Math 396 Problem set Updated Mar 26

(1) Draw the bifurcation diagram for $f_a(x) = x^3 + ax$. Make sure you indicate which segments correspond to stable and unstable periodic orbits. (Note: the bifurcation diagram contains all periodic points, not just the fixed points.)

(2) Use the octal decomposition given in the notes (Ch 1) to classify the orbit topology near a fixed point. Use our convention of L_i (R_j) indicating the i^{th} point in the orbit on the left (respectively right) of the fixed point, and $a < b$ ($a > b$) to indicate that the point a is closer to the fixed point (p) than b (respectively, a is further from the fixed point than b). You will not be able to unambiguously classify all (16) cases, but explain why not.

(3) Refer to Chapter 2 notes (pages 10,11) to prove that if $\bar{a}$ is a bifurcation point of $f_a(x)$, then $\partial_x f_a(x) = 1$ at $a = \bar{a}, x = p_{\bar{a}}$ (here we write the fixed point as $p(a)$ to exhibit its dependence on a).

Also, at a period doubling bifurcation point $\bar{a}$ of f_a, that necessarily $\partial_x f_a(x) = -1$ at $a = \bar{a}, x = p_{\bar{a}}$. And note that this is what we observe graphically (see the notes page 4 Lecture 2 and page 28 of the presentation that is posted in the 'Lectures' folder on Canvas (at the top)).

(4) Prove that if f is continuous, then (i) period $2^{k+1} \Longrightarrow$ period 2^k, and (ii) if f has a period **3** orbit then f has a fixed point (so, no fixed point then no periodic orbits!).

(5) Shadow lines. Refer to the final state diagram of the logistic equation (e.g. page 17 Lecture 2).

(a) Why does not the diagram have points from bottom ($x = 0$) to the top ($x = 1$)? That is, although $f_a(x)$ is defined on the entire interval $[0, 1]$, we only see points in the final state diagram in a smaller subinterval.

(b) The shadow lines are caused by points near the peak $x = \frac{1}{2}$ being 'squeezed' towards the value $f_a(\frac{1}{2})$. See the figure on page 44,45 of the posted presentation notes (see above, Q 3). Following this 'enhanced' density of points under iteration produces the other (weaker) shadow lines. Determine these curves and plot them to confirm.

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(6) Show that when $a > 4$ that the *prisoner set* of the logistic equation is a kind of Cantor set. The prisoner set are those points (in $[0, 1]$) that don't leave the interval $[0, 1]$ under iteration. Here you will have to do *backward* graphical iteration. To begin, note the interval I_1 around the peak $x = 1/2$ such that if $x \in I_1$ then $f_a(x) > 1$ and hence this orbit goes off to $-\infty$. Then let I_2 denote the (two) intervals such that if $x \in I_2$ then $f_a(x) \in I_1$, and hence these points also eventually escape. Continue in this fashion to find the (infinite) sequence of (disjoint) intervals $I_1, I_2, \ldots$; if $x \in I_k$, then $f_a^k(x) > 1$. What is the total lengths of all these intervals?

(7) Recall the Collatz discrete dynamical system; start with a positive integer x_0. If x_0 is even, then $x_1 = x_0/2$, otherwise (if x_0 is odd) $x_1 = 3x_0 + 1$.

(a) Show that $\{1, 4, 2\}$ is a period 3 orbit.

(b) Compute the orbits of $3, 5, 6, 7, 8, 9, \ldots..$

(c) Study this problem using graphical iteration. Make a sketch with the two lines $y = \frac{1}{2}x$ and $y = 3x + 1$ along with the diagonal $y = x$. Then the rectilinear curves of the orbits lie between the two extreme lines.

(d) Study this problem using symbolic dynamics (base 2 and base 3...)

The conjecture is: $\{1, 4, 2\}$ is a globally stable orbit, i.e., all orbits eventually become this one.

