < \Big(\frac{1}{s} - \frac{1}{\psi(\mathbf{x})}\Big)a.$$

Now let $t > \psi(\mathbf{x})$, $|\mathbf{v}| = 1$, and $|\mathbf{v}-\mathbf{x}| < \Big(\dfrac{1}{\psi(\mathbf{x})} - \dfrac{1}{t}\Big)a$. Choose $t' \in (\psi(\mathbf{x}), t)$ such that

$$|\mathbf{v}-\mathbf{x}| < \Big(\frac{1}{t'} - \frac{1}{t}\Big)a.$$

If $\psi(\mathbf{v}) \ge t$, we have *a fortiori* $\psi(\mathbf{v}) > t'$ and

$$|\mathbf{x}-\mathbf{v}| < \Big(\frac{1}{t'} - \frac{1}{\psi(\mathbf{v})}\Big)a,$$

which, as already shown, implies $\psi(\mathbf{x}) \ge t'$, contradicting the choice of t'. Therefore $\psi(\mathbf{v}) < t$.

To summarize, if $s < \psi(\mathbf{x}) < t$, then $s \le \psi(\mathbf{v}) < t$ provided

$$|\mathbf{x}-\mathbf{v}| < \min\bigg(\Big(\frac{1}{s} - \frac{1}{\psi(\mathbf{x})}\Big)a, \Big(\frac{1}{\psi(\mathbf{x})} - \frac{1}{t}\Big)a\bigg).$$

This proves that ψ is continuous. Specializing to the case where $s = \psi(\mathbf{x}) - \varepsilon$ and $t = \psi(\mathbf{x}) + \varepsilon$, we see that $|\psi(\mathbf{x}) - \psi(\mathbf{v})| \le \varepsilon$ provided $|\mathbf{x}-\mathbf{v}| < \dfrac{a\varepsilon}{\psi(\mathbf{x})(\psi(\mathbf{x})+\varepsilon)}$.

Again, *a fortiori*,

$$|\mathbf{x}-\mathbf{v}| < \frac{a\varepsilon}{b(b+\varepsilon)} \Rightarrow |\psi(\mathbf{x})-\psi(\mathbf{v})| \le \varepsilon.$$

We now claim that

$$|\mathbf{x}-\mathbf{v}| \le \frac{a\varepsilon}{2b^2} \Rightarrow |\psi(\mathbf{x})-\psi(\mathbf{v})| \le \varepsilon.$$

This follows from the previous statement and the continuity of ψ (together with the fact that a closed ball on the sphere S is the closure of the open ball with the same center and radius) if $\varepsilon \le b-a$. If $\varepsilon > b-a$, the second inequality automatically holds because $\psi(\mathbf{x})$ and $\psi(\mathbf{v})$ differ by at most $b-a$. Specializing to equality in the hypothesis, we deduce the Lipschitz inequality

$$|\psi(\mathbf{x})-\psi(\mathbf{v})| \le \frac{2b^2}{a}|\mathbf{x}-\mathbf{v}|.$$

We remark that the statement that $\mathbf{y}$ belongs to the interior, boundary, and exterior of C is equivalent to $|\mathbf{y}-\mathbf{z}| < \psi\Big(\frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|}\Big)$, $|\mathbf{y}-\mathbf{z}| = \psi\Big(\frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|}\Big)$, or $|\mathbf{y}-\mathbf{z}| > \psi\Big(\frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|}\Big)$.

Step 2. Use the function $\psi(\mathbf{x})$ to define a homeomorphism of R^k onto itself that maps S^{k-1} to ∂H and is Lipschitz in a neighborhood of S^{k-1}. Such a homeomorphism $\mathbf{T}(\mathbf{x})$ is defined for all $\mathbf{x} \in R^k$ as follows. We set $\mathbf{T}(\mathbf{0}) = \mathbf{z}$ and

$$\mathbf{T}(\mathbf{x}) = \mathbf{z} + \psi\Big(\frac{\mathbf{x}}{|\mathbf{x}|}\Big)\mathbf{x}$$

if $\mathbf{x} \ne \mathbf{0}$. Since $|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{0})| \le M|\mathbf{x}|$, where $M = \sup\{\psi(\mathbf{y}) : |\mathbf{y}| = 1\}$, it is clear that $\mathbf{T}$ is continuous at $\mathbf{0}$. At all other points it is a composition of continuous functions, hence continuous. Since $\frac{\mathbf{T}(\mathbf{x})-\mathbf{z}}{|\mathbf{T}(\mathbf{x})-\mathbf{z}|} = \frac{\mathbf{x}}{|\mathbf{x}|}$, we have the continuous inverse function

$$\mathbf{x} = \frac{\mathbf{T}(\mathbf{x})-\mathbf{z}}{\psi\big((\mathbf{T}(\mathbf{x})-\mathbf{z})/|\mathbf{T}(\mathbf{x})-\mathbf{z}|\big)},$$

That is, for $\mathbf{y} \ne \mathbf{z}$,

$$\mathbf{T}^{-1}(\mathbf{y}) = \frac{\mathbf{y}-\mathbf{z}}{\psi\big((\mathbf{y}-\mathbf{z})/|\mathbf{y}-\mathbf{z}|\big)},$$

which is not $\mathbf{0}$. Thus the mapping is one-to-one and onto.

The mapping also satisfies a Lipschitz condition on the exterior of each ball about $\mathbf{0}$, that is, on the set $E_\eta = \{\mathbf{x} : |\mathbf{x}| \ge \eta\}$ for each $\eta > 0$. To see this we observe that for any $\mathbf{x}$ and $\mathbf{y}$ in this set,

$$|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})| \;=\; \Big|\psi\Big(\frac{\mathbf{x}}{|\mathbf{x}|}\Big)\mathbf{x} - \psi\Big(\frac{\mathbf{y}}{|\mathbf{y}|}\Big)\mathbf{y}\Big|$$

$$\begin{aligned}
&\leq \left|\psi\Big(\frac{\mathbf{x}}{|\mathbf{x}|}\Big) - \psi\Big(\frac{\mathbf{y}}{|\mathbf{y}|}\Big)\right||\mathbf{y}| + \psi\Big(\frac{\mathbf{x}}{|\mathbf{x}|}\Big)\mathbf{x} - \mathbf{y}| \\
&\leq K\left|\frac{\mathbf{x}}{|\mathbf{x}|} - \frac{\mathbf{y}}{|\mathbf{y}|}\right| + M|\mathbf{x} - \mathbf{y}| \\
&\leq \Big(\frac{2K}{\eta} + M\Big)|\mathbf{x} - \mathbf{y}|.
\end{aligned}$$

Here we have used the fact that

$$\begin{aligned}
\left|\frac{\mathbf{x}}{|\mathbf{x}|} - \frac{\mathbf{y}}{|\mathbf{y}|}\right| &= \frac{1}{|\mathbf{x}|\,|\mathbf{y}|}\big||\mathbf{y}|\mathbf{x} - |\mathbf{x}|\mathbf{y}\big| \\
&= \frac{1}{|\mathbf{x}|\,|\mathbf{y}|}\big|(|\mathbf{y}| - |\mathbf{x}|)\mathbf{x} + |\mathbf{x}|(\mathbf{x} - \mathbf{y})\big| \\
&\leq \frac{1}{|\mathbf{y}|}\big||\mathbf{y}| - |\mathbf{x}|\big| + \frac{1}{|\mathbf{y}|}|\mathbf{x} - \mathbf{y}| \\
&= \frac{2}{|\mathbf{y}|}|\mathbf{y} - \mathbf{x}|.
\end{aligned}$$

The statements about the images of the inside of the unit ball, the unit sphere, and the outside are now obvious. For example, as remarked above, $\mathbf{y} \in \partial H$ if and only if $|\mathbf{y} - \mathbf{z}| = \psi\Big(\frac{\mathbf{y} - \mathbf{z}}{|\mathbf{y} - \mathbf{z}|}\Big)$. But this is equivalent to the statement that $\mathbf{T}^{-1}(\mathbf{y}) = \frac{\mathbf{y}-\mathbf{z}}{|\mathbf{y}-\mathbf{z}|}$, which says precisely that $\mathbf{T}^{-1}(\mathbf{y})$ belongs to the unit sphere.

We have now finished Stage 2 of the proof and are ready for the third and final stage.

Stage 3. For each $\delta > 0$, approximate a function $f(\mathbf{x})$ that is continuous on H by a function $f_\delta(\mathbf{x})$ that is continuous on all of R^k and such that the iterated integrals of f and f_δ differ by at most a fixed multiple of δ no matter what order they are taken in.

To that end, we first let $\delta \in (0, 1/\sqrt{k})$ be given. According to what was proved in Stage 1, the hypersphere S^{k-1} is contained in the interior of the set of hypercubes $I^k_{\delta\mathbf{m}}(\delta)$ that intersect it, and there are at most $6^{k^2}\delta^{1-k}$ of these hypercubes. In each hypercube $I^k_{\delta\mathbf{m}}(\delta)$ from this family we choose and keep fixed one point $\mathbf{x_m}$ belonging to S^{k-1}. The image of these hypercubes under $\mathbf{T}$ is a compact set containing ∂H in its interior, and each of them is contained in a hypercube of side at most $2L\sqrt{k}\delta$ centered at $\mathbf{T}(\mathbf{x_m}) \in \partial H$, where L is the Lipschitz constant for the mapping $\mathbf{T}$ on the set $E_{1-\delta}$, so that the total volume of these hypercubes is at most $6^{k^2}(2L\sqrt{k})^k\delta$. Let $c > 0$ be the distance from H to the complement of the union of these hypercubes.

We define $f_\delta(\mathbf{x})$ as a continuous function that equals $f(\mathbf{x})$ for $\mathbf{x} \in H$, while for $\mathbf{x}$ not in the interior of H we set $f_\delta(\mathbf{x}) = \max\big(0, 1 - \frac{d(\mathbf{x},H)}{c}\big)f\big(\theta(\mathbf{x})\big)$. Here $\theta(\mathbf{x})$ is the unique point of H closest to $\mathbf{x}$ and $d(\mathbf{x}, H)$ is the distance from $\mathbf{x}$ to H. On the boundary of H, where we have apparently given two definitions of f_δ we have $d(\mathbf{x}, H) = 0$, so that the two definitions are consistent. Hence the piecewise-defined function will be continuous if each of the pieces is. The piece defined on H is continuous by assumption, so that we need only concern ourselves with the second definition. It is well-known that $d(\mathbf{x}, H)$ is a continuous function of $\mathbf{x}$. It is somewhat less obvious that $\theta(\mathbf{x})$ is continuous, so that we must prove that fact.

First we show that there *is* a unique point $\theta(\mathbf{x})$ in H closest to $\mathbf{x}$. This is obvious if $\mathbf{x} \in H$, so we assume $\mathbf{x} \notin H$. Let $c = \min\{|\mathbf{x} - \mathbf{z}| : \mathbf{z} \in H\}$, and suppose $\mathbf{z}$ and $\mathbf{w}$ are two points of H such that $|\mathbf{x} - \mathbf{z}| = c = |\mathbf{x} - \mathbf{w}|$. Then the point $\mathbf{w} + t(\mathbf{z} - \mathbf{w})$ belongs to H for $0 \le t \le 1$, and so the quadratic function $|\mathbf{x} - \mathbf{w} - t(\mathbf{z} - \mathbf{w})|^2 = |\mathbf{x} - \mathbf{w}|^2 - 2t(\mathbf{x} - \mathbf{w}) \cdot (\mathbf{z} - \mathbf{w})) + t^2|\mathbf{z} - \mathbf{w}|^2$ has its minimum value c on $[0, 1]$ at *both* endpoints. But this is impossible for a non-constant quadratic function whose leading coefficient is positive. Hence the function is constant, that is, $\mathbf{z} = \mathbf{w}$. Now suppose $\mathbf{x}_n \to \mathbf{x}$. We claim $\theta(\mathbf{x}_n) \to \theta(\mathbf{x})$.

Since H is compact, we can pass to a subsequence if necessary and assume that $\theta(\mathbf{x}_n) \to \mathbf{z}$ for some point $\mathbf{z} \in H$. Certainly $|\mathbf{x}_n - \theta(\mathbf{x}_n)| \to |\mathbf{x} - \mathbf{z}|$. But $|\mathbf{x}_n - \theta(\mathbf{x}_n)| = d(\mathbf{x}_n, H) \to d(\mathbf{x}, H)$, so that $|\mathbf{x} - \mathbf{z}| = d(\mathbf{x}, H) = |\mathbf{x} - \theta(\mathbf{x})|$. As H contains only one point satisfying this equality, we must have $\mathbf{z} = \theta(\mathbf{x})$. Thus $\theta(\mathbf{x})$ is a continuous function, and therefore $f_\delta(\mathbf{x})$ is continuous on all of R^k.

It is now clear that $|f(\mathbf{x})|$ and $|f_\delta(\mathbf{x})|$ have the same maximum value, say J, and that f and f_δ differ only on the finite set of hypercubes covering ∂H. The iterated integrals of the two functions, taken in any order, over this finite set of hypercubes differ by at most $6^{k^2} JL(2\sqrt{k})^k \delta$. Thus the iterated integral of f differs from the iterated integral of f_δ by at most this amount, and since all the iterated integrals of f_δ are equal, it follows that any two iterated integrals of f differ by arbitrarily small amounts, hence are equal.

The proof is, at long last, complete.

Because this proof is so long and involved, it may be worthwhile to look at an alternative proof that works only for the case $k = 2$ and does not generalize to higher dimensions. To this end, let $k = 2$. we define two functions $m(x)$ and $M(x)$, as follows: The domain of both functions is the projection of H on the x-axis, that is, the set $\prod(H)$ consisting of x such that there exists y for which $(x, y) \in H$. By definition $m(x)$ is the minimal y for which $(x, y) \in H$, and $M(x)$ is the maximal y for which $(x, y) \in H$. We claim that these functions are continuous on $\prod(H)$. Indeed, suppose $(x^{(n)}, y^{(n)}) \in H$ and $x^{(n)} \to x$. Without loss of generality we can assume that $x^{(n)} > x$ for all x. (By passing to a subsequence if necessary, we can have either $x^{(n)} < x$ for all n or $x^{(n)} > x$ or $x^{(n)} = x$ for all n. The last case is trivial, and the other two cases are handled by identical arguments.) Some subsequence of $M(x^{(n)})$ converges to a value z. Since $(x^{(n)}, M(x^{(n)})) \in H$, and H is closed, it follows that (x, z) belongs to H. It is clear then that the assumption $z > M(x)$ contradicts the definition of $M(x)$ as the maximal number y for which $(x, y) \in H$. Hence it suffices to prove that $z \ge M(x)$. This will certainly be the case if $M(x^{(n)} \ge M(x)$ for all n. Hence assume that n_0 is an index for which $M(x^{(n_0)}) < M(x)$. Now $x^{(n_0)} > x$, since if the two were equal, $M(x^{(n_0)})$ would equal $M(x)$. We observe that if $t \in [0, 1]$, then the point $(tx^{(n_0)} + (1 - t)x, tM(x^{(n_0)}) + (1 - t)M(x))$ belongs to H. In particular, taking $t = \frac{x^{(n)}-x}{x^{(n_0)}-x}$, we find that $tx^{(n_0)} + (1-t)x = x^{(n)}$. It therefore follows that $M(x^{(n)}) \geq \frac{x^{(n)}-x}{x^{(n_0)}-x}M(x^{(n_0)} + \frac{x^{(n_0)}-x^{(n)}}{x^{(n_0)}-x}M(x) \to M(x)$. Therefore $z \geq M(x)$.

(It is this part of the argument that does not generalize to R^3, as shown by the the convex set

$$H = \{(1-t, ty, tz) : 0 \leq t \leq 1, -1 \leq y \leq 1, y^2 \leq z \leq 1\}.$$

On this set, if we define $M(y,z) = \sup\{x : (x,y,z) \in H\}$, we have $M(s,s^2) = 0$ for $s \neq 0$, but $M(0,0) = 1$.)

It now follows that $M(x)$ is continuous on H, and the proof that $m(x)$ is continuous is similar.

Now let H be a convex closed set in R^2 containing an interior point. For each $\delta > 0$, we let H_δ be the δ-neighborhood of H, that is, the set of points whose distance from H is at most δ. It is clear that H_δ is a convex set containing H in its interior. If f is a continuous function on H, we extend f to a function f_δ defined on all of R^2, as above.

By our definition

$$\int_H f(x,y)\,dy\,dx = \int_a^b \int_{m(x)}^{M(x)} f(x,y)\,dy\,dx,$$

where $[a,b]$ is the projection of H on the x-axis and for each $x \in [a,b]$

$$m(x) = \min\{t : (x,t) \in H\}$$

and

$$M(x,y) = \max\{t : (x,t) \in H\}.$$

We intend to show that the when these integrals are evaluated, the resulting value is the limit of the same integrals evaluated for f_δ, and of course the same for the integrals in reverse order. Hence these two iterated integrals are equal.

To that end, let A be the maximal value of $|f(x,y)|$, which is also the maximal value of $|f_\delta(x,y)|$. As we have set $f(x,y) = 0$ on the complement of H, the two functions $f(x,y)$ and $f_\delta(x,y)$ differ only on the set $H_\delta \setminus H$, and by no more than A at any point.

Let $P_\delta = [a - \lambda(\delta), b + \mu(\delta)]$ be the projection of H_δ on the x-axis. We claim that $\lambda(\delta)$ and $\mu(\delta)$ both tend to zero as δ tends to zero. For certainly $\lambda(\delta)$ decreases as δ decreases. Let its limit be c. There is a point $(a - \lambda(\delta), y_\delta) \in H_\delta$ for each $\delta > 0$. If y is a limit point of y_δ as $\delta \to 0$, then, since $(a - \lambda(\eta), y_\eta) \in H_\eta \subset H_\delta$ for $\eta < \delta$ and H_δ is closed, it follows that $(a-c, y) \in H_\delta$ for all $\delta > 0$, and therefore, since $\bigcap_{\delta>0} H_\delta = H$, that $(a-c, y) \in H$. By definition of a, it then follows that $a \leq a - c \leq a$, and so $c = 0$. The proof that $\mu \to 0$ is similar.

Let $m_\delta(x)$ and $M_\delta(x)$ be the functions corresponding to $m(x)$ and $M(x)$ for H_δ. For $x \in [a,b]$ we have $m_\delta(x) < m(x) \leq M(x) < M_\delta(x)$. An argument similar to the one just given shows that $m(x) - m_\delta(x)$ and $M_\delta(x) - M(x)$ tend monotonically to zero for each $x \in [a,b]$. Since these are continuous functions, this convergence is *uniform.* let $\varphi(\delta) = \max_{x\in[a,b]}\{m(x) - m_\delta(x), M_\delta(x) - M(x)\}$, so that $\varphi(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$.

