1. What is the support of X, and is X a discrete or continuous random variable?

X is a continuous random variable described by F such that:

$$F : \mathbb{R} \mapsto \mathbb{R}$$

$$\left(F(x) \doteq \begin{cases} 0 & \text{if } x \leq 0 \\ x^\alpha & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases} \right) : \alpha \in \mathbb{R}^+$$

X is clearly continuous. I have a proof of this in the "Extras" section, just for fun.

For any continuous random variable X described by a cdf F, that random variable's support can be described as:

$$S_X = \{x \mid 0 < \mathrm{F}(x) < 1\}$$

For our function F, that occurs at the interval $(0, 1)$.

The way the function is written excludes 0 and 1, as their probability density must be 0.

2. calculate the pdf, $f(x)$

pdf is the derivative of the cdf for a continuous random variable.

$$\frac{\delta F(x)}{\delta x} = \begin{cases} 0 & \text{if } x \leq 0 \\ \alpha x^{\alpha-1} & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

3. Calculate $E[X]$ and $V[X]$.

In our case (I'm assuming we're meant to assume *iid*):

$$E(X) = \int_0^1 x \cdot \alpha x^{\alpha-1} \delta x$$

$$\cdots = \int_0^1 \alpha x^\alpha \delta x$$

$$\cdots = \frac{\alpha \cdot 1^{\alpha+1}}{\alpha+1} - \frac{\alpha \cdot 0^{\alpha+1}}{\alpha+1} = \frac{\alpha \cdot 1^{\alpha+1} - \alpha \cdot 0^{\alpha+1}}{\alpha+1} = \frac{\alpha}{\alpha+1}$$

$$V(X) = \int_0^1 x^2 \cdot \alpha x^{\alpha-1} \delta x - \frac{\alpha^2}{(\alpha+1)^2}$$

$$\cdots = \int_0^1 \alpha x^{\alpha+1} \delta x - \frac{\alpha^2}{(\alpha+1)^2}$$

$$\cdots = \frac{\alpha}{\alpha+2} - \frac{\alpha^2}{(\alpha+1)^2} = \frac{\alpha}{a^3 + 4a^2 + 5a + 2}$$

Extra:

Given a random variable X described by a cdf F, X is continuous iff F is continuous.

Continuity, as a property in the reals ($\mathbb{R}$), can be expressed:

$$f : \mathbb{R} \mapsto \mathbb{R}$$

$$\forall y \in \mathbb{R}, F \text{ is continuous at } y \leftrightarrow \left(\lim_{x \to y^+} f(x) = \lim_{x \to y^-} f(x) = f(y) \right) \quad (A)$$

The following is a proof that F is continuous:

$$\forall \alpha \in \mathbb{R}, 1^\alpha = 1 \quad (B)$$

$$\forall \alpha \in \mathbb{R} \setminus \{0\}, 0^\alpha = 0 \quad (C)$$

$$\forall \alpha \in \mathbb{R} \setminus \{0\}, x^\alpha \text{ is continuous.} \quad (D)$$

$$\lim_{x \to 1^+} F(x) = \lim_{x \to 1^+} 1 = 1 \quad (1)$$

$$\lim_{x \to 1^-} F(x) = \lim_{x \to 1^-} x^\alpha = 1^\alpha = 1 \quad (\text{using } A, B, D) \quad (2)$$

$$\lim_{x \to 1^+} F(x) = \lim_{x \to 1^-} F(x) = 1 \quad (\text{using } 1, 2) \quad (3)$$

$$\therefore F(x) \text{ is continuous at } 1 \quad (\text{using } A, 3) \quad (I)$$

$$\lim_{x \to 0^-} F(x) = \lim_{x \to 0^-} 0 = 0 \quad (1)$$

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} x^\alpha = 0^\alpha = 0 \quad (\text{using } A, C, D) \quad (2)$$

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^-} F(x) = 0 \quad (\text{using } 1, 2) \quad (3)$$

$$\therefore F(x) \text{ is continuous at } 0 \quad (\text{using } A, 3) \quad (II)$$

$$F(x) : x \in (0, 1) = x^\alpha \quad (1)$$

$$x^\alpha \text{ is continuous.} \quad (\text{using } D) \quad (2)$$

$$\therefore F(x) \text{ is continuous over } (0, 1) \quad (\text{using } 1, 2) \quad (III)$$

$$F(x) : x \in (1, \infty) = 1 \quad (1)$$

$$1 \text{ is continuous.} \quad (\text{trivially}) \quad (2)$$

$$\therefore F(x) \text{ is continuous over } (1, \infty) \quad (\text{using } 1, 2) \quad (IV)$$

$$F(x) : x \in (-\infty, 0) = 0 \quad (1)$$

$$0 \text{ is continuous.} \quad \text{(trivially)} \quad (2)$$

$$\therefore F(x) \text{ is continuous over } (-\infty, 0) \quad (\text{using } 1, 2) \quad (V)$$

$$\therefore F \text{ is continuous.} \quad (\text{using } I, II, III, IV, V)$$

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