A solutions manual for Set Theory by Thomas Jech

https://github.com/9beach

In December 2017, for no special reason I started studying mathematics and writing a solutions manual for Set Theory by Thomas Jech.

GitHub repository here, HTML versions here, and PDF version here.

Contents

Part I: Basic Set Theory

1. Axioms of Set Theory

1.1. Verify (1.1) $\quad (a,b) = (c,d)$ if and only if $a = c$ and $b = d$.

Proof. If $a = c$ and $b = d$, then $(a,b) = \{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\} = (c,d)$. Assume that $(a,b) = (c,d)$ and $a = b$. Then $\{\{c\},\{c,d\}\} = \{\{a\}\}$; thus $\{c,d\} \in \{\{a\}\}$ and $\{c\} \in \{\{a\}\}$, so $\{c,d\} = \{a\} = \{c\}$. Hence $c = d = a$. Since it was assumed that $a = b$, $a = c$ and $b = d$. Assume that $(a,b) = (c,d)$ and $a \neq b$. Since $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$ and $\{a\} \neq \{a,b\}$, $\{c\} = \{a\}$ and $\{a,b\} = \{c,d\}$; thus $c = a$ and $d = b$. $\square$

1.2. There is no set X such that $P(X) \subset X$.

Proof. Suppose $P(X) \subset X$, then we have a function f from X onto $P(X)$. But the set $Y = \{x \in X : x \notin f(x)\}$ is not in the range of f: If $z \in X$ were such that $f(z) = Y$, then $z \in Y$ if and only if $z \notin Y$, a contradiction. Thus f is not a function of X onto $P(X)$; also a contradiction. $\square$

Let

$$\mathbb{N} = \bigcap\{X : X \text{ is inductive}\}.$$

$\mathbb{N}$ is the smallest inductive set. Let us use the following notation:

$$0 = \emptyset, \quad 1 = \{0\}, \quad 2 = \{0,1\}, \quad 3 = \{0,1,2\}, \quad \ldots.$$

If $n \in \mathbb{N}$, let $n+1 = n \cup \{n\}$. Let us define $<$ (on $\mathbb{N}$) by $n < m$ if and only if $n \in m$.

A set T is transitive if $x \in T$ implies $x \subset T$.

1.3. If X is inductive, then the set $\{x \in X : x \subset X\}$ is inductive. Hence $\mathbb{N}$ is transitive, and for each $n, n = \{m \in \mathbb{N} : m < n\}$.

Proof. Let $Y = \{x \in X : x \subset X\}$. Since $\emptyset \subset X$, and $\emptyset \in X$, $\emptyset \in Y$. Now let $y \in Y$. Since $Y \subset X$, and X is inductive, $y \in X$, i.e., $\{y\} \subset X$, and $y \cup \{y\} \in X$, and since $y \subset X$, $y \cup \{y\} \subset X$; thus $y \cup \{y\} \in Y$. Therefore, Y is inductive.

Let $Y_{\mathbb{N}} = \{x \in \mathbb{N} : x \subset \mathbb{N}\}$, then $Y_{\mathbb{N}} \subset \mathbb{N}$, and since $Y_{\mathbb{N}}$ is inductive, $\mathbb{N} \subset Y_{\mathbb{N}}$; thus $\mathbb{N} = Y_{\mathbb{N}}$, and so we have that $x \in \mathbb{N}$ implies $x \subset \mathbb{N}$. Therefore, $\mathbb{N}$ is transitive.

It's obvious that $k \in n \cup \{n\}$ if and only if $k \in n$ or $k = n$. So it follows that for all $k, n \in \mathbb{N}, k < n+1$ if and only if $k < n$ or $k = n$. Now we show that for each $n, n = \{m \in \mathbb{N} : m < n\}$ by induction. Let $P(x)$ be the property "$x = \{m \in \mathbb{N} : m < x\}$". $P(0)$ holds, and assume that $P(n)$ holds. $n + 1 = n \cup \{n\}$ $= \{m \in \mathbb{N} : m < n\} \cup \{n\} = \{m \in \mathbb{N} : m < n \text{ or } m = n\} = \{m \in \mathbb{N} : m < n+1\} =$ $P(n+1)$ holds. Therefore, for each $n, n = \{m \in \mathbb{N} : m < n\}$. $\square$

1.4. If X is inductive, then the set $\{x \in X : x \text{ is transitive}\}$ is inductive. Hence every $n \in \mathbb{N}$ is transitive.

Proof. Let $Y = \{x \in X : x \text{ is transitive}\}$. Since $\emptyset \in X$, and $\emptyset$ is transitive, $\emptyset \in Y$. Now let $y \in Y$. Since $Y \subset X$, and X is inductive, $y \in X$, and $y \cup \{y\} \in X$. Let $z \in y \cup \{y\}$, then either $z \in y$ or $z = y$; since y is transitive, in any case, $z \subset y \cup \{y\}$. Thus $y \cup \{y\}$ is transitive, and so $y \cup \{y\} \in Y$. Therefore, Y is inductive.

Let $Y_{\mathbb{N}} = \{x \in \mathbb{N} : x \text{ is transitive}\}$, then $Y_{\mathbb{N}} \subset \mathbb{N}$, and since $Y_{\mathbb{N}}$ is inductive, $\mathbb{N} \subset Y_{\mathbb{N}}$; thus $\mathbb{N} = Y_{\mathbb{N}}$. Therefore, every $n \in \mathbb{N}$ is transitive. $\square$

1.5. If X is inductive, then the set $\{x \in X : x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n + 1$ for each $n \in \mathbb{N}$.

Proof. Let $Y = \{x \in X : x \text{ is transitive and } x \notin x\}$. Since $\emptyset \in X$, and $\emptyset$ is transitive and $\emptyset \notin \emptyset$, $\emptyset \in Y$. Now let $y \in Y$. Since $Y \subset X$, and X is inductive, $y \in X$, and $y \cup \{y\} \in X$. We already have that $y \cup \{y\}$ is transitive. Suppose $y \cup \{y\} \in y \cup \{y\}$, then $y \cup \{y\} = y$ or $y \cup \{y\} \in y$, i.e., $y \cup \{y\} \subset y$; in any case, $\{y\} \subset y$, i.e., $y \in y$; a contradiction. Thus $y \cup \{y\} \notin y \cup \{y\}$, and so $y \cup \{y\} \in Y$. Therefore, Y is inductive.

