PHY 4910U TECHNIQUES OF MODERN ASTROPHYSICS | WINTER 2019

SUBMODULE 1.2
SOLVING ORDINARY DIFFERENTIAL EQUATIONS

Consider the Lane-Emden equation for polytropic models of stars:

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left[\eta^2\frac{d\varrho}{d\eta}\right] = -\varrho^n.$$

SETTING UP

Rewrite this equation as two first order ODEs. For simplicity and consistency with my work, define $x \equiv \eta$, $y \equiv \varrho$, and $z \equiv d\varrho/d\eta$.

subject to boundary conditions

$$\varrho(\eta_s) = 0, \quad \frac{d\varrho}{d\eta} = 0 \text{ at } \eta = 0,$$

where η_s is the surface of the star. Choose a value for n. Plot the solution. What is the radius of the star? How does the radii behave as you change n?

A. EULER'S METHOD

Choose either $n = 0, 1$, or 5, and solve the ODE subject to boundary conditions

$$\varrho(\eta_s) = 0, \quad \frac{d\varrho}{d\eta} = 0 \text{ at } \eta = 0,$$

where η_s is the surface of the star.

B. FOURTH ORDER RUNGE-KUTTA

Now solve it again, but use the fourth order Runger-Kutta method. Recall that

$$\begin{aligned}
k_1 &= hf(x_n, y_n) \\
k_2 &= hf(x_n + h/2, y_n + k_1/2) \\
k_3 &= hf(x_n + h/2, y_n + k_2/2) \\
k_4 &= hf(x_n + h, y_n + k_3) \\
y_{n+1} &= y_n + k_1/6 + k_2/3 + k_3/3 + k_4/6.
\end{aligned}$$

ANALYSIS

Explore the difference between the two methods by plotting them both, along with the analytical solution. Vary your step-size Δx; at what step size does Runge-Kutta become noticeably better? About how large is Δx when each method visibly breaks down?

CLEAN UP

Create a function for your ODE solver, with a pattern that looks like:

```
def ode_solver(x_start, x_stop, step_size, y_0, z_0, f, g),
```

where $f = f(x, y, z)$ and $g = g(x, y, z)$ are functions for the derivatives (e.g., $f = dy/dx$).

Finally, create a "library" file, called `phy4910.py`, which contains your two solvers (maybe other things, too!).