Math 396 Problem set #1 updated Jan 31

(1) Draw the bifurcation diagram for $f_a(x) = x^3 + ax$. Make sure you indicate which segments correspond to stable and unstable periodic orbits. (Note: the bifurcation diagram contains all periodic points, not just the fixed points.)

(2) Use the octal decomposition given in the notes (Ch 1) to classify the orbit topology near a fixed point. Use our convention of L_i (R_j) indicating the i^{th} point in the orbit on the left (respectively right) of the fixed point, and $a < b$ ($a > b$) to indicate that the point a is closer to the fixed point (p) than b (respectively, a is further from the fixed point than b). You will not be able to unambiguously classify all (16) cases, but explain why not.

(3) Refer to Chapter 2 notes (pages 10,11) to prove that if $\bar{a}$ is a bifurcation point of $f_a(x)$, then $\partial_x f_a(x) = 1$ at $a = \bar{a}, x = p_{\bar{a}}$ (here we write the fixed point as $p(a)$ to exhibit its dependence on a).

Also, at a period doubling bifurcation point $\bar{a}$ of f_a, that necessarily $\partial_x f_a(x) = -1$ at $a = \bar{a}, x = p_{\bar{a}}$. And note that this is what we observe graphically (see the notes page 4 Lecture 2 and page 28 of the presentation that is posted in the 'Lectures' folder on Canvas (at the top)).

(4) Prove that if f is continuous, then (i) period $2^{k+1} \implies$ period 2^k, and (ii) if f has a period **3** orbit then f has a fixed point (so, no fixed point then no periodic orbits!).

(5) Shadow lines. Refer to the final state diagram of the logistic equation (e.g. page 17 Lecture 2).

(a) Why does not the diagram have points from bottom ($x = 0$) to the top ($x = 1$)? That is, although $f_a(x)$ is defined on the entire interval $[0, 1]$, we only see points in the final state diagram in a smaller subinterval.

(b) The shadow lines are caused by points near the peak $x = \frac{1}{2}$ being 'squeezed' towards the value $f_a(\frac{1}{2})$. See the figure on page 44,45 of the posted presentation notes (see above, Q 3). Following this 'enhanced' density of points under iteration produces the other (weaker) shadow lines. Determine these curves and plot them to confirm.