Convergent Sequences

Part 1

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Definition 1

A *sequence* is a function $\mathbb{N} \to \mathbb{R}$.

The customary notation for functions—such as $a : \mathbb{N} \to \mathbb{R}$ with $a : n \mapsto a(n)$, where $a(n)$ denotes the image of $n \in \mathbb{N}$ under the function a—is not used for sequences. Instead, we write $a(n)$ as a_n, and instead of referring to the function a as a sequence, we say that $(a_n)_{n\in\mathbb{N}}$ is a sequence. Given $n \in \mathbb{N}$, we call a_n the *nth term* of the sequence $(a_n)_{n\in\mathbb{N}}$, and we call n the *index* of the term a_n.

Definition 2

A sequence $(a_n)_{n\in\mathbb{N}}$ *converges to* $a \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have $|a_n - a| < \varepsilon$. We write this in symbols as $a = \lim_{n\to\infty} a_n$. If there exists $a \in \mathbb{R}$ such that $a = \lim_{n\to\infty} a_n$, the sequence $(a_n)_{n\in\mathbb{N}}$ is said to be *convergent*.

The statement $a = \lim_{n\to\infty} a_n$ can be written symbolically as

$$\forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \in [N, \infty[\cap \mathbb{N} \qquad [|a_n - a| < \varepsilon], \tag{1}$$

where $[N, \infty[:= \{x \in \mathbb{R} \ : \ x \geq N\}$. The set in the third quantifier in (1) seems to be an unnecessary complication. Since it can be seen anyway from the rest of the statement that n appears as an index of a sequence term, n must be an element of $\mathbb{N}$. Henceforth, we adopt the less precise but simpler notation

$$\forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \ [|a_n - a| < \varepsilon],$$

as the symbolic equivalent of $a = \lim_{n\to\infty} a_n$.

Uniqueness of Sequence Limits

Theorem 3

If $(a_n)_{n\in\mathbb{N}}$ is convergent, then $\lim\limits_{n\to\infty} a_n$ is unique.

Proof of Theorem 3

Suppose $(a_n)_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$ and yet also to $b \in \mathbb{R}$. Each of these two assertions of convergence is equivalent to

$$\forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \ \left[|a_n - a| < \varepsilon\right], \tag{2}$$

$$\forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \ \left[|a_n - b| < \varepsilon\right], \tag{3}$$

respectively. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$, and (2) applies to $\frac{\varepsilon}{2}$ in place of ε, so there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |a - a_n| = |a_n - a| < \frac{\varepsilon}{2}. \tag{4}$$

But since (3) also holds for the case when ε is replaced by $\frac{\varepsilon}{2}$, we also find that for some $N_2 \in \mathbb{N}$,

$$n \geq N_2 \implies |a_n - b| < \frac{\varepsilon}{2}. \tag{5}$$

Now, any index $n \geq \max\{N_1, N_2\}$ satisfies the hypotheses of both (6) and (6), and so the conclusions of both (6) and (6) hold for this choice of n.

Proof of Theorem 3

We want an inequality relating a and b that is independent of n, and we accomplish this by the use of the triangle inequality:

$$|a-b| = |(a-a_n)+(a_n-b)| \leq |a-a_n| + |a_n-b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We have thus shown $|a-b| < \varepsilon$ for any $\varepsilon > 0$, which further implies $a = b$. $\square$

Proof.

