3
Neutrino Masses and Mixing

There are two types of massive fermions: Dirac or Majorana. A Majorana fermion is one that is its own antiparticle, and the concept was formulated by Ettore Majorana in 1937. In contrast, Dirac fermions are those that are not their own antiparticles. Except possibly for neutrinos in the standard model, no elementary fermions are known to be their own antiparticle. Indeed, massive charge-carrying fermions such as the electron, the muon, the tau or the quarks must all be of Dirac type. However, electrically neutral fermions, such as neutrinos, are expected to be Majorana type on general grounds, irrespective of how they acquire their mass.[1] Note that the argument in favour of Majorana neutrinos goes beyond any particular neutrino mass generation mechanism, for example, the seesaw, to be discussed in Chapters 7, 13 and 14. This fits also with the fact that the simplest effective source of neutrino mass is Weinberg's dimension-five operator depicted in Figure 3.1 [40].

Although little is known regarding the mechanism that induces Weinberg's operator, its characteristic scale or flavour structure, the generic fact is that it is lepton-number-violating. We now first discuss how to describe Majorana masses at the kinematical level, before introducing gauge interactions.

3.1
Two-Component Formalism

The most basic spin-1/2 fermion corresponding to the lowest representation of the Lorentz group is given in terms of a two-component spinor ρ_α. To write its Lagrangian, we also need to define its complex conjugate, $\rho_\alpha^* \equiv \overline{\rho}_{\dot\alpha}$. As explained in Appendix A, we can use the antisymmetric tensor $\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot\alpha\dot\beta} = i\sigma_2$ to raise and lower indices in spinor space:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \quad \overline{\psi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\psi_{\dot\beta}, \quad \overline{\psi}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\psi^{\dot\beta}. \tag{3.1}$$

1) The same holds for the electrically neutral fermions postulated in supersymmetric theories, such as the gravitino, the gluino and the neutralinos.

Neutrinos in High Energy and Astroparticle Physics, First Edition. José W. F. Valle and Jorge C. Romão.

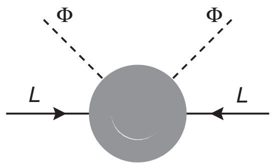

Figure 3.1 Lepton-number-violating dimension-five operator responsible for generating neutrino mass after the electroweak symmetry breaking takes place.

With these conventions, we can write the following free-field Lagrangian (we are following Ref. [46] but converting to our metric conventions):

$$\begin{aligned}\mathcal{L}_{\mathrm{M}} &= i\,\overline{\rho}_{\dot\alpha}\,\overline{\sigma}^{\mu\dot\alpha\beta}\partial_\mu\rho_\beta - \frac{1}{2}m(\rho^\alpha\rho_\alpha + \overline{\rho}_{\dot\alpha}\overline{\rho}^{\dot\alpha})\\ &\equiv i\,\overline{\rho}\,\overline{\sigma}^\mu\partial_\mu\rho - \frac{1}{2}m(\rho\rho + \overline{\rho}\,\overline{\rho}),\end{aligned} \tag{3.2}$$

where in the second line we defined a simplified notation, $\rho\rho \equiv \rho^\alpha\rho_\alpha, \overline{\rho}\,\overline{\rho} \equiv \overline{\rho}_{\dot\alpha}\overline{\rho}^{\dot\alpha}$ and where $\sigma^\mu \equiv (1,\vec{\sigma}), \quad \overline{\sigma}^\mu \equiv (1,-\vec{\sigma})$, see Appendix A for details (notice that dotted and undotted indices contract in different ways).

Under the usual Lorentz transformations $x'^\mu = \Lambda^\mu_\nu\, x^\nu$, the spinors transform as

$$\rho'_\alpha = S_\alpha^\beta\,\rho_\beta, \overline{\rho'}_{\dot\alpha} = S^{*\dot\beta}_{\dot\alpha}\,\overline{\rho}_{\dot\beta}, \rho'^\alpha = \rho^\beta S^{-1\alpha}_\beta, \overline{\rho'}^{\dot\alpha} = \overline{\rho}^{\dot\beta} S^{*-1\dot\alpha}_{\dot\beta}\ , \tag{3.3}$$

where S is unimodular (for proper Lorentz transformations); det $S = 1$ and obeys

$$S^\dagger\overline{\sigma}^\mu S = \Lambda^\mu_\nu\overline{\sigma}^\nu, \quad S\sigma^\mu S^\dagger = \Lambda^\mu_\nu\sigma^\nu. \tag{3.4}$$

Using Eqs. 3.3 and 3.4, one can show that indeed the Lagrangian in Eq. 3.2 is Lorentz-invariant.

The field equations that result from Eq. 3.2 are

$$i\,\overline{\sigma}^\mu\partial_\mu\rho = m\overline{\rho}\ , \quad i\,\partial_\mu\overline{\rho}\,\overline{\sigma}^\mu = -m\rho. \tag{3.5}$$

As a result of the conjugation and Clifford properties of the σ-matrices, one can verify that each component of the spinor ρ obeys the Klein–Gordon wave equation.

Note that the mass term in Eq. 3.2 and hence the field equation 3.5 are not invariant under a phase transformation of the spinor field ρ.

