Graph Theory

Chapter 3

More on special classes of graphs

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Graph

Trees

Def. A *tree* is a connected graph with no cycles.

Exmp. Give all trees with orders 1,2,3,4,5,6.

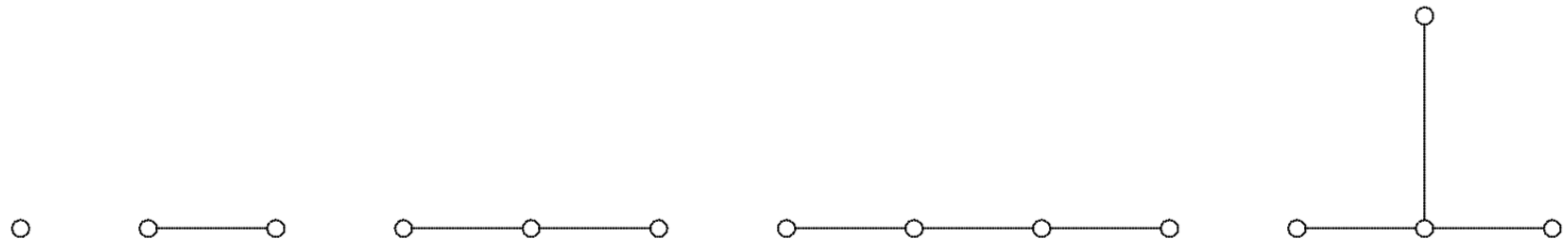


Figure 20: Trees of orders 1, 2, 3, 4

Graph

Trees

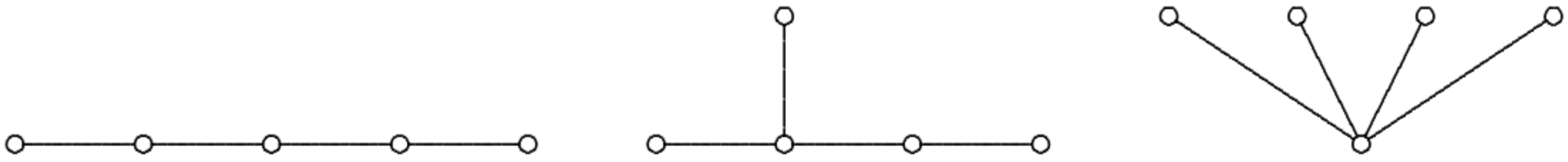


Figure 21: Trees of order 5

Graph

Trees

Remk. A graph with no cycles is called an *acyclic* graph. Thus, we say that a tree is a connected acyclic graph. Furthermore, a graph with no cycles is called a *forest*. This implies that the components of a forest is a tree.

Graph

Trees

Thm. Let G be a graph. The following are equivalent:

1. G is a tree.
2. For every pair of distinct vertices u and v in G, there is exactly one path from u to v.
3. G is connected and $p = q + 1$.
4. G is acyclic and $p = q + 1$.
5. G is acyclic and if two nonadjacent vertices of G are joined by an edge e, then $G + e$ has exactly one cycle.
6. G is connected, is not K_p for $p \geq 3$, and if any two nonadjacent vertices f G are joined by an edge e, then $G + e$ has exactly one cycle.
7. G is not $K_3 \cup K_1$ or $K_3 \cup K_2$, $|V| = |E| + 1$, and if any nonadjacent vertices of G are joined by an edge e, then $G + e$ has exactly one cycle.

Graph

Trees

Proof: $\{(1) \implies (2)\}$

{G is a tree then for every pair of distinct vertices u and v in G, there is exactly one path from u to v}

Suppose G is a tree. Thus, G is connected. Let u and v be distinct vertices in G and P_1 and P_2 be two distinct $u-v$ paths in G. Starting with the initial vertex u, and since the paths are distinct, there is a vertex w (this maybe u itself) in P_1 and P_2 whose successor are two different vertices, say x_1 and x_2. Thus we have P_1, the path $v, \ldots, w, x_1, \ldots v$ and P_2, the path $u, \ldots, w, x_2, \ldots, v$. Clearly, this will form a cycle as these two paths will meet at another vertex, say y (this could be v). This contradicts the assumption that \$G\$ is a tree and thus, do not contain any cycle.

