Unit 3: Multiple Integrals

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3.1: Double Integrals Over Rectangles

Review of Integrals

- With single integrals, we could split up the area into infinitely many rectangles with extremely small width to compute the area under a curve
- A single integral computes the *area under a curve*
- We can approximate area under a curve using a Riemann Sum

Volumes and Double Integrals

- A double integral computes the *volume under a surface*
- If we define f on a closed rectangle R that has dimensions $[a, b] \times [c, d]$, then we can subdivide R into an infinitely small number of rectangles and add up the volumes of the resulting (and really skinny) rectangular prisms
- We can approximate the volume under a surface using a Double Riemann Sum

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

> The **double integral** of f over the rectangle R is:
>
> $$\iint_R f(x, y)dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$
>
> if the limit exists.

- DOUBLE RIEMMAN SUM (w/ upper right corner rectangles): $\iint_R f(x, y)dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta A$
- If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is $V = \iint_R f(x, y)dA$
- **MIDPOINT RULE:** $\iint_R f(x, y)dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x_i}, \bar{y_j})\Delta A$
  - More accurate approximation for the volume under a surface

Average Value

> The average value of a function of two variables defined on a rectangle R is:
>
> $$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y)dA$$
>
> where $A(R)$ is equal to the area of R.

Properties of Double Integrals

- The double integral of a sum is the same as the sum of the double integrals
- The double integral of a constant times a function is equal to that constant times the double integral of the function
- If $f \geq g$ then the double integral of f is $\geq$ the double integral of g

3.2: Iterated Integrals

Definition of an Iterated Integral

- An **iterated integral** is the following: $\int_a^b \int_c^d f(x,y)\,dy\,dx$
  - We integrate with respect to y on the interval [c,d], then integrate what is left over with respect to x on the interval [a,b]
- You can integrate in any order (wrt y or wrt x)- just make sure that you don't accidentally integrate x over y's interval on accident if you flip around!
- When solving iterated integrals, we work *from the inside out*

> FUBINI'S THEOREM: If f is continuous on the rectangle R with dimensions $[a,b]\times[c,d]$, then:
>
> $$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$
>
> More generally, this theorem is true if it is assumed that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist

- $$\iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \quad R=[a,b]\times[c,d]$$

3.3: Double Integrals Over General Regions

The Double Integral Dilemma

- When computing single integrals, we are always computing over an interval
- With double integrals, we don't always want to compute integrals over rectangles, but rather over general regions of weird and interesting shapes
- $\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$ where $F(x,y) = \begin{cases} f(x,y); (x,y)\in D \\ 0; (x,y)\in R\,\&(x,y)\notin D \end{cases}$
- **Type I Region:** a domain that lies between the graphs of two continuous functions of x
- **Type 2 Region:** a domain that lies between the graphs of two continuous functions of y

> If f is continuous on a type I region D, then:
>
> $$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

> If f is continuous on a type II region D, then:
>
> $$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

- TIP: Drawing a diagram of D can help you figure out over what region you are integrating
  - Draw errors to see how you integrate over the region
    - Type I Region: arrow from bottom to top
    - Type 2 Region: arrow from left to right

Properties of Double Integrals

- All of the double integral properties from 3.2 also hold up here as well
- $$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA_+ \iint_{D_2} f(x,y)\,dA$$
    - The above property means that if you can split up a region D into two smaller regions, then you can take the double integral of f wrt the different regions and add the result together
- $$\iint_D 1\,dA = A(D)$$
- If $m \leq f(x,y) \leq M\,\forall\,(x,y) \in D$, then $mA(D) \leq \iint_D f(x,y)\,dA \leq MA(D)$

3.4: Double Integrals in Polar Coordinates

Review of Polar Coordinates

> To convert from rectangular coordinates to polar coordinates (and vice-versa):
>
> $$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r^2 &= x^2 + y^2 \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

Computing Double Integrals in Polar Coordinates

> If f is a continuous function on a polar rectangle R given by $0 \leq a \leq r \leq b$ and $\alpha \leq \theta \leq \beta$ where $0 \leq \beta - \alpha \leq 2\pi$, then
>
> $$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

