Contents

1 Grids and tables

In Olympiad combinatorics, especially in olympiads like IZhO, All-Russian, and Saint-Petersburg, very often encounters problems with grids. In this chapter, I have tried to write every single idea or method that I have encountered during my preparation for olympiad mathematics. Let us start with the simplest one:

1.1 Invariants

Invariants are the most popular and the most efficient method to crack grid problems, but invariants might be very tricky and extremely non-obvious. This thought will become clearer through numerous examples in this chapter. As usual, we will start with the simplest one.

1.1.1 Invariants as a method of solution

The most popular type of problem involving the grids and tables are formulated in the following way:

You have some board (not necessary rectangular) and you need to divide/tile it into some pieces (not necessary same type or rectangular by shape). Can you do it?

or like this

You have some board (again, not necessary rectangular) and you need to place/move figure(s) on the board, so that some criteria are met.

In this types of problems it is useful to color the board in such a way that some properties are constant – they are *invariant*.
One of such problems is the following one

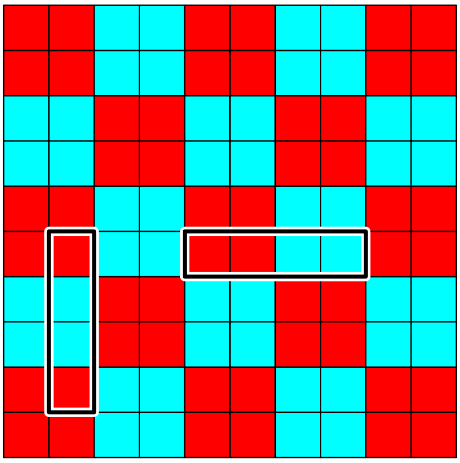
Example 1.1.1
Prove that a 10×10 *board cannot be cut along the grid lines into* 1×4 *rectangles.*

In this problem the thing we want to preserve – is the content of each 1×4 rectangle.
Solution 1:
Firstly, to not overcomplicate things, let's try to use as few color as possible – 2. With such a limitation in colors natural and obvious thing to attach to is the following *invariant*:

Every 1×4 *rectangle contains 2 of red and cyan cells.*

With this idea in the mind we need to create the coloring for which this is true. We will construct it dynamically (pretend that we are artist coloring the canvas as we go), start by doing the most general thing you can:
Place the 1×4 in the corner of the table and paint 2 cell red and other 2 of cyan. Now shift your 1×4 rectangle by 1, new rectangle already has 2 cyan cells and 1 red cell, so the remaining cell must be red. Continuing this logic we will get to one line of the table colored. But that is only one dimension, we need 2.

Fortunately, simply by rotating obtained coloring and sometimes inverting the colors we get to the following:



Now the board has 52 red cells and 48 cyan ones, i.e. not an equal number. This means that it will not be possible to cut the 10×10 board into 1×4 tetrominoes.

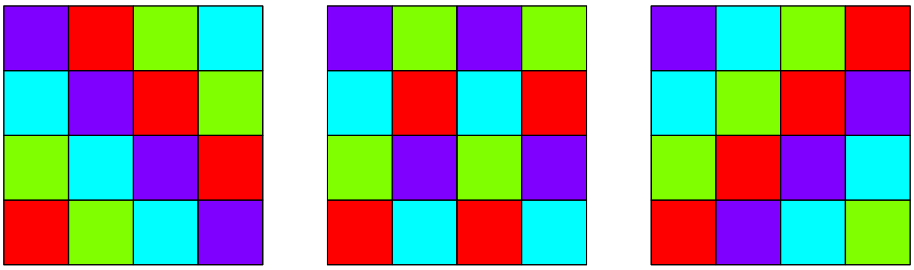
Solution 2:

As you may have noticed, after first 4 cells (in any direction) pattern repeats itself, so we can just focus on the one 4×4 board.

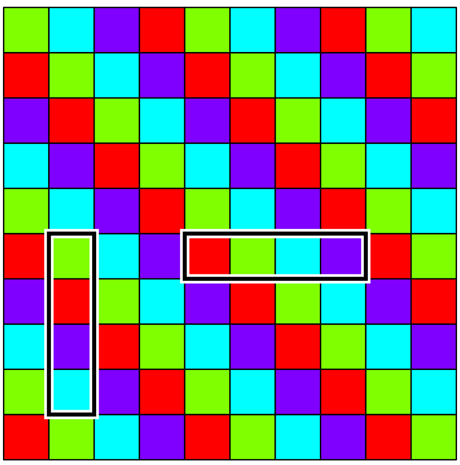
As you have the 4-celled figure you also may be tempted to use 4 colors, so if we change our invariant to the following:

Every 1×4 rectangle contains one cell of each of the 4 colors.

You can easily get to one of multiple working examples.



Now using the same logic as in the previous example, we can use one of such 4×4 tables as a blueprint and spam it to get this:



With this coloring, the board has 25 cells of the 1st and 3rd colors, 26 cells of the 2nd, and 24 cells of the 4th, i.e., not same number.

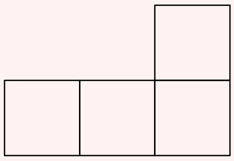
This means that it will not be possible to cut the 10×10 board into 1×4 tetrominoes.

This idea of coloring as we go and slightly changing the shape of our focus is the main idea for constructing such colorings.

In the previous example we encountered with the simplest colorings and invariants. The next example with the similar taste

Example 1.1.2

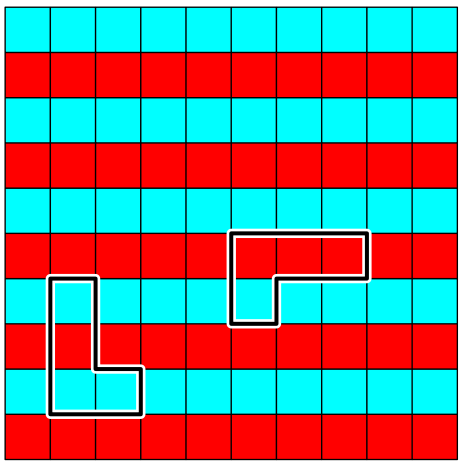
Is it possible to cover a 10×10 *board with the following pieces without them overlapping?*
Note: The pieces ***can*** *be flipped and/or turned.*



If you try to make the same steps as in the previous example (using 2 or 4 colors) you will quickly fail, only possible coloring being checkerboards pattern which is simply useless in this problem.

So we need to find a little more complex coloring:

Color the columns into red and cyan alternatively.



There are 50 cyan squares and 50 red squares. We can see that, regardless of how we place a piece, figure always covers 3 squares of one color and 1 of the other.

We call the piece cyan if it covers 3 cyan squares, and red if it covers three red squares.

The number of cyan pieces is equal to the number of red pieces. This tells us that the total number of pieces must be even. This would mean that the number of squares should be divisible by 8. Since there are 100 squares, there is no possible cover.

Example 1.1.3 (ISL 2023 C1)
Let m and n be positive integers greater than 1. In each unit square of an $m \times n$ grid lies a coin with its tail side up. A move consists of the following steps.

1. *select a 2×2 square in the grid;*
2. *flip the coins in the top-left and bottom-right unit squares;*
3. *flip the coin in either the top-right or bottom-left unit square.*

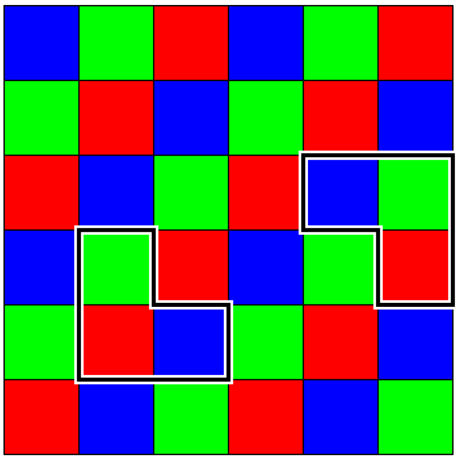
Determine all pairs (m, n) for which it is possible that every coin shows head-side up after a finite number of moves.

Answer: $\boxed{3 \mid m \text{ or } 3 \mid n}$.

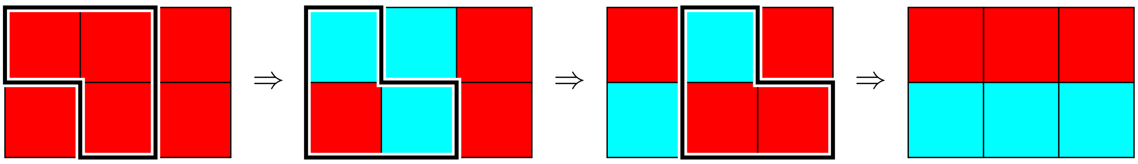
Only two possible moves consist of changing the state 3 cell corners with only difference between them being top-right or bottom-left cell.
And, as in the first example, we want to make a coloring such that every move is *invariant*, so we can make the top right and bottom left cells the same color. So we get to the following rule:

Make top right and bottom left cells in every 2×2 the same color

Continuously using this rule we get to the following coloring:



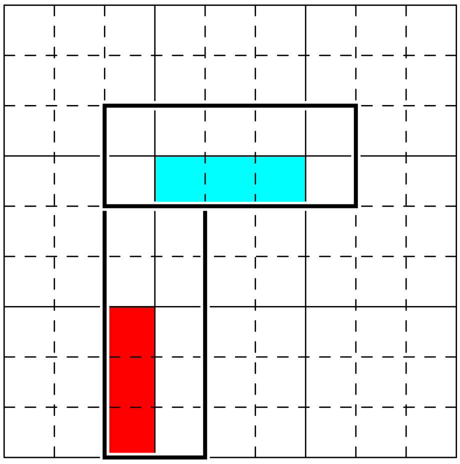
Note, that in every move one of each type of coin is flipped. Therefore, the amount of every color initially should have been the same. This is only achieved when $3 \mid mn$, so the bound is done.
For construction, it is easy to see we can always change the state of 2×3 rectangle. Now, the only thing we need to do is learn how to flip all the coins in a 1×3 array. This however can be done in the following way:



However, invariants are not only about colorings; rather, invariants about cuts and divisions. Next problem illustrates that idea more clearly.

