Calculations for [1]

Justification of the complex Langevin method with the gauge cooling procedure.

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2 Brief review of the Langevin method

System defined by $\boldsymbol{x} : (\mathbb{N}_n \longrightarrow \mathbb{R} = \mathbb{R}^n) = (x_k : \mathbb{R})_{k=1}^n$ and action $S : \mathbb{R}^n \longrightarrow \mathbb{K}$ with partition function

$$Z : (\mathbb{R}^n \longrightarrow \mathbb{K}) \longrightarrow \mathbb{K},\ Z(S) = \int d\boldsymbol{x} \exp(-S(\boldsymbol{x})) = \prod_k dx_k \exp(-S(\boldsymbol{x})) \tag{2.1}$$

2.1 $\mathbb{K} = \mathbb{R}$

Langevin equation

$$\frac{dx_k}{dt}(t;\boldsymbol{\eta}) = \eta_k(t) - \frac{\partial S}{\partial x_k}(\boldsymbol{x}(t;\boldsymbol{\eta})) \tag{2.2}$$

with gaussian noise η ($\mu = 0$ and $\sigma = \sqrt{2}$) obeying the distribution

$$\rho_{\text{noise}}(\boldsymbol{\eta}) = \rho_{\text{noise},0} \exp\left(-\frac{1}{4}(\boldsymbol{\eta}|\boldsymbol{\eta})\right),\ \rho_{\text{noise},0} = \int d\boldsymbol{\eta} \exp\left(-\frac{1}{4}(\boldsymbol{\eta}|\boldsymbol{\eta})\right) \text{ and } (\boldsymbol{\eta}|\boldsymbol{\eta}) = \int dt \sum_k \eta_k(t)\eta_k(t) \tag{2.3}$$

which can be used for evaluating expectation values of variables depending on the noise

$$\langle \ldots \rangle_{\text{noise}} = \int d\boldsymbol{\eta} \ldots (\boldsymbol{\eta}) \rho_{\text{noise}}(\boldsymbol{\eta}) \tag{2.4}$$

The random solution $\boldsymbol{x}$ is then picked with probability by definition

$$\rho_{\text{L}}(\boldsymbol{x};t) = \langle \delta(\boldsymbol{x} - \boldsymbol{x}(t))\rangle_{\text{noise}} = \int d\boldsymbol{\eta} \delta(\boldsymbol{x} - \boldsymbol{x}(t;\boldsymbol{\eta}))\rho_{\text{noise}}(\boldsymbol{\eta}) \tag{2.5}$$

The most generic requirement for noise can be defined

$$\langle \eta_k(t)\eta_{k'}(t')\rangle_{\text{noise}} = 2\delta_{kk'}\delta(t-t') \tag{2.6}$$

Byt derivating the noise probability distribution

$$\frac{\partial \rho_{\text{noise}}}{\partial \eta_k(t)}(\boldsymbol{\eta}) = \rho_{\text{noise},0} \frac{\partial}{\partial \eta_k(t)} \exp\left(-\frac{1}{4}(\boldsymbol{\eta}|\boldsymbol{\eta})\right) = -\frac{1}{4}\rho_{\text{noise},0} \exp\left(-\frac{1}{4}(\boldsymbol{\eta}|\boldsymbol{\eta})\right) \frac{\partial}{\partial \eta_k(t)}(\boldsymbol{\eta}|\boldsymbol{\eta}) \tag{2.7}$$

with

$$\frac{\partial}{\partial \eta_k(t)}(\boldsymbol{\eta}|\boldsymbol{\eta}) = \frac{\partial}{\partial \eta_k(t)} \int ds \sum_l (\eta_l(s))^2 = \int ds \sum_l \frac{\partial}{\partial \eta_k(t)}(\eta_l(s))^2 = 2\int ds \sum_l \delta_{kl}\delta(t-s)\eta_l(s) = 2\eta_k(t) \tag{2.8}$$

therefore

$$\frac{\partial \rho_{\text{noise}}}{\partial \eta_k(t)}(\boldsymbol{\eta}) = -\frac{1}{2}\eta_k(t)\rho_{\text{noise}}(\boldsymbol{\eta}) \text{ or } \eta_k(t)\rho_{\text{noise}}(\boldsymbol{\eta}) = -2\frac{\partial \rho_{\text{noise}}}{\partial \eta_k(t)}(\boldsymbol{\eta}) \tag{2.9}$$

Assuming we can integrate by parts (no boundary conditions)

$$\begin{aligned}\langle \eta_k(t)\eta_{k'}(t')\rangle_{\text{noise}} &= \int d\boldsymbol{\eta}\eta_k(t)\eta_{k'}(t')\rho_{\text{noise}}(\boldsymbol{\eta}) = -2\int d\boldsymbol{\eta}\eta_k(t)\frac{\partial \rho_{\text{noise}}}{\partial \eta_{k'}(t')}(\boldsymbol{\eta}) = 2\int d\boldsymbol{\eta}\frac{\partial \eta_k(t)}{\partial \eta_{k'}(t')}\rho_{\text{noise}}(\boldsymbol{\eta}) \\ &= 2\int d\boldsymbol{\eta}\delta_{kk'}\delta(t-t')\rho_{\text{noise}}(\boldsymbol{\eta}) = 2\delta_{kk'}\delta(t-t')\int d\boldsymbol{\eta}\rho_{\text{noise}}(\boldsymbol{\eta}) = 2\delta_{kk'}\delta(t-t')\end{aligned}$$

The other partial integration term integrates a total derivative

$$-2\int d\boldsymbol{\eta}\eta_k(t)\rho_{\text{noise}}(\boldsymbol{\eta}) = -4\int d\boldsymbol{\eta}\frac{\partial \rho_{\text{noise}}}{\partial \eta_k(t)}(\boldsymbol{\eta}) \tag{2.10}$$

resulting in a boundary term that is 0, if the distribution ρ_{noise} falls-off at infinity fast enough, holding true for the gaussian distribution.

References

[1] Keitaro Nagata, Jun Nishimura, Shinji Shimasaki.
Justification of the complex Langevin method with the gauge cooling procedure.
arXiv:1508.02377v2 [hep-lat] 18 Sep 2015.