UNIT 4: Vector Calculus

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4.1: Vector Fields

About Vector Fields

> Let D be a set in $\mathbb{R}^n$ ($\because$ a plane region). A **vector field on $\mathbb{R}^n$ is a function $\vec{F}$ that assigns to each point $(x_1, x_2, \ldots, x_i)$ in D a n-dimensional vector

- Each point in a vector field represents a vector in space that starts at the indicated point
- You can get a reasonable impression for F by drawing a few of the vectors in the vector field (since it is impossible to draw a vector for the infinite number of points in D)
- Because $\vec{F}$ is a vector, we can express F in it's components- except that in this case each component is an individual function itself

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} + R(x,y)\vec{k} = < P(x,y), Q(x,y), R(x,y) >$$
$$\vec{F}(x,y) = P\vec{i} + Q\vec{j} + R\vec{k}$$

  - P, Q, and R are called *scalar fields* to be distinguished from vector fields
- To sketch a vector field, simply sketch a few of the vectors that are created from the vector function $\vec{F}$
- Vector fields can be used to model gravitation, fluid flow, weather, electricity flow, and more

Gradient Fields

- Because the gradient of f is technically a vector whose components are functions (partial derivatives are functions), ∇f is technically a vector field and is called the **gradient vector field**

Vector Field Vocabulary

- A vector field $\vec{F}$ is called a **conservative vector field** if it is the gradient of some scalar function
  - if there exists a function f such that $\nabla f = \vec{F}$, then $\vec{F}$ is a conservative vector field
    - f would be considered the *potential function for $\vec{F}$
- Not all vector fields are conservative, but conservative vector fields do arise often in the application of physics
  - The gravitation vector field is a conservative vector field

4.2: Line Integrals

Definition of Line Integrals

- Line integrals are similar to single integrals except that rather than integrating over an interval, you integrate over a curve C defined by the parametric equations:

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$
$$OR$$
$$\vec{r} = x(t)\vec{i} + y(t)\vec{j}$$

> If f is defined on a smooth curve C using the parametric equations (see above), then the **line integral of f along C*** s:

> $$\int_C f(x,y)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$
>
> To evaluate the line integral, usethe following formula:
>
> $$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Computing Line Integrals in Different Ways

- If C is a *piecewise-smooth curve* (C is a union of a finite number of smooth curves $C_1, C_2, \cdots, C_i$) where the initial point of C_{i+1} is the final point of C_i, then:

  $$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_i} f(x,y)\,ds$$

- The interpretation of a line integral is based on the interpretation of the function f
  - Ex: if f represents the density of a wire, then the mass of the wire is the line integral of the density function

> To evaluate the **line integral of f along C wrt x**:
>
> $$\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt$$
>
> To evaluate the **line integral of f along C wrt y:**
>
> $$\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$$

- It happens frequently that the line integrals wrt x and y occur together, therefore it is OK to abbreviate using the following:

  $$\int_C P(x,y)dx + \int_c Q(x,y)dy = \int_C P(x,y)dx + Q(x,y)dy$$

  Even though by strict mathematical rules the above statement isn't true

- The hardest part of computing a line integral is figuring out how to set up the line integral with the correct parametric representation
  - Often helpful to parameterize a line segment that starts at $\vec{r}_0$ and ends at $\vec{r}_1$

    $$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \quad 0 \le t \le 1$$

- parameterization is important because it determines the **orientation** of a curve C with the positive direction corresponding to increasing values for t

Line Integrals in Space

- If C is a smooth curve defined parametrically or defined by a vector valued function, then:

  $$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

- line integrals in 2D and 3D can be expressed under the same vector notation:

$$\int_a^b f(\vec{r}(t))|\vec{r}'(t)|\,dt$$

- using this notation, if you know what $\vec{r}(t)$ is, then you know how many dimensions the vector has and you know whether your line integral lives in the plane or whether your line integral lives in space

- Fun fact: the special case of $f(x, y, z) = 1$ results in $\int_C |\vec{r}'(t)|dt = L$ which is the same as the formula for the length of a curve defined parametrically from single-variable calculus
- When evaluating line integrals in space, it is possible to write line integrals using the following:

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

Line Integrals Over Vector Fields

> If $\vec{F}$ is a continuous vector field defined on a smooth curve C given by a vector function $\vec{r}(t), a \leq t \leq b$, then the **line integral of $\vec{F}$ along C** is
>
> $$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}\vec{r}(t)) \cdot \vec{r}'(t)\,dt = \int_C \vec{F} \cdot \vec{T}\,ds$$

