Welcome Back! Differential Calculus

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Suppose x and y are related variables. So as one changes, the other changes. We can ask:

How much does y change per unit change in x?

Answer: The derivative of y with respect to x tells us, and it depends on the current value of x!

If we write y as a function of x like this: $y = f(x)$, then the derivative is written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{or} \quad \frac{\mathrm{d}f}{\mathrm{d}x} \quad \text{or} \quad f'(x)$$

It is the limit of "average rate of change" over shorter and shorter Δx:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

also known as "instantaneous rate of change"

A nice thing about derivatives...

$$\frac{d}{dx}(a \cdot f(x) + b \cdot g(x)) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$$
$$= a \cdot f'(x) + b \cdot g'(x)$$

For example...

$$\begin{aligned} \tfrac{d}{dx}(3x^2 + 5x) &= 3\tfrac{d}{dx}x^2 + 5\tfrac{d}{dx}x \\ &= 3(2x) + 5(1) \\ &= 6x + 5 \end{aligned}$$

A Warning!

$$\frac{d}{dx}\left(f(x)g(x)\right) \neq f'(x) \times g'(x)$$

Example: $5x^4 = \frac{d}{dx}\left(x^5\right) = \frac{d}{dx}\left(x^2 \cdot x^3\right) \neq (2x)(3x^2) = 6x^3$

Example: Find the derivative of $(x+1)(2x+3)$

Question: $\frac{d}{dx}\left((x^2+1)(x^3+1)\right) = ?$

A$= 6x^3$ B$= 5x^4 + 3x^2 + 2x$ C$= x^5 + x^3 + x^2 + 1$ D$=$ Other

Answer: $\boxed{\text{B}}$

Review Examples:

(1) What is the x-coordinate of the point on the graph of $y = 4x^2 - 3x + 7$ where the graph has slope 13?

A$= 0$ B$= 1$ C$= 2$ D$= 3$ E$= 4$ $\boxed{\text{C}}$

(2) A circle is expanding so that after R seconds it has radius R cm. What is the rate of increase of area inside the circle after 2 seconds?

A$= 4\pi$ B$= 2\pi R^2$ C$= 2$ D$= 2\pi R$ E$= \pi R^2$ $\boxed{\text{A}}$

Differentiating $f(x) = e^{kx}$

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

versus

$$\frac{d}{dx}\left(x^{n}\right) = nx^{n-1}$$

Do not get confused and write $\frac{d}{dx}\left(e^{kx}\right) = ke^{(k-1)x}$.

Question: Find $\frac{d}{dx}\left(4e^{3x} + 5x^3\right)$

A= $12e^{2x} + 15x^2$ B= $12e^{3x} + 15x^3$ C= $4e^{3x} + 15x^2$
D= $12e^{3x} + 15x^2$ E= Other $\boxed{\text{D}}$

Example

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

The temperature (in $^\circ$ C) of a cup of coffee t hours after it is made is $f(t) = 50 + 40e^{-2t}$.

(a) What is the initial temperature when the coffee is made?

A$=40$ B$=50$ C$=90$ D$=100$ $\boxed{\text{C}}$

(b) How quickly is the coffee cooling down initially? This means how many degrees per hour is the temperature going down instantaneously at $t = 0$?

A$=20$ B$=40$ C$=60$ D$=80$ E$=100$ $\boxed{\text{D}}$

More Examples

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

(1) $\frac{d}{dx}\left(\frac{3}{e^{2x}}\right) = ?$

A $= \frac{3}{2e^{2x}}$ B $= \frac{3}{2e^{x}}$ C $= \frac{6}{e^{2x}}$ D $= \frac{-6}{e^{2x}}$ $\boxed{\text{D}}$

(2) The number of grams of Einsteinium-253 after t days is $m(t) = 10e^{-t/30}$. How quickly is the mass changing (in grams per day) when $t = 0$?

A $= -1/30$ B $= -1/3$ C $= -10e^{-t/30}$ D $= -\frac{1}{3}\,e^{t/30}$ $\boxed{\text{B}}$

§8.12: The Second Derivative

Today: We can take the derivative of a function repeatedly!

