Lecture 11

The category of small categories

Recall definition of Cat:

- objects are all small categories
- ightharpoonup morphisms in Cat(C, D) are all functors $C \to D$
- composition and identity morphisms as for functors

Cat has a terminal object

The category

0 id_0

one object, one morphism

is terminal in Cat

Cat has binary products

Given small categories $C, D \in Cat$, their product $C \leftarrow C \times D \xrightarrow{\pi_2} D$ is:

Cat has binary products

Given small categories $C, D \in Cat$, their product $C \xleftarrow{\pi_1} C \times D \xrightarrow{\pi_2} D$ is:

- ▶ objects of $\mathbb{C} \times \mathbb{D}$ are pairs (X, Y) where $X \in \mathbb{C}$ and $Y \in \mathbb{D}$
- ► morphisms $(X, Y) \rightarrow (X', Y')$ in $\mathbb{C} \times \mathbb{D}$ are pairs (f, g) where $f \in \mathbb{C}(X, X')$ and $g \in \mathbb{D}(Y, Y')$
- composition and identity morphisms are given by those of C
 (in the first component) and D (in the second component)

$$\begin{cases}
\pi_1 \left((X, Y) \xrightarrow{(f,g)} (X', Y') \right) = X \xrightarrow{f} X' \\
\pi_2 \left((X, Y) \xrightarrow{(f,g)} (X', Y') \right) = Y \xrightarrow{g} Y'
\end{cases}$$

Cat not only has finite products, it is also cartesian closed.

Exponentials in Cat are called functor categories.

To define them we need to consider natural transformations, which are the appropriate notion of morphism between functors.

Natural transformations

Motivating example: fix a set $S \in Set$ and consider the two functors $F, G : Set \rightarrow Set$ given by

$$F\left(X \xrightarrow{f} Y\right) = S \times X \xrightarrow{\mathrm{id}_{S} \times f} S \times Y$$

$$G\left(X \xrightarrow{f} Y\right) = X \times S \xrightarrow{f \times \mathrm{id}_{S}} Y \times S$$

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For each $X \in \mathbf{Set}$ there is an isomorphism (bijection) $\theta_X : FX \cong GX$ in \mathbf{Set} given by $\langle \pi_2, \pi_1 \rangle : S \times X \to X \times S$.

These isomorphisms do not depend on the particular nature of each set X (they are "polymorphic in X"). One way to make this precise is...

...if we change from X to Y along a function $f: X \to Y$, then we get a commutative diagram in Set:

The square commutes because for all $s \in S$ and $x \in X$

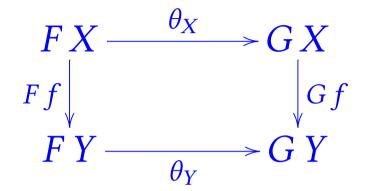
$$\langle \pi_2, \pi_1 \rangle ((id \times f)(s, x)) = \langle \pi_2, \pi_1 \rangle (s, f x)$$

$$= (f x, s)$$

$$= (f \times id)(x, s)$$

$$= (f \times id)(\langle \pi_2, \pi_1 \rangle (s, x))$$

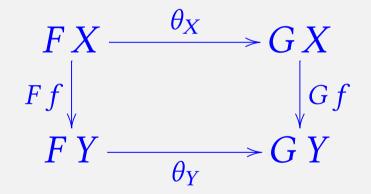
...if we change from X to Y along a function $f: X \to Y$, then we get a commutative diagram in Set:



We say that the family $(\theta_X \mid X \in \mathbf{Set})$ is natural in X.

Natural transformations

Definition. Given categories and functors $F, G : \mathbb{C} \to \mathbb{D}$, a natural transformation $\theta : F \to G$ is a family of D-morphisms $\theta_X \in D(FX, GX)$, one for each $X \in \mathbb{C}$, such that for all \mathbb{C} -morphisms $f : X \to Y$, the diagram



commutes in **D**, that is, $\theta_Y \circ F f = G f \circ \theta_X$.

