RESEARCH STATEMENT

TRENT LUCAS

My research lies at the intersection of low dimensional topology and geometric group theory; I study manifolds and their symmetries. The central object in my work is the *mapping class group* of a manifold M:

$$\mathrm{Mod}(M) \coloneqq \mathrm{Homeo}^+(M)/\text{isotopy},$$

i.e. the group of isotopy classes of (orientation-preserving) homeomorphisms M. This group was first studied in the early 20th century in the work of Dehn and Nielsen, and later played a central role in Thurston's geometrization program for 3-manifolds in the 1970s. Beyond just topology, the mapping class group arises in algebraic geometry and complex analysis, with further connections to number theory, representation theory, and more.

One reason for this ubiquity is the role it plays in the theory of *fiber bundles*, or in algebro-geometric terms, *families of varieties*. Given a bundle $E \to B$ with fiber M, there is a *monodromy homomorphism* $\pi_1(B) \to \mathrm{Mod}(M)$, which measures how the fiber changes as we travel around a loop in the base. An important problem is to determine the *image* of the monodromy; the bigger the image, the further the bundle is from being trivial.

In my research, I study questions about mapping class groups with applications to low dimensional topology and algebraic geometry via the lens of bundles and monodromy. I am especially interested in *equivariant* questions, i.e. questions about homeomorphisms that commute with some finite group action. My work has three main components:

(i) **Birman-Hilden theory for 3-manifolds:** We show that for most finite group actions on 3-manifolds, there are equivariant homeomorphisms that are isotopic but not equivariantly isotopic.

(ii) **Arithmetic Monodromy for low genus curves:** For a finite group action G on a surface S of genus $g \leq 3$, we show that the action of the centralizer $\mathrm{Mod}(S)^G$ on the homology $H_1(S)$ has "arithmetic" image.

(iii) **Geography of complex surfaces and flat geometry:** We explicitly compute the topological and algebro-gometric invariants of complex surfaces arising as families of translation surfaces.

1. Birman-Hilden theory for 3-manifolds

Suppose a finite group G acts on a closed oriented manifold M. A natural problem is to understand how the group $\mathrm{Homeo}^+_G(M)$ of G-equivariant homeomorphisms sits inside of the full group $\mathrm{Homeo}^+(M)$. For instance, one can ask whether the map on path components $\pi_0(\mathrm{Homeo}^+_G(M)) \to \pi_0(\mathrm{Homeo}^+(M))$ is injective. Since paths in $\mathrm{Homeo}^+(M)$ are precisely isotopies, we can rephrase this question as follows:

Question 1.1. *Suppose two equivariant homeomorphisms of M are isotopic. Need they be equivariantly isotopic?*

From the theory of classifying spaces, elements of $\pi_0(\mathrm{Homeo}^+(M))$ correspond to M-bundles over S^1, and elements of $\pi_0(\mathrm{Homeo}^+_G(M))$ correspond to M-bundles over S^1 with a fiberwise G-action. Thus, Question 1.1 asks: if two M-bundles over S^1 with equivalent fiberwise G-actions are topologically isomorphic, are they equivariantly isomorphic?

An important theorem of Birman-Hilden [BH73] and MacLachlan-Harvey [MH75] says that the answer to Question 1.1 is “yes” if M is a hyperbolic surface. Birman and Hilden applied this result to prove several theorems in 3-manifold topology. They also computed the first presentation of the mapping class group of a genus 2 surface by applying this result to the “hyperelliptic involution.” Inspired by their work, Margalit-Winarski [MW21] asked Question 1.1 for the case of 3-manifolds. Surprisingly, in contrast to the surface case, we prove that the answer to Question 1.1 is “no” for most finite group actions on 3-manifolds.

Theorem 1.2 ([Luc24a]). *Suppose a finite group G acts smoothly and non-freely on a closed oriented 3-manifold M. Let $B \subseteq M$ be the set of points with nontrivial G-stabilizers, and let M° be the complement of a G-invariant regular neighborhood of B. If M°/G has at least 3 prime factors, then there is an infinite collection of equivariant homeomorphisms of M that are pairwise isotopic, but pairwise not equivariantly isotopic.*

Given the unexpected contrast with the surface case, Theorem 1.2 opens a new direction of work in the study of 3-manifolds. We prove a similar result to Theorem 1.2 in the case that M°/G has 2 prime factors; a natural open problem is prove a complementary result in the case that M°/G is prime.