(8) A dynamical system depends on a parameter a. Initially, you observe a steady state (i.e., a period 1 orbit). As a increases you observe a period 2 oscillation appearing at $a = a_1 = 7$. Then at $a = a_2 = 10$ you observe that the period 2 orbits splits into a period 4 orbit. As a continues to increase a series of period-doublings occurs. Assuming Universality, at what a value would you expect to observe the onset of chaos? ('Assuming Universality' means assuming that the system will go through a series of period-doubling bifurcations as the parameter a changes, and that the distance (in a) between bifurcations is given by the Feigenbaum constant.)

(9) Can the Shadowing Lemma be used to justify the following two statements?

The *observed* (i.e., numerically) instability of a periodic point of $f(x)$ implies that the periodic point is *theoretically* (i.e., exactly) unstable.

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If $f(x)$ has an *observed* ergodic orbit, then $f(x)$ has an *exact* ergodic orbit.

(10) Use symbolic dynamics to prove the following statements about $f_4(x) = 4x(1-x)$.

(a) Find a period 3 orbit (all 3 points) of f_4 that is *not* period 1.

(b) Periodic points are dense in $[0,1]$. Ergodic points are also dense in $[0,1]$. There are 'only' a countable infinity of periodic points but an uncountable number of ergodic points. (Use this fact: In any base, a number is rational iff its base b expansion is eventually repeating, while a number is irrational if it doesn't). Are there any other 'types' of points than these?

(c) Show that f_4 is *mixing*. That is, for any two open subsets $I_1,\ I_2 \subset [0,1]$, there is a k such that $f_4^k(I_1) \cap I_2 \neq \emptyset$.

(Mixing, dense periodic points, and sensitive dependence on initial conditions are usually the criteria a dynamical system must satisfy in order to be called *chaotic*. What we haven't verified precisely yet is that f_4 exhibits sensitive dependence on initial conditions, and to do that we would need to measure the *average* rate of divergence of two orbits that begin with initial points close together; want to try that?)

(d) Can you establish the Shadowing Lemma by the use of symbolic dynamics?

(11) Show that the points in $[0,1]$ that have orbits that remain bounded (in fact, remain in $[0,1]$) under iteration by $f_a(x) = ax(1-x)$ for $a > 4$ is a kind of 'Cantor Set'. That is, it is an uncountable set that can be obtained by removing infinitely many subintervals from $[0,1]$ whose total length is 1.

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(12) Are these objects fractals or not? Explain.

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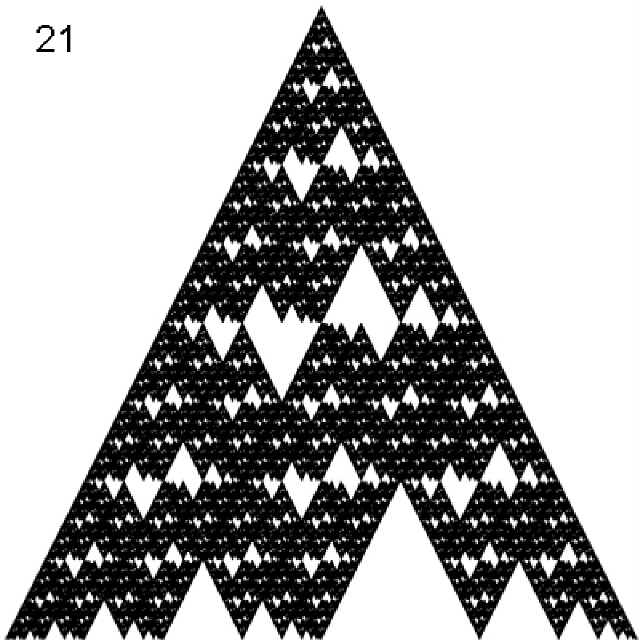

