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Eletromagnetismo I - UFSM00068

Lista de Exercícios 1

Álgebra Vetorial

1. Determine o vetor unitário ao longo da direção OP, se O for a origem e P o ponto $(4, -5, 1)$.

$$|OP| = \sqrt{4^2+5^2+1} = \sqrt{42}$$

$$\hat{a}OP = \left(\frac{4}{\sqrt{42}}\hat{a}_x + \frac{-5}{\sqrt{42}}\hat{a}_y + \frac{1\hat{a}_z}{\sqrt{42}}\right) =$$

$$(.6172\hat{a}_x - .7715\hat{a}_y + .1543\hat{a}_z)$$

2. Os vetores posição dos pontos M e N são $\hat{a}_x - 4\hat{a}_y - 2\hat{a}_z$ e $3\hat{a}_x + 5\hat{a}_y - \hat{a}_z$, respectivamente. Determine o vetor distância orientado de M a N.

$$\overrightarrow{MN} = (3\hat{a}_x + 5\hat{a}_y - \hat{a}_z) - (\hat{a}_x - 4\hat{a}_y - 2\hat{a}_z) =$$

$$(2\hat{a}_x + 9\hat{a}_y + \hat{a}_z)$$

3. Os vetores posição dos pontos P e Q são $4\hat{a}_x + 6\hat{a}_y - 2\hat{a}_z$ e $\hat{a}_x + 8\hat{a}_y + 3\hat{a}_z$, respectivamente. Determine o vetor distância orientado de P a Q,

$$\vec{PQ} = (\hat{a}_x + 8\hat{a}_y + 3\hat{a}_z) - (4\hat{a}_x + 6\hat{a}_y - 2\hat{a}_z) =$$

$$(-3\hat{a}_x + 2\hat{a}_y + 5\hat{a}_z)$$

4. Dados $\vec{A} = 4\hat{a}_y + 10\hat{a}_z$ e $\vec{B} = 2\hat{a}_x + 3\hat{a}_y$, encontre a projeção de $\vec{A}$ sobre $\vec{B}$.

$$\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|} = \frac{0+12+0}{\sqrt{2^2+3^2}} = \frac{12}{\sqrt{13}}$$

5. Determine o ângulo entre $\vec{A} = 2\hat{a}_x + 3\hat{a}_y + \hat{a}_z$ e $\vec{B} = -\hat{a}_x + 5\hat{a}_y + \hat{a}_z$.

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{-2+15+1}{\sqrt{14}\cdot\sqrt{27}} = \frac{14}{\sqrt{378}}$$

$$\theta = \arccos\left(\frac{14}{\sqrt{378}}\right)$$

6. Ache o menor ângulo entre $\vec{A} = 10\hat{a}_x + 2\hat{a}_z$ e $\vec{B} = -4\hat{a}_y + 0{,}5\hat{a}_z$ usando tanto o produto escalar quanto o produto vetorial.

PE) $$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{0+0+1}{\sqrt{104}\cdot\sqrt{16.25}} = \frac{1}{\sqrt{1690}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{1690}}\right)$$

PV) $$\sin\theta = \frac{\vec{A}\times\vec{B}}{|\vec{A}||\vec{B}|} = \frac{\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 10 & 0 & 2 \\ 0 & -4 & .5 \end{vmatrix}}{\sqrt{104}\,\sqrt{16.25}} = \frac{+8\hat{a}_x - 5\hat{a}_y - 40\hat{a}_z}{\sqrt{1690}} = \frac{\sqrt{1690}}{\sqrt{1690}} = 1$$

$$\theta = \arcsin(1)$$

7. Considere $\vec{A} = 4\hat{a}_x - 2\hat{a}_y + 5\hat{a}_z$ e $\vec{B} = 3\hat{a}_x + \hat{a}_y - \hat{a}_z$. Determine $\vec{A} \times \vec{B}$.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 4 & -2 & 5 \\ 3 & 1 & -1 \end{vmatrix} = (2-5)\hat{a}_x + (15+4)\hat{a}_y + (4+6)\hat{a}_z =$$

$$-3\hat{a}_x + 19\hat{a}_y + 10\hat{a}_z$$

8. Dados $\vec{A} = 2\hat{a}_x - \hat{a}_z$, $\vec{B} = 3\hat{a}_x + \hat{a}_y$ e $\vec{C} = -2\hat{a}_x + 6\hat{a}_y - 4\hat{a}_z$, mostre que $\vec{C}$ é perpendicular simultaneamente a $\vec{A}$ e $\vec{B}$.