Graph

Trees

Proof: $\{\{(2) \implies (3)\}\}$
{For every pair of distinct vertices u and v in G, there is exactly one path from u to v then G is connected and $p = q + 1$.}

Suppose every pair of distinct vertices u and v in G is in exactly one $u - v$ path. This implies that G is connected. We show that $p = q + 1$ by induction. If $p = 1$, clearly $q = 0$ (that is one vertex and zero edge) and if $p = 2$, then $q = 1$ (that is two vertices and one edge). Let $p = q + 1$ be true when $p = k \in Z$. Assume that $k = q + 1$ is true for all graphs of order k and size q with $k < p$. With a graph G of order p, we remove an edge. This together with the assumption will make G disconnected and having two components. Let these components be G_1 of order k_1 and size q_1 and G_2 of order k_2, and size q_2, with $k_1 > 0$ and $k_2 > 0$. Clearly $k_1 + k_2 = p$ and $k_1 < p$ and $k_2 < p$. Also, $q_1 + q_2 + 1 = q$. Thus, since G_1 and G_2 are of order less that p, bt the hypothesis of the induction, the equations $k_1 = q_1 + 1$ and $k_2 = q_2 + 1$ are true. Therefore,

$$p = k_1 + k_2 = (q_1 + 1) + (q_2 + 1) = (q_1 + q_2 + 1) + 1 = q + 1.$$

Graph

Trees

Proof: $\{(3) \implies (4)\}\{G$ is connected and $p = q + 1$ then G is acyclic and $p = q + 1\}$

Suppose G is connected and $p = q + 1$. We need to show that G is acyclic. Suppose G is not acyclic. Thus it contains a cycle. Let this cycle contain n vertices and of course all n edges. Each of the remaining $p - n$ vertices is adjacent to another vertex on a geodesic to a vertex on the said cycle. Each of these edges are different. Thus, the number of edges of G is at least $n + p - n = p$, that is $q \geq p$. This contradicts the assumption that $p = q + 1$.

Graph

Trees

Proof: $\{(4) \implies (5)\}\{G$ is acyclic and $p = q + 1$, then G is acyclic and if any two nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle.$\}$

Suppose G is acyclic and $p = q + 1$. Suppose G has k components, then each of these components is a tree. For $i = 1,2,\ldots k$, let p_i and q_i be the order and size, respectively of the k components. Since each component is a tree (and is thus connected), $p_i = q_i + 1$, is true for $i = 1,2,\ldots k$. We then have,

$$p = \sum_{i=1}^{k} p_i = \sum_{i=1}^{k} (q_i + 1) = q + k.$$

But by assumption, $p = q + 1$, thus $k = 1$ and G is connected and is a tree. Thus, for every distinct pair of vertices u and v in G there is a unique $u - v$ path. If we add the edge $e = uv$ to G, (this is $G + e$) we form a cycle, and this cycle is unique because of the uniqueness of the $u - v$ path.

Graph

Trees

Proof: $\{(5) \implies (6)\}\{G$ is acyclic and if any two nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle, implies that G is connected, is not K_p for $p \geq 3$, and if any two nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle.$\}$

Suppose G is acyclic and if any two nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle. We need to show that G can not be $K_p, p \geq 3$ and G is connected. Clearly G can not be $K_p, p \geq 3$ since K_p contains a cycle and G is assumed to be acyclic. Furthermore, G must be connected since if G contains two components say G_1 and G_2, then we can add an edge $e = x_1x_2$ where $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$ then G remains acyclic. This contradicts the assumption that $G + e$ should have exactly one cycle.