- NOTE: Even though $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\,ds$ and integrals wrt arc length are unchanged when orientation is reversed, it is still true that $\int_{-C} \vec{f} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$
  - This is because the unit tangent vector $\vec{T}$ is replaced by its negative when C is replaced by $-C$
- $$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$
  $$\text{where } \vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$$

3.3: The Fundamental Theorem for Line Integrals

Definition of the Fundamental Theorem for Line Integrals

- Just as there is a Fundamental Theorem of Calculus for single variable calculus, there is a Fundamental Theorem for Line Integrals which is the FToC applied to line integrals

> If C is a smooth curve given by the vector function $\vec{r}(t), a \leq t \leq b$ and f is a differentiable function of two or three variables whose gradient vector ∇f is continuous on C, then:
>
> $$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$
>
> In other words: we can evaluate the line integral of a conservative vector field simply by knowing the value of f at the endpoints of C.

- The line integral of ∇f is the net change in f

Independence of Path

- If C_1 and C_2 are two piecewise-smooth curves (called *paths*) that have the same initial point A and terminal point B, then:

$$\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$$

- IN GENERAL: If $\vec{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is **independent of path** if $\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$ for any two paths C_1 and C_2
    - $\therefore$ line integrals of conservative vector fields are independent of path
- a curve is **closed** if its terminal point is the same as its initial point ($\vec{r}(b) = \vec{r}(a)$)
- a curve is **open** if for every point P in D there is a disk with center P that lies entirely in D
- a domain D is **connected** if any two points in D can be joined by a path that lies in D
- a **simple curve** is a curve that doesn't intersect itself anywhere between its endpoints

> $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in D

- Work done by a conservative force field as it moves an object around a closed path is 0

> If $\vec{F}$ is a vector field that is continuous on an open connected region D and $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D and there exists a function f such that $\nabla f = \vec{F}$.

> If $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivative on a domain D, then throughout D:
>
> $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

> If $\vec{F} = P\vec{i} + Q\vec{j}$ is a vector field on an open simply-connected region D and P and Q have continuous first order partial derivatives and
>
> $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ throughout } D$$
>
> then $\vec{F}$ is conservative.

Application: Conservation of Energy

- See *Multivariable Calculus: 6th edition* by James Stewart to see a proof of how to use line integrals and the fundamental theorem for line integrals to prove the Law of Conservation of Energy

4.4: Green's Theorem

> Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then:

> $$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$$

- a curve has **positive orientation** if the direction of the motion along the curve is counterclockwise
- The integral $\oint$ is sometimes used as a notation to show that the line integral is being calculated using the positive orientation
- Green's Theorem is the equivalent of the Fundamental Theorem of Calculus for Double Integrals
  - In both cases there is an integral involving derivatives and the values of the original function only on the boundaries of the domain
- **simple region:** a region that is both a type I and a type II region
- using Green's Theorem, we can get the following formula for the area of D:
$$A = \oint_C x\,dy = -\oint_C y\,dy = \frac{1}{2}\oint_C x\,dy - y\,dx$$

4.5: Curl and Divergence

Curl

- Curl measures how much the fluid flow tends to rotate around a certain point
  - If the net direction of rotation is counterclockwise, the curl is positive
  - If the net direction of rotation is clockwise, the curl is negative
- Curl of a 2d function is a scalar, curl of a 3d function is a vector
  - If 3D, the magnitude of the vector represents how quickly something would rotate about the axis of rotation

> If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field in $\mathbb{R}^3$ and the partial derivative of P, Q, R all exist, then the curl of $\vec{F}$ is the vector field on $\mathbb{R}^3$ defined by:
>
> $$\text{curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

- It can be easier to remember curl if you think of it as $\text{curl } \vec{F} = \nabla \times \vec{F}$
- If f is a function of three or more variables that has continuous second-order partial derivatives, then $\text{curl } (\nabla f) = \vec{0}$

> If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\text{curl } \vec{F} = \vec{0}$, then $\vec{F}$ is a conservative vector field