Example

Recall forgetful (U) and free (F) functors:

$$\mathbf{Set} \underbrace{\overset{U}{\underbrace{\hspace{1cm}}}}_{F} \mathbf{Mon}$$

There is a natural transformation $\eta : id_{Set} \rightarrow U \circ F$, where for each $\Sigma \in Set$

$$\eta_{\Sigma}: \Sigma \to U(F\Sigma) = \operatorname{List}\Sigma$$
 $a \in \Sigma \mapsto [a] \in \operatorname{List}\Sigma$ (one-element list)

(Easy to see that
$$\Sigma$$
 \longrightarrow $U(F\Sigma)$ commutes.)
$$f \bigvee_{\Sigma'} U(Ff) \bigvee_{\Sigma'} U(F\Sigma')$$

Example

The covariant powerset functor $\mathscr{P}: \mathbf{Set} \to \mathbf{Set}$ is

$$\mathcal{P}X \triangleq \{S \mid S \subseteq X\}$$

$$\mathcal{P}\left(X \xrightarrow{f} Y\right) \triangleq \mathcal{P}X \xrightarrow{\mathcal{P}f} \mathcal{P}Y$$

$$S \mapsto \mathcal{P}fS \triangleq \{f \mid x \mid x \in S\}$$

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There is a natural transformation $\cup : \mathscr{P} \circ \mathscr{P} \to \mathscr{P}$ whose component at $X \in \mathbf{Set}$ sends $\mathcal{S} \in \mathscr{P}(\mathscr{P}X)$ to

$$\bigcup_{X} \mathcal{S} \triangleq \{x \in X \mid \exists S \in \mathcal{S}, \ x \in S\} \in \mathcal{P} X$$

(check that \cup_X is natural in X)

Non-example

The classic example of an "un-natural transformation" (the one that caused Eilenburg and MacLane to invent the concept of naturality) is the linear isomorphism between a finite dimensional real vectorspace V and its dual V^* (= vectorspace of linear functions $V \to \mathbb{R}$).

Both V and V^* have the same finite dimension, so are isomorphic by choosing bases; but there is no choice of basis for each V that makes the family of isomorphisms natural in V.

For a similar, more elementary non-example, see Ex. Sh. 5, question 4.

Composing natural transformations

Given functors $F, G, H : \mathbb{C} \to \mathbb{D}$ and natural transformations $\theta : F \to G$ and $\varphi : G \to H$,

we get $\varphi \circ \theta : F \to H$ with

$$(\varphi \circ \theta)_X = \left(FX \xrightarrow{\theta_X} GX \xrightarrow{\varphi_X} HX \right)$$

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Check naturality:

$$\begin{split} Hf \circ (\varphi \circ \theta)_X &\triangleq Hf \circ \varphi_X \circ \theta_X \\ &= \varphi_Y \circ Gf \circ \theta_X \\ &= \varphi_Y \circ \theta_Y \circ Ff \\ &\triangleq (\varphi \circ \theta)_Y \circ Ff \end{split} \qquad \text{naturality of } \theta \end{split}$$

Identity natural transformation

Given a functor $F: \mathbb{C} \to \mathbb{D}$, we get a natural transformation $id_F: F \to F$ with

$$(\mathrm{id}_F)_X = FX \xrightarrow{\mathrm{id}_{FX}} FX$$

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Check naturality:

$$Ff \circ (\mathrm{id}_F)_X \triangleq Ff \circ \mathrm{id}_{FX} = Ff = \mathrm{id}_{FY} \circ Ff \triangleq (\mathrm{id}_F)_Y \circ Ff$$

Functor categories

It is easy to see that composition and identities for natural transformations satisfy

$$(\psi \circ \varphi) \circ \theta = \psi \circ (\varphi \circ \theta)$$
$$id_G \circ \theta = \theta \circ id_F$$

so that we get a category:

Definition. Given categories C and D, the functor category D^C has

- ightharpoonup objects are all functors $C \rightarrow D$
- ▶ given $F, G : \mathbb{C} \to \mathbb{D}$, morphism from F to G in $\mathbb{D}^{\mathbb{C}}$ are the natural transformations $F \to G$
- composition and identity morphisms as above

If \mathcal{U} is a Grothendieck universe, then for each $X \in \mathcal{U}$ and $F \in \mathcal{U}^X$ we have that their dependent product and dependent function sets

$$\sum_{x \in X} F x \triangleq \{(x, y) \mid x \in X \land y \in F x\}$$

$$\prod_{x \in X} F x \triangleq \{f \subseteq \sum_{x \in X} F x \mid f \text{ is single-valued and total}\}$$

are also in \mathcal{U} ; and as a special case (of \prod , when F is a constant function with value Y) we also have that $X, Y \in \mathcal{U}$ implies $Y^X \in \mathcal{U}$.