Graph

Trees

Proof: $\{(6) \implies (7)\}$$\{G$ is connected, is not K_p for $p \geq 3$, and if any two nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle implies that G is not $K_3 \cup K_1$ or $K_3 \cup K_2$, $p = q + 1$, and if any nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle.$\}$

Since G is connected, then there is path connecting every pair of vertices. Suppose there are two paths connecting the same pair of vertices. Then, from the proof of $\{(1) \implies (2)\}$, there is a cycle in G. However, note that this cycle in G can not have more than three vertices, since if this were true, adding an edge e incident to two nonadjacent vertices in the cycle produces $G + e$ containing two cycles. Thus, the cycle must be K_3 and this is a proper subgraph of G, since it is assumed that G is not $K_p, p \geq 3$. This implies that there is at least one vertex adjacent to one of the vertices of K_3, since G is connected.

Clearly, that if any edge$e$ is added to G , then one may be added so as to form two cycles in $G + e$. If no more edges maybe added, we have formed $K_p, p \geq 3$. This contradicts the hypothesis, thus every pair of vertices in G are joined by a unique path and by $\{(2) \implies (3)\}$, $p = q + 1$. We note that G should contain K_3 as a proper subgraph, satisfy $p = q + 1$ and is connected. Thus it can not be $K_3 \cup K_1$ or $K_3 \cup K_2$.

Graph

Trees

Proof: $\{(7) \implies (1)\}\{G$ is not $K_3 \cup K_1$ or $K_3 \cup K_2$, $p = q + 1$, and if any nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle implies that G is a tree.$\}$

Suppose G is not $K_3 \cup K_1$ or $K_3 \cup K_2$, $p = q + 1$, and if any nonadjacent points of G are joined by an edge e, then $G + e$ has exactly one cycle. Suppose G contains a cycle. Then from the argument above, this cycle must be K_3, with three vertices and three edges. Since $p = q + 1$, G contains another component and this component must be a tree. If the other component is a path of on three vertices with two edges, and adding an edge to G to form $G + e$ with result in a graph with two cycles. A contradiction to the hypothesis and thus G can only be either $K_3 \cup K_1$ or $K_3 \cup K_2$. These are the graphs excluded. Thus, G is acyclic. But then $p = q + 1$ as well, so since $\{(3) \implies (4)\}$ and $\{(4) \implies (5)\}$, then G is connected as well. Therefore, G is a tree.

Graph

Trees

Remk. A graph $G = (V, E)$ with $|V| = 1$ and $|E| = 0$ is called a *trivial* graph.

Defn. Let G be a graph and v be a vertex of G. If $deg\ \ v = 0$, then v is called an isolated vertex. If $deg\ \ v = 1$, then we call v an endpoint of G.

Graph

Trees

Cor. 5.1.1 Every nontrivial tree has at least two endpoints.

Proof: We note that if $G = (V, E)$ where $V = \{x_1, x_2, \ldots, x_p\}$ and $|E| = q$. Suppose G is a tree, then G is connected and $p = q + 1$. Thus,

$$\sum_{i=1}^{p} deg(x_i) = 2q = 2(p-1) = 2p - 2.$$

This implies that there are at least two vertices with degree less than 2. Since G is connected, then these vertices are of degree 1. So these vertices are endpoints.

Graph

Eulerian graphs and Hamiltonian graphs

Defn. A *trail* in a graph G is a walk in which the edges are distinct. A closed trail in G is an *eulerian trail* that contains all vertices and edges of G. A graph G that contains an eulerian trail is called an *eulerian graph*.

Exmp. Consider the graph in Figure 22 below. The trail $x_1e_1x_2e_2x_3e_3x_4e_4x_5e_5x_3e_6x_6e_7x_1$ is an eulerian trail. Thus, the graph is Figure 22 is an eulerian graph.