- If the curl is zero, then there is no sort of rotation occurring at that point

Divergence

- Divergence is a measure of whether vectors generate motion out of nothingness

  - If the divergence is positive at a point, then all the vectors (and therefore the motion associated with it) point outwards (the point is a source)
    - In the context of fluid, fluid flows out of this point
    - Divergence can also be positive if the fluid flow goes from slow to fast
  - If the divergence is negative at a point, then all the vector (and the motion associated with it) point inwards (the point is a sink)
    - In the context of fluid, fluid flows into this point
    - Divergence can also be negative if the fluid flows from fast to slow
- The result of computing the divergence is a scalar
- If the divergence is 0, then in the context of fluids the fluid is incompressible

> If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field in $\mathbb{R}^3$ and $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ exist, then the divergence of $\vec{F}$ is defined by:
>
> $$\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- It can be easier to remember divergence as $\text{div } \vec{F} = \nabla \cdot \vec{F}$
- If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, R have continuous second order partial derivatives, then $\text{div curl } \vec{F} = 0$

Vector Forms of Green's Theorem

- Curl and divergence allows us to rewrite Green's Theorem in other forms that can potentially be useful when solving problems
- GREEN'S THEOREM (Vector Form): $\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot \vec{k} \, dA$
  - ine integral of the tangential component of $\vec{F}$ along C is equal to the double integral of the vertical component of $\vec{F}$ over the region D that is enclosed by C
- GREEN'S THEOREM (Another Vector Form): $\oint_C \vec{F} \cdot \vec{n} \, ds = \iint_D \text{div } \vec{F}(x, y) \, dA$
  - The line integral of the normal component of $\vec{F}$ along C is equal to the double integral of the divergence of $\vec{F}$ over the region D enclosed by C

4.6: Parametric Surfaces and Their Area

Definition of a Parametric Surface

- **Parametric Surface:** A surface where each component of the surface is defined in terms of two or more parameters
- **Parametric Equations:** Equations expressed in terms of the parameters that make up the parametric surface
- **Grid curves:** curves you get from parametric equations by holding one of the parameters constant

Identifying and Creating Parametric Surfaces

- To identify a parametric surface, write out and analyze the parametric equations to find similarities to already-learned surfaces
- To find a parametric representation of a multivariable function, look for other ways to express the function
  - Polar coordinates
  - Spherical coordinates
  - Cylindrical coordinates
  - z in terms of x and y (meaning $x = x$ and $y = y$ are parameters)

Surfaces of Revolution

If a surface S is obtained by rotating a curve $y = f(x)$ around the x-axis where $f(x) \geq 0$ and θ is the angle of rotation, then if (x, y, z) is a point on S,

$$x = x$$
$$y = f(x)\cos\theta$$
$$z = f(x)\sin\theta$$

Tangent Planes to Parametric Surfaces

To find the tangent plane to a parametric surface...

- If given the values for the parameter, plug those values into the parametric equations to obtain the point for which you must find the tangent plane
- Compute the partial derivatives of the parametric surface with respect to the parameters

$$\frac{\partial \vec{r}}{\partial u} = \vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$$
$$\frac{\partial \vec{r}}{\partial v} = \vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$$

- Take the cross-product of the partial derivatives ($\vec{r}_u \times \vec{r}_v$) and use the parameters to compute the resulting vector
  - The coefficients of the resulting vector are the slopes of the tangent plane in each direction
- Write the equation of the tangent plane

$$(\vec{r}_u \times \vec{r}_v)_x(x) + (\vec{r}_u \times \vec{r}_v)_y(y) + (\vec{r}_u \times \vec{r}_v)_z(z) = 0$$

  - Simplify as needed

Surface Area

> GENERAL DEFINITION OF SURFACE AREA:

If a smooth parametric surface S is given by the equation
$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$
and S is covered just once as (u, v) ranges throughout the parameter domain D,
then the surface area of S is:

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| dA$$

SURFACE AREA TO A GRAPH

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

- When evaluating Surface Area for a parametric surface (w/o using surface integrals), you will most likely have to express the surface given to you parametrically (such as expressing the surface in terms of spherical, polar, or cylindrical coordinates)

4.7: Surface Integrals

Definition of a Surface Integral

Suppose that a surface S has a vector equation

$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

If we assume that the parameter domain D is a rectangle, then we can split up the surface into infinitely small "patches" (similar to a line integral)

$$\iint_S f(x,y,z) dS = \iint f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| dA$$

Surface Integrals and Graphs

- The idea of a surface integral can be extended to graphs (where $f(x) = z$) because any surface S can be regarded as a parametric surface with parametric equations

$$x = x$$
$$y = y$$
$$z = g(x,y)$$

The surface integral for a surface S defined parametrically as a graph where $z = g(x,y)$ is:

$$\iint_S f(x,y,z) dS = \iint f(x,y,g(x,y)) \sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}\, dA$$

