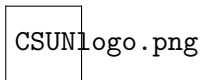


A Problem from Erdős About Products of 2 or 3 Primes

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Motivation

Take two prime numbers, say 2 and 3, and make a list of all the natural numbers which can be formed using only 2 and 3 as factors.

$$\left\{ \begin{array}{ccccccc} 2 & 3 & 4 & 6 & 8 & 9 & 12 & \dots \\ 2^1 & 3^1 & 2^2 & 2^1 3^1 & 2^3 & 3^2 & 2^2 3^1 & \dots \end{array} \right.$$

There is an interesting pattern here: there keep occurring pairs of numbers which have only 2 or only 3 as a factor.

$$\begin{array}{ccccccc} \dots & 16 & 18 & 24 & 27 & 32 & 36 & \dots \\ \dots & 2^4 & 2^1 3^2 & 2^3 3^1 & 3^3 & 2^5 & 2^2 3^2 & \dots \end{array}$$

Motivation

Does this keep happening?

$$\begin{array}{ccccccc}
 \dots & 1536 & 1728 & 1944 & 2048 & 2187 & 2304 & \dots \\
 \dots & 2^9 3^1 & 2^6 3^3 & 2^3 3^5 & 2^{11} & 3^7 & 2^8 3^2 & \dots
 \end{array}$$

Does it still happen with any primes?

$$\begin{array}{ccccccc}
 59 & 61 & 3481 & 3599 & 3721 & 205379 & 212341 & \dots \\
 59^1 & 61^1 & 59^2 & 59^1 61^1 & 61^2 & 59^3 & 59^2 61^1 & \dots
 \end{array}$$

$$\dots \quad 59^3 61^2 \quad 59^2 61^3 \quad 59^1 61^4 \quad 61^5 \quad 59^6 \quad 59^5 61^1 \quad \dots$$

Motivation

What if you do it with *three* primes (i.e. 2, 3, and 5)?

$$\begin{array}{ccccccc}
 \dots & 18 & 20 & 24 & 25 & 27 & 30 & \dots \\
 \dots & 2^1 3^2 & 2^2 5^1 & 2^3 3^1 & 5^2 & 3^3 & 2^1 3^1 5^1 & \dots
 \end{array}$$

Question!

Paul Erdős asked the following question about three distinct primes: If we construct a sequence of all the products of their powers, with the sequence arranged in increasing order, is it true infinitely often that consecutive terms in this sequence are both prime-powers?

Definitions

Let p, q be distinct primes, and let $m, n \in \mathbb{Z}^+$.

Definition

A *pure power* of p is an integer of the form p^m .

Definition

A *mixed power* of p and q is an integer of the form $p^m q^n$.

Definition

A *critical pair* of p and q is a pair of pure powers of p and q which do not have a mixed power between them.

Developing Intuition

Lemma 1

If $a_k = q^n$, then $a_{k+1} \neq q^{n+1}$.

Ex.

... 2^4 2^13^2 2^33^1 3^3 ...

Here, 3^4 can't come next, because $3^3 < 3^32^1 < 3^4$.

Early Stages

Lemma 2

There exist at most finitely many $a_k = p^m$ such that $a_{k+1} = p^{m+1}$.

Even in a dramatic example like $p = 2$, $q = 509$, the sequence starts out all powers of 2,

$$2^1 \quad 2^2 \quad 2^3 \quad 2^4 \quad 2^5 \quad \dots$$

but this can't continue forever.

$$2^7 \quad 2^8 \quad 509^1 \quad 2^9 \quad 2^1 509^1 \quad \dots$$

Early Stages

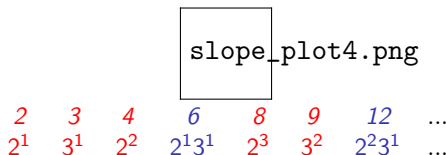
Lemma 3

If $a_k = p^m$ and $a_{k+1} = q^n$, then m and n are relatively prime.

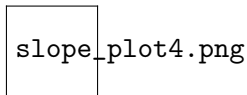
Sketch of Proof. Suppose for contradiction that $\gcd(m, n) = d$. Then $m = m'd$ and $n = n'd$ for some $m', n' \in \mathbb{N}$. Since $p^m = p^{m'd} < q^{n'd} = q^n$, we have $p^{m'} < q^{n'}$. Thus:

$$\begin{aligned} p^m &= p^{m'd} \\ &= p^{m'd-m'+m'} \\ &= p^{m'(d-1)} p^{m'} \\ &< \frac{p^{m'(d-1)} q^{n'}}{q^{n'(d-1)} q^{n'}} \\ &= q^{n'd} \\ &= q^n. \end{aligned}$$

The "Tail" of the Sequence of Exponents



The "Tail" of the Sequence of Exponents

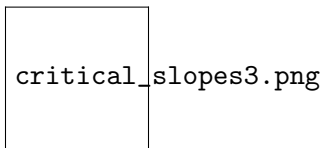


Notice how the lines seem to approach a constant slope!

