Lecture 14

Dependent Types

A brief look at some category theory for modelling type theories with dependent types.

Will restrict attention to the case of Set, rather than in full generality.

Further reading:

M. Hofmann, Syntax and Semantics of Dependent Types. In: A.M. Pitts and

P. Dybjer (eds), Semantics and Logics of Computation (CUP, 1997).

Simple types

$$\diamond, x_1: T_1, \ldots, x_n: T_n \vdash t(x_1, \ldots, x_n): T$$

Dependent types

$$\diamond, x_1 : T_1, \ldots, x_n : T_n \vdash t(x_1, \ldots, x_n) : T(x_1, \ldots, x_n)$$

and more generally

$$\diamond$$
, $x_1:T_1$, $x_2:T_2(x_1)$, $x_3:T_3(x_1,x_2)$, ... \vdash $t(x_1,x_2,x_3,\ldots):T(x_1,x_2,x_3,\ldots)$

If type expressions denote sets, then

a type $T_1(x)$ dependent upon x:T

should denote

an indexed family of sets $(E i | i \in I)$ (where I is the set denoted by type T)

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i.e. $E: I \rightarrow \mathbf{Set}$ is a set-valued function on a set I.

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For each $I \in Set$, let Set^{I} be the category with

- ▶ obj(Set^I) \triangleq (obj Set)^I, so objects are *I*-indexed families of sets, $X = (X_i \mid i \in I)$
- ► morphisms $f: X \to Y$ in \mathbf{Set}^I are I-indexed families of functions $f = (f_i \in \mathbf{Set}(X_i, Y_i) \mid i \in I)$
- ► composition: $(g \circ f) \triangleq (g_i \circ f_i \mid i \in I)$ (i.e. use composition of functions in **Set** at each index $i \in I$)
- ► identity: $id_X \triangleq (id_{X_i} \mid i \in I)$ (i.e. use identity functions in **Set** at each index $i \in I$)

For each $p:I\to J$ in Set, let $p^*:\operatorname{Set}^J\to\operatorname{Set}^I$ be the functor defined by:

$$p^* \begin{pmatrix} Y_j \\ | f_j \\ Y'_j \end{pmatrix} \stackrel{\triangleq}{=} \begin{pmatrix} Y_{pi} \\ | f_{pi} \\ Y'_{pi} \end{pmatrix}$$

i.e. p^* takes J-indexed families of sets/functions to I-indexed ones by precomposing with p

Dependent products

of families of sets

For $I, J \in \mathbf{Set}$, consider the functor $\pi_1^* : \mathbf{Set}^I \to \mathbf{Set}^{I \times J}$ induced by precomposition with the first projection function $\pi_1 : I \times J \to I$.

Theorem. π_1^* has a left adjoint $\Sigma : \mathbf{Set}^{I \times J} \to \mathbf{Set}^{I}$.

Proof. We apply the <u>Theorem</u> from Lecture 13: for each $E \in \mathbf{Set}^{I \times J}$ we define $\Sigma E \in \mathbf{Set}^{I}$ and $\eta_E : E \to \pi_1^*(\Sigma E)$ in $\mathbf{Set}^{I \times J}$ with the required universal property...

For each $E \in \mathbf{Set}^{I \times J}$, define $\Sigma E \in \mathbf{Set}^{I}$ to be the function mapping each $i \in I$ to the set

$$(\Sigma E)_i \triangleq \sum_{j \in J} E_{(i,j)} = \{ (j,e) \mid j \in J \land e \in E_{(i,j)} \}$$

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and define $\eta_E : E \to \pi_1^*(\Sigma E)$ in $\mathbf{Set}^{I \times J}$ to be the function mapping each $(i,j) \in I \times J$ to the function $(\eta_E)_{(i,j)} : E_{(i,j)} \to (\Sigma E)_i$ given by $e \mapsto (j,e)$.