Let $\varepsilon > 0$. If $(a_n)_{n\in\mathbb{N}}$ converges to a, then there exists $N_1 \in \mathbb{N}$ such that

$$|a - a_n| < \frac{\varepsilon}{2}, \qquad (n \geq N_1)$$

and if $(b_n)_{n\in\mathbb{N}}$ converges to b, then there exists $N_2 \in \mathbb{N}$ such that

$$|a_n - b| < \frac{\varepsilon}{2}, \qquad (n \geq N_2).$$

If $n \geq \max\{N_1, N_2\}$, then by the triangle inequality,

$$|a - b| = |(a - a_n) + (a_n - b)| \leq |a - a_n| + |a_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $a = b$. $\square$

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$, and let $c \in \mathbb{R}$. We define new sequences

$$
\begin{aligned}
(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} &:= (a_n + b_n)_{n\in\mathbb{N}}, && (6) \\
c\,(a_n)_{n\in\mathbb{N}} &:= (ca_n)_{n\in\mathbb{N}}. && (7)
\end{aligned}
$$

That is, $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}$ is the sequence with $a_n + b_n$ as the nth term, while $c\,(a_n)_{n\in\mathbb{N}}$ is the sequence the nth term of which is ca_n.

The algebra of convergent sequences

Theorem 4

If the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent, then so are the sequences $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}$ and $c\,(a_n)_{n\in\mathbb{N}}$ for any $c \in \mathbb{R}$. Furthermore,

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n, \tag{8}$$

$$\lim_{n\to\infty}(ca_n) = c \cdot \lim_{n\to\infty} a_n. \tag{9}$$

The above theorem is stated such that the claim that the sequences (6),(7) are convergent is stated before the claim of what their limits are. This is the logical manner of stating the theorem. But in terms of the logic in proving the theorem, we simply show that (8),(9) hold, and then we conclude that the sequences in question are convergent.

Proof of Theorem 4

Let $\varepsilon > 0$. Since $(a_n)_{n\in\mathbb{N}}$ converges to a, there exists $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies \left| a_n - \lim_{n\to\infty} a_n \right| < \frac{\varepsilon}{2},$$

and since $(b_n)_{n\in\mathbb{N}}$ converges to b, there exists $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \implies \left| b_n - \lim_{n\to\infty} b_n \right| < \frac{\varepsilon}{2}.$$

If $n \geq N := \max\{N_1, N_2\}$, then

$$\begin{aligned} \left|(a_n + b_n) - \left(\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n\right)\right| &= \left|\left(a_n - \lim_{n\to\infty} a_n\right) + \left(b_n - \lim_{n\to\infty} b_n\right)\right|, \\ &\leq \left|a_n - \lim_{n\to\infty} a_n\right| + \left|b_n - \lim_{n\to\infty} b_n\right|, \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$. This means that $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}$ is convergent.

Proof of Theorem 4

As for (9), we first consider the case $c = 0$. Here, an arbitrary term of $c\,(a_n)_{n\in\mathbb{N}}$ is $ca_n = 0 \cdot a_n = 0$. That is, $c\,(a_n)_{n\in\mathbb{N}} = (0)_{n\in\mathbb{N}}$, which we call the *zero sequence* in $\mathbb{R}$. If $\varepsilon > 0$, then

$$|ca_n - 0| = |0 - 0| = 0 < \varepsilon.$$

Therefore, $\lim\limits_{n\to\infty} ca_n = 0 = 0 \cdot \lim\limits_{n\to\infty} a_n = c \lim\limits_{n\to\infty} a_n$. Hence, the zero sequence is convergent. We now consider the case $c \neq 0$. This means $|c| > 0$, and if we consider an arbitrary $\varepsilon > 0$, then $\frac{\varepsilon}{|c|} > 0$, and by the convergence of $(a_n)_{n\in\mathbb{N}}$ to $\lim\limits_{n\to\infty} a_n$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies \left| a_n - \lim_{n\to\infty} a_n \right| < \frac{\varepsilon}{|c|}. \tag{10}$$

If indeed $n \geq N$, then the conclusion of (10) is true, and we further have

Proof of Theorem 4

$$\begin{aligned}
\left|a_n - \lim_{n\to\infty} a_n\right| &< \frac{\varepsilon}{|c|}, \\
|c| \cdot \left|a_n - \lim_{n\to\infty} a_n\right| &< |c| \cdot \frac{\varepsilon}{|c|}, \\
\left|ca_n - c \cdot \lim_{n\to\infty} a_n\right| &< \varepsilon.
\end{aligned}$$

We have thus shown $\lim_{n\to\infty} (ca_n) = c \cdot \lim_{n\to\infty} a_n$. Therefore, $c\,(a_n)_{n\in\mathbb{N}}$ is convergent. $\square$

Definition 5

Let $\mathfrak{c}(\mathbb{R})$ denote the set of all convergent sequences in $\mathbb{R}$.