Now the double integrals that we wish to evaluate are

$$\int_a^b \int_{m(x)}^{M(x)} f(x,y)\, dy\, dx$$

and

$$\int_{a-\lambda(\delta)}^{b+\mu(\delta)} \int_{m_\delta(x)}^{M_\delta(x)} f_\delta(x,y)\, dy\, dx.$$

Now fix a number N larger than twice the absolute value of any coordinate of any point in H_1, and assume $\delta < \min(1, N)$. We observe that the difference between the two integrals is

$$\int_{a-\lambda(\delta)}^{a} \int_{-N}^{N} f_\delta(x,y)\, dy\, dx + \int_a^b \int_{m_\delta(x)}^{m(x)} f_\delta(x,y)\, dy\, dx$$
$$+ \int_a^b \int_{M(x)}^{M_\delta(x)} f_\delta(x,y)\, dy\, dx + \int_b^{b+\mu(\delta)} \int_{-N}^{N} f_\delta(x,y)\, dy\, dx.$$

This expression is assuredly not larger than

$$2AN\big(\lambda(\delta) + \varphi(\delta) + \mu(\delta)\big),$$

and hence it tends to zero as $\delta \longrightarrow 0$. The same is true of the integral in the reverse order, and for the same reasons. Since the integral of f_δ is the same in either order, it follows that the integral of f is also the same in either order.

Exercise 10.2 For $i = 1, 2, 3, \ldots$, let $\varphi_i \in C(R^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x,y) = \sum_{i=1}^{\infty} \big[\varphi_i(x) - \varphi_{i+1}(x)\big]\varphi_i(y).$$

The f has compact support in R^2, f is continuous except at $(0,0)$, and

$$\int dy \int f(x,y)\, dx = 0 \text{ but } \int dx \int f(x,y)\, dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0,0)$.

Solution: The computation is straightforward:

$$\int f(x,y)\, dx = \sum_{i=1}^{\infty} \varphi_i(y)[1-1] = 0;$$

$$\int f(x,y)\,dy = \sum_{i=1}^{\infty} \left[\varphi_i(x) - \varphi_{i+1}(x)\right] = \varphi_1(x).$$

To justify the first of these, we observe that the sum is finite for each fixed y, since $\varphi_i(y) = 0$ for $i > -\log_2(y)$ if $y > 0$. Likewise the second sum is finite for each fixed x. The result now follows. The function must be unbounded, since the integral of φ_i must be 1, even though the support of that function has length 2^{-i}.

Exercise 10.3 (*a*) If $\mathbf{F}$ is as in Theorem 10.7, put $\mathbf{A} = \mathbf{F}'(0)$, $\mathbf{F}_1(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{F}(\mathbf{x})$. Then $\mathbf{F}_1'(\mathbf{0}) = I$. Show that

$$\mathbf{F}_1(\mathbf{x}) = \mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x})$$

in some neighborhood of $\mathbf{0}$, for certain primitive mappings $\mathbf{G}_1, \ldots, \mathbf{G}_n$. This gives another version of Theorem 10.7:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}'(\mathbf{0})\mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x}).$$

(*b*) Prove that the mapping $(x, y) \to (y, x)$ of R^2 onto R^2 is not the composition of any two primitive mappings, in any neighborhood of the origin. (This shows that the flips B_1 cannot be omitted from the statement of Theorem 10.7.)

Solution: (*a*) According to the proof of Theorem 10.7, the flips are needed only to interchange m and k, where k is the first index not less than m for which $D_m\alpha_k(\mathbf{0}) \neq 0$. Here

$$\mathbf{F}_m'(\mathbf{0})\mathbf{e}_m = \sum_{i=m}^{n} (D_m\alpha_i)(\mathbf{0})\mathbf{e}_i.$$

But in that proof $\mathbf{F}_1 = \mathbf{F}$, and since in the present case $\mathbf{F}'(\mathbf{0})$ is the identity, B_1 is the identity. But then the definition of $\mathbf{G}_1(\mathbf{x})$ as

$$\mathbf{G}_1(\mathbf{x}) = \mathbf{x} + \left[\alpha_1(\mathbf{x}) - x_1\right]\mathbf{e}_1$$

implies that $\mathbf{G}_1'(\mathbf{0})$ is also the identity. Suppose we know that B_j, $\mathbf{F}_m'(\mathbf{0})$, and $\mathbf{G}_j'(\mathbf{0})$ are all equal to the identity for $j \leq m$. Then the inductive definition of $\mathbf{F}_{m+1}$ as $\mathbf{F}_{m+1}(\mathbf{y}) = \mathbf{F}_m \circ \mathbf{G}_m^{-1}(\mathbf{y})$ implies that $\mathbf{F}_{m+1}'(\mathbf{0})$ is also the identity, from which it then follows that $\mathbf{F}_{m+1}'(\mathbf{0})$, B_{m+1}, and $\mathbf{G}_{m+1}'(\mathbf{0})$ are all equal to the identity. Thus the decomposition of $\mathbf{F}_1$ involves no flips, as asserted.

(*b*) If this map were a composition of two primitive maps, its derivative at $(0,0)$ would be the product of two matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} a+bc & bd \\ c & d \end{pmatrix}.$$

Since this matrix must be $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it follows that $c = 1$, $d = 0$. But then the second column of the product of the two matrices is zero, which is a contradiction.

Exercise 10.4 For $(x, y) \in R^2$, define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y).$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, where

$$\begin{aligned} \mathbf{G}_1(x, y) &= (e^x \cos y - 1, y) \\ \mathbf{G}_2(x, y) &= (u, (1+u)\tan v) \end{aligned}$$

are primitive in some neighborhood of $(0, 0)$.

Compute the Jacobians of $\mathbf{G}_1$, $\mathbf{G}_2$, and $\mathbf{F}$ at $(0, 0)$. Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ in some neighborhood of $(0, 0)$.

Solution: The equation $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ is a routine computation, and the fact that $\mathbf{G}_1$ and $\mathbf{G}_2$ are primitive is immediate.

The Jacobians of $\mathbf{G}_1$ and $\mathbf{G}_2$ are

$$\mathbf{G}_1'(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{pmatrix}, \quad \mathbf{G}_2'(u, v) = \begin{pmatrix} 1 & 0 \\ \tan v & (1+u)\sec^2 v \end{pmatrix},$$

so that each of them equals the identity at $(0, 0)$. It therefore follows that $\mathbf{F}'(0, 0) = I$ also.

If we take $h(u, v) = (\sqrt{e^{2u} - v^2} - 1, v)$, the primitive mapping $\mathbf{H}_1(u, v) = (h(u, v), v)$ will yield $\mathbf{H}_1 \circ \mathbf{H}_2 = \mathbf{F}$.

Exercise 10.5 Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space. (Replace the functions φ_i that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 22 of Chap. 4.)

Solution: We are given a compact set K in a metric space X (say with metric d) and a cover of K by open sets V_i, $i = 1, 2, \ldots, n$. (We may as well assume a finite number of sets, since we can find a finite subcover of any infinite cover.)

We need to construct continuous functions ψ_i, $i = 1, 2, \ldots, n$ such that $0 \le \psi_i(x) \le 1$ for all i and all $x \in X$, the support of $\psi_i(x)$ is contained in V_i, and $\sum_{i=1}^{n} \psi_i(x) = 1$ for all $x \in K$.

To do this, let $\eta > 0$ be a Lebesgue number for the covering of K by the sets V_i, that is such that the η-neighborhood of every point $x \in K$ is contained in some V_i. Let $\varepsilon \in (0, \eta)$, and let U_i be the set of points whose distance from the complement of V_i is larger than ε and W_i the set of points whose distance from the complement of V_i is larger than $\frac{\epsilon}{2}$. Since the distance from x to the complement of V_i is a continuous function of x, it follows that U_i and W_i are open sets. It is obvious that the closure of U_i is contained in W_i and the closure of W_i is contained in V_i. We note that $K \subset \bigcup_{i=1}^{n} U_i$. For if $x \in K$ there exists V_i such that the η-neighborhood of x is contained in V_i, and hence the distance from x to the complement of that V_i is at least η.

Now let A_i be the closure of U_i, and B_i the complement of W_i. Define

$$\varphi_i(x) = \frac{d(x, B_i)}{d(x, A_i) + d(x, B_i)}.$$

Then $\varphi_i(x)$ is 1 on A_i (and hence certainly on U_i) and 0 on B_i, $\varphi_i(x)$ is continuous, and $0 \leq \varphi_i(x) \leq 1$ for all x. Since the support of $\varphi_i(x)$ is the closure of W_i, it is contained in V_i. Since $\varphi_i(x) > 0$ for x in W_i, the sum $\varphi(x) = \sum_{i=1}^{n} \varphi_i(x)$ is positive on the open set $W = \bigcup_{i=1}^{n} W_i$, which contains K. Now let L be the complement of W, and define a continuous function $\psi(x)$ by

$$\psi(x) = \frac{d(x, L)}{d(x, K) + d(x, L)},$$

so that $0 \leq \psi(x) \leq 1$ for all x, $\psi(x) = 1$ if $x \in K$, and $\psi(x) = 0$ if $x \in L$. If we now define $\psi_i(x) = 0$ for $x \notin W$ and

$$\psi_i(x) = \frac{\varphi_i(x)\psi(x)}{\varphi(x)},$$

then $\psi_i(x)$ is continuous on the entire space. Its restriction to L is continuous. If we can show that its restriction to the closure of W is continuous, we shall be done. But it is obvious that it is continuous on W itself, and so we need only show that it is continuous at a point of ∂W. Hence let $x_n \to x \in \partial W$. Since $\varphi_i(x)/\varphi(x)$ is bounded, and $\psi(x_n) \to 0$, it follows that $\psi_i(x_n) \to 0 = \psi_i(x)$, and hence ψ_i is continuous at x.

The construction is now complete.

Exercise 10.6 Strengthen the conclusion of Theorem 10.8 by showing that the functions ψ_i can be made differentiable, and even infinitely differentiable. (Use Exercise 1 of Chap. 8 in the construction of the auxiliary functions φ_i.)

Solution: The function $\varphi_i(\mathbf{x})$ is required to have only three properties: 1) $\varphi_i(\mathbf{x}) = 1$ for $|\mathbf{x} - \mathbf{a}_i| \leq r_i$; 2) $\varphi_i(\mathbf{x}) = 0$ for $|\mathbf{x} - \mathbf{a}_i| \geq s_i$; 3) $0 =\leq \varphi_i(\mathbf{x}) \leq 1$ for all $\mathbf{x}$. These properties can be achieved with an infinitely differentiable function $\varphi_i(\mathbf{x})$. To construct such a function, we go to the function $f(t)$ in Exercise 1 of Chapter 8, namely

$$f(t) = e^{-\frac{1}{t^2}}$$

for $t \neq 0$ and $f(0) = 0 = f^{(n)}(0)$ for all positive integers n, $f^{(n)}(t)$ being the nth derivative of $f(t)$. It was established in that exercise that $f(t)$ is infinitely differentiable, and it is obvious that $f(t)$ is strictly increasing for nonnegative values of t.

Let

$$g(t) = \frac{f\big(f(1) - f(t)\big)}{f\big(f(1)\big)}.$$

Then it is obvious that $g(t)$ is an infinitely differentiable function that decreases from 1 to 0 as x increases from 0 to 1. If we show that $g^{(n)}(0) = 0 = g^{(n)}(1)$ for all positive integers n, it will follow that the function

$$h(t) = \begin{cases} 1, & t \leq 0, \\ g(t), & 0 \leq t \leq 1, \\ 0, & 1 \leq t \end{cases}$$

is also a $\mathcal{C}^\infty$ function, and we can then take

$$\varphi(\mathbf{x}) = h\Big(\frac{|\mathbf{x}|^2 - r_i^2}{s_i^2 - r_i^2}\Big).$$

But it is easy to prove these properties by showing inductively that for all integers j and k with $0 \leq j \leq n - k$ and $1 \leq k \leq n$ there exist infinitely differentiable functions $\theta_{j,k,n}(t)$ such that

$$g^{(n)}(t) = \sum_{\substack{0 \leq j \leq n-k \\ 1 \leq k \leq n}} \theta_{j,k,n}(t) f^{(k)}\big(f(1) - f(t)\big) f^{(n-k-j+1)}(t).$$

In fact the chain rule shows that

$$\theta_{0,1,1}(t) = -\frac{1}{f\big(f(1)\big)}.$$

Then, assuming there exist such functions $\theta_{j,k,n}(t)$, we find

$$\begin{aligned} g^{(n+1)}(t) \quad = \quad \sum_{\substack{0 \leq j \leq n-k \\ 1 \leq k \leq n}} \Big\{ & \theta'_{j,k,n}(t) f^{(k)}\big(f(1) - f(t)\big) f^{(n-k-j+1)}(t) \\ & + \theta_{j,k,n}(t)\big(-f'(t)\big) f^{(k+1)}\big(f(1) - f(t)\big) f^{n-k-j+1}(t) \\ & + \theta_{j,k,n}(t) f^{(k)}\big(f(1) - f(t)\big) f^{(n-k-j+2)}. \Big\} \end{aligned}$$

Each term in this expression contains a factor $f^{(s)}\big(f(1) - f(t)\big) f^{(n+1-s-r+1)}(t)$ with $0 \leq r \leq n + 1 - s$, $1 \leq s \leq n + 1$ and with a coefficient that is infinitely differentiable. Thus when suitably rearranged, this sum has the appropriate form

$$g^{(n+1)}(t) = \sum_{\substack{0 \leq j \leq n+1-k \\ 1 \leq k \leq n+1}} \theta_{j,k,n+1}(t) f^{(k)}\big(f(1) - f(t)\big) f^{(n-k-j+1)}(t)$$

with infinitely differentiable functions $\theta_{j,k,n+1}$. Since each term contains a factor $f^{(k)}(f(1)-f(t))f^{(l)}(t)$ with $k \geq 1$, it follows that each term vanishes when $t = 0$ or $t = 1$, and hence that $g^{(n)}(1) = 0 = g^{(n)}(0)$ for $n = 1, 2, \ldots$.

Exercise 10.7 (*a*) Show that the simplex Q^k is the smallest convex subset of R^k that contains $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k$.

(*b*) Show that affine mappings take convex sets to convex sets.

Solution: (*a*) By definition $Q^k = \{\mathbf{x} : x_1 + \cdots x_k \leq 1, x_j \geq 0, j = 1, \ldots, k\}$. It is obvious that Q^k contains all the points $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_k$. It is nearly obvious that Q^k is convex. Indeed, if $\mathbf{x}$ and $\mathbf{y}$ are points of Q^k and $0 < t < 1$, then $t\mathbf{x} + (1-t)\mathbf{y} = \mathbf{z}$, where $z_j = tx_j + (1-t)y_j$. Since $x_j \geq 0$ and $y_j \geq 0$ and $0 < t < 1$, it is clear that $z_j \geq 0$. Simple algebra shows that $z_1 + \cdots + z_k \leq t + (1-t) = 1$, so that $\mathbf{z} \in Q^k$ also. Thus Q^k is convex.

Now let C be any convex set containing these points, and let $\mathbf{x} \in Q^k$. We need to show that $\mathbf{x} \in C$. We shall show by induction that the point $x_1\mathbf{e}_1 + \cdots + x_j\mathbf{e}_j$ is in C whenever $x_1 \geq 0, \ldots, x_j \geq 0$ and $x_1 + \cdots + x_j \leq 1$. If $j = 1$, this is obvious, since $x_1\mathbf{e}_1 = x_1\mathbf{e}_1 + (1-x_1)\mathbf{0}$ and by assumption $0 \leq x_1 \leq 1$.

Suppose the theorem is true for j, and let $c = x_1 + \cdots + x_{j+1} \leq 1$, $x_1 \geq 0, \ldots, x_{j+1} \geq 0$. If $c = 0$, the point $x_1\mathbf{e}_1 + \cdots + x_{j+1}\mathbf{e}_{j+1}$ is $\mathbf{0}$, and hence belongs to C. Therefore we assume $c > 0$. Since $\mathbf{e}_{j+1} \in C$, we need only consider the case $x_{j+1} < 1$. By the induction assumption, taking $x'_l = \frac{x_l}{1-x_{j+1}}$ for $l = 1, \ldots, j$, we find that the point $\mathbf{y} = x'_1\mathbf{e}_1 + \cdots + x'_j\mathbf{e}_j$ belongs to C, and therefore the point $(1-x_{j+1})\mathbf{y} + x_{j+1}\mathbf{e}_{j+1} = x_1\mathbf{e}_1 + \cdots + x_j\mathbf{e}_j + x_{j+1}\mathbf{e}_{j+1}$ does also.

(*b*) Let $\mathbf{A}(\mathbf{x})$ be an affine mapping, that is, $\mathbf{A}(\mathbf{x}) = \mathbf{x}_0 + \mathbf{T}(\mathbf{x})$, where $\mathbf{T}(\mathbf{x})$ is a linear transformation, let C be any convex set, and let $\mathbf{u} \in \mathbf{A}(C)$, $\mathbf{v} \in \mathbf{A}(C)$. We need to show that $t\mathbf{u} + (1-t)\mathbf{v} \in \mathbf{A}(C)$ for all $t \in (0,1)$. But this is trivial, since if $\mathbf{u} = \mathbf{A}(\mathbf{x})$ and $\mathbf{v} = \mathbf{A}(\mathbf{y})$, then $t\mathbf{u} + (1-t)\mathbf{v} = \mathbf{A}(t\mathbf{x} + (1-t)\mathbf{y})$ and $t\mathbf{x} + (1-t)\mathbf{y} \in C$.

Exercise 10.8 Let H be the parallelogram in R^2 whose vertices are $(1,1)$, $(3,2)$, $(4,5)$, $(2,4)$. Find the affine map T which sends $(0,0)$ to $(1,1)$, $(1,0)$ to $(3,2)$, $(0,1)$ to $(2,4)$. Show that $J_T = 5$. Use T to convert the integral

$$\alpha = \int_H e^{x-y}\,dx\,dy$$

to an integral over I^2 and thus compute α.