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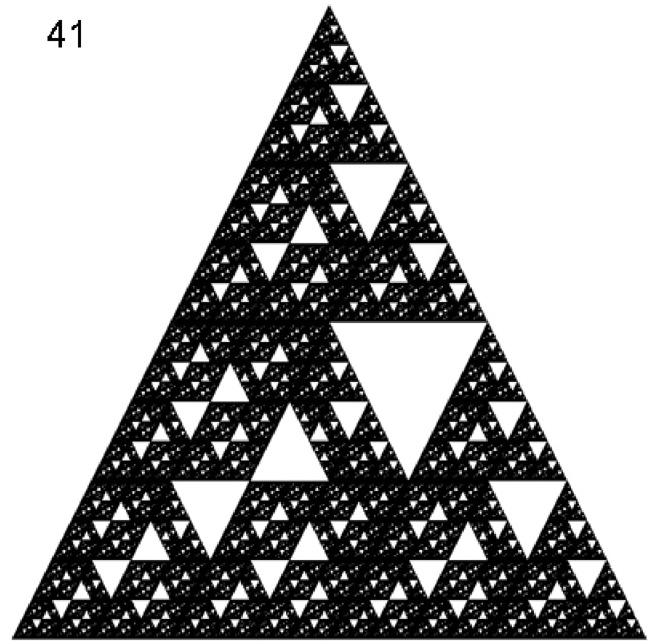

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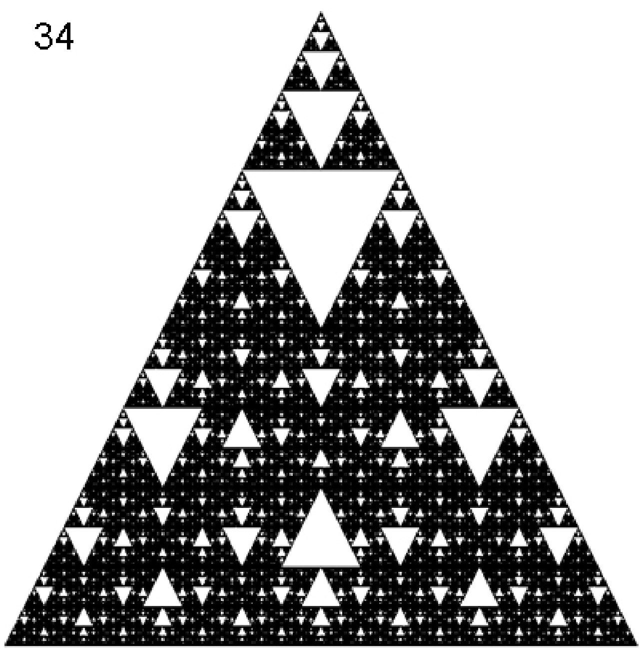

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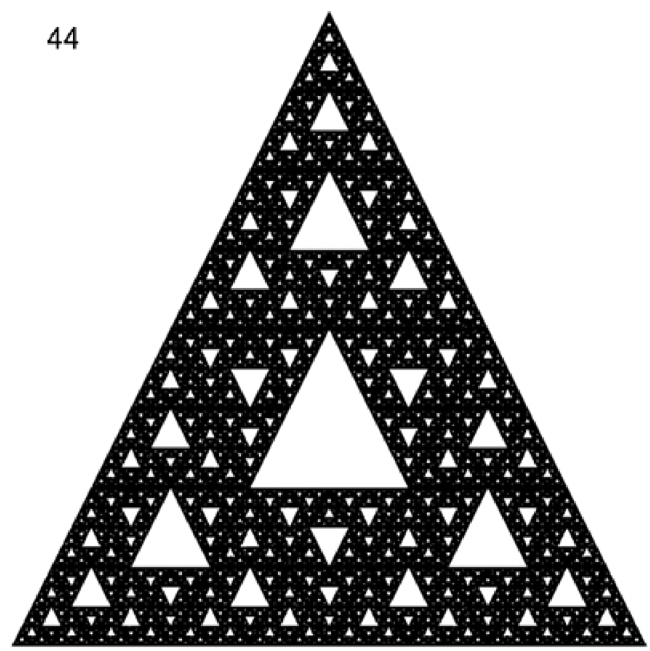

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(13) Explain these two images below that were created with chaos game using the full square IFS and unequal probabilities. Can you predict the relative intensities of the features?