Introduction to Probability

GENERAL ASSEMBLY

Definition

If you were to repeat an experiment infinitely many times, the **probability** of an event occurring is the proportion of times that event occured in those experiments.

For example, if you flip a fair coin infinitely many times, about half of those times you will get a heads.

More Definitions

- **Experiment**: Some clearly defined "process" with an outcome. Lab coat optional.
- **Sample Space**: The set of all possible outcomes of an experiment, usually denoted S (sometimes Ω if you're fancy).
- **Event**: Any collection of outcomes of an experiment. A **subset** of the sample space.

Examples

- **Experiment**: Flip a coin twice.
- **Sample Space**:

- **Event**:

- **Experiment**: Roll one die.
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- **Experiment**: Flip a coin twice.
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  S = {HH, HT, TH, TT}

- **Event**:

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  B = {At least one H}

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- **Experiment**: Roll one die.
- **Sample Space**:

  S = {1, 2, 3, 4, 5, 6}

- **Event**:

  A = {1}
  B = {An even number}
  C = {A prime number}

Definitions

- **Set**: An unordered collection of distinct objects.
  - { Chloe, 17, 😂}

- **Element**: An object that is a member of a set.
  - Chloe
  - 17
  - 😂

Set Operations

- **Intersection**: $A \cap B$ = the set of elements in set A **and** set B
- **Union**: $A \cup B$ = the set of elements in set A **or** in set B
- **Complement**: A^C = the set of elements **not** in A

Memory hint: A∩D

Examples

Experiment: Roll an 8-sided die

A = {2, 4, 6, 8}

B = {2, 3, 5, 7}

1. **$A \cap B$ =**
2. **$A \cup B$ =**
3. **A^C =**
4. **$(A \cup B)^C$ =**

Examples

Experiment: Roll an 8-sided die

A = {2, 4, 6, 8}

B = {2, 3, 5, 7}

1. **$A \cap B$ = {2}**
2. **$A \cup B$ = {2, 3, 4, 5, 6, 7, 8}**
3. **A^C = {1, 3, 5, 7}**
4. **$(A \cup B)^C$ = {1}**

Probability

$$P(A)$$

Probability that...

event A occurs

Key Probability Facts

1. **$P(S) = 1$** : The probability that *something* happens is 1

2. **$P(\varnothing) = 0$** : The probability that *nothing* happens is 0

3. **$0 \leq P(A) \leq 1$** : The probability of any given event is between 0 and 1

(Some) Probability Rules

Rule 1: Complements

$$P(A^C) = 1 - P(A)$$

Rule 2: Intersections (*and* Probabilities)

$$P(A \cap B) = P(A)P(B \mid A)$$

Probability that A *and* B occur

Probability that B occurs **given** that A has already occurred

Statisticians love urns #1

You have an urn with 12 red balls and 12 blue balls. You pull one ball, do not put it back in, and then you pull a second ball.

What's the probability that they're both blue?

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$$P(B_1 \cap B_2) = P(B_1)\,P(B_2 \mid B_1) = (12 / 24) \times (11 / 23) = 11 / 46 = 0.239$$

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$$P(R_1 \cap R_2) = P(R_1)\, P(R_2 \mid R_1) = (12 / 24) \times (11 / 23) = 11 / 46 = 0.239$$

Aside: Independence

The concept of independence shows up over and over again in our course.

Two events are said to be **independent** if the probability of one occurring does not affect the probability of the other occurring.

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Alternatively, A and B are **independent** if:

$$P(A \mid B) = P(A)$$

Which of the following are independent?

Scenario	A	B
1	Flipping a heads on a coin	Rolling a 1 on a six-sided die
2	Tim hits the snooze button on his alarm clock	Someone in Paris, whom Tim does not know, hits the snooze button on his alarm clock
3	Tim has pizza for lunch	Tim has pizza for dinner
4	The amount of shark attacks on a given day is high	The amount of ice cream sales on that same day is high
5	Today's high temperature is 76°	Tomorrow's high temperature is 76°

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Which of the following are independent?