Solution: Clearly the constant term in an affine mapping is the image of $(0,0)$, which in the present case is to be $(1,1)$. Thus we are looking for a linear transformation L such that $(3,2) = (1,1) + L(1,0)$ and $(2,4) = (1,1) + L(0,1)$, which is to say $L(1,0) = (2,1)$ and $L(0,1) = (1,3)$. Obviously $L(x,y) = (2x+y, x+3y)$. Then $J_T = 2\cdot 3 - 1\cdot 1 = 5$. The inverse of T is given by $T^{-1}(u,v) = L^{-1}((u,v)-(1,1))$. Simple algebra then reveals that

$$T^{-1}(u,v) = \Big(\frac{-2+3u-v}{5}, \frac{-1-u+2v}{5}\Big).$$

The parallelogram H is the image of the unit square S under T, and so

$$\alpha = \int_{T(S)} e^{x-y}\,dx\,dy = \int_S e^{T^{-1}(u,v)}|J_{T^{-1}}|\,du\,dv.$$

Thus

$$\begin{aligned}
\alpha &= \int_0^1\int_0^1 e^{\frac{-1+4u-3v}{5}}\frac{1}{5}\,du\,dv \\
&= \frac{e^{-\frac{1}{5}}}{5}\int_0^1 e^{\frac{4u}{5}}\,du\int_0^1 e^{\frac{-3v}{5}}\,dv \\
&= e^{-\frac{1}{5}}\cdot\frac{5}{4}\cdot(e^{\frac{4}{5}}-1)\cdot\Big(\frac{-5}{3}\Big)\cdot(e^{\frac{-3}{5}}-1).
\end{aligned}$$

Exercise 10.9 Define $(x,y) = T(r,\theta)$ on the rectangle

$$0 \le r \le a, \quad 0 \le \theta \le 2\pi$$

by the equations

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Show that T maps this rectangle onto the closed disc D with center at $(0,0)$ and radius a, that T is one-to-one on the interior of the rectangle, and that $J_T(r,\theta) = r$. If $f \in \mathcal{C}(D)$, prove the formula for integration in polar coordinates:

$$\int_D f(x,y)\,dx\,dy = \int_0^a\int_0^{2\pi} f(T(r,\theta))r\,dr\,d\theta.$$

Hint: Let D_0 be the interior of D, minus the interval from $(0,0)$ to $(0,a)$. As it stands, Theorem 10.9 applies to continuous functions whose support lies in D_0. To remove this restriction, proceed as in Example 10.4.

Solution: The simple geometry of this transformation allows a fairly straightforward proof. Let $\varepsilon \in (0, \min(\pi, a/2))$. Let $H_\varepsilon = \{(r,\theta) : \varepsilon \le r \le a-\varepsilon, \varepsilon \le \theta \le 2\pi-\varepsilon\}$. The transformation T is one-to-one on H_ε. Let $\varphi_\varepsilon(x,y)$ be a continuous function on all of R^2 such that $\varphi_\varepsilon(x,y) = 1$ for $(x,y) \in T(H_\varepsilon)$, $\varphi_\varepsilon(x,y) = 0$ for $(x,y) \notin T(H_{\varepsilon/2})$ and $0 \le \varphi_\varepsilon(x,y) \le 1$. Define $f_\varepsilon(x,y) = f(x,y)\varphi_\varepsilon(x,y)$ for $(x,y) \in D$ and $f_\varepsilon(x,y) = 0$ for $(x,y) \notin D$. Then $f_\varepsilon(x,y) = f(x,y)$ except for $(x,y) \in D \setminus T(H_\varepsilon)$. Hence $f_\varepsilon(T(x,y)) = f(x,y)$ on $[0,a]\times[0,2\pi]\setminus H_\varepsilon$. Let

M be the maximum of $|f(x,y)|$ on D. Since the support of f_ε is contained in $T(H_{\varepsilon/2})$, which in turn is contained in D_0, we certainly have

$$\int_{R^2} f_\varepsilon(x,y)\,dx\,dy = \int_{R^2} f_\varepsilon(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

We need to see how much each of these integrals differs from the corresponding integral of f. We first look at $f_\varepsilon(x,y)$. In evaluating its integral we can confine ourselves to the square $-a \le x \le a$, $-a \le y \le a$, since D is contained in that square. We first exclude the three intervals $-a \le y \le -a+\varepsilon$, $\min(-\varepsilon, -a\sin\varepsilon) \le y \le \max(\varepsilon, a\sin\varepsilon)$, and $a-\varepsilon \le y \le a$. When y is not in these intervals, we have $\varepsilon^2 \le y^2 \le (a-\varepsilon)^2$, and $f(x,y)$ and $f_\varepsilon(x,y)$ can differ only on the two intervals where $\sqrt{(a-\varepsilon)^2 - y^2} \le |x| \le \sqrt{a^2-y^2}$, each of which has length

$$\begin{aligned}\sqrt{a^2-y^2} - \sqrt{(a-\varepsilon)^2-y^2} &= \frac{(a^2-y^2) - ((a-\varepsilon)^2 - y^2)}{\sqrt{a^2-y^2} + \sqrt{(a-\varepsilon)^2-y^2}} \\ &\le \frac{2a\varepsilon}{\sqrt{a^2-y^2}} \le \frac{2a\varepsilon}{\sqrt{2a\varepsilon+\varepsilon^2}} \le \frac{2a}{\sqrt{2a+\varepsilon}}\sqrt{\varepsilon} \le \sqrt{2a\varepsilon}.\end{aligned}$$

Since the maximum possible difference between $f(x,y)$ and $f_\varepsilon(x,y)$ is M, we see that

$$\left|\int f(x,y)\,dx - \int f_\varepsilon(x,y)\,dx\right| \le 2M\sqrt{2a\varepsilon}$$

if y is not in one of the three excluded intervals.

If y is in one of the three excluded intervals, since f and f_ε can differ by at most M, we have

$$\left|\int f(x,y)\,dx - \int f_\varepsilon(x,y)\,dx\right| \le 2Ma.$$

Since the total length of the excluded y-intervals is at most $(2a+3)\varepsilon$, and the total length of the interval over which y varies is at most $2a$, we see that

$$\left|\int\int f(x,y)\,dx\,dy - \int\int f_\varepsilon(x,y)\,dx\,dy\right| \le 4Ma\sqrt{2a\varepsilon} + 2Ma(2a+3)\varepsilon.$$

Thus this approximation can be made arbitrarily good by taking ε sufficiently small.

As for the integral with respect to r, θ, we observe that we can confine ourselves to the rectangle $0 \le r \le a$, $0 \le \theta \le 2\pi$, and that $f_\varepsilon(r\cos\theta, r\sin\theta) = f(r\cos\theta, r\sin\theta)$ for $\varepsilon \le r \le a-\varepsilon$ and $\varepsilon \le \theta \le 2\pi - \varepsilon$. Thus, excluding the intervals $0 \le \theta \le \varepsilon$, and $2\pi - \varepsilon \le \theta \le 2\pi$, we find that for θ not in these intervals $f(r\cos\theta, r\sin\theta)$ and $f_\varepsilon(r\cos\theta, r\sin\theta)$ can differ (by at most M) only on the two intervals $0 \le r \le \varepsilon$ and $a-\varepsilon \le r \le a$. Hence as before, if θ is not in one of these two intervals, then

$$\left|\int f(r\cos\theta, r\sin\theta)r\,dr - \int f_\varepsilon(r\cos\theta, r\sin\theta)r\,dr\right| \le 2Ma\varepsilon.$$

On the other hand, if θ is in one of these two intervals, we have

$$\left| \int f(r\cos\theta, r\sin\theta) r\, dr - \int f_\varepsilon(r\cos\theta, r\sin\theta) r\, dr \right| \leq Ma^2.$$

Since the exceptional intervals have total length 2ε and the total length of the θ interval is 2π, we see that

$$\left| \iint f(r\cos\theta, r\sin\theta) r\, dr\, d\theta - \iint f_\varepsilon(r\cos\theta, r\sin\theta) r\, dr\, d\theta \right| \leq 4\pi Ma\varepsilon + 2Ma^2\varepsilon.$$

Hence these two integrals also can be made arbitrarily close together by choosing ε sufficiently small. Since the two integrals of f_ε are equal for each $\varepsilon > 0$, it follows that the other two are also equal.

Exercise 10.10 Let $a \to \infty$ in Exercise 9, and prove that

$$\int_{R^2} f(x,y)\, dx\, dy = \int_0^\infty \int_0^{2\pi} f(T(r,\theta)) r\, d\theta\, dr,$$

for continuous functions f that decrease sufficiently rapidly as $|x| + |y| \to \infty$. (Find a more precise formulation.) Apply this to

$$f(x,y) = \exp(-x^2 - y^2)$$

to derive formula (101) of Chap. 8.

Solution: Without striving for ultimate generality, we shall assume that there are positive numbers K and δ such that $|f(x,y)| \leq K(x^2+y^2)^{-1-\delta}$ for all $(x,y) \neq (0,0)$. (Such an estimate holds for (x,y) ranging over any *bounded* set merely because $f(x,y)$ is continuous.) Let $D_a = \{(x,y) : 0 \leq x^2 + y^2 \leq a^2\}$ and $S_a = \{(x,y) : |x| \leq a, |y| \leq a\}$. Since both D_a and S_a are convex sets, the functions $g_a(x,y) = \chi_{D_a}(x,y) f(x,y)$ and $h_a(x,y) = \chi_{S_a}(x,y) f(x,y)$ are both integrable over R^2. We shall show that

$$\lim_{a\to\infty} \int_{R^2} g_a(x,y)\, dx\, dy = \int_{R^2} f(x,y)\, dx\, dy = \lim_{a\to\infty} \int_{R^2} h_a(x,y)\, dx\, dy.$$

Our job is simpler if we first show that

$$\lim_{a\to\infty} \left(\int_{R^2} g_a(x,y)\, dx\, dy - \int_{R^2} h_a(x,y)\, dx\, dy \right) = 0.$$

As before, we let $M = \sup\{|f(x,y)|\}$. Since $g_a(x,y) = h_a(x,y)$ except for $(x,y) \in S_a \setminus D_a$, and on this set $g_a(x,y) = 0$ and $|h_a(x,y)| \leq Ka^{-2-2\delta}$, the maximum possible difference in these two integrals is $4Ka^{-2\delta}$, which does indeed tend to zero as $a \to \infty$.

It now suffices to show only the second of the two equalities given above, i.e., that

$$\int_{R^2} f(x,y)\,dx\,dy = \lim_{a\to\infty} \int_{R^2} h_a(x,y)\,dx\,dy.$$

To that end, we fix y. We then have, if $|y| \geq a$, so that $h_a(x,y) = 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} f(x,y) - h_a(x,y)\,dx &\leq K \int_{-\infty}^{\infty} \frac{1}{(x^2+y^2)^{1+\delta}}\,dx \\ &\leq \int_{-\infty}^{-|y|} \frac{K}{(x^2)^{1+\delta}}\,dx + \int_{-|y|}^{|y|} \frac{K}{(y^2)^{1+\delta}}\,dx + \int_{|y|}^{\infty} \frac{K}{(x^2)^{1+\delta}}\,dx \\ &\leq \frac{2K|y|^{-1-2\delta}}{1+2\delta} + 2K|y|^{-1-2\delta} \leq 4K|y|^{-1-2\delta}.\end{aligned}$$

If $|y| \leq a$, we note that $f(x,y) = h_a(x,y)$ for $-a \leq x \leq a$, and so

$$\begin{aligned}\int_{-\infty}^{\infty} f(x,y) - h_a(x,y)\,dx &\leq \int_{-\infty}^{-a} \frac{K}{(x^2)^{1+\delta}}\,dx + \int_{a}^{\infty} \frac{K}{(x^2)^{1+\delta}}\,dx \\ &\leq \frac{2Ka^{-1-2\delta}}{1+2\delta} \leq 2Ka^{-1-2\delta}.\end{aligned}$$

Applying these two inequalities we find that

$$\begin{aligned}\left|\int_{R^2} f(x,y) - h_a(x,y)\,dx\,dy\right| &\leq 4K \int_{-\infty}^{-a} |y|^{-1-2\delta}\,dy + 4Ka^{-2\delta} + \\ &+ 4K \int_{a}^{\infty} y^{-1-2\delta}\,dy \leq 4K\Big(1+\frac{1}{\delta}\Big)a^{-2\delta}.\end{aligned}$$

The desired formula is now proved by merely remarking that

$$\int_{R^2} h_a(x,y)\,dx\,dy = \int_0^a \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r\,d\theta\,dr.$$

The fact that the limit on the right-hand side exists as $a \to \infty$ follows from the fact that the limit on the left-hand side does, but can also be proved directly, since $|f(r\cos\theta, r\sin\theta)r| \leq Kr^{-1-2\delta}$.

Applying this formula with $f(x,y) = e^{-x^2-y^2}$, we find that

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-t^2}\,dt\right)^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\,dx\,dy = \int_0^{\infty} \int_0^{2\pi} e^{-r^2} r\,d\theta\,dr = \\ &= 2\pi \int_0^{\infty} e^{-r^2} r\,dr = \pi \int_0^{\infty} e^{-u}\,du = \pi.\end{aligned}$$

In other words,

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}.$$

Exercise 10.11 Define $(u, v) = T(s, t)$ on the strip

$$0 < s < \infty, \quad 0 < t < 1$$

by setting $u = s - st$, $v = st$. Show that T is a 1-1 mapping of the strip onto the positive quadrant Q in R^2. Show that $J_T(s,t) = s$.

For $x > 0$, $y > 0$ integrate

$$u^{x-1}e^{-u}v^{y-1}e^{-v}$$

over Q, use Theorem 10.9 to convert the integral to one over the strip, and derive formula (96) of Chap. 8 in this way.

(For this application, Theorem 10.9 has to be extended so as to cover certain improper integrals. Provide this extension.)

Solution: It is easy to compute the inverse of T, namely

$$s = u + v, \quad t = \frac{v}{u+v},$$

and this inverse is defined on the entire (u, v)-plane with the line $v = -u$ removed. It is obvious that v is positive if and only if s and t have the same sign, and that u is positive if and only if s and $1 - t$ have the same sign.

Thus if u and v are both positive, then t and $1-t$ have the same sign, which happens if and only if $0 < t < 1$. In this case s must also be positive. Conversely, the equations that give s and t show that if u and v are both positive, then $s > 0$ and $0 < t < 1$. The Jacobian matrix of T is

$$\begin{pmatrix} 1-t & -s \\ t & s \end{pmatrix},$$

so that $J_T(s,t) = s$.

The integral of $u^{x-1}e^{-u}v^{y-1}e^{-v}$ over the quadrant is

$$\int_0^\infty u^{x-1}e^{-u}\,du \int_0^\infty v^{y-1}e^{-v}\,dv = \Gamma(x)\Gamma(y).$$

According to Theorem 10.9

$$\int_0^\infty \int_0^1 f(s-st, st)s\,dt\,ds = \int_0^\infty \int_0^\infty f(u,v)\,du\,dv$$

for any function $f(u, v)$ having compact support contained in the open quadrant. Assuming this theorem remains valid for the particular function we have in mind, we get

$$\Gamma(x)\Gamma(y) = \int_0^\infty s^{x+y-1}e^{-s}\,ds \int_0^1 t^{y-1}(1-t)^{x-1}\,dt =$$

$$= \Gamma(x+y)\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

which is indeed formula (96) of Chapter 8.

Thus we need only justify the use of Theorem 10.9 with the function f in the unbounded regions. To do this, we first show that Theorem 10.9 applies to the function $f(u,v)\varphi_\delta(u,v)$, where $\varphi_\delta(u,v)$ is the characteristic function of the set

$$E_\delta = \{(u,v) : \delta \le u \le \delta^{-1},\ \delta \le v \le \delta^{-1}\}.$$

Since this function is positive on E_δ, it is easy to modify it and make it into a continuous nonnegative function f_η that vanishes outside the set $E_{\delta-\eta}$, for $\eta < \delta$ and this can be done without increasing its maximal value. Theorem 10.9 applies to f_η, and it is easy to see that the integral of f_η on both sides of the formula tends to the integral of $f\varphi_\delta$ as $\eta \longrightarrow \delta$. (Indeed, there is a constant ε such that $T^{-1}(u,v) = (s,t)$ lies in the strip $\varepsilon \le t \le 1-\varepsilon$ whenever $(u,v) \in E_{\delta-\eta}$ and $\eta \le \frac{\delta}{2}$. In that case, for each fixed t, the distance between the rightmost points (s,t) in $T^{-1}(E_{\delta-\eta})$ and in $T^{-1}(E_\delta)$ is at most $\frac{\delta-\eta}{\varepsilon}$. A similar statement applies to the leftmost points in the two regions, showing that the usual argument applies: The integrals of f and $f\varphi_\delta$ over each horizontal line differ by at most $\frac{2M(\delta-\eta)}{\varepsilon}$, except for a small range of t whose length tends to zero with $\delta - \eta$, on which the difference is bounded. It then follows that both of the integrals of f_η tend toward the corresponding integrals of $f\varphi_\delta$

It then remains only to prove that the integral of $f\varphi_\delta$ tends to the integral of f on both sides of the formula. Since these integrals increase as δ decreases, there is no question that the limit exists, and we need only show that in both cases the limit is the integral in the formula. This is nearly immediate in the case of the integral over the quadrant. As for the integral over the strip, the set $T^{-1}(E_\delta)$ contains the region $\delta^{1/2} \le s \le \frac{1}{\delta-\delta^{3/2}}$, $\delta^{1/2} \le t \le 1-\delta^{1/2}$. For these inequalities imply that $\delta \le st \le \frac{1}{\delta}$, and since $1-t$ satisfies the same inequalities as t, we also have $\delta \le s(1-t) \le \frac{1}{\delta}$. The integral of $f(s-st,st)s$ over the two strips $0 \le t \le \sqrt{\delta}$ and $1-\sqrt{\delta} \le t \le 1$ tends to zero with δ, and for each t with $\sqrt{\delta} \le t \le 1-\sqrt{\delta}$ the integral

$$\int_{\sqrt{\delta}}^{1/(\delta-\delta^{3/2})} f(s-st,st)s\,ds$$

differs from the integral from 0 to ∞ by less than

$$t^{1-x}(1-t)^{1-y}\Big(\int_0^{\sqrt{\delta}} s^{x+y-1}\,ds + \int_{1/(\delta-\delta^{3/2})}^{\infty} s^{x+y-1}e^{-s}\,ds\Big).$$

The first of these integrals is explicitly calculable and tends to zero as $\delta \longrightarrow 0$. In the second we use the fact that $e^{-s} < \frac{n!}{s^n}$ for all $s > 0$ and take $n \ge x+y+1$. It then follows that the integral of $f(s-st,st)\varphi_\delta(s-st,st)s$ over each of these horizontal line differs from the integral of $f(s-st,st)s$ by an amount that tends to zero uniformly for $\sqrt{\delta} \le t \le 1-\sqrt{\delta}$.

The proof is now complete.

