Investigation of Wave Propagation on Simple 2D Membranes

Trevor Ford
Department of Physics and Astronomy, West Virginia University, Morgantown, WV, 26505, USA

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Abstract: Wave behavior is fundamental to many areas of physics. Waves serve as a means by which energy (electric, magnetic, mechanical, etc.) is transferred through space, but the discussion of wave behavior is not limited only to these energetic quantities. The propagation of a wave allows us to determine how a property of a system is distributed at every point in space at any given time. More precisely, wave propagation models describe the oscillating nature of quantities and can describe properties such as dispersion, reflection, refraction, et cetera. Strings or membranes bouncing up and down, reflection of light, and even probability oscillations of bound states in quantum wells can be described in some capacity by the nature of waves. The aim of this project is to study mechanical waves, with the final goal being to model the propagation of waves on specifically shaped (rectangular, circular, and triangular) 2D membranes. Of greatest importance to this study is how the waves on these membranes reflect against the fixed-end membranes and interfere after such reflections.

1. Background

The undamped wave equation[1] is the simplest model for describing wave propagation:

$$\ddot{u} = c^2\nabla^2 u \quad (1)$$

This type of equation is known as a partial differential equation. u represents the value of some quantity (in the case of a string or membrane, its height) at all points in space and time, c is the speed of the wave, and ∇^2 is the Laplacian operator;[2] the Laplacian operator acting on a function u is equivalent to $(\nabla \cdot \nabla)u$. Note that this notation with the Laplacian only involves the spatial partial derivatives of u, not the time derivative. Time-derivatives are indicated by Newton's dot notation, as seen with $\ddot{u} = \frac{\partial^2 u}{\partial t^2}$.

The wave equation is an indispensable tool for describing wave propagation, but it alone is not able to describe systems of interest. We also need constraints known as boundary conditions to convey how waves should reflect or transmit upon reaching a membrane's edge. The type of boundary condition we will work with in this project are known as homogeneous Dirichlet conditions,[3] which can be expressed generally in the following form:

$$u(\overrightarrow{r_0}, t) = 0 \quad (2)$$

These boundary conditions add the constraint that our solution must remain zero on a prescribed curve given by the set of vectors $\overrightarrow{r_0}$, which can be in any number of spatial dimensions to preserve generality. This type of boundary condition leads to perfect reflection inversion of waves across boundaries, which is exactly what we want to characterize interference patterns that arise from propagation on these specific membranes.

Separation of variables is a common analytical technique employed in finding solutions to the wave equation on strings and rectangular and circular membranes with homogeneous Dirichlet conditions. This method leads to a solution set of basis vibrational modes that can be linearly combined to describe how nearly any initial distribution will propagate in space under the wave equation.[4] We will employ this technique to find the vibrational modes for strings and rectangular and circular membranes with the prescribed boundary conditions. However, this technique will not work for the triangular boundary.

In order to further investigate the propagation of waves on these boundaries, a numerical method for solving PDEs known as finite difference[5] will be employed. This will allow us to numerically describe how a wave will propagate given specific initial conditions. The initial conditions we will use in general are:

$$u(x, y, 0) = f(x, y) \quad (3)$$

$$\frac{\partial u}{\partial t}(x, y, 0) = 0 \quad (4)$$

These kinds of initial conditions correspond to releasing the membrane with an initial height given by $f(x, y)$ from rest and are the easiest to use for understanding how waves propagate.

2. Research Plan

The first step of this project will be to describe wave behavior using the 1D case of a string with fixed ends, which has many verifiable results from elementary classical mechanics.[6] We will develop the methods of separation of variables and finite difference for this simpler case so that extending to the 2D case will be less strenuous. NumPy and SciPy will be used for the bulk of the heavy computation and information storage; Matplotlib's animation capabilities will be utilized to show how the string's shape will change over time.

We will then move on to study membranes. The extension into an additional dimension will bring some added complexity, especially at the boundaries, but the process will be again accomplished using the capabilities of NumPy and SciPy; Matplotlib has 3D animation tools we can use, but it would also be worthwhile to invest some time looking into other packages.