In order to display clearly the relationship between the theory in Eq. 3.2 and the usual theory of a massive spin-1/2 Dirac fermion, we consider the familiar Lagrangian

$$\mathcal{L}_{\mathrm{D}} = i\,\overline{\Psi}\gamma^\mu\partial_\mu\Psi - m\,\overline{\Psi}\Psi, \tag{3.6}$$

where by convenience we use the chiral representation of the Dirac algebra

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,g^{\mu\nu}, \tag{3.7}$$

in which γ_5 is diagonal,

$$\gamma^0 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \gamma^i = \begin{bmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{bmatrix}, \quad \gamma_5 = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}. \tag{3.8}$$

In this representation, the charge conjugation matrix C obeying

$$C^T = -C, \quad C^\dagger = C^{-1}, \quad C^{-1}\,\gamma_\mu\,C = -\,\gamma_\mu^T \tag{3.9}$$

is simply given in terms of the basic conjugation matrix $i\,\sigma_2$ as

$$C = \begin{bmatrix} i\,\sigma_2 & 0 \\ 0 & i\,\sigma_2 \end{bmatrix}. \tag{3.10}$$

A Dirac spinor can then be written in terms of two-component spinors χ and ϕ as

$$\Psi_{\mathrm{D}} = \begin{bmatrix} \chi \\ i\,\sigma_2\,\phi^* \end{bmatrix} = \begin{bmatrix} \chi_\alpha \\ \epsilon^{\dot\alpha\dot\beta}\,\overline{\phi}_{\dot\beta} \end{bmatrix} = \begin{bmatrix} \chi_\alpha \\ \overline{\phi}^{\dot\alpha} \end{bmatrix}, \tag{3.11}$$

so that the corresponding charge-conjugate spinor $\Psi^c_{\mathrm{D}} = C\,\overline{\Psi}_{\mathrm{D}}^T$ is the same as Ψ_{D} but exchanging ϕ and χ; that is

$$\Psi^c_{\mathrm{D}} = \begin{bmatrix} \phi \\ i\,\sigma_2\,\chi^* \end{bmatrix}. \tag{3.12}$$

A four-component spinor is said to be self-conjugate (or Majorana-type) if $\Psi = C\overline{\Psi}^T$, which amounts to setting $\chi = \phi$. Using Eq. 3.11 , we can rewrite Eq. 3.6 as

$$\begin{aligned} \mathcal{L}_{\mathrm{D}} &= i\,\phi\sigma^\mu\partial_\mu\overline{\phi} + i\,\overline{\chi}\,\overline{\sigma}^\mu\partial_\mu\chi - m(\phi\chi + \overline{\chi}\,\overline{\phi}) \\ &= i\,\overline{\phi}\overline{\sigma}^\mu\partial_\mu\phi + i\,\overline{\chi}\,\overline{\sigma}^\mu\partial_\mu\chi - m(\phi\chi + \overline{\chi}\,\overline{\phi}) \\ &= i\sum_{a=1}^{2}\overline{\rho}_a\overline{\sigma}^\mu\partial_\mu\rho_a - \frac{1}{2}m\sum_{i=a}^{2}(\rho_a\rho_a + \overline{\rho}_a\overline{\rho}_a), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \chi &= \frac{1}{\sqrt{2}}(\rho_1 + i\rho_2), \\ \phi &= \frac{1}{\sqrt{2}}(\rho_1 - i\rho_2), \end{aligned} \tag{3.14}$$

are the left-handed components of Ψ_D and of the charge-conjugate field Ψ^c_D, respectively. In this way, the Dirac fermion is shown to be equivalent to two Majorana fermions of equal mass. The U(1) symmetry of the theory described by Eq. 3.6 under $\Psi_{\mathrm{D}} \to e^{i\alpha}\Psi_{\mathrm{D}}$ corresponds to the continuous rotation symmetry between the ρ_1 and ρ_2 fields

$$\begin{aligned} \rho_1 &\to \cos\alpha\ \ \rho_1 + \sin\alpha\ \ \rho_2, \\ \rho_2 &\to -\sin\alpha\ \ \rho_1 + \cos\alpha\ \ \rho_2, \end{aligned}$$

which follows from their mass degeneracy. This shows that, indeed, the concept of fermion number requires more than one fermion and, to this extent, is not basic.

3.2
Quantization of Majorana and Dirac Fermions

The solutions of the Majorana field equation in Eq. 3.2 can easily be obtained in terms of those of Eq. 3.6 , which are well known. The answer is

$$\Psi_{\mathrm{M}} = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{2E} \sum_{r=1}^{2} [e^{-ik\cdot x} A_r(k) u_L(k,r) + e^{ik\cdot x} A_r^\dagger(k) v_L(k,r)], \tag{3.15}$$

where $u = C\,\overline{v}^T$ and $E(k) = (\vec{k}^2 + m^2)^{1/2}$ is the mass-shell condition. Note that here the creation and annihilation operators obey canonical anti-commutation rules and, like the u's and v's, depend on the momentum k and helicity label r. The expression 3.15 describes the basic Fourier expansion of a massive Majorana fermion. It differs from the usual Fourier expansion for the Dirac spinor in Eq. 3.16 in two ways:

- the spinor wave functions are two-component, as there is a chiral projection acting in front of the u's and v's, and
- instead of the two independent Fock spaces characterizing the Dirac theory, corresponding to particle and anti-particle, in the Majorana case there is only one.

The u's and v's are the same wave functions that appear in the Fourier decomposition the Dirac field

$$\Psi_{\mathrm{D}} = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{2E} \sum_{r=1}^{2} [e^{-ik\cdot x} a_r(k) u_r(k) + e^{ik\cdot x} b_r^\dagger(k) v_r(k)]\ . \tag{3.16}$$

Hence, it becomes clear that a massive Majorana fermion corresponds to just 'one half' of a conventional massive Dirac fermion.