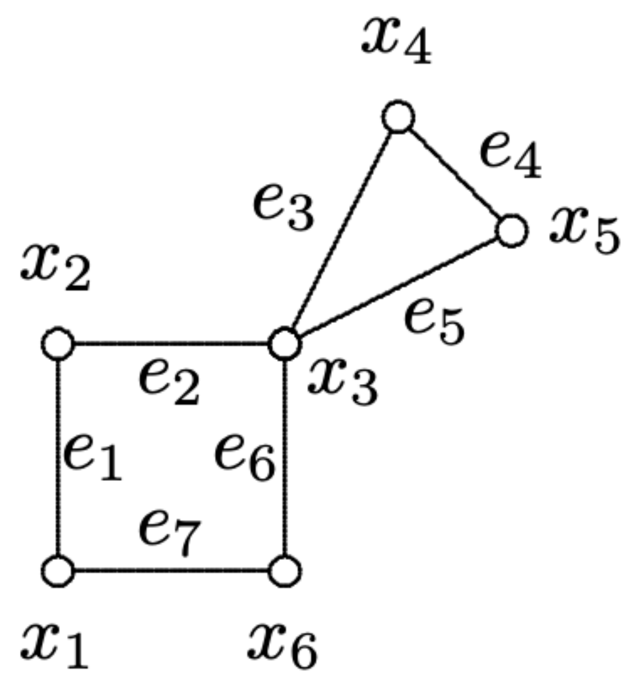


Figure 22: Example of Eulerian graph

Graph

Eulerian graphs and Hamiltonian graphs

Thm. Let G be a connected graph. The following statements are equivalent:

1. G is eulerian
2. Every vertex in G has even degree.
3. The set of vertices in G can be partitioned into cycles.

Graph

Eulerian graphs and Hamiltonian graphs

Proof: $\{(1) \implies (2)\}$$\{G$ is eulerian implies every vertex in G has even degree$\}$

Suppose G is eulerian thus, G contains an eulerian trail. Let T be this closed trail, then every occurrence of a vertex in T, contributes two units to the degree of that vertex. Also, since each edge in G occurs only once in T, then every vertex in G has even degree.

Graph

Eulerian graphs and Hamiltonian graphs

Proof: $\{(2) \implies (3)\}$\{Every vertex in G has even degree implies that the set of edges in G can be partitioned into cycles.\}

Suppose every vertex in G is of even degree. Since G is connected and each vertex of degree at least 2, G contains a cycle. Let this cycle be C_1. The removal of the edges in C_1 from G gives a spanning subgraph of G, say G_1. We note that the degree of the vertices in G_1 are still even (Some of the vertices may be of degree zero!). We continue this process to get cycles $C_2, C_3, \ldots, C_n$, of spanning subgraphs $G_2, G_3, \ldots, G_n$ respectively, until a totally disconnected graph G_n is obtained. The set of cycles $\{C_1, C_2, \ldots, C_n\}$ is a partition of the edges of G.

Graph

Eulerian graphs and Hamiltonian graphs

Proof: $\{(3) \implies (1)\}$ {The set of edges in G can be partitioned into cycles implies that G is eulerian}

Let the set of cycles $\mathscr{C} = \{C_1, C_2, \ldots, C_n\}$ be a partition of the edges of G. If there is only one cycle in the partition, then G is eulerian. Otherwise, if C_i is a cycle in $\mathscr{C}$ there is another cycle $C_j \in \mathscr{C}$, $i \neq j$ which has a common vertex, x with C_i. Then, the walk starting at x containing the edges of the cycles C_i and C_j in succession is a closed trail containing all the edges of these two cycles. We continue this process, until we obtain a closed trail in G containing all edges of G and each edge appearing only once in the trail. Thus, G is eulerian.

Graph

Eulerian graphs and Hamiltonian graphs

Remk. The last theorem suggests that if a connected graph G has no vertex of odd degree, then G contains a trail consisting of all vertices and edges of G.