Exercise 10.12 Let I^k be the set of all $\mathbf{u} = (u_1, \ldots, u_k) \in R^k$ with $0 \le u_i \le 1$ for all i; let Q^k be the set of all $\mathbf{x} = (x_1, \ldots, x_k) \in R^k$ with $x_i \ge 0$, $\sum x_k \le 1$. (I^k is the unit cube; Q^k is the standard simplex in R^k.) Define $\mathbf{x} = T(\mathbf{u})$ by

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= (1-u_1)u_2 \\ &\cdots\cdots\cdots\cdots \\ x_k &= (1-u_1)\cdots(1-u_{k-1})u_k. \end{aligned}$$

Show that

$$\sum_{i=1}^{k} x_i = 1 - \prod_{i=1}^{k}(1-u_i).$$

Show that T maps I^k onto Q^k, that T is 1-1 in the interior of I^k, and that its inverse S is defined in the interior of Q^k by $u_1 = x_1$ and

$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}}$$

for $i = 2, \ldots, k$. Show that

$$J_T(\mathbf{u}) = (1-u_1)^{k-1}(1-u_2)^{k-2}\cdots(1-u_{k-1}),$$

and

$$J_S(\mathbf{x}) = \left[(1-x_1)(1-x_1-x_2)\cdots(1-x_1-\cdots-x_{k-1})\right]^{-1}.$$

Solution: The first identity is easily proved by induction on k. It is obvious for $k=1$, and

$$\begin{aligned} \sum_{i=1}^{k+1} x_i &= x_{k+1} + \sum_{i=1}^{k} x_k \\ &= (1-u_1)\cdots(1-u_k)u_{k+1} + 1 - (1-u_1)\cdots(1-u_k) \\ &= 1 - (1-u_1)\cdots(1-u_k)(1-u_{k+1}). \end{aligned}$$

The defining formulas and the formula just proved show that $\mathbf{x} \in Q^k$ whenever $\mathbf{u} \in I^k$. In the process of showing that T is onto, we shall prove the inverse formula. Let $\mathbf{x} \in Q^k$, and assume for the moment that $\sum_{i=1}^{k-1} x_i < 1$. Then all of the equations given as inverse equations are defined. We need only show that the defining equations yield $\mathbf{x}$ when applied to the left-hand sides of these equations. Certainly we do have $x_1 = u_1$. Suppose that $x_r = (1-u_1)\cdots(1-u_{r-1})u_r$ for $r < j$. For the moment assume $u_r \ne 0$.

$$(1-u_1)\cdots(1-u_{r-1})(1-u_r)u_{r+1} = x_r\Big(1 - \frac{1}{u_r}\Big)u_{r+1} =$$

$$= x_r \cdot \frac{1 - x_1 - \cdots - x_r}{x_r} \cdot \frac{x_{r+1}}{1 - x_1 - \cdots - x_r} = x_{r+1}.$$

If $u_l \neq 0$, but $u_j = 0$ for $l < j \leq r$, then $x_j = 0$ also for these values, and $u_{r+1} = \frac{x_{r+1}}{1-x_1-\cdots-x_l}$. We then have

$$\begin{aligned} x_{r+1} &= (1-u_1)\cdots(1-u_l)u_{r+1} \\ &= x_l\Big(1-\frac{1}{u_l}\Big)u_{r+1} \\ &= x_l \cdot \frac{1-x_1-\cdots-x_l}{x_l} \cdot \frac{x_{r+1}}{1-x_1-\cdots-x_l} \\ &= x_{r+1}. \end{aligned}$$

Finally, if $u_1 = u_2 = \cdots = u_r = 0$, we have simply $u_{r+1} = x_{r+1}$ in both sets of equations. Thus in all cases the point $\mathbf{u} \in I^k$ is a preimage of the point $\mathbf{x} \in Q^k$.

It remains only to consider the case when $\sum_{i=1}^r x_i = 1$ for some $r < k$. For these points $x_{r+1} = \cdots = x_k = 0$.

To find preimages of these points, let r be the first index for which $\sum\limits_{i=1}^{r} x_i = 1$. If $r = 1$, we have $x_2 = \cdots = x_k = 0$, and this point is its own preimage. In general the preïmage of the point $\mathbf{x}$ for which $x_{r+1} = \cdots = x_k = 0$ is $\mathbf{u}$, where $u_1, \ldots, u_r$ are given by the formulas for S. The formulas imply $u_r = 1$. The values of $u_{r+1}, \cdots, u_k$ are then arbitrary, since the formulas that define T will automatically make the remaining x_i equal to zero.

The Jacobian matrix is a triangular matrix whose diagonal consists of the entries 1, $(1-u_1)$, $(1-u_1)(1-u_2)$,..., $(1-u_1)\cdots(1-u_{k-1})$, and this fact yields the formula for $J_T(\mathbf{u})$ immediately.

Likewise, the Jacobian of S is triangular and has diagonal entries 1, $\frac{1}{1-x_1}$, $\frac{1}{1-x_1-x_2}, \ldots, \frac{1}{1-x_1-x_2\ldots-x_{k-1}}$, from which again the formula for $J_S(\mathbf{x})$ is immediate.

Exercise 10.13 Let $r_1, \ldots, r_k$ be nonnegative integers, and prove that

$$\int_{Q^k} x_1^{r_1} \cdots x_k^{r_k}\, dx = \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots + r_k)!}.$$

Hint: Use Exercise 12, Theorems 10.9 and 8.20.

Note that the special case $r_1 = \cdots = r_k = 0$ shows that the volume of Q^k is $1/k!$.

Solution: Following the hint, we rewrite the integral in terms of $\mathbf{u}$, getting

$$\int_{I^k} u_1^{r_1} \cdots u_k^{r_k} (1-u_1)^{r_2+\cdots+r_k}(1-u_2)^{r_3+\cdots+r_k}\cdots$$
$$(1-u_{k-1})^{r_k}(1-u_1)^{k-1}(1-u_2)^{k-2}\cdots(1-u_{k-1})\,du_1\cdots du_k.$$

This integral is the product

$$\prod_{i=1}^{k} \int_0^1 u_i^{r_i}(1-u_i)^{k-i+r_{i+1}\cdots+r_k}\, du_i,$$

which by formula (96) of Chapter 8 (just proved in Exercise 11 above) equals the product

$$\prod_{i=1}^{k} \frac{\Gamma(r_i+1)\Gamma(k+1-i+r_{i+1}\cdots+r_k)}{\Gamma(k+2-i+r_i+r_{i+1}\cdots+r_k)}.$$

When this product is evaluated, the numerator $\Gamma(k+1-i+r_{i+1}\cdots+r_k)$ in each factor cancels the denominator $\Gamma(k+2-(i+1)+r_{i+1}\cdots+r_k)$ in the next factor. Thus the product "telescopes" to the product of the factors $\Gamma(r_i+1)$ in the numerators divided by the first denominator $\Gamma(k+1+r_1+\cdots+r_k)$. Considering that $\Gamma(n+1)=n!$ for integers n, we therefore get the required formula.

Theoretically we ought to be worried about the fact that T is not 1-1 on the entire cube I^k. This problem, however, is handled by the same reasoning used in Exercises 9, 10, and 11, and need not be repeated.

Exercise 10.14 Prove formula (46).

Solution: Formula (46) asserts that $\prod\limits_{p<q} \operatorname{sgn}(j_q-j_p)$ is -1 if the permutation $j_1,\ldots,j_k$ is odd and 1 if the permutation is even. We observe that this product is $(-1)^k$, where k is the number of pairs (j_p,j_q) for which $j_p>j_q$. Since $\operatorname{sgn}(j_q-j_p)=1$ if $j_p<j_q$ and $\operatorname{sgn}(j_q-j_p)=-1$ if $j_p>j_q$, we need to show that the parity of k is the same as the parity of the number of interchanges that will be used in converting this permutation to the identity. (As a corollary, that parity will be the same, no matter what particular sequence of interchanges is used to get to the identity.) This equality is obvious if the permutation is the identity to begin with. Suppose then that $j_m>j_n$ and $m<n$. The elements j_i, $m<i<n$ are of three kinds: Set A, those for which $j_i<j_n$; set B, those for which $j_n<j_i<j_m$; and set C, those for which $j_m<j_i$. Before j_m and j_n are interchanged, there is one out-of-order pair (j_m,j_i) for each $j_i\in A$, one out-of-order pair (j_i,j_n) for each $j_i\in C$, and two out-of-order pairs (j_m,j_i) and (j_i,j_n) for each $j_i\in B$. After the switch there is one out-of-order pair (j_n,j_i) for each $j_i\in A$, and one pair (j_i,j_m) for each $j_i\in C$. There are no pairs involving any $j_i\in B$. Hence, *when an out-of-order pair (j_m,j_n) is put in the right order by interchanging its elements, the number of out-of-order pairs decreases by $2|B|+1$, where B is the set of elements j_i between j_m and j_n that are in the wrong order relative to both j_m and j_n and $|B|$ is the number of elements in B.*

Of course the number would *increase* by an odd number if we foolishly interchanged a pair that were not out-of-order relative to each other. (The number would increase by $2|B|+1$, where $|B|$ is the number of elements between them that were in the correct order relative to both elements of the interchanged pair.) In any case, each interchange of two elements changes the number of inversions (out-of-order pairs) by an odd number, so that an odd number of interchanges, starting from the identity, will result in an odd number of inversions, and an even number of interchanges will result in an even number of inversions.

Exercise 10.15 If ω and λ are k- and m-forms, respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega$$

Solution: Because of the associative and distributive laws, it suffices to prove this in the case when $\omega = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and $\lambda = g\, dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_{k+m}}$. In that case

$$\omega \wedge \lambda = fg\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_{k+m}}.$$

For each $j = 1, 2, \ldots, k$ exactly m interchanges of adjacent basic one-forms will move $dx_{i_{k+1-j}}$ to the position just right of $dx_{i_{k+m}}$, if these moves are made in increasing order of j. Thus a total of km interchanges will exactly reverse λ and ω. The result now follows from the alternating property of the wedge product on basis elements.

Exercise 10.16 If $k \geq 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$ is an oriented affine k-simplex, prove that $\partial^2\sigma = 0$, directly from the definition of the boundary operator ∂. Deduce from this that $\partial^2\Psi = 0$ for every chain Ψ.

Hint: For orientation, do it first for $k = 2$, $k = 3$. In general, if $i < j$, let σ_{ij} be the $(k-2)$-simplex obtained by deleting $\mathbf{p}_i$ and $\mathbf{p}_j$ from σ. Show that each σ_{ij} occurs twice in $\partial^2\sigma$ with opposite sign.

Solution. For $k = 2$ we have

$$\partial\sigma = [\mathbf{p}_1, \mathbf{p}_2] - [\mathbf{p}_0, \mathbf{p}_2] + [\mathbf{p}_0, \mathbf{p}_1],$$

so that

$$\partial^2\sigma = (\mathbf{p}_2 - \mathbf{p}_1) - (\mathbf{p}_2 - \mathbf{p}_0) + (\mathbf{p}_1 - \mathbf{p}_0) = 0.$$

For $k = 3$ we have

$$\partial\sigma = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3] - [\mathbf{p}_0, \mathbf{p}_2, \mathbf{p}_3] + [\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_3] - [\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2],$$

so that

$$\begin{aligned}
\partial^2\sigma &= ([\mathbf{p}_2, \mathbf{p}_3] - [\mathbf{p}_1, \mathbf{p}_3] + [\mathbf{p}_1, \mathbf{p}_2]) \\
&\quad -([\mathbf{p}_2, \mathbf{p}_3] - [\mathbf{p}_0, \mathbf{p}_3] + [\mathbf{p}_0, \mathbf{p}_2]) \\
&\quad +([\mathbf{p}_1, \mathbf{p}_3] - [\mathbf{p}_0, \mathbf{p}_3] + [\mathbf{p}_0, \mathbf{p}_1]) \\
&\quad -([\mathbf{p}_1, \mathbf{p}_2] - [\mathbf{p}_0, \mathbf{p}_2] + [\mathbf{p}_0, \mathbf{p}_1]) \\
&= 0.
\end{aligned}$$

In general the order in which $\mathbf{p}_i$ and $\mathbf{p}_j$ are omitted from σ determines the sign that σ_{ij} will have. If $\mathbf{p}_j$ is omitted first, the resulting $(k-1)$-simplex $\sigma_j = [\mathbf{p}_0, \ldots, \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, \ldots, \mathbf{p}_k]$ will acquire the sign $(-1)^j$. If $\mathbf{p}_i$ is then omitted, the resulting $(k-2)$-simplex will acquire a factor of $(-1)^i$, resulting in $(-1)^{i+j}\sigma_{ij}$.

However, if $\mathbf{p}_i$ is omitted first, $\mathbf{p}_j$ will move forward one position in the resulting $(k-1)$-simplex σ_i, and when it is subsequently omitted, a factor of

$(-1)^{j-1}$ will be affixed, resulting in $(-1)^{i+j-1}\sigma_{ij}$. Hence the two occurrences of σ_{ij} in the second boundary will cancel each other.

The linearity of the boundary operator, operating on a base of simplexes, then shows that ∂^2 is the zero operator on all chains.

Exercise 10.17 Put $J^2 = \tau_1 + \tau_2$, where

$$\tau_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2], \quad \tau_2 = -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

Explain why it is reasonable to call J^2 the positively oriented unit square in R^2. Show that ∂J^2 is the sum of 4 oriented affine simplexes. Find these. What is $\partial(\tau_1 - \tau_2)$?

Solution: Although J^2 is really a collection of two affine mappings, the *ranges* of these mappings cover the unit square, the diagonal from $(0,0)$ to $(1,1)$ being covered twice with opposite orientations in the two mappings. In both cases, the sense of orientation is such that the cross product of the last two vertices of the simplex is $\mathbf{e}_3$, which is a reasonable definition of the positive orientation on the unit square.

By routine computation,

$$\begin{aligned}
\partial J^2 &= ([\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] \\
&\qquad -[\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1]) - ([\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_2]) \\
&= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}] + [\mathbf{0}, \mathbf{e}_1].
\end{aligned}$$

Again, by routine computation,

$$\begin{aligned}
\partial(\tau_1 - \tau_2) &= ([\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1]) \\
&\qquad +([\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_2]) \\
&= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] - [\mathbf{e}_2, \mathbf{0}] + [\mathbf{0}, \mathbf{e}_1] - 2[\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2].
\end{aligned}$$

Exercise 10.18 Consider the oriented affine 3-simplex

$$\sigma_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

in R^3. Show that σ_1 (regarded as a linear transformation) has determinant 1. Thus σ_1 is positively oriented.

Let $\sigma_2, \ldots, \sigma_6$ be five other oriented simplexes, obtained as follows: There are five permutations (i_1, i_2, i_3) of $(1,2,3)$ distinct from $(1,2,3)$. Associate with each (i_1, i_2, i_3) the simplex

$$s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

where s is the sign that occurs in the definition of the determinant. (This is how τ_2 was obtained from τ_1 in Exercise 17.)

Show that $\sigma_2, \ldots, \sigma_6$ are positively oriented.

Put $J^3 = \sigma_1 + \cdots + \sigma_6$. Then J^3 may be called the positively oriented unit cube in R^3.

Show that ∂J^3 is the sum of 12 oriented affine 2-simplexes. (These 12 triangles cover the surface of the unit cube I_3.)

Show that $\mathbf{x} = (x_1, x_2, x_3)$ is in the range of σ_1 if and only if $0 \le x_3 \le x_2 \le x_1 \le 1$.

Show that the ranges of $\sigma_1, \ldots, \sigma_6$ have disjoint interiors, and that their union covers I^3. (Compare with Exercise 13; note that $3! = 6$.)

Solution. We first show that each of these simplexes is positively oriented. To that end, it is convenient to refer to the simplex $[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$ corresponding to the permutation (i_1, i_2, i_3) as $\sigma^{(i_1, i_2, i_3)}$.

The simplex $\sigma^{(i_1,i_2,i_3)}$, regarded as a linear transformation, maps (x, y, z) to $(x+y+z)\mathbf{e}_{i_1} + (y+z)\mathbf{e}_{i_2} + z\mathbf{e}_{i_3}$. Its matrix therefore has $(1\ \ 1\ \ 1)$ as row i_1, $(0\ \ 1\ \ 1)$ as row i_2, and $(0\ \ 0\ \ 1)$ as row i_3. By interchanging rows in correspondence with the interchanges needed to convert the permutation (i_1, i_2, i_3) to the identity, we can convert this matrix to an upper-triangular matrix with 1's on the main diagonal. The determinant of the matrix is therefore $s(i_1, i_2, i_3)$, so that the simplex $s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$ is positively oriented.

The boundary of $\sigma^{(i_1,i_2,i_3)}$ consists of four terms, two of which ($[\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$ and $-[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}]$) are not shared with any other $\sigma^{(i)}$. The other two terms ($-[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$ and $[\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$ are shared with $\sigma^{(i_1,i_3,i_2)}$ and $\sigma^{(i_2,i_1,i_3)}$ respectively. As these two permutations each differ from (i_1, i_2, i_3) by a single interchange, the sign of each of these terms will be opposite in its two occurrences, and hence they will cancel out. Thus the boundary of J^3 will consist of a total of 12 oriented affine 2-simplexes.

A point $\mathbf{x} = (x_1, x_2, x_3)$ is in the range of σ_1 if and only if there are numbers $r, s, t \in [0, 1]$ such that $r + s + t \le 1$ and $\mathbf{x} = r\mathbf{e}_1 + s(\mathbf{e}_1 + \mathbf{e}_2) + t(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$, that is, $x_1 = r + s + t$, $x_2 = s + t$, and $x_3 = t$. If such numbers r, s, t exist, obviously $0 \le x_3 \le x_2 \le x_1 \le 1$. Conversely, if these conditions hold, there will be such numbers r, s, t, namely $t = x_3$, $s = x_2 - x_3$, and $r = x_1 - x_2$.

The interior of the range of $\sigma^{(i_1,i_2,i_3)}$ is the set of all $\mathbf{x} = (x_1, x_2, x_3)$ such that $0 < x_{i_3} < x_{i_2} < x_{i_1} < 1$. For the range of this simplex is the set of $\mathbf{x}$ for which each of these inequalities or the corresponding equality holds. If equality holds in any of them, the point can be approached by points outside the range, as one can easily see. That the union covers I^3 is also obvious. Indeed, the characterization of the range of σ_1 applies to all $\sigma^{(i_1,i_2,i_3)}$ and shows that this range is contained in I^3. Thus we need only show the reverse inclusion.

If $\mathbf{x} = (x_1, x_2, x_3) \in I^3$, let i_1 be the smallest subscript i for which $x_i = \max\{x_1, x_2, x_3\}$. Let i_2 be the first subscript for which $x_i = \max\left(\{x_1, x_2, x_3\} \setminus \{x_{i_1}\}\right)$. Finally, let i_3 be such that $\{x_{i_3}\} = \{x_1, x_2, x_3\} \setminus \{x_{i_1}, x_{i_2}\}$. By construction $0 \le x_{i_3} \le x_{i_2} \le x_{i_1} \le 1$, and so, by the argument given above, $\mathbf{x}$ belongs to the range of $\sigma^{(i_1,i_2,i_3)}$. Symmetry shows that all of these simplexes have the same volume, which must therefore be $1/6$. (Remember that we showed back in Exercise 1 that the boundary of a convex set in R^k has k-dimensional volume 0, so that the volume of each of these sets equals the volume of its interior. As the interiors are disjoint, the sum of their volumes is at most 1. Since the simplexes together cover I^3, the sum of their volumes is at least 1. Therefore it is exactly 1.)

Exercise 10.19 Let J^2 and J^3 be as in Exercise 17 and 18. Define

$$\begin{aligned} B_{01}(u,v) &= (0,u,v), & B_{11}(u,v) &= (1,u,v), \\ B_{02}(u,v) &= (u,0,v), & B_{12}(u,v) &= (u,1,v), \\ B_{03}(u,v) &= (u,v,0), & B_{13}(u,v) &= (u,v,1). \end{aligned}$$

These are affine and map R^2 into R^3.