Scenario	A	B	Why are these not independent?
3	Tim has pizza for lunch	Tim has pizza for dinner	If I have pizza for lunch, I will probably not have pizza for dinner.
4	The amount of shark attacks on a given day is high	The amount of ice cream sales on that same day is high	While one does *cause* the other, if the amount of shark attacks is high, it is because the weather is nice, which means ice cream sales are likely to be high
5	Today's high temperature is 76°	Tomorrow's high temperature is 76°	If it's nice today, it will more likely be nice tomorrow, too

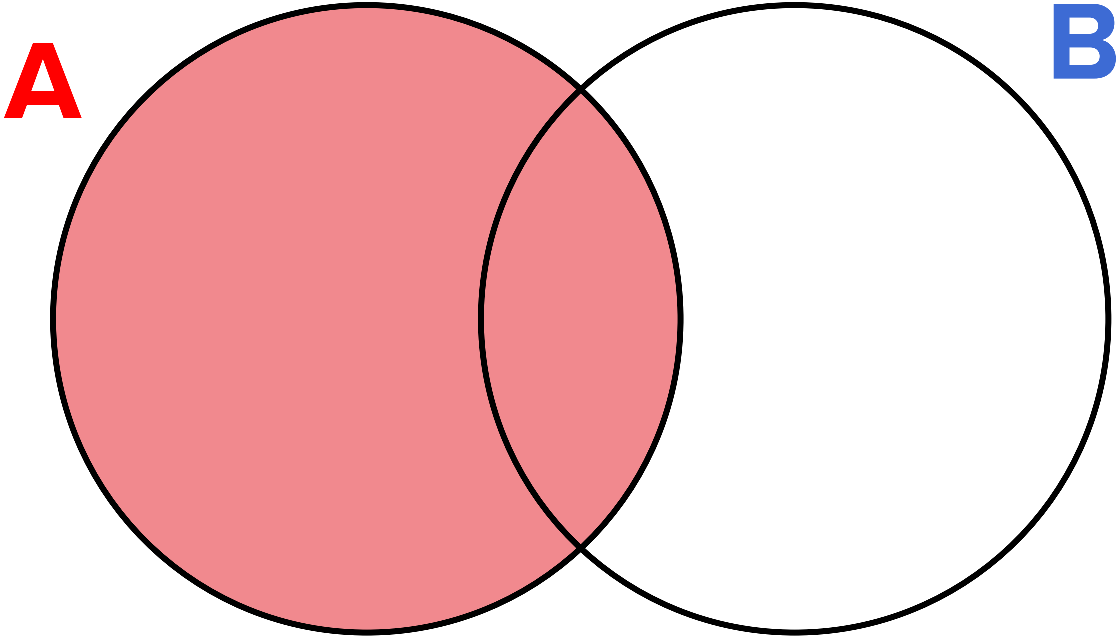
Rule 3: Union (*or* Probabilities)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Probability that A *or* B occur

??? what???

Probability Unions



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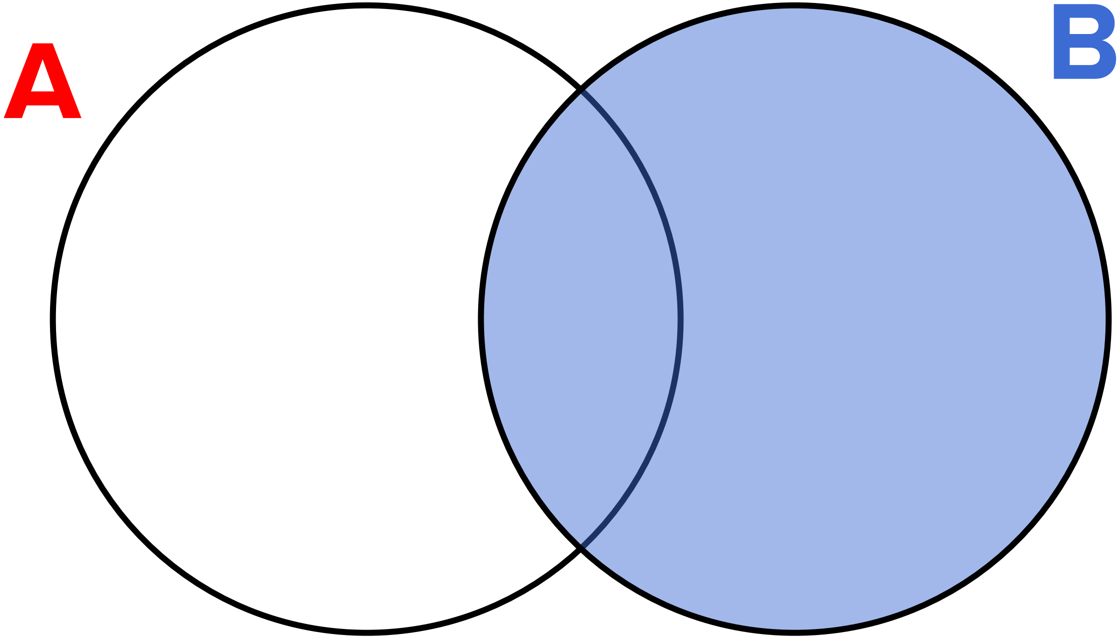

