Solutions Manual to Walter Rudin's *Principles of Mathematical Analysis*

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Chapter 3

Numerical Sequences and Series

Exercise 3.1 Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

Solution. Let $\varepsilon > 0$. Since the sequence $\{s_n\}$ is a Cauchy sequence, there exists N such that $|s_m - s_n| < \varepsilon$ for all $m > N$ and $n > N$. We then have $\big||s_m| - |s_n|\big| \leq |s_m - s_n| < \varepsilon$ for all $m > N$ and $n > N$. Hence the sequence $\{|s_n|\}$ is also a Cauchy sequence, and therefore must converge.

The converse is not true, as shown by the sequence $\{s_n\}$ with $s_n = (-1)^n$.

Exercise 3.2 Calculate $\lim_{n \to \infty} (\sqrt{n^2+n} - n)$.

Solution. Multiplying and dividing by $\sqrt{n^2+n}+n$ yields

$$\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}.$$

It follows that the limit is $\frac{1}{2}$.

Exercise 3.3 If $s_1 = \sqrt{2}$ and

$$s_{n+1} = \sqrt{2+\sqrt{s_n}} \quad (n = 1, 2, 3 \ldots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3 \ldots$.

Solution. Since $\sqrt{2} < 2$, it is manifest that if $s_n < 2$, then $s_{n+1} < \sqrt{2+2} = 2$. Hence it follows by induction that $\sqrt{2} < s_n < 2$ for all n. In view of this fact, it also follows that $(s_n - 2)(s_n + 1) < 0$ for all $n > 1$, i.e., $s_n > s_n^2 - 2 = s_{n-1}$. Hence the sequence is an increasing sequence that is bounded above (by 2) and so converges. Since the limit s satisfies $s^2 - s - 2 = 0$, it follows that the limit is 2.

Exercise 3.4 Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Solution. We shall prove by induction that

$$s_{2m} = \frac{1}{2} - \frac{1}{2^m} \text{ and } s_{2m+1} = 1 - \frac{1}{2^m}$$

for $m = 1, 2, \ldots$. The second of these equalities is a direct consequence of the first, and so we need only prove the first. Immediate computation shows that $s_2 = 0$ and $s_3 = \frac{1}{2}$. Hence assume that both formulas hold for $m \le r$. Then

$$s_{2r+2} = \frac{1}{2}s_{2r+1} = \frac{1}{2}\Big(1 - \frac{1}{2^r}\Big) = \frac{1}{2} - \frac{1}{2^{r+1}}.$$

This completes the induction. We thus have $\limsup_{n\to\infty} s_n = 1$ and $\liminf_{n\to\infty} s_n = \frac{1}{2}$.

Exercise 3.5 For any two real sequences $\{a_n\}$, $\{b_n\}$ prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Solution. Since the case when $\limsup_{n\to\infty} a_n = +\infty$ and $\limsup_{n\to\infty} b_n = -\infty$ has been excluded from consideration, we note that the inequality is obvious if $\limsup_{n\to\infty} a_n = +\infty$. Hence we shall assume that $\{a_n\}$ is bounded above.

Let $\{n_k\}$ be a subsequence of the positive integers such that $\lim_{k\to\infty}(a_{n_k} + b_{n_k}) = \limsup_{n\to\infty}(a_n + b_n)$. Then choose a subsequence of the positive integers $\{k_m\}$ such that

$$\lim_{m\to\infty} a_{n_{k_m}} = \limsup_{k\to\infty} a_{n_k}.$$

The subsequence $a_{n_{k_m}} + b_{n_{k_m}}$ still converges to the same limit as $a_{n_k} + b_{n_k}$, i.e., to $\limsup_{n\to\infty}(a_n + b_n)$. Hence, since a_{n_k} is bounded above (so that $\limsup_{k\to\infty} a_{n_k}$ is finite), it follows that $b_{n_{k_m}}$ converges to the difference

$$\lim_{m\to\infty} b_{n_{k_m}} = \lim_{m\to\infty}(a_{n_{k_m}} + b_{n_{k_m}}) - \lim_{m\to\infty} a_{n_{k_m}}.$$

Thus we have proved that there exist subsequences $\{a_{n_{k_m}}\}$ and $\{b_{n_{k_m}}\}$ which converge to limits a and b respectively such that $a+b = \limsup_{n\to\infty}(a_n + b_n)$. Since a is the limit of a subsequence of $\{a_n\}$ and b is the limit of a subsequence of $\{b_n\}$, it follows that $a \le \limsup_{n\to\infty} a_n$ and $b \le \limsup_{n\to\infty} b_n$, from which the desired inequality follows.

Exercise 3.6 Investigate the behavior (convergence or divergence) of $\sum a_n$ if
(a) $a_n = \sqrt{n+1} - \sqrt{n}$;
(b) $a_n = \dfrac{\sqrt{n+1}-\sqrt{n}}{n}$;
(c) $a_n = (\sqrt[n]{n} - 1)^n$;
(d) $a_n = \dfrac{1}{1+z^n}$ for complex values of z.

Solution. (a) Multiplying and dividing a_n by $\sqrt{n+1}+\sqrt{n}$, we find that $a_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, which is larger than $\frac{1}{2\sqrt{n+1}}$. The series $\sum a_n$ therefore diverges by comparison with the p series $(p = \frac{1}{2})$.

Alternatively, since the sum telescopes, the nth partial sum is $\sqrt{n+1} - 1$, which obviously tends to infinity.

(b) Using the same trick as in part (a), we find that $a_n = \frac{1}{n[\sqrt{n+1}+\sqrt{n}]}$, which is less than $\frac{1}{n^{3/2}}$. Hence the series converges by comparison with the p series $(p = \frac{3}{2})$.

(c) Using the root test, we find that $a_n^{\frac{1}{n}} = \sqrt[n]{n} - 1$, which tends to zero as $n \to \infty$. Hence the series converges. (Alternatively, since by part (c) of Theorem 3.20 $\sqrt[n]{n}$ tends to 1 as $n \to \infty$, we have $a_n \le 2^{-n}$ for all large n, and the series converges by comparison with a geometric series.)

(d) If $|z| \le 1$, then $|a_n| \ge \frac{1}{2}$, so that a_n does not tend to zero. Hence the series diverges. If $|z| > 1$, the series converges by comparison with a geometric series with $r = \frac{1}{|z|} < 1$.