Problem 1.3. *For a finite group G acting on a 3-manifold M, determine whether isotopy implies equivariant isotopy when M°/G is prime.*

Another new direction suggested by Theorem 1.2 concerns the *lifting map*. The quotient map $p : M \to M/G$ is a *branched cover*, meaning that it's a covering map away from the set $B \subseteq M$ of points with nontrivial G-stabilizers. By lifting homeomorphisms from M/G to M, we get a lifting map

$$\mathcal{L} : \mathrm{LMod}(M/G) \to \mathrm{SMod}(M)/\Gamma,$$

where $\mathrm{LMod}(M/G)$ is the *liftable subgroup* of $\mathrm{Mod}(M/G, p(B))$, the subgroup $\mathrm{SMod}(M) \leq \mathrm{Mod}(M)$ is the subgroup of lifts, and Γ is the image of G in $\mathrm{Mod}(M)$. If the lifting map is not injective, then the answer to Question 1.1 is “no.” We prove Theorem 1.2 by showing that $\mathcal{L}$ is not injective for most finite regular branched covers of 3-manifolds. Therefore, the natural next step is to understand the kernel:

Question 1.4. *If the lifting map $\mathcal{L}$ is not injective, what is a natural generating set for its kernel?*

Finding generators of $\mathrm{Ker}(\mathcal{L})$ is difficult in general, since $\mathrm{Ker}(\mathcal{L})$ is an infinite index subgroup of $\mathrm{LMod}(M/G)$. Still, we answer Question 1.4 for a rich family of examples.

Let $n \geq 2$, and let $p_n : M_n \to S^3$ be the double cover branched over the n-component unlink $C_n \subseteq S^3$. In this case, $\mathrm{LMod}(S^3)$ is the full mapping class group $\mathrm{Mod}(S^3, C_n)$; we denote the lifting map by

$$\mathcal{L}_n : \mathrm{Mod}(S^3, C_n) \to \mathrm{SMod}_n(M_n)/\Gamma.$$

The cover p_n is a direct generalization of the hyperelliptic cover studied by Birman-Hilden. In our work on Theorem 1.2, we show that $\mathrm{Ker}(\mathcal{L}_n)$ is infinite for $n \geq 3$. Furthermore, we answer Question 1.4 for this cover.

Theorem 1.5 ([Luc24a]). *Let $\rho \in \mathrm{Mod}(S^3, C_n)$ be the involution which reverses the orientation of C_n. Then for $n \geq 3$, $\mathrm{Ker}(\mathcal{L}_n)$ is the $\mathrm{Mod}(S^3, C_n)$-normal closure of $\{\rho\}$.*

It is not hard to show that the element ρ lies in $\mathrm{Ker}(\mathcal{L}_n)$. Theorem 1.5 then says that $\mathrm{Ker}(\mathcal{L}_n)$ is as small as possible given the "obvious" element it contains. From Theorem 1.5, we compute $\mathrm{Ker}(\mathcal{L}_n)$ explicitly for small n.

Corollary 1.6 ([Luc24a])**.** *For $n \in \{2,3\}$, $\mathrm{Ker}(\mathcal{L}_n)$ is as follows.*

(i) For $n=2$, $\mathrm{Ker}(\mathcal{L}_2) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and is generated by ρ and the element σ which swaps the two components of C_2.

(ii) For $n=3$, $\mathrm{Ker}(\mathcal{L}_3)$ splits as a semidirect product $F_\infty \rtimes \mathbb{Z}/2\mathbb{Z}$, where F_∞ is an infinite rank free group. In particular, $\mathrm{Ker}(\mathcal{L}_3)$ is not finitely generated.

Thus, for $n=3$, the infinite generating set in Theorem 1.5 cannot be upgraded to a finite generating set. One naturally asks whether the same is true in general.

Problem 1.7. *For $n \geq 4$, determine whether the lifting kernel $\mathrm{Ker}(\mathcal{L}_n)$ finitely generated.*

2. Arithmetic monodromy for low genus curves

Let $E \to B$ be a family of smooth varieties over $\mathbb{C}$ with fiber F. We can post-compose the monodromy $\pi_1(B) \to \mathrm{Mod}(F)$ with the action on homology $\mathrm{Mod}(F) \to \mathrm{Aut}(H_i(F;\mathbb{Q}))$ to get the *algebraic monodromy*

$$\pi_1(B) \to \mathrm{Aut}(H_i(F;\mathbb{Q})).$$

Griffiths-Schmid [GS75] asked the following question:

Question 2.1. *For a family of smooth varieties over $\mathbb{C}$, is the algebraic monodromy* arithmetic*? That is, does the image have finite index in the integer points of its Zariski closure?*

Arithmetic subgroups are "large" (they are *lattices* in the $\mathbb{R}$-points of the Zariski closure), and so Question 2.1 refines the question of whether the monodromy image is "big."