Let $Y_{\mathbb{N}} = \{x \in \mathbb{N} : x \text{ is transitive and } x \notin x\}$, then $Y_{\mathbb{N}} \subset \mathbb{N}$, and since $Y_{\mathbb{N}}$ is inductive, $\mathbb{N} \subset Y_{\mathbb{N}}$; thus $\mathbb{N} = Y_{\mathbb{N}}$, and so $n \notin n$. Suppose $n + 1 = n$, i.e., $n \cup \{n\} = n$, then $\{n\} \subset n$, i.e., $n \in n$; a contradiction. Therefore, $n \notin n$ and $n \neq n + 1$ for each $n \in \mathbb{N}$. $\square$

1.6. If X is inductive, then $\{x \in X : x$ is transitive and every nonempty $z \subset x$ has an $\in$-minimal element$\}$ is inductive (t is $\in$-*minimal* in z if there is no $s \in z$ such that $s \in t$).

Proof. Let $P(x)$ be the property "every nonempty $z \subset x$ has an $\in$-minimal element"; let $Y = \{x \in X : x \text{ is transitive and } P(x)\}$. $\emptyset \in X$, and is transitive, and has no nonempty subset, thus $\emptyset \in Y$. Let $y \in Y$. Since $Y \subset X$, and X is inductive, $y \in X$, and $y \cup \{y\} \in X$. We already have that $y \cup \{y\}$ is transitive. Now we show that $P(y \cup \{y\})$ holds. $y \notin y$; otherwise $\{y\} \subset y$ does not have an $\in$-minimal element $(\cdots y \in y \in y \cdots)$, a contradiction. There is no $a \in y$ such that $y \in a$; otherwise $y \in a \in y \Rightarrow y \in a \subset y \Rightarrow y \in y$. Hence for every nonempty $z \subset y$ if m is an $\in$-minimal element in z then so is in $z \cup \{y\}$; otherwise $y \in m$, a contradiction. Similarly, $P(\{y\})$ holds; otherwise $\cdots y \in y \in y \cdots$. Therefore, $P(y \cup \{y\})$ holds, and so Y is inductive. $\square$

1.7. Every nonempty $X \subset \mathbb{N}$ has an $\in$-minimal element.

[Pick $n \in X$ and look at $X \cap n$.]

Proof. Since $\mathbb{N}$ is the smallest inductive set, from 1.6, we have that every $n \in \mathbb{N}$ has an $\in$-minimal element. Let $n \in X$. If $n \cap X = \emptyset$, then n is an $\in$-minimal element. Suppose not. There exists $m \in X \setminus n$ such that $m \in n$, but then since $n = \{m \in \mathbb{N} : m < n\}$, $n \cap X \neq \emptyset$; a contradiction. If $n \cap X \neq \emptyset$, then $n \cap X \subset n$ has an $\in$-minimal element, and it's an $\in$-minimal element of X; otherwise similarly to the previous, a contradiction. $\square$

1.8. If X is inductive then so is $\{x \in X : x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$. Hence each $n \neq 0$ is $m + 1$ for some m.

Proof. Let $A = \{x \in X : x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$; let $a \neq \emptyset \in A$. Since $a = y \cup \{y\}$ for some y, so is $a \cup \{a\}$ for a; thus $a \cup \{a\} \in A$. Therefore, A is inductive, and each $n \neq 0$ is $m + 1$ for some m. □

1.9 (Induction). Let A be a subset of $\mathbb{N}$ such that $0 \in A$, and if $n \in A$ then $n+1 \in A$. Then $A = \mathbb{N}$.

Proof. By definition, A is an inductive subset of $\mathbb{N}$. Therefore, $A = \mathbb{N}$. □

A set X *has n elements* (where $n \in \mathbb{N}$) if there is a one-to-one mapping of n onto X. A set is *finite* if it has n elements for some $n \in \mathbb{N}$, and *infinite* if it is not finite.

A set S is *T-finite* if every nonempty $X \subset P(S)$ has a $\subset$-maximal element, i.e., $u \in X$ such that there is no $v \in X$ with $u \subset v$ and $u \neq v$. S is T-*infinite* if it is not T-finite. (T is for Tarski.)

1.10. Each $n \in \mathbb{N}$ is T-finite.

Proof. Let $A = \{n \in \mathbb{N} : n \text{ is T-finite}\}$. We show that $A = \mathbb{N}$ by induction.

$P(\emptyset) = \{\emptyset\}$ has the only nonempty subset $\{\emptyset\}$ which has a $\subset$-maximal element $\emptyset$.

Let $n \in A$; let $X \subset P(n+1)$. For some $Y \subset P(n)$, X is either Y or $Z = \{x \cup \{n\} : x \in Y\}$. For the latter case, let a be a $\subset$-maximal element of Y. Then it's obvious that $a \cup \{n\}$ is a $\subset$-maximal element of Z; thus X is T-finite. □

1.11. $\mathbb{N}$ is T-infinite; the set $\mathbb{N} \subset P\ (\mathbb{N})$ has no $\subset$-maximal element.

Proof. For any $n \in \mathbb{N}$, there exists $n + 1$ such that $n \subsetneq n + 1$; thus $\mathbb{N} \subset P(\mathbb{N})$ has no $\subset$-maximal element. □

Note that $\mathbb{N} \in P(\mathbb{N})$*,* $\mathbb{N} \subset P(\mathbb{N})$*, and* $\bigcup \mathbb{N} = \mathbb{N}$*.*

1.12. Every finite set is T-finite.

Proof. Let F be a finite set, then there is a one-to-one mapping f of F onto $n \in \mathbb{N}$. Let $A \subset P(F)$ be a nonempty set. Then $B = \{f(X) \subset P(n) : X \in A\}$ is nonempty, and has a $\subset$-maximal element. It's obvious that $\forall X, Y \in A(X \subset Y \iff f(X) \subset f(Y))$; A has a $\subset$-maximal element. □

1.13. Every infinite set is T-infinite.

[If S is infinite, consider $X = \{u \subset S : u \text{ is finite}\}$.]