Put $\beta_{ri} = B_{ri}(J^2)$, for $r = 0, 1$, $i = 1, 2, 3$. Each β_{ri} is an affine-oriented 2-chain. (See Sec. 10.30.) Verify that

$$\partial J^3 = \sum_{i=1}^{3}(-1)^i(\beta_{0i} - \beta_{1i}),$$

in agreement with Exercise 18.

Solution. Although we did not spell it out in our solution of Exercise 18, the boundary of J^3 is the 2-chain

$$\sum_{i_1,i_2,i_3} s(i_1,i_2,i_3)\big([\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] - [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}]\big).$$

This sum can be rearranged as a sum of three terms, each of which consists of four terms. For example, the terms in the sum for which $i_=1$ can be written as

$$\begin{aligned} &\big([\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] \\ &\qquad - [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]\big) - \big([\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3]\big). \end{aligned}$$

For $i_1 = 2$ we get a similar set of four terms, namely,

$$\begin{aligned} &\big(-[\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] + [\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]\big) \\ &\qquad + \big([\mathbf{0}, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3]\big). \end{aligned}$$

Finally, for $i_1 = 3$ we have

$$\begin{aligned} &\big([\mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]\big) \\ &\qquad - \big([\mathbf{0}, \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_1] - [\mathbf{0}, \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_2]\big). \end{aligned}$$

Now consider the 2-chain β_{01}. According to the notation of Eq. (88), it is $B_{01}(\tau_1) + B_{01}(\tau_2)$. Letting $(u,v) = \tau_1(x,y) = (x+y)\mathbf{e}_1 + y\mathbf{e}_2$, and then $(u,v) = \tau_2(x,y) = (x+y)\mathbf{e}_2 + y\mathbf{e}_1$ (and keeping in mind the orientation assigned to τ_2), we see that $\beta_{01}(x,y) = B_{01}(x+y,y) - B_{01}(y,x+y) = (0,x+y,y) -$

$(0, y, x+y) = [\mathbf{0}, \mathrm{e}_2, \mathrm{e}_2+\mathrm{e}_3] - [\mathbf{0}, \mathrm{e}_3, \mathrm{e}_3+\mathrm{e}_2]$. Notice that these two terms occur in the expression for ∂J^3, in the groupings for $i_1 = 2$ and $i_1 = 3$ respectively, but each occurs with the opposite sign. Hence these terms can be accounted for in ∂J^3 by being grouped together and written as $-\beta_{01}$. Similarly when we look at β_{11}, we find that it is the 2-chain whose points are $(1, x+y, y) - (1, y, x+y)$, which is $[\mathrm{e}_1, \mathrm{e}_1+\mathrm{e}_2, \mathrm{e}_1+\mathrm{e}_2+\mathrm{e}_3] - [\mathrm{e}_1, \mathrm{e}_1+\mathrm{e}_3, \mathrm{e}_1+\mathrm{e}_2+\mathrm{e}_3]$. Again these terms occur in the expression for e_1, this time with exactly the same signs, so that they can be accounted for by grouping them and writing them as the term β_{11}. Thus four of the twelve simplexes in ∂J^3 are accounted for by the expression $(-1)^1(\beta_{01} - \beta_{11})$. The other 8 simplexes are accounted for similarly.

Exercise 10.20 State conditions under which the formula

$$\int_\Phi f\,d\omega = \int_{\partial\Phi} f\omega - \int_\Phi (df) \wedge \omega$$

is valid, and show that it generalizes the formula for integration by parts.
Hint: $d(f\omega) = (df) \wedge \omega + f\,d\omega$.

Solution. Given the formula in the hint, we need only invoke Stokes' Theorem. For any chain Φ satisfying the hypotheses of that theorem we shall have

$$\int_\Phi d(f\omega) = \int_{\partial\Phi} f\omega,$$

which is precisely the given theorem. The ordinary formula for integration by parts follows by considering a 0-form fg.

Exercise 10.21 As in Example 10.36, consider the 1-form

$$\eta = \frac{x\,dy - y\,dx}{x^2+y^2}$$

in $R^2 \setminus \{\mathbf{0}\}$.
(*a*) Carry out the computation that leads to formula (113), and prove that $d\eta = 0$.
(*b*) Let $\gamma(t) = (r\cos t, r\sin t)$, for some $r > 0$, and let Γ be a $\mathcal{C}''$-curve in $R^2 \setminus \{\mathbf{0}\}$, with parameter interval $[0, 2\pi]$, with $\Gamma(0) = \Gamma(2\pi)$, such that the intervals $[\gamma(t), \Gamma(t)]$ do not contain $\mathbf{0}$ for any $t \in [0, 2\pi]$. Prove that

$$\int_\Gamma \eta = 2\pi.$$

Hint: For $0 \le t \le 2\pi$, $0 \le u \le 1$, define

$$\Phi(t, u) = (1-u)\Gamma(t) + u\gamma(t).$$

then Φ is a 2-surface in $R^2 \setminus \{\mathbf{0}\}$ whose parameter domain is the indicated rectangle. Because of cancellations (as in Example 10.32),

$$\partial\Phi = \Gamma - \gamma.$$

Use Stokes' theorem to deduce that

$$\int_\Gamma \eta = \int_\gamma \eta$$

because $d\eta = 0$.

(*c*) Take $\Gamma(t) = (a\cos t, b\sin t)$ where $a > 0$, $b > 0$ are fixed. Use part (*b*) to show that

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2\sin^2 t}\,dt = 2\pi.$$

(*d*) Show that

$$\eta = d\Big(\arctan\frac{y}{x}\Big)$$

in any convex open set in which $x \neq 0$, and that

$$\eta = d\Big(-\arctan\frac{x}{y}\Big)$$

in any convex open set in which $y \neq 0$.

Explain why this justifies the notation $\eta = d\theta$, in spite of the fact that η is not exact in $R^2 \setminus \{\mathbf{0}\}$.

(*e*) Show that (*b*) can be derived from (*d*).

(*f*) If Γ is any closed C'-curve in $R^2 \setminus \{\mathbf{0}\}$, prove that

$$\frac{1}{2\pi}\int_\Gamma \eta = \mathrm{Ind}\,(\Gamma).$$

(See Exercise 23 of Chap. 8 for the definition of the index of a curve.)

Solution. (*a*) By the rules for computing line integrals, given that $x = r\cos t$ and $y = r\sin t$,

$$\int_\gamma \eta = \int_0^{2\pi} \frac{(r\cos t)(r\cos t)\,dt - (r\sin t)(-r\sin t)\,dt}{r^2\cos^2 t + r^2\sin^2 t} = \int_0^{2\pi} dt = 2\pi.$$

(*b*) Let $\Gamma(t) = \big(X(t), Y(t)\big)$ and $\gamma(t) = \big(x(t), y(t)\big)$. Following the hint, observing that the hypothesis that the interval from $\Gamma(t)$ to $\gamma(t)$ does not pass through $\mathbf{0}$, we find that $\Phi(t,u)$ is indeed a 2-surface in $R^2 \setminus \{\mathbf{0}\}$, and making it into a singular 2-chain by regarding the domain as an affine 2-chain, as in Exercise 17, we find by Stokes' theorem that

$$\begin{aligned} 0 &= \int_\Phi d\eta = \int_{\partial\Phi} \eta \\ &= -\int_\Gamma \eta + \int_\gamma \eta + \int_\delta \eta - \int_\varepsilon \eta, \end{aligned}$$

where δ is the curve $\delta(u) = \Phi(2\pi, u) = (1-u)\Gamma(2\pi) + u\gamma(2\pi)$ and ε is the curve $\varepsilon(u) = \Phi(0,u) = (1-u)\Gamma(0) + u\gamma(0)$. Since δ and ε are the same curve, the last two terms in this expression cancel each other, yielding the required result.

(*c*) We need only verify that $\Phi(t,u) \neq \mathbf{0} = (0,0)$. But this is clear: If $((1-u)a + ur)\cos t = 0$, then $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$, since $(1-u)a + ur \geq \min(a,r) > 0$. But this means that $((1-u)b + ur)\sin t \neq 0$, since t is not a multiple of π. The result now follows.

(*d*) It is a routine computation that the differential of $\arctan\frac{y}{x}$ is η in the entire right or left half-plane, and similarly for $\pi - \arctan\frac{x}{y}$, which is after all $\operatorname{arccot}\frac{x}{y}$, which in turn is $\arctan\frac{y}{x}$ wherever both functions are defined. Thus *locally* we have $\eta = d\theta$, even though θ is not defined *globally* in $R^2 \setminus \{\mathbf{0}\}$.

(*e*) Break the integral over γ into five parts: $0 \leq t \leq \frac{\pi}{4}$, $\frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$, $\frac{3\pi}{4} \leq t \leq \frac{5\pi}{4}$, $\frac{5\pi}{4} \leq t \leq \frac{7\pi}{4}$, $\frac{7\pi}{4} \leq t \leq 2\pi$. In the first, third, and fifth parts we have $\eta = d\big(\arctan\frac{y}{x}\big)$, and in the second and fourth we have $\eta = d\big(-\arctan\frac{x}{y}\big)$. Now in the first, third, and fifth parts, $\frac{y}{x} = \frac{\sin t}{\cos t} = \tan t$, so that either $t = \arctan\frac{y}{x}$ or $t = \pi + \arctan\frac{y}{x}$ on these arcs. In either case the integral over the these parts is just the difference in t at the endpoints. Hence these three integrals contribute $\frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4} = \pi$ to the integral. On the other parts $\frac{x}{y} = \cot t = \tan\big(\frac{\pi}{2} - t\big)$. Hence, once again, $\arctan\frac{x}{y}$ is either $\frac{\pi}{2} - t$ or $\frac{3\pi}{2} - t$. In either case, these two integrals contribute $\frac{\pi}{2} + \frac{\pi}{2} = \pi$ to the integral, and provides the result of (*b*).

(*f*) The definition of $\operatorname{Ind}(\Gamma)$ is defined by regarding $\Gamma(t)$ as a curve $X(t) + Y(t)i$ in the complex plane, in which case

$$\operatorname{Ind}(\Gamma) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\Gamma'(t)}{\Gamma(t)}\,dt = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(X(t) - Y(t)i)(X'(t) + Y'(t)i)}{(X(t))^2 + Y(t))^2}\,dt.$$

Since we know the imaginary part is zero, we consider only the real part, which is

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{X(t)Y'(t) - Y(t)X'(t)}{(X(t))^2 + Y(t))^2}\,dt = \frac{1}{2\pi}\int_\Gamma \eta.$$

(Incidentally, it follows from Stokes' theorem that the imaginary part of this complex integral is zero, since it is

$$-\frac{1}{2\pi}\int_0^{2\pi} \frac{X(t)X'(t) + Y(t)Y'(t)}{(X(t))^2 + (Y(t))^2}\,dt = -\frac{1}{4\pi}\int_\Gamma d\zeta = -\frac{1}{4\pi}\int_{\partial\Gamma} \zeta$$

where $\zeta(x,y) = \ln\big(x^2 + y^2\big)$. This last integral is zero, since Γ is a closed curve.)

Exercise 10.22 As in Example 10.37, define ζ in $R^3 - \mathbf{0}$ by

$$\zeta = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{r^3},$$

where $r = (x^2 + y^2 + z^2)^{1/2}$, let D be the rectangle given by $0 \le u \le \pi$, $0 \le v \le 2\pi$, and let Σ be the 2-surface in R^3, with parameter domain D, given by

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u.$$

(a) Prove that $d\zeta = 0$ in $R^3 \setminus \mathbf{0}$.

(b) Let S denote the restriction of Σ to a parameter domain $E \subset D$. Prove that

$$\int_S \zeta = \int_E \sin u \, du \, dv = A(S),$$

where A denotes area, as in Sec. 10.43. Note that this contains (115) as a special case.

(c) Suppose g, h_1, h_2, h_3, are C''-functions on $[0,1]$, $g > 0$. Let $(x, y, z) = \Phi(s,t)$ define a 2-surface Φ, with parameter domain I^2, by

$$x = g(t)h_1(s), \quad y = g(t)h_2(s), \quad z = g(t)h_3(s).$$

Prove that

$$\int_\Phi \zeta = 0,$$

directly from (35).

Note the shape of the range of Φ: For fixed s, $\Phi(s,t)$ runs over an interval on a line through $\mathbf{0}$. The range of Φ thus lies in a "cone" with vertex at the origin.

(d) Let E be a closed rectangle in D, with edges parallel to those of D. Suppose $f \in C''(D)$, $f > 0$. Let Ω be the 2-surface with parameter domain E, defined by

$$\Omega(u,v) = f(u,v)\Sigma(u,v).$$

Define S as in (b) and prove that

$$\int_\Omega \zeta = \int_S \zeta = A(S).$$

(Since S is the "radial projection" of Ω onto the unit sphere, this result makes it reasonable to call $\int_\Omega \zeta$ the "solid angle" subtended by the range of Ω at the origin.)

Hint: Consider the 3-surface Ψ given by

$$\Psi(t,u,v) = [1 - t + tf(u,v)]\Sigma(u,v),$$

where $(u,v) \in E$, $0 \le t \le 1$. For fixed v, the mapping $(t,u) \to \Psi(t,u,v)$ is a 2-surface Φ to which (c) can be applied to show that $\int_\Phi \zeta = 0$. The same thing holds when u is fixed. By (a) and Stokes' theorem,

$$\int_{\partial\Psi} \zeta = \int_\Psi d\zeta = 0.$$

(e) Put $\lambda = -(z/r)\eta$, where

$$\eta = \frac{x\,dy - y\,dx}{x^2 + y^2},$$

as in Exercise 21. Then λ is a 1-form in the open set $V \subset R^3$ in which $x^2+y^2 > 0$. Show that ζ is *exact in* V by showing that

$$\zeta = d\lambda.$$

(f) Derive (d) from (e), without using (c).

Hint: To begin with, assume $0 < u < \pi$ on E. By (e),

$$\int_\Omega \zeta = \int_{\partial\Omega} \lambda \quad \text{and} \quad \int_S \zeta = \int_{\partial S} \lambda.$$

Show that the two integrals of λ are equal, by using part (d) of Exercise 21, and by noting that z/r is the same at $\Sigma(u,v)$ as at $\Omega(u,v)$.

(g) Is ζ exact in the complement of every line through the origin?

Solution. (a) We note that, since $\frac{\partial r}{\partial x} = xr^{-1}$, we have

$$\frac{\partial}{\partial x}\frac{x}{r^3} = r^{-3} - 3x^2r^{-5} = r^{-5}(r^2 - 3x^2).$$

By symmetry we have analogous relations for the partial derivatives of yr^{-3} and zr^{-3} with respect to y and z respectively. Since $dx \wedge dy \wedge dz = dy \wedge dz \wedge dx = dz \wedge dx \wedge dy$, we find that

$$d\zeta = r^{-5}(r^2 - 3x^2 + r^2 - 3y^2 + r^2 - 3z^2)\,dx \wedge dy \wedge dz = 0.$$

(b) Since $r\big(\Sigma(u,v)\big) = 1$, we have only to note that the differentials pull back to D as $dy \wedge dz = \frac{\partial(y,z)}{\partial(u,v)}\,du \wedge dv = \sin^2 u \cos v\,du \wedge dv$, $dz \wedge dx = \sin^2 u \sin v\,du \wedge dv$ and $dx \wedge dy = \sin u \cos u\,du \wedge dv$. The integrand then pulls back as $(\sin^3 u + \sin u \cos^2 u)\,du \wedge dv = \sin u\,du \wedge dv$. The reference to Sec. 10.43 must be a misprint for Sec. 10.46.

(c) For the application to be made in part (d) below we actually need to allow the function $g(t)$ to depend on s also. Thus we consider $g(s,t)$ instead of $g(t)$. Using only the definition (35) for the integral, we need to get the pullbacks of the wedge products to the parameter domain $[0,1] \times [0,1]$. Since $dx = \frac{\partial g}{\partial t} h_1(s)\,dt + \big(g(s,t)h_1'(s) + h_1(s)\frac{\partial g}{\partial s}\,ds$, with similar expressions for dy and dz, we find that $dy \wedge dz = g(s,t)\frac{\partial g}{\partial t}\big(h_3(s)h_2'(s) - h_3'(s)h_2(s)\big)\,ds \wedge dt$, $dz \wedge dx = g(s,t)\frac{\partial g}{\partial t}\big(h_3'(s)h_1(s) - h_3(s)h_1'(s)\big)\,ds \wedge dt$, and $dx \wedge dy = g(s,t)\frac{\partial g}{\partial t}\big(h_1'(s)h_2(s) - h_1(s)h_2'(s)\big)\,ds \wedge dt$. Thus, *assuming $h_1(t)$, $h_2(t)$, and $h_3(t)$ do not vanish simultaneously,* we have

$$\int_\Phi \zeta = \int_0^1 \int_0^1 \frac{\frac{\partial g}{\partial t}}{g(s,t)} \frac{(h_1(s)h_2(s)h_3(s))' - (h_1(s)h_2(s)h_3(s))'}{(h_1(s))^1 + (h_2(s))^2 + (h_3(s))^2}\,ds\,dt = 0.$$

(*d*) Using part (*c*), as amended, we note that $\partial\Psi$ consists of six mappings $\Psi(1,u,v) = \Omega(u,v)$, $\Psi(0,u,) = S(u,v)$, $\Psi(t,b,v)$, $\Psi(t,a,v)$, $\Psi(t,u,d)$, and $\Psi(t,u,c)$, where $E = [a,b] \times [c,d]$. By part (*c*) the integrals over each of the last 4 surfaces are all zero. Since $d\zeta = 0$, Stokes' theorem implies that

$$\int_\Omega \zeta - \int_S \zeta = 0.$$

(*e*) By straightforward computation,

$$\begin{aligned} d\lambda &= -d(z/r) \wedge \eta - (z/r)\, d\eta \\ &= \frac{xz\,dx + yz\,dy + (z^2 - r^2)\,dz}{r^3} \wedge \eta \\ &= \frac{(x^2 z + y^2 z)\,dx \wedge dy}{r^3(x^2+y^2)} - \frac{x\,dz \wedge dy - y\,dz \wedge dx}{r^3} \\ &= \zeta. \end{aligned}$$

(*f*) Again by Stokes' theorem we must have

$$\int_\Omega \zeta = \int_\Omega d\lambda = \int_{\partial\Omega} \lambda.$$

But η is independent of z, and z/r is the same for both $S(u,v)$ and $\Omega(u,v)$. Therefore

$$\int_{\partial\Omega} \lambda = \int_{\partial S} \lambda.$$

(*g*) Yes, ζ is exact on the complement of every line through the origin. Indeed, for every line through the origin there is a rotation T that maps that line to the z-axis. By Theorem 10.22, part (*c*) we have $d(\lambda_T) = (d\lambda)_T = \zeta_T$. However, $\zeta_T = \zeta$, as one can easily compute. Indeed, since r is invariant under T, we need only show that $x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$ is rotation-invariant. To that end, suppose $(u,v,w) = T(x,y,z)$, say $u = t_{11}x + t_{12}y + t_{13}z$, so that $du = t_{11}\,dx + t_{12}\,dy + t_{13}\,dz$, etc. We then have $dv \wedge dw = (t_{22}t_{33} - t_{32}t_{23})\,dy \wedge dz + (t_{23}t_{31} - t_{33}t_{21})\,dz \wedge dx + (t_{21}t_{32} - t_{31}t_{22})\,dx \wedge dy$, etc. and so $u\,dv \wedge dw + v\,dw \wedge du + w\,du \wedge dv$ works out (after tedious computation) to precisely $x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$.

