MAT 275 Laboratory 6
Forced Equations and Resonance

In this laboratory we take a deeper look at second-order nonhomogeneous equations. We will concentrate on equations with a periodic harmonic forcing term. This will lead to a study of the phenomenon known as resonance. The equation we consider has the form

$$\frac{d^2y}{dt^2} + c\frac{dy}{dt} + \omega_0^2 y = \cos \omega t. \tag{1}$$

This equation models the motion of a mass-spring system similar to the one described in Laboratory 5. The forcing term on the right-hand side of (1) models a vibration, with amplitude 1 and frequency ω (in radians per second $= \frac{1}{2\pi}$ rotation per second $= \frac{60}{2\pi}$ rotations per minute, or RPM) of the plate holding the mass-spring system. All physical constants are assumed to be positive.

Let $\omega_1 = \sqrt{\omega_0^2 - c^2/4}$. When $c < 2\omega_0$ the general solution of (1) is

$$y(t) = e^{-\frac{1}{2}ct}(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)) + C \cos(\omega t - \alpha) \tag{2}$$

with

$$C = \frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + c^2\omega^2}}, \tag{3}$$

$$\alpha = \begin{cases} \arctan\left(\frac{c\omega}{\omega_0^2 - \omega^2}\right) & \text{if} \quad \omega_0 > \omega \\ \pi + \arctan\left(\frac{c\omega}{\omega_0^2 - \omega^2}\right) & \text{if} \quad \omega_0 < \omega \end{cases} \tag{4}$$

and c_1 and c_2 determined by the initial conditions. The first term in (2) represents the complementary solution, that is, the general solution to the homogeneous equation (independent of ω), while the second term represents a particular solution of the full ODE.

Note that when $c > 0$ the first term vanishes for large t due to the decreasing exponential factor. The solution then settles into a (forced) oscillation with amplitude C given by (3). The objectives of this laboratory are then to understand

1. the effect of the forcing term on the behavior of the solution for different values of ω, in particular on the amplitude of the solution.

2. the phenomena of resonance and beats in the absence of friction.

The Amplitude of Forced Oscillations

We assume here that $\omega_0 = 2$ and $c = 1$ are fixed. Initial conditions are set to 0. For each value of ω, the amplitude C can be obtained numerically by taking half the difference between the highs and the lows of the solution computed with a MATLAB ODE solver after a sufficiently large time, as follows: (note that in the M-file below we set $\omega = 1.5$).

<div style="border:1px solid">LAB06ex1.m</div>

```matlab
1  clear all; %this deletes all variables
2  omega0 = 2; c = 1; omega = 1.5;
3  param = [omega0,c,omega];
4  t0 = 0; y0 = 0; v0 = 0; Y0 = [y0;v0]; tf = 60;
5  options = odeset('AbsTol',1e-10,'relTol',1e-10);
6  [t,Y] = ode45(@f,[t0,tf],Y0,options,param);
7  y = Y(:,1); v = Y(:,2);
8  figure(1)
9  plot(t,y,'b-'); ylabel('y'); grid on;
10 t1 = 25; i = find(t>t1);
11 C = (max(Y(i,1))-min(Y(i,1)))/2;
12 disp(['computed amplitude of forced oscillation = ', num2str(C)]);
13 Ctheory = 1/sqrt((omega0^2-omega^2)^2+(c*omega)^2);
14 disp(['theoretical amplitude = ', num2str(Ctheory)]);
15 %------------------------------------------------------------
16 function dYdt = f(t,Y,param)
17 y = Y(1); v = Y(2);
18 omega0 = param(1); c = param(2); omega = param(3);
19 dYdt = [ v ; cos(omega*t)-omega0^2*y-c*v ];
20 end
```

When executing this program we get the plot in Figure 1. and the following output in the command window:

```
computed amplitude of forced oscillation = 0.43386
theoretical amplitude = 0.43386
```
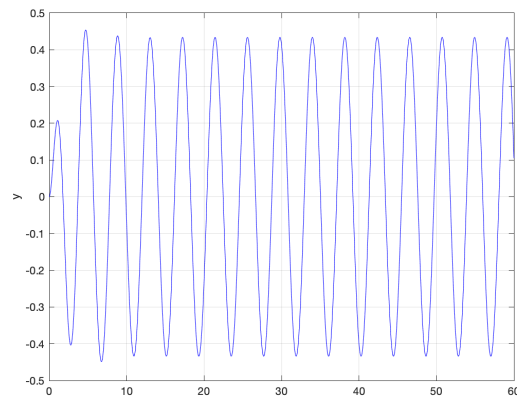

Figure 1: Forced oscillation.