- **WARNING: Be careful not to forget the additional factor of r when computing a double integral in polar coordinates**

> If f is continuous over a general polar region, then:
>
> $$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

3.5: Applications of Double Integrals

Density and Mass

- mass is equal to the integral of a density function

$$m = \iint_D \rho(x, y)\, dA$$

- the total charge is the integral of the charge density function

$$Q = \iint_D \sigma(x, y)\, dA$$

Moments and Centers of Mass

- **Moment of the entire lamina about the a-axis:** $\iint_D y\, \rho(x, y)\, dA$
- **Moment of the entire lamina about the y-axis:** $\iint_D x\, \rho(x, y)\, dA$
- **Coordinates of the center of mass of a lamina** is:

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\, \rho(x, y)\, dA$$
$$\bar{y} = \frac{M_y}{m} = \frac{1}{m} \iint_d y\, \rho(x, y)\, dA$$
$$\text{where } m = \iint_D \rho(x, y)\, dA$$

Moment of Inertia

- **Moment of inertia about the x-axis:** $I_x = \iint_D y^2 \rho(x, y) dA$
- **Moment of inertia about the y-axis:** $I_y = \iint_D x^2 \rho(x, y)\, dA$
- **Moment of inertia about the origin:** $I_0 = \iint_D (x^2 + y^2) \rho(x, y) dA$
- $I_0 = I_x + I_y$
- **radius of gyration of a lamina about an axis** is a number R such that $mR^2 = I$
- **radius of gyration wrt x-axis:** $m\bar{\bar{y}}^2 = I_x$
- **radius of gyration wrt y-axis:** $m\bar{\bar{x}}^2 = I_y$

Probability

- **joint density function of X and Y:** $P((X, Y) \in D) = \iint_D f(x, y) dA$
- $P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x, y)\, dy\, dx$
- The joint density function has the following properties (because probabilities can't be negative and are measured on a scale from 0 to 1):

$$f(x, y) \ge 0 \qquad \iint_{\mathbb{R}^2} f(x, y)\, dA = 1$$
$$\iint_{\mathbb{R}^2} f(x, y)\, dA = \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

Expected Values

- **expected value** is a *synonym for mean* and is represented in statistics by μ
- $\mu_x = \iint_{\mathbb{R}^2} x f(x, y)\, dA \qquad \mu_y = \iint_{\mathbb{R}^2} y f(x, y) dA$
- **Normal distribution:** $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

3.6: Triple Integrals

Definition of the Triple Integral

> The **triple integral** of f over box B is:
>
> $$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$$

- Fubini's Theorem also works for triple integrals (it doesn't matter the order that you integrate as long as you match the bounds to what you are integrating)

> The triple integral of f over a general region E is...
>
> *Over a Type I region:*
>
> $$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$
>
> *Over a Type II region:*
>
> $$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

Applications of Triple Integrals

- A triple integral is the volume of a solid that is "floating" somewhere in $\mathbb{R}^3$
  - Double integrals compute volumes of solids over regions, whereas with triple integrals each dimension of the solid is defined and is NOT just projected from a domain lying in a specific plane
- **mass:** $m = \iiint_E \rho(x,y,z)\,dV$
- **moments:** $M_{yz} = \iiint_E x\rho(x,y,z)\,dV \quad M_{xz} = \iiint_E y\rho(x,y,z)\,dV \quad M_{xy} = \iiint_E z\rho(x,y,z)\,dV$
- **center of mass:** $\bar{x} = \frac{M_{yz}}{m} \quad \bar{y} = \frac{M_{xz}}{m} \quad \bar{z} = \frac{M_{xy}}{m}$
- **moments of inertia:** $I_x = \iiint_E (y^2+z^2)\rho(x,y,z)\,dV \quad I_y = \iiint_E (x^2+z^2)\rho(x,y,z)\,dV \quad I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV$
- **total electric charge:** $Q = \iiint_E \sigma(x,y,z)\,dV$
- **joint density function of X, Y, and Z:** $P((X,Y,Z)\in E) = \iiint_E f(x,y,z)\,dV$
  - $P(a\le X\le b, c\le Y\le d, r\le Z\le s) = \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx$
  - joint density function satisfies $f(x,y,z)\ge 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx = 1$

3.7: Triple Integrals in Cylindrical Coordinates

What Are Cylindrical Coordinates?