In the final implementation, we hope to allow users the capability to change the side lengths of rectangular membranes and the interior angles of triangular membranes. This will allow an extension from simple equilateral triangle and square membranes into a much larger array of shapes of interest, such as general scalene triangles. These boundaries will be designed in such a way that the membrane will be centered at the origin of the coordinate system. The initial condition we plan to use for this implementation is

$$f(x, y) = \begin{cases} -100(x^2 + y^2) + 1 & x^2 + y^2 \leq \dfrac{1}{100} \\ 0 & x^2 + y^2 > \dfrac{1}{100} \end{cases} \quad (5)$$

The final goal is to build a program to animate many different instances of wave propagation on simple membranes, which we are confident will be fulfilled by this proposed plan.

3. Project Extensions

Future work on this project could include the generalization from homogeneous Dirichlet conditions to nonhomogeneous Robin[7] boundary conditions of the general form

$$a(\overrightarrow{r_0}, t) * u(\overrightarrow{r_0}, t) + \sum_n b_n(\overrightarrow{r_0}, t) * \frac{\partial u}{\partial q_n}(\overrightarrow{r_0}, t) = c(\overrightarrow{r_0}, t) \quad (6)$$

where q_n is the nth independent spatial dimension of the problem. This increase in generality would greatly impact the number of scenarios that can be investigated. Similarly, another extension can be made by allowing users to give initial conditions either in terms of a functional form or an array of n-dimensional points with statistical fit models to fill the gaps, all at some specific, and possibly nonzero, time t_0.

Another possible development of this problem is to program solutions to the wave equation on regular polygons with $n \geq 3$ sides. This can then be used to show how greatly increasing n will lead to more circular membrane-like wave behavior. To make this extension even more general, inputs could be included to give specific internal angles and side lengths of desired membranes.

An even greater implementation to understand real-life wave behavior is to use the damped wave equation[8] model:

$$\ddot{u} + b\dot{u} = c^2\nabla^2 u \quad (7)$$

Not only would using this model allow us to more accurately model real-life wave behavior than what the undamped equation allows, but it will also show how damping can affect reflections and interference.

4. Conclusion

In summary, our project wishes to visually investigate wave behavior on different membranes. Starting with a simple 1D case and generalizing to another dimension seems the most straightforward for this task, and along the way we will develop computational methods that can be applied to a slew of different PDEs. There is plenty of room available for improvement of the project beyond what is planned to be implemented over the next month, and there are always new cases that can be considered for initial conditions, membrane shapes, et cetera. The applications of this material can also be extended beyond string and membrane motion into fields that heavily rely on wave behavior and can be generalized to as many spatial dimensions as desired.

5. References

[1]Nave C. R. "The Wave Equation" [Online]. 1999. http://hyperphysics.phy-astr.gsu.edu/hbase/Waves/waveq.html [Oct. 2020].

[2]"Laplace Operator" [Online]. 2020. https://en.wikipedia.org/wiki/Laplace_operator [Oct. 2020].

[3]Hattori H. *Partial Differential Equations: Methods, Applications, and Theories*, pp. 33. Singapore: World Scientific Publishing Co. Pte. Ltd., 2013.

[4]Haberman R. *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, 5e, pp. 137-140,272-282,295-307. Boston, MA, USA: Pearson Education, Inc., 2019.

[5]Asmar N. H. *Partial Differential Equations with Fourier Series and Boundary Value Problems*, 3e, pp. 515-544. Mineola, NY, USA: Dover Publications, Inc., 2016.

[6]Young H. D. et al. *University Physics*, 14e, pp. 485-488. USA: Pearson Education, Inc., 2016.

[7]Hillen T. et al. *Partial Differential Equations: Theory and Completely Solved Problems*, 2e, pp. 15. Victoria, BC, Canada: Friesens Corporation, 2019.

[8]Vajiac M., Tolosa J. "An Introduction to Partial Differential Equations in the Undergraduate Curriculum, Lecture 7: The Wave Equation" [Online]. https://www.math.hmc.edu/~ajb/PCMI/lecture7.pdf [Nov. 2020].