UNIT 1: Transcendental Functions and Their Integrals

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MTH202A

Integrals of trigonometric functions

1. $\displaystyle\int \sin u \; du = -\cos u + C$
2. $\displaystyle\int \cos u \; du = \sin u + C$
3. $\displaystyle\int \sec^2 u \; du = \tan u + C$
4. $\displaystyle\int \csc^2 u \; du = -\cot u + C$
5. $\displaystyle\int \sec u \tan u \; du = \sec u + C$
6. $\displaystyle\int \csc u \cot u \; du = -\csc u + C$

Integrals yielding the natural logarithmic function

Recall (from MTH201A):

If u is a differentiable function of x, then

$$D_x(\ln|u|) = \frac{1}{u} \cdot D_x u$$

From this, we obtain the following.

Theorem 1.1

$$\int \frac{1}{u}\, du = \ln|u| + C$$

From this result and a known result from your earlier calculus, we have the following.

For any rational number n,

$$\int u^n \; du = \begin{cases} \dfrac{u^{n+1}}{n+1} + C & \text{if } n \neq -1 \\ \ln|u| + C & \text{if } n = -1. \end{cases}$$

Evaluate the following integral.

1. $\displaystyle\int \frac{x^2}{x^3+1}\ dx$
2. $\displaystyle\int_0^2 \frac{x^2+2}{x+1}\ dx$
3. $\displaystyle\int \frac{\ln x}{x}\ dx$
4. $\displaystyle\int \tan x\ dx$

Theorem 1.2

$$\int \tan u \ du = \ln|\sec u| + C$$

Example: Evaluate

$$\int \tan 3x \ dx.$$

Theorem 1.3

$$\int \cot u \ du = \ln|\sin u| + C$$

Proof. (Exercise)

Theorem 1.4

$$\int \sec u \; du = \ln \left| \sec u + \tan u \right| + C$$

Proof. We multiply the numerator and denominator of the integrand by $\sec u + \tan u$:

$$\begin{aligned} \int \sec u \; du &= \int \frac{\sec u (\sec u + \tan u)}{\sec u + \tan u} \; du \\ &= \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u} \; du \end{aligned}$$

Let

$$v = \sec u + \tan u \qquad dv = (\sec u \tan u + \sec^2 u) \; du.$$

Proof (cont'd).

Thus,

$$\begin{aligned} \int \sec u \; du &= \int \frac{dv}{v} \\ &= \ln|v| + C \\ &= \ln|\sec u + \tan u| + C. \quad \square \end{aligned}$$

Theorem 1.5

$$\int \csc u \; du = \ln|\csc u - \cot u| + C$$

"Proof"

Multiply the numerator and denominator of the integrand by $\csc u - \cot u$ and proceed as the previous proof.

Examples

1. Evaluate $\int \frac{dx}{\sin 2x}$.
2. Find the exact value of

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} (\csc 4x - \cot 4x)\ dx.$$

Exercises

A. Evaluate the indefinite integral.

1. $\displaystyle\int \frac{3x^2}{5x^3 - 1}\ dx$
2. $\displaystyle\int \frac{dy}{y \ln y}$
3. $\displaystyle\int \frac{\sin 3t}{\cos 3t - 1}\ dt$
4. $\displaystyle\int (\cot 5x + \csc 5x)\ dx$
5. $\displaystyle\int \frac{5 - 4y^2}{3 + 2y}\ dy$
6. $\displaystyle\int \frac{\tan(\ln x)}{x}\ dx$

B. Find the exact value of the definite integral.

7. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos t}{1 + 2\sin t}\ dt$
8. $\displaystyle\int_0^{\frac{\pi}{6}} (\tan 2x + sec2x)\ dx$

Integrals of Exponential Functions

Recall:

If u is a differentiable function of x,

$$D_x(e^u) = e^u \cdot D_x u.$$

Consequently, we have

Theorem 1.6

$$\int e^u \, du = e^u + C.$$

Evaluate the indefinite integral.

1. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$
2. $\displaystyle\int e^{2x+1}\, dx$
3. $\displaystyle\int \frac{1+e^{2x}}{e^{x}}\, dx$
4. $\displaystyle\int x^2 e^{2x^3}\, dx$
5. $\displaystyle\int \frac{dx}{1+e^{x}}$

Other Exponential and Logarithmic Functions

Recall:

If a is any positive number and u is a differentiable function of x, then

$$D_x(a^u) = a^u \cdot \ln a \cdot D_x u.$$

Example: Find $f'(x)$ if $f(x) = 3^{x^2}$.

It follows from the above result that

Theorem 1.7

If a is any positive number except 1,

$$\int a^u \, du = \frac{a^u}{\ln a} + C.$$

Example: Evaluate

$$\int \sqrt{10^{3x}} \, dx.$$

Integrals involving logarithms to the base a

To evaluate integrals involsing logarithms to the base a, we first apply the following change-of-base formula:

$$\log_a x = \frac{\ln x}{\ln a}$$

Example: Evaluate

$$\int \frac{\log_{10} x}{x}\, dx.$$

Exercises

Evaluate the indefinite integral.

1. $\int 3^{2x}\ dx$
2. $\int a^t \cdot e^t\ dt$
3. $\int x^2 10^{x^3}\ dx$
4. $\int a^{z \ln z}(\ln z + 1)\ dz$
5. $\int \frac{\log_2 x^2}{x}\ dx$

Integrals yielding trigonometric functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \Rightarrow \int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$
$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \Rightarrow \int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C$$

Evaluate the indefinite integral.

1. $\displaystyle\int \frac{1}{1-4x^2}\,dx$
2. $\displaystyle\int \frac{dx}{4x\sqrt{(x+1)(x-1)}}$
3. Find the exact area of the region in the first quadrant bounded by the curve
$$y = \frac{1}{1+x^2}$$
the x axis, the y axis, and the line $x = 1$.

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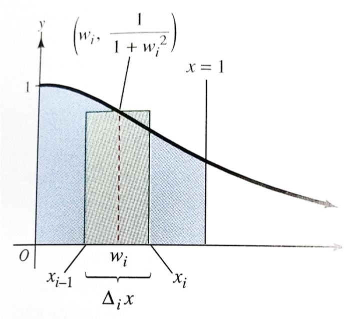

$$\text{Area} = \lim_{||\Delta||\to 0} \sum_{i=0}^{n} \frac{1}{1+w_i^2} \Delta_i x = \int_0^1 \frac{dx}{1+x^2}$$

Integrals yielding trigonometric functions

The following theorem provides some more general formulas.

Theorem 1.8

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C \qquad \textit{where } a > 0$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \qquad \textit{where } a \neq 0$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C \qquad \textit{where } a > 0$$

Evaluate:

1. $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}}$
2. $\displaystyle\int \frac{dx}{3x^2-2x+5}$

Find the exact value of the definite integral:

3. $\displaystyle\int_0^1 \frac{2x+7}{x^2+2x+5}\ dx$
4. $\displaystyle\int_0^{2-\sqrt{2}} \frac{6\ dx}{(2-x)\sqrt{x^2-4x+3}}$