- Cylindrical coordinates are represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane to $P
- r is the radius of a circle in the xy-plane
- θ is the angle of rotation in the xy-plane
- To convert from cylindrical coordinates to rectangular coordinates (and vice-versa), use the following equations:

> $$x = r\cos\theta$$
> $$y = r\cos\theta$$
> $$z = z$$
> $$r^2 = x^2 + y^2$$
> $$\theta = \tan^{-1}\frac{y}{x}$$

Evaluating Triple Integrals with Cylindrical Coordinates

> To evaluate a triple integral in cylindrical coordinates, use the following:
>
> $$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2\theta} \int_{u_1(r\cos\theta)}^{u_2(r\cos\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

- **WARNING: Don't forget to add in the extra factor of r when converting from rectangular coordinates to spherical coordinates**
- Cylindrical coordinates make evaluating certain solids (such as cylinders) much easier to compute their integral

3.8: Triple Integrals in Spherical Coordinates

What Are Spherical Coordinates?

- Spherical coordinates are represented by the ordered triple (ρ, θ, ϕ)
  - ρ is the radius
  - θ is the angle of rotation in the xy-plane
  - ϕ is the angle between the positive z-axis and ρ

> To convert between rectangular coordinates and spherical coordinates (and vice-versa), use the following formulas:
>
> $$x = \rho\sin\phi\cos\theta$$
> $$y = \rho\sin\phi\sin\theta$$
> $$z = \rho\cos\phi$$
> $$\rho^2 = x^2 + y^2 + z^2$$

- **WARNING:** There is not a universal agreement on the notation of spherical coordinates
    - Most books reverse the meanings of θ and ϕ and use r in place of ρ

Evaluating Triple Integrals With Spherical Coordinates

> To evaluate a triple integral in spherical coordinates, use the following formula:
>
> $$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
>
> Where E is a spherical wedge given by:
>
> $$E = \{(\rho,\theta,\phi) | a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

- Triple integral over a general region: $\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(\phi)}^{h_2(\phi)} \int_{u_1(\theta\phi)}^{u_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$
- **WARNING: Don't forget to add in the factor $\rho^2\sin\phi$ when converting from rectangular to spherical coordinates**
- Triple integrals make evaluating integrals with cones or spheres as their boundary significantly easier

3.9: Change of Variables in Multiple Integrals

Taking What We Know and Extending To Generality

- We've already tried changing variables before!
- In 2D calculus, we would substitute x for u using the Substitution Rule
- In 3D calculus, we would substitute x and y using the polar coordinates conversion equations
- In this section, we learn a general way to change variables that is given by a **linear transformation** T from the uv-plane to the xy-plane: $T(u,v) = (x,y)$
    - This is a linear algebra question at its heart
- x and y are related by the equations $x = g(u,v)$ and $y = h(u,v)$
- We usually assume that T is a C^1 transformation, which means that g and h have continuous first-order partial derivatives
- A linear transformation T is just a function whose domain and range are both subsets of $\mathbb{R}^2$
- If $T(u_1,v_1) = (x_1,y_1)$, then the point (x_1,y_1) is in the **range** of T
- If no two points have the same associated range point, then T is **one-to-one**
- If T is a one-to-one transformation, then the **inverse transformation** T^{-1} from the xy-plane
- to the uv-plane can possibly be solved using the equations $u = G(x,y)$ and $v = H(x,y)$

The Jacobian

> The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is:
>
> $$\frac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$
>
> Using this notation, we can approximate ΔA with:
>
> $$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \, \Delta v$$

> CHANGE OF VARIABLES IN A JACOBIAN: Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S into the uv-plane onto a region R in the xy-plane. If F is continuous on R and that R and S are type I or type II plane regions AND that T is one-to-one, except perhaps on the boundary of S, then:
>
> $$\iint_R f(x, y) \, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \, \Delta v$$

The Jacobian and Triple Integrals

- The Jacobian in $\mathbb{R}^3$ is the following determinant:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \Delta u \, \Delta v \, \Delta w$$