$$\vec{A} \cdot \vec{C} = (2,0,-1) \cdot (-2,6,-4) = -4 + 0 + 4 = 0 \checkmark$$

$$\vec{B} \cdot \vec{C} = (3,1,0) \cdot (-2,6,-4) = -6 + 6 + 0 = 0 \checkmark$$

9. Determine o produto escalar, o produto vetorial e o ângulo entre os vetores $\vec{P} = 2\hat{a}_x - 6\hat{a}_y + 5\hat{a}_z$ e $\vec{Q} = 3\hat{a}_y + \hat{a}_z$.

$$\vec{P} \cdot \vec{Q} = (2,-6,5) \cdot (0,3,1) = 0 - 18 + 5 = -13$$

$$\vec{P} \times \vec{Q} = \begin{vmatrix} a_x & a_y & a_z \\ 2 & -6 & 5 \\ 0 & 3 & 1 \end{vmatrix} = (-6-15)\hat{a}_x + (-2)\hat{a}_y + (6+6)\hat{a}_z =$$

$$-21\hat{a}_x - 2\hat{a}_y + 12\hat{a}_z$$

$$\cos\theta = \frac{\vec{P} \cdot \vec{Q}}{|\vec{P}||\vec{Q}|} = \frac{-13}{\sqrt{4+36+25} \cdot \sqrt{10}} = \frac{-13}{\sqrt{650}}$$

$$\theta = \arccos\left(\frac{-13}{\sqrt{650}}\right)$$

10. Dados os vetores $\vec{A} = 2\hat{a}_x + 5\hat{a}_z$ e $\vec{B} = \hat{a}_x - 3\hat{a}_y + 4\hat{a}_z$, determine $|\vec{A} \times \vec{B}| + \vec{A} \cdot \vec{B}$.

$$|\vec{A} \times \vec{B}| + \vec{A} \cdot \vec{B} = \left\| \begin{matrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & 0 & 5 \\ 1 & -3 & 4 \end{matrix} \right\| + (2,0,5) \cdot (1,-3,4) =$$

$$|15\hat{a}_x + (5-8)\hat{a}_y - 6\hat{a}_z| + 2 + 20 = \sqrt{15^2 + 9 + 36} + 22 =$$

$$\sqrt{270} + 22$$

11. Considere $\vec{A} = \hat{a}_x - \hat{a}_z$, $\vec{B} = \hat{a}_x + \hat{a}_y + \hat{a}_z$, $\vec{C} = \hat{a}_y + 2\hat{a}_z$ e determine:

a) $\vec{A} \cdot (\vec{B} \times \vec{C}) = (1,0,-1) \cdot \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = (1,0,-1) \cdot (1,-2,1) =$

$1 + 0 - 1 = 0$

b) $(\vec{A} \times \vec{B}) \cdot \vec{C} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix} \cdot (0,1,2) = (1,-2,1) \cdot (0,1,2) =$

$0 - 2 + 2 = 0$

c) $\vec{A} \times (\vec{B} \times \vec{C}) = (1,0,-1) \times (1,-2,1) = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{vmatrix} = (-2,-2,-2) =$

$-2\hat{a}_x - 2\hat{a}_y - 2\hat{a}_z$

d) $(\vec{A} \times \vec{B}) \times \vec{C} = (1,-2,1) \times (0,1,2) = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 1 & -2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = (-5,-2,1) =$

$-5\hat{a}_x - 2\hat{a}_y + 1\hat{a}_z$

12. Demonstre que $(\vec{A} \cdot \vec{B})^2 + |\vec{A} \times \vec{B}|^2 = (|\vec{A}||\vec{B}|)^2$.

Propriedade: $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cdot \cos\theta$

$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \cdot \sin\theta$

Então: $(|\vec{A}||\vec{B}|\cos\theta)^2 + (|\vec{A}||\vec{B}|\sin\theta)^2 = (|\vec{A}||\vec{B}|)^2$

$(|\vec{A}||\vec{B}|)^2 \cos^2\theta + (|\vec{A}||\vec{B}|)^2 \sin^2\theta = (|\vec{A}||\vec{B}|)^2$