Exercise 3.7 Prove that the convergence of Σa_n implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \ge 0$.

Solution. Since $(\sqrt{a_n} - \frac{1}{n})^2 \ge 0$, it follows that

$$\frac{\sqrt{a_n}}{n} \le \frac{1}{2}\Big(a_n^2 + \frac{1}{n^2}\Big).$$

Now Σa_n^2 converges by comparison with Σa_n (since Σa_n converges, we have $a_n < 1$ for large n, and hence $a_n^2 < a_n$). Since $\Sigma \frac{1}{n^2}$ also converges (p series, $p = 2$), it follows that $\Sigma \frac{\sqrt{a_n}}{n}$ converges.

Exercise 3.8 If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.

Solution. We shall show that the partial sums of this series form a Cauchy sequence, i.e., given $\varepsilon > 0$ there exists N such that $\left| \sum_{k=m+1}^{n} a_k b_k \right| < \varepsilon$ if $n > m \geq N$. To do this, let $S_n = \sum_{k=1}^{n} a_k$ ($S_0 = 0$), so that $a_k = S_k - S_{k-1}$ for $k = 1, 2, \ldots$. Let M be an uppper bound for both $|b_n|$ and $|S_n|$, and let $S = \sum a_n$ and $b = \lim b_n$. Choose N so large that the following three inequalities hold for all $m > N$ and $n > N$:

$$|b_n S_n - bS| < \frac{\varepsilon}{3}; \quad |b_m S_m - bS| < \frac{\varepsilon}{3}; \quad |b_m - b_n| < \frac{\varepsilon}{3M}.$$

Then if $n > m > N$, we have, from the formula for summation by parts,

$$\sum_{k=m+1}^{n} a_n b_n = b_n S_n - b_m S_m + \sum_{k=m}^{n-1} (b_k - b_{k+1}) S_k.$$

Our assumptions yield immediately that $|b_n S_n - b_m S_m| < \frac{2\varepsilon}{3}$, and

$$\left| \sum_{k=m}^{n-1} (b_k - b_{k+1}) S_k \right| \leq M \sum_{k=m}^{n-1} |b_k - b_{k+1}|.$$

Since the sequence $\{b_n\}$ is monotonic, we have

$$\sum_{k=m}^{n-1} |b_k - b_{k+1}| = \left| \sum_{k=m}^{n-1} (b_k - b_{k+1}) \right| = |b_m - b_n| < \frac{\varepsilon}{3M},$$

from which the desired inequality follows.

Exercise 3.9 Find the radius of convergence of each of the following power series

$$(a) \ \sum n^3 z^n, \qquad (b) \ \sum \frac{2^n}{n!} z^n,$$

$$(c) \ \sum \frac{2^n}{n^2} z^n, \qquad (d) \ \sum \frac{n^3}{3^n} z^n.$$

Solution. (a) The radius of convergence is 1, since $a_n = n^3$ satisfies $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}} =$ 1.

(b) The radius of convergence is infinite, since $a_n = \frac{2^n}{n!}$ satisfies $\lim\limits_{n\to\infty} \dfrac{a_n}{a_{n+1}} = \lim\limits_{n\to\infty} \dfrac{n+1}{2} = \infty.$

(c) The radius of convergence is $\frac{1}{2}$, since $a_n = \frac{2^n}{n^2}$ satisfies

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{1}{2}\Big(1+\frac{1}{n}\Big)^2 = \frac{1}{2}.$$

(d) The radius of convergence is 3, since $a_n = \frac{n^3}{3^n}$ satisfies

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} 3\Big(\frac{n}{n+1}\Big)^3 = 3.$$

Exercise 3.10 Suppose that the coefficients of the power series $\Sigma a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Solution. The series diverges if $|z| > 1$, since its general term does not tend to zero. (Infinitely many terms are larger than 1 in absolute value.)

Exercise 3.11 Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and Σa_n diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \quad \text{and} \quad \sum \frac{a_n}{1_n^2 a_n}?$$

Solution. (a) If a_n does not remain bounded, then $\frac{a_n}{1+a_n}$ does not tend to zero, and hence the series $\sum \frac{a_n}{1+a_n}$ diverges. If $a_n \leq M$ for all n, then $\dfrac{a_n}{1+a_n} \geq \dfrac{1}{1+M} a_n$, and hence again the series is divergent.

(b) Replacing each denominator on the left by s_{N+k}, we have

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{1}{s_{N+k}}(a_{N+1} + a_{N+2} + \cdots + a_{N+k}) =$$

$$= \frac{1}{s_{N+k}}(s_{N+k} - s_N) = 1 - \frac{s_N}{s_{N+k}}.$$

It follows that the partial sums of the series $\sum \frac{a_n}{s_n}$ do not form a Cauchy sequence. For, no matter how large N is taken, if N is held fixed, the right hand side can be made larger than $\frac{1}{2}$ by taking k sufficiently large (since $S_{N+k} \to \infty$).

(c) We observe that if $n \geq 2$, then

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{a_n}{s_{n-1}s_n} \geq \frac{a_n}{s_n^2}.$$

Since the series $\sum\limits_{n=2}^{\infty} \frac{1}{s_{n-1}} - \frac{1}{s_n}$ converges to $\frac{1}{a_1}$, it follows by comparison that $\sum \frac{a_n}{s_n^2}$ converges.

(d) The series $\sum \frac{a_n}{1+na_n}$ may be either convergent or divergent. If the sequence $\{na_n\}$ is bounded above or has a positive lower bound, it definitely diverges. Thus if $na_n \leq M$, each term is at least $\frac{1}{1+M}a_n$, and so the series diverges. If $na_n \geq \varepsilon > 0$ for all n, then each term is at least $\frac{\varepsilon}{1+\varepsilon}\frac{1}{n}$, and once again the series is divergent.

In general, however, the series $\sum \frac{a_n}{1+na_n}$ may converge. For example let $a_n = \frac{1}{n^2}$ if n is not a perfect square and $a_n = \frac{1}{\sqrt{n}}$ if n is a perfect square. The sum of $\frac{a_n}{1+na_n}$ over the nonsquares obviously converges by comparison with the p series, $p = 2$. As for the sum over the square integers it is $\sum \frac{1}{n+n^2}$, which converges by comparison with the p series, $p = 2$.

Finally, the series $\sum \frac{a_n}{1+n^2a_n}$ is obviously majorized by the p series with $p = 2$, hence converges.