Example 1.1.4
A 60×60 checkered square is divided into 2×5 tiles. Prove that it is possible to partition the square into 1×3 rectangles such that each 2×5 tile contains at least one rectangle entirely.

First, let us draw a marking that divides the original 60×60 square into 3×3 squares.



Note that each 2×5 tile has a side of length 5 divided by this marking into either two parts (their lengths are 2 and 3) or three parts (with lengths 1, 3, 1).

Therefore, on each tile there is a 1×3 rectangle that is not divided into parts by the marking lines (the figure shows examples of such rectangles), we will paint one such rectangle at each point. Obviously, the painted rectangles do not intersect, since they lie in different tiles.

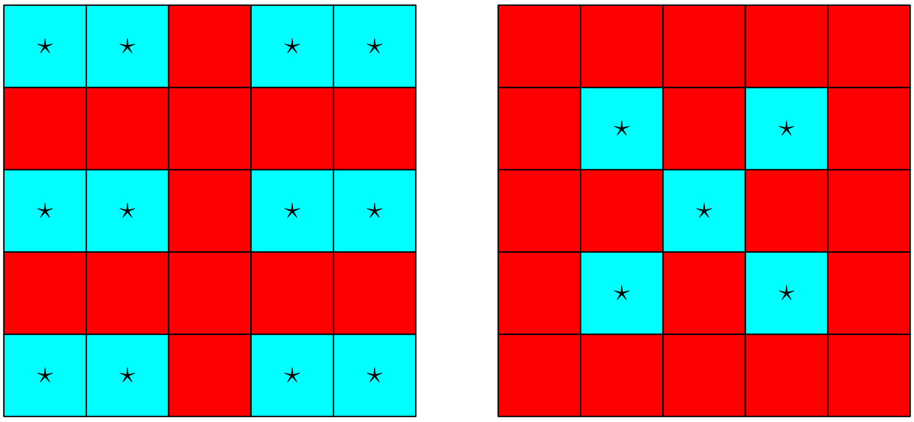
If in some 3×3 square more than one 1×3 rectangle is painted, then they lie in the same direction (all horizontally or all vertically), so they do not intersect.

Therefore, each square of the 3×3 marking can be divided into 1×3 rectangles in such a way that all the painted rectangles will belong to this division.

It might seem that invariant problems are kinda easy: just spam one of the basic colorings or cuts. However, invariant problems frequently can be extremely non-obvious and tricky, with mind-blowing colorings or cuts. In next 2 examples that idea will be clearer.

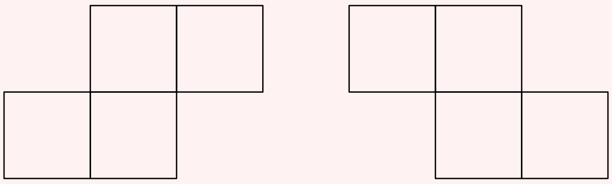
Example 1.1.5 (APMO 2007)
A regular 5×5 array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off.
After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.

Consider the following coloring:



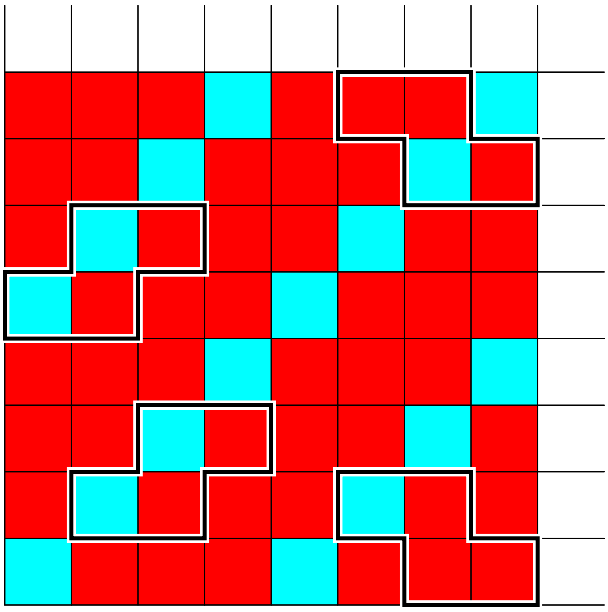
There are 12 cyan cells, and it is easily verifiable that any move on the board will affect 0 or 2 of these cells. Thus none of these can remain alone.
Rotating this coloring by 90 degrees eliminates all except for the 5 points.

Example 1.1.6 (ISL 2014 C4)
Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them $S-$ and $Z-$tetrominoes, respectively.
Assume that a lattice polygon P can be tiled with $S-$ tetrominoes. Prove that no matter how we tile P using only $S-$ and $Z-$tetrominoes, we always use an even number of $Z-$tetrominoes.
Note: tetrominoes cannot be rotated.



$S-$tetromino $Z-$tetromino

Color some cells of grid cyan, as a repeating of coloring below.



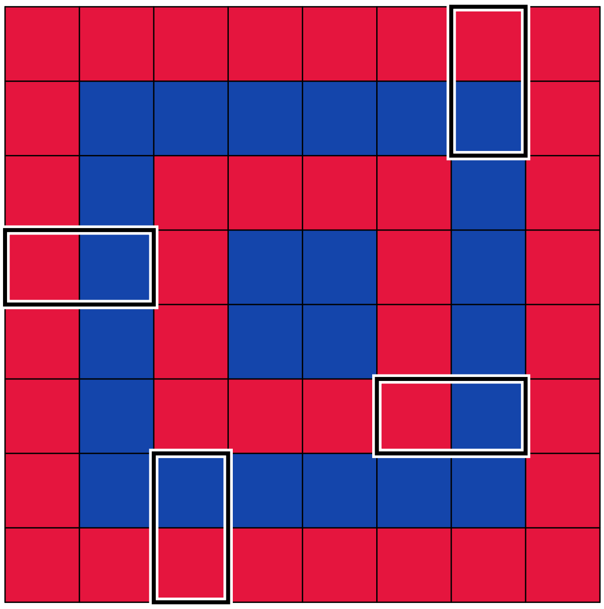
Observe that any $S-$tetramino covers even number of cyan cells, so P also cover even number of cyan. But any $Z-$tetramino covers an odd number of cyan cells, so the conclusion follows.

1.1.2 Invariants as a method of counting or bounding

Example 1.1.7 (EGMO 2019)
Given a natural number n, find the largest possible number of dominoes that can be placed on a $2n \times 2n$ checkered board so that any cell is adjacent to or belongs to exactly one domino.

Bound:

Consider the following coloring



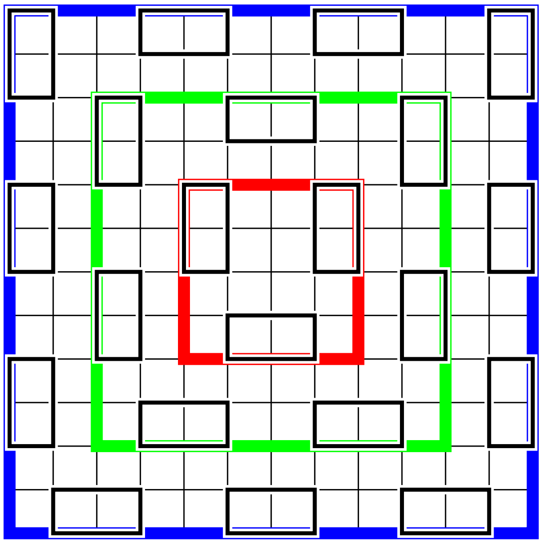
Note that every domino covers or is adjacent to exactly 4 red cells. So the maximum number of dominoes $= \frac{\text{number of red cells}}{4}$.

If $2 \mid n \implies$ number of red cells $= 3 + 7 + \cdots + (2n-1) = \frac{(2n-1)+3}{2} \cdot \frac{n}{2} = \frac{n(n+1)}{2}$.

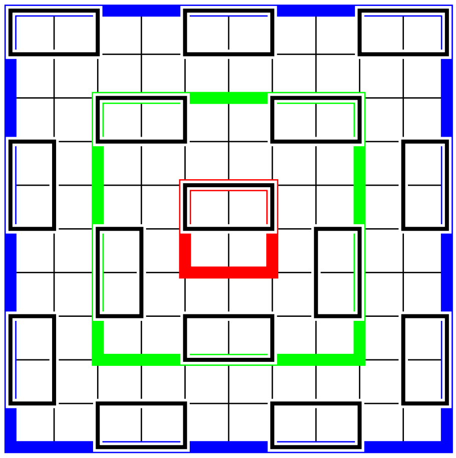
If $2 \nmid n \implies$ number of red cells $= 1 + 5 + \cdots + (2n-1) = \frac{(2n-1)+1}{2} \cdot \frac{n+1}{2} = \frac{n(n+1)}{2}$.

So the number of dominoes $= \frac{n(n+1)}{8}$

Example:



for even n



for odd n

Example 1.1.8 (All-Russian 2023)
Let n be an odd integer. In a $2n \times 2n$ board, we color $2(n-1)^2$ cells. What is the largest number of three-square corners that can surely be cut out of the uncolored figure?

Answer: $2n - 1$.

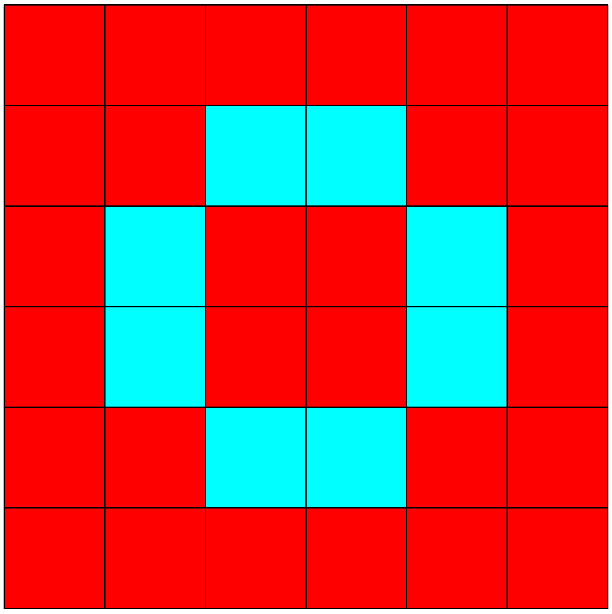
Bound:

Divide the grid into 2×2 squares. In order to stop a three corner from being cut out from one of these squares, there must be at least two squares shaded. Notice, there can only by $(n-1)^2$ such squares with at least two squares shaded. Thus, we can cut out three corners form at least $n^2 - (n-1)^2 = 2n - 1$ of them.