$(|\vec{A}||\vec{B}|)^2 \cdot (\cos^2\theta + \sin^2\theta) = (|\vec{A}||\vec{B}|)^2$

$(|\vec{A}||\vec{B}|)^2 \cdot 1 = (|\vec{A}||\vec{B}|)^2$

13. Considere $\vec{A} = \alpha\hat{a}_x + 3\hat{a}_y - 2\hat{a}_z$ e $\vec{B} = 4\hat{a}_x + \beta\hat{a}_y + 8\hat{a}_z$, determine:

a) Os valores de α e β se $\vec{A}$ e $\vec{B}$ forem paralelos.

$$\vec{B} = k\vec{A}$$

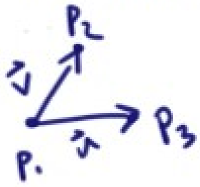
$$\frac{4}{\alpha} = \frac{\beta}{3} = \frac{8}{-2} \rightarrow \frac{8}{-2} = \boxed{-4 = k} \quad \begin{cases} \frac{4}{\alpha} = -4 \rightarrow \alpha = \frac{4}{-4} = -1 \\ \frac{\beta}{3} = -4 \rightarrow \beta = -12 \end{cases}$$

b) A relação entre α e β se $\vec{B}$ for perpendicular a $\vec{A}$.

$$\vec{A} \cdot \vec{B} = 0$$

$$4\alpha + 3\beta - 16 = 0$$

14. Calcule a área de um triângulo determinado pelos pontos $P_1 = (1,1,1)$, $P_2 = (2,3,4)$ e $P_1 = (3,0,-1)$m.



$$\vec{v} = \overrightarrow{P_1P_2} = (2-1, 3-1, 4-1) = (1,2,3)$$

$$\vec{u} = \overrightarrow{P_1P_3} = (3-1, 0-1, -1-1) = (2,-1,-2)$$

$$\vec{v} \times \vec{u} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 1 & 2 & 3 \\ 2 & -1 & -2 \end{vmatrix} = (-4+3)\hat{a}_x + (6+2)\hat{a}_y + (-4-1)a_z =$$

$$-1\hat{a}_x + 8\hat{a}_y - 5\hat{a}_z$$

$$A = \frac{|\vec{v} \times \vec{u}|}{2} = \frac{\sqrt{1+64+25}}{2} = \frac{\sqrt{90}}{2}$$

Lista de Exercícios 2

Sistemas e Transformação de Coordenadas

1. Expresse os seguintes pontos em coordenadas cilíndricas e esféricas

	cilíndricas	esféricas
a) $P = (1, -4, -3)$	$\rho = \sqrt{17}$ $\varphi = \arctan(-4)$ $z = -3$	$r = \sqrt{26}$ $\theta = \arccos(-3/\sqrt{26})$ $\varphi = \arctan(-4)$
b) $Q = (3{,}0{,}5)$	$\rho = 3$ $\varphi = \arctan(0)$ $z = 5$	$r = \sqrt{34}$ $\theta = \arccos(5/\sqrt{34})$ $\varphi = \arctan(0)$
c) $R = (-2{,}6{,}0)$	$\rho = \sqrt{40}$ $\varphi = \arctan(-3)$ $z = 0$	$r = \sqrt{40}$ $\theta = \arccos(0)$ $\varphi = \arctan(-3)$

2. Dados dois pontos em coordenadas cilíndricas $P = (10{,}60°{,}2)$ e $Q = (5{,}30°, -4)$, determine a distância entre eles.

$$P = (10 \cdot \cos 60°, 10 \cdot \sin 60°, 2)$$

$$Q = (5 \cdot \cos 30°, 5 \cdot \sin 30°, -4)$$

$$d = \sqrt{(5\cos 30° - 10\cos 60°)^2 + (5\sin 30° - 10\sin 60°)^2 - 6^2}$$

$$d = \sqrt{(5\cos 30° - 5)^2 + (2.5 - 10\sin 60°)^2 - 36}$$

3. Se $\vec{A} = 5\hat{a}_\rho + 2\hat{a}_\phi - \hat{a}_z$ e $\vec{B} = \hat{a}_\rho - 3\hat{a}_\phi + 4\hat{a}_z$, determine:

