Spring 2020 **Math 221C Final Exam** Trevor Klar

① **Def:** Let X^k, Y^k compact, without boundary, connected, and oriented. For $f: X \xrightarrow[\text{smooth}]{} Y$, suppose $y \in Y$ is regular for f. Note that for all $x \in f^{-1}\{y\}$, df_x is an isomorphism $T_x X \longrightarrow T_y Y$.†

ⓐ Define **the sign of a linear isomorphism** between oriented vector spaces

$$\operatorname{sign}(df_x) = \begin{cases} +1 & \text{if } df_x \text{ is orientation-preserving} \\ -1 & \text{if not.} \end{cases}$$

ⓑ Define **the degree of f at y**

$$\deg(f, y) = \sum_{x \in f^{-1}\{y\}} \operatorname{sign}(df_x)$$

Thm: We proved in class that $\forall\, y_1, y_2$ regular values for f,

$$\deg(f, y_1) = \deg(f, y_2)$$

so the following invariant is well-defined:

ⓒ Define **the degree of f**

$$\deg(f) = \sum_{x \in f^{-1}\{y\}} \operatorname{sign}(df_x),$$

where y is any regular value of f.

①ⓘ **Show that** if $f: A \to B$ and $g: B \to C$ are smooth maps between orientable manifolds of the same dimension, then

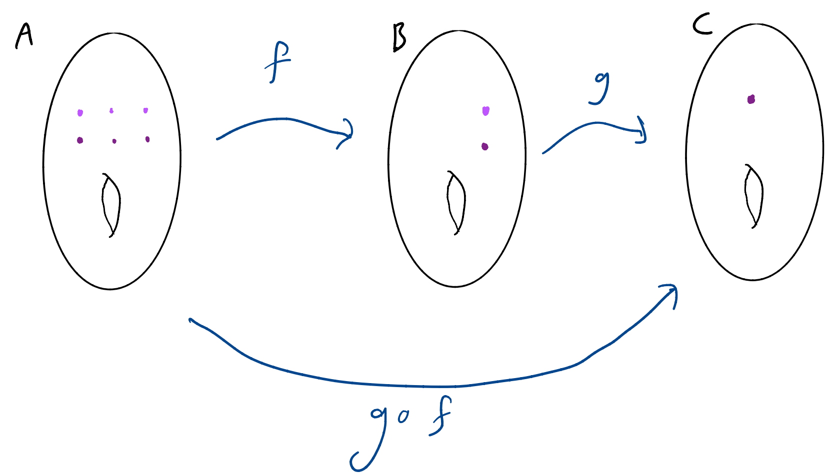
$$\deg(g \circ f) = \deg(f)\deg(g).$$

† since $y = f(x)$ being regular gives us that df_x is onto and $\dim X = \dim Y$ gives us 1-1.

Proof: Let $c \in C$ be a regular value of $g \circ f$. Since $d(g\circ f)_a$ is an isomorphism for each $a \in (g\circ f)^{-1}\{c\}$ and

$$d(g\circ f)_a = dg_b \circ df_a \quad \text{where } b = f(a) \text{ and } c = g(b),$$

then these maps are both isomorphisms as well.† This means that c is regular for g, and every $b \in g^{-1}\{c\}$ is regular for f.



Note that a composition of two linear isomorphisms is orientation-preserving exactly when both isomorphisms are, or neither is. Thus

$$\operatorname{sign}(d(g\circ f)_a) = \operatorname{sign}(dg_b)\operatorname{sign}(df_a).$$

Therefore

$$\begin{aligned}
\deg(g\circ f) &= \sum_{a \in (g\circ f)^{-1}\{c\}} \operatorname{sign}(d(g\circ f)_a) \\
&= \sum_{a \in (g\circ f)^{-1}\{c\}} \operatorname{sign}(dg_{f(a)})\operatorname{sign}(df_a) \\
&= \sum_{a \in f^{-1}\{b_1\}} \operatorname{sign}(dg_{b_1})\operatorname{sign}(df_a) + \cdots + \sum_{a \in f^{-1}\{b_k\}} \operatorname{sign}(dg_{b_k})\operatorname{sign}(df_a) \quad ‡ \\
&= \sum_{i=1}^{k}\left(\operatorname{sign}(dg_{b_i})\sum_{a \in f^{-1}\{b_i\}} \operatorname{sign}(df_a)\right)
\end{aligned}$$

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† See the footnote on page 1, the same reasoning applies.

‡ Where we denote $\{b_1, \dots, b_k\} = g^{-1}(c)$.

(continuing from page 2)

$$\sum_{i=1}^{k}\left(\operatorname{sign}(dg_{b_i})\sum_{a\in f^{-1}\{b_i\}}\operatorname{sign}(df_a)\right)$$
$$=\sum_{i=1}^{k}\left(\operatorname{sign}(dg_{b_i})\deg(f)\right)$$
$$=\deg(f)\sum_{i=1}^{k}\operatorname{sign}(dg_{b_i})$$
$$=\deg(f)\deg(g)$$

and (i) is proved. $\blacksquare$

1 (ii) Given $f: A\to B$ with $\deg(f)=1$ and such that the index

$$[\pi_1(B) : f_*(\pi_1(A))] < \infty,$$

prove that $f_*: \pi_1(A)\to\pi_1(B)$ is surjective.