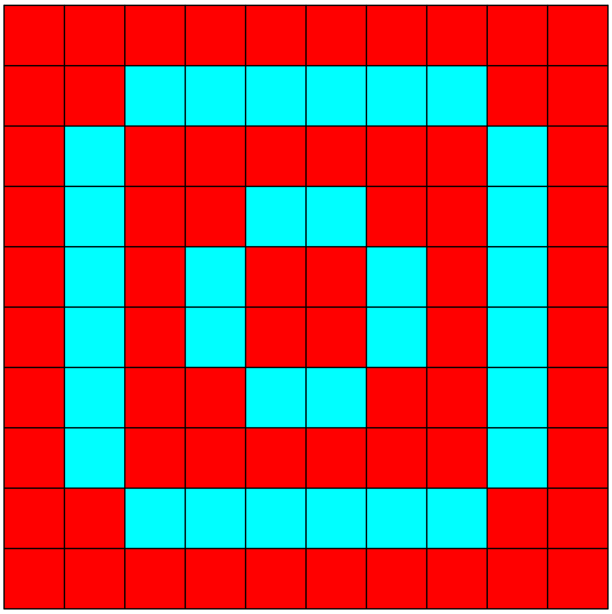
Construction:

We will proceed by induction. For $n = 3$, by bound we *should* be able to place only 5 three-square corners. so we need to distribute 5 empty 2×2 squares in such a way, that we cannot place any additional three-square corners.

Note that if we have 2 of empty 2×2 squares next to each other, it is very hard to avoid extra three-square corners from appearing, so trying to avoid it we get to the following construction:



When going from $n \to n+2$ since we are using induction it makes sense to use previous construction to our advantage, so placing a copy for the construction for $2n \times 2n$ in the center of the $2n+2 \times 2n+2$ grid feel natural. Then with the additional $2(n+1)^2 - 2(n-1)^2 = 4n$ squares surround the central $2n \times 2n$ construction. Then you are able to cut out 4 additional three corner pieces from the corners in addition to the $2n-1$ three corners from the center.

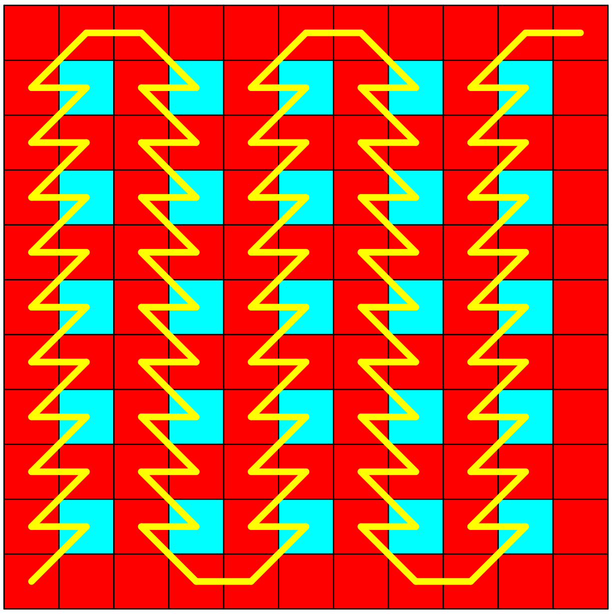


Example 1.1.9 (239 Olympiad 2022)
A chip is placed in the lower left-corner cell of the 15×15 board. We can move to the cells that are adjacent to the sides or the corners of its current cell. We must also alternate between horizontal and diagonal moves the first move must be diagonal What is the maximum number of moves it can make without stepping on the same cell twice

Answer: the chip can make 196 moves, having visited 197 cells (including the original one).

Bound:

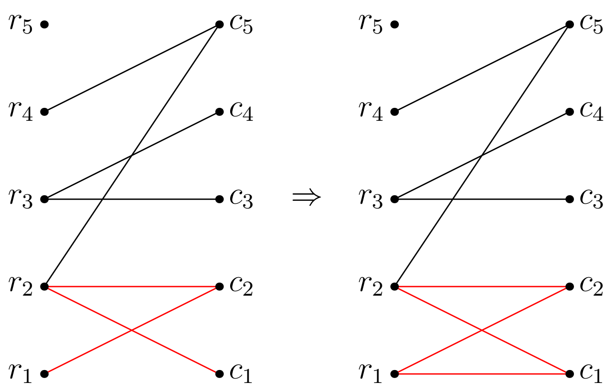
For the motivation, notice that we can change vertical position only by moving diagonally, so after 2 moves you change your horizontal position by 0 or 2, and your vertical position by 1. So to get to your initial position you must spend al least 4 moves. From this point you should consider all positions that you can get to after even number of moves. And from that point notice pattern in every 4th move.
Let's paint all 49 intersections of rows and columns with even numbers cyan. After the first move, the chip will end up in a cyan cell. After any 4 moves in a row from a cyan cell, the chip will end up on a cyan cell again. Thus, having made $1 + 49 \cdot 4 = 197$ moves, the chip will have to visit $1 + 49 = 50$ cyan cells. Such a number of cyan cells does not exist, so it will not be possible to make more than 196 moves.
Example:
First, the chip (moving up) goes around the two left columns without two cells, then the next two columns without two cells (this time moving down), then the next two columns, etc. At the end of the path, the chip will visit one cell from the last column. In total, it will visit 14 cells in each column except the last one, and one more cell in the last column, a total of 197 cells, i.e., it will make 196 moves.



1.2 Graphs

Example 1.2.1
In the 100×100 grid k cells are colored in black color. If at any moment 3 of the 4 cells that lie as the corners of rectangle with sides parallel to sides of the initial square are black, after a minute the fourth cell will also become black. What is the minimum k, such that after some time the whole square will turn black?

Answer: $\boxed{199}$.
First idea in graphs is turning table into bipartite graph, by assigning a vertex to each row and column and drawing an edge between them if corresponding intersection of row and column is colored.
Because of the way we set up rows and columns this idea works well if we are working with something related to horizontal or vertical movements. And this problem is perfect for demonstrating it.
So, formally:
Let us construct a bipartite graph with vertices $r_1, \ldots, r_{100}$ corresponding to the rows of the board, and vertices $c_1, \ldots, c_{100}$ corresponding to its columns. We connect vertices r_i and c_j by an edge if the cell at the intersection of the corresponding row and column is black.
So the move now consists of completing the cycle of length 4.



Notice that connectivity remains constant. That is, if any two vertices were connected before, they remain connected. And if not, then after the move they still remain unconnected.
As the final graph is connected, the initial should also be. So the total number of edges must be no less than 199 (cuz the tree).

Example 1.2.2 (All-Russian 2013)
Petya put several chips on a 50×50 board, no more than one in each cell. Prove that Vasya has a way to put no more than 99 new chips (possibly none) on the free squares of this same board so that there is still no more than one chip in each cell, and each row and each column of this board has an even number of chips.

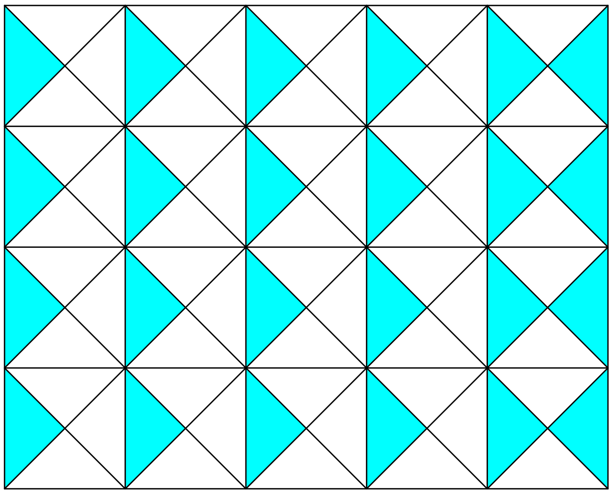
Construct the same bipartite graph as in previous example, but we connect vertices r_i and c_j by an edge if the cell is empty. Then Vasya's goal is reformulated as follows: it is required to mark no more than 99 edges so that an even number of unmarked edges emerge from each vertex. Indeed, if Vasya places chips in the cells corresponding to the marked edges, then an even number of free cells will remain in each row and each column.
We will prove a more general fact: in any graph on $n \geq 2$ vertices, it is possible to mark no more than $n-1$ edges so that an even number of unmarked edges emerge from each vertex.
Induction on n. The base ($n = 2$) is obvious.

The induction step. Let $n > 2$. If the graph contains a vertex of degree 0, then it suffices to throw it away and apply the induction hypothesis.
If there is a vertex of degree 1, then we can mark the only edge emanating from it, throw it away together with this edge, and apply the induction hypothesis to the remaining graph.
Now let the degree of each vertex be at least 2. Let us leave an arbitrary vertex along an edge, leave the vertex we arrived at along another edge, and so on; this process can be continued until we return to a vertex we have already visited. Thus, a cycle has been found in the graph. Throwing its edges out of the graph does not change the parities of the vertex degrees; therefore, we can just erase those edges. Applying the same process to it, sooner or later we will obtain a graph in which the degree of some vertex does not exceed 1; and for such graphs the assertion has already been proved.

1.3 Double counting

Example 1.3.1 (Saint Petersburg 2017)
In each cell of an $m \times n$ rectangle, two diagonals were drawn, resulting in the rectangle being divided into $4mn$ triangles. All triangles were painted black or white so that each white triangle has a common side with at least one black triangle. What is the smallest number of black triangles that could be in such a coloring?

Answer: $\boxed{mn + \min(m, n)}$.
Example:
WLOG assume that table has m rows and n columns, where $m < n$. In each cell we paint the left triangle black, and in each cell of the right column we paint another right triangle. In total we get $mn + m$ black triangles, and it is easy to see that this example satisfies the condition of the problem.