We study this question for families of curves over $\mathbb{C}$. Suppose a finite group G acts on closed surface S. Then the action of $\mathrm{Mod}(S)$ on $H_1(S;\mathbb{Q})$ restricts to a map on centralizers

$$\Phi_G : \mathrm{Mod}(S)^G \to \mathrm{Aut}(H_1(S;\mathbb{Q}))^G.$$

If G is cyclic and S/G is a sphere, then this is precisely the algebraic monodromy for a family of curves

$$y^d = (x_1 - b_1)^{k_1} \cdots (x_n - b_n)^{k_n}$$

where $b_1, \ldots, b_n$ range over distinct points of $\mathbb{C}$. In general, if S/G has genus h and n branch points, then Φ_G is the monodromy for the "universal" family of curves over a certain finite cover of the moduli space of curves $\mathcal{M}_{h,n}$.

Following Griffiths-Schmid, we ask: is the image of Φ_G arithmetic? More generally, it is of interest to determine whether the image of Φ_G is "big". For instance, Putman and Wieland [PW13] conjecture that $\mathrm{Im}(\Phi_G)$ never has finite orbits on $H_1(S;\mathbb{Q})$ when S/G has genus $h \geq 3$; moreover, they prove that if this is true, then the genus h mapping class group does not virtually surject onto $\mathbb{Z}$ for $h \geq 4$, which would answer a long-open question of Ivanov.

We can slightly refine the arithmeticity question. For each irreducible $\mathbb{Q}$-representation V of G, the homology $H_1(S;\mathbb{Q})$ has an associated *isotypic component* $H_1(S;\mathbb{Q})_V$. The group $\mathrm{Aut}(H_1(S;\mathbb{Q}))^G$ preserves each isotypic component; we can therefore project to representations

$$\Phi_V : \mathrm{Mod}(S)^G \to \mathrm{Aut}(H_1(S;\mathbb{Q})_V)^G.$$

for each irreducible $\mathbb{Q}$-representation V of G.

Question 2.2. *Is the image of* $\Phi_V : \mathrm{Mod}(S)^G \to \mathrm{Aut}(H_1(S;\mathbb{Q})_V)^G$ *arithmetic?*

Question 2.2 is difficult in general, and many authors have given partial answers, including A'Campo [ACa79], Deligne-Mostow [DM86], Looijenga [Loo97; Loo21], McMullen [McM13], Venkataramana [Ven14], and Grunewald-Larsen-Lubotzky-Malestein [Gru+15]. The answer is "yes" in many cases, but Deligne-Mostow surprisingly proved the answer is "no" for certain cyclic branched covers of S^2. In their examples, the images of Φ_V for various V yield *non-arithmetic lattices* in $U(1,2)$ and $U(1,3)$, which are rare and poorly understood objects. Towards the goal of understanding how often the answer is "yes" versus "no," we give a complete answer to Question 2.2 in all low genus cases.

Theorem 2.3 ([Luc24b])**.** *If* S *has genus* $g \leq 3$, *then for every action of a finite group* G *on* S *and for every irreducible* $\mathbb{Q}$*-representation* V *of* G, *the image of the representation*

$$\Phi_V : \mathrm{Mod}(S)^G \to \mathrm{Aut}(H_1(S;\mathbb{Q})_V)^G$$

is an arithmetic subgroup.

One aspect that makes the arithmeticity question difficult is that the target group $\mathrm{Aut}(H_1(S;\mathbb{Q})_V)^G$ can be isomorphic to many different types of algebraic groups. In the low genus cases of Theorem 2.3, the group $\mathrm{Aut}(H_1(S;\mathbb{Q})_V)^G$ is often isomorphic to $\mathrm{SL}(2,\mathbb{Q})$, which makes the these cases amenable to computation. Still, even when the target is $\mathrm{SL}(2,\mathbb{Q})$, it is difficult to decide what may happen in general.

Problem 2.4. *Determine whether there exist a group action where* $\mathrm{Aut}(H_1(S;\mathbb{Q})_V)^G$ *is isomorphic to* $\mathrm{SL}(2,\mathbb{Q})$ *for some* V, *but* $\mathrm{Im}(\Phi_V)$ *is not commensurable to* $\mathrm{SL}(2,\mathbb{Z})$.

3. Geography of complex surfaces and flat geometry

In joint work with Sam Freedman, we study certain smooth compact complex surfaces that we call *Veech fibrations*. These complex surfaces are built from *translation surfaces*: a translation surface is a Riemann surface obtained by starting with a polygon in $\mathbb{R}^2$, and then gluing pairs of parallel edges together via translations. A Veech fibration is then a family of (possibly singular) Riemann surfaces $\mathbb{X} \to B$, where B is a Riemann surface and the generic fiber is a *Veech surface*, a type of highly symmetric translation surface. Veech fibrations generalize *elliptic fibrations*, which are families of elliptic curves. Moreover, they are examples of *Lefschetz fibrations*, which are spaces that are central in the study of 4-manifolds.