Exercise 3.12 Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_n.$$

(a) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Solution. (*a*) Replacing all the denominators on the left-hand side by the largest one (r_m), we find

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m + \cdots + a_n}{r_m} = \frac{r_m - r_{n+1}}{r_m} > 1 - \frac{r_n}{r_m},$$

since $r_n > r_{n+1}$.

As in the previous problem, this keeps the partial sums of the series $\sum \frac{a_n}{r_n}$ from forming a Cauchy sequence. No matter how large m is taken, one can choose n larger so that the difference $\sum_{k=m}^{n} \frac{a_k}{r_k}$ is at least $\frac{1}{2}$, since $r_n \longrightarrow 0$ as $n \longrightarrow \infty$.

(*b*) We have

$$\frac{a_n}{\sqrt{r_n}}(\sqrt{r_n} + \sqrt{r_{n+1}}) = a_n + a_n\frac{\sqrt{r_{n+1}}}{\sqrt{r_n}} < 2a_n = 2(r_n - r_{n+1}).$$

Dividing both sides by $\sqrt{r_n} + \sqrt{r_{n+1}}$ now yields the desired inequality.

Since the series $\sum(\sqrt{r_n} - \sqrt{r_{n+1}})$ converges to $\sqrt{r_1}$, it follows by comparison that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Exercise 3.13 Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Solution. Since both the hypothesis and conclusion refer to absolute convergence, we may assume both series consist of nonnegative terms. We let $S_n = \sum_{k=0}^{n} a_n$, $T_n = \sum_{k=0}^{n} b_n$, and $U_n = \sum_{k=0}^{n}\sum_{l=0}^{k} a_l b_{k-l}$. We need to show that U_n remains bounded, given that S_n and T_n are bounded. To do this we make the convention that $a_{-1} = T_{-1} = 0$, in order to save ourselves from having to separate off the first and last terms when we sum by parts. We then have

$$\begin{aligned}
U_n &= \sum_{k=0}^{n}\sum_{l=0}^{k} a_l b_{k-l} \\
&= \sum_{k=0}^{n}\sum_{l=0}^{k} a_l (T_{k-l} - T_{k-l-1}) \\
&= \sum_{k=0}^{n}\sum_{j=0}^{k} a_{k-j} (T_j - T_{j-1}) \\
&= \sum_{k=0}^{n}\sum_{j=0}^{k} (a_{k-j} - a_{k-j-1}) T_j \\
&= \sum_{j=0}^{n}\sum_{k=j}^{n} (a_{k-j} - a_{k-j-1}) T_j
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{n} a_{n-j} T_j \\
&\leq T \sum_{m=0}^{n} a_m \\
&= TS_n \\
&\leq ST.
\end{aligned}$$

Thus U_n is bounded, and hence approaches a finite limit.

Exercise 3.14 If $\{s_n\}$ is a complex sequence, define its arithmetic mean σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \qquad (n = 0, 1, 2, \ldots).$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, even though $\lim \sigma_n = 0$?

(d) Put $a_n = s_n - s_{n-1}$ for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

Assume that $\lim(na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $na_n \to 0$.]

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$ by completing the following outline:

If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^{n} (s_n - s_i).$$

For these i,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

then $(m+1)/(n-m) \leq 1/\varepsilon$ and $|s_n - s_i| < M\varepsilon$. Hence

$$\limsup_{n\to\infty} |s_n - \sigma| \leq M\varepsilon.$$

Since ε was arbitrary, $\lim s_n = \sigma$.

Solution. Let $\varepsilon > 0$. Let $M = \sup\{|s_n|\}$, and let N_0 be the first integer such that $|s_n - s| < \frac{\varepsilon}{2}$ for all $n > N_0$. Let $N = \max\left(N_0, \left[\frac{2(N_0+1)(M+|s|)}{\varepsilon}\right]\right)$. Then if $n > N$, we have

$$\begin{aligned}
|\sigma_n - s| &= \left|\frac{(s_0 - s) + (s_1 - s) + \cdots + (s_n - s)}{n+1}\right| \\
&\leq \left|\frac{(s_0 - s) + \cdots + (s_{N_0} - s)}{n+1}\right| + \\
&\quad + \left|\frac{(s_{N_0+1}) - s) + \cdots + (s_n - s)}{n+1}\right|.
\end{aligned}$$

The first sum on the right-hand side here is at most $\frac{(N_0+1)(M+|s|)}{n+1}$, and since $n+1 > \frac{2(N_0+1)(M+|s|)}{\varepsilon}$, this sum is at most $\frac{\varepsilon}{2}$. The second sum is at most $\frac{(n-N_0)\frac{\varepsilon}{2}}{n+1}$, which is at most $\frac{\varepsilon}{2}$. Thus $|\sigma_n - s| < \varepsilon$ if $n > N$, which was to be proved.

(*b*) Let $s_n = (-1)^n$. Here σ_n is 0 if n is odd and $\frac{1}{n+1}$ if n is even. Thus $\sigma_n \longrightarrow 0$, though s_n has no limit.

(*c*) Let $s_n = \frac{1}{n}$ if n is not a perfect cube and $s_n = \sqrt[3]{n}$ if n is a perfect cube. Then if $k^3 \leq n < (k+1)^3$ we have

$$\begin{aligned}
\sigma_n &\leq \frac{1}{n+1}\sum_{m=1}^{n}\frac{1}{m} + \frac{1}{n+1}\sum_{j=1}^{k} j \\
&= \frac{1}{n+1}\left(\sum_{m=1}^{n}\frac{1}{m}\right) + \frac{1}{n+1}\cdot\frac{k(k+1)}{2}.
\end{aligned}$$

The first sum on the right tends to zero by part (*a*) applied to the sequence $s_0 = 0$, $s_n = \frac{1}{n}$ for $n \geq 1$. As for the last term, since $n \geq k^3$, it is less than $\frac{1}{2k} + \frac{1}{2k^2}$, which tends to zero as $k \longrightarrow \infty$. Since $(k+1)^3 > n$, it follows that k tends to infinity as n tends to infinity, and hence we have $\sigma_n \longrightarrow 0$, even though $s_{n^3} \longrightarrow \infty$.

