Chapter 2

Properties of neutrinos

In quantum field theory spin-$\frac{1}{2}$ particles are described by four-component wavefunctions $\psi(x)$ (spinors) which obey the Dirac equation. The four independent components of $\psi(x)$ correspond to particles and antiparticles with the two possible spin projections $J_Z = \pm 1/2$ equivalent to the two helicities $\mathcal{H} = \pm 1$. Neutrinos as fundamental leptons are spin-$\frac{1}{2}$ particles like other fermions; however, it is an experimental fact that only left-handed neutrinos ($\mathcal{H} = -1$) and right-handed antineutrinos ($\mathcal{H} = +1$) are observed. Therefore, a two-component spinor description should, in principle, be sufficient (Weyl spinors). In a four-component theory they are obtained by projecting out of a general spinor $\psi(x)$ the components with $\mathcal{H} = +1$ for particles and $\mathcal{H} = -1$ for antiparticles with the help of the operators $P_{L,R} = \frac{1}{2}(1 \mp \gamma_5)$. The two-component theory of the neutrinos will be discussed in detail later. The discussion will be quite general; for a more extensive discussion see [Bjo64, Bil87, Kay89, Kim93, Sch97, Fuk03a].

2.1 Helicity and chirality

The Dirac equation is the relativistic wave equation for spin-$\frac{1}{2}$ particles and given by (using Einstein conventions)

$$\left(\mathrm{i}\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)\psi = 0. \tag{2.1}$$

Here ψ denotes a four-component spinor and the 4×4 γ-matrices are given in the form[1]

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{2.2}$$

where σ_i correspond to the 2×2 Pauli matrices. Detailed introductions and treatments can be found in [Bjo64]. The matrix γ_5 is given by

$$\gamma_5 = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

[1] Other conventions of the γ-matrices are also commonly used in the literature, which leads to slightly different forms for the following expressions.

and the following anticommutator relations hold:

$$\{\gamma_\alpha, \gamma_\beta\} = 2g_{\alpha\beta} \tag{2.4}$$

$$\{\gamma_\alpha, \gamma_5\} = 0 \tag{2.5}$$

with $g_{\alpha\beta}$ as the metric $(+1, -1, -1, -1)$. Multiplying the Dirac equation from the left with γ_0 and using $\gamma_i = \gamma_0\gamma_5\sigma_i$ results in

$$\left(\mathrm{i}\gamma_0^2 \frac{\partial}{\partial x_0} - \mathrm{i}\gamma_0^2\gamma_5\sigma_i \frac{\partial}{\partial x_i} - m\gamma_0\right)\psi = 0 \qquad i = 1, \ldots, 3. \tag{2.6}$$

Another multiplication of (2.6) from the left with γ_5 and using $\gamma_5\sigma_i = \sigma_i\gamma_5$ (which follows from (2.5)) leads to ($\gamma_0^2 = 1$, $\gamma_5^2 = 1$)

$$\left(\mathrm{i}\frac{\partial}{\partial x_0}\gamma_5 - \mathrm{i}\sigma_i \frac{\partial}{\partial x_i} + m\gamma_0\gamma_5\right)\psi = 0. \tag{2.7}$$

Subtraction and addition of the last two equations result in the following system of coupled equations:

$$\left(\mathrm{i}\frac{\partial}{\partial x_0}(1+\gamma_5) - \mathrm{i}\sigma_i \frac{\partial}{\partial x_i}(1+\gamma_5) - m\gamma_0(1-\gamma_5)\right)\psi = 0 \tag{2.8}$$

$$\left(\mathrm{i}\frac{\partial}{\partial x_0}(1-\gamma_5) + \mathrm{i}\sigma_i \frac{\partial}{\partial x_i}(1-\gamma_5) - m\gamma_0(1+\gamma_5)\right)\psi = 0. \tag{2.9}$$

Now let us introduce left- and right-handed components by defining two projection operators P_L and P_R given by

$$P_L = \tfrac{1}{2}(1-\gamma_5) \qquad \text{and} \qquad P_R = \tfrac{1}{2}(1+\gamma_5) \tag{2.10}$$

Because they are projectors, the following relations hold:

$$P_L P_R = 0 \qquad P_L + P_R = 1 \qquad P_L^2 = P_L \qquad P_R^2 = P_R. \tag{2.11}$$

With the definition

$$\psi_L = P_L\psi \qquad \text{and} \qquad \psi_R = P_R\psi \tag{2.12}$$

it is obviously valid that

$$P_L\psi_R = P_R\psi_L = 0. \tag{2.13}$$

Then the following eigenequation holds:

$$\gamma_5\psi_{L,R} = \lambda \pm \psi_{L,R}. \tag{2.14}$$

The eigenvalues $\pm$ 1 to γ_5 are called chirality and $\psi_{L,R}$ are called chiral projections of ψ. Any spinor ψ can be rewritten in chiral projections as

$$\psi = (P_L + P_R)\psi = P_L\psi + P_R\psi = \psi_L + \psi_R. \tag{2.15}$$

The equations (2.8) and (2.9) can now be expressed in these projections as

$$\left(\mathrm{i}\frac{\partial}{\partial x_0} - \mathrm{i}\sigma_i\frac{\partial}{\partial x_i}\right)\psi_R = m\gamma_0\psi_L \tag{2.16}$$

$$\left(\mathrm{i}\frac{\partial}{\partial x_0} + \mathrm{i}\sigma_i\frac{\partial}{\partial x_i}\right)\psi_L = m\gamma_0\psi_R. \tag{2.17}$$