Example: If $f(x) = x^3 - 3x + 2$, then

- $\dfrac{df}{dx} = f'(x) = 3x^2 - 3$
- The second derivative of $f(x)$ is $\dfrac{d}{dx}\left(\dfrac{df}{dx}\right) = f''(x) = 6x$.

  This is written $f''(x)$ or $\dfrac{d^2 f}{dx^2}$.
- The third derivative of $f(x)$ is $\dfrac{d}{dx}\left(\dfrac{d^2 f}{dx^2}\right) = f'''(x) = 6$.

  This is written $f'''(x)$ or $\dfrac{d^3 f}{dx^3}$.
- Keep Going! The fourth derivative is $\dfrac{d^4 f}{dx^4} = f''''(x) = 0$.
- The fun ends here, for this $f(x)$ all higher derivatives are zero.

Examples

General idea: Differentiating the function n times gives us the nth derivative of f. It is written as

$$f''''\cdots''''(x) = f^{(n)}(x) = \frac{d^n f}{dx^n}.$$

(1) What is the second derivative of $3x^2 - 5x + 7$?

A$=0$ B$=7$ C$=6$ D$=3$ E$=-5$ $\boxed{\text{C}}$

(2) $\dfrac{d^2}{dx^2}\left(x^5\right) = ?$

A$=20$ B$=5x^4$ C$=0$ D$=20x^4$ E$=20x^3$ $\boxed{\text{E}}$

(3) $\dfrac{d^2}{dx^2}\left(\sqrt{x}\right) = ?$

A$=\frac{1}{4}x^{-3/2}$ B$=\frac{-1}{4}x^{-1/2}$ C$=\frac{-1}{4}x^{-3/2}$ D$=\frac{1}{2}x^{-1/2}$ E$=0$ $\boxed{\text{C}}$

More Examples

(4) $\dfrac{d^2}{dt^2}\left(e^{3t}\right) = ?$

A$= e^{3t}$ B$= 3e^{2t}$ C$= 9e^{3t}$ D$= 3e^{3t}$ E$= 9e^{t}$ $\boxed{\text{C}}$

(5) Find $f'''(x)$ when $f(x) = x^3$.

A$= 6x^2$ B$= 0$ C$= 3x$ D$= 3x^2$ E$= 6$ $\boxed{\text{E}}$

(6) If $f(x) = x^3 - 4x^2 + 7x - 31$, then $f''(10) = ?$

A$= 6$ B$= 3x^2 - 8x$ C$= 6x$ D$= 60$ E$= 52$ $\boxed{\text{E}}$

Example: Acceleration

The acceleration due to gravity is

$$32 \text{ feet per second per second} = 32 \text{ ft/sec}^2.$$

This means:

every second you fall,
your speed increases by 32 ft/sec $\approx$ 22 mph.

$$\text{acceleration} = \text{rate of change of velocity} = \text{derivative of velocity}.$$
$$\text{velocity} = \text{rate of change of distance} = \text{derivative of distance}.$$

Therefore

$$\boxed{\text{acceleration} = \text{second derivative of distance}}$$

Example: Height of ball is $h(t) = 20t - 5t^2$ meters after t seconds.
(a) Velocity of ball after t seconds is $h'(t) = 20 - 10t$ m/sec
(b) Acceleration of ball after t seconds is $h''(t) = -10$ m/sec^2

It's not the speed that kills

Suppose you hit a brick wall at 60 mph.
Question: What is your (sudden!) acceleration?

$$\begin{pmatrix} \text{Average rate of} \\ \text{change of velocity} \\ \text{in stopping} \end{pmatrix} = \frac{\Delta \text{ velocity}}{\Delta \text{ time}} = \frac{-60 \text{ mph}}{1/10 \text{ sec}}$$

$$\approx \frac{-88 \text{ ft/sec}}{1/10 \text{ sec}} = -880 \text{ ft/sec}^2.$$

Since 1 gravity $= 32$ ft/sec^2, this is about

$$880 \text{ ft/sec}^2 = \left(880 \text{ ft/sec}^2\right) \times \frac{1 \text{ gravity}}{32 \text{ ft/sec}^2} \approx 28 \text{ ``g''}.$$

The force at which the brick wall pushes you is 28 times your weight.
If you weigh 110 pounds, this force is about 3000 pounds = 1.5 tons.