Recall from the proof of Theorem 4, the zero sequence $(0)_{n\in\mathbb{N}}$, which we have proven to be convergent. Hence, $(0)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$, and so $\mathfrak{c}(\mathbb{R}) \neq \emptyset$. The importance of Theorem 4 is that we can view addition of convergent sequences as an operation, or a function

$$\begin{aligned} + \quad : \quad \mathfrak{c}(\mathbb{R}) \times \mathfrak{c}(\mathbb{R}) &\to \mathfrak{c}(\mathbb{R}) \\ \left((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}\right) &\mapsto (a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}. \end{aligned} \tag{11}$$

That is, the operation $+$ of addition of convergent sequences takes, as an input, an ordered pair of sequences $\left((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}\right)$. Any input to the function $+$ always results to an output because $\mathfrak{c}(\mathbb{R}) \neq \emptyset$. Since addition in $\mathbb{R}$ is well-defined, so is the termwise addition performed in constructing the sequence $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}$, the output of the operation $+$ of addition of convergent sequences. The result of the operation, as Theorem 4 guarantees us, resides in $\mathfrak{c}(\mathbb{R})$.

Hence, addition $+$ of convergent sequences is indeed a function with codomain $\mathfrak{c}(\mathbb{R})$, as indicated in (11). This is the sophisticated version of the *closure* property of an operation, which in this case is (11). Other properties of the addition operation for sequences in $\mathfrak{c}(\mathbb{R})$ are

$$\begin{aligned}
((a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}) + (c_n)_{n\in\mathbb{N}} &= (a_n)_{n\in\mathbb{N}} + ((b_n)_{n\in\mathbb{N}} + (c_n)_{n\in\mathbb{N}}), && (12)\\
(a_n)_{n\in\mathbb{N}} + (0)_{n\in\mathbb{N}} &= (a_n)_{n\in\mathbb{N}} = (0)_{n\in\mathbb{N}} + (a_n)_{n\in\mathbb{N}}, && (13)\\
(a_n)_{n\in\mathbb{N}} + (-a_n)_{n\in\mathbb{N}} &= (0)_{n\in\mathbb{N}} = (-a_n)_{n\in\mathbb{N}} + (a_n)_{n\in\mathbb{N}}, && (14)\\
(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} &= (b_n)_{n\in\mathbb{N}} + (a_n)_{n\in\mathbb{N}}, && (15)
\end{aligned}$$

for any $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$. The properties (12)–(15) are readily verifiable from the properties of operations in $\mathbb{R}$ performed termwise, and are, respectively, referred to as *associativity*, *existence of an identity* [which in this case is $(0)_{n\in\mathbb{N}}$], *existence of inverses* [Theorem 4 guarantees that $-1 \cdot (a_n)_{n\in\mathbb{N}} = (-a_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$], and *commutativity*. The five properties mentioned from closure to commutativity gives the set $\mathfrak{c}(\mathbb{R})$ the algebraic structure of an *abelian group* under the operation of addition of convergent sequences.