Both equations decouple in the case of a vanishing mass $m = 0$ and can then be depicted as

$$\mathrm{i}\frac{\partial}{\partial x_0}\psi_R = \mathrm{i}\sigma_i\frac{\partial}{\partial x_i}\psi_R \tag{2.18}$$

$$\mathrm{i}\frac{\partial}{\partial x_0}\psi_L = -\mathrm{i}\sigma_i\frac{\partial}{\partial x_i}\psi_L. \tag{2.19}$$

But this is identical to the Schrödinger equation ($x_0 = t$, $\hbar = 1$)

$$\mathrm{i}\frac{\partial}{\partial t}\psi_{L,R} = \mp\mathrm{i}\sigma_i\frac{\partial}{\partial x_i}\psi_{L,R} \tag{2.20}$$

or in momentum space ($\mathrm{i}\frac{\partial}{\partial t} = E$, $-\mathrm{i}\frac{\partial}{\partial x_i} = p_i$)

$$E\psi_{L,R} = \pm\sigma_i p_i\psi_{L,R}. \tag{2.21}$$

The latter implies that the $\psi_{L,R}$ are also eigenfunctions to the helicity operator $\mathcal{H}$ given by (see Chapter 1)

$$\mathcal{H} = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \tag{2.22}$$

ψ_L is an eigenspinor with helicity eigenvalues $\mathcal{H} = +1$ for particles and $\mathcal{H} = -1$ for antiparticles. Correspondingly ψ_R is the eigenspinor to the helicity eigenvalues $\mathcal{H} = -1$ for particles and $\mathcal{H} = +1$ for antiparticles. Therefore, in the case of massless particles, chirality and helicity are identical.[2] For $m > 0$ the decoupling of (2.16) and (2.17) is no longer possible. This means that the chirality eigenspinors ψ_L and ψ_R no longer describe particles with fixed helicity and helicity is no longer a good conserved quantum number.

The two-component theory now states that the neutrino spinor ψ_ν in weak interactions always reads as

$$\psi_\nu = \tfrac{1}{2}(1-\gamma_5)\psi = \psi_L \tag{2.23}$$

meaning that the interacting neutrino is always left-handed and the antineutrino always right-handed. For $m = 0$, this further implies that ν always has $\mathcal{H} = -1$ and $\bar{\nu}$ always $\mathcal{H} = +1$. The proof that indeed the Dirac spinors ψ_L and ψ_R can be written as the sum of two independent 2-component Weyl spinors can be found in [Sch97].

[2] May be of opposite sign depending on the representation used for the γ-matrices.

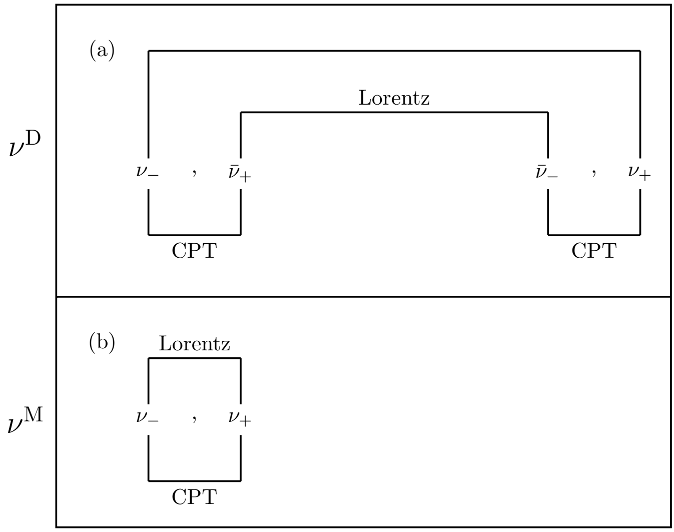

Figure 2.1. Schematic drawing of the difference between massive Dirac and Majorana neutrinos. (*a*) The Dirac case: ν_L is converted via CPT into a $\bar{\nu}_R$ and via a Lorentz boost into a ν_R. An application of CPT on the latter results in $\bar{\nu}_L$ which is different from the one obtained by applying CPT on ν_L. The result is four different states. (*b*) The Majorana case: Both operations CPT and a Lorentz boost result in the same state ν_R, there is no difference between particle and antiparticle. Only two states emerge (from [Kay89]). © World Scientific Publishing Company.

2.2 Charge conjugation

While for all fundamental fermions of the Standard Model (see Chapter 3) a clear discrimination between particle and antiparticle can be made by their electric charge, for neutrinos it is not so obvious. If particle and antiparticle are not identical, we call such a fermion a Dirac particle which has four independent components. If particle and antiparticle are identical, they are called Majorana particles (Figure 2.1). The latter requires that all additive quantum numbers (charge, strangeness, baryon number, lepton number, etc.) have to vanish. Consequently, the lepton number is violated if neutrinos are Majorana particles.

