ESERCIZI F.G.M.

Calcolare la **FGM** delle variabili aleatorie:

Discrete:

- $X \in \{1, 2, 3, 4, 5, 6\}$ Uniforme Discreta

$$G_X(t) = E[e^{tX}] = \sum_{a \in \{1,\dots,6\}} e^{ta} \cdot f(a) = \sum_{a \in \{1,\dots,6\}} e^{ta} \cdot \frac{1}{6} = \frac{1}{6} \cdot \sum_{a \in \{1,\dots,6\}} e^{ta} = \boxed{\frac{1}{6} \cdot \frac{e^t(1-e^{6t})}{1-e^t}}.$$

- $Y \sim Ber(p)$

$$G_Y(t) = E[e^{tX}] = \sum_{a \in \{0,1\}} e^{ta} \cdot P_X(a) = e^{t0} \cdot (1-p) + e^{t1} \cdot p = \boxed{e^t p + (1-p)}.$$

- $Z \sim Geom(p)$

$$G_Z(t) = E[e^{tX}] = \sum_{K=1}^{\infty} e^{tK} \cdot \underbrace{P_X(K)}_{(1-p)^{K-1} \cdot p} = p \cdot \sum_{K=1}^{\infty} e^{tK} \cdot (1-p)^{K-1}$$

$$= p \cdot \sum_{K=1}^{\infty} (e^t)^{K-1+1} \cdot (1-p)^{K-1} = p \cdot e^t \cdot \sum_{K=1}^{\infty} (e^t \cdot (1-p))^{K-1}$$

$$= p \cdot e^t \cdot \sum_{m=0}^{\infty} (e^t \cdot (1-p))^m \overset{e^t(1-p)<1}{=} p \cdot e^t \cdot \frac{1}{1-e^t(1-p)} = \boxed{\begin{cases} \dfrac{p \cdot e^t}{1-e^t(1-p)} & \text{SE} \quad t < \ln\left(\frac{1}{1-p}\right) \\ \infty & \text{SE} \quad t \geq \ln\left(\frac{1}{1-p}\right) \end{cases}}.$$

Continue:

- $Z \sim N(0,1)$

$\boxed{\mu = 0, \ \sigma^2 = 1}$

$$G_Z(t) = E[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} \cdot f(x) \cdot dx = \int_{-\infty}^{+\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \cdot dx = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{tx} \cdot e^{-\frac{x^2}{2}} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{tx - \frac{x^2}{2}} \cdot dx = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2 - tx)} \cdot dx = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[(x-t)^2 - t^2]} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{(x-t)^2}{2}} \cdot e^{\frac{t^2}{2}} \cdot dx = e^{\frac{t^2}{2}} \cdot \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-t)^2}{2}} \cdot dx}_{} = \boxed{e^{\frac{t^2}{2}}}.$$

$$\boxed{x^2 - tx = x^2 - tx + t^2 - t^2 = (x-t)^2 - t^2}$$

N(0,1) con $y = x - t$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} \cdot dy = 1$$

- $X \sim N(\mu,\sigma^2)$
- $Y \sim Exp(\lambda)$
- $X \sim Par(\alpha)$, per quali valori di t, G(t) è finita?

Calcolare il momento terzo di Z ~ N(0,1)

$$G_X(t) = e^{\frac{t^2}{2}}$$

$E[X^3] = G'''_X(0)$

$G'_X(t) = \left(e^{\frac{t^2}{2}}\right)' = e^{\frac{t^2}{2}} \cdot \frac{2t}{2} = t \cdot e^{\frac{t^2}{2}}$.

$G''_X(t) = \left(t \cdot e^{\frac{t^2}{2}}\right)' = 1 \cdot e^{\frac{t^2}{2}} + t \cdot t e^{\frac{t^2}{2}}$.