a) $\vec{A} \cdot \vec{B} = 5 - 6 - 4 = -5$

b) $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi & \hat{a}_z \\ 5 & 2 & -1 \\ 1 & -3 & 4 \end{vmatrix} = 4\hat{a}_\rho - 21\hat{a}_\phi - 17\hat{a}_z$

c) O ângulo entre $\vec{A}$ e $\vec{B}$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{-5}{\sqrt{30}\cdot\sqrt{26}} = \frac{-5}{\sqrt{780}}$$

$$\theta = \arccos\left(\frac{-5}{\sqrt{780}}\right)$$

d) O vetor unitário normal ao plano que contém ambos $\vec{A}$ e $\vec{B}$

$$\vec{u} = \frac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|} = \frac{4\hat{a}_\rho - 21\hat{a}_\phi - 17\hat{a}_z}{\sqrt{16+441+289}} = \frac{4}{\sqrt{755}}\hat{a}_\rho + \frac{-21}{\sqrt{755}}\hat{a}_\phi + \frac{-17}{\sqrt{755}}\hat{a}_z$$

e) O vetor projeção de $\vec{A}$ em $\vec{B}$.

$$\left(\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\right)\cdot\vec{B} = \frac{-5}{\sqrt{26}^2}\cdot(1,-3,4) = \frac{-5}{26}\hat{a}_\rho + \frac{15}{26}\hat{a}_\phi + \frac{-20}{26}\hat{a}_z$$

4. Dados os vetores $\vec{A} = 2\hat{a}_x + 4\hat{a}_y + 10\hat{a}_z$ e $\vec{B} = -5\hat{a}_\rho + \hat{a}_\phi - 3\hat{a}_z$, determine:

a) $\vec{A} + \vec{B}$ em $P = (0{,}2,-5)$

$\rho = \sqrt{4} = 2 \qquad \varphi = \arctan\left(\frac{2}{0}\right)$

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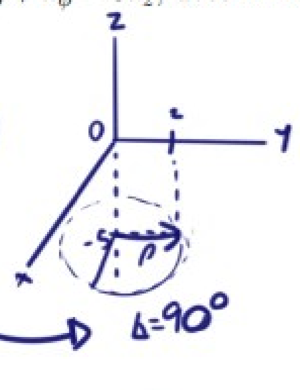

$$\vec{B} = (-5\cos 90° - 1\sin 90°)\hat{a}_x + (-5\sin 90° + 1\cos 90°)\hat{a}_y - 3\hat{a}_z$$

$$\vec{B} = (-5\cdot 0 - 1\cdot 1)\hat{a}_x + (-5\cdot 1 + 1\cdot 0)\hat{a}_y - 3\hat{a}_z = (-1,-5,-3$$

$$\vec{A}+\vec{B} = (2,4,10) + (-1,-5,-3) = \hat{a}_x - \hat{a}_y + 7\hat{a}_z$$

b) O ângulo entre $\vec{A}$ e $\vec{B}$ em P

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{-2-20-30}{\sqrt{120}\cdot\sqrt{35}} = \frac{-52}{2\sqrt{1050}} = \frac{-52}{10\sqrt{42}} = \frac{-26}{5\sqrt{42}}$$

$$\theta = \arccos\left(\frac{-26}{5\sqrt{42}}\right)$$

c) A componente escalar de $\vec{A}$ ao longo de $\vec{B}$ em P.

$$\frac{\vec{A}\cdot\vec{B}}{|\vec{B}|} = \frac{-52}{\sqrt{35}}$$

5. Dado um ponto $P = (-2{,}6{,}3)$ e um vetor $A = 6\hat{a}_x + \hat{a}_y$, determine P em coordenadas cilíndricas e esféricas.

$$\rho = \sqrt{4+36} = \sqrt{40}$$

$$\varphi = \arctan\left(\frac{6}{-2}\right) = \arctan(-3)$$

$$r = \sqrt{4+36+9} = \sqrt{49} = 7$$

$$\theta = \arccos\left(\frac{3}{7}\right)$$

$$P_{(\rho,\phi,z)} = \left(\sqrt{40}, \tan^{-1}(-3), 3\right)$$

$$P(r,\theta,\phi) = \left(7, \cos^{-1}(3/7), \tan^{-1}(-3)\right)$$

Ainda, expresse o vetor $\vec{A}$ no ponto P em coordenadas cilíndricas e esféricas.