As $k \rightarrow \infty$, we see that $\frac{n_k}{m_k} \rightarrow -\frac{\log p}{\log q}$.

Converging Ratios

As $k \rightarrow \infty$, we see that $\frac{n_k}{m_k} \rightarrow -\frac{\log p}{\log q}$.



Two Primes

After developing an understanding for the problem, we began to analyze the details that Erdős glossed over. This lead us to a proof for the following theorem:

Theorem 1

For any two distinct prime numbers p , and q , there exist infinitely many critical pairs.

Key Lemmas

Lemma 4

Consider the pure powers p^a, q^b with $p^a < q^b$ and $a, b \in \mathbb{Z}^+$. If, for all critical pairs p^s, q^t with $s < a$ and $t < b$,

$$1 < \frac{q^b}{p^a} < \frac{q^t}{p^s}, \quad s, t \in \mathbb{Z}^+$$

then p^a, q^b is a critical pair.

A sketch of the proof follows.

Assume that for all critical pairs p^s, q^t with $s < a$ and $t < b$,

$$1 < \frac{q^b}{p^a} < \frac{q^t}{p^s},$$

and suppose for contradiction that p^a, q^b is not a critical pair. Since p^a, q^b is not a critical pair, then there exists an intermediate mixed power of the form

$$p^a < q^{b-\tilde{b}} p^{\tilde{a}} < q^b$$

So, since $p^a < q^{b-\tilde{b}} p^{\tilde{a}}$,

$$\frac{q^b}{p^a} > \frac{q^b}{q^{b-\tilde{b}} p^{\tilde{a}}} = \frac{q^{\tilde{b}}}{p^{\tilde{a}}}.$$

If $p^{\tilde{a}}$ and $q^{\tilde{b}}$ are a critical pair, we have a contradiction. If not, repeat.


Key Lemmas

Dirichlet's Approximation Theorem

Let α be a real number. Given any $\epsilon > 0$, there exist $M, N \in \mathbb{Z}$ such that

$$\left| \alpha - \frac{M}{N} \right| < \epsilon.$$

The "Tail" of the Sequence of Exponents



slope_plot4.png

By the way, the fact that as $k \rightarrow \infty$, $\frac{n_k}{m_k} \rightarrow -\frac{\log p}{\log q}$.
follows from Dirichlet's Approximation Theorem and Lemma 4.

Key Lemmas

Dirichlet's Approximation Theorem

Let α be a real number. Given any $\epsilon > 0$, there exist $M, N \in \mathbb{Z}$ such that

$$\left| \alpha - \frac{M}{N} \right| < \epsilon.$$

Lemma 5

Let α be an irrational number. Given any $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $n\alpha - \lfloor n\alpha \rfloor < \epsilon$.

Proof of the Result

Let p and q be distinct primes. Suppose for contradiction that there exist finitely many critical pairs. Of these critical pairs, consider the critical pair with $p^k < q^\ell$ such that $\frac{q^\ell}{p^k}$ is smallest. This means that

$$1 < \frac{q^\ell}{p^k}.$$

Choose some $\epsilon \in \mathbb{R}$ such that

$$1 < p^\epsilon < \frac{q^\ell}{p^k}.$$

Now, consider the irrational number $\log_p q$. By Lemma 5, there exists some $\Omega \in \mathbb{N}$ such that

$$\Omega \log_p q - \lfloor \Omega \log_p q \rfloor < \epsilon.$$

Proof of the Result

To simplify the notation, let $a = \lfloor \Omega \log_p q \rfloor$. Thus, with a little algebra,

$$\Omega \log_p q - a < \epsilon$$

$$\Omega \log_p q < a + \epsilon$$

$$q^\Omega < p^a p^\epsilon$$

$$\frac{q^\Omega}{p^a} < p^\epsilon < \frac{q^\ell}{p^k}$$

Hence,

$$1 < \frac{q^\Omega}{p^a} < \frac{q^\ell}{p^k}.$$

Therefore, by Lemma 4, (q^Ω, p^a) is a critical pair with ratio closer to 1, which is a contradiction.

Redeveloped Proof

Problem

Everything depends on

$$1 < \frac{q^{\Omega}}{p^a} < \frac{q^{\ell}}{p^k}.$$

Issues with 3 Primes

We have attempted the problem with various techniques: Mean Value Theorem/Taylor Polynomial bounds, Geometric Arguments, Linear Programming, Special cases (59,61,3601, $(p,q,r = pq + 2)$), etc.

Moving Forward

Currently, we are continuing to look into our geometric argument in three dimensions.

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