$G'''_X(t) = \left(e^{\frac{t^2}{2}} + t^2 \cdot e^{\frac{t^2}{2}}\right)' = t \cdot e^{\frac{t^2}{2}} + 2t \cdot e^{\frac{t^2}{2}} + t^3 \cdot e^{\frac{t^2}{2}}$

$= 3t \cdot e^{\frac{t^2}{2}} + t^3 \cdot e^{\frac{t^2}{2}}$

$G'''_X(0) = 3 0 \cdot e^{\frac{0^2}{2}} + 0^3 \cdot e^{\frac{0^2}{2}} = \boxed{0}$.

Calcolare il momento terzo di Y ~ Exp(λ)

$$G_X(t) = \begin{cases} \frac{\lambda}{\lambda - t} & t < \lambda \\ +\infty & t \geq \lambda \end{cases}$$

$E[X^3] = G'''_X(0) \qquad t < \lambda \qquad \lambda > 0$

$G'_X(t) = \left(\frac{\lambda}{\lambda - t}\right)' = \lambda \cdot \left(\frac{1}{\lambda - t}\right)' = \lambda \cdot \left(-\frac{(\lambda - t)'}{(\lambda - t)^2}\right) = \lambda \cdot \left(-\frac{-1}{(\lambda - t)^2}\right) = \frac{\lambda}{(\lambda - t)^2}$

$G''_X(t) = \left(\frac{\lambda}{(\lambda - t)^2}\right)' = \lambda \cdot \left(-2 \cdot \frac{1}{(\lambda - t)^3} \cdot (-1)\right) = \frac{2\lambda}{(\lambda - t)^3}$

$G'''_X(t) = \left(\frac{2\lambda}{(\lambda - t)^3}\right)' = 2\lambda \cdot \left(-3 \cdot \frac{1}{(\lambda - t)^4} \cdot (-1)\right) = \frac{6\lambda}{(\lambda - t)^4}$

$G'''_X(0) = \frac{6\lambda}{(\lambda - 0)^4} = \boxed{\frac{6}{\lambda^3}}$ CON $\lambda > 0$.

FATTORE COSTANTE

$(c \cdot g(x))' = c \cdot (g(x))'$

REGOLA DELLA FUNZIONE RECIPROCA

$D\left[\frac{1}{v}\right] = D[v^{-1}] = D[-1 \cdot v^{-2}] = D\left[-\frac{1}{v^2}\right]$

Calcolare il valore atteso di X ~ U(0,1) col metodo della FGM

$G_X(t) = E[e^{tx}] = \int_0^1 e^{tx} \cdot f(x) \cdot dx = \int_0^1 e^{tx} \cdot \frac{1}{1-0} \cdot dx = \int_0^1 e^{tx} \cdot dx = \left.\frac{e^{tx}}{t}\right|_0^1 = \boxed{\frac{e^t - 1}{t}} \quad t \neq 0.$

$G'_X(t) = \frac{t \cdot e^t - (e^t - 1)}{t^2} = \frac{e^t(t-1) + 1}{t^2}$, $\lim_{t \to 0} G'_X(t) = \frac{(1 + \frac{t^2}{2} + \ldots)(t-1) + 1}{t^2}$

$t \to 0$

NUMERATORE: $(1 + t + \frac{t^2}{2})(t-1) + 1 = (t - 1 + t^2 - t + \frac{t^3}{2} + \frac{t^2}{2}) + 1 = \frac{t^2}{2} \pm o(t^2)$

$G'_X(t) \approx \frac{\frac{t^2}{2}}{t^2} = \boxed{\frac{1}{2}}$.