Bound:
Consider any row of a rectangle. It contains $2n$ "vertical" triangles (triangles with a vertical side). If such a triangle is painted white, draw an arrow leading from it to the adjacent black triangle (it lies in the same row). If the vertical triangle is black, draw an arrow from it to itself.
We drew $2n$ arrows in total. Note that no more than two arrows lead to each triangle in this row: arrows can lead to a horizontal triangle only from two vertical triangles in the same cell, and to a vertical triangle only from itself and the vertical triangle adjacent to it. Therefore, there are at least n black triangles in this row. Similarly, there are at least m black triangles in each column.
Let's go further. Note that if there are exactly n black triangles in some row, then exactly two arrows enter each of them. In particular, the leftmost vertical triangle is not painted black (otherwise the arrow would lead to it only from itself), so it is white. This means that the arrow from it leads to one of the neighboring horizontal triangles, i.e. this horizontal triangle is black. But then another arrow must lead to it, so the right vertical triangle in this cell is also white. Continuing in the same way, we find that in each cell of this row there is exactly one black triangle, and it is horizontal. Similarly, if in some column there are exactly m black triangles, then all of them are vertical, and there is one in each cell.
However, a row with exactly n black triangles and a column with exactly m black triangles cannot exist simultaneously: the cell at their intersection would contain exactly one black triangle, which must be both horizontal and vertical. In other words, either in each row there are at least $n+1$ black triangles, or in each column there are at least $m+1$ black triangles. In one case, there are no less than $m(n+1) = mn + m$ black triangles in total, in the other, no less than $n(m+1) = mn + n$, in any case, no less than $mn + \min(m, n)$.

Example 1.3.2 (All-Russian 2017)
Each cell of a 100×100 board is painted either black or white, and all cells adjacent to the board's border are black. It turns out that there is no single-color 2×2 checkered square anywhere on the board. Prove that there is a 2×2 checkered square on the board whose cells are painted in a checkerboard pattern.

FTSoC assume that there are no single-colored or checkerboard-colored 2×2 squares on the board.

Consider all the grid segments separating two cells of different colors (let's call them separators); let their number be N.

In any 2×2 square, there is either exactly one cell of one color and three cells of the other, or two adjacent white cells and two adjacent black cells. In both cases, there are exactly two separators inside the square.

There are 99^2 2×2 squares in total, and each separator lies inside exactly two of them (since the separators do not adjoin the border).

Therefore, $N = \frac{2(99^2)}{2} = 99^2$. In each row and each column, the first and last cell are black; therefore, there must be an even number of color changes, so the total number of separators must be even. Contradiction.

Example 1.3.3 (BMOSL 2019)
Determine the largest natural number N having the following property: every 5×5 array consisting of pairwise distinct natural numbers from 1 to 25 contains a 2×2 subarray of numbers whose sum is, at least, N.

Solution:
Answer: $k_{max} = 45$. First if all, we will enumerate the columns and rows. Then, we will select all possible $3^2 = 9$ choices for the odd column with an odd row. By collecting all such pairs of an odd column with an odd row, we double count some squares. Indeed, we took some 3^2 squares 5 times, some 12 squares 3 times, and there are some 4 squares (namely all the intersections of an even column with an even row) that we did not take in such pairs. It follows that the maximum total sum over all 32 choices of an odd column with an odd row is $5 \times (17+18+\cdots+25)+3 \times (5+6+\cdots+16) = 1323$. So, by averaging argument, there exists a pair of an odd column with an odd row $\frac{1323}{9}$ with sum at most $= 147$. Then all the other squares of the array will have sum at least $(1+2+\cdots+25) - 147 = 178$. However, for these squares there is a tiling with $2 \times 2 arrays$, which are 4 in total. So there is an 2 x 2 array, whose numbers have a sum at least $\frac{178}{4} > 44$. So, there is a 2×2 array whose numbers have a sum at least 45. This argument gives that $k_{max} \geq 45$. Here, we have a bound that $k_{max} \geq 45$, but now we should prove that $k = 45$ is optimal. Fortunately, this part looks less impressive than the previous part: we can do this by providing an example:

$$\begin{bmatrix} 23 & 13 & 19 & 12 & 24 \\ 7 & 2 & 9 & 4 & 5 \\ 18 & 16 & 17 & 15 & 21 \\ 8 & 1 & 10 & 3 & 6 \\ 22 & 14 & 20 & 11 & 25 \end{bmatrix}$$

1.4 Induction

Example 1.4.1
On an $n \times n$ chart where $n \geq 4$, stand n "$+$" signs in cells of one diagonal and a '$-$' sign in all the other cells. In a move, one can change all the signs in one row or in one column, ($-$ changes to $+$ and $+$ changes to $-$). Prove that it is impossible to reach a stage where there are fewer than n pluses on the board.

Note that operating twice on a row is equivalent to not operating on it at all. So we can assume that each row and column has been operated upon 0 or 1 times. Now we use induction on n.
The base case $n = 4$ is not entirely trivial, but is left to the reader in keeping with my general habit of dismissing base cases.
Now passing to the induction step, given an $n \times n$ board there are at least $(n-1)$ pluses in the bottom right $(n-1) \times (n-1)$ square by the induction hypothesis. If we have a plus in the first row or column we are done. Suppose there is no plus in the first column or row. Then either the first row or the first column (but not both) has been operated upon (otherwise the top left square would have a plus).
WLOG the first row has been operated upon. Then columns $2, 3, \ldots, n$ have all been operated upon (otherwise row 1 would have a plus). Also no other row has been operated upon (otherwise the first column would have a plus). But in this case, the lower right $(n-1) \times (n-1)$ square has had all its columns and none of its rows operated upon, and hence each column has $(n-2)$ pluses. In total it has $(n-2)(n-1) > n$ pluses, so in this case as well we are done.

1.5 Complex numbers

Example 1.5.1 (IMO 2016)
Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:

- *in each row and each column, one third of the entries are I, one third are M and one third are O*
- *on any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.*

We claim that $9 \mid n$ is the only answer.
For $n = 9$, consider the following table

I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M
I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M
I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M

Bound:
Since each of I, M, O appears the same number of times, $3 \mid n^2$ and so $3 \mid n \implies n = 3k$.
Tile the board with 3×3 sub-grids. For each 3×3 grid call the central square its representative cell.
Assign the numbers $1, \omega$ and ω^2 to I, M and O, respectively.
Add the numbers assigned to all rows and columns indexed with numbers $\equiv 2 \pmod 3$ and all numbers assigned to the diagonals with number of entries a multiple of 3. This total sum must be zero since each of the summands is zero for a row and a column and any diagonal having number of elements a multiple of 3.
However, the total sum is the sum of all entries of the $n \times n$ board and the sum of all numbers assigned to the representative cells. As the sum of all entries is zero, it follows that the sum of all numbers assigned to the representative cells is zero. Let x of them be labeled I, y of them be labeled M, z of them be labelled O. We have $x + y + z = k^2$ and $x + y\omega + z\omega^2 = 0$.
There are two ways to conclude that $x = y = z$ from here.
Notice that multiplying the last equation by ω and ω^2 gives $z + x\omega + y\omega^2 = 0$ and $y + z\omega + x\omega^2 = 0$.
Considering $1, \omega, \omega^2$ as constants and x, y, z as parameters, we see that this system of equations has a solution if and only if

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = \frac{1}{2}(x + y + z)\left((x - y)^2 + (y - z)^2 + (z - x)^2\right) = 0.$$

As $x + y + z = k^2 > 0$, we have $x = y = z$.
Alternatively, note that the polynomial $1+x+x^2$ is irreducible and has no double roots. The polynomial $x+yx+zx^2$ has a common root with $1+x+x^2$ so if $1+x+x^2 \nmid x+yx+zx^2$ then $x-\omega$ is a polynomial with integer coefficients. This is clearly false, so the former must hold and we have $x = y = z$.
Thus, $3 \mid x + y + z = k^2$ and $3 \mid k$. It follows that $9 \mid n$.

1.6 Problems

Problem 1.6.1 (All-Russian 2016) 1950 dominoes were cut out from 100×100 sheet of paper. Prove that it is possible to cut out a four-cell figure from the remaining part.
(If such a figure is already among the remaining parts, it is considered that it was cut out.)

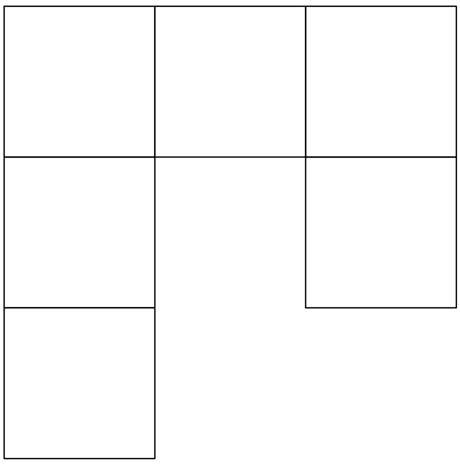
Problem 1.6.2 (EGMO 2016) Let m be a positive integer. Consider a $4m \times 4m$ array of square unit cells. Two different cells are *related* to each other if they are either in the same row or in the same column. No cell is related to itself. Some cells are colored blue, such that every cell is related to at least two blue cells. Determine the minimum number of blue cells.

Problem 1.6.3 On a 11×11 checkered board, 22 squares are marked so that exactly two squares are marked on each vertical and each horizontal. Two arrangements of marked squares are equivalent if, by changing the verticals and horizontals between each other any number of times, we can obtain the other arrangement from one. How many nonequivalent arrangements of marked squares are there?

Problem 1.6.4 (Italian TST 1995) An 8×8 board is tiled with 21 trominoes (3×1 tiles), so that exactly one square is not covered by a tromino. No two trominoes can overlap and no tromino can stick out of the board. Determine all possible positions of the square not covered by a tromino.

Problem 1.6.5 (IMO 1993) On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side n The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?

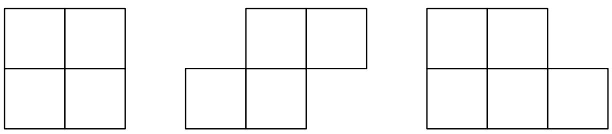
Problem 1.6.6 (IMO 2004) Define a "hook" to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



Determine all $m \times n$ rectangles that can be covered without gaps and without overlaps with hooks such that

- the rectangle is covered without gaps and without overlaps
- no part of a hook covers area outside the rectangle.