Oriented Surfaces

- We consider only orientable (two-sided) surfaces (so not Mobius strips)
- If we have a surface that has a tangent plane at every point (except at any boundary point), if it is possible to choose a unit normal vector $\vec{n}$ at every such point (x, y, z) so that $\vec{n}$ varies continuously over the surface, then S is called an *oriented surface* and the given choice of $\vec{n}$ provides S with an *orientation***
    - There are two possible orientations
        - Oriented **outward** (positive)
        - Oriented **inward** (negative)

- It is possible to determine the orientation of the surface if you know the normal vector $\vec{n}$
    - POSITIVE $\vec{n}$ means orientation is **outward**
    - NEGATIVE $\vec{n}$ means orientation is **inward**
    - NOTE: This only works for *closed surfaces*

Surface Integrals of Vector Fields

Definition of a surface integral over a vector field:

> If $\vec{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$, then the _surface integral of $\vec{F}$ over S is defined as
>
> $$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$$
>
> This integral is also called the **flux of $\vec{F}$ across S**

- In more English-speak: the surface integral of a vector field over S is equal to the surface integral of its normal component over S
- If S is given by a vector function $\vec{r}(u, v)$, then the surface integral of S over the parameter domain D is:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

- If the surface S is given by a graph $z = g(x, y)$, then the surface integral of the vector field becomes:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R)\, dA$$

  NOTE: This formula assumes positive orientation. For negative orientation, multiply the above by -1

17.8: Stokes' Theorem

About Stokes' Theorem

- Stokes' Theorem: Green's Theorem for *Surface Integrals*
- CONDITIONS FOR STOKES' THEOREM
  - Oriented- out/in (two sides)
  - Piecewise Smooth- Continuous/ smooth sides for each part of the function
  - Simple- No intersections
  - Closed- Starts and ends at the same point

Definition of Stokes' Theorem

> Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then...
>
> $$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

- In English: Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\vec{F}$ is equal to the surface integral of the normal component of the curl of $\vec{F}$
- Keep in mind that you can go either way when solving Stokes' Theorem (starting with the surface **or** the curve)
- If you have two surfaces that have the same boundary curve C, then **the surface integrals will be the same!**

Relationship Between Stokes' Theorem and the curl Operator

- **circulation**: A measure of the tendency of a fluid (or vector field) to rotate around a curve C
  - $$\text{circulation } = \int_C \vec{v} \cdot d\vec{r}$$
- Using Stokes' Theorem, we can make an approximation for curl:

> The relationship between circulation and curl is defined by the following equation:
>
> $$\operatorname{curl} \vec{v}(P_0) \cdot \vec{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \vec{v} \cdot d\vec{r}$$

- In English: $\operatorname{curl} \vec{v} \cdot \vec{n}$ is a measure of the rotating effect of the vector field about the axis $\vec{n}$
  - The curling effect is greatest about the axis parallel to $\operatorname{curl} \vec{v}$

4.9: The Divergence Theorem

About the Divergence Theorem

- Since Green's Theorem was able to be rewritten as the double integral over the region D of the divergence of $\vec{F}$, then the Divergence Theorem will allow us to extend that into three dimensions
  - ... so the surface integral can be rewritten as the triple integral over a region E of the divergence of $\vec{F}$

Definition of the Divergence Theorem

> If E is a simple solid region and S is the boundary of the surface of E given with the positive (outward) orientation and $\vec{F}$ is a vector field whose component functions have continuous partial derivatives on an open region that contains E, then...
>
> $$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} \, dV$$

- In English: Under certain conditions, the flux of $\vec{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\vec{F}$ over E

Relationship With the Divergence Theorem and the div operator

> $$\text{div } \vec{F}(P_0) = \lim_{a \to 0} \frac{1}{V(B_a)} \iint_{S_a} \vec{F} \cdot d\vec{S}$$

- In English: $\text{div } \vec{F}(P_0)$ is the net rate of outward flux per unit volume at the point P_0
  - Hence the reason for the name "divergence"
- If $\text{div } \vec{F}(P) > 0$ the net flow is outward near P and P is called a **source**
- If $\text{div } \vec{F}(P) < 0$ the net flow is inward near P and P is called a **sink**