$U_1 \times U_2 \times \mathbb{R} \times \mathbb{R} \times \cdots$

3. Let $\mathbb{R}^\omega$ be the product of a countable # of copies of $\mathbb{R}$ which can be thought of as the set of sequences of real numbers. Let A be the set of sequences that have only finitely many non-zero terms. What is the closure of A if $\mathbb{R}^\omega$ has the product topology? What if $\mathbb{R}^\omega$ has the box topology? Prove your answers. (Recall, a basis for the box topology is the collection of sets of the form $\prod_n U_n$ where $U_1, U_2, \ldots$ are open subsets of $\mathbb{R}$).

The closure of A in the product topology is the whole space. Let $\vec{x} \in \mathbb{R}^\omega$ & let $\mathcal{U}$ be an open set containing $\vec{x}$ in the product topology. Then $\exists$ a basis element B containing x with $B \subset \mathcal{U}$. In the product topology a basis element is a infinite product of open sets of $\mathbb{R}$ where only finitely many of them are not $\mathbb{R}$. Say $B = \prod \mathcal{U}_i$

$\mathcal{U}_i \subseteq \mathbb{R}$ open
$\mathcal{U}_i = \mathbb{R}$ except $i = \{i_1, \ldots, i_m\}$

For convenience, write $\vec{x} = (x_n)$. Then $x_{i_j} \in \mathcal{U}_{i_j}$. Then define $\vec{y} = (y_n)$ where
$$y_n = \begin{cases} x_n & \text{if } n = i_1, \ldots, i_m \\ x_n + 1 & \text{if } n = \max\{i_1, \ldots, i_m\} + 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $y_n \in A$ & $y_n \in \mathcal{U}_n$ $\forall n = \{i_1, \ldots, i_m\}$ & $y_n \in U_n = \mathbb{R}$ $\forall n \notin \{i_1, \ldots, i_m\}$, so $\vec{y} \in B$ & $\vec{y} \neq \vec{x}$, so $\mathcal{U} \cap A \setminus \{\vec{x}\} \neq \emptyset$ so $\overline{A} = \mathbb{R}^\omega$.

On the other hand, in the box topology $\overline{A} = A$. Let $\vec{x} \notin A$. Then $\vec{x}$ has infinitely many nonzero terms. Say $\vec{x} = (x_n)$. Then $\mathcal{U}_n = (x_n - \frac{|x_n|}{2}, x_n + \frac{|x_n|}{2})$ is an open set of x_n & $0 \notin \mathcal{U}_n$, so $\prod \mathcal{U}_n$ is an open set in the box topology that contains $\vec{x}$, but does not contain any $\vec{y} = (y_n)$ where $y_n = 0$ for any n. So $\prod U_n \cap A = \emptyset$ so $A = \overline{A}$. □