How to Model Stars!

Note Title 1/9/2019

Equations of Stellar Structure

① $\rho \cancel{\frac{D\underset{\sim}{u}}{Dt}} = -\underset{\sim}{\nabla} p - \rho \underset{\sim}{\nabla} \phi$ (Euler).

② $\frac{D\rho}{Dt} + \rho \underset{\sim}{\nabla} \cdot \underset{\sim}{u} = 0$ (conservation of mass).

③ $\nabla^2 \phi = 4\pi G \rho$ (gravity)

④ $\frac{DU}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = \epsilon - \frac{1}{\rho}\underset{\sim}{\nabla} \cdot \underset{\sim}{F}$ (energy).

Some basic ("bad") assumptions:

- time independence and static: $\underset{\sim}{u} = 0$.
- spherical symmetry.

① $\frac{dp}{dr} = -\rho \frac{d\phi}{dr}$

② $0 = 0$

③ $\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 4\pi G \rho$

④ $\rightarrow \quad 0 = \epsilon - \frac{1}{\rho}\underset{\sim}{\nabla} \cdot \underset{\sim}{F}$.

Integrate ③ : $r^2 \frac{d\phi}{dr} = 4\pi G \int_0^r \rho(r')r'^2 dr'$

But the mass enclosed inside a radius r is

$$M = 4\pi \int_0^r \rho(r') r'^2 d$$

③ $\rightarrow$ $\frac{d\phi}{dr} = \frac{GM(r)}{r^2}$

① $\longrightarrow$ $\boxed{\frac{dp}{dr} = -\frac{GM\rho}{r^2}}$ Hydrostatic equilibrium.

But with another relationship between ρ and p, we can't solve this equation.

Polytropic Model.

Assume $p = k\rho^{\gamma}$ $\quad k, \gamma$ are constants.

(e.g. for an adiabatic gas, $p\rho^{-\gamma} = \text{constant}$)

From ①, $\dfrac{dp}{dr} = -\dfrac{GM\rho}{r^2}$

$$\Rightarrow \frac{r^2}{\rho}\frac{dp}{dr} = -GM$$

Take derivative of both sides:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -G\frac{dM}{dr}$$

From ③, $dM/dr = 4\pi r^2 \rho$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho$$

Use $p = k\rho^{-\gamma} \Rightarrow \dfrac{dp}{dr} = k\gamma\,\rho^{\gamma-1}\dfrac{d\rho}{dr}$

$$\frac{k\gamma}{r^2}\frac{d}{dr}\left(r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right) = -4\pi G\rho$$

This is an ODE involving just density.

Clear it up: • define a dimensionless parameter

$$\rho(r) = \rho_c \theta_n^n(r)$$

• Define n such that

$$\gamma = \frac{n+1}{n}$$

$$\Rightarrow \left(\frac{(n+1)}{4\pi G} k \rho_c^{\frac{1-n}{n}} \right) \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta_n}{dr} \right) = -\theta_n^n$$

• Define $\eta = \frac{r}{\lambda_n}$, $\lambda_n = \left[\frac{(n+1)}{4\pi G} k \rho_c^{\frac{(1-n)}{n}} \right]^{\frac{1}{2}}$

$$\Rightarrow \boxed{\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\theta_n}{d\eta} \right) = -\theta_n^n}$$

Lane-Emden equation.

Boundary Conditions:

• We want the density to go to zero at some radius - call it η_s.

$$\boxed{\theta_n(\eta_s) = 0}$$

• At the centre of the star, we want the density to be finite:

$$\boxed{\theta_n(0) = 1}$$

[So that ρ_c is the density at the centre of the star.]

- Consider a small volume at the centre with radius δ

$$V = \frac{4}{3}\pi\delta^3$$

The mass in this radius is

$$M = \frac{4\pi}{3}\bar{\rho}\,\delta^3$$

↳ average density

From ① we have

$$\frac{dp}{dr} = -\frac{GM\rho}{r^2} = -\frac{4\pi}{3}G\bar{\rho}^2\delta$$

so as $\delta \to 0$, $dp/dr \to 0$

so

$$\frac{dp}{dr} \to 0 \quad \text{as} \quad r \to 0$$

or

$$\boxed{\frac{d\rho_n}{d\eta} = 0 \quad \text{at} \quad \eta = 0.}$$

Analytic solutions:

- $n = 0$, $\rho_0(\eta) = 1 - \frac{\eta^2}{6}$

The surface is at $\eta_s = \sqrt{6}$

- $n = 1$, $\rho_1(\eta) = \frac{\sin\eta}{\eta}$, $\eta_s = \pi$
- $n = 5$, $\rho_5(\eta) = \frac{1}{\sqrt{1 + \frac{1}{3}\eta^2}}$, $\eta_s = \infty$.

Solving ODEs (Press et al.)

Consider a general 2nd order ODE:

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$

We can always turn this into 2 1st order ODEs:

$$\begin{cases} \dfrac{dy}{dx} = z(x) \\ \dfrac{dz}{dx} = r(x) - q(x)\,z(x) \end{cases}$$

So our general problem is to solve a set of coupled 1st order ODEs:

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \dots, y_N), \quad i = 1 \dots N.$$

We'll also need BCs – but we'll assume we have an "initial value problem" – we have the y_i's at some starting point, and are looking for y_i's at some point.

Euler's Method

Consider just 1 ODE,

$$\frac{dy}{dx} = f(x,y).$$

write $dy \to \Delta y$, $dx \to \Delta x$

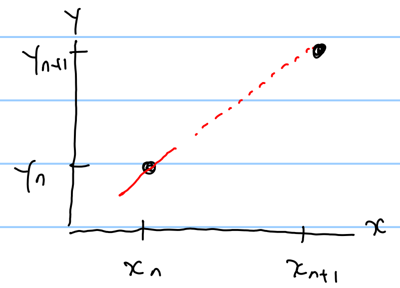
$$\frac{\Delta y}{\Delta x} = f(x, y)$$

$$\Delta y = f(x,y)\,\Delta x.$$

$$\Rightarrow \quad y_{n+1} = y_n + \Delta y = y_n + f(x,y)\,\Delta x.$$

Write $\Delta x = x_{n+1} - x_n = h$

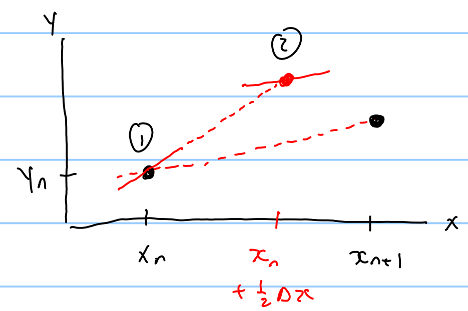
$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$



Error is $\mathcal{O}(h^2)$

MidPoint Method

Let's take a "trial" step to midpoint and use the derivative at that point :



- compute the derivative at start ①

$$k_1 = h f(x_n, y_n)$$

- point (2) is at $(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$
  Compute the derivative at that point

$$k_2 = h f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)$$

So $y_{n+1} = y_n + k_2$

Midpoint is $\mathcal{O}(h^3)$

4th Runge - Kutta

Why stop there? Use 4 derivatives:

- one at the start
- two at the midpoint
- one at the end

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + \tfrac{h}{2}, y_n + \tfrac{k_1}{2})$$

$$k_3 = h f(x_n + \tfrac{h}{2}, y_n + \tfrac{k_2}{2})$$

$$k_4 = h f(x_n + h, y_n + k_3)$$

$$\Rightarrow \quad y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}$$

i Order $\mathcal{O}(h^5)$