$$\vec{A}_{(\rho,\phi,z)} = \left(6\cdot\frac{-1}{\sqrt{10}} + 1\cdot\frac{3}{\sqrt{10}}\right)a_\rho + \left(-6\cdot\frac{3}{\sqrt{10}} + \frac{-1}{\sqrt{10}}\right)a_\phi + 0 = \frac{-3}{\sqrt{10}}a_\rho - \frac{19}{\sqrt{10}}a_\phi$$

$$\vec{A}_{(r,\theta,\varphi)} = \left(6\cdot\frac{\sqrt{40}^{\,2}}{7}\cdot\frac{-1}{\sqrt{10}} + \frac{\sqrt{40}^{\,2}}{7}\cdot\frac{3}{\sqrt{10}}\right)\hat{a}_r + \left(6\cdot\frac{3}{7}\cdot\frac{-1}{\sqrt{10}} + \frac{3}{7}\cdot\frac{3}{\sqrt{10}}\right)\hat{a}_\theta + \left(-6\cdot\frac{3}{\sqrt{10}} + \frac{-1}{\sqrt{10}}\right)\hat{a}_\phi =$$

$$\left(\frac{-6}{7}\right)\hat{a}_\rho + \left(\frac{-9}{7\sqrt{10}}\right)\hat{a}_\theta + \left(\frac{-19}{\sqrt{10}}\right)a_\phi$$

Por fim, calcule o módulo do vetor $\vec{A}$ nos três sistemas.

$$|A_{(x,y,z)}| = \sqrt{36+1+0} = \sqrt{37}$$

$$|A_{(\rho,\varphi,z)}| = \sqrt{\frac{9}{10} + \frac{361}{10}} = \sqrt{\frac{370}{10}} = \sqrt{37}$$

$$|A_{(r,\theta,\varphi)}| = \sqrt{\left(\frac{36}{49} + \frac{81}{49\cdot 10} + \frac{361}{10}\right)} = \sqrt{37}$$

Lista de Exercícios 3

Cálculo Vetorial

1. Utilizando o comprimento diferencial dl, determine o comprimento de cada uma das seguintes curvas:

   a) $\rho = 3; \pi/4 < \phi < \pi/2; z =$ constante.

$$d\vec{l} = d\rho\hat{a}_\rho + \rho d\phi\hat{a}_\phi + dz\hat{a}_z = 0 + 3d\phi\hat{a}_\phi + 0$$

$$L = \int_{\pi/4}^{\pi/2} 3d\phi = 3\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{3\pi}{4}$$

   b) $r = 1; \theta = 30°; 0 < \phi < 60°$.

$$d\vec{l} = dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta d\phi\hat{a}_\phi = 0 + 0 + 1 \cdot .5 \cdot d\phi\hat{a}_\phi$$

$$L = \int_0^{60° = \pi/3} .5 d\phi = \frac{1}{2}\left(\frac{\pi}{3}\right) = \pi/6$$

   c) $r = 4; 30° < \theta < 90°; \phi =$ constante.

$$d\vec{l} = dr\hat{a}_r + rd\theta\hat{a}_\theta + r\sin\theta d\phi\hat{a}_\phi = 0 + 4d\theta\hat{a}_\theta + 0$$

$$L = \int_{\pi/6}^{\pi/2} 4d\theta = 4 \cdot \left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{4\pi}{3}$$

2. Determine o gradiente dos seguintes campos escalares:

   a) $U = 5y - x^3y^2$.

$$\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z = -3x^2y^2\hat{a}_x + (5 - 2yx^3)\hat{a}_y$$

   b) $U = x^2y + xyz$.

$$\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z = (2xy + yz)\hat{a}_x + (x^2 + xz)\hat{a}_y + (xy)\hat{a}_z$$

   c) $V = \rho z \sin\phi + z^2\cos^2\phi + \rho^2$.

$$\vec{\nabla}V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z = (z \cdot \sin\varphi + 2\rho)\hat{a}_\rho + (\rho\sin\varphi + 2z \cdot \cos^2\varphi)\hat{a}_z$$

$$1/\rho(\rho z\cos\varphi + z^2 \cdot 2\cos\varphi \cdot (-\sin\varphi)\hat{a}_\varphi =$$

$$(z \cdot \sin\varphi + 2\rho)\hat{a}_\rho + (z \cdot \cos\varphi - 2z^2 \cdot \cos\varphi \cdot \sin\varphi)\hat{a}_\varphi + (\rho\sin\varphi + 2z \cdot \cos^2\varphi)\hat{a}_z$$

3. Determine o divergente dos seguintes campos vetoriais e os calcule nos pontos especificados:

a) $\vec{A} = yz\hat{a}_x + 4xy\hat{a}_y + y\hat{a}_z$ em $(1, -2, 3)$.

