I want to use age (X_1) and years of professional experience (X_2) to predict income. (Y)

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

X_1 and X_2 are likely to be correlated, so we want to add an interaction term.

What if our interaction term added X_1, X_2?

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 + X_2)$$

$$\Rightarrow Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 + \beta_3 X_2$$

$$\Rightarrow Y = \beta_0 + (\beta_1 + \beta_3) X_1 + (\beta_2 + \beta_3) X_2$$

$$\Rightarrow Y = \beta_0 + \delta_1 X_1 + \delta_2 X_2$$

$\delta_1 = \beta_1 + \beta_3$

$\delta_2 = \beta_2 + \beta_3$

We get the same model as before!
They are algebraically the same.

What if our interaction term subtracted X_1, X_2?

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 - X_2)$$

$$\Rightarrow Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 - \beta_3 X_2$$

$$\Rightarrow Y = \beta_0 + (\beta_1 + \beta_3) X_1 + (\beta_2 - \beta_3) X_2$$

$$\Rightarrow Y = \beta_0 + \delta_1 X_1 + \delta_2 X_2$$

$\delta_1 = \beta_1 + \beta_3$

$\delta_2 = \beta_2 - \beta_3$

We get the same model as before!

They are algebraically the same.

What if our interaction term divided X_1, X_2?

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 \frac{X_1}{X_2}$$

If $X_2 = 0$, then $\frac{X_1}{X_2}$ is undefined.

↳ We can't divide by zero!

Also, $\frac{X_1}{X_2}$ has a different interpretation than $\frac{X_2}{X_1}$. This isn't "wrong," but may not be desirable.

What if our interaction term multiplied X_1, X_2?

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 \cdot X_2$$

If $X_2 = 0$, then that's okay! $X_1 \cdot X_2$ is defined.

Multiplication is commutative, so $X_1 \cdot X_2 = X_2 \cdot X_1$.

If I want to interpret a one-unit change in X_1, I can! See next page.

How does a one-unit change in X_1 affect Y?

$$\Big[\underbrace{\beta_0 + \beta_1(X_1+1) + \beta_2 X_2 + \beta_3(X_1+1)X_2}_{X_1+1}\Big] - \Big[\underbrace{\beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2}_{X_1}\Big]$$

$$\Rightarrow \beta_0 + \beta_1 X_1 + \beta_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + \beta_3 X_2 - \beta_0 - \beta_1 X_1 - \beta_2 X_2 - \beta_3 X_1 X_2$$

$$\Rightarrow \cancel{\beta_0} + \cancel{\beta_1 X_1} + \beta_1 + \cancel{\beta_2 X_2} + \cancel{\beta_3 X_1 X_2} + \beta_3 X_2 - \cancel{\beta_0} - \cancel{\beta_1 X_1} - \cancel{\beta_2 X_2} - \cancel{\beta_3 X_1 X_2}$$

$$\Rightarrow \beta_1 + \beta_3 X_2$$

$\Rightarrow$ As X_1 increases by 1, we expect Y to increase by $\beta_1 + \beta_3 \cdot X_2$.