Theorem 4 also tells us that left-multiplication of a convergent sequence by a constant is an operation $\mathbb{R} \times \mathfrak{c}(\mathbb{R}) \to \mathfrak{c}(\mathbb{R})$, with additional properties

$$\begin{aligned} c_1((a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}) &= c_1 (a_n)_{n\in\mathbb{N}} + c_1 (b_n)_{n\in\mathbb{N}}, \\ (c_1 + c_2) (a_n)_{n\in\mathbb{N}} &= c_1 (a_n)_{n\in\mathbb{N}} + c_2 (a_n)_{n\in\mathbb{N}}, \\ c_1(c_2 (a_n)_{n\in\mathbb{N}}) &= (c_1 c_2) (a_n)_{n\in\mathbb{N}}, \\ 1 \cdot (a_n)_{n\in\mathbb{N}} &= (a_n)_{n\in\mathbb{N}}, \end{aligned}$$

for any $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$ and any $c_1, c_2 \in \mathbb{R}$. These properties are immediate from the properties of operations in $\mathbb{R}$ applied to the termwise operations on the sequences. More importantly, these properties give the abelian group $\mathfrak{c}(\mathbb{R})$ the algebraic structure of a *vector space over* $\mathbb{R}$, with addition of convergent sequences as *vector addition*, and left-multiplication of a convergent sequence by a constant as *scalar multiplication*. We refer to vector addition and scalar multiplication in $\mathfrak{c}(\mathbb{R})$ as the *vector space operations* in $\mathfrak{c}(\mathbb{R})$.

Theorem 6

For any $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$, *the sequence*

$$(a_n)_{n\in\mathbb{N}} (b_n)_{n\in\mathbb{N}} := (a_n b_n)_{n\in\mathbb{N}},$$

is convergent [i.e., also in $\mathfrak{c}(\mathbb{R})$*], and furthermore,*

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n. \tag{16}$$

Proof of Theorem 6

Let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Symbolically,

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \ [|a_n - a| < \varepsilon]\,, \tag{17}$$

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \ [|b_n - b| < \varepsilon]\,. \tag{18}$$

Let $\varepsilon > 0$. The first trick in our proof is to use the Archimedean property of $\mathbb{R}$: there exists a positive real number [actually an integer]

$$M > \max\{1 + |a|, |b|\}. \tag{19}$$

The second trick is to instantiate (17) twice [for two values in place of ε, which are $1 > 0$ and $\frac{\varepsilon}{2M} > 0$] and (18) once [also for an 'epsilon value' of $\frac{\varepsilon}{2M} > 0$]. Thus, there exist $N_1, N_2, N_3 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_1 \quad &\Longrightarrow \quad |a_n - a| < 1, \\ n \geq N_2 \quad &\Longrightarrow \quad |a_n - a| < \frac{\varepsilon}{2M}, \\ n \geq N_3 \quad &\Longrightarrow \quad |b_n - b| < \frac{\varepsilon}{2M}. \end{aligned} \tag{20}$$

The third trick is to use the reverse triangle inequality on (20): if $n \geq N_1$, then

$$\begin{aligned} && |a_n - a| &< 1, \\ ||a_n| - |a|| &\leq & |a_n - a| &< 1, \\ && ||a_n| - |a|| &< 1, \\ -1 &< & |a_n| - |a| &< 1, \\ && |a_n| - |a| &< 1, \\ && |a_n| &< 1 + |a|. \end{aligned} \tag{21}$$

With (21) as a new conclusion for (20), we have the following new conditions for N_1, N_2, N_3:

$$n \geq N_1 \implies |a_n| < 1 + |a|, \tag{22}$$

$$n \geq N_2 \implies |a_n - a| < \frac{\varepsilon}{2M}, \tag{23}$$

$$n \geq N_3 \implies |b_n - b| < \frac{\varepsilon}{2M}. \tag{24}$$

This sets the stage for the main trick, which is the simple equation

$$a_n b_n - ab = (a_n - a)b + (b_n - b)a_n. \tag{25}$$

The doubtful reader need only simplify the right-hand side. We take the absolute values of both sides of (25), and using the triangle inequality and the multiplicativity of absolute value, we have

$$\begin{aligned} |a_n b_n - ab| &= |(a_n - a)b + (b_n - b)a_n|, \\ &\leq |(a_n - a)b| + |(b_n - b)a_n|, \\ &= |a_n - a| \cdot |b| + |b_n - b| \cdot |a_n|. \end{aligned}$$