The following derivations are taken from [Bil87, Sch97]. The operator connecting particle $f(\boldsymbol{x},t)$ and antiparticle $\bar{f}(\boldsymbol{x},t)$ is charge conjugation C:

$$C|f(\boldsymbol{x},t)\rangle = \eta_c|\bar{f}(\boldsymbol{x},t)\rangle. \tag{2.24}$$

If $\psi(x)$ is a spinor field of a free neutrino then the corresponding charge conjugated field ψ^c is defined by

$$\psi \xrightarrow{C} \psi^c \equiv C\psi C^{-1} = \eta_c C\bar{\psi}^T \tag{2.25}$$

with η_c as a phase factor with $|\eta_c| = 1$. The 4×4 unitary charge conjugation matrix

C obeys the following general transformations:

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T \qquad C^{-1}\gamma_5 C = \gamma_5^T \qquad C^\dagger = C^{-1} = C^T = -C. \tag{2.26}$$

A possible representation is given as $C = \mathrm{i}\gamma_0\gamma_2$. Using the projection operators $P_{L,R}$, it follows that

$$P_{L,R}\psi = \psi_{L,R} \xrightarrow{C} P_{L,R}\psi^c = (\psi^c)_{L,R} = (\psi_{R,L})^c. \tag{2.27}$$

It is easy to show that if ψ is an eigenstate of chirality; ψ^c is an eigenstate too but it has an eigenvalue of opposite sign. Furthermore, from (2.27) it follows that the charge conjugation C transforms a right(left)-handed particle into a right(left)-handed antiparticle, leaving the helicity (chirality) untouched. Only the additional application of a parity transformation changes the helicity as well. However, the operation of (2.25) converts a right(left)-handed particle into a left(right)-handed antiparticle. Here helicity and chirality are converted as well.

To include the fact that $\psi_{L,R}$ and $\psi^c_{L,R}$ have opposite helicity, one avoids calling $\psi^c_{L,R}$ the charge conjugate of $\psi_{L,R}$. Instead it is more frequently called the CP (or CPT) conjugate with respect to $\psi_{L,R}$ [Lan88]. In the following sections we refer to ψ^c as the CP or CPT conjugate of the spinor ψ, assuming CP or CPT conservation correspondingly.

2.3 Parity transformation

A parity transformation P operation is defined as

$$\psi(\boldsymbol{x}, t) \xrightarrow{P} P\psi(\boldsymbol{x}, t)P^{-1} = \eta_p\gamma_0\psi(-\boldsymbol{x}, t). \tag{2.28}$$

The phase factor η_P with $|\eta_p| = 1$ corresponds for real $\eta_p = \pm 1$ to the inner parity. Using (2.25) for the charge conjugated field, it follows that

$$\psi^c = \eta_C C\bar{\psi}^T \xrightarrow{P} \eta_C\eta_P^* C\gamma_0^T\bar{\psi}^T = -\eta_P^*\gamma_0\psi^c. \tag{2.29}$$

This implies that a fermion and its corresponding antifermion have opposite inner parity, i.e. for a Majorana particle $\psi^c = \pm\psi$ holds which results in $\eta_P = -\eta_P^*$.

Therefore, an interesting point with respect to the inner parity occurs for Majorana neutrinos. A Majorana field can be written as

$$\psi_M = \frac{1}{\sqrt{2}}(\psi + \eta_C\psi^c) \qquad \text{with } \eta_C = \lambda_C e^{2\mathrm{i}\phi},\ \lambda_C = \pm 1 \tag{2.30}$$

where λ_C is sometimes called creation phase. By applying a phase transformation

$$\psi_M \to \psi_M e^{-\mathrm{i}\phi} = \frac{1}{\sqrt{2}}(\psi e^{-\mathrm{i}\phi} + \lambda_C\psi^c e^{\mathrm{i}\phi}) = \frac{1}{\sqrt{2}}(\psi + \lambda_C\psi^c) \equiv \psi_M \tag{2.31}$$

it can be achieved that the field ψ_M is an eigenstate with respect to charge conjugation C

$$\psi_M^c = \frac{1}{\sqrt{2}}(\psi^c + \lambda_C \psi) = \lambda_C \psi_M \tag{2.32}$$

with eigenvalues $\lambda_C = \pm 1$. This means the Majorana particle is identical to its antiparticle; i.e., ψ_M and ψ_M^c cannot be distinguished. With respect to CP, one obtains

$$\begin{aligned}\psi_M(\boldsymbol{x}, t) \xrightarrow{C} \psi_M^c = \lambda_C \psi_M \xrightarrow{P} \frac{\lambda_C}{\sqrt{2}}(\eta_P \gamma_0 \psi - \lambda_C \eta_P^* \gamma_0 \psi^c) \\ = \lambda_C \eta_P \gamma_0 \psi_M = \pm \mathrm{i} \gamma_0 \psi_M(-\boldsymbol{x}, t)\end{aligned} \tag{2.33}$$

because $\eta_P^* = -\eta_P$. This means that the inner parity of a Majorana particle is imaginary, $\eta_P = \pm \mathrm{i}$ if $\lambda_C = \pm 1$. Finally, from (2.31) it follows that

$$(\gamma_5 \psi_M)^c = \eta_C C (\bar{\gamma}_5 \bar{\psi}_M)^T = -\eta_C C \gamma_5^T \bar{\psi}_M^T = -\gamma_5 \psi_M^c = -\lambda_C \gamma_5 \psi_M \tag{2.34}$$

because $\gamma_5 \bar{\psi}_M = (\gamma_5 \psi_M)^\dagger \gamma_0 = \psi_M^\dagger \gamma_5 \gamma_0 = -\bar{\psi}_M \gamma_5$. Using this together with (2.27) one concludes that an eigenstate to C cannot be at the same time an eigenstate to chirality. A Majorana neutrino, therefore, has no fixed chirality. However, because ψ and ψ^c obey the Dirac equation, ψ_M will also do so.