Proof. Since $\emptyset \in X$, X is nonempty. Suppose X has a $\subset$-maximal element m. Then $S \setminus m \neq \emptyset$; otherwise S is a subset of a finite set; a contradiction, and so there exists $x \in S \setminus m$. Then $m \subsetneq m \cup \{x\} \in X$; a contradiction. □

1.14. The Separation Axioms follow from the Replacement Schema.

[Given ϕ, let $F = \{(x, x) : \phi(x)\}$. Then $\{x \in X : \phi(x)\} = F(X)$, for every X.]

Proof. Let $\varphi(x, y)$ be $x = y \wedge \phi(x)$. Then $F = \{(x, x) : \phi(x)\} = \{(x, y) : \varphi(x, y)\}$. Since $\forall x \forall y \forall z(\varphi(x, y) \wedge \varphi(x, z) \to y = z)$ holds, $\varphi(x, y)$ is a functional formula. Therefore, we have that The Separation Axioms follow from the Replacement Schema.

$F(X) = \{y : (\exists x \in X)\varphi(x, y)\} = \{y : (\exists x \in X)x = y \wedge \phi(x)\} = \{x : (\exists x \in X)\phi(x)\} = \{x \in X : \phi(x)\}$. $\square$

1.15. Instead of Union, Power Set, and Replacement Axioms consider the following weaker versions:

(1.8) $\forall X \exists Y \bigcup X \subset Y$, i.e., $\forall X \exists Y (\forall x \in X)(\forall u \in x) u \in Y$,
(1.9) $\forall X \exists Y P(X) \subset Y$, i.e., $\forall X \exists Y \forall u (u \subset X \to u \in Y)$,
(1.10) If a class F is a function, then $\forall X \exists Y F(X) \subset Y$.

Then axioms 1.4, 1.5, and 1.7 can be proved from (1.8), (1.9), and (1.10), using the Separation Schema (1.3).

Proof. Using the Separation Schema,
(1.8) $\implies \{x \in Y : (\exists a \in X) x \in a\} = \bigcup X$,
(1.9) $\implies \{x \in Y : x \subset X\} = P(X)$,
(1.10) $\implies \{y \in Y : (\exists x \in X)\varphi(x, y, p)\} = F(X)$. $\square$

2. Ordinal Numbers

2.1. The relation "$(P, <)$ is isomorphic to $(Q, <)$" is an equivalence relation (on the class of all partially ordered sets).

Proof. Let $(P, <)$, $(Q, <)$, and $(G, <)$ be partially ordered sets. P is isomorphic to P by $f : x \mapsto x$. If P is isomorphic to Q by f, then Q is isomorphic to P by f^{-1}. If P is isomorphic to Q by f, and Q is isomorphic to G by g, then P is isomorphic to G by $g \circ f$. $\square$

2.2. α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$, for every β.

Proof. Suppose $\beta + 1 \geq \alpha$. Since $\beta < \alpha$, $\alpha = \beta + 1$; thus α is a successor ordinal, a contradiction. Conversely, suppose that α is a successor ordinal. Then $\alpha = \beta + 1$ for some $\beta < \alpha$, a contradiction. $\square$

2.3. If a set X is inductive, then $X \cap Ord$ is inductive. The set $\mathbb{N} = \bigcap\{X : X \text{ is inductive}\}$ is the least limit ordinal $\neq 0$.

Proof. $\emptyset \in X \cap Ord$. If $x \in X \cap Ord$, then $x + 1 = x \cup \{x\} \in X \cap Ord$. Therefore, $X \cap Ord$ is inductive.
By exercise 1.3, $\mathbb{N}$ is transitive. Since $\mathbb{N} \cap Ord = \mathbb{N}$, by exercise 1.7 and lemma 2.11, $\mathbb{N}$ is well-ordered by $\in$. Thus $\mathbb{N}$ is an ordinal number. And, if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$. Therefore, $\mathbb{N}$ is the the least limit ordinal $\neq \emptyset$. $\square$

2.4. (Without the Axiom of Infinity). Let ω = least limit $\alpha \neq 0$ if it exists, $\omega = Ord$ otherwise. Prove that the following statements are equivalent:

(i) There exists an inductive set.
(ii) There exists an infinite set.
(iii) ω is a set.
[For (ii) $\to$ (iii), apply Replacement to the set of all finite subsets of X.]

Proof. *Case (i) $\Longrightarrow$ (ii).* By exercise 1.11, there exists an inductive T-infinite set $\mathbb{N}$, and by exercise 1.12 and 1.13, $\mathbb{N}$ is infinite.

Case (ii) $\Longrightarrow$ (iii). Let $Y = \{x \subset X : x \text{ is a finite set}\}$. Then $Y \subset P(X)$ is a set, and since $x \in Y$ is finite, there is a mapping f from Y into ω given by $x \mapsto n$, n such that x has n elements. Since X is infinite, if $x \in Y$ has n elements, then there exists $y \in Y$ which has $n + 1$ elements; thus $ran(f) = \omega$. Therefore, ω is a set.

Case (iii) $\Longrightarrow$ (i). By definition, $\emptyset \in \omega$, and if $\alpha \in \omega$, then $\alpha + 1 \in \omega$. Therefore, ω is inductive. $\square$

2.5. If W is a well-ordered set, then there exists no sequence $\langle a_n : n \in \mathbb{N}\rangle$ in W such that $a_0 > a_1 > a_2 > \dots$.

Proof. Otherwise there is no least element. $\square$

2.6. There are arbitrarily large limit ordinals; i.e., $\forall\alpha\exists\beta > \alpha$ (β is a limit).
[Consider $\lim_{n\to\omega} \alpha_n$, where $\alpha_{n+1} = \alpha_n + 1$.]

Proof. Given $\alpha_0 \in Ord$, we define $\alpha_{n+1} = \alpha_n + 1$, and $\beta = \lim_{n\to\omega} \alpha_n$, i.e., $\beta = \bigcup\{\alpha_n : n < \omega\} = \sup\,\{\alpha_n : n < \omega\}$. Then since the union of ordinals is an ordinal, β is an ordinal. And for every $\gamma < \beta$, there exists α_n such that $\alpha_n > \gamma$; otherwise $\gamma \geq \sup\,\{\alpha_n : n < \omega\}$, a contradiction. Thus $\gamma + 1 < \alpha_n + 1 = \alpha_{n+1} < \beta$, and so β is a limit ordinal. Therefore, there are arbitrarily large limit ordinals. $\square$

2.7. Every normal sequence $\langle \gamma_\alpha : \alpha \in Ord\rangle$ has arbitrarily large *fixed points*, i.e., α such that $\gamma_\alpha = \alpha$.