Using the helicity eigenstate wave functions

$$\vec{\sigma}\cdot\vec{k}\ u_L^{\pm}(k) = \pm|\vec{k}|u_L^{\pm}(k) \tag{3.17}$$

and

$$\vec{\sigma}\cdot\vec{k}\ v_L^{\pm}(k) = \mp|\vec{k}|v_L^{\pm}(k), \tag{3.18}$$

one can show that, out of the four linearly independent wave functions $u_L^{\pm}(k)$ and $v_L^{\pm}(k)$, only two survive as the mass approaches zero, namely, $u_L^{-}(k)$ and $v_L^{+}(k)$ [169]. This way, we recover the Lee–Yang two-component massless neutrino theory, namely as the massless limit of the Majorana theory.

In contrast to the Dirac theory, there are independent propagators that follow from Eq. 3.2 . Using Eqs. A.38 and A.39, one can show that

$$\langle 0|T(\rho_\alpha(x)\overline{\rho}_{\dot\beta}(y))|0\rangle = i(\sigma^\mu)_{\alpha\dot\beta}\ \partial_\mu \Delta_F(x-y;m), \tag{3.19}$$

$$\langle 0|T(\rho_\alpha(x)\rho_\beta(y))\ |0\rangle = -\,m\ \epsilon_{\alpha\beta}\Delta_F(x-y;m) = m(i\sigma_2)_{\alpha\beta}\ \Delta_F(x-y;m), \tag{3.20}$$

where $\Delta_F(x-y;m)$ is the usual Feynman function. The first one is the standard propagator that characterizes fermion-number-conserving processes, while the one in Eq. 3.20 describes the virtual propagation of Majorana fermions. An important example of the latter is the neutrinoless double-beta decay (Chapter 6). As will be discussed below, this processes is induced by the exchange of Majorana neutrinos, violating lepton number by two units ($\Delta L = 2$).

Taking into account the free Lagrangian described above and the gauge interactions associated with any given theory, one can derive all Feynman rules for processes involving Majorana (as well as Dirac) fermions from first principles.

A summary of Feynman rules associated with the standard $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ theory is given in Appendix C. The corresponding rules associated with additional scalar and/or gauge interactions in extended gauge theories can be treated exactly in the same way and the corresponding Feynman rules derived.

3.3 The Lepton Mixing Matrix

We now turn to the structure of the charged and neutral current weak interactions associated with massive neutrinos within a gauge theory. To determine these, the standard procedure in any theory is to diagonalize all relevant mass matrices that result from spontaneous gauge symmetry-breaking and then to rewrite the gauge interactions in the mass eigenstate basis, where physical particles are clearly identified. Indeed, we saw in Chapter 2 how the Cabibbo–Kobayashi–Maskawa (CKM) matrix in Eq. 2.34 characterizing quark weak interactions arises from a mismatch between up- and down-type quark Yukawa couplings in Eq. 2.20. As will be thoroughly discussed in Chapter 6, mechanisms giving mass to neutrinos generally imply interactions associated with new Yukawa couplings (like Y_ν) that do not commute with that of the charged leptons, Y_l, in a way similar to what happens for quarks. As a result, like quarks, the charged current weak interactions of massive neutrinos will be described by a mixing matrix $K \equiv \mathbf{V}^{\text{LEP}}$, which also follows from the mismatch between such Yukawa couplings. However, we will see that lepton mixing is in general more complex in structure than quark mixing.

Whatever be the ultimate high energy gauge theory of Nature, it must be broken to the SM at low scales, so one should characterize the structure of the lepton mixing matrix in terms of the $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ group. While the details of $\mathbf{V}^{\text{LEP}}$ will depend on the possible existence of an underlying flavour theory, here for generality we assume that no special theory of flavour is present, the only symmetries being the standard model $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ gauge symmetry and Lorentz invariance.

3.3.1 Lepton Mixing Matrix for Dirac Neutrinos

Let us first consider the simplest possibility for the lepton mixing matrix describing the charged current in theories in which total lepton number is assumed to be conserved. From the start, such lepton mixing matrix of massive Dirac neutrinos is a unitary matrix $\boldsymbol{V}^{\text{LEP}}$ obtained in a way analogous to $\boldsymbol{V}^{\text{CKM}}$ as

$$\boldsymbol{V}^{\text{LEP}} = \boldsymbol{R}_L^{e\dagger}\, \boldsymbol{R}_L^{\nu}, \tag{3.21}$$

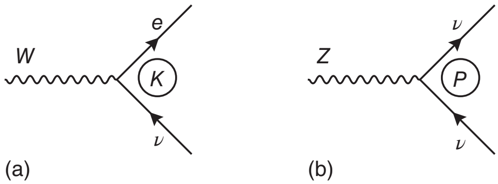

Figure 3.2 (a,b) Generic charged and neutral current couplings of mass eigenstate neutrinos in seesaw theories, discussed in Section 3.3.3. The lepton mixing matrix K is rectangular and the neutral current coupling $P = K^{\dagger}K$. Particular cases in which K is unitary and $P = I$, the identity matrix, are discussed in Sections 3.3.1 and 3.3.2.