For a discussion of T transformation and C, CP and CPT properties, see [Kay89, Kim93].

2.4 Dirac and Majorana mass terms

Consider the case of free fields without interactions and start with the Dirac mass. The Dirac equation can then be deduced with the help of the Euler–Lagrange equation from a Lagrangian [Bjo64]:

$$\mathcal{L} = \bar{\psi} \left(\mathrm{i} \gamma_\mu \frac{\partial}{\partial x_\mu} - m_D \right) \psi \tag{2.35}$$

where the first term corresponds to the kinetic energy and the second is the mass term. The Dirac mass term is, therefore,

$$\mathcal{L} = m_D \bar{\psi} \psi \tag{2.36}$$

where the combination $\bar{\psi}\psi$ has to be Lorentz invariant and Hermitian. Requiring $\mathcal{L}$ to be Hermitian as well, m_D must be real ($m_D^* = m_D$). Using the following relations valid for two arbitrary spinors ψ and ϕ (which follow from (2.10) and (2.11))

$$\bar{\psi}_L \phi_L = \bar{\psi} P_R P_L \phi = 0 \qquad \bar{\psi}_R \phi_R = 0 \tag{2.37}$$

it follows that

$$\bar{\psi}\phi = (\bar{\psi}_L + \bar{\psi}_R)(\phi_L + \phi_R) = \bar{\psi}_L \phi_R + \bar{\psi}_R \phi_L. \tag{2.38}$$

In this way the Dirac mass term can be written in its chiral components (Weyl spinors) as

$$\mathcal{L} = m_D(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) \qquad \text{with } \bar{\psi}_R\psi_L = (\bar{\psi}_L\psi_R)^\dagger. \tag{2.39}$$

Applying this to neutrinos, it requires both a left- and a right-handed Dirac neutrino to produce such a mass term. In the Standard Model of particle physics only left-handed neutrinos exist; that is the reason why neutrinos remain massless as will be discussed in Chapter 3.

In a more general treatment including ψ^c one might ask which other combinations of spinors behaving like Lorentz scalars can be produced. Three more are possible: $\bar{\psi}^c\psi^c$, $\bar{\psi}\psi^c$ and $\bar{\psi}^c\psi$. $\bar{\psi}^c\psi^c$ is also hermitian and equivalent to $\bar{\psi}\psi$; $\bar{\psi}\psi^c$ and $\bar{\psi}^c\psi$ are hermitian conjugates, which can be shown for arbitrary spinors

$$(\bar{\psi}\phi)^\dagger = (\psi^\dagger\gamma_0\phi)^\dagger = \phi^\dagger\gamma_0\psi = \bar{\phi}\psi. \tag{2.40}$$

With this we have an additional hermitian mass term, called the Majorana mass term and given by

$$\mathcal{L} = \frac{1}{2}(m_M\bar{\psi}\psi^c + m_M^*\bar{\psi}^c\psi) = \frac{1}{2}m_M\bar{\psi}\psi^c + h.c.^3 \tag{2.41}$$

m_M is called the Majorana mass. Now using again the chiral projections with the notation

$$\psi^c_{L,R} = (\psi^c)_{R,L} = (\psi_{R,L})^c \tag{2.42}$$

one gets two hermitian mass terms:

$$\mathcal{L}^L = \frac{1}{2}m_L(\bar{\psi}_L\psi^c_R + \bar{\psi}^c_R\psi_L) = \frac{1}{2}m_L\bar{\psi}_L\psi^c_R + h.c. \tag{2.43}$$

$$\mathcal{L}^R = \frac{1}{2}m_R(\bar{\psi}^c_L\psi_R + \bar{\psi}_R\psi^c_L) = \frac{1}{2}m_R\bar{\psi}^c_L\psi_R + h.c. \tag{2.44}$$

with $m_{L,R}$ as real Majorana masses because of (2.40). Let us define two Majorana fields (see (2.30) with $\lambda_C = 1$)

$$\phi_1 = \psi_L + \psi^c_R \qquad \phi_2 = \psi_R + \psi^c_L \tag{2.45}$$

which allows (2.43) to be rewritten as

$$\mathcal{L}^L = \frac{1}{2}m_L\bar{\phi}_1\phi_1 \qquad \mathcal{L}^R = \frac{1}{2}m_R\bar{\phi}_2\phi_2. \tag{2.46}$$

While $\psi_{L,R}$ are interaction eigenstates, $\phi_{1,2}$ are mass eigenstates to $m_{L,R}$.