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 + 4x + 0 = 4x \quad \text{em } 4$$

b) $\vec{B} = \rho z \sin\phi \hat{a}_\rho + 3\rho z^2 \cos\phi \hat{a}_\phi$ em $(5, \pi/2, 1)$.

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{\rho}\cdot(\rho\cdot\rho)' z \sin\varphi + \frac{1}{\rho}\left(3\rho z^2(-\sin\varphi)\right) + 0 =$$

$$= 2z\sin\varphi - 3z^2\sin\varphi = z\sin\varphi(2-3z) \quad \text{em } -1$$

c) $\vec{C} = 2r\cos\theta\cos\phi\hat{a}_r + r^{1/2}\hat{a}_\phi$ em $(1, \pi/6, \pi/3)$.

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$= \frac{(r^2\cdot 2r)'\cos\theta\cos\varphi}{r^2} + 0 + 0 = 6\cos\theta\cdot\cos\varphi \quad \text{em } 6\cos\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right)$$

4. Determine o rotacional dos seguintes campos vetoriais e os calcule nos pontos especificados:

a) $\vec{A} = yz\hat{a}_x + 4xy\hat{a}_y + y\hat{a}_z$ em $(1, -2, 3)$.

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \hat{a}_x(1-0) + \hat{a}_y(y-0) + \hat{a}_z(4y-z) = \hat{a}_x + y\hat{a}_y + (4y-z)\hat{a}_z \quad \text{em } \hat{a}_x - 2\hat{a}_y - 11\hat{a}_z$$

b) $\vec{B} = \rho z \sin\phi \hat{a}_\rho + 3\rho z^2 \cos\phi \hat{a}_\phi$ em $(5, \pi/2, 1)$.

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \frac{\hat{a}_\rho}{\rho}\left(\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi \rho}{\partial z}\right) + \hat{a}_\varphi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\frac{\rho}{\rho} + \frac{\hat{a}_z}{\rho}\left(\frac{\partial A_\varphi \rho}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right) =$$

$$= (0 - 6z\rho\cos\varphi)\hat{a}_\rho + (\rho\sin\varphi - 0)a_\varphi + \left(6\frac{\rho}{\rho}z^2\cos\varphi - \frac{\rho}{\rho}z\cos\varphi\right)a_z =$$

$$= (-6z\rho\cos\varphi)\hat{a}_\rho + (\rho\sin\varphi)a_\varphi + \left(z\cos\varphi(6z-1)\right)a_z \quad \text{em } 5\hat{a}_\varphi$$

c) $\vec{C} = 2r\cos\theta\cos\phi\hat{a}_r + r^{1/2}\hat{a}_\phi$ em $(1, \pi/6, \pi/3)$.

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$= \frac{\hat{a}_r}{r^2\sin\theta}\left(\frac{\partial r\sin\theta A_\varphi}{\partial\theta} - \frac{\partial rA_\theta}{\partial\varphi}\right) + \frac{\hat{a}_\theta}{r\sin\theta}\left(\frac{\partial A_r}{\partial\varphi} - \frac{\partial r\sin\theta A_\varphi}{\partial r}\right) + \frac{\hat{a}_\varphi}{r}\left(\frac{\partial rA_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)$$

$$= \frac{\hat{a}_r}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta} r\sin\theta \cdot r^{1/2} + 0\right) + \frac{\hat{a}_\theta}{r\sin\theta}\left(-2r\cos\theta\sin\varphi - \frac{\partial}{\partial r} r^{3/2}\sin\theta\right) + \frac{\hat{a}_\varphi}{r}\left(\frac{\partial rA_\theta}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)$$