Consequently,

$$|a_n b_n - ab| \leq |a_n - a| \cdot |b| + |b_n - b| \cdot |a_n|. \tag{26}$$

The task boils down to finding bounds on the four absolute values in the right-hand side of (26). The conclusions of (22)–(24) offer three of such bounds. For such conclusions to hold, we consider only the indices

Proof of Theorem 6

$$n \geq N := \max\{N_1, N_2, N_3\},$$

so that the hypotheses of (22)–(24) are satisfied, and so (26) becomes

$$|a_n b_n - ab| < \frac{\varepsilon}{2M} \cdot |b| + \frac{\varepsilon}{2M} \cdot (1 + |a|)\,. \tag{27}$$

Finally, we want ε to neatly 'come out.' To accomplish this, we use the inequalities $1 + |a| < M$ and $|b| < M$ which are direct consequences of (19), and so (27) becomes

$$|a_n b_n - ab| < \frac{\varepsilon}{2M} \cdot M + \frac{\varepsilon}{2M} \cdot M = \varepsilon.$$

In summary, we have shown

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \;\; [|a_n b_n - ab| < \varepsilon]\,.$$

Therefore, $\lim_{n\to\infty} (a_n b_n) = ab$ as desired. $\square$

Theorem 6 asserts that $\mathfrak{c}(\mathbb{R})$ is closed under multiplication of sequences. Thus, we have tihis new operation $\mathfrak{c}(\mathbb{R}) \times \mathfrak{c}(\mathbb{R}) \to \mathfrak{c}(\mathbb{R})$ on top of the vector space operations.
Consider four arbitrary sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ in $\mathfrak{c}(\mathbb{R})$ and let $c_1, c_2, c_3, c_4 \in \mathbb{R}$. By routine computations involving the algebraic properties of operations in $\mathbb{R}$, it can be shown that the product

$$\lambda = \left[c_1 (a_n)_{n\in\mathbb{N}} + c_2 (b_n)_{n\in\mathbb{N}}\right] \cdot \left[c_3 (c_n)_{n\in\mathbb{N}} + c_4 (d_n)_{n\in\mathbb{N}}\right]$$

of the sequence $c_1 (a_n)_{n\in\mathbb{N}} + c_2 (b_n)_{n\in\mathbb{N}}$ and the sequence $c_3 (c_n)_{n\in\mathbb{N}} + c_4 (d_n)_{n\in\mathbb{N}}$ is such that

$$\begin{aligned}\lambda \;\; = \;\; & c_1 c_3 \left[(a_n)_{n\in\mathbb{N}} (c_n)_{n\in\mathbb{N}}\right] + c_1 c_4 \left[(a_n)_{n\in\mathbb{N}} (d_n)_{n\in\mathbb{N}}\right] \\ & + c_2 c_3 \left[(b_n)_{n\in\mathbb{N}} (c_n)_{n\in\mathbb{N}}\right] + c_2 c_4 \left[(b_n)_{n\in\mathbb{N}} (d_n)_{n\in\mathbb{N}}\right].\end{aligned}$$

This property of multiplication of sequences is called *bilinearity*.
The presence of this bilinear multiplication operation on the vector space $\mathfrak{c}(\mathbb{R})$ makes $\mathfrak{c}(\mathbb{R})$ an *algebra over* $\mathbb{R}$.

Just like the zero sequence in $\mathfrak{c}(\mathbb{R})$, we take interest in another special element of $\mathfrak{c}(\mathbb{R})$:

Example 7

Consider the sequence $(1)_{n\in\mathbb{N}}$, i.e., $a_n = 1$ for any $n \in \mathbb{N}$. If $\varepsilon > 0$, then $|a_n - 1| = |1 - 1| = 0 < \varepsilon$. Thus, $(1)_{n\in\mathbb{N}}$ converges to 1 and hence, $(1)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$.