Cor. Let G be a connected graph with exactly $2n$ vertices of odd degree, $n \geq 1$. Then, the set of edges of G can be partitioned into n open trails.

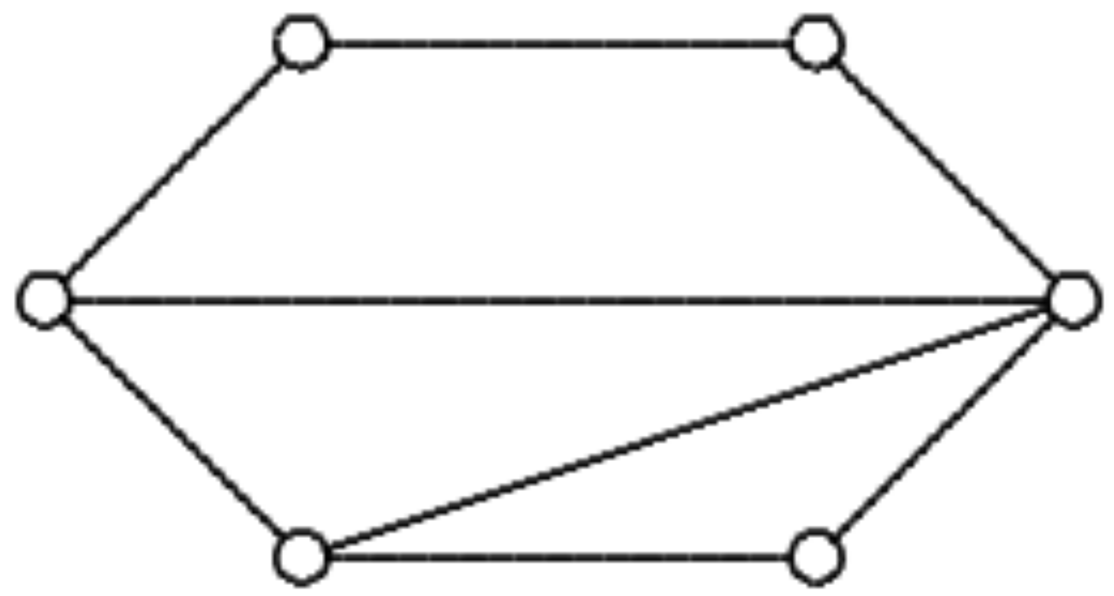
Cor. Let G be a connected graph with exactly two vertices of odd degree. Then, G has an open trail containing all vertices and edges of G. This open trail begins at one of the vertices with an odd degree and ends on the other.

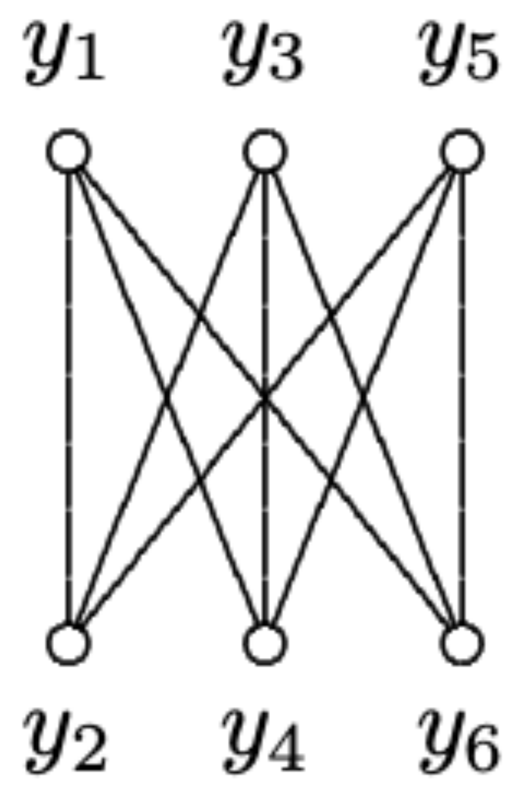
Graphs

Eulerian graphs and Hamiltonian graphs

Defn. Let *G* be a graph. If *G* has a spanning cycle, then *G* is called a *hamiltonian graph*. Suppose Z is the spanning cycle go G, then Z is called a *hamiltonian cycle.*

Exmp. The following graphs are hamiltonian





Graph

Planar graphs

Defn. A graph is said to be *embedded* in a surface S when it is drawn on S so that no two edges intersect. A graph is *planar,* if it can be embedded in the plane.