$$= \frac{\hat{a}_r}{r\sin\theta}\left(r^{1/2}\cdot\cos\theta\right) + \frac{\hat{a}_\theta}{\sin\theta}\left(-2r\cos\theta\cdot\sin\varphi - \frac{3}{2}r^{1/2}\sin\theta\right) + \frac{\hat{a}_\varphi}{r}\left(+2r\sin\theta\cos\varphi\right)$$

$$= \hat{a}_r\left(r^{-1/2}\cdot\frac{\cos\theta}{\sin\theta}\right) + \hat{a}_\theta\left(-2r\frac{\cos\theta}{\sin\theta}\sin\varphi - \frac{3r^{1/2}}{2}\right) + \hat{a}_\varphi\left(2\sin\theta\cos\varphi\right)$$

$$\text{em } \hat{a}_r\left(2\cos\left(\tfrac{\pi}{6}\right)\right) + \hat{a}_\theta\left(-4\cos\left(\tfrac{\pi}{6}\right)\sin\left(\tfrac{\pi}{3}\right) - \tfrac{3}{2}\right) + \hat{a}_\varphi\left(\tfrac{1}{2}\right)$$

5. Determine o laplaciano dos seguintes campos escalares:

a) $U = x^2y + xyz$.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 2y$$

b) $V = \rho z\sin\phi + z^2\cos^2\phi + \rho^2$.

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho z\sin\varphi + 2\rho^2\right) + \frac{1}{\rho^2}\left(\frac{\partial}{\partial\varphi}\rho z\cos\phi + z^2(2\cos\phi(-\sin\phi))\right) + 2\cos^2\varphi$$

$$= \frac{1}{\rho}\left(z\sin\varphi + 4\rho\right) + \frac{1}{\rho^2}\left(-\rho z\sin\varphi + 2z^2\sin\varphi\cdot -\cos\varphi\right) + 2\cos^2\varphi$$

$$= \frac{z\sin\varphi}{\rho} + 4 + \frac{1}{\rho^2}\left(-\rho z\sin\varphi + 2z^2\cdot(\cos\varphi\cdot -\cos\varphi + \sin\varphi\cdot\sin\varphi\right) + 2\cos^2\varphi$$

$$= \frac{z\sin\varphi}{\rho} + 4 - \frac{z\sin\varphi}{\rho} + \frac{2z^2}{\rho^2}\left(-\cos^2\varphi + \sin^2\varphi\right) + 2\cos^2\varphi$$

$$= 4 + \frac{2z^2}{\rho^2}\left(\sin^2\varphi - \cos^2\varphi\right) + 2\cos^2\varphi$$

6. Dado que $\vec{H} = x^2\hat{a}_x + y^2\hat{a}_y$, calcule $\int \vec{H} \cdot d\vec{l}$, considerando L ao longo da curva $y = x^2$, de $(0,0,0)$ até $(1,1,0)$.

$dl = dx\,\hat{a}_x + dy\,\hat{a}_y = dx\,\hat{a}_x + 2x\,dx\,\hat{a}_y$

$H dl = x^2 dx + y^2(2x dx) = x^2 dx + x^4 \cdot 2x dx = x^2 dx + 2x^5 dx$

$\int_0^1 (x^2 + 2x^5) dx = \left(\frac{x^3}{3} + \frac{2x^6}{6}\right)\Big|_0^1 = 1/3 + 1/3 = 2/3$

7. Se $\vec{H} = (x-y)\hat{a}_x + (x^2 + zy)\hat{a}_y + 5yz\hat{a}_z$, calcule $\int \vec{H} \cdot d\vec{l}$ ao longo do caminho $(1,0,0) \to (0,0,0) \to (0,0,1) \to (0,2,0)$.

$\int_1^0 x dx + \int_0^1 z d_z + \int_0^1 (6t, 6t^2) dt =$

$\frac{x^2}{2}\Big|_0^1 + 0 + (-3t^2 + 2t^3)\Big|_0^1 = -\frac{1}{2} - 3 + 2 = -3/2$

8. Determine o vetor unitário normal à superfície $S = x^2 + y^2 - z$ no ponto (1,3,0).

$\nabla S = (2x, 2y, -1)$

$|\nabla S| = \frac{2}{\sqrt{41}}\hat{a}_x + \frac{6}{\sqrt{41}}\hat{a}_y - \frac{1}{\sqrt{41}}\hat{a}_z$

9. Mostre que o campo vetorial $\vec{F} = y^2z\hat{a}_x + 2xyz\hat{a}_y + (2z + xy^2)\hat{a}_z$ é um campo conservativo e encontre o campo escalar V associado tal que $\vec{F} = \vec{\nabla}V$.