A Rocket

A rocket is fired vertically upwards. The height after t seconds is $2t^3 + 5t^2$ meters.

Question: What is the acceleration in m/sec^2 after t seconds?

A$= 2t^3 + 5t^2$ B$= 6t^2 + 10t$ C$= 12t + 10$ D$= 12$ E$= 0$ $\boxed{\text{C}}$

Idea:

- $h(t) =$ height in meters at time t seconds
- $h'(t) =$ velocity in m/sec at time t seconds
- $h''(t) =$ acceleration in m/sec^2 at time t seconds

More Questions:

(a) What can we say about $f(t)$ if $f'(t) = 0$ for all t?

(b) What can we say about $f(t)$ if $f''(t) = 0$ for all t?

Application 2: Concavity

$$\frac{df}{dx} = \text{rate of change of } f(x)$$

$$\text{and so } \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \text{rate of change of } \frac{df}{dx}$$

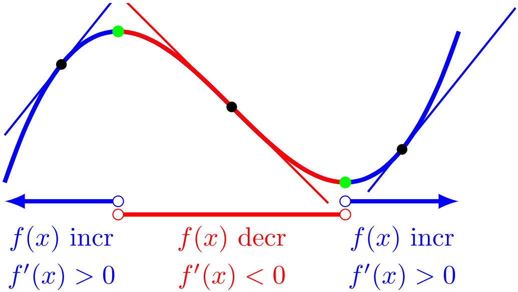
Conclusion:

> The second derivative tells you how quickly the rate of change is changing.

Uses of second derivative:

- We've seen: acceleration is the rate of change of velocity
  So: acceleration is the second derivative of distance traveled.
- Is the graph concave up or concave down?
- Are things changing for better or worse?

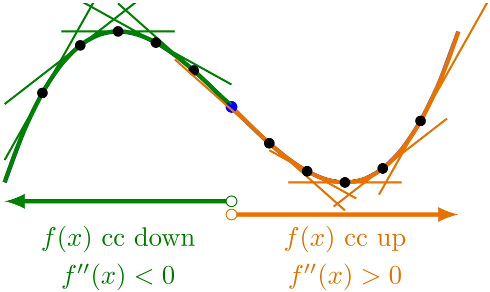
Meanings: The First Derivative



Point:

$$f'(x) > 0 \iff f(x) \text{ is increasing}$$
$$f'(x) < 0 \iff f(x) \text{ is decreasing}$$

Meanings: The Second Derivative



Point:

$$\begin{aligned} f''(x) > 0 &\iff f'(x) \text{ is increasing} \\ &\iff f(x) \text{ is concave up} \\[4pt] f''(x) < 0 &\iff f'(x) \text{ is decreasing} \\ &\iff f(x) \text{ is concave down} \end{aligned}$$

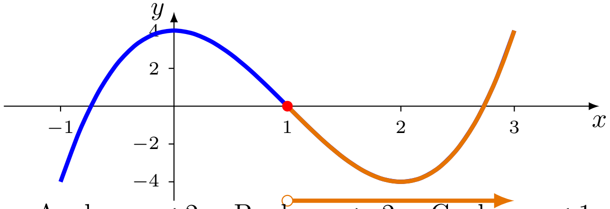
Concavity

$$f''(x) > 0 \iff f(x) \text{ is concave up}$$
$$f''(x) < 0 \iff f(x) \text{ is concave down}$$

(1) For which values of x is $f(x) = x^3 - 6x^2 + 3x + 2$ concave up?

A when $x = 0$ B when $x < 6$ C when $x > 6$
D when $x < 2$ E when $x > 2$ $\boxed{\text{E}}$

(2) Where is $f''(x) > 0$?



A when $x < 2$ B when $x > 2$ C when $x < 1$
D when $x > 1$ E when $-0.7 < x < 1$ $\boxed{\text{D}}$