Exercise 10.23 Fix n. Define $r_k = (x_1^2 + \cdots + x_k^2)^{1/2}$ for $1 \le k \le n$, let E_k be the set of all $\mathbf{x} \in R^n$ at which $r_k > 0$, and let ω_k be the $(k-1)$-form defined in E_k by

$$\omega_k = (r_k)^{-k} \sum_{i=1}^{k} (-1)^{i-1} x_i\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k.$$

Note that $\omega_2 = \eta$, $\omega_3 = \zeta$ in the notation of Exercises 21 and 22. Note also that

$$E_1 \subset E_2 \subset \cdots \subset E_n = R^n \setminus \{\mathbf{0}\}.$$

(a) Prove that $d\omega_k = 0$ in E_k.
(b) For $k = 2, \ldots, n$, prove that ω_k is exact in E_{k-1}, by showing that

$$\omega_k = d(f_k\omega_{k-1}) = (df_k) \wedge \omega_{k-1},$$

where $f_k(\mathbf{x}) = (-1)^k g_k(x_k/r_k)$ and

$$g_k(t) = \int_{-1}^{t} (1-s)^{(k-3)/2}\, ds \quad (-1 < t < 1).$$

Hint: f_k satisfies the differential equations

$$\mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) = 0$$

and

$$(D_k f_k)(\mathbf{x}) = \frac{(-1)^k (r_{k-1})^{k-1}}{(r_k)^k}.$$

(c) Is ω_n exact in E_n?
(d) Note that (b) is a generalization of part (e) of Exercise 22. Try to extend some of the other assertions of Exercises 21 and 22 to ω_n, for arbitrary n.

Solution. (a) Computation shows that $d\Big(\sum_{i=1}^{k}(-1)^{i-1}x_i\, dx_1\wedge\cdots\wedge dx_{i-1}\wedge dx_{i+1}\wedge \cdots \wedge dx_k\Big) = k dx_1 \wedge \cdots \wedge dx_k$, and $\frac{\partial r_k}{\partial x_j} = \frac{x_j}{r_k}$ for $j \leq k$, so that $d(r_k) = -k(r_k)^{-k-2} \sum_{j=1}^{k} x_j\, dx_j$, we find that

$$\begin{aligned} d\omega_k &= k(r_k)^{-k}\, dx_1 \wedge \cdots \wedge dx_k - k(r_k)^{-k-2} \sum_{j=1}^{k} x_j^2 dx_1 \wedge \cdots \wedge dx_k = \\ &= k(r_k)^{-k-2}\Big(r_k^2 - \sum_{j=1}^{k} x_j^2\Big)\, dx_1 \wedge \cdots \wedge dx_k = 0. \end{aligned}$$

This argument shows, incidentally, that $d\omega_k = 0$ in $E_n = R^n \setminus \{\mathbf{0}\}$.
(b) We compute that

$$\begin{aligned} df_k &= (-1)^k(1 - x_k^2/r_k^2)^{(k-3)/2}\Big(\big(r_k^{-1} - x_k^2 r_k^{-3}\big)dx_k - \sum_{i=1}^{k-1} x_k x_i r_k^{-3}\, dx_i\Big) \\ &= (-1)^k(r_{k-1}/r_k)^{k-3}\Big(\big(r_k^{-3} r_{k-1}^2\big)dx_k - r_k^{-3}\sum_{i=1}^{k-1} x_i x_k\Big)\, dx_i \\ &= (-1)^k(r_k)^{-k}\Big(r_{k-1}^{k-1}dx_k - r_{k-1}^{k-3}\sum_{i=1}^{k-1} x_i x_k\, dx_i\Big). \end{aligned}$$

Hence, since $(df_k) \wedge \omega_{k-1} = (-1)^{k-2}\omega_{k-1} \wedge (df_k)$, the first term in this last expression contributes

$$(r_k)^{-k} \sum_{i=1}^{k-1} (-1)^{i-1} x_i \, dx_i \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{k-1} \wedge dx_k$$

to the wedge product. As this contribution is all of ω_k except the last term $r_k^{-k}(-1)^{k-1}x_k \, dx_1 \wedge \cdots \wedge dx_{k-1}$, we must endeavor to show that the contribution of the remaining terms amounts to this expression. Since any term containing a repeated factor dx_j is zero, we see that the rest of the expression is

$$(-1)^{k-1}x_k(r_k)^{-k}(r_{k-1})^{k-3}\Big(\sum_{i=1}^{k-1} x_i \, dx_i\Big) \wedge (r_{k-1})^{k-1} \times$$
$$\times \sum_{i=1}^{k-1} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{k-1},$$

which is easily seen to be the same as

$$(-1)^{k-1} r_k^{-k} x_k r_{k-1}^{-2} \sum_{i=1}^{k-1} x_i^2 dx_1 \wedge \cdots \wedge dx_{k-1} = (-1)^{k-1} r_k^{-k} x_k \, dx_1 \wedge \cdots \wedge dx_{k-1},$$

exactly as required. Thus we have computed this result by "brute force," arrogantly ignoring the hint.

For the benefit of those who wish to *use* the hint, here is an alternative approach. The wedge product $(df_k) \wedge \omega_{k-1}$ is the sum of $D_k f_k(\mathbf{x}) \, dx_k \wedge \omega_{k-1}$ and

$$r_{k-1}^{-k-1}\Big(\sum_{i=1}^{k-1} x_i D_i f(\mathbf{x})\Big) \, dx_1 \wedge \cdots \wedge dx_k =$$
$$= r_{k-1}^{-k-1}(\mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) - x_k D_k f_k(\mathbf{x})) \, dx_1 \wedge \cdots \wedge dx_k,$$

and hence, by the first differential equation, equals

$$D_k f_k(\mathbf{x}) \, dx_k \wedge \omega_{k-1} - r_{k-1}^{-k-1} x_k D_k f_k(\mathbf{x}) \, dx_1 \wedge \cdots dx_{k-1},$$

so that the second equation yields the result immediately. The two differential equations themselves are routine computations.

(*c*) No, ω_n is not exact in E_n for any n, since its integral over the $(n-1)$-sphere equals $\frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$, as will be shown below in the answer to part (*d*). (If it were exact, say the differential of λ, this integral would equal thte integral of λ over the boundary of the $(n-1)$-sphere, which is the 0 $(n-2)$-chain.)

(*d*) We can parameterize the $(n-1)$-sphere Σ^{n-1} by the mapping T_n defined by

$$x_1 \quad = \quad \sin t_1 \sin t_2 \cdots \sin t_{n-1},$$

$$
\begin{aligned}
x_2 &= \cos t_1 \sin t_2 \cdots \sin t_{n-1}, \\
x_3 &= \cos t_2 \sin t_3 \cdots \sin t_{n-1}, \\
\cdots &\cdots \cdots \\
x_{n-1} &= \cos t_{n-2} \sin t_{n-1} \\
x_n &= \cos t_{n-1},
\end{aligned}
$$

where $0 \leq t_1 \leq 2\pi$ and $0 \leq t_j \leq \pi$ for $2 \leq j \leq n-1$ That is, the domain of T_n is the parallelepiped $D = [0, 2\pi] \times [0, \pi]^{n-2}$. This is known to be true for $n = 2$ and $n = 3$, and follows easily by induction on n. Suppose, for example, we know it is true for $n-1$, and suppose $x_1^2 + \cdots + x_n^2 = 1$. If $x_n = \pm 1$, we can take $t_{n-1} = 0$ or π, and the values of the other angles can be anything. If $-1 < x_n < 1$, there is precisely one angle $t_{n-1} \in (0, \pi)$ such that $x_n = \cos t_{n-1}$. But then the point $(x_1/\sin t_{n-1}, \ldots, x_{n-1}/\sin t_{n-1})$ belongs to Σ^{n-2}, and hence, by induction, can be written as

$$
\begin{aligned}
x_1/\sin t_{n-1} &= \sin t_1 \cdots \sin t_{n-2}, \\
x_2/\sin t_{n-1} &= \cos t_1 \cdots \sin t_{n-2}, \\
\cdots &\cdots \cdots \\
x_{n-2}/\sin t_{n-1} &= \cos t_{n-3} \sin t_{n-2} \\
x_{n-1}/\sin t_{n-1} &= \cos t_{n-2}.
\end{aligned}
$$

This completes the induction. Observe that the angle t_1 requires the entire range $[0, 2\pi]$. That is, all points on the unit circle in R^2 can be written as $(\cos t, \sin t)$ only if t is allowed to range from 0 to 2π. Otherwise put, the $(n-1)$-sphere is parameterized by $n-2$ latitude angles and one longitude angle.

We can easily show by induction that the pullback of ω_n is

$$
(\omega_n)_{T_n} = (-1)^{n-1} \sin t_2 \sin^2 t_3 \cdots \sin^{n-2} t_{n-1}\, dt_1 \wedge \cdots \wedge dt_{n-1}.
$$

To make the induction work, we need to distinguish the x_i's in various numbers of dimensions; hence let the transformation T_n be defined by giving its components $x_i^{(n)}$, $i \leq n$, by the equations

$$
\begin{aligned}
x_1^{(n)} &= \sin t_1 \sin t_2 \cdots \sin t_{n-1}, \\
x_2^{(n)} &= \cos t_1 \sin t_2 \cdots \sin t_{n-1}, \\
x_3^{(n)} &= \cos t_2 \sin t_3 \cdots \sin t_{n-1}, \\
\cdots &\cdots \cdots \\
x_{n-1}^{(n)} &= \cos t_{n-2} \sin t_{n-1}, \\
x_n^{(n)} &= \cos t_{n-1},
\end{aligned}
$$

Thus we have $x_n^n = \cos t_{n-1}$ and $x_j^{(n)} = x_j^{(n-1)} \sin t_{n-1}$ for $j < n$. Suppose we have proved that

$$
\left(\sum_{i=1}^{n-1} (-1)^{i-1} x_i^{(n-1)} dx_1^{(n-1)} \wedge \cdots \wedge dx_{i-1}^{(n-1)} \wedge dx_{i+1}^{(n-1)} \wedge \cdots \wedge dx_{n-1}^{(n-1)} \right)_{T_{n-1}} =
$$

$$= (-1)^{n-2}(\sin t_1 \sin^2 t_2 \cdots \sin^{n-3} t_{n-2}\, dt_1 \wedge \cdots \wedge dt_{n-2}.$$

We observe that the Jacobian matrix of the transformation T_n is the $n \times (n-1)$ matrix

$$\frac{\partial(x_1^{(n)}, \ldots, x_n^{(n)})}{\partial(t_1, \ldots, t_{n-1})} = \begin{pmatrix} \frac{\partial x_1^{(n-1)}}{\partial t_1} \sin t_{n-1} & \cdots & \frac{\partial x_1^{(n-1)}}{\partial t_{n-2}} \sin t_{n-1} & x_1^{(n-1)} \cos t_{n-1} \\ \frac{\partial x_2^{(n-1)}}{\partial t_1} \sin t_{n-1} & \cdots & \frac{\partial x_2^{(n-1)}}{\partial t_{n-2}} \sin t_{n-1} & x_2^{(n-1)} \cos t_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_{n-1}^{(n-1)}}{\partial t_1} \sin t_{n-1} & \cdots & \frac{\partial x_{n-1}^{(n-1)}}{\partial t_{n-2}} \sin t_{n-1} & x_{n-1}^{(n-1)} \cos t_{n-1} \\ 0 & \cdots & 0 & -\sin t_{n-1} \end{pmatrix}$$

It follows immediately, when we expand the determinant of the first $n-1$ rows along the last column, that

$$\begin{aligned}(dx_1^{(n)} \wedge \cdots \wedge dx_{n-1}^{(n)})_{T_n} &= \sin^{n-2} t_{n-1} \cos t_{n-1} \times \\ &\times \sum_{i=1}^{n-1} (-1)^{n-1+i} x_i^{(n-1)} \frac{\partial(x_1^{(n-1)}, \ldots, x_{n-1}^{(n-1)})}{\partial(t_1, \ldots, t_{n-2})} dt_1 \wedge \cdots \wedge dt_{n-1} r = \\ &= (-1)^n \sin^{n-2} t_{n-1} \cos t_{n-1} \Big(\sum_{i=1}^{n-1} (-1)^{i-1} x_i^{(n-1)}\, dx_1^{(n-1)} \wedge \cdots \\ &\qquad \cdots \wedge dx_{i-1}^{(n-1)} \wedge dx_{i+1}^{(n-1)} \wedge \cdots \wedge dx_{n-1}^{(n-1)} \Big)_{T_{n-1}} \wedge dt_{n-1} \\ &= \sin^{n-2} t_{n-1} \cos t_{n-1} \big(\sin t_2 \sin^2 t_3 \cdots \sin^{n-3} t_{n-2} \big) dt_1 \wedge \cdots \wedge dt_{n-1}.\end{aligned}$$

Hence

$$\begin{aligned}&(-1)^{n-1} \big(x_n^{(n)}\, dx_1^{(n)} \wedge \cdots \wedge dx_{n-1}^{(n)}\big)_{T_n} = \\ &\qquad = (-1)^{n-1} \cos^2 t_{n-1} \sin t_2 \sin^2 t_3 \cdots \sin^{n-2} t_{n-1} \big)\, dt_1 \wedge \cdots \wedge dt_{n-1}.\end{aligned}$$

Next, omitting row i ($i < n$) and expanding the resulting determinant along the last row, we find that

$$\begin{aligned}&(dx_1^{(n)} \wedge \cdots \wedge dx_{i-1}^{(n)} \wedge dx_{i+1}^{(n)} \wedge \cdots \wedge dx_n^{(n)})_{T_n} = \\ &= -\sin^{n-1} t_{n-1} (dx_1^{(n-1)} \wedge \cdots \wedge dx_{i-1}^{(n-1)} \wedge dx_{i+1}^{(n-1)} \wedge \cdots \wedge dx_{n-1}^{(n-1)})_{T_{n-1}} \wedge dt_{n-1},\end{aligned}$$

so that

$$\begin{aligned}&\Big(\sum_{i=1}^{n-1} (-1)^{i-1} x_i^{(n)} dx_1^{(n)} \wedge \cdots \wedge dx_{i-1}^{n} \wedge dx_{i+1}^{(n)} \wedge \cdots \wedge dx_n^{(n)} \Big)_{T_n} = \\ &\quad = -\sin^n t_{n-1} \Big(\sum_{i=1}^{n-1} (-1)^{i-1} x_i^{(n-1)} dx_1^{(n-1)} \wedge \cdots \wedge dx_{i-1}^{(n-1)} \wedge dx_{i+1}^{(n-1)} \wedge \cdots \\ &\qquad\qquad \cdots \wedge dx_{n-1}^{(n-1)} \Big)_{T_{n-1}} \wedge dt_{n-1},\end{aligned}$$

and again by induction this is

$$(-1)^{n-1}\sin^2 t_{n-1}\big(\sin t_2 \sin^2 t_3 \cdots \sin^{n-2} t_{n-1}\big)dt_1 \cdots dt_{n-1}.$$

Combining these results we find that

$$\Big(\sum_{i=1}^{n}(-1)^{i-1}x_i^{(n)}dx_1^{(n)} \wedge \cdots \wedge dx_{i-1}^{(n)} \wedge dx_{i+1}^{(n)} \wedge \cdots \wedge dx_n^{n)}\Big)_{T_n} =$$

$$= (-1)^{n-1}\sin t_2 \sin^2 t_3 \cdots \sin^{n-2} t_{n-1} dt_1 \wedge \cdots \wedge dt_{n-1}.$$

The induction is now complete.

Except for the unimportant factor of -1, this formula gives results consistent with the known results for the area of the $(n-1)$-sphere, namely a total area of

$$A_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\big(\frac{n}{2}\big)}.$$

This is easily verified for $n = 2$ and $n = 3$. In general

$$A_{n-1} = A_{n-2}\int_0^{\pi} \sin^{n-2} t_{n-1}\, dt_{n-1} =$$

$$2A_{n-2}\int_0^{\frac{\pi}{2}} \sin^{n-2} s\, ds = 2A_{n-2}\frac{\Gamma\big(\frac{1}{2}\big)\Gamma\big(\frac{n-1}{2}\big)}{\Gamma\big(\frac{n}{2}\big)}.$$

It easily follows, since $\Gamma\big(\frac{1}{2}\big) = \sqrt{\pi}$, that the formula for the surface area of $\Sigma^{(n-1)}$ is valid for all n.

Similarly we can show the analog of part (c) of Exercise 22, namely that

$$\int_{\Phi} \omega_n = 0$$

for any $(n-1)$-dimensional surface given by a mapping of the form

$$\Phi(s_1, \ldots, s_{n-2}, t) = \big(g(s_1, \ldots, s_{n-2}, t)h_1(s_1, \ldots, s_{n-2}), \ldots$$

$$\ldots, g(s_1, \ldots, s_{n-2}, t)h_n(s_1, \ldots, s_{n-2})\big).$$

Indeed, the pullback of ω_n is

$$(\omega_n)_{\Phi} =$$

$$= \begin{vmatrix} gh_1 & \frac{\partial g}{\partial s_1}h_1 + g\frac{\partial h_1}{\partial s_1} & \cdots & \frac{\partial g}{\partial s_{n-2}}h_1 + g\frac{\partial h_1}{\partial s_{n-2}} & \frac{\partial g}{\partial t}h_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ gh_n & \frac{\partial g}{\partial s_1}h_n + g\frac{\partial h_n}{\partial s_1} & \cdots & \frac{\partial g}{\partial s_{n-2}}h_n + g\frac{\partial h_n}{\partial s_{n-2}} & \frac{\partial g}{\partial t}h_n \end{vmatrix} ds_1 \wedge \cdots \wedge ds_{n-2} \wedge dt.$$

But this determinant is zero, since the first and last columns are proportional.