Def: A covering space $p:\widetilde{X}\to X$ is a map where every $x\in X$ has an open neighborhood which is evenly covered.

Galois Correspondence thm: Suppose X is path connected, locally path connected, and semilocally simply connected. For every subgroup $G\subset\pi_1 X$, $\exists$ covering space $p:\widetilde{X}\to X$ s.t. $\operatorname{im} p_* = G$.

Lifting Criterion: Let $p:\widetilde{X}\to X$ be a covering space, Y path connected and locally path connected, and f a ct's map $Y\to X$. Then a lift $\tilde{f}: Y\to\widetilde{X}$ exists iff $\operatorname{im}(f_*)\subseteq\operatorname{im}(p_*)$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Thm: Every covering map of d sheets is regular everywhere and has degree d.

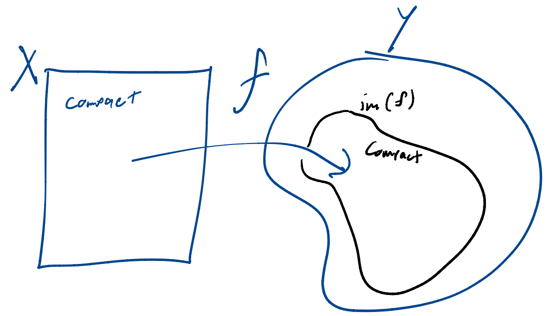
Thm: $\forall$ covering maps p, the subgroup $\operatorname{im}(p_*)\subset\pi_1(X)$ is $\{[\gamma]\in\pi_1(X) \mid \tilde{\gamma} \text{ is a loop}\}$

Proof: Note that the image of the induced map $\operatorname{im}(f_*)$ is always a subgroup of $\pi_1(B)$ since $f_*(\gamma)^{-1} = f_*(\gamma^{-1})$ and $f(\gamma \cdot \eta) = f(\gamma) \cdot f(\eta)$ for all loops $\gamma, \eta \in \pi_1(A)$.

$$\begin{tikzcd} & \tilde{B} \arrow[d, "p \text{ s.t. } \operatorname{im} p_* = \operatorname{im} f_*"] \\ A \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"', "\deg f = 1"'] & B \end{tikzcd}$$

By the Galois Correspondence theorem, $\exists$ covering map p s.t. $\operatorname{im}(p_*) = \operatorname{im}(f_*)$, and we will choose p by rearr-anging coordinates if necessary so that p preserves orientation. By the Lifting Criterion, we can lift f to a map $\tilde{f}: A \to \tilde{B}$, and note that $1 = \deg f = \deg \tilde{f} \deg p$, so $\deg p = 1$. This means that p is a 1-sheeted cover, so every $\gamma \in \pi_1(B)$ lifts to a loop $\tilde{\gamma} \in \pi_1 \tilde{B}$, and $p_*: \pi_1(\tilde{B}) \to \pi_1(B)$ is surjective. And now we're done, since $\operatorname{im} f_* = \operatorname{im} p_*$, and so $\forall\, b \in \pi_1(B)$ $\exists$ some $a \in \pi_1(A)$ s.t. $f_*(a) = p_*(\tilde{b}) = b$. $\blacksquare$

② (i) Suppose that X compact, Y connected. **Show** that if $f: X \to Y$ submersion then f surjective.



<u>**Local Submersion Thm:**</u> If $f: X \to Y$ is a submersion, then for every $x \in X$, there exist charts ϕ, ψ of X, Y near $x, f(x)$ respectively such that $U \xrightarrow{\psi^{-1}\circ f \circ \phi} V$ by the

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi \uparrow & & \uparrow \psi \\ U^{\subset \mathbb{R}^k} & & V^{\subset \mathbb{R}^\ell} \end{array}$$

canonical submersion, which is the projection $(x_1, \cdots, x_\ell, \cdots, x_k) \longmapsto (x_1, \cdots, x_\ell)$ restricted to U.

<u>**Proof:**</u> First we show that the image $\operatorname{im} f$ is open. Let $x \in X$. Since f is a submersion, apply the Local Submersion Theorem to obtain ϕ, ψ and U, V as above. Since $\phi(x) \in U$ is open, take an open rectangle $R \subset \mathbb{R}^k$ with $\phi^{-1}(x) \in R \subset U$, and observe that its projection $\psi^{-1} \circ f \circ \phi(R) = R' \subset \psi^{-1}(\operatorname{im} f)$ is an open rectangle, so $\psi^{-1}(\operatorname{im} f)$ is open. Since ψ is a diffeomorphism, then $\operatorname{im} f$ is open.