Problem 1.6.7 (Vietnam 1993) With the following shapes we tile a 1993×2000 board. Let s be the number of shapes used of the first two types. Find the largest possible value of s.



Problem 1.6.8 (Romania 2007) In an $n \times n$ board the squares are painted black or white. Three of the squares in the corners are white and one is black. Show that there is a 2×2 square with an odd number of white unit squares.

Problem 1.6.9 (USAJMO 2023) Consider an n-by-n board of unit squares for some odd positive integer n. We say that a collection C of identical dominoes is a maximal grid-aligned configuration on the board if C consists of $(n^2-1)/2$ dominoes where each domino covers exactly two neighboring squares and the dominoes don't overlap: C then covers all but one square on the board. We are allowed to slide (but not rotate) a domino on the board to cover the uncovered square, resulting in a new maximal grid-aligned configuration with another square uncovered. Let $k(C)$ be the number of distinct maximal grid-aligned configurations obtainable from C by repeatedly sliding dominoes. Find the maximum value of $k(C)$ as a function of n

Problem 1.6.10 (IZhO 2024) We are given $m \times n$ table tiled with 3×1 stripes and we are given that $6 \mid mn$. Prove that there exists a tiling of the table with 2×1 dominoes such that each of these stripes contains one whole domino.

Problem 1.6.11 (ISL 2010) 2500 chess kings have to be placed on a chessboard so that

- no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);
- each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

Problem 1.6.12 (ISL 2023) Let $n \geqslant 2$ be a positive integer. Paul has a $1 \times n^2$ rectangular strip consisting of n^2 unit squares, where the i^{th} square is labelled with i for all $1 \leqslant i \leqslant n^2$. He wishes to cut the strip into several pieces, where each piece consists of a number of consecutive unit squares, and then translate (without rotating or flipping) the pieces to obtain an $n \times n$ square satisfying the following property: if the unit square in the i^{th} row and j^{th} column is labelled with a_{ij}, then $a_{ij} - (i + j - 1)$ is divisible by n.
Determine the smallest number of pieces Paul needs to make in order to accomplish this.

Problem 1.6.13 (ISL 2022) Let n be a positive integer. A Nordic square is an $n \times n$ board containing all the integers from 1 to n^2 so that each cell contains exactly one number. Two different cells are considered adjacent if they share a common side. Every cell that is adjacent only to cells containing larger numbers is called a valley. An uphill path is a sequence of one or more cells such that:

- the first cell in the sequence is a valley,
- each subsequent cell in the sequence is adjacent to the previous cell,
- the numbers written in the cells in the sequence are in increasing order.

Find, as a function of n, the smallest possible total number of uphill paths in a Nordic square.

Problem 1.6.14 (All-Russian 2023) A 100×100 square is divided into 2×2 squares. It is then broken into dominoes. What is the smallest number of dominoes that could be inside the squares of the partition?

Problem 1.6.15 (All-Russian 2016) A 100×100 grid is given, the cells of which are painted black and white. In all columns there are equal numbers of black cells, while in all rows there are different numbers of black cells. What is the maximum possible number of pairs of adjacent multi-colored cells?

Problem 1.6.16 (Tuymaada 2021) In a $n \times n$ table $(n > 1)$ k unit squares are marked. One wants to rearrange rows and columns so that all the marked unit squares are above the main diagonal or on it. For what maximum is it always possible?

Problem 1.6.17 (All-Russian 2013) 400 three-cell corners (rotated as desired) and another 500 cells were cut out of 55×55 checkered square along the borders of the cells. Prove that some two cut out figures have common boundary segments.

Problem 1.6.18 (IZhO 2020) Some squares of a $n \times n$ table $(n > 2)$ are black, the rest are white. In every white square we write the number of all the black squares having at least one common vertex with it. Find the maximum possible sum of all these numbers.

2 Algorithms

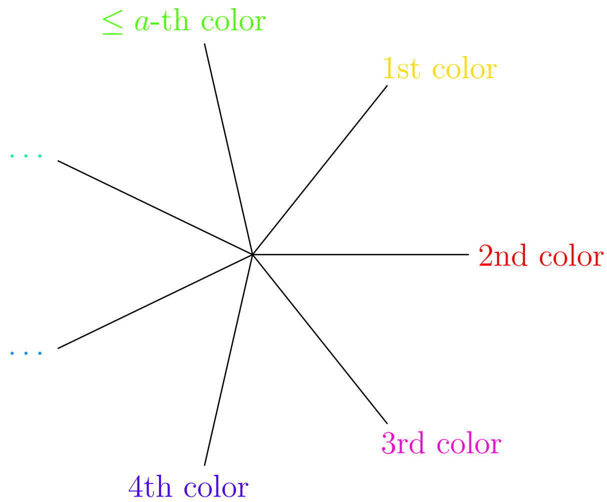
Algorithms are vast part of olympiad combinatorics. It might seem that algorithms are not so important, or even algorithms are full intuition, but this is only part of the story. In this chapter, we will present crucial ideas for cracking complicated algorithm problems. However, as usual, we will start with the most intuitive approach to crack algorithms.

2.1 Greedy algorithms

Greedy algorithms are algorithms that make the best possible short – term choices, hence in each step maximizing short – term gain. These words will seem clearer in next chapter:

Example 2.1.1
In a graph G with n vertices, there is no vertex with degree greater than a. Show that one can color the vertices using at most $a+1$ colors, such that no two neighboring vertices of the same color.

Solution:
We will provide clear algorithm to this problem. At first, we will arrange the vertices in an arbitrary order. Let the colors be $1, 2, 3 \ldots,$ and color the first vertex with color 1. Here is a greedy part of our algorithm: in each stage, take the next vertex in the order and color it with the smallest color that has not yet been used on any of its neighbors.

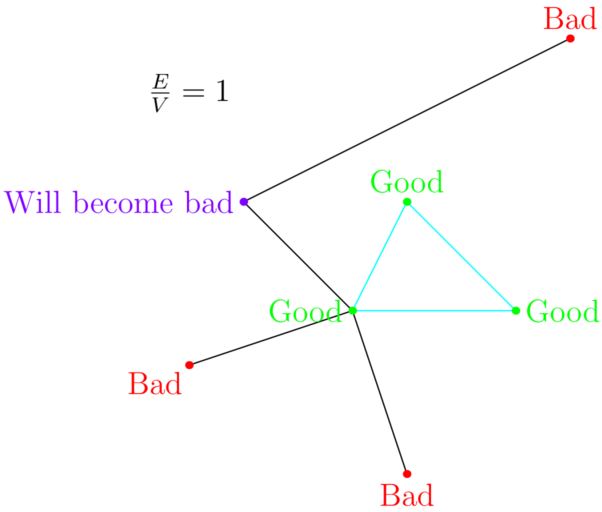


Now we will prove that our algorithm indeed works. Clearly this algorithm ensures that two adjacent vertices will not be the same color. It also ensures that at most $a+1$ colors are used: each vertex has at most a neighbors, so when coloring a particular vertex v, at most a colors have been used by its neighbors, so at least one color in the set $\{1, 2, 3, \ldots, a+1\}$ has not been used. The minimum such color will be used for the vertex v. Hence, all vertices are colored using colors in the set $\{1, 2, 3, \ldots, a+1\}$ and the problem is solved.

To recall, in this problem we employed the simplest, almost foolish algorithm - yet it proved remarkably effective. We have only one obstacle, and we have tried to overcome only this obstacle in the most effective way, without paying attention to what happens to others. Of course, this approach will not be convenient, if we have several important quantities. However, greedy algorithms might be surprisingly helpful, even in complex problems, like next two examples.

Example 2.1.2
In a graph G with V vertices and E edges, show that there exists an induced subgraph H with each vertex having degree at least $\frac{E}{V}$.
(In other words, a graph with average degree d has an induced subgraph with minimum degree at least $\frac{d}{2}$).

Solution:
Note that the average degree of a vertex is $\frac{2E}{V}$. Intuitively, we should remove the 'bad' vertices: vertices that have degree $< \frac{E}{V}$. Thus, a natural algorithm for finding such a subgraph is as follows: start with graph G, and as long as there exists a vertex with degree $< \frac{E}{V}$, delete it.



However, remember that while deleting a vertex we are also deleting the edges incident to it, and in the process vertices that were initially not 'bad' may become bad in the subgraph formed. What if we end up with a graph with all vertices bad? Fortunately, this will not happen: notice that the ratio of $\frac{\text{edges}}{\text{vertices}}$ is strictly increasing (it started at $\frac{E}{V}$ and each time we deleted a vertex, less than $\frac{E}{V}$ edges were deleted by the condition of our algorithm).

Hence, it is impossible to reach a stage when only one vertex is remaining, since in this case the $\frac{\text{edges}}{\text{vertices}}$ ratio is 0. So, at some point, our algorithm must terminate, leaving us with a graph with more than one vertex, all of whose vertices have degree at least $\frac{E}{V}$.

Example 2.1.3 (ISL 2001)
A set of three nonnegative integers x, y, z with $x < y < z$ satisfying $z - y, y - x = a, b$, where a and b are distinct positive integers and $a < b$, is called a historic set. Show that the set of all nonnegative integers can be written as a disjoint union of historic sets.

Solution:
Note that a historic set is in the form $\{x, x + a, x + a + b\}$ (which we will call a small set) or $\{x, x + b, x + a + b\}$ (which we will call a large set).

We can use the following algorithm to achieve this task: We let x be the smallest number not yet covered. If none of the numbers in the small set have been covered already, then we use a small set, otherwise we use a large set.

Now we prove that this algorithm works. We prove that if the small set does not work, then the large set will always work.

Assume for the sake of contradiction that the large set does not work.

Clearly the x element can not have already been covered.
If the $x + a + b$ element has been covered, then that means that on the previous step, we either used the small set on $x + b$ or the large set on $x + a$. However, this is impossible as $x < x + a < x + b$.