[Let $\alpha_{n+1} = \gamma_{\alpha_n}$, and $\alpha = \lim_{n\to\omega} \alpha_n$.]

Proof. Since $\langle \gamma_\alpha : \alpha \in Ord\rangle$ is increasing, for $\beta \in Ord$, there exists $m \in Ord$ such that $\gamma_m > \beta$. Let $\alpha_0 = \gamma_m$, $\alpha_{n+1} = \gamma_{\alpha_n}$. Then $\langle \alpha_n : n \in \omega\rangle$ is increasing, so we let $\alpha = \lim_{n\to\omega} \alpha_n$; similarly to 2.6, α is a limit ordinal. Hence we have that $\alpha = \lim_{n\to\omega} \alpha_{n+1} = \lim_{n\to\omega} \gamma_{\alpha_n} = \lim_{\xi\to\alpha} \gamma_\xi = \gamma_{\lim_{\xi\to\alpha} \xi} = \gamma_\alpha$. Therefore, $\gamma_\alpha = \alpha$ $\square$

2.8. For all α, β and γ,

(i) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$,

(ii) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$,

(iii) $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$.

Proof. *Case (i).* We show by induction on γ. $\alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + \alpha \cdot 0$. $\alpha\cdot(\beta+(\gamma+1)) = \alpha\cdot((\beta+\gamma)+1) = \alpha\cdot(\beta+\gamma)+\alpha = \alpha\cdot\beta+\alpha\cdot\gamma+\alpha = \alpha\cdot\beta+\alpha\cdot(\gamma+1)$. For all limit $\gamma > 0$, $\alpha\cdot(\beta+\gamma) = \alpha\cdot\lim_{\xi\to\gamma}(\beta+\xi) = \lim_{\xi\to\gamma} \alpha\cdot(\beta+\xi) = \lim_{\xi\to\gamma}(\alpha\cdot\beta+\alpha\cdot\xi)$ $= \alpha \cdot \beta + \lim_{\xi\to\gamma}(\alpha \cdot \xi) = \alpha \cdot \beta + \alpha \cdot \lim_{\xi\to\gamma} \xi = \alpha \cdot \beta + \alpha \cdot \gamma$

Case (ii) and (iii). Similarly to the previous. $\square$

2.9. (i) Show that $(\omega + 1) \cdot 2 \neq \omega \cdot 2 + 1 \cdot 2$.

(ii) Show that $(\omega \cdot 2)^2 \neq \omega^2 \cdot 2^2$.

Proof. *Case (i).* $(\omega+1)\cdot2 = \omega+1+\omega+1 = \omega+\omega+1 = \omega\cdot2+1 < \omega\cdot2+2 = \omega\cdot2+1\cdot2$

Case (ii). $(\omega \cdot 2)^2 = \omega \cdot 2 \cdot \omega \cdot 2 = \omega \cdot (2 \cdot \omega) \cdot 2 = \omega \cdot \omega \cdot 2 < \omega \cdot \omega \cdot 4 = \omega^2 \cdot 2^2$ $\square$

2.10. If $\alpha < \beta$ then $\alpha + \gamma \leq \beta + \gamma$, $\alpha \cdot \gamma \leq \beta \cdot \gamma$, and $\alpha^\gamma \leq \beta^\gamma$.

Proof. We show that if $\alpha < \beta$ then $\alpha + \gamma \leq \beta + \gamma$ by induction on γ.
$\alpha + 0 \leq \beta + 0$. $\alpha + \gamma + 1 \leq \alpha + 1 + \gamma + 1 \leq \beta + \gamma + 1$. Let a limit ordinal > 0 be γ. For all $\xi < \gamma$, if $\alpha + \xi < \beta + \xi$ then $\sup\,\{\alpha + \xi : \xi < \gamma\} \leq \sup\,\{\beta + \xi : \xi < \gamma\}$. Therefore, $\alpha + \gamma \leq \beta + \gamma$.

Similarly to the previous, $\alpha \cdot \gamma \leq \beta \cdot \gamma$, and $\alpha^\gamma \leq \beta^\gamma$. $\square$

2.11. Find α, β, γ such that

(i) $\alpha < \beta$ and $\alpha + \gamma = \beta + \gamma$,

(ii) $\alpha < \beta$ and $\alpha \cdot \gamma = \beta \cdot \gamma$,

(iii) $\alpha < \beta$ and $\alpha^\gamma = \beta^\gamma$.

Proof. *Case (i).* $0 + \omega = 1 + \omega$

Case (ii). $1 \cdot \omega = 2 \cdot \omega$

Case (iii). $2^\omega = 3^\omega$ $\square$

2.12. Let $\varepsilon_0 = \lim_{n\to\omega} \alpha_n$ where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$ for all n. Show that ε_0 is the least ordinal ε such that $\omega^\varepsilon = \varepsilon$.

Proof. $\varepsilon_0 = \sup\ \{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\} = \sup\ \{\omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots\} = \omega^{\sup\ \{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}} = \omega^{\varepsilon_0}$.

Suppose that there exists $\varepsilon < \varepsilon_0$ such that $\omega^\varepsilon = \varepsilon$. Then since for every finite number a, $a \neq \omega^a$, $\varepsilon \geq \omega$, and so there exists the least n such that $n > 0, n \in \omega$, and $\alpha_n > \varepsilon$. Then $\alpha_n = \omega^{\alpha_{n-1}} > \varepsilon = \omega^\varepsilon$. But since $\alpha_{n-1} < \varepsilon$, a contradiction. $\square$

A limit ordinal $\gamma > 0$ is called *indecomposable* if there exist no $\alpha < \gamma$ and $\beta < \gamma$ such that $\alpha + \beta = \gamma$.

2.13. A limit ordinal $\gamma > 0$ is indecomposable if and only if $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$ if and only if $\gamma = \omega^\alpha$ for some α.