Lines 10-14 deserve some explanation. Line 10 defines a time `t1` after which we think the contribution of the first term in (2) has become negligible compared to the second term. This depends of course on the parameter values, in particular c. With $c = 1$ we obtain $e^{-\frac{1}{2}ct} \approx 3.7 \times 10^{-6}$ for $t = 25$, so this is certainly small enough compared to the amplitude seen on Figure 1. The index `i` of time values larger than `t1` is then determined. The quantity `Y(i,1)` refers to the values of y associated to times larger than `t1` only. The computed amplitude is

simply half the difference between the max and the min values. This value is compared to the theoretical value (3).

1. (a) What is the period of the forced oscillation? What is the numerical value (modulo 2π) of the angle α defined by (4)?

   (b) In this question you are asked to modify the file `LAB06ex1.m` in order to plot the complementary solution of (1), that is, the first term in (2). First define in the file the angle α (alpha) using (4), then evaluate the complementary solution `yc` by subtracting the quantity $C\cos(\omega t - \alpha)$ from the numerical solution `y`. Plot the resulting quantity. Does it look like an exponentially decreasing oscillation? Why or why not? Include the modified M-file and the corresponding plot.

2. We now consider C as a function of ω. We use again $\omega_0 = 2, c = 1$ and $y(0) = y'(0) = 0$. The previous problem determined C for a specific value of ω. Here we consider a range of values for ω and determine numerically the corresponding amplitude C. We then plot the result as a function of ω, together with the theoretical amplitude from (3). The plot is shown in Figure 2. You may need the following MATLAB program.

   LAB06ex2

```
clear all; %this deletes all variables
omega0 = 2; c = 1;
OMEGA =0.5:0.01:3;
C = zeros(size(OMEGA));
Ctheory = zeros(size(OMEGA));
t0 = 0; y0 = 0; v0 = 0; Y0 = [y0;v0]; tf =60; t1 = 25;
for k = 1:length(OMEGA)
    omega = OMEGA(k);
    param = [omega0,c,omega];
    [t,Y] = ode45(@f,[t0,tf],Y0,[],param);
    i = find(t>t1);
    C(k) = (max(Y(i,1))-min(Y(i,1)))/2;
    Ctheory(k) = ??; % FILL-IN the formula for Ctheory
end
figure(2)
plot(??); grid on; % FILL-IN to plot C and Ctheory as a function of OMEGA
xlabel('\omega'); ylabel('C');
legend('computed numerically','theoretical')
%----------------------------------------------------------
function dYdt = f(t,Y,param)
y = Y(1); v = Y(2);
omega0 = param(1); c = param(2); omega = param(3);
dYdt = [ v ; cos(omega*t)-omega0^2*y-c*v ];
end
```

   (a) Fill in the missing parts in the M-file `LAB06ex2.m` and execute it. You should get a figure like Figure 2. Include the modified M-file in your lab report.

   (b) Examine the graph obtained by running `LAB06ex2.m` and determine for what (approximate) value of ω the amplitude of the forced oscillation, C, is maximal. This value of ω is called the *practical resonance frequency*. Give the corresponding maximum value of C.

   (c) Determine analytically the value of ω for which the amplitude of the forced oscillation, C, is maximal by differentiating the expression for C in (3) as a function of ω. Compare this value of ω with the value obtained in part (b).
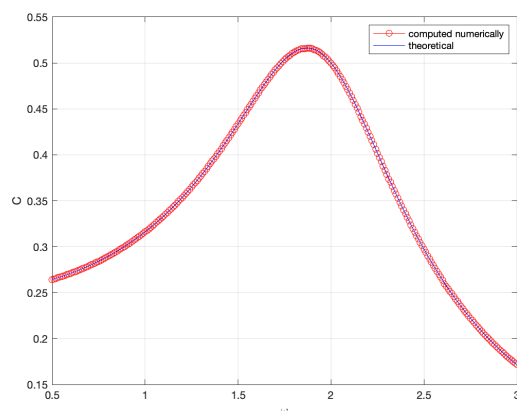
Figure 2: Amplitude as a function of ω

(d) Run `LAB06ex1.m` with the value of ω found in part (c) (include the graph). What is the amplitude of the forced oscillation? How does it compare with the amplitude of the forced oscillation in problem 1? If you run `LAB06ex1.m` with any other value of ω, how do you expect the amplitude of the solution to be?

(e) Are the results affected by changes in the initial conditions? Answer this question both numerically (by modifying the initial conditions in `LAB06ex2.m`) and theoretically (by analyzing the expression for C in (3)). Note that the initial conditions for the DE are y_0 and v_0.

Resonance

We now investigate what happens to the solution (2), and more specifically to the maximal amplitude C of the forced oscillation, when we let $c \to 0$. The value of ω corresponding to this maximal amplitude is called *pure resonance frequency*. When a mechanical system is stimulated by an external force operating at this frequency the system is said to be *resonant*.