Veech surfaces are rare and remarkable objects; the study of Veech surfaces has lead to a rich theory of dynamics on the moduli space of Riemann surfaces $\mathcal{M}_g$. Veech fibrations were pioneered in the work of Möller [Möl06] as a tool to study the $\mathrm{SL}(2,\mathbb{R})$-orbits of Veech surfaces. Given the remarkable nature of Veech surfaces, we expect Veech fibrations to be rich examples in the geometry and topology of 4-manifolds. As a start, we study the *geography problem:*

Question 3.1. *Which compact complex surfaces arise as Veech fibrations?*

There is a basic recipe to construct Veech fibrations. Given a Veech surface X, its $\mathrm{SL}(2,\mathbb{R})$-orbit $C \subseteq \mathcal{M}_g$ is a finite area hyperbolic surface. For each integer $m \geq 3$, there is a finite manifold cover $\mathcal{M}_g[p] \to \mathcal{M}_g$. Over the preimage of C in $\mathcal{M}_g[p]$, we can restrict the "universal family" and compactify to get a Veech fibration $\mathbb{X}_m \to B_m$.

Following work of Chen and Möller [CM12], we derive formulas for the topological and algebro-geometric invariants of $\mathbb{X}_m$ in terms of the geometry of the fiber X. However, computing these invariants explicitly requires understanding the base B_m; this amounts to computing the image of the mod m monodromy representation

$$\rho_m : \mathrm{Aff}^+(X) \to \mathrm{Aut}(H_1(X; \mathbb{Z}/m\mathbb{Z})).$$

Here $\mathrm{Aff}^+(X)$ is the group of "affine automorphisms," i.e. the homeomorphisms of X that preserve its flat structure. Computing this image of ρ_m is difficult in general; it is closely related to computing the "Kontsevich-Zorich monodromy" of X, which is a central problem in the study of translation surfaces.

In our work with Sam Freedman, we study this representation in the case that X is "algebraically primitive." In particular, we compute the image of ρ_m for infinitely many m for all known examples of algebraically primitive Veech surfaces.

Theorem 3.2 ([FL23]). *Let X be a genus g Veech surface in one of the following families:*

(i) the genus 2 Weierstrass eigenforms with nonsquare discriminant,
(ii) the regular n-gon surfaces with n prime, twice a prime, or a power of 2, or
(iii) the sporadic Veech surfaces E_7 and E_8.

Then, for a prime $p \geq 3$ in an explicit infinite set depending on X, the image of

$$\rho_m : \mathrm{Aff}^+(X) \to \mathrm{Aut}(H_1(X; \mathbb{F}_p))$$

is isomorphic to $\mathrm{SL}(2, \mathbb{F}_{p^g})$.

This allows us to answer Question 3.1 in the algebraically primitive case. For instance:

Theorem 3.3 ([FL23]). *Let Y_q be the double regular q-gon surface of genus $g = (q-1)/2$ where $q \geq 5$ is a prime. Fix $p \geq 3$ a prime such that the minimal polynomial of $4\cos(\pi/q)^2$ is irreducible over $\mathbb{F}_p$, and let $\mathbb{X}_{q,p} \to B_{q,p}$ be the Veech fibration. Then $\mathbb{X}_{q,p}$ is a minimal general type complex surface with $\pi_1(\mathbb{X}_{q,p}) \cong \pi_1(B_{q,p})$, and its Euler characteristic and signature are*

$$e(\mathbb{X}_{q,p}) = dg + d(q-3)\left(\frac{1}{2} - \frac{1}{q} - \frac{1}{p}\right) \quad \textit{and} \quad \sigma(\mathbb{X}_{q,p}) = -\frac{d(q^2-1)}{4q},$$

where $d = |\,\mathrm{PSL}(2,5)|$ if $(p, g) = (3, 2)$ or $|\,\mathrm{PSL}(2, \mathbb{F}_{p^g})|$ otherwise.

We obtain similar results for each family listed in Theorem 3.2. These results give very explicit answers to Question 3.1; for example, Theorem 3.3 identifies the Veech fibration $\mathbb{X}_{5,3}$ as a well-known *Horikawa surface* with Euler characteristic 116 and signature -72. We therefore obtain a new geometric perspective of certain complex surfaces.

Given the rich geometry of the fibers, Veech fibrations appear to be promising examples in the study of mapping class groups of 4-manifolds recently pioneered by Farb-Looijenga [FL24].

Problem 3.4. *Construct explicit homeomorphisms of Veech fibrations using the flat geometry of the fibers.*

In a different direction, computing the monodromy ρ_m in Theorem 3.2 proved to be an interesting problem itself. There is little known about the image of this representation in general.

Problem 3.5. *Determine the image of $\rho_m : \mathrm{Aff}^+(X) \to \mathrm{Aut}(H_1(X; \mathbb{Z}/m\mathbb{Z}))$ for other Veech surfaces X.*

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