Proof. $\gamma > 0$ is indecomposable if and only if $\alpha + \beta < \gamma$ for all $\alpha < \gamma$ and $\beta < \gamma$; otherwise $\alpha < \gamma < \alpha + \beta$, and so there is δ such that $\alpha + \delta = \gamma$ and $\delta < \gamma$; a contradiction. Hence for all $\alpha < \gamma$, $\alpha + \gamma = \sup\ \{\alpha + \xi : \xi < \gamma\} \leq \gamma$, but we know that $\sup\ \{\alpha + \xi : \xi < \gamma\} \geq \sup\ \{\xi : \xi < \gamma\} = \gamma$. Thus $\alpha + \gamma = \gamma$. Conversely, if $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$ then $\sup\ \{\alpha + \xi : \xi < \gamma\} = \gamma$. It follows that for all $\alpha < \gamma$ and $\beta < \gamma$, $\alpha + \beta < \gamma$.

Let $\gamma = \omega^{\beta_1} \cdot k_1 + \ldots + \omega^{\beta_n} \cdot k_n$ be Cantor's normal form. If a limit ordinal $\gamma \neq \omega^\alpha$ for all α, i.e., $n > 1$ or $k_n > 1$ for some n then clearly not indecomposable, i.e., *decomposable.* Conversely, let $\beta_1, \beta_2 < \gamma = \omega^\alpha$ for all $\alpha > 0$. There exist $\alpha' < \alpha$ and $k < \omega$ such that $\beta_1, \beta_2 < \omega^{\alpha'} \cdot k$ (Consider Cantor's normal forms of β_1 and β_2). Hence $\beta_1 + \beta_2 < \omega^{\alpha'} \cdot (k + k) < \omega^\alpha = \gamma$. $\square$

2.14. If E is a well-founded relation on P, then there is no sequence $\langle a_n : n \in \mathbb{N}\rangle$ in P such that $a_1 \mathrel{E} a_0$, $a_2 \mathrel{E} a_1$, $a_3 \mathrel{E} a_2$, ...

Proof. Otherwise, $\cdots\ E\ a_3 \mathrel{E} a_2 \mathrel{E} a_1 \mathrel{E} a_0$; there is no E-minimal elements. $\square$

2.15. (Well-Founded Recursion). Let E be a well-founded relation on a set P, and let G be a function. Then there exists a function F such that for all $x \in P, F(x) = G(x, F \restriction \{y \in P : y \mathrel{E} x\})$.

Proof. A set $B \subset A$ is called E-transitive in A if $\{y \in A : y \mathrel{E} x\} \subset B$ holds for all $x \in B$. Let $T = \{g : g$ is a function. $\mathrm{dom}(g)$ is a E-transitive in P, and $(\forall x \in \mathrm{dom}(g))g(x) = G(x, g \restriction x)\}$. T is nonempty, since for every E-miminal element $m \in P$,{m} is E-transitive, and a function g of $\{m\}$ is given by $m \mapsto G(m, \emptyset)$.

We claim that $\bigcup T$ is a function. Suppose not. There is a E-minimal element m of the set $\{x \in \mathrm{dom}(g_1) \cap \mathrm{dom}(g_2) : g_1(x) \neq g_2(x)$ for some $g_1, g_2 \in T\}$. Then $g_1(m) = G(m, g_1 \restriction \{y \in \mathrm{dom}(g_1) : y \mathrel{E} m\}) = G(m, g_1 \restriction \{y \in \mathrm{dom}(g_2) : y \mathrel{E} m\}) = G(m, g_2 \restriction \{y \in \mathrm{dom}(g_2) : y \mathrel{E} m\}) = g_2(m)$, a contradiction. Similarly, $\mathrm{dom}(\bigcup T) = P$. Therefore, $\bigcup T = F$. $\square$

3. Cardinal Numbers

3.1. (i) A subset of a finite set is finite.
(ii) The union of a finite set of finite sets is finite.
(iii) The power set of a finite set is finite.
(iv) The image of a finite set (under a mapping) is finite.

Proof. (i) Let X be a finite set, and $Y \subset X$. Suppose that Y is infinite. Then Y is T-infinite, so there is $S \subset P(Y)$ such that S has no $\subset$-maximal element. But by definition, $P(Y) \subset P(X)$, and so $S \subset P(X)$, a contradiction.

(ii) For p such that $0 < p < \omega$, let S_i be a finite set, and f_i be a function of S_i onto a finite ordinal n_i for each $i < p$. Let $S = \bigcup_{i<p} S_i$; let $f : S \to \sum_{i<p} n_i$ given by $x \mapsto \sum_{i<k} n_i + f_k(x)$ where k is the least number such that $x \in S_k$. Then f is one-to-one function of S into $\sum_{i<p} n_i$ which is bounded. Thus S is finite.

(iii) Let X be a finite set. $|P(X)| = 2^{|X|} < \aleph_0$.

(iv) Let f be a function of a finite set X onto Y. Then there is a one-to-one function g of X onto $n < \omega$. Clearly, a function h of $f(X)$ into n given by $y \mapsto \bigcap g_{-1} \circ f_{-1}(y)$ exists. $\square$

3.2. (i) A subset of a countable set is at most countable.
(ii) The union of a finite set of countable sets is countable.
(iii) The image of a countable set (under a mapping) is at most countable.

Proof. (i) Let X be a countable set, and $Y \subset X$. Then there is a one-to-one function f of X onto ω. Let id_Y be a function of Y into X given by $x \mapsto x$. Clearly the function $f \cdot id_Y$ is a function of Y into ω, and so $|Y| \leq \aleph_0$. Therefore, by definition of $\aleph_0$, Y is at most countable.

(ii) For some n such that $0 < n < \omega$, Let $S = \bigcup_{i<n} S_i$ where S_i is a countable set; for each $I < n$, let f_i be a function of S_i onto ω. Let $S = \bigcup_{i<n} S_i$; let $f : S \to \omega$ given by $x \mapsto 2^i 3^{f_i(x)}$ where i is the least number $x \in S_i$. Then f is one-to-one function of S into ω. Thus S is countable.