(*d*) If we set $a_0 = s_0$, we have $s_n = \sum\limits_{k=0}^{n} a_k$. Then

$$\begin{aligned}
s_n - \sigma_n &= s_n - \frac{s_0 + s_1 + \cdots + s_n}{n+1} \\
&= (a_0 + a_1 + \cdots + a_{n-1} + a_n) - \\
&\quad \frac{(n+1)a_0 + na_1 + \cdots + 2a_{n-1} + a_n}{n+1} \\
&= \frac{a_1 + 2a_2 + \cdots + (n-1)a_{n-1} + na_n}{n+1},
\end{aligned}$$

which was to be proved. If $na_n \longrightarrow 0$, then the expression on the right-hand side tends to zero by part (*a*) with s_n replaced by na_n. Hence $s_n - \sigma_n \longrightarrow 0$.

(e) If $m < n$ we have

$$\begin{aligned}
\sigma_n - \sigma_m &= \frac{s_0 + \cdots + s_n}{n+1} - \frac{s_0 + \cdots + s_m}{m+1} \\
&= (s_0 + \cdots + s_n)\Big(\frac{1}{n+1} - \frac{1}{m+1}\Big) + \sum_{i=m+1}^{n} \frac{s_i}{m+1} \\
&= \frac{m-n}{m+1}\sigma_n + \frac{1}{m+1}\sum_{i=m+1}^{n} s_i.
\end{aligned}$$

If we multiply both sides of this equation by $\frac{m+1}{m-n}$, and then transpose the left-hand side to the right and the term σ_n to the left, we obtain

$$-\sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) - \frac{1}{n-m}\sum_{i=m+1}^{n} s_i.$$

Adding $s_n = \frac{1}{n-m}\sum_{i=m+1}^{n} s_n$ to both sides then yields the result.

We then have

$$|s_n - s_i| = |a_{i+1} + \cdots + a_n| \leq M\Big(\frac{1}{i+1} + \cdots + \frac{1}{n}\Big) \leq \frac{(n-i)M}{i+1}.$$

Since the function $\frac{n-x}{x+1} = \frac{n+1}{x+1} - 1$ is decreasing, the maximal value of the right-hand side here is reached with $i = m+1$, so that $|s_n - s_i| \leq \frac{(n-m-1)M}{m+2}$, as asserted.

When we choose m to be the largest integer in $\frac{n-\varepsilon}{1+\varepsilon}$, we clearly have $m < n$. Since ε is fixed, we can assume $m > \varepsilon$. The inequality $\frac{n-\varepsilon}{1+\varepsilon} < m+1$ can easily be converted to $\frac{n-m-1}{m+2} < \varepsilon$, and the inequality $m \leq \frac{n-\varepsilon}{1+\varepsilon}$ likewise becomes $\frac{m+1}{n-m} \leq \frac{1}{\varepsilon}$. The first of these implies that $m \to \infty$ as $n \to \infty$, and we have

$$|s_n - \sigma_n| \leq \frac{1}{\varepsilon}|\sigma_n - \sigma_m| + M\varepsilon$$

for all n. This implies that the limit of any subsequence of $|s_n - \sigma_n|$ is at most $M\varepsilon$, and since ε is arbitrary, every convergent subsequence of $|s_n - \sigma_n|$ converges to zero. This, of course, implies that $s_n - \sigma_n$ tends to zero, so that if $\sigma_n \to s$, then $s_n \to s$.

Exercise 3.15 Definition 3.21 can be extended to the case in which the a_n lie in some fixed R^k. Absolute convergence is defined as convergence of $\sum |\mathbf{a}_n|$. Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)

Solution. (Theorem 3.22). $\sum \mathbf{a}_n$ *converges if and only if for every* $\varepsilon > 0$ *there is an integer* N *such that*

$$\left|\sum_{k=n}^{m} \mathbf{a}_k\right| \le \varepsilon$$

if $m \ge n \ge N$.

It is a trivial remark that, since $|a_j - b_j| \le |\mathbf{a} - \mathbf{b}| \le |a_1 - b_1| + \cdots + |a_k - b_k|$, the sequence $\{\mathbf{a}_n\}$ converges if and only if each sequence of components $\{a_{nj}\}$ converges, $j = 1, \ldots, k$. Hence the sequence of vector-valued functions converges if and only if each sequence of its components is a Cauchy sequence, and by the same inequalities, this is equivalent to saying that the vector-valued sequence is a Cauchy sequence.

(Theorem 3.23) *If* $\sum \mathbf{a}_n$ *converges, then* $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{0}$.

Using the remark made in the previous paragraph, if $\sum \mathbf{a}_n$ converges, then each sum of components $\sum a_{nj}$ converges. Hence for each j we have $a_{nj} \to 0$, which, again by the remark, means $\mathbf{a}_n \to \mathbf{0}$.

(Theorem 3.25 (a)) *If* $|\mathbf{a}_n| \le c_n$ *for* $n \ge N_0$, *where* N_0 *is some fixed integer, and if* $\sum c_n$ *converges, then* $\sum \mathbf{a}_n$ *converges.*

Again, the hypothesis implies that $|a_{nj}| \le c_n$ for $n \ge N_0$, so that $\sum a_{nj}$ converges for each $j = 1, 2, \ldots, k$. Once again, by the remark, this means that $\sum \mathbf{a}_n$ converges.

(Theorem 3.33) *Given* $\sum \mathbf{a}_n$, *put* $\alpha = \limsup\limits_{n\to\infty} \sqrt[n]{|\mathbf{a}_n|}$. *Then*

(a) *if* $\alpha < 1$, $\sum \mathbf{a}_n$ *converges;*

(b) *if* $\alpha > 1$, $\sum \mathbf{a}_n$ *diverges;*

(c) *if* $\alpha = 1$, *the test gives no information.*

Part (a) follows from the remarks made above, since $\sqrt[n]{|a_{nj}|} \le \sqrt[n]{|\mathbf{a}_n|}$. (If $\alpha < 1$, then each component series converges.)

As for part (b), if $\alpha > 1$, then $|\mathbf{a}_n| > 1$ for infinitely many n, and hence the series diverges.

(Theorem 3.34) *The series* $\sum \mathbf{a}_n$

(a) *converges if* $\limsup\limits_{n\to\infty} \dfrac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|} < 1$,

(b) *diverges if* $\dfrac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|} \ge 1$ *for* $n \ge n_0$, *where* n_0 *is some fixed integer.*

(a) The inequality implies that for some constant A and some fixed $r < 1$ we have $|\mathbf{a}_n| < Ar^n$, so that $\sum |\mathbf{a}_n|$ converges. Therefore by 3.25 the series $\sum \mathbf{a}_n$ also converges.