and can always be parameterized as

$$V^{\mathrm{LEP}} \equiv K = \omega_0(\gamma) \prod_{i<j}^{n} \omega_{ij}(\eta_{ij}), \tag{3.22}$$

where

$$\omega_0(\gamma) = \exp \quad i(\sum_{a=1}^{n} \gamma_a A_a^a) \tag{3.23}$$

is a diagonal unitary matrix described by $n-1$ real parameters γ_a and the matrices A_a^b are defined by

$$(A_a^b)_{ij} = \delta_{ai}\delta_{bj}. \tag{3.24}$$

By choosing an overall relative phase between charged leptons and Dirac neutrinos, we can take $\mathbf{V}^{\mathrm{LEP}}$ as unimodular, that is, det $\mathbf{V}^{\mathrm{LEP}} = 1$, so that the phases in the 'Cartan' matrix ω_0 obey $\sum_{a=1}^{n} \gamma_a = 0$. On the other hand, each ω factor

$$\omega_{ab}(\eta_{ab}) = \exp \quad \sum_{a=1}^{n} (\eta_{ab} A_a^b - \eta_{ab}^* A_b^a), \tag{3.25}$$

is a complex rotation in ab with parameter $\eta_{ab} = |\eta_{ab}| \exp \quad i\theta_{ab}$. For example,

$$\omega_{12}(\eta_{12}) = \begin{bmatrix} c_{12} & e^{i\theta_{12}} s_{12} & 0 \ \dots \\ -e^{-i\theta_{12}} s_{12} & c_{12} & 0 \ \dots \\ 0 & 0 & 1 \ \dots \\ \dots & \dots & \dots \ \dots \end{bmatrix}. \tag{3.26}$$

Note, however, that, once the charged leptons and Dirac neutrino mass matrices are diagonal, one can still rephase the corresponding fields by $\omega_0(\alpha)$ and $\omega_0(\gamma - \alpha)$, respectively, keeping invariant the form of the free Lagrangian. This results in the simplified form

$$\mathbf{V}^{\mathrm{LEP}} = \omega_0(\alpha) \prod_{i<j}^{n} \omega_{ij}(\eta_{ij})\, \omega_0^{\dagger}(\alpha), \tag{3.27}$$

where $n - 1\alpha$-values associated with Dirac neutrino phase redefinitions are still free to eliminate, using the conjugation property

$$\omega_0(\alpha)\omega_{ab}(|\eta_{ab}| \exp \ i\theta_{ab})\, \omega_0^{\dagger}(\alpha) = \omega_{ab}[|\eta_{ab}| \exp \ i(\alpha_a + \theta_{ab} - \alpha_b)]. \tag{3.28}$$

This way, we arrive at the final Dirac lepton mixing matrix, which is, of course, identical in form to that describing quark mixing $\mathbf{V}^{\mathrm{CKM}}$ and is specified by a set of

$$n(n-1)/2 \qquad \text{mixing angles} \quad \theta_{ij} \quad \text{and} \tag{3.29}$$

$$n(n-1)/2-(n-1) \qquad \text{independent} \quad CP \quad \text{phases.} \tag{3.30}$$

This is the parameterization as originally given in [46], with unspecified ordering of the factors. The reader is referred to the Particle Data Group (PDG) for a further factor-ordering prescription [21]. In summary, if neutrino masses were added *a la Dirac* in such simple way, their charged current weak interaction $\mathbf{V}^{\mathrm{LEP}}$ would have exactly the same structure as the matrix $\mathbf{V}^{\mathrm{CKM}}$ describing quark mixing.

3.3.2
Lepton Mixing Matrix for Majorana Neutrinos: Unitary Approximation

The imposition of lepton number conservation in a gauge theory would be *ad hoc* and, to this extent, neutrinos are expected to be Majorana. From the theoretical point of view, if lepton number is conserved, there must be a symmetry reason, in whose absence the Majorana hypothesis prevails. Ultimately, the issue of whether neutrinos are Majorana or Dirac must be settled experimentally, the most relevant tool being the search for neutrinoless double-beta decay, see Chapter 6.

Here we consider the form of the lepton mixing matrix in models where neutrino masses arise in the absence of $\mathrm{SU(3)_c} \otimes \mathrm{SU(2)_L} \otimes \mathrm{U(1)_Y}$ singlets, the so-called 'right-handed' neutrinos. These include, for example, radiative models and constitute a good approximation for high-scale seesaw models. For n Majorana neutrino types, the lepton mixing matrix has exactly the same form as given in Eq. 3.27. The main difference is that now the mass term is not invariant under rephasings of the neutrino fields. As a result, there is no freedom to eliminate $n-1$ phases as we just did in Section 3.3.1 by choosing the phase parameters α in Eq. 3.27 . Consequently, these are additional sources of CP violation in the currents of gauge theories with Majorana neutrinos. Such so-called Majorana phases already exist for $n = 2$, that is, in a theory with just two generations of Majorana neutrinos [46].[2] For the case of three neutrinos, the lepton mixing matrix can be parameterized as ($n = 3$) [46] (Figure 3.2):

$$K = \omega_{23}(\theta_{23}, \phi_{23})\omega_{13}(\theta_{13}, \phi_{13})\omega_{12}(\theta_{12}, \phi_{12}), \tag{3.31}$$

where each factor in the product of the ω's is effectively 2×2, characterized by an angle and a CP phase. Even though the parameterization of the lepton mixing matrix K is fully 'symmetrical', there is a basic difference between Dirac and Majorana phases. The rephasing invariant combination

$$\delta \equiv \phi_{12} + \phi_{23} - \phi_{13} \tag{3.32}$$

2) Such 'Majorana' phase is, in a sense, mathematically more 'fundamental' than the Dirac phase whose existence, like the curl of two vectors, requires three generations at least.

(Section 4.2) corresponds to the 'Dirac phase' present in the CKM matrix characterizing quark mixing. The other two phases are associated with the Majorana nature of neutrinos. CP violation already exists in a two-neutrino scheme, but shows up only in lepton-number-violating processes, like neutrinoless double-beta decay [87], neutrino electromagnetic properties [169, 170] or the 'neutrino oscillation thought-experiment' described in [171]. In contrast, they do not affect (lepton-number-conserving) neutrino oscillations [171–174] which involve only the Dirac phase.

