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Trevor Klar

Homework 1

X, Y always smooth manifolds, maps are smooth etc.

1. Show that if $f : X \to X'$ and $g : Y \to Y'$ are both smooth, then so is the map $f \times g : X \times Y \to X' \times Y'$ defined by

$$(f \times g)(x, y) = (f(x), g(y)).$$

Proof. Since f, g are both smooth, then partials of $f \times g$ are given by

$$\frac{\partial}{\partial x} f \times g = \begin{cases} \left(\frac{\partial f}{\partial x_i}, \vec{0}_{\mathbb{R}^b} \right) & \text{if } 1 \le i \le a \\ \left(\vec{0}_{\mathbb{R}^a}, \frac{\partial g}{\partial x_i} \right) & \text{if } a+1 \le i \le a+b \end{cases}$$

and the mixed partials are similar. Thus all partials of $f \times g$ exist, so it is smooth. ■

2. Prove that the projection map $X \times Y \to X$ given by $(x, y) \mapsto x$ is smooth.

Proof. X is trivially diffeomorphic to $X \times \{\vec{0}\}$,[†] so we can think of the projection π as $\iota_X \times 0$, so $(x, y) \mapsto (x, 0)$. We know that inclusions and the zero map are smooth, so π is smooth by problem 1. ■

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† It doesn't really matter if Y has a point called "zero", it can be any point.

3. Suppose that U is an open subset of the manifold X. Show that for all $p \in U$

$$T_p(U) = T_p(X)$$

Proof. There is almost nothing to check. Let ϕ be a chart of U at p. Since $U \subset X$ then ϕ is a chart of X at p, so

$$T_p(U) = d_0\phi(\mathbb{R}^n) = T_p(X)$$

and we're done. ∎

4. If $f : X \to Y$ is a diffeomorphism, then df_x is an isomorphism for all $x \in X$. Deduce that if $\mathbb{R}^a$ is diffeomorphic to $\mathbb{R}^b$, then $a = b$.

Proof. Let $\phi : U \subseteq \mathbb{R}^a \to X$, $\psi : V \subseteq \mathbb{R}^b \to Y$ be charts mapping $0 \overset{\phi}{\longmapsto} x$, and $0 \overset{\psi}{\longmapsto} f(x)$. Then $h := \psi^{-1} \circ f \circ \phi$ as usual as shown in the following diagrams:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
U \subseteq \mathbb{R}^a \arrow[u, "\phi"] \arrow[r, dashed, "h"] & V \subseteq \mathbb{R}^b \arrow[u, "\psi"']
\end{tikzcd}$$

$$\begin{tikzcd}
T_xX \arrow[r, dashed, "df_x"] & T_{f(x)}Y \\
\mathbb{R}^a \arrow[u, "d\phi_0"] \arrow[r, "dh_0"] & \mathbb{R}^b \arrow[u, "d\psi_0"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T_xX & T_{f(x)}Y \arrow[l, dashed, "d(f^{-1})_{f(x)}"'] \\
\mathbb{R}^a \arrow[u, "d\phi_0"] & \mathbb{R}^b \arrow[u, "d\psi_0"'] \arrow[l, "d(h^{-1})_0"']
\end{tikzcd}$$

And now we observe using the chain rule that

$$df_x = d\psi_0 \circ dh_0 \circ d\phi_x^{-1} \quad \text{and}$$

$$\begin{aligned} d(f^{-1})_y &= d(\phi \circ h^{-1} \circ \psi^{-1})_y \\ &= d\phi_0 \circ dh_0^{-1} \circ d\psi_y^{-1}, \end{aligned}$$

so

$$d f_x \circ d(f^{-1})_y = d\psi_0 \circ dh_0 \circ d\phi_x^{-1} \circ d\phi_0 \circ dh_0^{-1} \circ d\psi_y^{-1} \quad \text{and}$$

$$d(f^{-1})_y \circ d f_x = d\phi_0 \circ dh_0^{-1} \circ d\psi_y^{-1} \circ d\psi_0 \circ dh_0 \circ d\phi_x^{-1},$$

and since $d\psi_y^{-1}, d\psi_0$ and dh_0^{-1}, dh_0 and $d\phi_0, d\phi_x^{-1}$ are respectively inverses by the chain rule, then $(d f_x)^{-1} = d(f^{-1})_y$ $\forall x \in X$ with $f(x) = y$ $\square$

In particular if f is a diffeomorphism from $\mathbb{R}^a \to \mathbb{R}^b$ thought of as manifolds, then their identity maps can serve as charts, and so $h = f$, which means $d f_x$ is an isomorphism $\mathbb{R}^a \to \mathbb{R}^b$, so they have the same dimension. ∎

5. A *curve* in a manifold X is smooth map $c : (-1, 1) \to X$. Define the *velocity vector* of c at $t_0 \in (-1, 1)$ to be

$$dc_{t_0}(1) \in T_{c(t_0)}(X)$$

(The point being that one should think of 1 as the unit vector in the one-dimensional vector space $\mathbb{R}$.)

Show that $T_p(X)$ is the set of velocity vectors of curves through the point p.

Proof. Let Φ be a chart of X^n at the point p. Since Φ is a diffeomorphism, then $d\Phi_0$ is an isomorphism, which means for any $v \in T_p X$, there exists $\vec{x} \in \mathbb{R}^n$ such that $d\Phi_0(\vec{x}) = v$. Then the desired curve is

$$c(t) = \Phi(t\vec{x})$$

and we're done. ∎

6. Prove that if $f : X \longrightarrow Y$ is a submersion and U is an open subset of X, then $f(U)$ is an open subset of Y.

Proof. Let $y \in f(U)$, $f(x) = y$. By the Local Immersion Thm, there exist local coordinates such that $0 \overset{\phi}{\longmapsto} x$, $0 \overset{\psi}{\longmapsto} y$, and $\phi^{-1}(U) \longmapsto \psi^{-1}(f(U))$ by the canonical submersion. Thus since ϕ and ψ are diffeomorphisms and the canonical submersion is a projection, which is an open map. $\blacksquare$