(b) As in the numerical case, this inequality implies that $\mathbf{a}_n$ does not tend to zero, so that the series must diverge.

(Theorem 3.42) *Suppose*

(a) *the partial sums* $\mathbf{A}_n$ *of* $\sum \mathbf{a}_n$ *form a bounded sequence;*
(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;
(c) $\lim_{n\to\infty} b_n = 0$.
Then $\sum b_n \mathbf{a}_n$ *converges.*

We reduce this to Theorem 3.22 by showing that the partial sums of the series $\sum b_n \mathbf{a}_n$ form a Cauchy sequence. In fact

$$\begin{aligned}\left|\sum_{n=p}^{q} b_n \mathbf{a}_n\right| &= \left|\sum_{n=p}^{q-1}(b_n - b_{n+1})\mathbf{A}_n + b_q\mathbf{A}_q - b_p\mathbf{A}_{p-1}\right| \\ &\leq M\Big(\sum_{n=p}^{q-1}|b_n - b_{n+1}| + b_q + b_p\Big) \\ &\leq 2Mb_p.\end{aligned}$$

Now, given $\varepsilon > 0$ choose N so large that $b_p < \frac{\varepsilon}{2M}$ for all $p > N$. Then if $q \geq p > N$, we have

$$\left|\sum_{n=p}^{q} b_n \mathbf{a}_n\right| \leq 2Mb_p < \varepsilon.$$

This proves that the partial sums form a Cauchy sequence, as required.

(Theorem 3.45) *If* $\sum \mathbf{a}_n$ *converges absolutely, then* $\sum \mathbf{a}_n$ *converges.*

Again this is a consequence of 3.25, with $c_n = |\mathbf{a}_n|$.

(Theorem 3.47) *If* $\sum \mathbf{a}_n = \mathbf{A}$ *and* $\sum \mathbf{b}_n = \mathbf{B}$, *then* $\sum(\mathbf{a}_n + \mathbf{b}_n) = \mathbf{A} + \mathbf{B}$ *and* $\sum c\mathbf{a}_n = c\mathbf{A}$ *for any fixed* c.

This theorem holds for each component of the vectors involved, hence it holds for the vectors themselves.

(Theorem 3.55) *If* $\sum \mathbf{a}_n$ *is a series of vectors which converges absolutely, then every rearrangement of* $\sum \mathbf{a}_n$ *converges, and they all converge to the same sum.*

Let $\mathbf{A}$ be the sum of the series in its original arrangement, and let $\varepsilon > 0$. Choose N so large that $\sum_{k=m}^{n} |\mathbf{a}_k| < \frac{\varepsilon}{2}$ if $n \geq m > N$. Then of course $\left|\sum_{k=1}^{n} \mathbf{a}_k - \mathbf{A}\right| \leq \frac{\varepsilon}{2}$ if $n > N$. For any arrangement of the series $\sum \mathbf{a}_{n_k}$, Choose N_1 so large that $\{1, 2, \ldots, N\} \subseteq \{n_1, n_2, \ldots, n_{N_1}\}$. Then if $m > N_1$ and N_2 is such that $\{n_1, \ldots, n_m\} \subseteq \{1, \ldots, N_2\}$ have,

$$\begin{aligned}\left|\sum_{k=1}^{m} \mathbf{a}_{n_k} - \mathbf{A}\right| &\leq \left|\sum_{k=1}^{m} \mathbf{a}_{n_k} - \sum_{k=1}^{m} \mathbf{a}_k\right| + \left|\sum_{k=1}^{m} \mathbf{a}_k - \mathbf{A}\right| \\ &\leq \sum_{k=N+1}^{m} |\mathbf{a}_k| + \frac{\varepsilon}{2} \\ &< \varepsilon\end{aligned}$$

Exercise 3.16 Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define x_1, x_2, $x_3, \ldots$, by the recursion formula

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{\alpha}{x_n}\Big).$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.
(b) Put $\varepsilon = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta\Big(\frac{\varepsilon_1}{\beta}\Big)^{2^n} \qquad (n = 1, 2, 3, \ldots,).$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < \frac{1}{10}$, and that therefore

$$\varepsilon_5 < 4 \cdot 10^{-16}, \quad \varepsilon_6 < 4 \cdot 10^{-32}.$$

Solution. (a) We note that x_n is always positive, and that if $x_n > \sqrt{\alpha}$, then $x_{n+1}^2 - \alpha = \frac{1}{4}\big(x_n - \frac{\alpha}{x_n}\big)^2 > 0$. Thus $x_n > \sqrt{\alpha}$ for all n. Since $x_n > \sqrt{\alpha}$, it follows that $\frac{\alpha}{x_n} < \sqrt{\alpha} < x_n$. Hence $x_n - x_{n+1} = \frac{1}{2}\Big(x_n - \frac{\alpha}{x_n}\Big) > 0$, and so $\{x_n\}$ decreases to a limit $\lambda \geq \sqrt{\alpha}$, which must satisfy $\lambda = \frac{\alpha}{\lambda}$, i.e., $\lambda = \sqrt{\alpha}$.
(b) We have $\frac{\varepsilon_n^2}{2x_n} = \frac{x_n^2 - 2x_n\sqrt{\alpha}+\alpha}{2x_n} = \frac{1}{2}\big(x_n + \frac{\alpha}{x_n}\big) - \sqrt{\alpha} = x_{n+1} - \sqrt{\alpha} = \varepsilon_{n+1}$. The inequality then results from the simple fact that $x_n > \sqrt{\alpha}$. Thus $\varepsilon_2 < \frac{\varepsilon_1^2}{\beta} = \beta\big(\frac{\varepsilon_1}{\beta}\big)^2$. By induction, if we suppose that $\varepsilon_n < \beta\big(\frac{\varepsilon_1}{\beta}\big)^{2^{n-1}}$, we find $\varepsilon_{n+1} < \frac{\varepsilon_n^2}{\beta} < \beta\big(\frac{\varepsilon_1}{\beta}\big)^{2^n}$.
(d) Taking $x_1 = 2$, $\alpha = 3$, we certainly have $\beta < 4$. And, since $\sqrt{3} > \frac{5}{3}$, we deduce that $12\sqrt{3} > 20$, so that $2\sqrt{3} > 10(2 - \sqrt{3})$, i.e., $\varepsilon_1 = 2 - \sqrt{3}$ and $\beta = 2\sqrt{3}$ satisfy $\varepsilon_1/\beta < \frac{1}{10}$, as asserted. It follows that $\varepsilon_n < 4 \cdot 10^{-2^{n-1}}$. In particular $\varepsilon_5 < 4 \cdot 10^{-16}$ and $\varepsilon_6 < 4 \cdot 10^{-32}$.

