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Assignment 3

Problem 1

$\mathcal{C}_n$ is a family of regular ($J = 2$, K) LDPC codes. n is the length of $\mathcal{C}_n$, the distance is d_n.

So a sequence of codes $\mathcal{C}_i : \mathcal{F}^{k_i} \to \mathcal{F}^{n_i}$ with distances d_i is asymptotically good iff exists constants δ such that:

$$\frac{d_i}{n_i} \geq \delta,\, n \to \infty$$

Basically we need to derive formula for d depending on n. If we try to construct Tanner graph, there will be nothing we can take out of it. Let's try instead represent Tanner graph in the different way. As we remember there are two types of nodes in the graph, the first type is check node (with K terms), the second one is symbol nodes (each of symbol node is contained in two check nodes).

Let's pick some symbol node and depict two branches coming out (which marked as check nodes). In fact each of two branches connects $K - 1$ nodes (in a loop free way, as there are no cycles so far). Then we proceed to the next layer and obtain $2(K-1)^2$ distinct symbol nodes. On the m-th layer, we will have $2(K-1)^m$ distinct symbol nodes for each digit.

The procedure is repeated till we get a loop in a tree (Fig. 1). So, let $(m+1)$-st level be a level with a loop, hence the following statement should hold for the m-th layer:

$$2(K-1)^m \leq n$$

If we derive m:

$$m \leq \frac{\ln(\frac{n}{2})}{\ln(k-1)}$$

Now, let's try to connect minimum distance and depth m. To construct a codeword with maximum weight, we simply need to iterate vertically through the tree (remember, each node is met one time in path due to specificity of construction) till we get a loop and set the value of each symbol (or digit respectively) to be equal 1. So the maximum weight we can expect is $2m + 2$ (1 for base node, 1 for loop, m for forward path, m for backward path).

Therefore:

$$d_{min} \leq 2m + 2$$

Putting all together we get:

$$d_{min} \leq \frac{2\ln(\frac{n}{2})}{\ln(k-1)} + 2$$

So, to check if it is asymptotically good, we need to divide d_{min} by n and check if it goes to zero whilst $n \to \infty$:

$$\frac{d_{min}}{n} \leq \frac{\frac{2\ln(\frac{n}{2})}{\ln(k-1)} + 2}{n}$$

$$\lim_{n\to\infty}\left(\frac{\frac{2\ln(\frac{n}{2})}{\ln(k-1)} + 2}{n}\right) = 0$$

Finally:

$$\boxed{\frac{d_{min}}{n} \to 0,\, n \to \infty}$$

This family is not asymptotically good.

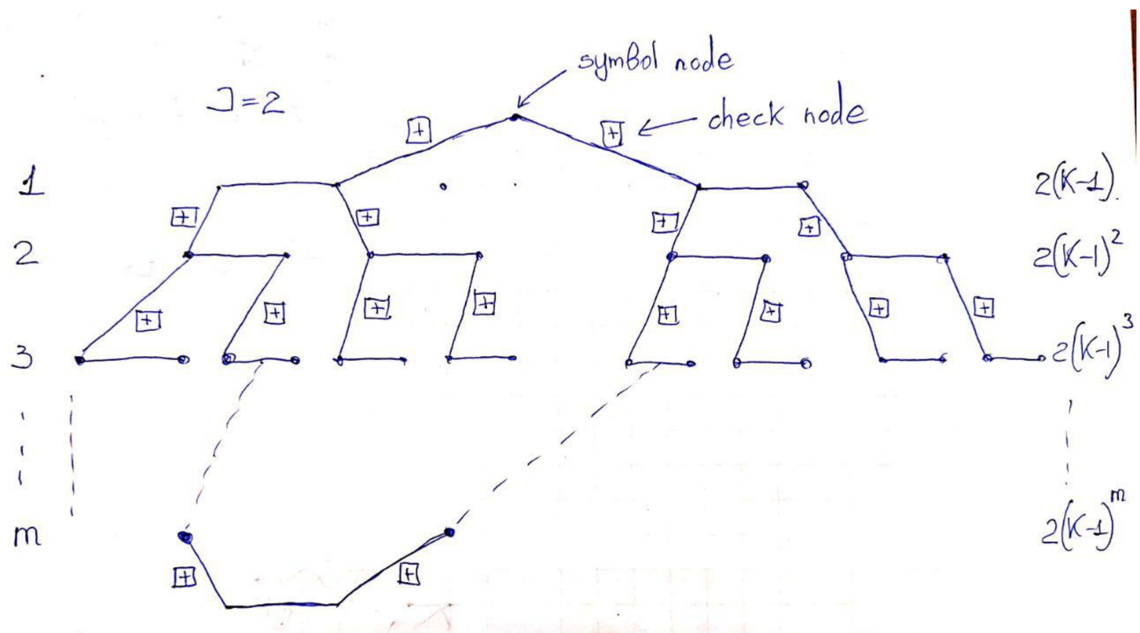

Figure 1: Unrolled Tanner graph

Problem 2

To remind, the bit-wise MAP decoding has the following form:

$$x_i = \arg\max_{x_i=\{\pm 1\}} \sum_{\neg x_i} \left(\prod_j \Pr(y_j|x_j)\right)$$

Compared to MAP bit-wise decoding algorithm we need to consider the whole vector (this is in fact called block-wise decoding):

$$\boldsymbol{x} = \arg\max_{x \in \mathcal{C}} = \Pr(x|y)$$

Using the fact that the channel is discrete and memoryless we can expand it to:

$$\boldsymbol{x} = \arg\max_{x \in \mathcal{C}} \Pr(x|y) = \arg\max_{x} \Pr(y|x)\Pr(x) = \arg\max_{x} \Big(\prod_j \Pr(y_j|x_j)\Big)\mathbb{1}_{x \in \mathcal{C}}$$

Let's consider i-th bit of x vector:

$$(x)_i = \arg\max_{x_i \in \{\pm 1\}} \max_{\neg x_i} \Big(\prod_j \Pr(y_j|x_j)\Big)\mathbb{1}_{x \in \mathcal{C}}$$

Using log-likelihood we obtain:

$$(x)_i = \arg\max_{x_i \in \{\pm 1\}} \max_{\neg x_i} \sum_j \log \Pr(y_j|x_j) + \log(\mathbb{1}_{x \in \mathcal{C}})$$

Then we need to change update rule from this one:

$$\mu_i(x) = \sum_{\neg x_i} f(x_1, ..., x_J) \prod_{j=1}^{J} \mu_j(x_j)$$

To that one:

$$\mu_i(x) = \max_{\neg x_i} f(x_1, ..., x_J) \prod_{j=1}^{J} \mu_j(x_j)$$

Problem 3

For vector y to be a codeword with a given H it should satisfy:

$$Hy = 0$$

We know the structure of H and can represent it in a following way:

$$P_{i1}a_1 + P_{i2}a_2 + .. + P_{in}a_n = 0$$

where a_i is denoted as s successive rows in vector y.

Keeping in mind the structure of permutation matrix we need to set at least two different a_i to ones to satisfy this equation. This implies y_j for these a_i being ones.

d_{min} is a minimum weight of codeword. So minimum number of ones in vector y for the equation to be satisfied is $2s$. Thereby:

$$\boxed{d_{min} \leq 2s}$$