Universal property-

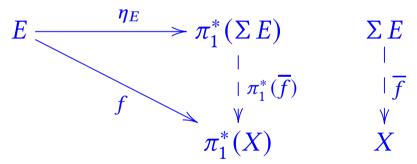
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Universal property–*existence part*: given any $X \in \mathbf{Set}^I$ and $f : E \to \pi_1^*(X)$ in

 $\mathbf{Set}^{I \times J}$, we have



where for all $i \in I$, $j \in J$ and $e \in E_{(i,j)}$ $\overline{f}_i(j,e) \triangleq f_{(i,j)}(e)$

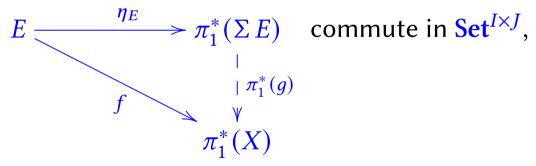
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Universal property–uniqueness part: given $g: \Sigma E \to X$ in \mathbf{Set}^I making



then for all $i \in I$, and $(j, e) \in (\Sigma E)_i$ we have

$$\overline{f}_{i}(j,e) \triangleq f_{(i,j)}(e) = (\pi_{1}^{*}g \circ \eta_{E})_{(i,j)} e = (\pi_{1}^{*}g)_{(i,j)}((\eta_{E})_{(i,j)} e) \triangleq g_{i}(j,e)$$

so
$$g = \overline{f}$$
. \square

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Dependent functions

of families of sets

We have seen that the left adjoint to $\pi_1^* : \mathbf{Set}^I \to \mathbf{Set}^{I \times J}$ is given by dependent products of sets.

Dually, dependent function sets give:

Theorem. π_1^* has a right adjoint $\Pi : \mathbf{Set}^{I \times J} \to \mathbf{Set}^{I}$.

Proof. We apply the <u>Theorem</u> from Lecture 13: for each $E \in \mathbf{Set}^{I \times J}$ we define $\Pi E \in \mathbf{Set}^I$ and $\varepsilon_E : \pi_1^*(\Pi E) \to E$ in $\mathbf{Set}^{I \times J}$ with the required universal property...

For each $E \in \mathbf{Set}^{I \times J}$, define $\Pi E \in \mathbf{Set}^{I}$ to be the function mapping each $i \in I$ to the set

$$(\Pi E)_i \triangleq \prod_{j \in J} E_{(i,j)} = \{ f \subseteq (\Sigma E)_i \mid f \text{ is single-value and total} \}$$

where $f \subseteq (\Sigma E)_i$ is

single-valued if $\forall j \in J, \forall e, e' \in E_{(i,j)}, \ (j,e) \in f \land (j,e') \in f \Rightarrow e = e'$ total if $\forall j \in J, \exists e \in E_{(i,j)} \ (j,e) \in f$

Thus each $f \in (\Pi E)_i$ is a dependently typed function mapping elements $j \in J$ to elements of $E_{(i,j)}$ (result set depends on the argument j).

For each $E \in \mathbf{Set}^{I \times J}$, define $\Pi E \in \mathbf{Set}^{I}$ to be the function mapping each $i \in I$ to the set

$$(\Pi E)_i \triangleq \prod_{j \in J} E_{(i,j)} = \{ f \subseteq (\Sigma E)_i \mid f \text{ is single-value and total} \}$$

and define $\varepsilon_E : \pi_1^*(\Pi E) \to E$ in $\mathbf{Set}^{I \times J}$ to be the function mapping each $(i,j) \in I \times J$ to the function $(\varepsilon_E)_{(i,j)} : (\Pi E)_i \to E_{(i,j)}$ given by $f \mapsto f f = \mathbf{I}$ unique $e \in E_{(i,j)}$ such that $(j,e) \in f$.

Universal property-

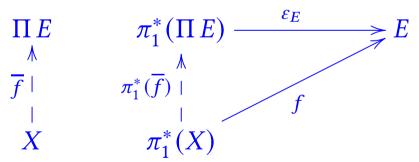
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Universal property–existence part: given any $X \in \mathbf{Set}^I$ and $f: \pi_1^*(X) \to E$ in

$$\mathbf{Set}^{I \times J}$$
, we have



where for all
$$i \in I$$
 and $x \in X_i$ $\overline{f}_i x \triangleq \{(j, f_{(i,j)} x) \mid j \in J\}$