(iii) Let f be a function of a countable set X onto Y. Then there is a one-to-one function g of X onto ω. Clearly, a function h of $f(X)$ into ω given by $y \mapsto \bigcap g_{-1} \circ f_{-1}(y)$ exists. $\square$

3.3. $\mathbb{N} \times \mathbb{N}$ is countable.
$[f(m, n) = 2^m(2n + 1) - 1.]$

Proof. (i) Let f be a function of $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$ given by $(m, n) \mapsto 2^m(2n+1) - 1$. Let $x \in \omega$, and $m = \sup \{a \in \omega : 2^a \text{ divides } x + 1\}$. Then $(x + 1)/2^m$ is odd, so there is $n \in \omega$ such that $2n+1 = (x+1)/2^m$. Thus f is a function onto $\mathbb{N}$. Suppose that $2^{m_1}(2n_1 + 1) = 2^{m_2}(2n_2 + 1)$. Since $2x + 1 \neq 2y$ for all $x, y \in \mathbb{N}$, the prime factorization of $2x+1$ does not have 2 as a factor. Thus $m_1 = m_2$ and $n_1 = n_2$, and so f is a one-to-one function onto $\mathbb{N}$. Therefore, $\mathbb{N} \times \mathbb{N}$ is countable. $\square$

3.4. (i) The set of all finite sequences in $\mathbb{N}$ is countable.
(ii) The set of all finite subsets of a countable set is countable.

Proof. (i) Let f be a function of all finite sequences in $\mathbb{N}$ into $\mathbb{N}$ given by, for some $k \in \mathbb{N}$, $\langle s_i \in \mathbb{N} : i < k\rangle \mapsto \prod_{i<k} p_{i+1}^{s_i+1} - 1$ where p_i is the i-th prime number. Clearly, f is a one-to-one function onto $\mathbb{N}$.

(ii) Let X be a countable set; let Y be a set of all finite subsets of X. Then there is a one-to-one function f of X onto $\mathbb{N}$ given by $x \mapsto n$ for some $n < \omega$, and so for $S \in Y$, there is a unique increasing finite sequence $\langle f(x) : x \in S\rangle$. Thus there is a one-to-one function of Y into all finite sequences in $\mathbb{N}$; $Y \leq \aleph_0$, and $\aleph_0 = |X| = \{S \in Y : S \text{ is singleton}\} \subset Y$; thus $\aleph_0 \leq Y$. Therefore, $Y = \aleph_0$. □

3.5. Show that $\Gamma(\alpha \times \alpha) \leq \omega^\alpha$.

Proof. We show this by induction of α. $\Gamma(0 \times 0) \leq \omega^0$. $\Gamma(\alpha \times \alpha) \leq \omega^\alpha \Leftrightarrow \Gamma((\alpha+1) \times (\alpha+1)) = \Gamma(\alpha \times \alpha) + \alpha \cdot 2 + 1 \leq \omega^\alpha + \omega^\alpha \cdot 2 + \omega^\alpha = \omega^\alpha \cdot 4 \leq \omega^{\alpha+1}$. For a limit ordinal $\gamma > 0$, by definition $\Gamma(\gamma \times \gamma) = \sup\{\Gamma(\alpha \times \alpha) : \alpha < \gamma\} \leq \omega^\gamma$. □

3.6. There is a well-ordering of the class of all finite sequences of ordinals such that for each α, the set of all finite sequences in ω_α is an initial segment and its order-type is ω_α.

Proof. We define:

$$
\begin{aligned}
&\langle \alpha_0, \dots\rangle \prec \langle \beta_0, \dots\rangle \leftrightarrow \\
&\qquad \text{there is } k \text{ such that} \\
&\qquad\qquad \alpha_k < \beta_k \text{ and } \alpha_i = \beta_i \text{ for all } i < k, \\
&\langle \alpha_i : i < m\rangle < \langle \beta_i : i < n\rangle \leftrightarrow \\
&\qquad \text{either } \sum_{i<m} \alpha_i + m < \sum_{i<n} \beta_i + n \\
&\qquad \text{or } \sum_{i<m} \alpha_i + m = \sum_{i<n} \beta_i + n \\
&\qquad\qquad \text{and } \langle \alpha_i : i < m\rangle \prec \langle \beta_i : i < n\rangle.
\end{aligned}
$$

Let X be the class of all finite sequences of ordinals. The relation $<$ defined above is a linear ordering of X. Moreover, if $S \subset X$ is nonempty, then S has a least element. If we let $\Gamma(\alpha) =$ the order-type of the set $\{\beta \in X : \beta < \alpha\}$ for $\alpha \in X$, then Γ is a one-to-one mapping of X onto *Ord*. Note that for a finite sequence α in ω, $\Gamma(\alpha) \in \omega$, and so $\langle\omega\rangle$ is the least element α of X such that α is not a finite sequence in ω; thus $\Gamma(\langle\omega\rangle) = \omega$.

Let $\gamma(\alpha) = \Gamma(\langle\alpha\rangle)$. Note that $\gamma(\alpha)$ is an increasing function of α, and also that since each infinite cardinal is indecomposable, by definition of $(X, <)$, $\gamma(\omega_\alpha)$ is the set of all finite sequences in ω_α. Let $\eta(\alpha) =$ the order-type of the set of all finite sequences in α. Then $\gamma(\alpha) \leq \eta(\alpha)$ and $\gamma(\omega_\alpha) = \eta(\omega_\alpha)$ for each α. We show that $\gamma(\omega_\alpha) = \omega_\alpha$ by induction of α. This is true for $\alpha = 0$. Thus let α be the least ordinal such that $\gamma(\omega_\alpha) \neq \omega_\alpha$. Since γ is increasing, $\gamma(\omega_\alpha) \geq \omega_\alpha$; thus $\gamma(\omega_\alpha) > \omega_\alpha$, and so there is a sequence β such that $\Gamma(\beta) = \omega_\alpha$ and $\beta < \langle\omega_\alpha\rangle$. Then there is an ordinal δ such that $\beta < \langle\delta\rangle < \langle\omega_\alpha\rangle$; thus $\Gamma(\beta) = \omega_\alpha < \gamma(\delta) \leq \eta(\delta)$

$\Leftrightarrow \aleph_\alpha \leq |\eta(\delta)| = |\eta(|\delta|)| \leq \eta(|\delta|)$. But since $\delta < \omega_\alpha$, by the minimality of α, $\eta(|\delta|) = |\delta| < \aleph_\alpha$. A contradiction. Finally, by definition of γ, for each nonzero limit ordinal α, $\gamma(\omega_\alpha) = \sup\{\gamma(\omega_\xi) : \xi < \alpha\} = \omega_\alpha$. $\square$

We say that a set B is a *projection* of a set A if there is a mapping of A onto B. Note that B is a projection of A if and only if there is a partition P of A such that $|P| = |B|$. If $|A| \geq |B| > 0$, then B is a projection of A. Conversely, using the Axiom of Choice, one shows that if B is a projection of A, then $|A| \geq |B|$. This, however, cannot be proved without the Axiom of Choice.