3.3.3 General Seesaw-Type Lepton Mixing Matrix

We now turn to the effective form of the lepton mixing matrix in general Majorana neutrino mass schemes, such as the seesaw mechanism, where isosinglet and isodoublet mass terms coexist [46]. In addition to the existence of Majorana phases, discussed in Section 3.3.2, one also has doublet–singlet mixing parameters, which are in general complex. As a result, one finds that leptonic mixing [52] as well as CP violation may take place even in the massless neutrino limit [53, 54].

The most general effective model is described by (n, m), n being the number of $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ isodoublet and m the number of isosinglet leptons. Here, m is completely arbitrary, as $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ singlets carry no anomaly. These two-component $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ singlet leptons have, in general, a gauge- and Lorentz-invariant Majorana mass term, breaking the total lepton number symmetry. The resulting structure of the weak currents is substantially more complex, since the heavy isosinglets will now mix with the ordinary SU(2) doublet neutrinos in the charged current weak interaction. As a result, the mixing matrix describing the charged leptonic weak interaction is a rectangular matrix, called K. Consequently, the charged weak interactions of the light (mass-eigenstate) neutrinos in such (n, m) seesaw models are described by an effective mixing matrix which is non-unitary. For example, the coupling of a given light neutrino to the corresponding charged lepton is decreased by a certain factor. If light enough, these neutral heavy leptons could be searched for directly, say at the large electron-positron collider (LEP) [175].[3] Their possible existence could also be inferred from low-energy weak decay processes, where the neutrinos that can be kinematically produced are only the light ones; for a compilation of constraints on their mass and coupling strengths, see [21].

3.3.3.1 Symmetrical Parametrization of the General Lepton Mixing Matrix

An explicit parameterization of the weak charged current mixing matrix K that covers the most general situation present in these $(n, m)SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$ models is easily obtained as a generalization of Eq. 3.27 and has also been given

3) At the large hadron collider (LHC), they may be copiously produced if there is a 'production portal' involving extra kinematically accessible gauge bosons that may be present, for example, in left–right symmetric models [176, 177].

in Ref. [46]. The three rows of the mixing matrix determining the ν_e, ν_μ and ν_τ, respectively, are given as (Figure 3.2)

$$K^{(1)} = \prod_{a=2}^{n+m} \omega(\eta_{1a}) e^{(1)}, \tag{3.33}$$

$$K^{(2)} = \prod_{a=2}^{n+m} \omega(\eta_{1a}) \prod_{b=3}^{n+m} \omega(\eta_{2b}) e^{(2)}, \tag{3.34}$$

$$K^{(3)} = \prod_{a=2}^{n+m} \omega(\eta_{1a}) \prod_{b=3}^{n+m} \omega(\eta_{2b}) \prod_{c=4}^{n+m} \omega(\eta_{3c}) e^{(3)}, \tag{3.35}$$

where $\eta_{ab} = |\eta_{ab}| \exp\ i\theta_{ab}$ and we define the basis vectors as $e_b^{(a)} = \delta_{ab}$. Clearly, the weak eigenstate neutrinos are orthonormal and determined by

$$n(n + 2m - 1)/2 \tag{3.36}$$

mixing angles θ_{ij} and

$$n(n + 2m - 1)/2 \tag{3.37}$$

CP-violating phases ϕ_{ij}.

This parameterization covers the case where the $\mathrm{SU(3)_c \otimes SU(2)_L \otimes U(1)_Y}$ singlets (not necessarily sequentially assigned) are heavy states and decouple from the oscillations, as in seesaw schemes. Moreover, it also covers the case where, due to some symmetry such as lepton number, (some of the) isosinglets remain light enough as to take part in low-energy phenomenology [178–180], possibly including neutrino oscillations. Although there are stringent constraints on the existence of such light states [82, 181, 182], they have been invoked for a long time [183] in connection with the LSND (liquid scintillator neutrino detector) and MiniBooNE anomalies, as well as cosmology, for different choices for their mass and mixing parameters [184].

The number of physical parameters in Eqs. 3.36 and 3.37 far exceeds those describing the CKM matrix characterizing the charged current weak interaction of quarks, Eq. 2.34. The reason is twofold:

- neutrinos are Majorana particles, so their mass terms are not invariant under phase transformations, and
- the isodoublet neutrinos in general mix with the $\mathrm{SU(3)_c \otimes SU(2)_L \otimes U(1)_Y}$ singlets and CP may also be violated in this mixing.

For example, the (3, 3) seesaw model is characterized, from Eqs. 3.36 and 3.37 , by 12 mixing angles and 12 CP phases (both Dirac and Majorana-type) [46].

Let us mention that the above parameterization of the lepton mixing matrix was originally given in [46], but with unspecified factor ordering. In what follows, we tacitly employ the ordering prescription now adopted by the PDG [21]. Further discussion on the advantages of such 'symmetrical parameterization' of the rectangular mixing matrix describing the weak charged current interaction of general schemes can be found in Ref. [58]. See Problem 3.6.

3.4
The Neutrino Neutral Current in Seesaw-Type Schemes

Another important feature in any theory based on $\mathrm{SU}(3)_c \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$ where isosinglet and isodoublet lepton mass terms coexist, such as any seesaw model, is the existence of non-diagonal couplings of the Z boson to the mass-eigenstate neutrinos. These are expressed as a projective Hermitian matrix (Figure 3.2)

$$P = K^\dagger K. \tag{3.38}$$

In contrast, the neutral current couplings of mass-eigenstate neutrinos in theories where there are no isosinglet neutrinos (i.e. $m = 0$) is just the identity matrix. The same happens in a theory with $m = n$ and lepton number conservation (Dirac neutrinos); the mixing matrix would have the same structure as the CKM matrix in the quark sector. As a result, the neutral current in Eq. 3.38 is again trivial, just the identity matrix. This is known as the *Glashow–Iliopoulos–Maiani mechanism* [129].