Exercise 3.17 Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(a) Prove that $x_1 > x_3 > x_5 > \cdots$.

(b) Prove that $x_2 < x_4 < x_6 < \cdots$.

(c) Prove that $\lim x_n = \sqrt{\alpha}$.

(d) Compare the rapidity of convergence of this process with the one described in Exercise 16.

Solution. Most of the work in this problem is done by the following three identities, whose proofs are routine computations:

$$(1 + x_n)(1 + x_{n+1}) = 2(1 + x_n) + (\alpha - 1),$$

$$x_{n+1}^2 - \alpha = -\left[\frac{(\alpha - 1)}{(1 + x_n)^2}\right](x_n^2 - \alpha),$$

$$x_{n+1}^2 - \alpha = \frac{(\alpha - 1)^2}{(1 + x_n)^2(1 + x_{n-1})^2}(x_{n-1}^2 - \alpha) =$$

$$= \left[\frac{\alpha - 1}{(\alpha - 1) + 2(1 + x_{n-1})}\right]^2 (x_{n-1}^2 - \alpha).$$

The second of these identities shows that x_n and x_{n+1} lie on opposite sides of $\sqrt{\alpha}$. The third shows that x_{n+1} is closer to $\sqrt{\alpha}$ than x_{n-1}. Hence, since $x_1 > \sqrt{\alpha}$ by hypothesis, parts (a) and (b) are proved. As for (c), the third relation shows that $|x_{n+1}^2 - \alpha| \le r^2 |x_{n-1}^2 - \alpha|$, where $r = \frac{\alpha - 1}{2 + \alpha - 1} < 1$. It follows that $|x_{n+2k}^2 - \alpha| \le r^{2k} |x_n^2 - \alpha|$, and the right-hand side of this expression tends to zero as $k \to \infty$. Thus $\lim\limits_{k \to \infty} x_{n+2k} = \sqrt{\alpha}$ whether n is odd or even, and so $\lim\limits_{n \to \infty} x_n = \sqrt{\alpha}$.

The convergence in this case is geometric, but not quadratically geometric, as in Exercise 16. The rate of convergence will depend on the size of α. For $1 < \alpha \le 2$ we certainly have $x_n \ge \alpha - 2$ for all n, and so in this case $r < \frac{1}{3}$, i.e., $|x_{n+1}^2 - \alpha| < \frac{1}{9}|x_{n-1}^2 - \alpha|$. This implies that $|x_{n+1} - \sqrt{\alpha}| < \frac{1}{9}\frac{x_{n-1} + \sqrt{\alpha}}{x_{n+1} + \sqrt{\alpha}}|x_{n-1} - \sqrt{\alpha}|$. If n is odd, we have $x_{n-1} < x_{n+1}$, and so $|x_{n+1} - \sqrt{\alpha}| < \frac{1}{9}|x_{n-1} - \sqrt{\alpha}|$. If n is even, we can at least assume $x_1 < 1.5$ (since $\alpha \le 2$), and so $\frac{x_{n-1} + \sqrt{\alpha}}{x_{n+1} + \sqrt{\alpha}} < 1.5$, so that $|x_{n+1} - \sqrt{\alpha}| < \frac{1.5}{9}|x_{n-1} - \sqrt{\alpha}|$.

Exercise 3.18 Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p - 1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1},$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.

Solution. (Exercise 16 is the case $p = 2$, of course.) The main work is done by the following easily derived formulas, which hold if $x_n > \alpha^{\frac{1}{p}}$.

$$x_{n+1} - \alpha^{\frac{1}{p}} = (x_n - \alpha^{\frac{1}{p}})\left[\left(\frac{p-1}{p}\right) - \frac{1}{p}\left(\left(\frac{\alpha^{\frac{1}{p}}}{x_n}\right) + \cdots + \left(\frac{\alpha^{\frac{1}{p}}}{x_n}\right)^{p-1}\right)\right]$$

$$
\begin{aligned}
&< (x_n - \alpha^{\frac{1}{p}})\Big(\frac{p-1}{p}\Big)\Big(1 - \Big(\frac{\alpha^{\frac{1}{p}}}{x_n}\Big)^{p-1}\Big) \\
&= (x_n - \alpha^{\frac{1}{p}})\Big(\frac{p-1}{px_n^{p-1}}\Big)(x_n^{p-1} - (\alpha^{\frac{1}{p}})^{p-1}) \\
&= (x_n - \alpha^{\frac{1}{p}})^2 \cdot \frac{p-1}{px_n^{p-1}} \cdot [x_n^{p-2} + x_n^{p-3}\alpha^{\frac{1}{p}} + \cdots + \alpha^{\frac{p-2}{p}}] \\
&< (x_n - \alpha^{\frac{1}{p}})^2 \cdot \frac{(p-1)^2}{px_n} \\
&< (x_n - \alpha^{\frac{1}{p}})^2 \cdot \frac{(p-1)^2}{p\alpha^{\frac{1}{p}}}.
\end{aligned}
$$

Thus we can guarantee quadratic-geometric convergence if we start with $x_1 - \alpha^{\frac{1}{p}} = \varepsilon_1 < \beta = \frac{p\alpha^{\frac{1}{p}}}{(p-1)^2}$. In that case we obtain the same inequalities as in Exercise 16, and $x_n \longrightarrow \alpha^{\frac{1}{p}}$.

Exercise 3.19 Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.