If the $x + b$ has already been covered, then either we used the small set on $x + b - a$, or we used any set on $x - a$. Clearly the first one is impossible as $x + b - a > x$. Now we consider the second case. If we used the small set on $x - a$, then it would imply that we would have already covered x, a contradiction. If we used the large set, then that means that some element in the small set was already covered. If $x + b$ was already covered, then this implies that x was already covered, contradiction. Otherwise, this implies that x was already covered, another contradiction.

Therefore, a situation where both sets do not work, and the algorithm always works

2.2 Induction + Recursion

In the previous chapter, we have considered the simplest algorithms, without any additional ideas. In this chapter, we will look through various examples that will deal with the reference to smaller cases.

Example 2.2.1
Cards numbered 1 to n are arranged at random in a row with $n \geq 5$. In a move, one may choose any block of consecutive cards whose numbers are in ascending or descending order and switch the block around.
For example, if $n = 9$, then
(9 1 $\underline{6\ 5\ 3}$ 2 7 4 8) might be changed to (9 1 $\underline{3\ 5\ 6}$ 2 7 4 8).
Prove that in at most $2n - 6$ moves, one can arrange the n cards so that their numbers are in ascending or descending order.

Solution:
Let $f(n)$ be the minimum number of moves required to 'arrange' any permutation of the n cards. Suppose we have a permutation with starting card k. In $f(n-1)$ moves we can arrange the remaining $(n-1)$ cards to get either the sequence $(k, 1, 2, \ldots, k-1, k+1, \ldots, n)$ or $(k, n, n-1, \ldots, k+1, k-1, \ldots, 2, 1)$. In one move, we can make the former sequence $(k, k-1, k-2, \ldots, 1, k+1, k+2, \ldots, n)$ and with one more move we get the sequence $(1, 2, 3, \ldots, n)$ and we are done. Similarly in the latter case we need only two additional moves to get $(n, n-1, \ldots, 1)$. Thus, in either case, we can complete the task using $f(n-1) + 2$ moves, so $f(n) \leq f(n-1) + 2$.

Now, to prove the bound for general $n \geq 5$, it suffices to prove it for $n = 5$ and then induct using $f(n) \leq f(n-1) + 2$. The proof of $f(5) = 4$ we left to the reader as an exercise.

Example 2.2.2 (IMO 2017)
An integer $N \geq 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:

1. *no one stands between the two tallest players*
2. *no one stands between the third and fourth tallest players*

   $\vdots$

N. *no one stands between the two shortest players.*

Show that this is always possible.

Solution:
We will crack this problem by induction on N with an algorithm.

Base: The case when $n = 1$ is trivial; the original two soccer players work.

Induction hypothesis: Sir Alex has an algorithm for $N(N-1)$ soccer players.

Induction step: Split the row of players into N segments, each with $N + 1$ people. Consider the $N + 1$ tallest people, at least two of them will be in one of the segments by Pigeonhole principle. WLOG, let it be the first segment. Call these two people Timur and Amir.

Then we can remove all the other players in the first segment (apart from Amir and Timur), which gives $N - 1$ players deleted. Next, we remove the rest of the $N + 1$ tallest players (again, excluding Timur and Amir), which gives at most another $N-1$ players deleted. Hence, at most $2(N-1)$ players were removed and we still have at least $N(N-1)$ people. (excluding Timur and Amir), so we can apply the induction hypothesis to get $2(N-1)$ players, and we can get the desired result. Lastly, since Amir and Timur will now be taller than these $2(N-1)$ players (and they will be next to each other), we can add them in and all the conditions still hold. This completes the induction.

2.3 Invariants and Monovariants

Here, we will introduce another crucial idea in algorithm problems: invariants and monovariants. To reiterate from previous chapter: invariant is a quantity that does not change, and monovariant is a quantity that changes monotonically: it is either always increasing or decreasing. We will start with a simple, introductory example:

Example 2.3.1
Some numbers are written in the cells of the table $m \times n$. It is allowed to change the sign of all numbers in one column or in one row at the same time. Prove that by several such operations it is possible to ensure that the sums of the numbers in any row and in any column are nonnegative.

Solution:
Consider the following algorithm: if the sum of the numbers in a row (or in a column) is negative, then we will change the sign of the numbers in this row (column). To prove that this process stops, we will find some characteristics of the table, which increases monotonously with each step. The desired characteristic is the sum of all cells. This amount increases at each step. The process will end, since the number of character arrangements for numbers is finite.

Example 2.3.2 (Canada 2014)
A number of robots are placed on the squares of a finite, rectangular grid of squares. A square can hold any number of robots. Every edge of each square of the grid is classified as either passable or impassable. All edges on the boundary of the grid are impassable. You can give any of the commands up, down, left, or right.
All of the robots then simultaneously try to move in the specified direction. If the edge adjacent to a robot in that direction is passable, the robot moves across the edge and into the next square. Otherwise, the robot remains on its current square. You can then give another command of up, down, left, or right, then another, for as long as you want. Suppose that for any individual robot, and any square on the grid, there is a finite sequence of commands that will move that robot to that square. Prove that you can also give a finite sequence of commands such that all of the robots end up on the same square at the same time.

Solution:
Clearly, it suffices to consider the case where two robots are on the grid. Define the distance between two robots to be the minimum number of commands needed for a robot to travel from one of the squares to the other.

Let A and B be the two robots on the grid. We will provide an algorithm that, when repeatedly invoked, eventually decreases the distance between A and B. In particular, the algorithm is to give the commands that would move A to B's position with the minimum distance. This works since A will not hit a wall (otherwise there is a 'faster' way to move A), but since the rectangular grid is finite, B will eventually hit a wall after a few runs of this algorithm. Therefore, the distance between A and B after running the algorithm eventually decreases, and we are done.

Example 2.3.3 (IMO 1986)
To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y, -y, z + y$ respectively. This iteration is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Solution:
Answer: This procedure is always terminates.
First, if all our numbers on the pentagon are labeled x_i, all the indices have taken the undo module 5. Let in the first step $(x_1, x_2, x_3, x_4, x_5) \to (x_1, x_2 + x_3, -x_3, x_4 + x_3, x_5)$ We will solve this problem using monovariants. It is not hard to see that sum of all numbers written on the pentagon remains invariant. Because the sum is positive and remains invariant and at the end all numbers are also positive, this leads to the thought that pairwise differences should be small, so it can be an idea to consider pairwise differnces. After several trials we can end with such monovariant: $f_0(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2}\sum_{i=1}^{5}(x_{i+1} - x_i)^2$. WlOG assume that $x_4 < 0$, and after meticulous opening brackets it is not hard to see that our quantitiy is decrasing step by step. Because: $f_{\text{new}} - f_{\text{old}} = Sx_4 < 0$, where S is sum of all numbers in pentagon. Because our quantity is decreasing, the process terminates after finite number of steps.
In previous examples we have seen the problems related to monovariant, and of course we can solve certain algorithmical problems with invariants,

Example 2.3.4 (All-Russian 1997)
There are some stones placed on an infinite (in both directions) row of squares labeled by integers. (There can be more than one stone on a given square). There are two types of moves:
(i) Remove one stone from each of the squares n and n – 1 and place one stone on n + 1
(ii) Remove two stones from square n and place one stone on each of the squares n + 1 and n – 2. Show that at some point no more moves can be made.

Solution:

Give a stone in square k weight φ^k with $\varphi = \frac{1+\sqrt{5}}{2}$, and it is not hard to see that the sum of all weights remains invariant due to the relation $\varphi^k = \varphi^{k-1} + \varphi^{k-2}$. Suppose that the process never terminates, from which there exists at least one square that is operated on infinitely many times. Consider the square of least value that is operated on finitely many times, and note that the square to the left of it is operated on infinitely, so arbitrarily many stones build up on the former square, so it must be operated on infinitely, contradiction. Thus all squares are operated on infinitely, so the index of the rightmost square that contains a stone grows arbitrarily large. At some point, the weight of the stone in that square exceeds the initial total weight, contradicting the invariant.

Example 2.3.5 (APMO 1997)
n people are seated in a circle. A total of nk coins have been distributed among them, but not necessarily equally. A move is the transfer of a single coin between two adjacent people. Find an algorithm for making the minimum possible number of moves which result in everyone ending up with the same number of coins.

Solution:
We want each person to end up with k coins. Let the people be labeled from $1, 2, \ldots, n$ in order (note that n is next to 1 since they are sitting in a circle). Suppose that person i has c_i coins. Now, we introduce the variable $d_i = c_i - k$, since this indicates how close a person is to having the desired number of coins. Consider the quantity $X = |d_1| + |d_1 + d_2| + |d1 + d_2 + d_3| + \cdots + |d_1 + d_2 + \cdots + d_{n-1}|$

Clearly $X = 0$ if and only if everyone has k coins, so our goal is to make $X = 0$. The reason for this choice of X is that moving a coin between person j and person $j+1$ for $1 \leq j \leq n-1$ changes X by exactly 1 as only the term $|d_1 + d_2 + \cdots + d_j|$ will be affected. Hence, X is a monovariant and is fairly easy to control (except when moving a coin from 1 to n or vice versa). Let $s_j = d_1 + d_2 + \cdots + d_j$.

We claim that as long as $X > 0$ it is always possible to reduce X by 1 by a move between j and $j+1$ for some $1 \leq j \leq n-1$. We use the following algorithm. Assume WLOG $d_1 \geq 1$. Take the first j such that $d_{j+1} < 0$. If $s_j > 0$, then simply make a transfer from j to $j+1$. This reduces X by one since it reduces the term $|s_j|$ by one. The other possibility is $s_j = 0$, which means $d_1 = d_2 = \cdots = d_j = 0$ (recall that d_{j+1} is the first negative term). In this case, take the first m > i+1 such that $d_m \geq 0$. Then, $d_{m-1} < 0$ by the assumption on m, so we move a coin from m to $(m-1)$. Note that all terms before dm were either 0 or less than 0 and $d_{m-1} < 0$, so s_{m-1} was less than 0. Our move has increased sm-1 by one, and has hence decreased $|s_{m-1}|$ by one, so we have decreased X by one.