The most general mass term (the Dirac–Majorana mass term) is a combination of (2.39) and (2.43) (Figure 2.2):

$$\begin{aligned} 2\mathcal{L} &= m_D(\bar{\psi}_L\psi_R + \bar{\psi}^c_L\psi^c_R) + m_L\bar{\psi}_L\psi^c_R + m_R\bar{\psi}^c_L\psi_R + h.c. \\ &= (\bar{\psi}_L, \bar{\psi}^c_L)\begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}\begin{pmatrix}\psi^c_R \\ \psi_R\end{pmatrix} + h.c. \\ &= \bar{\Psi}_L M \Psi^c_R + \bar{\Psi}^c_R M \Psi_L \end{aligned} \tag{2.47}$$

[3] $h.c.$ throughout the book signifies Hermitian conjugate.

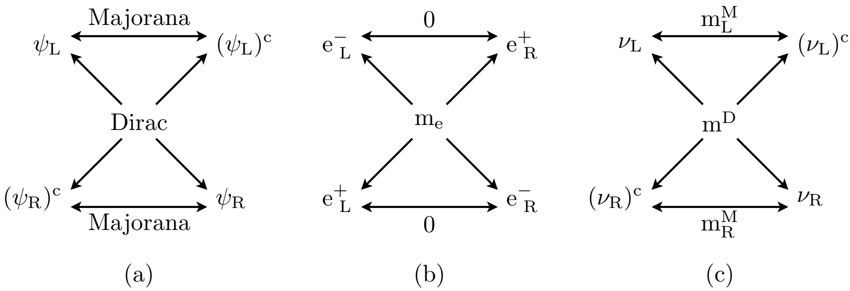

Figure 2.2. Coupling schemes for fermion fields via Dirac and Majorana masses: (*a*) general scheme for left- and right-handed fields and the charge conjugate fields; (*b*) the case for electrons (because of its electric charge only Dirac-mass terms are possible) and (*c*) coupling scheme for neutrinos. It is the only fundamental fermion that allows all possible couplings (From [Kun01]).

where, in the last step, the following was used:

$$M = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \qquad \Psi_L = \begin{pmatrix} \psi_L \\ \psi_L^c \end{pmatrix} = \begin{pmatrix} \psi_L \\ (\psi_R)^c \end{pmatrix} \tag{2.48}$$

implying

$$(\Psi_L)^c = \begin{pmatrix} (\psi_L)^c \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_R^c \\ \psi_R \end{pmatrix} = \Psi_R^c.$$

In the case of CP conservation the elements of the mass matrix M are real. Coming back to neutrinos, in the known neutrino interactions only ψ_L and ψ_R^c are present (active neutrinos) and not the fields ψ_R and ψ_L^c (called sterile neutrinos, they are not participating in weak interaction); it is quite common to distinguish between both types in the notation: $\psi_L = \nu_L$, $\psi_R^c = \nu_R^c$, $\psi_R = N_R$, $\psi_L^c = N_L^c$. With this notation, (2.47) becomes

$$\begin{aligned} 2\mathcal{L} &= m_D(\bar{\nu}_L N_R + \bar{N_L^c}\nu_R^c) + m_L\bar{\nu}_L\nu_R^c + m_R\bar{N_L^c}N_R + h.c. \\ &= (\bar{\nu}_L, \bar{N_L^c}) \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} \nu_R^c \\ N_R \end{pmatrix} + h.c. \end{aligned} \tag{2.49}$$

The mass eigenstates are obtained by diagonalizing M and are given as

$$\psi_{1L} = \cos\theta\psi_L - \sin\theta\psi_L^c \qquad \psi_{1R}^c = \cos\theta\psi_R^c - \sin\theta\psi_R \tag{2.50}$$

$$\psi_{2L} = \sin\theta\psi_L + \cos\theta\psi_L^c \qquad \psi_{2R}^c = \sin\theta\psi_R^c + \cos\theta\psi_R \tag{2.51}$$

while the mixing angle θ is given by

$$\tan 2\theta = \frac{2m_D}{m_R - m_L}. \tag{2.52}$$

The corresponding mass eigenvalues are

$$\tilde{m}_{1,2} = \frac{1}{2}\left[(m_L + m_R) \pm \sqrt{(m_L - m_R)^2 + 4m_D^2}\right]. \tag{2.53}$$

To get positive masses,[4] we use [Lan88, Gro90]

$$\tilde{m}_k = \epsilon_k m_k \qquad \text{with } m_k = |\tilde{m}_k| \text{ and } \epsilon_k = \pm 1 \ (k = 1, 2). \tag{2.54}$$

To get a similar expression as (2.45), two independent Majorana fields with masses m_1 and m_2 (with $m_k \geq 0$) are introduced via $\phi_k = \psi_{kL} + \epsilon_k \psi_{kR}^c$ or, explicitly,

$$\phi_1 = \psi_{1L} + \epsilon_1 \psi_{1R}^c = \cos\theta(\psi_L + \epsilon_1 \psi_R^c) - \sin\theta(\psi_L^c + \epsilon_1 \psi_R) \tag{2.55}$$

$$\phi_2 = \psi_{2L} + \epsilon_2 \psi_{2R}^c = \sin\theta(\psi_L + \epsilon_2 \psi_R^c) + \cos\theta(\psi_L^c + \epsilon_2 \psi_R) \tag{2.56}$$

and, as required for Majorana fields,

$$\phi_k^c = (\psi_{kL})^c + \epsilon_k \psi_{kL} = \epsilon_k(\epsilon_k \psi_{kR}^c + \psi_{kL}) = \epsilon_k \phi_k \tag{2.57}$$