Exmp. The complete graph K_4 is planar because it can be drawn in the plane where no two edges intersect.

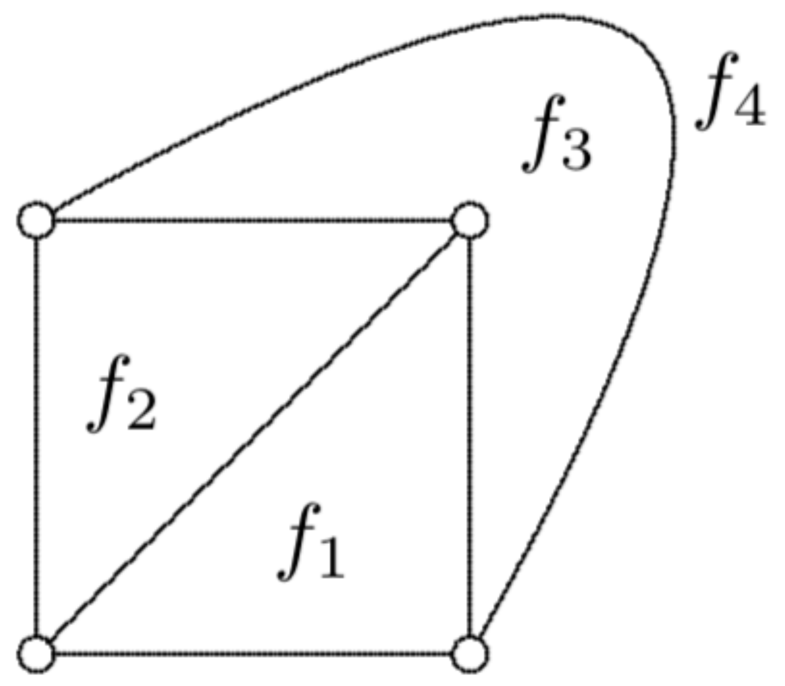


Figure 21: An embedding of K_4 in the plane and its faces

Graph

Planar graphs

Defn. The regions defined by the plane graph are called its *interior faces* and the unbounded region is called its *exterior face.*

Remk. If G is a tree, then G is planar and the number of faces of G is 1. The boundary of an interior face is the set of edges surrounding it. Every edge is a boundary of two faces.

Graph

Planar graphs

Thm. If a connected planar graph G has p vertices and q edges and f faces then

$$p - q + f = 2.$$

Cor. Let G be a planar graph of order p and size q. If each face of G is in an n-cycle, then

$$q = \frac{n(p-2)}{n-2}.$$

Graph

Planar graphs

Defn. A *maximal planar graph* is a graph in which no edge can be added without losing planarity.

Cor. Let G be a graph of order p and size q.

1. If G is a maximal planar graph, every face is a triangle and $q = 3p - 6$.
2. If G is planar in which every face is a 4-cycle, then $q = 2p - 4$.

Graph

Planar graphs

Remk. From the previous corollary, the maximum number of edges in a place occurs when each face is a triangle, we have a necessary condition for planarity of a graph in terms of the number of edges as given in the next corollary.

Cor. If G is a planar graph of order p and size q with $p \geq 3$, then $q \leq 3p - 6$. Furthermore, if G has no triangles, then $q \leq 2p - 4$.

Cor. The graphs K_5 and $K_{3,3}$ are nonplanar.

