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§8.12: The Second Derivative

Today: We can take the derivative of a function repeatedly!

Example: If $f(x) = x^3 - 3x + 2$, then

- $\dfrac{df}{dx} = f'(x) = 3x^2 - 3$
- The second derivative of $f(x)$ is $\dfrac{d}{dx}\left(\dfrac{df}{dx}\right) = f''(x) = 6x$.

  This is written $f''(x)$ or $\dfrac{d^2 f}{dx^2}$.
- The third derivative of $f(x)$ is $\dfrac{d}{dx}\left(\dfrac{d^2 f}{dx^2}\right) = f'''(x) = 6$.

  This is written $f'''(x)$ or $\dfrac{d^3 f}{dx^3}$.
- Keep Going! The fourth derivative is $\dfrac{d^4 f}{dx^4} = f''''(x) = 0$.
- The fun ends here, for this $f(x)$ all higher derivatives are zero.

Examples

General idea: Differentiating the function n times gives us the nth derivative of f. It is written as

$$f''''\cdots''''(x) = f^{(n)}(x) = \frac{d^n f}{dx^n}.$$

1. What is the second derivative of $3x^2 - 5x + 7$?

(A) 0 (B) 7 (C) 6 (D) 3 (E) -5 C

2. $\dfrac{d^2}{dx^2}\left(x^5\right) = ?$

(A) 20 (B) $5x^4$ (C) 0 (D) $20x^4$ (E) $20x^3$ E

3. $\dfrac{d^2}{dx^2}\left(\sqrt{x}\right) = ?$

(A) $\frac{1}{4}x^{-3/2}$ (B) $\frac{-1}{4}x^{-1/2}$ (C) $\frac{-1}{4}x^{-3/2}$ (D) $\frac{1}{2}x^{-1/2}$ (E) 0 C

More Examples

4. $\dfrac{d^2}{dt^2}\left(e^{3t}\right) =?$

(A) e^{3t} (B) $3e^{2t}$ (C) $9e^{3t}$ (D) $3e^{3t}$ (E) $9e^{t}$ $\boxed{\text{C}}$

5. Find $f'''(x)$ when $f(x) = x^3$.

(A) $6x^2$ (B) 0 (C) $3x$ (D) $3x^2$ (E) 6 $\boxed{\text{E}}$

6. If $f(x) = x^3 - 4x^2 + 7x - 31$, then $f''(10) =?$

(A) 6 (B) $3x^2 - 8x$ (C) $6x$ (D) 60 (E) 52 $\boxed{\text{E}}$

Example: Acceleration

The acceleration due to gravity is

$$32 \text{ feet per second per second} = 32 \text{ ft/sec}^2.$$

This means:

every second you fall,
your speed increases by 32 ft/sec $\approx$ 22 mph.

acceleration = rate of change of velocity = derivative of velocity.
velocity = rate of change of distance = derivative of distance.

Therefore

acceleration = second derivative of distance

Example: Height of ball is $h(t) = 20t - 5t^2$ meters after t seconds.
(a) Velocity of ball after t seconds is $h'(t) = 20 - 10t$ m/sec
(b) Acceleration of ball after t seconds is $h''(t) = -10$ m/sec^2

It's not the speed that kills

Suppose you hit a brick wall at 60 mph.
Question: What is your (sudden!) acceleration?

$$\begin{pmatrix} \text{Average rate of} \\ \text{change of velocity} \\ \text{in stopping} \end{pmatrix} = \frac{\Delta \text{ velocity}}{\Delta \text{ time}} = \frac{-60 \text{ mph}}{1/10 \text{ sec}}$$

$$\approx \frac{-88 \text{ ft/sec}}{1/10 \text{ sec}} = -880 \text{ ft/sec}^2.$$

Since 1 gravity $= 32$ ft/sec^2, this is about

$$880 \text{ ft/sec}^2 = \left(880 \text{ ft/sec}^2\right) \times \frac{1 \text{ gravity}}{32 \text{ ft/sec}^2} \approx 28 \text{ ``g''}.$$

The force which pushes you at the windshield is about 28 times your weight.
If you weigh 110 pounds, this force is about 3000 pounds $=$ 1.5 tons.

Rocket!

A rocket is fired vertically upwards. The height after t seconds is $2t^3 + 5t^2$ meters.

Question: What is the acceleration in $\mathrm{m/sec}^2$ after t seconds?

(A) $2t^3 + 5t^2$ (B) $6t^2 + 10t$ (C) $12t + 10$ (D) 12 (E) 0 $\boxed{\text{C}}$

Idea:

- $h(t)$ = height in meters at time t seconds
- $h'(t)$ = velocity in m/sec at time t seconds
- $h''(t)$ = acceleration in $\mathrm{m/sec}^2$ at time t seconds

More Questions:

(a) What can we say about $f(t)$ if $f'(t) = 0$ for all t?