We can now prove that if $f(t_1, \ldots, t_{n-1}) > 0$ and

$$\Omega(t_1, \ldots, t_{n-1}) = f(t_1, \ldots, t_{n-1})\Sigma^{(n-1)}(t_1, \ldots, t_{n-1}),$$

then

$$\int_{\Omega} \omega_n = \int_S \omega_n = \pm A_{n-1}(S).$$

To do so, we consider the n-surface in R^n given by

$$\Psi(t_1, \ldots, t_{n-1}, t) = [1 - t + tf(t_1, \ldots, t_{n-1})]\Sigma^{(n-1)}(t_1, \ldots, t_{n-1}),$$

for $0 \le t \le 1$ and $t_1, \ldots, t_n$ ranging over a parallelepiped contained in the interior of D with boundary faces parallel to those of D. For each fixed t_j, this Ψ is an $(n-1)$-surface of the form just considered, and hence the integral of ω_n over it is zero. This applies in particular to the faces of the closed parallelepiped E. Since $\int_{\partial\Psi} \omega_n = \int_{\Psi} d\omega_n = 0$, it then follows that, up to a factor of ± 1,

$$\int_{\Omega} \omega_n = \int_S \omega_n = A_{n-1}(S).$$

Finally, as in Exercise 22, ω_n is exact in the complement of every $(n-2)$-hyperplane through the origin, since there is a rotation that maps the complement of that hyperplane to E_{n-1}, while ω_n is rotation-invariant.

Exercise 10.24 Let $\omega = \sigma a_i(\mathbf{x})\, dx_i$ be a 1-form of class C'' in a convex open set $E \subset R^n$. Assume $d\omega = 0$ and prove that ω is exact in E by completing the following outline:

Fix $\mathbf{p} \in E$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p},\mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

Apply Stokes' theorem to affine-oriented 2-simplexes $[\mathbf{p}, \mathbf{x}, \mathbf{y}]$ in E. Deduce that

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^{n} (y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y})\, dt$$

for $\mathbf{x} \in E$, $\mathbf{y} \in E$. Hence $D_i f(\mathbf{x}) = a_i(\mathbf{x})$.

Solution. Because $d\omega = 0$, the integral of ω over the boundary of the oriented 2-simplex $[\mathbf{p}, \mathbf{x}, \mathbf{y}]$ is zero. That is

$$\int_{[\mathbf{x},\mathbf{y}]} \omega - \int_{[\mathbf{p},\mathbf{y}]} \omega + \int_{[\mathbf{p},\mathbf{x}]} \omega = 0,$$

which can be rewritten as

$$f(\mathbf{y}) - f(\mathbf{x}) = \int_{[\mathbf{x},\mathbf{y}]} \omega = \sum_{i=1}^{n} (y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y})\, dt.$$

Differentiating with respect to y_i, we find

$$D_i f(\mathbf{y}) = \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y})\,dt + \sum_{j=1}^n (y_j - x_j) \int_0^1 tD_i a_j((1-t)\mathbf{x} + t\mathbf{y})\,dt.$$

The fact that $d\omega = 0$ says that $D_i a_j = D_j a_i$, so that we have

$$\begin{aligned} D_i f(\mathbf{y}) &= \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y})dt + \int_0^1 \sum_{j=1}^n t(y_j - x_j) D_j a_i((1-t)\mathbf{x} + t\mathbf{y})dt \\ &= \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y})\,dt + \int_0^1 t\frac{d}{dt} a_i((1-t)\mathbf{x} + t\mathbf{y})\,dt \\ &= \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y})\,dt + ta_i((1-t)\mathbf{x} + t\mathbf{y})\Big|_0^1 \\ &\qquad - \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y})\,dt \\ &= a_i(\mathbf{y}). \end{aligned}$$

Thus $\omega = df$.

Exercise 10.25 Assume that ω is a 1-form in an open set $E \subset R^n$ such that

$$\int_\gamma \omega = 0$$

for every closed curve γ in E of class $\mathcal{C}'$. Prove that ω is exact in E, by imitating part of the argument sketched in Exercise 24.

Solution. We first observe that Stokes' theorem and the argument of Theorem 10.15 show that $d\omega = 0$ in E. (Theorem 10.15 actually shows that if some component of $d\omega$ is nonzero at some point of E, then there is a 2-surface Φ in E whose domain is a 2-cell in R^2 for which $\int_\Phi d\omega \neq 0$. Then by Stokes' theorem, $\int_{\partial\Phi} \omega \neq 0$ also, contradicting the assumption of the problem.

In each connected component E_α of E, we choose a fixed point $\mathbf{x}_\alpha$. There is a ball of some positive radius r_α centered at $\mathbf{x}_\alpha$ and contained in E. Let this ball be B_α. Exercise 24 shows that there is a function $f(\mathbf{x})$ such that $\omega(\mathbf{x}) = df(\mathbf{x})$ inside B_α. By subtracting a constant from f we can assume that $f(\mathbf{x}_\alpha) = 0$.

Now consider the set S of all points $\mathbf{x} \in E_\alpha$ having the property that there exist a connected open set $F_\mathbf{x}$ containing $\mathbf{x}$ and $\mathbf{x}_\alpha$ and a function $f_\mathbf{x}$ defined on $F_\mathbf{x}$ such that $df_\mathbf{x} = \omega$ on $F_\mathbf{x}$ and $f_{\mathbf{x}_\alpha} = 0$. It is clear that S is an open connected subset of E_α, being the union of all the connected open sets $F_\mathbf{x}$, which have the common point $\mathbf{x}_\alpha$. It is also clear that there is a function f defined on S such that $df = \omega$ on S. In fact we can define $f(\mathbf{x}) = f_\mathbf{x}(\mathbf{x})$, and this definition is unambiguous, since if $f_\mathbf{x}$ and $f_\mathbf{y}$ are both defined at $\mathbf{z}$, then

$$f_\mathbf{x}(\mathbf{z}) = \int_\gamma df_\mathbf{x} = \int_\gamma \omega = \int_\delta \omega = \int_\delta df_\mathbf{y} = f_\mathbf{y}(\mathbf{z}).$$

Here γ is a path in $E_{\mathbf{x}}$ from $\mathbf{x}_\alpha$ to $\mathbf{z}$, and δ is a path in $E_{\mathbf{y}}$ from $\mathbf{x}_\alpha$ to $\mathbf{z}$. The path $\gamma - \delta$ lies in E and is a closed loop, so that

$$\int_{\gamma-\delta} \omega = 0.$$

We need only show that $S = E_\alpha$. But if not, then E_α contains a boundary point $\mathbf{x} \in S$. Some open ball B about $\mathbf{x}$ is contained in E, and this open ball contains a point $\mathbf{y} \in S$. But then there exists a function g such that $dg = \omega$ in B, and subtracting a constant makes it possible to ensure that $g(\mathbf{y}) = f_{\mathbf{y}}(\mathbf{y}) = f(\mathbf{y})$. We claim that $g(\mathbf{z}) = f(\mathbf{z})$ on the entire set $S \cap B$. In fact this argument merely repeats the argument just given to show that f is unambiguously defined. It then follows that $\mathbf{y}$ is contained in the connected open set $S \cap B$ and that the function h defined to be f on S and g on B has the property that $dh = \omega$ on $S \cap B$. By definition, this means $\mathbf{y} \in S$, which contradicts the assumption that $\mathbf{y}$ is a boundary point of S. Therefore $S = E_\alpha$.

Thus we can find a primitive for ω on each connected component of E. These primitives can be pieced together to provide a single primitive for ω on E.

Exercise 10.26 Assume ω is a 1-form in $R^3 \setminus \{\mathbf{0}\}$, of class C' and $d\omega = 0$. Prove that ω is exact in $R^3 \setminus \{\mathbf{0}\}$.

Hint: Every closed continuously differentiable curve in $R^3 \setminus \{\mathbf{0}\}$ is the boundary of a 2-surface in $R^3 \setminus \{\mathbf{0}\}$. Apply Stokes' theorem and Exercise 25.

Solution. Given the assumption in the hint, the solution is easy. By Exercise 25 we need only show that the integral of ω over every closed curve is zero. By the assertion in the hint, this closed curve is the boundary of a two-surface. By Stokes' theorem, the integral of ω over the curve equals the integral of $d\omega$ over the 2-surface.

To prove the claim that every continuously differentiable curve in $R^3 \setminus \{\mathbf{0}\}$ is the boundary of a two-surface, we may assume that the curve is of the form $\mathbf{x}(t)$, $0 \le t \le 1$ and $\mathbf{x}(0) = \mathbf{x}(1)$. Let $\mathbf{x}(t) = \big(x(t), y(t), z(t)\big)$. We shall show first of all that there is some line through the origin in R^3 that does not intersect the curve.

To that end, we observe that the intersection of a sphere of radius ρ in R^3 with a ball of radius r ($r \le 2\rho$) about a point of the sphere is a spherical cap whose area is πr^2. (Note that this result is independent of ρ. It is a remarkable fact, whose proof is a routine computation.) Since the area of the whole sphere is $4\pi\rho^2$, it follows that half of any given hemisphere cannot be covered by fewer than ρ^2/r^2 such spherical caps. Now, since $\mathbf{x}(t) \ne \mathbf{0}$ and $\mathbf{x}'(t)$ is continuous, it follows that $\mathbf{v}(t) = \mathbf{x}(t)/|\mathbf{x}(t)|$ is a Lipschitz function, that is, there exists a constant M such that $|\mathbf{v}(s) - \mathbf{v}(t)| \le M|t - s|$ for all s and t. In particular the image of each interval $[k/n, (k+1)/n]$ is contained in a spherical cap of radius M/n. Thus the complete curve is contained in a set of n spherical caps of radius at most M/n. But to cover the half of any given hemisphere of the unit sphere requires at least $\frac{n^2}{M^2}$ such caps. Hence, if $n > M^2$, the projection of the curve $\mathbf{x}(t)$ on the unit sphere is contained in a set of spherical caps covering less than half of the upper hemisphere and less than half of the lower hemisphere. Hence there are two antipodal points $\mathbf{x}_0$ and $-\mathbf{x}_0$ on the unit sphere not in its image. That means there is at least one line through the origin that the curve does not intersect.

This line through the origin gives us a sense of positive rotation from $\mathbf{x}(t)$ to $\mathbf{x}(t+\frac{1}{2})$ for each $t \in [0, \frac{1}{2}]$. We can then construct a $\mathcal{C}'$-curve $\gamma_t(s)$ in $R^3\backslash\{\mathbf{0}\}$ that goes from $\mathbf{x}(t)$ to $\mathbf{x}(t+\frac{1}{2})$ by letting cylindrical coordinates vary linearly with respect to s. To be specific, we can assume without loss of generality that the line is the z-axis. In that case, the radial coordinate $r(t) = \sqrt{x^2(t) + y^2(t)}$ is never zero and is a continuously differentiable function of position. We choose $\theta(t)$ as the cylindrical polar coordinate of $\mathbf{x}(t)$ in a continuously differentiable manner for $0 \le t \le 1$. (This is possible by piecing together sections of this function over sufficiently small intervals.) We then define $\gamma(s,t) = \big(x(s,t), y(s,t), z(s,t)\big)$ for $0 \le s \le 1$, $0 \le t \le 1/2$ by

$$
\begin{aligned}
x(t,u) &= (1-u)r(t)\cos\big((1-u)\theta(t)\big) + ur(1-t)\cos\big(u\theta(1-t)\big)\big),\\
y(t,u) &= (1-u)r(t)\sin\big((1-u)\theta(t)\big) + ur(1-t)\sin\big(u\theta(1-t)\big)\big),\\
z(t,u) &= (1-u)z(t) + uz(1-t).
\end{aligned}
$$

We let the boundary of this cell be $\delta_1+\delta_2+\delta_3+\delta_4$. Here δ_1 is $\gamma(t,0)$, $0 \le t \le 1/2$, which is just $\mathbf{x}(t)$ over the same interval; δ_2 is $\gamma(1/2, u)$, which is the "line segment" from $\mathbf{x}(1/2)$ to $\mathbf{x}(1/2)$, whose range is just a point, and hence counts as 0 when regarded as a 1-chain; δ_3 is $\gamma(1/2-t, 1)$ which is just $\mathbf{x}(t+\frac{1}{2})$, so that $\delta_1 + \delta_3$ represents $\mathbf{x}(t)$ as t goes from 0 to 1. Finally δ_4 is $\gamma(0,u)$, which is the line segment from $\mathbf{x}(1)$ to $\mathbf{x}(0)$, and since $\mathbf{x}$ is a closed curve, these two points are the same. Hence once again δ_4 counts as 0 when regarded as a 1-chain. Thus the boundary of γ is indeed the curve $\mathbf{x}$.

Exercise 10.27 Let E be an open 3-cell in R^3, with edges parallel to the coordinate axes. Suppose $(a,b,c) \in E$, $f_i \in \mathcal{C}'(E)$ for $i = 1,2,3$,

$$\omega = f_1\,dy \wedge dz + f_2\,dz \wedge dx + f_3\,dx \wedge dy,$$

and assume that $d\omega = 0$ in E. Define

$$\lambda = g_1\,dx + g_2\,dy$$

where

$$
\begin{aligned}
g_1(x,y,z) &= \int_c^z f_2(x,y,s)\,ds - \int_b^y f_3(x,t,c)\,dt\\
g_2(x,y,z) &= -\int_c^z f_1(x,y,s)\,ds,
\end{aligned}
$$

for $(x, y, z) \in E$. Prove that $d\lambda = \omega$ in E.

Evaluate these integrals when $\omega = \zeta$ and thus find the form λ that occurs in part (e) of Exercise 22.

Solution. Since

$$d\lambda = -\frac{\partial g_2}{\partial z}\, dy \wedge dz + \frac{\partial g_1}{\partial z}\, dz \wedge dx + \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right) dx \wedge dy,$$

we need only show that

$$\begin{aligned} \frac{\partial g_2}{\partial z} &= -f_1, \\ \frac{\partial g_1}{\partial z} &= f_2, \\ \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} &= f_3. \end{aligned}$$

The first two equations are immediate. As for the third, direct computation shows that

$$D_1 g_2(x,y,z) - D_2 g_1(x,y,z) = \int_c^z -\big(D_1 f_1(x,y,s) + D_2 f_2(x,y,s)\big)\, ds + f_3(x,y,c).$$

Now the assumption that $d\omega = 0$ says that

$$D_1 f_1(x,y,s) + D_2 f_2(x,y,s) = -D_3 f_3(x,y,s),$$

Substituting this value into the last expression and evaluating the integral using the fundamental theorem of calculus yields the result $d\lambda = \omega$.

Taking

$$\begin{aligned} f_1(x,y,z) &= \frac{x}{(x^2+y^2+z^2)^{3/2}}, \\ f_2(x,y,z) &= \frac{y}{(x^2+y^2+z^2)^{3/2}}, \\ f_3(x,y,z) &= \frac{z}{(x^2+y^2+z^2)^{3/2}}, \end{aligned}$$

we get

$$\begin{aligned} g_2(x,y,z) &= -\int_c^z \frac{x}{(x^2+y^2+s^2)^{3/2}}\, ds = \\ &= \frac{1}{x^2+y^2}\left(\frac{cx}{\sqrt{x^2+y^2+c^2}} - \frac{zx}{\sqrt{x^2+y^2+z^2}}\right), \\ g_1(x,y,z) &= \int_c^z \frac{y}{(x^2+y^2+s^2)^{3/2}}\, ds - \int_b^y \frac{c}{(x^2+t^2+c^2)^{3/2}}\, dt \\ &= \frac{1}{x^2+y^2}\left(\frac{yz}{\sqrt{x^2+y^2+z^2}} - \frac{yc}{\sqrt{x^2+y^2+c^2}}\right) \\ &\quad - \frac{1}{x^2+c^2}\left(\frac{cy}{\sqrt{x^2+y^2+c^2}} - \frac{bc}{\sqrt{x^2+b^2+c^2}}\right). \end{aligned}$$

It is a routine computation to verify that these functions do indeed provide a primitive for ω.

Exercise 10.28 Fix $b > a > 0$, define

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta)$$

for $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$. (The range of Φ is an annulus in R^2.) Put $\omega = x^3\, dy$, and compute both

$$\int_\Phi d\omega \text{ and } \int_{\partial\Phi} \omega$$

to verify that they are equal.

Solution. Since $d\omega = 3x^2\, dx \wedge dy$, we have $(d\omega)_\Phi = -r\, dr \wedge d\theta$, and

$$\int_\Phi d\omega = -\int_a^b \int_0^{2\pi} 3r^3 \cos^2\theta\, d\theta\, dr = \frac{3\pi}{4}(a^4 - b^4).$$

For the integral over the boundary we have $dy = r\cos\theta\, d\theta + \sin\theta\, dr$, and we get

$$\int_0^{2\pi} (a^4 - b^4)\cos^4\theta\, d\theta = \frac{3\pi}{4}(a^4 - b^4).$$

Exercise 10.29 Prove the existence of a function α with the properties needed in the proof of Theorem 10.38, and prove that the resulting function F is of class C'. (Both assertions become trivial if E is an open cell or an open ball, since α can then be taken to be a constant. Refer to Theorem 9.42.)

Solution. We are given a convex open set $V \subseteq R^p$ whose projection on R^{p-1} is the convex open set U. We need to show that there is a continuously differentiable function $\alpha : U \longrightarrow R$ whose graph is contained in V. If V is a cell or an open ball, there exists a section $x_p = c$ of it (many sections, if it is a cell), whose projection is U, and we can simply define $\alpha(\mathbf{y}) = c$ for all $\mathbf{y} \in U$.

Now write V as a countable union of open balls $V = \bigcup_{i=1}^{\infty} B_i$. Also write V as the union of an increasing sequence of compact sets K_n such that $K_n \subseteq \operatorname{int}(K_{n+1})$.

We claim that, as in Theorem 10.8, there exist continuous functions ψ_i such that the support of ψ_i is contained in the projection of B_i on R^{p-1}, $0 \leq \psi_i(\mathbf{y}) \leq 1$ for all $\mathbf{y}$, and $\sum_{i=1} \psi_i(\mathbf{y}) = 1$ for all $\mathbf{y} \in U$. Moreover, this sum is *locally finite*, that is, each point $\mathbf{y}$ has a neighborhood $U_\mathbf{y}$ such that the set of indices i for which $\psi_i(\mathbf{z}) \neq 0$ for some $\mathbf{z} \in U_\mathbf{y}$ is finite.

To construct such functions, for each $\mathbf{x} \in V$, let $i(\mathbf{x})$ be the smallest index r such that $\mathbf{x} \in B_i$. Then, as in the proof of Theorem 10.8, for each $\mathbf{x} \in K_1$, choose open balls $B(\mathbf{x})$ and $W(\mathbf{x})$ centered at $\mathbf{x}$ such that

$$\overline{B(\mathbf{x})} \subset W(\mathbf{x}) \subset \overline{W(\mathbf{x})} \subset B_{i(\mathbf{x})}.$$

Since K_1 is compact, there are points $\mathbf{x}_{11}, \ldots, \mathbf{x}_{1N_1}$ such that

$$K_1 \subseteq B(\mathbf{x}_{11}) \cup \cdots \cup B(\mathbf{x}_{1N_1}).$$

For later convenience we define $L_1 = K_1$.