Graph

Planar graphs

Defn. Let G be a graph. A graph H is said to be a *subdivision* of a graph G if H can be obtained from G by successively inserting a vertex in an edge of G.

Exmp. The graph H is a subdivision of G given in Figure 24 below.

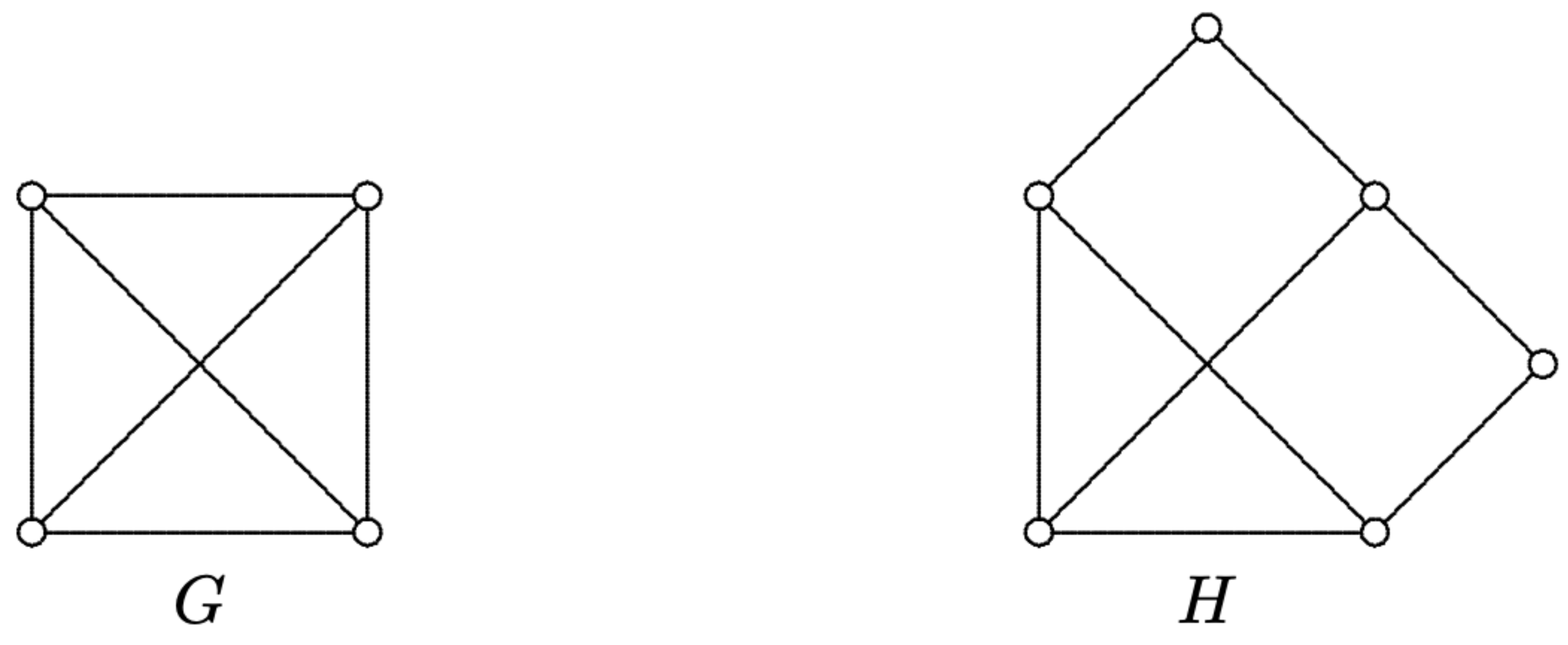


Figure 24: Subdivision H of G

Graph

Planar graphs

Thm. 5.4 (Kuratowski's Theorem)
Let G be a graph. Then, $G$is planar if and only if G contains a subgraph that is a subdivision of either $K_{3,3}$ or K_5.