3. Set $c = 0$ in `LAB06ex2.m`.
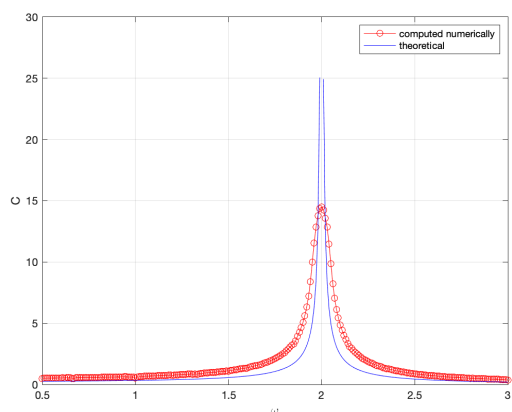
(a) Explain what happens. What is the maximal amplitude? What is the value of ω yielding the maximal amplitude in the forced solution? How does this value compare to ω_0?

(b) Run `LAB06ex1.m` with $c = 0$ and ω equal to the value found in part (a). Comment on the behavior of the solution. Include the graph.

Beats

When $c = 0$ and $\omega \neq \omega_0$, the solution (2) to (1) reduces to

$$y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + C \cos(\omega t - \alpha)$$

with $C = \frac{1}{\left|\omega_0^2 - \omega^2\right|}$. If the initial conditions are set to zero, the solution reduces to

$$y(t) = C(\cos(\omega t) - \cos(\omega_0 t))$$

which can be rewritten as

$$y(t) = 2C \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right) \sin\left(\frac{1}{2}(\omega_0 + \omega)t\right).$$

When ω is close to ω_0 we have that $\omega + \omega_0$ is large in comparison to $|\omega_0 - \omega|$. Then $\sin\left(\frac{1}{2}(\omega_0 + \omega)t\right)$ is a very rapidly varying function, whereas $\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)$ is a slowly varying function. If we define $A(t) = 2C \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)$, then the solution can be written as

$$y(t) = A(t) \sin\left(\frac{1}{2}(\omega_0 + \omega)t\right)$$

and we may interpret it as a rapidly oscillating function with period $T = \frac{4\pi}{\omega_0 + \omega}$, but with a slowly varying amplitude $A(t)$. This is the phenomenon known as *beats*. Note that $A(t)$ and $-A(t)$ are the so-called "envelope functions". The period of $A(t)$ is $\frac{4\pi}{|\omega_0 - \omega|}$, thus the length of the beats is $\frac{2\pi}{|\omega_0 - \omega|}$.

4. To see the beats phenomenon, set $c = 0$ and $\omega = 1.8$ in `LAB06ex1`. Also extend the interval of simulation to 80.
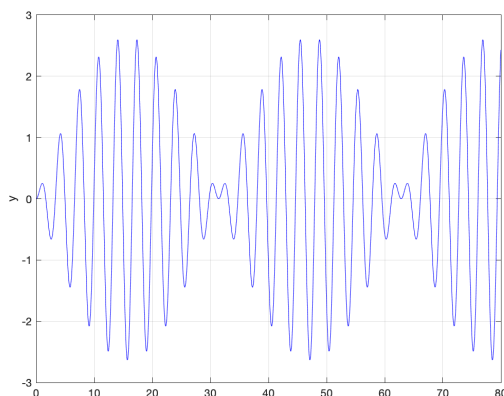
(a) In `LAB06ex1` define the "envelope" function $A = 2C \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)$ with $C = \frac{1}{\left|\omega_0^2 - \omega^2\right|}$.
Plot A in red and $-A$ in green, together with the solution. You should obtain Figure 3. Include the modified M-file.

(b) What is the period of the fast oscillation (that is, the period of $\sin\left(\frac{1}{2}(\omega_0 + \omega)t\right)$)? Confirm your answer by zooming in on the graph of the solution.

(c) What is the length of the beats? Determine the length analytically using the envelope functions, and numerically from the graph.
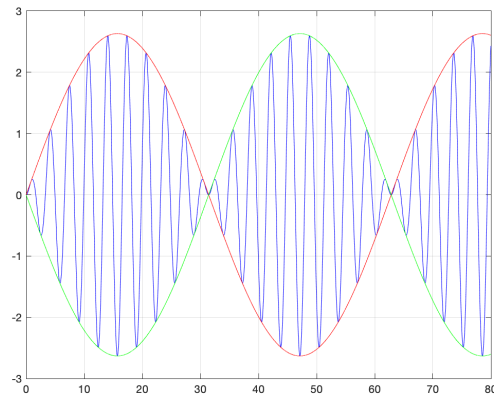
Figure 3: Solution and envelope functions

(d) Change the value of ω in `LAB06ex1` to 1.9 (a value closer to ω_0) and then $\omega = 1.5$ (a value farther away from ω_0). Include the two graphs. For each of these two values of ω find the period of the fast oscillation and the length of the beats. How do the periods change compared to parts (b) and (c)?

(e) If you let $\omega = 1$, is the beats phenomenon still present? Why or why not?