3.7. If B is a projection of ω_α, then $|B| \leq \aleph_\alpha$.

Proof. Let f be a function of ω_α onto B. Then a one-to-one function g of B into ω_α is given by $x \mapsto \min f_{-1}(x)$. $\square$

3.8. The set of all finite subsets of ω_α has cardinality $\aleph_\alpha$.

[The set is a projection of the set of finite sequences.]

Proof. Let X be the set of all finite sequences in ω_α; let Y be the set of all finite subsets of ω_α. Then there is a function of X onto Y given by $\langle\alpha_0, \ldots \alpha_n\rangle \mapsto \{\alpha_0, \ldots \alpha_n\}$. Thus $\aleph_\alpha = |X| \geq |Y|$. But there is a one-to-one mapping of $S \subset Y$ such that each $x \in S$ is singleton onto a set of cardinality $\aleph_\alpha$. Thus $Y \geq \aleph_\alpha$. Therefore, $|Y| = \aleph_\alpha$. $\square$

3.9. If B is a projection of A, then $|P(B)| \leq |P(A)|$.

[Consider $g(X) = f_{-1}(X)$, where f maps A onto B.]

Proof. Since for each $S \subset B$, there is unique $f_{-1}(S) \subset A$, there is a one-to-one function of $P(B)$ into $P(A)$ given by $S \mapsto f_{-1}(S)$. $\square$

3.10. $\omega_{\alpha+1}$ is a projection of $P(\omega_\alpha)$.

[Use $|\omega_\alpha \times \omega_\alpha| = \omega_\alpha$ and project $P(\omega_\alpha \times \omega_\alpha)$: If $R \subset \omega_\alpha \times \omega_\alpha$ is a well-ordering, let $f(R)$ be its order-type.]

Proof. Since $\omega_{\alpha+1}$ is a set of possible well-orderings of subsets of X such that $|X| = \aleph_\alpha$, there is $R \in P(\omega_\alpha \times \omega_\alpha)$ such that $f(R) = \beta$ for each $\beta \in \omega_{\alpha+1}$. Let $g(R) = f(R)$ if R is a well-ordering; otherwise $g(R) = 0$. Then we have a mapping of $P(\omega_\alpha \times \omega_\alpha)$ onto $\omega_{\alpha+1}$ given by $R \mapsto g(R)$. $\square$

3.11. $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$.

[Use Exercises 3.10 and 3.9.]

Proof. By exercises 3.10 and 3.9, $\aleph_{\alpha+1} \leq 2^{\aleph_\alpha}$, and by Cantor's theorem, $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$. $\square$

3.12. If $\aleph_\alpha$ is an uncountable limit cardinal, then $\mathrm{cf}\ \omega_\alpha = \mathrm{cf}\ \alpha$; ω_α is the limit of a cofinal sequence $\langle\omega_\xi : \xi < \mathrm{cf}\ \alpha\rangle$ of cardinals.

Proof. $\mathrm{cf}\ \omega_\alpha = \mathrm{cf}\ \mathrm{cf}\ \alpha = \mathrm{cf}\ \alpha$. $\square$

3.13 (ZF). Show that ω_2 is not a countable union of countable sets.

[Assume that $\omega_2 = \bigcup_{n<\omega} S_n$ with S_n countable and let α_n be the order-type of S_n. Then $\alpha = \sup_n \alpha_n \leq \omega_1$ and there is a mapping of $\omega \times \alpha$ onto ω_2.]

Proof. We can assume that S_n is disjoint for each $n \leq \omega$. Then we have a one-to-one function of $\omega \times \alpha$ onto ω_2 given by $(n, \beta) \mapsto$ the β-th element of S_n if $\beta \in \alpha_n$ otherwise 0. Thus $\aleph_2 = |\omega_2| \leq |\omega \times \alpha| \leq \aleph_0 \cdot \aleph_1 = \aleph_1$. A contradiction. □

A set S is *Dedekind-finite* (D-finite) if there is no one-to-one mapping of S onto a proper subset of S. Every finite set is D-finite. Using the Axiom of Choice, one proves that every infinite set is D-infinite, and so D-finiteness is the same as finiteness. Without the Axiom of Choice, however, one cannot prove that every D-finite set is finite.

The set $\mathbb{N}$ of all natural numbers is D-infinite and hence every S such that $|S| \geq \aleph_0$, is D-infinite.

3.14. S is D-infinite if and only if S has a countable subset.

[If S is D-infinite, let $f : S \to X \subset S$ be one-to-one. Let $x_0 \in S - X$ and $x_{n+1} = f(x_n)$. Then $S \supset \{x_n : n < \omega\}$.]

Proof. If S is D-infinite, since f is one-to-one, for each m and n such that $0 \leq m < n < \omega$, $x_m \neq x_n$. Thus we have a countable set $X = \{x_n : n < \omega\} \subsetneq S$.

Conversely, if S has a countable subset $X = \{x_n : n < \omega\}$. We have a one-to-one mapping of S onto $S \setminus \{x_0\}$ given by $x \mapsto x$ if $x \notin X$; otherwise $x_n \mapsto x_{n+1}$. □

3.15. (i) If A and B are D-finite, then $A \cup B$ and $A \times B$ are D-finite.

(ii) The set of all finite one-to-one sequences in a D-finite set is D-finite.

(iii) The union of a disjoint D-finite family of D-finite sets is D-finite.