Thus, at any stage, we can always decrease X by at least one by moving between j and $j+1$ for some $1 \leq j \leq n-1$. We have not yet considered the effect of a move between 1 and n. Thus our full algorithm is as follows: At any point of time, if we can decrease X by moving a coin from 1 to n or n to 1, do this. Otherwise, decrease X by 1 by the algorithm described in the above paragraph.

2.4 Crazy algorithms

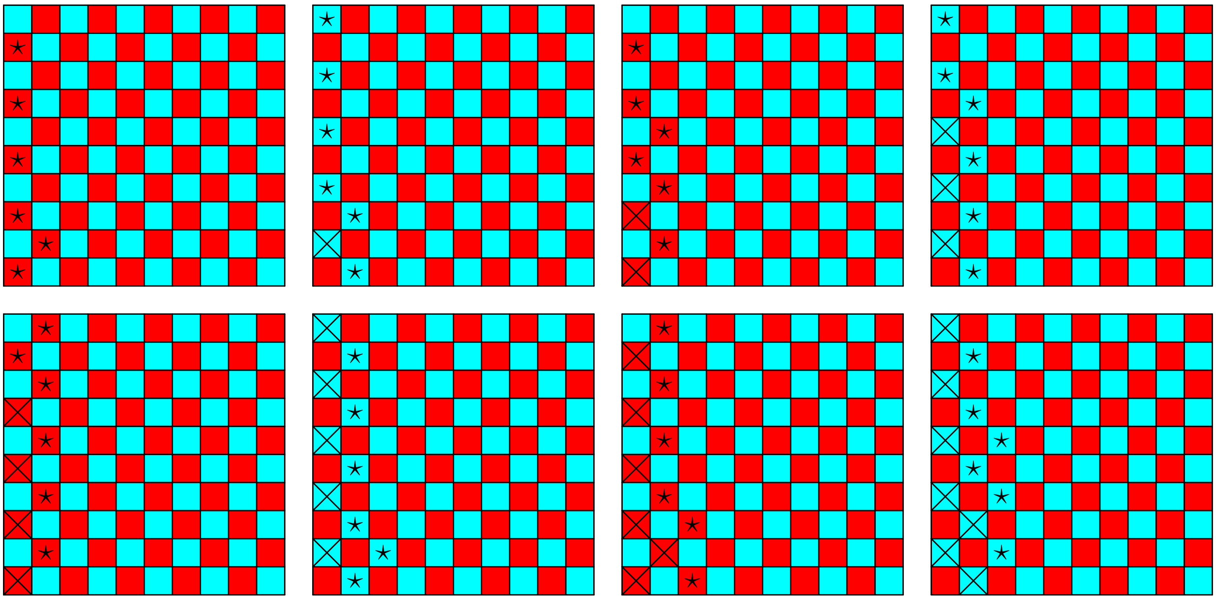
Example 2.4.1 (239 Olympiad 2017)
An invisible tank is on a 100×100 table. A cannon can fire at any 60 cells of the board after that the tank will move to one of the adjacent cells (by side). Then the process is repeated. Can the cannon grantee to shoot the tank?

Answer: Yes, it can.

Solution:

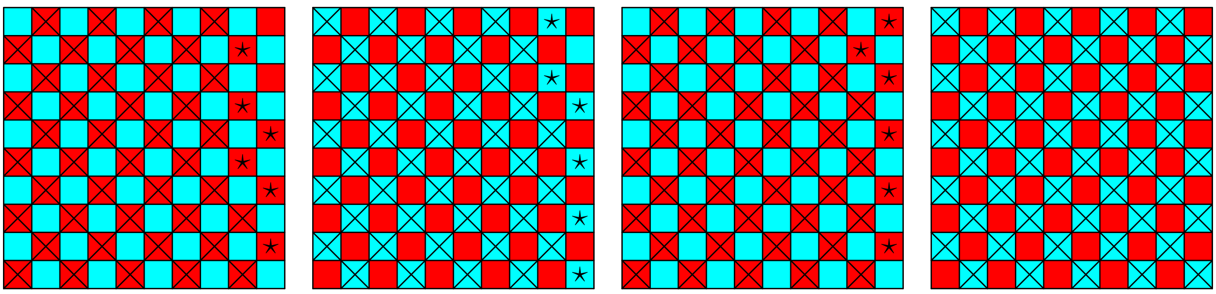
Actually to shoot the tank on the $2n \times 2n$ board $n + 1$ cells at a time is enough.

Let's consider a checkerboard coloring. WLOG tank is on the red tile. In the pictures, the stars show the squares at which the next shot is fired, and the crosses mark the squares on which the tank cannot be *before this shot.*



After each move tank changes the color of the cell it is in, and 1 more cross is placed.

Here are 4 last moves for $n = 5$:



After some (finite) amount of moves every cell of one color (in our case – cyan) is marked with cross. So if tank was not hit by the cannon, then our original assumption (tank is on the red) was wrong. But we can just repeat the process with the opposite color and now surely hit the tank.

Example 2.4.2 (JetBrains 2024)
In the vertices of a polygon $A_1, A_2, \ldots, A_{100}$, there are initially 30 zeros and 70 ones. There are 100 tokens numbered from 1 to 100. In one move, it is allowed to perform one of the following actions:

1. *If token i has not been used yet, then it can be placed on vertex A_i.*
2. *If there is a token on vertex A_j, then it is possible to change the number in this vertex (from 0 to 1 or from 1 to 0). If the number changes from 0 to 1, the token is moved to vertex A_{j+1} (where $A_{101} = A_1$); if the number changes from 1 to 0, the token is discarded completely.*

At no point may two tokens occupy the same vertex. After several moves, all 100 tokens have been discarded. How many zeros can be present in the vertices at this moment?

Answer: $\boxed{30}$.
Redefine the problem statement:
We have 3×100 board with the top row filled with tokens and initially 30 zeros and 70 ones on the second row. On any move tokens changes the number of the cell (from 0 to 1 or from 1 to 0). If the number changes from 0 to 1, the token is moved to the right, if the number changes from 1 to 0, the token moves down. After some number of moves every token is now on the third row. How many zeros can be present in the second row at this moment?
Note: we allow for the collisions between tokens.

1	1	1	1	1	1	$\cdots$	1	1
0	0	1	0	1	1	$\cdots$	0	0
1	1	1	1	1	1	$\cdots$	1	1

We assign the value to each token depending on the cell token is in as follows:

100	99	98	97	96	95	$\cdots$	2	1
101	100	99	98	97	96	$\cdots$	3	2
102	101	100	99	98	97	$\cdots$	4	3

If we count the total sum of values of tokens and values in the cells of the table we will get

$$100 + 70 + 100 + (100 + 99 + 98 + \cdots + 2 + 1) = 5320.$$

Note that on any move of the token total sum remains constant $\pmod{100}$.

Claim: at the end, no 2 tokens lie in the same cell.

So if we prove the claim the total sum at the end will be

$$0 + \# \text{ of ones in the 2nd row} + 100 + (102 + 101 + 100 + \cdots + 4 + 3) =$$

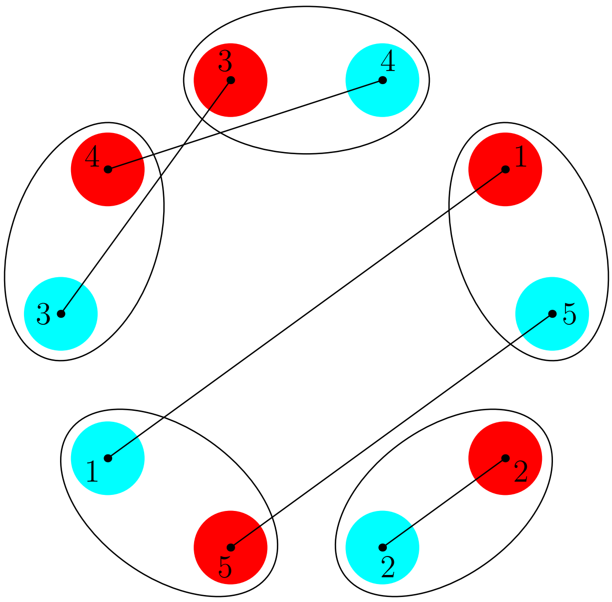
$$5350 + \# \text{ of ones in the 2nd row} \equiv 5320 \pmod{100}$$

So the # of ones in the 2nd row will be 70.

FTSoC assume that at some point 2 tokens lie in the same cell on the 3rd row. Now observe that both of them need to come in that space by moving downwards and changing the number on the 2nd row from 1 to 0. Between them we must have a token to change the number from 0 to 1, so at least 3 tokens visited the cell on the 2nd row.
Now look at the first time 3 tokens visited some cell (call it A, and the cell on the left of it – B), only one of them could have come from above, so at least 2 of them came from B. But by the same logic, for 2 tokens to move right, we need 1 token to move down, so at least 3 tokens visited B – contradiction.

Example 2.4.3 (All-Russian 2005)
There are 100 representatives of 50 countries sitting at the rounded table, two from each country. Prove that they can be divided into two groups in such a way that each group will have one representative from each country, and each person was in the same group with no more than one of his neighbors.

Solution:
At first, we will label 100 people by numbers from 1 to 100 clockwise, and let's pair them in groups of two consecutive guys: $\{1, 2\}, \{3, 4\}, \ldots, \{99, 100\}$.



Now we will present algorithm to solve our problem: Color 1 black, then color his countryman white, then color 1's countryman's neighbor black, then his countryman white, and so on, until we reach 2, whom we color white. We have colored several of our pairs completely, and we can initiate another process, starting from another pair.

Example 2.4.4 (Kazakhstan 2023)
Let G be a graph whose vertices are 2000 points in the plane, no three of which are collinear, that are colored in red and blue and there are exactly 1000 red points and 1000 blue points. Given that there exist 100 red points that form a convex polygon with every other point of G lying inside of it. Prove that one can connect some points of the same color such that segments connecting vertices of different colors do not intersect, and one can move from a vertex to any vertex of the same color using these segments.