Before we close, note that, in a scheme with $m < n$, $n - m$ neutrinos will remain massless, while $2m$ neutrinos will acquire Majorana masses, m light and m heavy. For example, in a model with $n = 3$ and $m = 1$, one has one light and one heavy Majorana neutrino in addition to the two massless ones. In this case, clearly there will be fewer parameters than in a model with $m = n$. Note also that for $m > n$, the case $m = 2n$ corresponds to the linear and inverse seesaw models, which will be discussed in Chapter 7.

3.5
CP Properties of Majorana Fermions

In any theory of neutrino masses, the free-field Lagrangian is given as

$$\mathcal{L}_\mathrm{M} = i \sum_{a=1}^{n} \overline{\rho}_a \overline{\sigma}_\mu \partial^\mu \rho_a - \frac{1}{2} \sum_{a,b=1}^{n} (\mathcal{M}_{\nu ab}\ \rho_a \rho_b + \text{h.c.}), \tag{3.39}$$

where the sum runs over the indices a and b. This generalizes Eq. 3.2 for an arbitrary number of Majorana neutrinos. By Fermi statistics, the mass coefficients $\mathcal{M}_{\nu ab}$ must form a symmetric matrix, which is in general complex. This matrix can always be diagonalized by a complex $n \times n$ unitary matrix $\mathcal{U}_\nu$ [46]

$$\mathcal{U}_\nu^T \mathcal{M}_\nu \mathcal{U}_\nu = \mathrm{diag}(m_1, m_2, \dots, m_n). \tag{3.40}$$

An important subtlety arises regarding the conditions for CP conservation in gauge theories of massive Majorana neutrinos. Consider the general complex mass matrix $\mathcal{M}_\nu$ in Eq. 3.39 , which is symmetric due to the Pauli principle. If this matrix is taken to be real (CP conservation), then its diagonalizing matrix U may be chosen to be orthogonal and, in general, its mass eigenvalues will have different signs. These may be assembled as a signature matrix

$$\eta = \mathrm{diag}(+, +, -, +, \dots). \tag{3.41}$$

As an example, take the simple case of just two fermion types whose mass matrices are proportional to the forms

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.42}$$

These define two non-equivalent classes of models; the first one is with $\eta =$ diag$(+,+)$, and the other, characterized by $\eta =$ diag$(+,-)$, is the one to which the standard Dirac neutrino belongs.

These signs acquire their full meaning once interactions are added, as described in Section 3.3. Unlike the case of Dirac fermions, where CP-invariance implies that the mixing matrix describing the charged current weak interaction should be real, in the Majorana case the requirement is

$$K^* = K\eta, \tag{3.43}$$

where $\eta =$ diag$(+,+,\cdots,,,..)$ is the signature matrix in Eq. 3.41 describing the relative signs of the neutrino mass eigenvalues that follow from diagonalizing Eq. 3.39 if one chooses to use real diagonalizing matrices. Notice, however, that one can always make all masses positive by introducing appropriate phase factors in the wave functions, such as the factors of i in Eq. 3.14 .

3.5.1 CP Properties and Neutrinoless Double-Beta Decay

Let us start be noting also that the condition for CP conservation, namely reality of the Lagrangian, for instance, Eq. 3.2 , can be consistent with some special cases of complexity of the lepton mixing matrix that appears in the generalized CP invariance condition for the case of Majorana fermions, given by Eq. 3.43 [169]. For example, in addition to the usual CP-conserving phase value $\phi_{12} = 0$, Eq. 3.43 is consistent with the value $\phi_{12} = \pi/2$.

CP properties of the neutrinos play a crucial role in processes involving propagators of Majorana neutrinos. As an example, consider the case of the neutrinoless double-beta decay process, which is induced by the exchange of massive Majorana neutrinos.

As will be seen in Chapter 6, the associated amplitude is proportional to

$$m_{\beta\beta} = \sum_i K_{ei}^2 m_i, \tag{3.44}$$

where the parameters K_{ei} form the first row in the lepton mixing matrix.

For the simple case of just two mass-degenerate Majorana neutrinos with opposite CP signs, $\eta =$ diag$(+,-)$, there is exact cancellation in the $0\nu\beta\beta$ amplitude. The full destructive interference is, of course, expected in this case, as it corresponds to the case of Dirac neutrino, for which $0\nu\beta\beta$ is forbidden by the lepton number symmetry. For the opposite case of $\eta =$ diag$(+,+)$, the two individual amplitudes would just add up. One sees how the CP properties play a crucial role in the discussion of neutrinoless double-beta decay [185].