$(\nabla \times F)_x = \partial_y(2z + xy^2) - \partial_z(2xyz) = 2xy - 2xy = 0$

$(\nabla \times F)_y = \partial_z(y^2 z) - \partial_x(2z + xy^2) = y^2 - y^2 = 0$

$(\nabla \times F)_z = \partial_x(2xyz) - \partial_y(y^2 z) = 2yz - 2yz = 0$

$V = xy^2z + g(y,z)$

$g = g(z) = z^2 + c$

$V(x,y,z) = xy^2z + z^2 + c$

Use o resultado para calcular $\int \vec{F} \cdot d\vec{l}$ do ponto (2,1,1) até o ponto (3,2,2).

$\int F d_v = V(3,2,2) - V(2,1,1) = (3 \cdot 4 \cdot 2 + 4) - (2 \cdot 1 \cdot 1 + 1) = 28 - 3 = 25$

10. Seja $\vec{D} = \rho^2 \cos^2 \phi \hat{a}_\rho + z \sin \phi \hat{a}_\phi$, determine o fluxo líquido de $\vec{D}$ sobre a superfície fechada de um cilindro definido por $0 \le z \le 1$, $\rho = 4$. Verifique o teorema da divergência para este caso.

$$\iint \vec{D} \cdot d\vec{S} = \iiint \nabla \cdot \vec{D} \, dV = \int_0^{2\pi} \int_0^4 \int_0^1 3\rho^2 \cos^2\varphi \, dz \, d\rho \, d\varphi + \int_0^{2\pi} \int_0^4 \int_0^1 \left(z \cos^2\varphi + \frac{z}{\rho} \cos\varphi \right) \rho \, dz \, d\rho \, d\varphi$$

$$= 1 \cdot 3 \cdot \frac{4^3}{3} \cdot \pi + \frac{1}{2} \cdot 4 \cdot 0 = 64\pi$$

11. Verifique o teorema da divergência para a função $A = r^2 \hat{a}_r + r \sin \theta \cos \phi \hat{a}_\phi$ sobre a superfície de um quadrante de hemisfério definido por $0 < r < 3; 0 < \phi < \pi/2; 0 < \theta < \pi/2$.

$$\int_0^{2\pi} \int_0^{2\pi} \int_0^3 (4r - \sin\varphi) r^2 \sin\theta \, dr \, d\theta \, d\varphi = \frac{81\pi}{2} - 9$$

12. Demonstre que $\vec{B} = (y + z\cos(xz))\hat{a}_x + x\hat{a}_y + x\cos(xz)\hat{a}_z$ é conservativo, sem calcular nenhuma integral.

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{\partial}{\partial y} x\cos(xz) - \frac{\partial}{\partial z} x, \; + \frac{\partial}{\partial z}(y + z\cos(xz)) - \frac{\partial}{\partial x}(x\cos(xz)),$$

$$+ \frac{\partial}{\partial x} x - \frac{\partial}{\partial y}(y + z\cos xz) =$$

$$= (0-0)\hat{a}_x + \left(\cos(xz) - xz\sin(xz) - \cos(xz) + xz\sin(xz) \right) \hat{a}_y + (1-1)\hat{a}_z =$$

$$0\hat{a}_x + 0\hat{a}_y + 0\hat{a}_z \quad \Rightarrow \quad \nabla \times \vec{B} = 0 \quad \text{CONSERVATIVO!}$$

13. Seja a função vetorial $\vec{A} = \rho \cos \phi \hat{a}_\rho + z \sin \phi \hat{a}_z$, verifique o teorema de Stokes para o caminho fechado L definido por $0 \le \rho \le 2$; $0 \le \phi \le 60°$; $z = 0$.

$$\oint \vec{A} \, d\vec{l} = \int_0^2 \rho \, d\rho + \int_0^{\pi/3} 0 \, d\phi + \int_2^0 \frac{\rho}{2} \, d\rho = 2 + 0 - 1 = 1$$

$$\nabla \times \vec{A} = \sin\varphi \, \hat{a}_z = 1$$