CS229PS0

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1.

(a)

$$\nabla f(x) = Ax + b$$

(b)

$$\begin{aligned}
\frac{\delta f(x)}{\delta x_i} &= \frac{\delta g(h(x))}{\delta h(x)} \cdot \frac{\delta h(x)}{\delta x_i} = g'(h(x))\nabla h(x)_i \\
\nabla f(x) &= g'(h(x))\nabla h(x)
\end{aligned}$$

(c)

$$\nabla^2 f(x) = A$$

(d)

$$\begin{aligned}
\frac{\delta f(x)}{\delta x_i} &= g'(a^T x)\frac{\delta a^T x}{\delta x_i} = g'(a^T x)a_i \\
\nabla f(x) &= g'(a^T x)a \\
\nabla^2 f(x) &= \left[\tfrac{\delta g'(a^T x)a}{\delta x_1} \dots \tfrac{\delta g'(a^T x)a}{\delta x_n}\right] \\
&= \left[g''(a^T x)aa_1 \dots g''(a^T x)aa_n\right] \\
&= g''(a^T x)aa^T
\end{aligned}$$

2.

(a)

Now, $A^T = (zz^T)^T = zz^T = A$, so A is symmetric.
For all $\vec{x} \in R^n$, $x^T Ax = x^T zz^T x = y^T y = \sum_i^n y_i^2 \geq 0$.
Hence A is positive semidefinite.

(b)

$zz^T = \begin{bmatrix} z_1 z & z_2 z & \dots & z_n z \end{bmatrix}$
Let $A\vec{x} = \vec{0}$:
$\vec{z} \sum_i^n z_i x_i = \vec{0}$
So $x \in R^n, x^T z = \vec{0}$
Rank of A is 1 as all columns are multiples of $\vec{z}$.

(c)

Yes.
$x^T BAB^T x = y^T Ay$ where $y = B^T x$.
Since A is PSD, $y^T Ay \geq 0$.
Hence BAB^T is PSD.

3.

(a)

$AT = T\Lambda, \begin{bmatrix} At^{(0)} & At^{(1)} & \dots & At^{(n)} \end{bmatrix} = \begin{bmatrix} T\Lambda^{(0)} & T\Lambda^{(1)} & \dots & T\Lambda^{(n)} \end{bmatrix} = \begin{bmatrix} \lambda_i t^{(0)} & \lambda_i t^{(1)} & \dots & \lambda_i t^{(n)} \end{bmatrix}$

(b)

$AU = U\Lambda U^T U, AU = U\Lambda$. Following steps are similar to the part (a).

(c)

$x^T Ax = x^T U\Lambda U^T x = y^T \Lambda y = \sum y_i^2 \lambda_i \geq 0$
Hence, $\lambda_i(A) \geq 0$ for all i.