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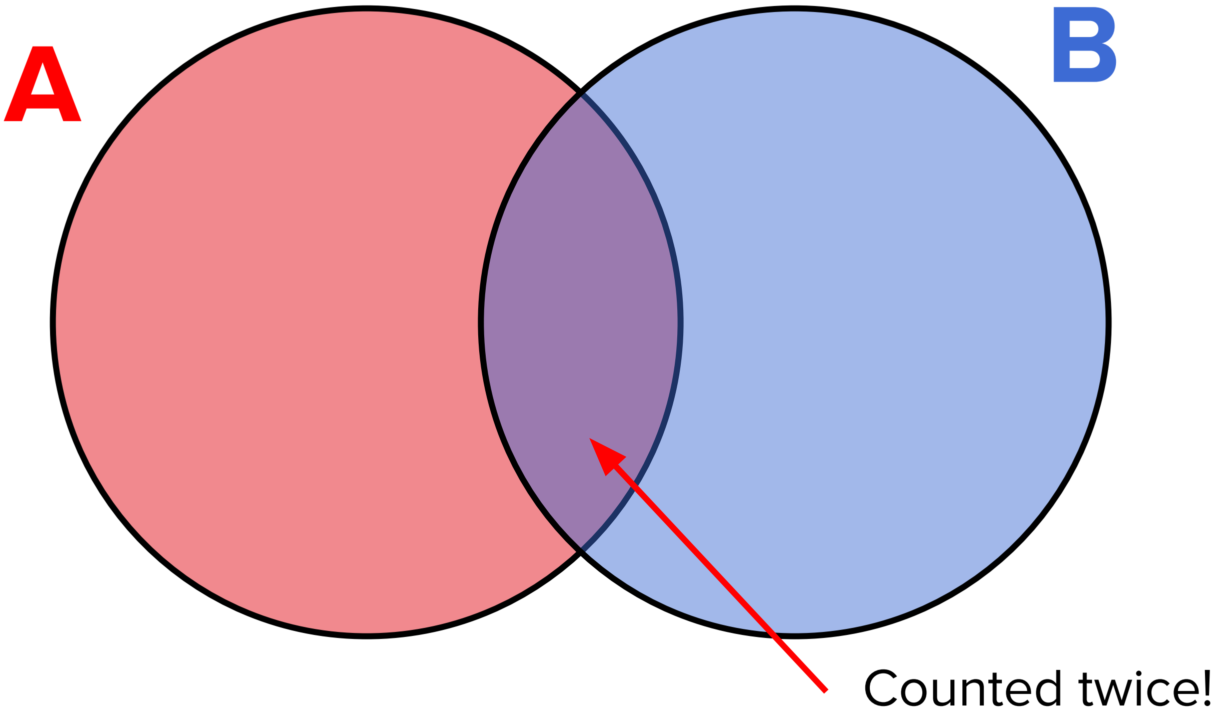

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0

P(Both Blue ∪ Both Red) = P(Both Blue) + P(Both Red) - P(Both Blue ∩ Both Red)

These events are disjoint!

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P(Both Blue ∪ Both Red) = P(Both Blue) + P(Both Red) = 0.239 + 0.239 = 0.478

When by hand is tough...

When these problems get too hard, why do the math? We can often find the approximate answers via **simulation**.

Why does this work?

The Law of Large Numbers (LLN)

The **Law of Large Numbers** states that if you were to repeat an experiment infinitely many times and some other conditions which are always met in practice, then the simulated probability of an event approaches the true probability.*

*Ok, the real LLN is more complicated and has many more implications than this, and there are many different kinds of LLNs. This is a good gist, though.

Let’s do it!