For any $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$, the properties

$$(a_n)_{n\in\mathbb{N}} \cdot \left[(b_n)_{n\in\mathbb{N}} (c_n)_{n\in\mathbb{N}}\right] = \left[(a_n)_{n\in\mathbb{N}} (b_n)_{n\in\mathbb{N}}\right] \cdot (c_n)_{n\in\mathbb{N}}, \tag{28}$$

$$(a_n)_{n\in\mathbb{N}} (1)_{n\in\mathbb{N}} = (a_n)_{n\in\mathbb{N}} = (1)_{n\in\mathbb{N}} (a_n)_{n\in\mathbb{N}}, \tag{29}$$

$$(a_n)_{n\in\mathbb{N}} (b_n)_{n\in\mathbb{N}} = (b_n)_{n\in\mathbb{N}} (a_n)_{n\in\mathbb{N}}, \tag{30}$$

can be readily verified using the properties of operations on $\mathbb{R}$ applied to the sequence terms. The property (28) means that the algebra $\mathfrak{c}(\mathbb{R})$ is an *associative algebra over* $\mathbb{R}$.

The presence of $(1)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$ such that (29) holds makes the associative algebra $\mathfrak{c}(\mathbb{R})$ *unital*, with $(1)_{n\in\mathbb{N}}$ as the *[multiplicative] identity*, while property (30) means that the associative algebra $\mathfrak{c}(\mathbb{R})$ is *commutative*. An element $(b_n)_{n\in\mathbb{N}}$ of $\mathfrak{c}(\mathbb{R})$ is a *unit* if there exists $(c_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$ such that $(b_n)_{n\in\mathbb{N}}(c_n)_{n\in\mathbb{N}} = (1)_{n\in\mathbb{N}} = (c_n)_{n\in\mathbb{N}}(b_n)_{n\in\mathbb{N}}$. The following gives us an insight about the units in $\mathfrak{c}(\mathbb{R})$.

Theorem 8

Let $(b_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$*. If* $\lim\limits_{n\to\infty} b_n \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$*, then* $\left(\frac{1}{b_n}\right)_{n\in\mathbb{N}}$ *is convergent, and*

$$\lim_{n\to\infty}\left(\frac{1}{b_n}\right) = \frac{1}{\lim\limits_{n\to\infty} b_n}. \tag{31}$$

Proof of Theorem 8

Let $b = \lim_{n\to\infty} b_n$, which in terms of symbolic logic, is equivalent to

$$\forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \ \left[|b_n - b| < \varepsilon\right]. \tag{32}$$

Let $\varepsilon > 0$. The first trick in our proof is to instantiate (32) twice, for two 'epsilon values' which are $\frac{1}{2}|b| > 0$ and $\frac{1}{2}|b|^2\varepsilon > 0$. Thus, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N_1 \quad &\Longrightarrow \quad |b - b_n| < \frac{1}{2}|b|, \\ n \geq N_2 \quad &\Longrightarrow \quad |b_n - b| < \frac{1}{2}|b|^2\varepsilon. \end{aligned} \tag{33}$$

Switching b_n and b [as symmetry of the distance function in $\mathbb{R}$ would allow] from (32) to (33), has a strategic effect as we shall see.

The next trick in our proof is to use the reverse triangle inequality to modify (33) so that we obtain a *lower* bound for $|b_n|$, which gives an upper bound for $\frac{1}{|b_n|}$. If $n \geq N_1$, then we have:

$$\begin{aligned}
&& |b - b_n| &< \frac{1}{2}|b|, \\
||b| - |b_n|| &\leq & |b - b_n| &< \frac{1}{2}|b|, \\
&& ||b| - |b_n|| &< \frac{1}{2}|b|, \\
-\frac{1}{2}|b| &< & |b| - |b_n| &< \frac{1}{2}|b|, \\
&& |b| - |b_n| &< \frac{1}{2}|b|, \\
&& \frac{1}{2}|b| &< |b_n|, & \quad (34) \\
&& \frac{1}{|b_n|} &< \frac{2}{|b|}, & \quad (35)
\end{aligned}$$