3.5.2
Electromagnetic Properties of Majorana Neutrinos

The most general Lorentz-invariant effective electromagnetic interaction of massive neutrinos takes the form [169]

$$\mathcal{L}_{\rm em} = \sum_{a,b=1}^{n} j_{ab}\ \overline{\nu}_a[(\overline{\sigma}_\mu \Box - \overline{\sigma}_\nu \partial^\nu \partial_\mu)A^\mu]\nu_b + \frac{1}{2}[h_{ab}\ \nu_a \sigma^{\mu\nu} \nu_b F_{\mu\nu} + \text{h.c.}], \quad (3.45)$$

where A^μ is the photon field, $F_{\mu\nu}$ is the usual Maxwell tensor and ν_a represents one of the mass-eigenstate two-component Majorana neutrinos. The matrix $\sigma^{\mu\nu}$ is defined by (Appendix A)

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4}(\sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu)_\alpha{}^\beta, \quad (3.46)$$

implying that, for anti-commuting fermion fields

$$\begin{aligned} \nu_a \sigma^{\mu\nu} \nu_b &\equiv \nu_a^\alpha (\sigma^{\mu\nu})_\alpha{}^\beta \nu_{b\beta} = \nu_a^\alpha (\sigma^{\mu\nu})_\alpha{}^\beta \epsilon_{\beta\gamma} \nu^{b\gamma} = \nu_a^\alpha (\sigma^{\mu\nu}\epsilon)_{\alpha\beta} \nu_b^\beta \\ &= \nu_a^\alpha (\sigma^{\mu\nu}\epsilon)_{\beta\alpha} \nu_b^\beta = -\nu_b \sigma^{\mu\nu} \nu_a, \end{aligned} \quad (3.47)$$

where we have used Eq. A.30 (or equivalently Eq. A.35). This property implies that the form factors obey the following symmetry properties:

$$j^*_{\alpha\beta} = j_{\beta\alpha}, \quad h_{\alpha\beta} = -h_{\beta\alpha}. \quad (3.48)$$

The first follows from hermiticity, while the second comes from the Pauli principle.

The explicit form of the diagonal matrix element of the electromagnetic current of a Majorana fermion can be easily found by substituting the Fourier decomposition of the Majorana field, Eq. 3.15 , into Eq. 3.45 . It is worthwhile to explain this in some detail, first in two-component spinor notation and then in the four-component one. We define the current in momentum space through

$$\begin{aligned} J_\mu(q^2) \equiv \int & \frac{d^3k_1}{(2\pi)^3 2k_1^0} \frac{d^3k_2}{(2\pi)^3 2k_2^0} \frac{d^3k_3}{(2\pi)^3 2k_3^0} \quad \cdots \\ & [\langle 0|a(k_3)a^\dagger(q)A(p_2)A^\dagger(k_1)\overline{u}_L(k_1)(-\overline{\sigma}_\mu k_3^2 + \overline{\sigma}_\nu k_3^\nu k_{3\mu})u_L(k_2)A(k_2)A^\dagger(p_1)|0\rangle \\ & + \langle 0|\ a(k_3)a^\dagger(q)A(p_2)A(k_1)\overline{v}_L(k_1)(-\overline{\sigma}_\mu k_3^2 + \overline{\sigma}_\nu k_3^\nu k_{3\mu})v_L(k_2)A^\dagger(k_2)A^\dagger(p_1)|0\rangle], \end{aligned} \quad (3.49)$$

where $A, A^\dagger$ are the annihilation and creation operators for the Majorana field as in Eq. 3.15 or Eqs. A.36 and A.37; $a, a^\dagger$ are the same for the photon and the dots refer to the *plane waves* with the appropriate normalization. The momentum of the photon satisfies $q = p_1 - p_2$. Using the commutation relations we get, after doing the integrations, that the current is (up to normalization factors)

$$\begin{aligned} J_\mu(q^2) =& j(q^2)[-\overline{u}_L(p_2)(\overline{\sigma}_\mu q^2 - \overline{\sigma}_\nu q^\nu q_\mu)u_L(p_1) \\ & + \overline{v}_L(p_1)(\overline{\sigma}_\mu q^2 - \overline{\sigma}_\nu q^\nu q_\mu)v_L(p_2)], \end{aligned} \quad (3.50)$$

where the relative minus sign comes from the anti-commutation relations and the spinors u_L, v_L are two-component spinors connected to the usual four-component spinors by

$$P_L u = \begin{bmatrix} u_L \\ 0 \end{bmatrix}, \quad P_L v = \begin{bmatrix} v_L \\ 0 \end{bmatrix}. \tag{3.51}$$

The form of the electromagnetic neutrino diagonal coupling in Eq. 3.50 can now be written in four-component notation. Using Eq. A.57, we obtain

$$J_\mu(q^2) = j(q^2)\ [-\overline{u}(p_2)(\gamma_\mu q^2 - \not{q} q_\mu)P_L u(p_1) + \overline{v}(p_1)(\gamma_\mu q^2 - \not{q} q_\mu)P_L v(p_2)]. \tag{3.52}$$

Now we have

$$\begin{aligned}
\overline{v}(p_1)(\gamma_\mu q^2 - \not{q} q_\mu)P_L v(p_2) &= v^T(p_2)P_L^T[\gamma_\mu q^2 - \not{q} q_\mu]^T \overline{v}^T(p_1) \\
&= -\overline{u}(p_2)CP_L^T C^{-1} C[\gamma_\mu q^2 - \not{q} q_\mu]^T C^{-1} u(p_1) \\
&= \overline{u}(p_2)P_L[\gamma_\mu q^2 - \not{q} q_\mu]u(p_1) \\
&= \overline{u}(p_2)[\gamma_\mu q^2 - \not{q} q_\mu]P_R u(p_1).
\end{aligned} \tag{3.53}$$

Therefore

$$\begin{aligned}
J_\mu(q^2) &= j(q^2)\overline{u}(p_2)[\gamma_\mu q^2 - \not{q} q_\mu](-P_L + P_R)u(p_1) \\
&= j(q^2)\overline{u}(p_2)[\gamma_\mu q^2 - \not{q} q_\mu]\gamma_5 u(p_1) \\
&= j(q^2)\overline{u}(p_2)[\gamma_\mu q^2 + 2m\ q_\mu]\gamma_5 u(p_1),
\end{aligned} \tag{3.54}$$

where the last form results from the fact that $\not{q} = \not{p}_1 - \not{p}_2$ and the use of the Dirac equation for the on-shell external spinors. This form factor is purely axial, since a vector form factor would vanish identically, as a result of the Majorana property and Fermi statistics, combined.