Solution. We note that the open middle third removed at the first stage of the construction is precisely the set of points whose ternary expansions *must* have a 1 as their first digit. (The numbers $\frac{1}{3}$ and $\frac{2}{3}$ *can* be written with a 1 in this place, since

$$
\begin{aligned}
\frac{1}{3} &= \frac{1}{3} + \frac{0}{9} + \cdots + \frac{0}{3^n} + \cdots \\
\frac{2}{3} &= \frac{1}{3} + \frac{2}{9} + \cdots + \frac{2}{3^n} + \cdots .
\end{aligned}
$$

However, these numbers can also be written as

$$
\begin{aligned}
\frac{1}{3} &= \frac{0}{3} + \frac{2}{9} + \cdots + \frac{2}{3^n} + \cdots \\
\frac{2}{3} &= \frac{2}{3} + \frac{0}{9} + \cdots + \frac{0}{3^n} + \cdots .
\end{aligned}
$$

Thus the points retained in the Cantor set after the first dissection are precisely those whose ternary expansions *may* be written without a 1 in the first digit. The same argument shows that the points retained in the Cantor set after the nth dissection are precisely those whose ternary expansions *may* be written without using a 1 in any of the first n digits. It then follows that the Cantor set is the set of points in $[0, 1]$ whose ternary expanions can be written without using any 1's, i.e., it is precisely the set of numbers $x(a)$ just described.

Exercise 3.20 Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_n\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Solution. Let $\varepsilon > 0$. Choose N_1 so large that $d(p_m, p_n) < \frac{\varepsilon}{2}$ if $m > N_1$ and $n > N_1$. Then choose $N \geq N_1$ so large that $d(p_{n_k}, p) < \frac{\varepsilon}{2}$ if $k > N$. Then if $n > N$, we have

$$d(p_n, p) \leq d(p_n, p_{n_{N+1}}) + d(p_{n_{N+1}}, p) < \varepsilon.$$

For the first term on the right is less than $\frac{\varepsilon}{2}$ since $n > N_1$ and $n_{N+1} > N+1 > N_1$. The second term is less than $\frac{\varepsilon}{2}$ by the choice of N.

Exercise 3.21 Prove the following analogue of Theorem 3.10(*b*): If $\{E_n\}$ is a sequence of closed and bounded sets in a *complete* metric space X, if $E_n \supset E_{n+1}$, and if

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0,$$

then $\bigcap_1^\infty E_n$ consists of exactly one point.

Solution. Choose $x_n \in E_n$. (We use the axiom of choice here.) The sequence $\{x_n\}$ is a Cauchy sequence, since the diameter of E_n tends to zero as n tends to infinity and E_n contains E_{n+1}. Since the metric space X is complete, the sequence x_n converges to a point x, which must belong to E_n for all n, since E_n is closed and contains x_m for all $m \geq n$. There cannot be a second point y in all of the E_n, since for any point $y \neq x$ the diameter of E_n is less than $d(x, y)$ for large n.

Exercise 3.22 Suppose X is a complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in X.) *Hint:* Find a shrinking sequence of neighborhoods E_n such that $\overline{E}_n \subset G_n$, and apply Exercise 21.

Solution. Let F_n be the complement of G_n, so that F_n is closed and contains no open sets. We shall prove that any nonempty open set U contains a point not in any F_n, hence in all G_n. To this end, we note that U is not contained in F_1, so that there is a point $x_1 \in U \setminus F_1$. Since $U \setminus F_1$ is open, there exists $r_1 > 0$ such that B_1, defined as the open ball of radius r_1 about x_1, is contained in $U \setminus F_1$. Let E_1 be the open ball of radius $\frac{r_1}{2}$ about x_1, so that the closure of E_1 is contained in B_1. Now F_2 does not contain E_1, and so we can find a point $x_2 \in E_1 \setminus F_2$. Since $E_1 \setminus F_2$ is an open set, there exists a positive number r_2 such that B_2, the open ball of radius R_2 about x_2, is contained in $E_1 \setminus F_2$, which in turn is contained in $U \setminus (F_1 \cup F_2)$. We let E_2 be the open ball of radius $\frac{r_2}{2}$ about x_2, so that $\overline{E}_2 \subseteq B_2$. Proceeding in this way, we construct a sequence of open balls E_j, such that $E_j \supseteq \overline{E}_{j+1}$, and the diameter of E_j tends to zero. By the previous exercise, there is a point x belonging to all the sets $\overline{E}_j$, hence to all the sets $U \setminus (F_1 \cup F_2 \cup \cdots \cup F_n)$. Thus the point x belongs to $U \cap \left(\bigcap_1^\infty G_n\right)$.

Exercise 3.23 Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. *Hint:* For any m, n,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

Solution. The inequality in the hint, which is an extension of the triangle inequality, shows that

$$d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n);$$

and since the same inequality holds with m and n reversed, it follows that

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n).$$

Now if $\varepsilon > 0$, choose N_1 and N_2 so that $d(p_n, p_m) < \frac{\varepsilon}{2}$ if $m > N_1$, $n > N_1$, and $d(q_n, q_m) < \frac{\varepsilon}{2}$ if $m > N_2$, $n > N_2$. Then let $N = \max(N_1, N_2)$. It follows immediately that $|d(p_n, q_n) - d(p_m, q_m)| < \varepsilon$ if $m > N$ and $n > N$. Since the real numbers are a complete metric space, it follows that $\{d(p_n, q_n)\}$ converges.

Exercise 3.24 Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X *equivalent* if

$$\lim_{n \to \infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$ and $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

(c) Prove that the resulting metric space X^* is complete.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^*.

(e) Prove that $\varphi(X)$ is dense in X, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X

Solution. (a) We need to show that: 1) $\{p_n\}$ is equivalent to itself; 2) if $\{p_n\}$ is equivalent to $\{q_n\}$, then $\{q_n\}$ is equivalent to $\{p_n\}$; and 3) if $\{p_n\}$ is equivalent to $\{q_n\}$ and $\{q_n\}$ is equivalent to $\{r_n\}$, then $\{p_n\}$ is equivalent to $\{r_n\}$. These follow from the properties of any metric. Thus 1) follows, since $d(p_n, p_n) = 0$ for all n; 2) follows since $d(p_n, q_n) = d(q_n, p_n)$; and 3) follows from the triangle inequality, i.e., $d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n)$, so that if $d(p_n, q_n) \longrightarrow 0$ and $d(q_n, r_n) \longrightarrow 0$, then $d(p_n, r_n) \longrightarrow 0$.

(b) Let $\{p_n\}$ be equivalent to $\{p'_n\}$ and $\{q_n\}$ equivalent to $\{q'_n\}$. Then, since we know in advance that all the limits exist, we have

$$\lim_{n\to\infty} d(p'_n, q'_n) \leq \lim_{n\to\infty} \left(d(p'_n, p_n) + d(p_n, q_n) + d(q_n, q'_n)\right) = \lim_{n\to\infty} d(p_n, q_n).$$

By symmetry, however, we must also have the opposite inequality, so that the two limits are actually equal.

