Calculations for [1]

Justification of the complex Langevin method with the gauge cooling procedure.

---

2 Brief review of the Langevin method

System defined by $\boldsymbol{x} : \mathbb{R}$ and action $S : \mathbb{R} \longrightarrow \mathbb{K}$ with partition function

$$Z : (\mathbb{R} \longrightarrow \mathbb{K}) \longrightarrow \mathbb{K},\ Z(S) = \int dx \exp(-S(x)) \tag{2.1}$$

2.1 $\mathbb{K} = \mathbb{R}$

Langevin equation

$$\frac{d}{dt} x(t;\eta) = \alpha(x(t;\eta)) + \eta(t) \tag{2.2}$$

with kernel $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ gaussian noise η ($\mu = 0$ and $\sigma = \sqrt{2}$) obeying the distribution

$$\rho_{\text{noise}}(\boldsymbol{\eta}) = \rho_{\text{noise},0} \exp\left(-\frac{1}{4}|\eta|^2\right),\ \rho_{\text{noise},0} = \int d\boldsymbol{\eta} \exp\left(-\frac{1}{4}|\eta|^2\right) \text{ and } |\eta|^2 = \int dt |\eta(t)|^2 \tag{2.3}$$

which can be used for evaluating expectation values of functions f of noise

$$\langle f(\eta) \rangle_\eta = \int d\eta f(\eta) \rho_{\text{noise}}(\eta) \tag{2.4}$$

The random solution x is then picked with probability by definition

$$\rho(x;t) = \langle \delta(x - x(t)) \rangle_{\text{noise}} = \int d\eta \delta(x - x(t;\eta)) \rho_{\text{noise}}(\eta) \tag{2.5}$$

The most generic requirement for noise can be defined

$$\langle \eta(t)\eta(t') \rangle_{\text{noise}} = 2\delta(t - t') \tag{2.6}$$

But derivating the noise probability distribution

$$\frac{\partial}{\partial \eta(t)} \rho_{\text{noise}}(\eta) = \rho_{\text{noise},0} \frac{\partial}{\partial \eta(t)} \exp\left(-\frac{1}{2\sigma^2}|\eta|^2\right) = -\frac{1}{2\sigma^2} \rho_{\text{noise},0} \exp\left(-\frac{1}{2\sigma^2}|\eta|^2\right) \frac{\partial}{\partial \eta_k(t)} |\eta|^2 \tag{2.7}$$

with

$$\frac{\partial}{\partial \eta(t)} |\eta|^2 = \frac{\partial}{\partial \eta(t)} \int ds |\eta(s)|^2 = \int ds \frac{\partial}{\partial \eta(t)} |\eta(s)|^2 = 2 \int ds \delta(t - s) \eta(s) = 2\eta(t) \tag{2.8}$$

therefore

$$\frac{\partial}{\partial \eta(t)}\rho_{\mathrm{noise}}(\eta) = -\frac{1}{\sigma^2}\eta(t)\rho_{\mathrm{noise}}(\eta) \text{ or } \eta(t)\rho_{\mathrm{noise}}(\eta) = -\sigma^2\frac{\partial}{\partial \eta(t)}\rho_{\mathrm{noise}}(\eta) \tag{2.9}$$

Assuming we can integrate by parts (no boundary conditions)

$$\begin{aligned}\langle \eta(t)\eta(t')\rangle_{\mathrm{noise}} = \int d\eta\eta(t)\eta(t')\rho_{\mathrm{noise}}(\eta) = -\sigma^2\int d\eta\eta(t)\frac{\partial}{\partial\eta(t')}\rho_{\mathrm{noise}}(\eta) = \sigma^2\int d\eta\frac{\partial\eta(t)}{\partial\eta(t')}\rho_{\mathrm{noise}}(\eta)\\ = \sigma^2\int d\eta\delta(t-t')\rho_{\mathrm{noise}}(\eta) = \sigma^2\delta(t-t')\int d\eta\rho_{\mathrm{noise}}(\eta) = \sigma^2\delta(t-t')\end{aligned}$$

where

$$-\sigma^2\int d\eta\frac{\partial}{\partial\eta(t)}(\eta(t')\rho_{\mathrm{noise}}(\eta)) = 0 \tag{2.10}$$

The probability distribution ρ of sulutions to the Langevin equatino defines expectation values

$$\langle O(x)\rangle_x(t) = \int dxO(x)\rho(x;t) \tag{2.11}$$

Stochastic Quantization postulation

$$\langle O(x)\rangle_x(t) = \int dxO(x)\rho(x;t) = \int d\eta O(x(t;\eta))\rho_{\mathrm{noise}}(\eta) = \langle O(x(t;\eta))\rangle_\eta \tag{2.12}$$

Taking the full time derivative of (2.12) and subistuting the Langevin equation (2.2)

$$\begin{aligned}\frac{d}{dt}\langle O(x(t;\eta))\rangle_\eta = \left\langle\frac{d}{dt}O(x(t;\eta))\right\rangle_\eta = \left\langle\frac{\partial}{\partial x}O(x(t;\eta))\frac{dx}{dt}(t;\eta)\right\rangle_\eta = \left\langle\frac{\partial}{\partial x}O(x(t;\eta))(\alpha(x(t;\eta))+\eta(t))\right\rangle_\eta\\ = \left\langle\alpha(x(t;\eta))\frac{\partial}{\partial x}O(x(t;\eta))\right\rangle_\eta + \left\langle\eta(t)\frac{\partial}{\partial x}O(x(t;\eta))\right\rangle_\eta\end{aligned}$$

Spesifically, for any function $f:\mathbb{R}\longrightarrow\mathbb{K}$

$$\begin{aligned}\langle\eta(t)f(x(t;\eta))\rangle_\eta = \int d\eta f(x(t;\eta))\eta(t)\rho_{\mathrm{noise}}(\eta) = -\sigma^2\int d\eta f(x(t;\eta))\frac{\partial}{\partial\eta(t)}\rho_{\mathrm{noise}}(\eta)\\ = \sigma^2\int d\eta\frac{\partial}{\partial\eta(t)}f(x(t;\eta))\rho_{\mathrm{noise}}(\eta)\end{aligned}$$

where

References

[1] Keitaro Nagata, Jun Nishimura, Shinji Shimasaki.
Justification of the complex Langevin method with the gauge cooling procedure.
arXiv:1508.02377v2 [hep-lat] 18 Sep 2015.