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Calcolare la FGM delle variabili aleatorie:

Discrete:

- $X \in \{1, 2, 3, 4, 5, 6\}$ Uniforme Discreta

$$G_X(t) = E[e^{tx}] = \sum_{x \in \{1, \dots, 6\}} e^{tx} \cdot f(x) = \sum_{x \in \{1, \dots, 6\}} e^{tx} \cdot \frac{1}{6} = \frac{1}{6} \cdot \sum_{x \in \{1, \dots, 6\}} e^{tx} = \boxed{\frac{1}{6} \cdot \frac{e^t(1-e^{6t})}{1-e^t}}$$

- $Y \sim \text{Ber}(p)$

$$G_Y(t) = E[e^{ty}] = \sum_{y \in \{0, 1\}} e^{ty} \cdot p_y(y) = e^{t \cdot 0} \cdot (1-p) + e^{t \cdot 1} \cdot p = \boxed{e^t p + (1-p)}$$

- $Z \sim \text{Geom}(p)$

$$\begin{aligned} G_Z(t) &= E[e^{tz}] = \sum_{k=1}^{\infty} e^{tk} \cdot P_Z(k) = p \cdot \sum_{k=1}^{\infty} e^{tk} \cdot (1-p)^{k-1} \\ &= p \cdot \sum_{k=1}^{\infty} (e^t)^{k-1+1} \cdot (1-p)^{k-1} = p \cdot e^t \cdot \sum_{k=1}^{\infty} (e^t \cdot (1-p))^{k-1} \\ &\quad e^t \cdot (1-p) < 1 \\ &= p \cdot e^t \cdot \sum_{m=0}^{\infty} (e^t \cdot (1-p))^m \stackrel{?}{=} p \cdot e^t \cdot \frac{1}{1 - e^t \cdot (1-p)} = \boxed{\begin{cases} \frac{p \cdot e^t}{1 - e^t \cdot (1-p)} & \text{se } t < \ln(\frac{1}{1-p}) \\ \infty & \text{se } t \geq \ln(\frac{1}{1-p}) \end{cases}} \end{aligned}$$

Continue:

- $Z \sim N(0, 1)$

$$\mu = 0, \sigma^2 = 1$$

$$\begin{aligned} G_Z(t) &= E[e^{tz}] = \int_{-\infty}^{+\infty} e^{tx} \cdot f(x) \cdot dx = \int_{-\infty}^{+\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \cdot dx = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{tx} \cdot e^{-\frac{x^2}{2}} \cdot dx \\ &= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{t x - \frac{x^2}{2}} \cdot dx = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x^2 - tx)} \cdot dx \stackrel{?}{=} \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[(x-t)^2 - t^2]} \cdot dx \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{(x-t)^2}{2}} \cdot e^{\frac{t^2}{2}} \cdot dx = e^{\frac{t^2}{2}} \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-t)^2}{2}} \cdot dx = \boxed{e^{\frac{t^2}{2}}}.$$

$$x^2 - tx = x^2 - tx + t^2 - t^2 = (x-t)^2 - t^2$$

$$N(0, 1) \text{ con } y = x - t$$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} \cdot dy = 1$$

- $X \sim N(\mu, \sigma^2)$

- $Y \sim \text{Exp}(\lambda)$

- $X \sim \text{Par}(\alpha)$, per quali valori di t , $G(t)$ è finita?

Calcolare il momento terzo di $Z \sim N(0,1)$

$$E[X^3] = G_x'''(0)$$

$$G'_x(t) = \left(e^{\frac{t^2}{2}}\right)' = e^{\frac{t^2}{2}} \cdot \frac{2t}{2} = t \cdot e^{\frac{t^2}{2}}.$$

$$G''_x(t) = \left(t \cdot e^{\frac{t^2}{2}}\right)' = 1 \cdot e^{\frac{t^2}{2}} + t \cdot te^{\frac{t^2}{2}}.$$