On the other hand, Eq. 3.48 implies that all the diagonal moments vanish; hence a Majorana particle, by itself, cannot have a nonzero magnetic moment. However, Majorana neutrinos can have nonzero *transition* magnetic moments [169].

These transition moments can cause the spin-flavour precession effect of neutrinos in a magnetic field, which has been extensively discussed in connection with the propagation of solar neutrinos [186, 187]. Radiative neutrino decays $\nu_\alpha \to \nu_\beta + \gamma$ may also be induced by *transition* magnetic moments. The amplitude

$$i \sum_\sigma \mathcal{I}m\ [K_{\sigma\alpha}K^*_{\sigma\beta}]\overline{u}_\beta \mathcal{M}_\nu(\sigma)u_\alpha + \sum_\sigma \mathcal{R}e\ [K_{\sigma\alpha}K^*_{\sigma\beta}]\overline{u}_\beta \mathcal{M}_\nu(\sigma)\gamma_5 u_\alpha, \tag{3.55}$$

where the sum is over the charged leptons of the theory, and the amplitudes $\overline{u}_\beta \mathcal{M}_\nu(\sigma)u_\alpha$ and $\overline{u}_\beta \mathcal{M}_\nu(\sigma)\gamma_5 u_\alpha$ are calculable within the gauge theory of interest [188].

3.5.3
Majorana–Dirac 'Confusion Theorem'

Phenomenological differences between Dirac and Majorana neutrinos are very hard to probe, first, because neutrinos are known to be very light and, second, because of the chiral nature of the weak interaction. This is sometimes referred to as the *confusion theorem*.[4)]

The most direct way to probe the Majorana nature of neutrinos is to search for processes involving lepton number violation, such as neutrinoless double-beta decay. The essence of this deep connection is contained in the 'black-box theorem' [87, 88], which will be considered in Chapter 6.

Similarly, as we saw above, the electromagnetic properties of neutrinos [169, 170] can probe their Majorana nature, see Sections 3.3.2 and 3.3.3, and Problem 3.4. Finally, certain CP-violation phenomena involving Majorana phases also provide a 'conceptual' way to distinguish Dirac from Majorana [171], but they are, unfortunately, not realistic, given the smallness of neutrino mass and the effective V – A form of the weak interaction, as mentioned above. This is an example of the 'confusion theorem', see Problem 4.3.

3.6 Summary

In summary, in this chapter we have given the general model-independent description of basic massive neutrino concepts. In the next chapter, we will give a dedicated description of neutrino oscillation phenomena, using solar and atmospheric neutrinos as well as neutrinos produced in reactors and accelerators. In particular, we will describe how these data can be analysed in a global way in order to extract information on fundamental neutrino mass and mixing parameters.

3.7 Problems for Chapter 3

3.1 Using the definition $\rho'_\alpha \equiv S_\alpha^{\beta}\rho_\beta$ and the relations in the Appendix A, verify Eqs. 3.19 and 3.4 and show that the Lagrangian in Eq. 3.2 is Lorentz-invariant.

3.2 Decomposing a massive Dirac into two two-component Majorana neutrinos, show that the resulting amplitude for neutrinoless double- beta decay vanishes.

3.3 Derive the explicit form of the Majorana fermion propagators in Eqs. 3.19 and 3.20 . Compare with the results for Majorana propagators using four-component spinors as given in Appendix B.

3.4 Analyse the conditions for CP invariance of the transition magnetic moments given in Eq. 3.55.

3.5 Show that, in the unitary approximation, for three generations of massive Majorana neutrinos one can write the relation between weak (ν_e, ν_μ, ν_τ) and mass-eigenstate neutrinos (ν_1, ν_2, ν_3) as

4) As far as we know, this terminology is due to Boris Kayser.

$$\nu_e = c_{12}c_{13}\nu_1 + s_{12}c_{13}e^{i\phi_{12}}\nu_2 + s_{13}e^{i\phi_{13}}\nu_3 \tag{3.56}$$

$$\begin{aligned}\nu_\mu = {} & [-s_{12}c_{23}e^{-i\phi_{12}} - c_{12}s_{13}s_{23}e^{i(\phi_{23}-\phi_{13})}]\nu_1 + \\ & [c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i(\phi_{12}+\phi_{23}-\phi_{13})}]\nu_2 + c_{13}s_{23}e^{i\phi_{23}}\nu_3\end{aligned} \tag{3.57}$$

$$\begin{aligned}\nu_\mu = {} & [s_{12}s_{23}e^{-i(\phi_{12}+\phi_{23})} - c_{12}s_{13}c_{23}e^{-i\phi_{13}}]\nu_1 \\ & - [c_{12}s_{23}e^{-i\phi_{23}} + s_{12}s_{13}c_{23}e^{i(\phi_{12}-\phi_{13})}]\nu_2 + c_{13}c_{23}\nu_3,\end{aligned} \tag{3.58}$$

where $c_{ij} \equiv \cos\theta_{ij}, s_{ij} \equiv \sin\theta_{ij}$ and ϕ_{ij} are the CP-violating phases in the symmetrical parametrization of the lepton mixing matrix.

3.6 Compare the symmetrical parameterization of the unitary lepton mixing matrix [46, 58] with the form given in the Particle Data Book [21].