ϵ_k is the CP eigenvalue of the Majorana neutrino ϕ_k. So we finally get the analogous expression to (2.45):

$$2\mathcal{L} = m_1 \bar{\phi}_1 \phi_1 + m_2 \bar{\phi}_2 \phi_2. \tag{2.58}$$

From this general discussion one can take some interesting special aspects:

(1) $m_L = m_R = 0$ ($\theta = 45°$), resulting in $m_{1,2} = m_D$ and $\epsilon_{1,2} = \mp 1$. As Majorana eigenstates, two degenerated states emerge:

$$\phi_1 = \frac{1}{\sqrt{2}}(\psi_L - \psi_R^c - \psi_L^c + \psi_R) = \frac{1}{\sqrt{2}}(\psi - \psi^c) \tag{2.59}$$

$$\phi_2 = \frac{1}{\sqrt{2}}(\psi_L + \psi_R^c + \psi_L^c + \psi_R) = \frac{1}{\sqrt{2}}(\psi + \psi^c). \tag{2.60}$$

These can be used to construct a Dirac field ψ:

$$\frac{1}{\sqrt{2}}(\phi_1 + \phi_2) = \psi_L + \psi_R = \psi. \tag{2.61}$$

The corresponding mass term (2.58) is (because $\bar{\phi}_1 \phi_2 + \bar{\phi}_2 \phi_1 = 0$)

$$\mathcal{L} = \frac{1}{2} m_D(\bar{\phi}_1 + \bar{\phi}_2)(\phi_1 + \phi_2) = m_D \bar{\psi}\psi. \tag{2.62}$$

We are left with a pure Dirac field. As a result, a Dirac field can be seen, using (2.61), to be composed of two degenerated Majorana fields; i.e., a Dirac ν can be seen as a pair of degenerated Majorana ν. The Dirac case is, therefore, a special solution of the more general Majorana case.

[4] An equivalent procedure for $\tilde{m}_k < 0$ would be a phase transformation $\psi_k \rightarrow i\psi_k$ resulting in a change of sign of the $\bar{\psi}^c \psi$ terms in (2.43). With $m_k = -\tilde{m}_k > 0$, positive m_k terms in (2.43) result.

(2) $m_D \gg m_L, m_R$ ($\theta \approx 45°$): In this case the states $\phi_{1,2}$ are almost degenerated with $m_{1,2} \approx m_D$ and such an object is called a pseudo-Dirac neutrino.
(3) $m_D = 0$ ($\theta = 0$): In this case $m_{1,2} = m_{L,R}$ and $\epsilon_{1,2} = 1$. So $\phi_1 = \psi_L + \psi_R^c$ and $\phi_2 = \psi_R + \psi_L^c$. This is the pure Majorana case.
(4) $m_R \gg m_D, m_L = 0$ ($\theta = (m_D/m_R) \ll 1$): One obtains two mass eigenvalues:

$$m_\nu = m_1 = \frac{m_D^2}{m_R} \qquad m_N = m_2 = m_R \left(1 + \frac{m_D^2}{m_R^2}\right) \approx m_R \tag{2.63}$$

and

$$\epsilon_{1,2} = \mp 1.$$

The corresponding Majorana fields are

$$\phi_1 \approx \psi_L - \psi_R^c \qquad \phi_2 \approx \psi_L^c + \psi_R. \tag{2.64}$$

The last scenario is especially popular within the seesaw model of neutrino mass generation and will be discussed in more detail in Chapter 5.

2.4.1 Generalization to n flavors

The discussion so far has related to only one neutrino flavor. The generalization to n flavors will not be discussed in greater detail; only some general statements are made—see [Bil87, Kim93, Sch97] for a more complete discussion. A Weyl spinor is now an n-component vector in flavor space, given, for example, as

$$\nu_L = \begin{pmatrix} \nu_{1L} \\ \cdot \\ \cdot \\ \cdot \\ \nu_{nL} \end{pmatrix} \qquad N_R = \begin{pmatrix} N_{1R} \\ \cdot \\ \cdot \\ \cdot \\ N_{nR} \end{pmatrix} \tag{2.65}$$

where every ν_{iL} and N_{iR} are normal Weyl spinors with flavor i. Correspondingly, the masses m_D, m_L, m_R are now $n \times n$ matrices M_D, M_L and M_R with complex elements and $M_L = M_L^T, M_R = M_R^T$. The general symmetric $2n \times 2n$ matrix is then, in analogy to (2.48),

$$M = \begin{pmatrix} M_L & M_D \\ M_D^T & M_R \end{pmatrix}. \tag{2.66}$$

The most general mass term (2.47) is now

$$2\mathcal{L} = \bar{\Psi}_L M \Psi_R^c + \bar{\Psi}_R^c M^\dagger \Psi_L \tag{2.67}$$
$$= \bar{\nu}_L M_D N_R + \bar{N}_L^c M_D^T \nu_R^c + \bar{\nu}_L M_L \nu_R^c + \bar{N}_L^c M_R N_R + h.c. \tag{2.68}$$

where

$$\Psi_L = \begin{pmatrix} \nu_L \\ N_L^c \end{pmatrix} \qquad \text{and} \qquad \Psi_R^c = \begin{pmatrix} \nu_R^c \\ N_R \end{pmatrix}. \tag{2.69}$$