$$G'''_x(t) = \left(e^{\frac{t^2}{2}} + t^2 \cdot e^{\frac{t^2}{2}}\right)' = t \cdot e^{\frac{t^2}{2}} + 2t \cdot e^{\frac{t^2}{2}} + t^3 \cdot e^{\frac{t^2}{2}} \\ = 3t \cdot e^{\frac{t^2}{2}} + t^3 \cdot e^{\frac{t^2}{2}}$$

$$G'''_x(0) = 3 \cdot 0 \cdot e^{\frac{0^2}{2}} + 0^3 \cdot e^{\frac{0^2}{2}} = \boxed{0}.$$

$$G_x(t) = e^{\frac{t^2}{2}}$$

Calcolare il momento terzo di $Y \sim \text{Exp}(\lambda)$

$$E[X^3] = G_x'''(0) \quad t < \lambda \quad \lambda > 0$$

$$G'_x(t) = \left(\frac{\lambda}{\lambda-t}\right)' = \lambda \cdot \left(\frac{1}{\lambda-t}\right)' = \lambda \cdot \left(-\frac{1}{(\lambda-t)^2}\right) = \lambda \cdot \left(-\frac{1}{(\lambda-t)^2}\right) = \frac{\lambda}{(\lambda-t)^2}$$

$$G''_x(t) = \left(\frac{\lambda}{(\lambda-t)^2}\right)' = \lambda \cdot \left(-2 \cdot \frac{1}{(\lambda-t)^3} \cdot (-1)\right) = \frac{2\lambda}{(\lambda-t)^3}$$

$$G'''_x(t) = \left(\frac{2\lambda}{(\lambda-t)^3}\right)' = 2\lambda \cdot \left(-3 \cdot \frac{1}{(\lambda-t)^4} \cdot (-1)\right) = \frac{6\lambda}{(\lambda-t)^4}$$

$$G'''_x(0) = \frac{6\lambda}{(\lambda-0)^4} = \boxed{\frac{6}{\lambda^3}} \quad \text{CON } \lambda > 0.$$

$$G_x(t) = \begin{cases} \frac{\lambda}{\lambda-t} & t < \lambda \\ +\infty & t \geq \lambda \end{cases}$$

FATTORE COSTANTE

$$(c \cdot g(x))' = c \cdot (g(x))'$$

REGOLA DELLA FUNZIONE RECIPROCA

$$D\left[\frac{1}{v}\right] = D[v^{-1}] = D[-1 \cdot v^{-2}] = D[-\frac{1}{v^2}]$$

Calcolare il valore atteso di $X \sim U(0,1)$ col metodo della FGM

$$G_x(t) = E[e^{tx}] = \int_0^1 e^{tx} \cdot f(x) \cdot dx = \int_0^1 e^{tx} \cdot \frac{1}{1-0} \cdot dx = \int_0^1 e^{tx} \cdot dx = \frac{e^{tx}}{t} \Big|_0^1 = \frac{e^t - 1}{t} \quad t \neq 0.$$

$$G'_x(t) = \frac{t \cdot e^t - (e^t - 1)}{t^2} = \frac{e^t(t-1)+1}{t^2}, \quad \lim_{t \rightarrow 0} G'_x(t) = \frac{(1 + \frac{t^2}{2} + \dots)(t-1) + 1}{t^2}$$

$t \rightarrow 0$
NUMERATORE: $(1 + t + \frac{t^2}{2})(t-1) + 1 = (t-1 + t^2 - t + \frac{t^3}{2} + \frac{t^2}{2}) + 1 = \frac{t^2}{2} + o(t^2)$

$$G'_x(t) \approx \frac{\frac{t^2}{2}}{t^2} = \boxed{\frac{1}{2}}.$$

MOMENTI ESPONENZIALE

$$E[X^m \sim \text{Exp}(\lambda)] = \frac{m! \cdot \lambda}{(\lambda-t)^{m+1}} \xrightarrow{t=0} \frac{m!}{\lambda^m}$$

MOMENTI NATURALE STANDARD

$$E[Z^{2k} \sim N(0,1)] = (2k-1)!! = \frac{(2k)!}{2^k \cdot k!}$$

$$E[Z^{2k+1} \sim N(0,1)] = 0.$$