Solution:
The problem might seem as tough, and indeed it is tough. However, the problem statement gives an exact clue to construct our algorithm. Let's call a triangle $2B$ if it has two vertices in black (connected by an edge) and one in red. It is similar to $2R$ if two vertices are red (connected by an edge) and one is black.
Let's define the operation $\oplus$ on a triangle $2B$ or $2R$ as follows:

- WLOG consider a $2R$ triangle.
- If there are no black dots inside this triangle, then connect all the red dots inside the triangle with one of the red vertices of $2R$ and finish the operation. Please note that the new red dots will be connected to at least one previous red dot (to connect the red dots).
- If there is at least one black dot inside $2R$, then take any of them and connect it to the black vertex of $2R$. Note that this black dot is connected to the previous black vertex of the triangle.2R (for connecting black dots). Then we will divide the triangle into three parts as shown in the figure above or belowThus, 2R is divided into two 2B and one 2R, and then we make $\oplus$ for each of the resulting parts.
- Since any triangle $2R$ or $2B$ contains only a finite number of points inside, the operation will end at some point.

Let's take the original graph and connect all neighboring vertices of the convex hull with an edge. Then we take any black point inside a convex 100-gon and divide the graph into 100 triangles of the form 2R, and perform $\oplus$ for each of them. From the construction, it can be understood that the newly found points inside the triangles have an edge with at least one previous vertex of the same color. From this we can conclude that the red and black dots are connected. In addition, no two multicolored edges intersect because we have performed a triangulation that does not allow for intersection. no segments (two segments can intersect only if they have a common vertex, but then all the ends of these two segments are the same color).

2.5 Information

In this chapter, we will see problems that needs to count some information that we can get after several moves to be able to solve problems.

Example 2.5.1
Magician and magician helper perform a magic trick as follows. A spectator writes on a board a sequence of N (decimal) digits. Magician helper covers two adjacent digits with a black disc. Then, a magician comes and says both closed digits (and their order). For which minimal N can this trick always work?

Solution:
Suppose that we can perform such a trick on a certain N. If so, a magician can uniquely restore the original sequence of digits from each variant that can be written by the spectator.
In other words, the number of variants that can be written on the board should be less than number of sequences with 2 hidden digits. Magician helper may have in front of magician $(N-1)10^{N-2}$ different configurations ($N-1$ positions for the disc and 10^{N-2} for the reminder), and spectator can choose 10^N configurations. So $(N-1)10^{N-2} \geq 10^N$ and $N \geq 101$
Strategy for $n = 101$:
Let $a_0, a_1, \ldots, a_{99}, a_{100}$ be the first 101 ordered numbers written by the spectator. At first, helper computes $S_1 = 10a_0 + a_1 + 10a_2 + a_3 + \cdots + 10a_{100} \pmod{100}$. Next, helper puts the black disc on numbers a_{S_1} and $a_{(S_1+1)}$

Magician first looks at the position of the black disc and knows S_1. Next, magician computes $S_2 = 10a_0 + a_1 + 10a_2 + a_3 + \cdots + 10a_{100} \pmod{100}$, using 0 and 0 for the two hidden positions. Magician computes $S_3 = S_1 - S_2 \pmod{100}$ and the final step is elementary : S_3 is a number between 0 and 99 and so $S_3 = 10a + b$

Then, if S_1 is even, the ordered two hidden numbers are (a, b) And if S_1 is odd, the ordered two hidden numbers are (b, a)

Example 2.5.2 (IMSC 2025)
There are $3^n + 1$ pebbles of 11 grams in a row. Two of them are fat and weigh 12 grams, and they are neighbors. In one query, it is possible to ask the weight of any set of pebbles. Find the minimum number of queries to determine the fat pair regardless of the answers to the queries. In addition, queries must be determined in advance. In other words, it is not possible to change queries depending on the answers for them.

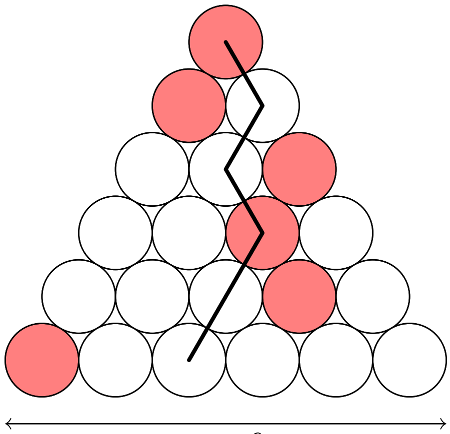
Solution:

2.6 Problems

Problem 2.6.1 (APMO 2022) Let n and k be positive integers. Cathy is playing the following game. There are n marbles and k boxes, with the marbles labeled 1 to n. Initially, all marbles are placed inside one box. Each turn, Cathy chooses a box and then moves the marbles with the smallest label, say i, to either any empty box or the box containing marble $i+1$. Cathy wins if at any point there is a box containing only marble n. Determine all pairs of integers (n, k) such that Cathy can win this game.

Problem 2.6.2 (All-Russian 2014) There are n cities in the state, and an express runs between each two of them (in both directions). For any express train, the prices of round-trip and round-trip tickets are equal, and for any different express trains these prices are different. Prove that a traveler can choose the starting city, leave it and travel sequentially on $n-1$ express trains, paying less for the fare on each next one than for the fare on the previous one. (A traveler may enter the same city several times.)

Problem 2.6.3 (IMO 2023) Let n be a positive integer. A Japanese triangle consists of $1+2+\cdots+n$ circles arranged in an equilateral triangular shape such that for each $i=1, 2, \ldots, n$, the i-th row contains exactly i circles, exactly one of which is colored red. A ninja path in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n=6$, along with a ninja path in that triangle containing two red circles.



In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

Problem 2.6.4 (IMO 2020) There are $4n$ pebbles of weights $1, 2, 3, \ldots, 4n$. Each pebble is colored in one of n colors and there are four pebbles of each color. Show that we can arrange the pebbles into two piles so that the following two conditions are both satisfied:

- The total weights of both piles are the same.
- Each pile contains two pebbles of each color.

Problem 2.6.5 (All-Russian 2005) 100 people from 25 countries, four from each country, sit in a circle. Prove that one may partition them onto 4 groups in such way that no two countrymen, nor two neighboring people in the circle, are in the same group.

Problem 2.6.6 (IMO 2010) Each of the six boxes B_1, B_2, B_3, B_4, B_5, B_6 initially contains one coin. The following operations are allowed

- Choose a non-empty box B_j, $1 \le j \le 5$, remove one coin from B_j and add two coins to B_{j+1}

- Choose a non-empty box B_k, $1 \leq k \leq 4$, remove one coin from B_k and swap the contents (maybe empty) of the boxes B_{k+1} and B_{k+2}.

Determine if there exists a finite sequence of operations of the allowed types, such that the five boxes B_1, B_2, B_3, B_4, B_5 become empty, while box B_6 contains exactly $2010^{2010^{2010}}$ coins.

Problem 2.6.7 (ISL 2024 C3) Let n be a positive integer. There are $2n$ knights sitting at a round table. They consist of n pairs of partners, each pair of which wishes to shake hands. A pair can shake hands only when next to each other. Every minute, one pair of adjacent knights swaps places. Find the minimum number of exchanges of adjacent knights such that, regardless of the initial arrangement, every knight can meet her partner and shake hands at some time.

Problem 2.6.8 (USAJMO 2020) Let $n \geq 2$ be an integer. Carl has n books arranged on a bookshelf. Each book has a height and a width. No two books have the same height, and no two books have the same width. Initially, the books are arranged in increasing order of height from left to right. In a move, Carl picks any two adjacent books where the left book is wider and shorter than the right book, and swaps their locations. Carl does this repeatedly until no further moves are possible. Prove that regardless of how Carl makes his moves, he must stop after a finite number of moves, and when he does stop, the books are sorted in increasing order of width from left to right.

Problem 2.6.9 (IMO 2024) Turbo the snail plays a game on a board with 2024 rows and 2023 columns. There are hidden monsters in 2022 of the cells. Initially, Turbo does not know where any of the monsters are, but he knows that there is exactly one monster in each row except the first row and the last row, and that each column contains at most one monster.
Turbo makes a series of attempts to go from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then repeatedly moves to an adjacent cell sharing a common side. (He is allowed to return to a previously visited cell.) If he reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move, and Turbo remembers whether or not each cell he has visited contains a monster. If he reaches any cell in the last row, his attempt ends and the game is over.
Determine the minimum value of n for which Turbo has a strategy that guarantees reaching the last row on the n-th attempt or earlier, regardless of the locations of the monsters.

Problem 2.6.10 (Kazakhstan 2025, Regional) A and B are playing a game on a 100×100 checkered board. Each player has a chip. At the beginning of the game, player A's chip is in the lower-left corner, and player B's chip is in the lower-right corner. The players take turns making moves, starting with A. In one turn, the player moves his chip to any square of the board adjacent to the square of the previous position. Prove that player A can achieve in a finite number of moves that at some point his chip will be on the same square as player B chip, regardless of the moves of the second player.

Problem 2.6.11 (All-Russian 2015) The field is a 41×41 checkered square, in one of the cells of which a tank is disguised. The fighter fires at one cell in one shot. If a hit occurs, the tank crawls over to the cell next to the side of the field, if not, it remains in the same cell. At the same time, after the shot, the fighter pilot does not know if a hit has occurred. To destroy a tank, you need to hit it twice. What is the least number of shots you can do to ensure that the tank is destroyed?

Problem 2.6.12 (Saint-Petersburg 1994) The pluses and minuses are placed in the cells of the 1995×1995 table. It is allowed to select 1995 cells, no two of which are in the same row or column, and change the signs in the selected cells. Prove that using such operations it is possible to ensure that no more than 1994 pluses remain in the table.

Problem 2.6.13 (Saint-Petersburg 2018) In a 9×9 table, all cells contain zeros. The following operations can be performed on the table:

1. Choose an arbitrary row, add one to all the numbers in that row, and shift all these numbers one cell to the right (and place the last number in the first position).
2. Choose an arbitrary column, subtract one from all its numbers, and shift all these numbers one cell down (and place the bottommost number in the top cell).

Is it possible to obtain a table in which all cells, except two, contain zeros, with 1 in the bottom-left cell and -1 in the top-right cell after several such operations?

3 Games

In the olympiads usually we only considered games with ”complete information”. That means that every player (if there are multiple ones) knows fully what he (and other players) can do at any moment of time, so there is no hidden cards or random events. Some examples of such games are chess, tic-tac-toe and checkers.