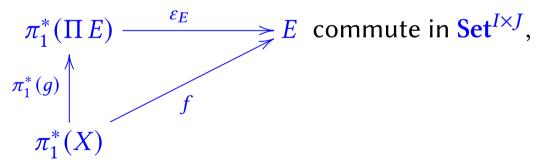
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For each $E \in \mathbf{Set}^{I \times J}$, define $\Pi E \in \mathbf{Set}^{I}$ to be the function mapping each $i \in I$ to the set

$$(\prod E)_i \triangleq \prod_{j \in J} E_{(i,j)} = \{ f \subseteq (\sum E)_i \mid f \text{ is single-value and total} \}$$

and define $\varepsilon_E : \pi_1^*(\Pi E) \to E$ in $\mathbf{Set}^{I \times J}$ to be the function mapping each $(i,j) \in I \times J$ to the function $(\varepsilon_E)_{(i,j)} : (\Pi E)_i \to E_{(i,j)}$ given by $f \mapsto f f = \mathbf{I}$ unique $e \in E_{(i,j)}$ such that $(j,e) \in f$.

Universal property–uniqueness part: given $g: X \to \Pi E$ in Set^I making



then for all $i \in I$, $j \in J$ and $x \in X_i$ we have

$$\overline{f}_i x j \triangleq f_{(i,j)} x = (\varepsilon_E \circ \pi_1^* g)_{(i,j)} x = (\varepsilon_E)_{(i,j)} (g_i x) \triangleq g_i x j$$

so
$$g = \overline{f}$$
. \square

Isomorphism of categories

Two categories C and D are isomorphic if they are isomorphic objects in the category of all categories of some given size, that is, if there are functors

$$C \xrightarrow{F} D$$
 with $id_C = G \circ F$ and $F \circ G = id_D$.

In which case, as usual, we write $\mathbb{C} \cong \mathbb{D}$.

Equivalence of categories

Two categories C and D are equivalent if there are functors $C \overset{F}{\longrightarrow} D$ and natural isomorphisms $\eta: \mathrm{id}_C \cong G \circ F$ and $\varepsilon: F \circ G \cong \mathrm{id}_D$. In which case, one writes $C \cong D$.

Equivalence of categories

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Some deep results in mathematics take the form of equivalences of categories. E.g.

Stone duality:
$$\binom{\text{category of }}{\text{Boolean algebras}}^{\text{op}} \simeq \binom{\text{category of compact}}{\text{totally disconnected}}$$

Hausdorff spaces

Gelfand duality: $\binom{\text{category of }}{\text{abelian } C^* \text{ algebras}}^{\text{op}} \simeq \binom{\text{category of compact}}{\text{Hausdorff spaces}}$

Example: $Set^I \simeq Set/I$

Set/*I* is a slice category:

- ▶ objects are pairs (E, p) where $E \in \text{obj Set}$ and $p \in \text{Set}(E, I)$
- ► morphisms $g:(E,p) \to (E',p')$ are $f \in Set(E,E')$ satisfying $p' \circ f = p$ in Set
- composition and identities as for Set

Example: $Set^I \simeq Set/I$

There are functors $F: \mathbf{Set}^I \to \mathbf{Set}/I$ and $G: \mathbf{Set}/I \to \mathbf{Set}^I$, given on objects and morphisms by:

$$FX \triangleq (\{(i,x) \mid i \in I \land x \in X_i\}, fst)$$
 $Ff(i,x) \triangleq (i,f_ix)$
 $G(E,p) \triangleq (\{e \in E \mid pe=i\} \mid i \in I)$
 $(Gf)_i e \triangleq f e$

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 $(Gf)_i e \triangleq f e$

There are natural isomorphisms

$$\eta : \mathrm{id}_{\mathrm{Set}^I} \cong G \circ F \text{ and } \varepsilon : F \circ G \cong \mathrm{id}_{\mathrm{Set}/I}$$

defined by...[exercise]

FACT Given $p: I \rightarrow J$ in **Set**, the composition

$$\mathbf{Set}/J \simeq \mathbf{Set}^J \xrightarrow{p^*} \mathbf{Set}^I \simeq \mathbf{Set}/I$$

is the functor "pullback along p".

One can generalize from Set to any category C with pullbacks and model Σ/Π types by left/right adjoints to pullback functors – see locally cartesian closed categories in the literature.