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If C and D are small categories, then so is D^C.

because

obj
$$(\mathbf{D}^{\mathbf{C}}) \subseteq \sum_{F \in (\text{obj} D)^{\text{obj} \mathbf{C}}} \prod_{X,Y \in \text{obj} \mathbf{C}} \mathbf{D}(FX, FY)^{\mathbf{C}(X,Y)}$$

 $\mathbf{D}^{\mathbf{C}}(F,G) \subseteq \prod_{X \in \text{obj} \mathbf{C}} \mathbf{D}(FX, GX)$

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Aim to show that functor category D^{C} is the exponential of C and D in Cat...

Cat is cartesian closed

Theorem. There is an application functor

$$app: D^C \times C \to D$$

that makes D^C the exponential for C and D in Cat.

Given $(F, X) \in \mathbf{D}^{\mathbb{C}} \times \mathbf{C}$, we define

$$app(F, X) \triangleq F X$$

and given $(\theta, f) : (F, X) \to (G, Y)$ in $\mathbf{D}^{\mathbf{C}} \times \mathbf{C}$, we define

$$\operatorname{app}\left((F,X) \xrightarrow{(\theta,f)} (G,Y)\right) \triangleq FX \xrightarrow{Ff} FY \xrightarrow{\theta_Y} GY$$
$$= FX \xrightarrow{\theta_X} GX \xrightarrow{Gf} GY$$

Check:
$$\begin{cases} \mathsf{app}(\mathsf{id}_F,\mathsf{id}_X) &= \mathsf{id}_{FX} \\ \mathsf{app}(\varphi \circ \theta, g \circ f) &= \mathsf{app}(\varphi, g) \circ \mathsf{app}(\theta, f) \end{cases}$$

Cat is cartesian closed

Theorem. There is an application functor

$$\texttt{app}: \mathbf{D}^{\mathbf{C}} \times \mathbf{C} \to \mathbf{D}$$

that makes D^{C} the exponential for C and D in Cat.

Definition of currying: given functor $F : \mathbf{E} \times \mathbf{C} \to \mathbf{D}$, we get a functor $\mathbf{cur} F : \mathbf{E} \to \mathbf{D}^{\mathbf{C}}$ as follows. For each $Z \in \mathbf{E}$, $\mathbf{cur} F Z \in \mathbf{D}^{\mathbf{C}}$ is the functor

$$\operatorname{cur} F Z \begin{pmatrix} X \\ \downarrow f \\ X' \end{pmatrix} \triangleq \begin{cases} F(Z, X) \\ \downarrow F(\operatorname{id}_Z, f) \\ F(Z, X') \end{cases}$$

For each $g: Z \to Z'$ in \mathbf{E} , $\operatorname{cur} F g: \operatorname{cur} F Z \to \operatorname{cur} F Z'$ is the natural transformation whose component at each $X \in \mathbf{C}$ is

$$(\operatorname{cur} F q)_X \triangleq F(q, \operatorname{id}_X) : F(Z, X) \to F(Z', X)$$

(Check that this is natural in X; and that $\operatorname{cur} F$ preserves composition and identities in E.)

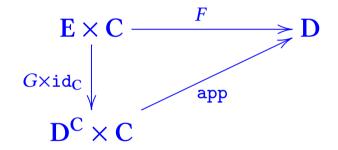
Cat is cartesian closed

Theorem. There is an application functor

$$\mathsf{app}:\mathsf{D}^\mathsf{C} imes\mathsf{C} o\mathsf{D}$$

 $app: \overset{\cdot}{D}^{C} \times C \to D$ that makes $\overset{\cdot}{D}^{C}$ the exponential for C and D in Cat.

Have to check that $\operatorname{cur} F$ is the unique functor $G : E \to D^{\mathbb{C}}$ that makes



commute in Cat (exercise).