Proof. (i) Suppose that $X \subset A \cup B$ is countable. Then since a subset of a countable set is at most countable, $X \cap A$ and $X \cap B$ are at most countable. Since $X = (X \cap A) \cup (X \cap B)$, and the union of a finite set of finite sets is finite, $X \cap A$ or $X \cap B$ are countable. Thus A or B are D-infinite. A contradiction. Suppose that $X = \{(x_i, y_i) : i < \omega\} \subset A \times B$ is countable. Consider $C = \{x \in A : (x, y) \in X$ for some $y\}$ and $D = \{y \in A : (x, y) \in X$ for some $x\}$. Since $\aleph_0 = |X| \leq |C| \times |D|$, $|C| \leq |X| = \aleph_0$, and, $|D| \leq |X| = \aleph_0$, C or D are countable. But $C \subset A$ and $D \subset B$, a contradiction.

(ii) Let A be a D-finite set; let $X = \{X_i : i < \omega\}$ be a subset of all finite one-to-one sequences in A. Suppose that X is countable. Consider the cardinality of $S = \bigcup_{i<\omega} \operatorname{ran}(X_i)$. Since X_i is a finite one-to-one sequence for all $i < \omega$, $|S| \leq |\omega| \cdot |\omega| = \aleph_0$. So suppose that $|S| = n < \omega$. then $X_i \in \bigcup\{S^1, S^2, \dots S^n\}$ for all $i < \omega \Leftrightarrow X \subset \bigcup\{S^1, S^2, \dots S^n\}$. By (i), and induction of n, the union of a finite family of D-finite sets is D-finite, and a finite product of D-finite sets is D-finite, thus $\bigcup\{S, S^2, \dots S^n\}$ is D-finite. But $X \subset \bigcup\{S, S^2, \dots S^n\}$, a contradiction. Thus S is countable. But $S \subset A$, also a contradiction.

(iii) Let $X = \bigcup_{i \in I} X_i$ for some D-finite I be a union of a disjoint D-finite family of D-finite sets. Suppose that $S = \{\alpha_n : n < \omega\} \subset X$ is countable. Consider the cardinality of $T = \{i \in I : i$ such that $\alpha \in X_i$ for some $\alpha \in S\}$. Since X_i is disjoint for each $i \in I$, $|T| \leq |S| = \aleph_0$. Now suppose that $|T| = n < \omega$. Then S is a union of a finite family of D-finite set, thus finite; a contradiction. So T is countable. But $T \subset I$, also a contradiction. $\square$

On the other hand, one cannot prove without the Axiom of Choice that a projection, power set, or the set of all finite subsets of a D-finite set is D-finite, or that the union of a D-finite family of D-finite sets is D-finite.

3.16. If A is an infinite set, then $PP(A)$ is D-infinite.
[Consider the set $\{\{X \subset A : |X| = n\} : n < \omega\}$.]

Proof. The set $\{\{X \subset A : |X| = n\} : n < \omega\} \subset PP(A)$ is countable. $\square$

4. Real Numbers

4.1. The set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $\mathfrak{c}$ (while the set of all functions has cardinality $2^{\mathfrak{c}}$).

[A continuous function on $\mathbb{R}$ is determined by its values at rational points.]

4.2. There are at least $\mathfrak{c}$ countable order-types of linearly ordered sets.

[For every sequence $a = \langle a_n : n \in \mathbb{N} \rangle$ of natural numbers consider the order-type

$$\tau_a = a_0 + \xi + a_1 + \xi + a_2 + \dots$$

where ξ is the order-type of the integers. Show that if $a \neq b$, then $\tau_a \neq \tau_b$.]

A real number is algebraic if it is a root of a polynomial whose coefficients are integers. Otherwise, it is transcendental.

4.3. The set of all algebraic reals is countable.

4.4. If S is a countable set of reals, then $|\mathbb{R} - S| = \mathfrak{c}$.

[Use $\mathbb{R} \times \mathbb{R}$ rather than $\mathbb{R}$ (because $|\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0}$).]

4.5. (i) The set of all irrational numbers has cardinality $\mathfrak{c}$.

(ii) The set of all transcendental numbers has cardinality $\mathfrak{c}$.

4.6. The set of all open sets of reals has cardinality $\mathfrak{c}$.

4.7. The Cantor set is perfect.

4.8. If P is a perfect set and (a, b) is an open interval such that $P \cap (a, b) \neq \emptyset$, then $|P \cap (a, b)| = \mathfrak{c}$.

4.9. If $P_2 \not\subset P_1$ are perfect sets, then $|P_2 - P_1| = \mathfrak{c}$.

[Use Exercise 4.8.]

If A is a set of reals, a real number a is called a *condensation point* of A if every neighborhood of a contains uncountably many elements of A. Let A^* denote the set of all condensation points of A.

4.10. If P is perfect then $P^* = P$.

[Use Exercise 4.8.]

4.11. If F is closed and $P \subset F$ is perfect, then $P \subset F^*$.

[$P = P^* \subset F^*$.]

4.12. If F is an uncountable closed set and P is the perfect set constructed in Theorem 4.6, then $F^* \subset P$; thus $F^* = P$.

[Every $a \in F^*$ is a limit point of P since $|F - P| \leq \aleph_0$.]

4.13. If F is an uncountable closed set, then $F = F^* \cup (F - F^*)$ is the unique partition of F into a perfect set and an at most countable set.

[Use Exercise 4.9.]

4.14. $\mathbb{Q}$ is not the intersection of a countable collection of open sets.
[Use the Baire Category Theorem.]

4.15. If B is Borel and f is a continuous function then $f_{-1}(B)$ is Borel.

4.16. Let $f : \mathbb{R} \to \mathbb{R}$. Show that the set of all x at which f is continuous is a G_δ set.

4.17. (i) $\mathbb{N} \times \mathbb{N}$ is homeomorphic to $\mathbb{N}$.
(ii) $\mathbb{N}^\omega$ is homeomorphic to $\mathbb{N}$.

4.18. The tree T_F in (4.6) has no maximal node, i.e., $s \in T$ such that there is no $t \in T$ with $s \subset t$. The map $F \mapsto T_F$ is a one-to-one correspondence between closed sets in $\mathbb{N}$ and sequential trees without maximal nodes.

4.19. Every perfect Polish space has a closed subset homeomorphic to the Cantor space.

4.20. Every Polish space is homeomorphic to a G_δ subspace of the Hilbert cube.
[Let $\{x_n : n \in \mathbb{N}\}$ be a dense set, and define $f(x) = \langle d(x, x_n) : n \in \mathbb{N}\rangle$.]