where going from (34) to (35) is made possible by the assumptions $b \neq 0$ and $b_n \neq 0$. We now have the condtions

$$n \geq N_1 \implies \frac{1}{|b_n|} < \frac{2}{|b|}, \tag{36}$$

$$n \geq N_2 \implies |b_n - b| < \frac{1}{2}|b|^2\varepsilon. \tag{37}$$

Our main trick is a simple consequence of combining fractions:

$$\frac{1}{b_n} - \frac{1}{b} = \frac{1}{b_n} \cdot \frac{1}{b}(b_n - b).$$

Taking absolute values,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{1}{|b_n|} \cdot \frac{1}{|b|} \cdot |b_n - b|. \tag{38}$$

The bounds for the right-hand side of (38) are available from the conclusions of (36)–(37). To make the hypotheses true, we consider only those indices

$$n \geq N := \max\{N_1, N_2\},$$

and so (38) becomes

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \frac{2}{|b|} \cdot \frac{1}{|b|} \cdot \frac{1}{2}|b|^2\varepsilon = \varepsilon.$$

Therefore, $\lim_{n\to\infty} \left(\frac{1}{b_n}\right) = \frac{1}{b}$, and hence $\left(\frac{1}{b_n}\right)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$. $\square$

Writing Exercise

Write proofs of Theorems 4, 6 and 8 with MINIMAL explanation or references to Logic, 'tricks,' strategy and properties of inequalities. This was done earlier for the proof of Theorem 3.

Theorem 8 tells us that any sequence in $\mathfrak{c}(\mathbb{R})$ with nonzero terms and nonzero limit is a unit in $\mathfrak{c}(\mathbb{R})$. Other algebraic properties of limits of sequences may be shown as immediate consequences of what we have proven so far. We collect some of them in the following.

Corollary 9

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$*, and let* $c \in \mathbb{R}$*. Assume that* $\lim_{n\to\infty} c_n \neq 0$*, and that* $c_n \neq 0$ *for any* $n \in \mathbb{N}$*. Then the sequences*

$$(c)_{n\in\mathbb{N}}, \quad (|a_n|)_{n\in\mathbb{N}}, \quad (a_n - b_n)_{n\in\mathbb{N}}, \quad \left(\frac{a_n}{c_n}\right)_{n\in\mathbb{N}}, \quad (a_n \vee b_n)_{n\in\mathbb{N}}, \quad (a_n \wedge b_n)_{n\in\mathbb{N}}, \tag{39}$$

are convergent, and furthermore,

1. $\lim_{n\to\infty} c = c$,
2. $\lim_{n\to\infty} |a_n| = \left|\lim_{n\to\infty} a_n\right|$,
3. $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$, *and*
4. $\lim_{n\to\infty} \frac{a_n}{c_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} c_n}$.
5. $\lim_{n\to\infty} (a_n \vee b_n) = a \vee b$.
6. $\lim_{n\to\infty} (a_n \wedge b_n) = a \wedge b$.

Prove Corollary 9. "Epsilonics" may not be necessary. Some hints:

(1) View $(c)_{n\in\mathbb{N}} = c\,(1)_{n\in\mathbb{N}}$.

(2) Use the reverse triangle inequality in the form
$\left| |a_n| - \left| \lim_{n\to\infty} a_n \right| \right| \leq \left| a_n - \lim_{n\to\infty} a_n \right|$.

(3) Use the fact that $a_n - b_n = a_n + (-1)b_n$.

(4) Use the fact that $\frac{a_n}{c_n} = a_n \cdot \frac{1}{c_n}$.

(5) and (6) Express the lattice operations in terms of previously known ones.