Diagonalization of M results in $2n$ Majorana mass eigenstates with associated mass eigenvalues $\epsilon_i m_i (\epsilon_i = \pm 1, m_i \geq 0)$. In the previous discussion, an equal number of active and sterile flavors ($n_a = n_s = n$) is assumed. In the most general case with $n_a \neq n_s$, M_D is an $n_a \times n_s$, M_L an $n_a \times n_a$ and M_R an $n_s \times n_s$ matrix. So the full matrix M is an $(n_a + n_s) \times (n_a + n_s)$ matrix whose diagonalization results in $(n_a + n_s)$ mass eigenstates and eigenvalues.

In seesaw models light neutrinos are given by the mass matrix (still to be diagonalized)

$$M_\nu = M_D M_R^{-1} M_D^T \tag{2.70}$$

in analogy to m_ν in (2.63).

Having discussed the formal description of neutrinos in some detail, we now take a look at the concept of lepton number.

2.5 Lepton number

Conserved quantum numbers arise from the invariance of the equation of motion under certain symmetry transformations. Continuous symmetries (e.g., translation) can be described by real numbers and lead to additive quantum numbers, while discrete symmetries (e.g., spatial reflections through the origin) are described by integers and lead to multiplicative quantum numbers. For some of them the underlying symmetry operations are known, as discussed in more detail in Chapter 3. Some quantum numbers, however, have not yet been associated with a fundamental symmetry such as baryon number B or lepton number L and their conservation is only motivated by experimental observation. The quantum numbers conserved in the individual interactions are shown in Table 2.1. Lepton number was introduced to characterize experimental observations of weak interactions. Each lepton is defined as having a lepton number $L = +1$, each antilepton $L = -1$. Moreover, each generation of leptons has its own lepton number L_e, L_μ, L_τ with $L = L_e + L_\mu + L_\tau$. Individual lepton number is not conserved, as has been established with the observation of neutrino oscillations (see Chapter 8).

Consider the four Lorentz scalars discussed under a global phase transformation $\mathrm{e}^{i\alpha}$:

$$\psi \to \mathrm{e}^{i\alpha}\psi \qquad \bar{\psi} \to \mathrm{e}^{-i\alpha}\bar{\psi} \qquad \text{so that} \qquad \bar{\psi}\psi \to \bar{\psi}\psi \tag{2.71}$$

$$\psi^c \to (\mathrm{e}^{i\alpha}\psi)^c = \eta_C C \mathrm{e}^{i\bar{\alpha}} \bar{\psi}^T = \mathrm{e}^{-i\alpha}\psi^c \qquad \bar{\psi}^c \to \mathrm{e}^{i\alpha}\bar{\psi}^c. \tag{2.72}$$

As can be seen, $\bar{\psi}\psi$ and $\bar{\psi}^c\psi^c$ are invariant under this transformation and are connected to a conserved quantum number, namely lepton number: ψ annihilates a lepton or creates an antilepton, $\bar{\psi}$ acts oppositely. $\bar{\psi}\psi$ and $\bar{\psi}^c\psi^c$ result in transitions $\ell \to \ell$ or $\bar{\ell} \to \bar{\ell}$ with $\Delta L = 0$. This does not relate to the other two Lorentz scalars $\bar{\psi}\psi^c$ and $\bar{\psi}^c\psi$ which force transitions of the form $\ell \to \bar{\ell}$ or $\bar{\ell} \to \ell$ corresponding to $\Delta L = \pm 2$ according to the assignment made earlier. For charged leptons such lepton-number-violating transitions are forbidden (i.e. $\mathrm{e}^- \to \mathrm{e}^+$) and they have to be Dirac particles. But if one associates a mass to neutrinos both types of transitions are, in principle, possible.

Table 2.1. Summary of conservation laws. B corresponds to baryon number and L to total lepton number.

Conservation law	Strong	Electromagnetic	Weak
Energy	yes	yes	yes
Momentum	yes	yes	yes
Angular momentum	yes	yes	yes
B, L	yes	yes	yes
P	yes	yes	no
C	yes	yes	no
CP	yes	yes	no
T	yes	yes	no
CPT	yes	yes	yes

Table 2.2. Some selected experimental limits on lepton-number-violating processes. The values are taken from [PDG00, PDG08] and [Kun01]. © 2000 by the American Physical Society.