(b) What can we say about $f(t)$ if $f''(t) = 0$ for all t?

Application 2: Concavity

$$\frac{df}{dx} = \text{rate of change of } f(x)$$

$$\text{and so } \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \text{rate of change of } \frac{df}{dx}$$

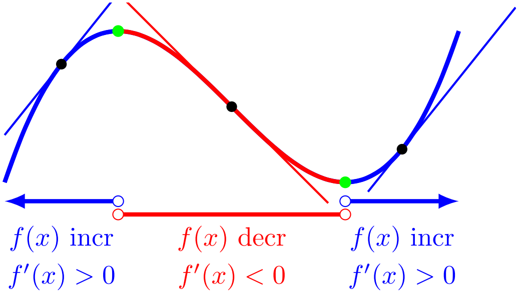
Conclusion:

> The second derivative tells you how quickly the rate of change is changing.

Uses of second derivative:

- We've seen: acceleration is the rate of change of velocity
  So: acceleration is the second derivative of distance traveled.
- Is the graph concave up or concave down?
- Are things changing for better or worse?

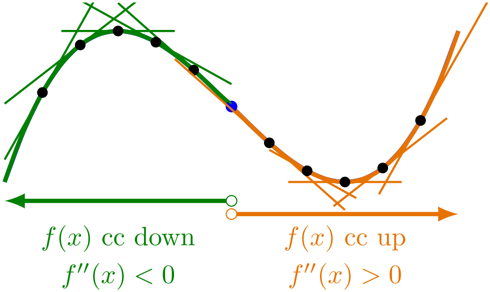
Meanings: The First Derivative



Point:

$$f'(x) > 0 \iff f(x) \text{ is increasing}$$
$$f'(x) < 0 \iff f(x) \text{ is decreasing}$$

Meanings: The Second Derivative



Point:

$$
\begin{aligned}
f''(x) > 0 &\iff f'(x) \text{ is increasing} \\
&\iff f(x) \text{ is concave up} \\
f''(x) < 0 &\iff f'(x) \text{ is decreasing} \\
&\iff f(x) \text{ is concave down}
\end{aligned}
$$

Concavity

$$f''(x) > 0 \iff f(x) \text{ is concave up}$$
$$f''(x) < 0 \iff f(x) \text{ is concave down}$$

7. For which values of x is $f(x) = x^3 - 6x^2 + 3x + 2$ concave up?

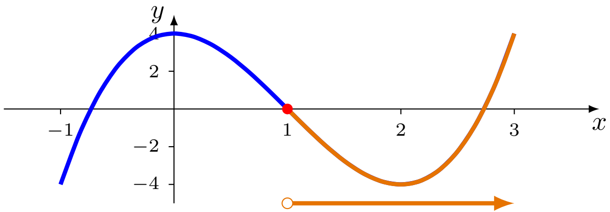
(A) when $x = 0$ (B) when $x < 6$ (C) when $x > 6$

(D) when $x < 2$ (E) when $x > 2$ $\boxed{\text{E}}$

Concavity

$$f''(x) > 0 \iff f(x) \text{ is concave up}$$
$$f''(x) < 0 \iff f(x) \text{ is concave down}$$

8. Where is $f''(x) > 0$?



(A) when $x < 2$ (B) when $x > 2$ (C) when $x < 1$

(D) when $x > 1$ (E) when $-0.7 < x < 1$ $\boxed{\text{D}}$

Review Problems

9. An oil slick in the shape of a rectangle is expanding. After t hours the length is $30t$ meters and the width is $50t$ meters. How quickly is the area increasing in m^2/hour after 2 hours?

(A) 800 (B) 1500 (C) 3200 (D) 6000 (E) Other $\boxed{\text{D}}$

10. Suppose $f'(1) = 4$ and $g'(1) = 3$. What is the rate of change of $f(x) + 2g(x)$ when $x = 1$?

(A) 3 (B) 4 (C) 7 (D) 10 (E) 14 $\boxed{\text{D}}$

More Review Problems

11. What is the x-coordinate of the point on the graph $y = 2x^2 + 5x - 7$ where the slope is 11?

(A) 1 (B) 3/2 (C) 2 (D) 5/3 (E) 0 $\boxed{\text{B}}$

12. What is the value of x at the point on the graph $y = 4x^2 + 16x$ where the tangent line is horizontal?

(A) 2 (B) 0 (C) -2 (D) -4 $\boxed{\text{C}}$

13. $\dfrac{d}{dx}\left(\dfrac{3}{x^4}\right) = ?$

(A) $\dfrac{3}{4x^3}$ (B) $\dfrac{12}{x^5}$ (C) $-\dfrac{3}{4x^3}$ (D) $-\dfrac{12}{x^5}$ $\boxed{\text{D}}$