Formal Explanation of Hair Color Blending Formulas (Revised for Realism)

Introduction

This document presents a mathematically rigorous model for simulating the blending of natural and artificial (dyed) hair colors. The model reflects the physical limitations of hair dyeing: lightness (brightness) of natural hair strongly limits the achievable lightness after dyeing, while hue and saturation are more blendable, especially for light hair or yellow-based dyes. The formulation is suitable for computer graphics implementations (e.g., OpenGL, GLSL).

Variables and Ranges

Let:

- **Natural hair color in HSL:**
  $N = (N_h, N_s, N_l)$
  - N_h: Hue in radians, $[0, 2\pi)$
  - N_s: Saturation, $[0, 1]$
  - N_l: Lightness, $[0, 1]$
- **Natural hair color in RGB:**
  (N_r, N_g, N_b) from HSL$\to$RGB, each $[0, 1]$
- **Natural brightness:**
  $NB = 0.299N_r + 0.587N_g + 0.114N_b$
- **Artificial color in HSL:**
  $A = (A_h, A_s, A_l)$, same ranges as above
- **Final blended color in HSL:**
  $F = (F_h, F_s, F_l)$

Blending Equations

If the hair is bleached (i.e., `bleached = true`), the final blended color is simply the artificial color: $F = A$

Otherwise, the final blended color is a weighted average of natural and artificial HSL components, but with **distinct blending factors for hue, saturation, and lightness**:

$$F_h = \alpha \cdot A_h + (1-\alpha) \cdot N_h$$
$$F_s = \beta \cdot A_s + (1-\beta) \cdot N_s$$
$$F_l = \gamma \cdot A_l + (1-\gamma) \cdot N_l$$

Where

- `bleached`: boolean, true if hair is bleached
- α: blending factor for hue
- β: blending factor for saturation
- γ: blending factor for lightness (special constraints, see below)

- **Special handling for achromatic colors (white, black, gray):**
  - If both natural and artificial saturation are zero (i.e., both are achromatic), set the final hue to any value (e.g., 0 or the natural hue), as it will not affect the RGB result. Do not perform hue blending in this case to avoid NaN.
  - If only one color is achromatic, use the hue of the chromatic color for blending.

$$F_h = \begin{cases} \text{any value (e.g., } N_h\text{)}, & \text{if } N_s = 0 \text{ and } A_s = 0 \\ A_h, & \text{if } A_s > 0 \text{ and } N_s = 0 \\ N_h, & \text{if } N_s > 0 \text{ and } A_s = 0 \\ \alpha \cdot A_h + (1-\alpha) \cdot N_h, & \text{otherwise} \end{cases}$$
$$F_s = \beta \cdot A_s + (1-\beta) \cdot N_s$$
$$F_l = \gamma \cdot A_l + (1-\gamma) \cdot N_l$$

$$\alpha = S(NB) + (1 - S(NB)) \left(\frac{1 + \cos\left(A_h - \frac{\pi}{3}\right)}{2} \right)^p$$
$$\gamma = S_{\text{dark}}(A_l - N_l) + (1 - S_{\text{dark}}(A_l - N_l)) NB^q$$

where $S(NB) = \frac{1}{1+e^{-k(NB-c)}}$ $S_{\text{dark}}(x) = \frac{1}{1+e^{dx}} \quad (d \gg 1)$

Parameter Recommendations

- $p = 2$: sharp proximity for yellow range
- $k = 5$, $c = 0.5$: sigmoid transition at mid-brightness
- $q = 1$ (linear) or 2 (quadratic, more realistic for lightness anchoring)
- $d = 20$: steepness for darkening transition
- All HSL values normalized to $[0, 1]$ except hue in $[0, 2\pi)$

Practical Effects

- **Dark hair**:
  - Final lightness stays low, regardless of dye lightness.
  - Only yellow-based dyes shift hue/saturation noticeably.
- **Light hair**:
  - Artificial color can smoothly darken hair (e.g., blonde to black becomes black), with no abrupt transition.
  - Artificial color cannot lighten hair beyond its natural brightness.
- **Intermediate hair**:
  - Gradual, nonlinear, and smooth transition in dominance.

Implementation Notes

- Always handle hue blending circularly (e.g., via trigonometric interpolation or modular arithmetic).
- **For achromatic colors (saturation = 0), skip hue blending and set hue to any value (e.g., 0 or the natural hue) to avoid NaN.**
- For graphics code, ensure all variables are clamped to valid ranges after computation.

Summary

This revised model ensures realistic simulation:

- **Hue and saturation** can shift with dye, especially for compatible colors and light hair.
- **Lightness** is strongly anchored to the natural base, reflecting real-world dye limitations.
- **Nonlinear transitions** ensure that extreme results (like light yellow on black hair) are impossible.

This model is robust for digital hair color preview and simulation applications.