Process	Exp. limit on BR
$\mu \to \mathrm{e}\gamma$	$< 1.2 \times 10^{-11}$
$\mu \to 3\mathrm{e}$	$< 1.0 \times 10^{-12}$
$\mu(A, Z) \to \mathrm{e}^-(A, Z)$	$< 6.1 \times 10^{-13}$
$\mu(A, Z) \to \mathrm{e}^+(A, Z)$	$< 1.7 \times 10^{-12}$
$\tau \to \mu\gamma$	$< 4.5 \times 10^{-8}$
$\tau \to \mathrm{e}\gamma$	$< 1.1 \times 10^{-7}$
$\tau \to 3\mathrm{e}$	$< 3.6 \times 10^{-8}$
$\tau \to 3\mu$	$< 3.2 \times 10^{-8}$
$K^+ \to \pi^-\mathrm{e}^+\mathrm{e}^+$	$< 6.4 \times 10^{-10}$
$K^+ \to \pi^-\mathrm{e}^+\mu^+$	$< 5.0 \times 10^{-10}$
$K^+ \to \pi^+\mathrm{e}^+\mu^-$	$< 5.2 \times 10^{-10}$
$K^+ \to \pi^-\mu^+\mu^+$	$< 3.0 \times 10^{-9}$

If the lepton number is related to a global symmetry which has to be broken spontaneously, a Goldstone boson is associated with the symmetry breaking. In this case it is called a majoron (see [Moh86, 92, Kim93] for more details).

2.5.1 Experimental status of lepton number violation

As no underlying fundamental symmetry is known to conserve lepton number, one might think about observing lepton flavor violation (LFV) at some level. Several searches for LFV are associated with muons. A classic test for the conservation of individual lepton numbers is the muon conversion on nuclei:

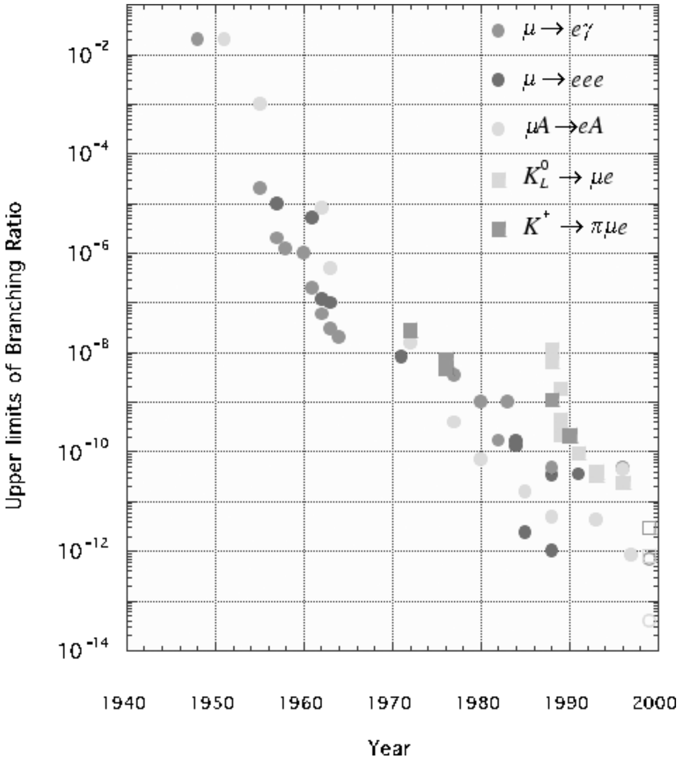

Figure 2.3. Time evolution of experimental limits of branching ratios on some rare LFV muon and kaon decays (from [Kun01]).

$$\begin{array}{lllll} & \mu^- & + {}^A_Z\mathrm{X} & \to {}^A_Z\mathrm{X} & + \mathrm{e}^- \\ L_e & 0 & +0 & \to 0 & +1 \\ L_\mu & 1 & +0 & \to 0 & +0 \end{array}$$

This would violate both L_e and L_μ conservation but would leave the total lepton number unchanged. It has not yet been observed and the current experimental limit for this decay is [Win98]

$$BR(\mu^- + \mathrm{Ti} \to \mathrm{e}^- + \mathrm{Ti}) < 6.1 \times 10^{-13} \qquad (90\%\ \mathrm{CL}). \tag{2.73}$$

A next generation of experiments is about to come. The MEG experiment has started data taking to improve the limit on the $\mu \to \mathrm{e}\gamma$ decay [Ada10] and a proposal (Mu2e) to go down to 10^{-16} or even lower in the muon-electron conversion channel has been made [Car08], (for a general discussion see [Kun01]). Other processes studied intensively with muons are $\mu \to 3\mathrm{e}$, muon–positron conversion on nuclei ($\mu^-(A,Z) \to \mathrm{e}^+(A,Z-2)$) and muonium–antimuonium conversion ($\mu^+\mathrm{e}^- \to \mu^-\mathrm{e}^+$). The evolution over time of experimental progress of some of the searches is shown in Figure 2.3. Searches involving τ-leptons, e.g., $\tau \to \mu\gamma$, are also performed but are not as sensitive yet, but at the LHC improvements can be made. A compilation of obtained limits on some selected searches is given in Table 2.2. Another LFV process is neutrino oscillation, discussed in Chapter 8. For a comprehensive list see [PDG08].

The 'gold-plated' reaction to distinguish between Majorana and Dirac neutrinos and therefore establish total lepton number violation is the process of neutrinoless double β-decay

$$(A, Z) \rightarrow (A, Z+2) + 2\mathrm{e}^-. \tag{2.74}$$

This process is possible only if neutrinos are massive Majorana particles and it is discussed in detail in Chapter 7. A compilation of searches for $\Delta L = 2$ processes is given in Table 7.6. On the other hand, in the case of CPT conservation the observation of a static electric or magnetic moment of the neutrino would prove its Dirac character (see Section 6.6).