Now let $L_2 = K_2 \setminus \bigcup_{j=1}^{N_1} B(\mathbf{x}_{1j})$. For each $\mathbf{x} \in L_2$ there are open balls $B(\mathbf{x})$ and $W(\mathbf{x})$ centered at $\mathbf{x}$ such that

$$\overline{B(\mathbf{x})} \subset W(\mathbf{x}) \subset \overline{W(\mathbf{x})} \subset B_{i(\mathbf{x})} \setminus K_1.$$

Since L_2 is compact, we choose a finite set of points $\mathbf{x}_{21}, \ldots, \mathbf{x}_{2N_2}$ such that

$$L_2 \subseteq B(\mathbf{x}_{21}) \cup \cdots \cup B(\mathbf{x}_{2N_2}).$$

Notice that $K_2 \subset \bigcup_{k=1}^{2} \bigcup_{j=1}^{N_k} B(\mathbf{x}_{kj})$.

Now suppose we have chosen a (possibly empty) collection of open balls $B(\mathbf{x}_{kj})$ and $W(\mathbf{x}_{kj})$, $1 \le j \le N_k$, $1 \le k \le r$, centered at $\mathbf{x}_{kj} \in L_k = K_k \setminus \bigcup_{i=1}^{k-1} \bigcup_{j=1}^{N_i} B(\mathbf{x}_{ij})$, and such that

$$\overline{B(\mathbf{x}_{kj})} \subset W(\mathbf{x}_{kj}) \subset \overline{W(\mathbf{x}_{kj})} \subset B_{i(\mathbf{x}_{kj})} \setminus K_{k-1},$$

and

$$K_r \setminus \bigcup_{k=1}^{r-1} \bigcup_{j=1}^{N_k} B(\mathbf{x}_{kj}) \subset \bigcup_{j=1}^{N_r} B(\mathbf{x}_{rj}),$$

and

$$K_s \subset \bigcup_{k=1}^{s} \bigcup_{j=1}^{N_k} B(\mathbf{x}_{kj}),$$

for $1 \le s \le r-1$. It then follows from the last two relationships that the last one also holds with $s = r$. By then considering the compact set $L_{r+1} = K_{r+1} \setminus \bigcup_{k=1}^{r} \bigcup_{j=1}^{N_k} B(\mathbf{x}_{kj})$ and repeating the argument, we can assume that the sets $B(\mathbf{x}_{kj})$ and $W(\mathbf{x}_{kj})$ with these properties have been chosen for all k and all j, $1 \le k < \infty$, $1 \le j \le N_k$. It follows in particular that

$$V = \bigcup_{n=1}^{\infty} K_n \subset \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{N_k} B(\mathbf{x}_{kj}).$$

Now let $\widetilde{K}_k$, $\widetilde{B}(\mathbf{x}_{kj})$, and $\widetilde{W}(\mathbf{x}_{kj})$ be respectively the projections on R^{p-1} of K_k, $B(\mathbf{x}_{kj})$, and $W(\mathbf{x}_{kj})$, and let

$$\widetilde{L}_r = \widetilde{K}_k \setminus \bigcup_{k=1}^{r-1} \bigcup_{j=1}^{N_k} \widetilde{B}(\mathbf{x}_{kj}).$$

We then choose functions φ_{jk} as smooth as we like such that $\varphi_{kj}(\mathbf{y}) = 1$ on $\widetilde{B}(\mathbf{x}_{kj})$ (and hence also on $\overline{\widetilde{B}(\mathbf{x}_{kj})}$), $\varphi_{kj}(\mathbf{y}) = 0$ outside $\widetilde{W}(\mathbf{x}_{kj})$, and $0 \le \varphi_{kj}(\mathbf{y}) \le 1$ on R^{p-1}. Let $\varphi_j(\mathbf{y}) = \varphi_{1j}(\mathbf{y})$ for $1 \le j \le N_1$ and $\varphi_j(\mathbf{y}) =$

$\varphi_{k,j-(N_1+\cdots+N_{k-1})}(\mathbf{y})$ for $N_1+\cdots+N_{k-1} < j \le N_1+\cdots+N_k$, $2 \le k < \infty$. We define $\mathbf{x}_j$ analogously. Let $\mathbf{x}_j = (\mathbf{y}_j, c_j)$.

We then proceed to define $\psi_1(\mathbf{y}) = \varphi_1(\mathbf{y})$ and

$$\psi_{j+1}(\mathbf{y}) = \big(1-\varphi_1(\mathbf{y})\big)\cdots\big(1-\varphi_j(\mathbf{y})\big)\varphi_{j+1}(\mathbf{y})$$

for $j = 1, 2, \ldots$, as in Theorem 10.8. It is obvious that the support of ψ_j is contained in the closure of $\widetilde{W}(\mathbf{x}_j)$ and hence in $\widetilde{B}_{i(\mathbf{x}_j)} \setminus \widetilde{K}_{k-1} \subseteq U \setminus \widetilde{K}_{k-1}$ when $N_1+\cdots+N_{k-1} < j \le N_1+\cdots+N_k$.

Now by the choice of the sets K_n, if $\mathbf{y} \in U$, there is some n such that $\mathbf{y} \in \widetilde{K}_n \subset \operatorname{int}\big(\widetilde{K}_{n+1}\big)$, and hence $\psi_j(\mathbf{y}) = 0$ on the open neighborhood $\operatorname{int}\widetilde{K}_{n+1}$ of $\mathbf{y}$ if $j > N_1+\cdots+N_{n+1}$. Therefore the sum and product

$$\sum_{j=1}^{\infty}\psi_j(\mathbf{y}) = 1 - \prod_{i=1}^{\infty}[1-\varphi_i(\mathbf{y})]$$

are both locally finite at each point. (Local finiteness of the product means all but a finite number of factors equal 1 on a neighborhood of each point.) However, if $y \in U$, then $\mathbf{y} \in \widetilde{B}(\mathbf{x}_j)$ for some j, and so $\varphi_j(\mathbf{y}) = 1$, from which it then follows that

$$\sum_{j=1}^{\infty}\psi_j(\mathbf{y}) = 1$$

for all $\mathbf{y} \in U$.

Since we have defined c_j so that $\mathbf{x}_j = (\mathbf{y}_j, c_j)$, it follows that the projection of the c_j-section of $B(\mathbf{x}_j)$ on R^{p-1}, which we denote C_j, is the same as the projection of $B(\mathbf{x}_j)$ on this subspace. That is, it is $\widetilde{B}(\mathbf{x}_j)$. We can now let $\alpha(\mathbf{y}) = \sum_{j=1}^{\infty} c_j\psi_1(\mathbf{y})$. For then at each $\mathbf{y} \in U$ there is a finite integer n such that

$$(\mathbf{y}, \alpha(\mathbf{y})) = \psi_1(\mathbf{y})(\mathbf{y}, c_1) + \cdots + \psi_n(\mathbf{y})(\mathbf{y}, c_n).$$

Since $\psi_k(\mathbf{y}) = 0$ if $\mathbf{y} \notin C_k$ and $(\mathbf{y}, c_k) \in B_k \subset V$ if $\mathbf{y} \in C_k$, it follows that $(\mathbf{y}, \alpha(\mathbf{y}))$ is a weighted average of points in V, hence belongs to V for all $\mathbf{y} \in U$.

Exercise 10.30 If $\mathbf{N}$ is the vector given by (135), prove that

$$\det\begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_2\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |\mathbf{N}|^2.$$

Also, verify Eq. (137).

Solution. The equation in the problem is a straightforward computation, and amounts merely to expanding the determinant along the last column. Likewise Eq. (137), which merely asserts that a cross product is perpendicular to each of the factors, is routine. The two inner products in the equation can be obtained by replacing the last column of this determinant by either $(\alpha_1, \alpha_2, \alpha_3)$ or $(\beta_1, \beta_2, \beta_3)$. In each case, the result is a determinant with two equal columns, which is therefore zero.

Exercise 10.31 Let $E \subset R^3$ be open, suppose $g \in C''(E)$, $h \in C''(E)$, and consider the vector field

$$\mathbf{F} = g\nabla h.$$

(a) Prove that

$$\nabla \cdot \mathbf{F} = g\nabla^2 h + (\nabla g) \cdot (\nabla h)$$

where $\nabla^2 h = \nabla \cdot (\nabla h) = \sum \partial^2 h/\partial x_i^2$ is the so-called "Laplacian" of h. (b) If Ω is a closed subset of E with positively oriented boundary $\partial\Omega$ (as in Theorem 10.51), prove that

$$\int_\Omega \left[g\nabla^2 h + (\nabla g) \cdot (\nabla h)\right] dV = \int_{\partial\Omega} g\frac{\partial h}{\partial n}\, dA$$

where (as is customary) we have written $\partial h/\partial n$ in place of $(\nabla h) \cdot \mathbf{n}$. (Thus $\partial h/\partial n$ is the directional derivative of h in the direction of the outward normal to $\partial\Omega$, the so-called *normal derivative* of h.) Interchange g and h, subtract the resulting formula from the first one, to obtain

$$\int_\Omega (g\nabla^2 h - h\nabla^2 g)\, dV = \int_{\partial\Omega} \left(g\frac{\partial h}{\partial n} - h\frac{\partial g}{\partial n}\right) dA.$$

These two formulas are usually called *Green's identities*.

(c) Assume that h is *harmonic* in E; this means that $\nabla^2 h = 0$. Take $g = 1$ and conclude that

$$\int_{\partial\Omega} \frac{\partial h}{\partial n}\, dA = 0.$$

Take $g = h$, and conclude that $h = 0$ in Ω if $h = 0$ on $\partial\Omega$.

(d) Show that Green's identities are also valid in R^2.

Solution. Part (a) is simply the product rule for derivatives.

The main equation in part (b) is simply the divergence theorem applied to $\mathbf{F}$. Green's identities then follow by completely routine computation.

(c) Taking $g = 1$ forces $\partial g/\partial n = 0$ and $\nabla^2 g = 0$. Since $\nabla^2 h = 0$ by the assumption that h is harmonic, the result follows. For the other assertion of this part we have to go back to the main equation before taking $g = h$. When we do, we actually get a slightly stronger assertion: $\nabla h = 0$ in Ω, and so h is constant on each component of Ω, if either $h = 0$ or $\partial h/\partial n = 0$ on all of $\partial\Omega$. When $h = 0$ on $\partial\Omega$, obviously the constant value of h must be 0.

(d) The "two-dimensional" divergence theorem is simply Green's theorem. That is, the assertion that

$$\int_{\Omega} \nabla \cdot \mathbf{F} = \int_{\partial\Omega} \mathbf{k} \times \mathbf{F}$$

follows upon applying Green's theorem to the one-form $\omega = -F_2\,dx + F_1\,dy$ corresponding to the vector field $\mathbf{k}\times\mathbf{F} = -F_2\mathbf{i}+F_1\mathbf{j}$. Because the dot and cross operations can be interchanged in the scalar triple product, integrating $\mathbf{k} \times \mathbf{F}$ along a curve, that is, taking the product $\mathbf{k} \times \mathbf{F} \cdot \mathbf{r}$, where $\mathbf{r}$ is the tangent to the curve, and then integrating, is the same as integrating $\mathbf{F} \cdot \mathbf{k} \times \mathbf{r}$, which is the normal component of $\mathbf{F}$. All the same identities now follow.

Exercise 10.32 Fix δ, $0 < \delta < 1$. Let D be the set of all $(\theta, t) \in R^2$ such that $0 \leq \theta \leq \pi$, $-\delta \leq t \leq \delta$. Let Φ be the 2-surface in R^3 with parameter domain D given by

$$\begin{aligned} x &= (1 - t\sin\theta)\cos 2\theta \\ y &= (1 - t\sin\theta)\sin 2\theta \\ z &= t\cos\theta \end{aligned}$$

where $(x, y, z) = \Phi(\theta, t)$. note that $\Phi(\pi, t) = \Phi(0, -t)$ and that Φ is one-to-one on the rest of D.

The range $M = \Phi(D)$ is known as a *Möbius band.* It is the simplest example of a nonorientable surface.

Prove the various assertions made in the following description: Put $\mathbf{p}_1 = (0, -\delta)$, $\mathbf{p}_2 = (\pi, -\delta)$, $\mathbf{p}_3 = (\pi, \delta)$, $\mathbf{p}_4 = (0, \delta)$, $\mathbf{p}_5 - \mathbf{p}_1$. Put $\gamma_i = [\mathbf{p}_i, \mathbf{p}_{i-1}]$, $i = 1, 2 \ldots, 4$, and put $\Gamma_i = \Phi \circ \gamma_i$. Then

$$\partial\Phi = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4.$$

Put $\mathbf{z} = (1, 0, -\delta)$, $\mathbf{b} = (1, 0, \delta)$. Then

$$\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a}, \quad \Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b},$$

and $\partial\Phi$ can be described as follows.

Γ_1 spirals up from $\mathbf{a}$ to $\mathbf{b}$; its projection into the (x, y)-plane has winding number $+1$ around the origin. (See Exercise 23, Chap. 8).

$\Gamma_2 = [\mathbf{b}, \mathbf{a}]$.

Γ_3 spirals up from $\mathbf{a}$ to $\mathbf{b}$; its projection into the (x, y)-plane has winding number -1 around the origin.

$\Gamma_4 = [\mathbf{b}, \mathbf{a}]$.

Thus $\partial\Phi = \Gamma_1 + \Gamma_3 + 2\Gamma_2$.

If we go from $\mathbf{a}$ to $\mathbf{b}$ along Γ_1 and continue along the "edge" of M until we return to $\mathbf{a}$, the curve traced out is

$$\Gamma = \Gamma_1 - \Gamma_3,$$

which may also be represented on the parameter interval $[0, 2\pi]$ by the equations

$$\begin{aligned} x &= (1 + \delta \sin \theta) \cos 2\theta, \\ y &= (1 + \delta \sin \theta) \sin 2\theta, \\ z &= -\delta \cos \theta. \end{aligned}$$

It should be emphasized that $\Gamma \neq \partial\Phi$: Let η be the 1-form discussed in Exercises 21 and 22. Since $d\eta = 0$, Stokes' theorem shows that

$$\int_{\partial\Phi} \eta = 0,$$

But although Γ is the "geometric" boundary of M, we have

$$\int_{\Gamma} \eta = 4\pi.$$

In order to avoid this possible source of confusion, Stokes' formula (Theorem 10.50) is frequently stated only for orientable surfaces Φ.

Solution. The claim about the boundary $\partial\Phi$ follows immediately from the definition of a boundary. The domain D is a cell whose boundary is $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, so that by definition $\partial\Phi = \Phi(\gamma_1) + \Phi(\gamma_2) + \Phi(\gamma_3) + \Phi(\gamma_4)$.

The claims that $\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a}$ and $\Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b}$ are routine computations.

The description of Γ_1 follows from the fact that γ_1 can be described as the set $(\theta, -\delta)$, $0 \leq \theta \leq \pi$, so that the projection of Γ_1 in the (x, y)-plane is the set of all points $\big(x(\theta), y(\theta)\big)$, where

$$\begin{aligned} x(\theta) &= (1 + \delta \sin \theta) \cos 2\theta \\ y(\theta) &= (1 + \delta \sin \theta) \sin 2\theta. \end{aligned}$$

Regarding the pair $\big(x(\theta), y(\theta)\big)$ as the complex number $z(\theta) = x(\theta) + iy(\theta) = (1 + \delta \sin \theta)(\cos 2\theta + i \sin 2\theta)$, and using the definition of the winding number, we find this winding number to be

$$n = \frac{1}{2\pi i} \int_0^{\pi} \frac{z'(\theta)}{z(\theta)}\, d\theta.$$

Now, $z'(\theta) = 2(1 + \delta \sin \theta)(-\sin 2\theta + i \cos 2\theta) + \delta \cos \theta(\cos 2\theta + i \sin 2\theta)$, so that we get

$$n = \frac{1}{\pi i}\left(\int_0^{\pi} \frac{-\sin 2\theta + i \cos 2\theta}{\cos 2\theta + i \sin 2\theta}\, d\theta + \delta \int_0^{\pi} \frac{\cos\theta}{1 + \delta \sin \theta}\, d\theta\right).$$

But $-\sin 2\theta + i \cos 2\theta = i(\cos 2\theta + i \sin 2\theta)$, so that the first integral is just πi, and that term contributes $+1$ to the winding number. The second integral is just $\ln(1 + \delta \sin \theta)$, and since this function has the value 0 at both $\theta = 0$ and $\theta = \pi$, it contributes nothing.

As for Γ_2, since $\theta = \pi$, it is given by $\big(x(t), y(t), z(t)\big)$, $-\delta \le t \le \delta$, where $x(t) = 1$, $y(t) = 0$, $z(t) = -t$. It therefore describes the line segment from $\mathbf{b}$ to $\mathbf{a}$ as t goes from $-\delta$ to δ.

The descriptions of Γ_3 and Γ_4 are justified exactly as was just done for Γ_1 and Γ_2.

As both Γ_1 and Γ_3 spiral upward from $\mathbf{a}$ to $\mathbf{b}$, it is manifest that $\Gamma_1 - \Gamma_3$ represents a spiral that goes from $\mathbf{a}$ to $\mathbf{b}$ and back again. It is also easy to see that this spiral does not intersect itself, as the ranges of Γ_1 and Γ_3 meet only in $\mathbf{a}$ and $\mathbf{b}$. For suppose θ and φ are such that $\Gamma_1(\theta) = \Gamma_3(\varphi)$. This means in particular that $-\delta\cos\theta = \delta\cos\varphi$, and so $\theta = \pi - \varphi$. It then follows that $(1 + \delta\sin\theta)\cos 2\theta = (1 - \delta\sin\theta)\cos 2\theta$, so that either $\cos 2\theta = 0$ or $\sin\theta = 0$. Since we also have $(1 + \delta\sin\theta)\sin 2\theta = -(1 - \delta\sin\theta)\sin 2\theta$, the possibility that $\cos 2\theta = 0$ is ruled out, and so $\sin\theta = 0$, i.e., $\theta = 0$ or $\theta = \pi$, meaning the point in common is either $\mathbf{a}$ or $\mathbf{b}$, as asserted.

As for the description of $\Gamma_1 - \Gamma_3$, it is clear that the mapping $T(\theta)$ given by the equations

$$\begin{aligned} x &= (1 + \delta\sin\theta)\cos 2\theta \\ y &= (1 + \delta\sin\theta)\sin 2\theta \\ z &= -\delta\cos\theta \end{aligned}$$

has the property that $T(\theta + \pi)$ is given by the equations

$$\begin{aligned} x &= (1 - \delta\sin\theta)\cos 2\theta \\ y &= (1 - \delta\sin\theta)\sin 2\theta \\ z &= \delta\cos\theta. \end{aligned}$$

Hence it equals describes $\Gamma_1(-\delta, \theta)$ on the interval $[0, \pi]$ and $-\Gamma_3(\delta, \theta)$ (since Γ_3 is given by the latter formulas, but is traversed with θ decreasing from π to 0).

Since $x^2 + y^2 = (1 - t\sin\theta)^2 \ge (1 - \delta)^2 > 0$ on all of M, it follows that η is defined on M. On Γ_1 and Γ_3 we have $\eta = 2\,d\theta$, so that

$$\int_\Gamma \eta = 4\pi.$$