MOMENTI ESPONENZIALE

$$E[X^m \sim Exp(\lambda)] = \frac{m! \cdot \lambda}{(\lambda - t)^{m+1}} \xrightarrow{t=0} \frac{m!}{\lambda^m}$$,

MOMENTI NATURALE STANDARD

$$E[Z^{2k} \sim N(0,1)] = (2k-1)!! = \frac{(2k)!}{2^k \cdot k!}$$

$$E[Z^{2k+1} \sim N(0,1)] = 0.$$

La prof all'esame può chiedere di calcolare la **FGM**, Gx(t), per una variabile aleatoria **non nota**! Quindi é importante esercitarsi anche su questi.

$$\boxed{G_X(t) = E[e^{tx}] = \ldots}$$

Calcolare la FGM della v.a. X ~ Pois(u)

$$G_X(t) = E[e^{tX}] = \sum_{m=0}^{\infty} e^{tm} \cdot P_X(m) = \sum_{m=0}^{\infty} e^{tm} \cdot \frac{\mu^m \cdot e^{-\mu}}{m!} = e^{-\mu} \cdot \underbrace{\sum_{m=0}^{\infty} \frac{(\mu \cdot e^t)^m}{m!}}_{\text{SERIE DI TAYLOR}}$$

$$= e^{-\mu} \cdot e^{\mu e^t} = \boxed{e^{\mu(e^t-1)} \quad \forall t \in \mathbb{R}}$$

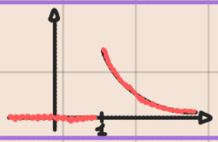
POISSON

$$P_X(m) = \frac{\mu^m}{m!} \cdot e^{-\mu}$$

Calcolare la FGM della v.a. X ~ Par(a)

PARETO

$$f(x) = \frac{\alpha}{x^{\alpha+1}} \cdot \mathbb{1}_{(1,+\infty)}(x) = \begin{cases} \frac{\alpha}{x^{\alpha+1}} & \text{SE } x \in (1,+\infty) \\ 0 & \text{SE } x \notin (1,+\infty) \end{cases}$$



$$G_X(t) = E[e^{tX}] = \int_1^{+\infty} e^{tx} \cdot \frac{\alpha}{x^{\alpha+1}} \cdot dx$$

SE $t \geq 0$:

$$\lim_{x \to \infty} \frac{e^x}{x^{\alpha+1}} = +\infty \implies \text{ALLORA L'INTEGRALE SICURAMENTE DIVERGE} \implies \alpha \cdot \int_1^{+\infty} \frac{e^{tx}}{x^{\alpha+1}} \cdot dx = +\infty \implies \boxed{G_X(t) = +\infty}$$

SE $t < 0$:

$$\alpha \cdot \int_1^{+\infty} \frac{e^{tx}}{x^{\alpha+1}} \cdot dx < +\infty \implies \boxed{G_X(t) < +\infty}.$$

ABBIAMO DIMOSTRATO CHE: $X \sim Par(\alpha) \implies G_X(t) < +\infty \iff t < 0$.

Calcolare la FGM della v.a. X ~ Exp(λ)

ESPONENZIALE

$$f(x) = \lambda e^{-\lambda x} \cdot \mathbb{1}_{(0,+\infty)}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{SE } x \in (0,+\infty) \\ 0 & \text{SE } x \notin (0,+\infty) \end{cases}$$

$$G_X(t) = E[e^{tX}] = \int_0^{+\infty} e^{tx} \cdot \lambda e^{-\lambda x} \cdot dx = \lambda \cdot \int_0^{+\infty} e^{-(\lambda-t)x} \cdot dx$$

CASO 1) $\lambda - t > 0$:

$$\lambda \cdot \int_0^{+\infty} e^{-(\lambda-t)x} \cdot dx = \frac{\lambda}{(\lambda-t)} \cdot \int_0^{+\infty} (\lambda-t) \cdot e^{-(\lambda-t)x} \cdot dx = \frac{\lambda}{\lambda-t} \cdot \overbrace{\text{Exp}(\lambda-t)}^{1} = \boxed{\frac{\lambda}{\lambda-t}}.$$

CASO 2) $\lambda - t \leq 0$:

$$\lambda \cdot \int_0^{+\infty} e^{-(\lambda-t)x} \cdot dx = \boxed{+\infty}.$$

QUINDI:

$$G_X(t) = \begin{cases} \frac{\lambda}{\lambda-t} & \text{SE } \lambda > t \\ 0 & \text{SE } \lambda \leq t \end{cases}$$