Now X^* is a metric space; for $\Delta(P, Q) \geq 0$, by definition $\Delta(P, Q) = 0$ means $P = Q$, and symmetry and the triangle inequality on X^* follow from the same properties on X.

(c) Suppose $\{P_k\}$ is a Cauchy sequence in X^*. Choose Cauchy sequences $\{p_{kn}\}$ in X such that $\{p_{kn}\} \in P_k$, $k = 1, 2, \ldots$. For each k, let N_k be the first positive integer such that $d(p_{kn}, p_{km}) < 2^{-k}$ if $m \geq N_k$ and $n \geq N_k$. Let $p_k = p_{kN_k}$. Observe that $d(p_k, p_{kn}) < 2^{-k}$ for any $n \geq N_k$, so that $\lim_{n\to\infty} d(p_k, p_{kn}) \leq 2^{-k}$. (This limit exists since the sequence all of whose terms equal p_k is a Cauchy sequence.) Also, for any k, l, and n we have

$$d(p_k, p_l) \leq d(p_k, p_{kn}) + d(p_{kn}, p_{ln}) + d(p_{ln}, p_l).$$

Hence, taking n sufficiently large and assuming $k < l$, we obtain

$$d(p_k, p_l) \leq 2^{-k} + \Delta(P_k, P_l) + 2^{-k} + 2^{-l} < 3 \cdot 2^{-k} + \Delta(P_k, P_l).$$

It follows that $\{p_k\}$ is a Cauchy sequence. Let P be the element of X^* containing $\{p_k\}$. We claim $P_k \to P$ in X^*. For

$$\begin{aligned}
\Delta(P_k, P) &= \lim_{n\to\infty} d(p_{kn}, p_n) \\
&\leq \lim_{n\to\infty} \left(d(p_{kn}, p_k) + d(p_k, p_n)\right) \\
&\leq 2^{-k} + \limsup_{n\to\infty} \Delta(P_k, P_n) + 3 \cdot 2^{-k}.
\end{aligned}$$

Thus if $\varepsilon > 0$, choose $N_1 = 2 + \left[\dfrac{-\log \varepsilon}{\log 2}\right]$, and N_2 such that $\Delta(P_k, P_l) < \frac{\varepsilon}{2}$ if $k > N_2$ and $l > N_2$. Let $N = \max(N_1, N_2)$. We claim that if $k > N$, then $d(P_k, P) < \varepsilon$. Indeed this follows, since we then have $2^{-k+2} < \frac{\varepsilon}{2}$ and $\limsup_{n\to\infty} \Delta(P_k, P_n) \leq \frac{\varepsilon}{2}$. We have thus finally proved that X^* is complete.

(*d*) The assertion $\Delta(P_p, P_q) = d(p,q)$ is the trivial assertion that if $p_n = p$ and $q_n = q$ for all n, then

$$\lim_{n\to\infty} d(p_n, q_n) = d(p,q).$$

(*e*) Let P be any element of X^*, and let $\varepsilon > 0$. We shall find $p \in X$ such that $\Delta(P, P_p) < \varepsilon$. To this end, let $\{p_n\} \in P$ and let N be such that $d(p_n, p_m) < \frac{\varepsilon}{2}$ if $n > N$ and $m > N$. Let $p = p_{N+1}$. Then $\Delta(P, P_p) = \lim d(p_n, p) \leq \frac{\varepsilon}{2}$, and we are done.

If X is already complete, then for each $P \in X^*$ and $\{p_n\} \in P$ there exists $p \in X$ such that $p_n \to p$. This p is obviously the same for any sequence equivalent to $\{p_n\}$, and it is clear that $P = P_p$. Hence $\varphi(X) = X^*$ when X is complete.

It should be remarked that X^* is unique, in the sense that if Y and Z are any two complete metric spaces, each containing a dense subset isometric to X, then Y is isometric to Z. Indeed let φ and ψ be isometries of X into Y and Z respectively, such that $\varphi(X)$ is dense in Y and $\psi(X)$ is dense in Z. We construct an isometry of Y onto Z as follows. For each $y \in Y$, there is a sequence $\{x_n\} \subset X$ such that $\varphi(x_n) \to y$. The sequence $\{x_n\}$ is a Cauchy sequence in X, and hence $\{\psi(x_n)\}$ is a Cauchy sequence in Z (since ψ preserves distance). Since Z is complete, there is an element z such that $\psi(x_n) \to z$. We define $\theta(y) = z$. We claim first of all that this definition is unambiguous. For if y is given and some other sequence $\{x'_n\}$ in X is such that $\{\varphi(x'_n)\}$ converges to y, then $d_Z(\psi(x_n), \psi(x'_n)) = d_X(x_n, x'_n) = d_Y(\varphi(x_n), \varphi(x'_n) \to 0$, and hence $\psi(x'_n) \to z$ also. The mapping θ is an isometry, since if $y_1 = \lim \varphi(x_{1n})$ and $y_2 = \lim \varphi(x_{2n})$, then

$$\begin{aligned} d_Z(\theta(y_1), \theta(y_2)) &= \lim d_Z(\psi(x_{1n}), \psi(x_{2n})) \\ &= \lim d_X(x_{1n}, x_{2n}) \\ &= \lim d_Y(\varphi(x_{1n}), \varphi(x_{2n})) \\ &= d_Y(y_1, y_2). \end{aligned}$$

(Here we have used the fact that if $p_n \to p$ and $q_n \to q$, then $d(p_n, q_n) \to d(p,q)$, which in turn follows from the inequality

$$|d(p,q) - d(p_n, q_n)| \leq d(p, p_n) + d(q, q_n)$$

proved in Exercise 23 above.)

Finally $\theta(Y) = Z$, since one can easily define an inverse mapping $\eta : Z \to Y$ by merely reversing the steps used to define θ.

Exercise 3.25 Let X be the metric space whose points are the rational numbers, with the metric $d(x,y) = |x - y|$. What is the completion of this space? (Compare Exercise 24.)

Answer. By the remarks at the end of Exercise 24, the completion of a metric space X is any complete metric space containing a dense subset isometric to the space X. Since the real numbers have this property, the completion of the rational numbers is the real numbers. A Cauchy sequence of rational numbers converges to a unique real number, of course, and two sequences are equivalent if and only if they converge to the same real number. Hence we have also a more direct reason for claiming that the completion of the rational numbers is the real numbers.