Now we show that $\operatorname{im} f$ is closed. Let $y \in \partial(\operatorname{im} f)$, and let $\{y_n\}_{n=1}^{\infty}$ be a sequence of points converging topologically to y. For all n, choose $x_n \in f^{-1}(y_n)$, so that $\{x_n\}_{n=1}^{\infty}$ is a sequence in X. Since X is compact, X is sequentially compact so $\exists$ a convergent subsequence $x_{n_k} \to x$. Since f is smooth, then it is continuous, so

$$f(x) = f(\lim_k x_{n_k}) = \lim_k (f(x_{n_k})) = \lim_k y_{n_k} = y$$

so $y \in \operatorname{im} f$. Thus $\operatorname{im} f$ contains its boundary and is therefore closed.

We have shown that $\operatorname{im} f$ is a clopen set in Y, and since Y is connected then its only clopen subsets are Y and $\emptyset$. So Assuming X is not an empty manifold, then $\operatorname{im} f \neq \emptyset$ so $\operatorname{im} f = Y$ and we're done. $\blacksquare$

<u>**Thm:**</u> Let $f: X^k \xrightarrow{\text{smooth}} Y^\ell$. If y is a regular point of f, then $f^{-1}(y)$ is a smooth manifold of codimension ℓ in X, and $\forall x \in f^{-1}(y)$, $T_x(f^{-1}(y)) = \ker(df_x)$.

② **Def:** The **Special Linear Group** is

$$SL(n) = \{A \in M_n\mathbb{C} \mid \det(A) = 1\}$$

Def: The **special unitary group** is

$$SU(n) = \{A \in M_n\mathbb{C} \mid A^*A = I,\ \det(A) = 1\}$$

Def: The **Special Orthogonal group** is

$$SO(n) = \{A \in M_n\mathbb{R} \mid A^TA = I,\ \det(A) = 1\}$$

(ii) **Show** that there is a homomorphism

$$\pi : SU(2) \longrightarrow SO(3)$$

by considering the action of $SU(2)$ on it's tangent space by conjugacy. **Prove** that π is surjective.

Proof: Suppose $V_t \in SU(2)$ is a path with $t \in (0-\varepsilon, 0+\varepsilon)$, and $V_0 = I$. Then $\forall t$, $V_t^* V_t = I$, so taking the derivative of both sides,

$$\frac{d}{dt}(V_t^* V_t) = \dot{V}_t^* V_t + V_t^* \dot{V}_t$$

$$= 0.$$

This means that for $t=0$,

$$\boxed{\dot{V}_0^* + \dot{V}_0 = 0} \qquad (1)$$

But also since $V_t \in SU(2)\ \forall t$, then $\det(V_t) = 1 = a_t d_t - b_t c_t$, where $V_t = \begin{bmatrix} a_t & b_t \\ c_t & d_t \end{bmatrix}$. Taking $\frac{d}{dt}$ yields

$$0 = \dot{a}_t d_t + a_t \dot{d}_t - \dot{b}_t c_t - b_t \dot{c}_t$$

and when $t=0$, $V_t = I$ so

$$0 = \dot{a}_0 (1) + (1)\dot{d}_0 - \dot{b}_0 (0) - (0)\dot{c}_0$$

$$\Rightarrow \boxed{0 = \dot{a}_0 + \dot{d}_0} \qquad \textbf{(2)}$$

Thus, for every vector $\dot{V}_0$ in $T_I SU(2)$, equations (1) and (2) hold. So $\forall$ elements of $T_I SU(2)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -\bar{a} & -\bar{c} \\ -\bar{b} & -\bar{d} \end{bmatrix}$$

by (1), so a,b,c,d are imaginary, and $-b=c$. We also have $-a=d$, by (2). Thus we can write a general element of $T_I SU(2)$ as

$$\begin{bmatrix} \alpha i & \beta i \\ -\beta i & -\alpha i \end{bmatrix} \qquad \alpha, \beta \in \mathbb{R} \qquad \textbf{(3)}$$

Note that for $x,y \in T_I SU(2)$, the trace $tr(x^* y)$ is an inner product, since

$$tr(x^* y) = \begin{bmatrix} -x_1 i & x_2 i \\ -x_2 i & x_1 i \end{bmatrix} \begin{bmatrix} y_1 i & y_2 i \\ -y_2 i & -y_1 i \end{bmatrix}$$
$$= 2(x_1 y_1 + x_2 y_2)$$

which is exactly $2\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$, and $T_I SU(2) \cong \mathbb{R}^2$ as vector spaces, as one can clearly see in (3).

We seek a function $\pi : SU(2) \to SO(3)$, which means $d\pi_I : T_I SU(2) \to T_I SO(3)$. By the same reasoning as in equation (1) any $v \in T_I O(3)$ (Not $SO(3)$) has $v^T + v = 0$, so $T_I O(3) \subset \Xi_3(\mathbb{R})$, the 3x3 antisymmetric real matrices. A general matrix in $\Xi_3(\mathbb{R})$ can be written as

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

so $\dim(O(3)) = \dim(T_I O(3)) \le \dim(\Xi_3) = 3$. However, since $\det : O(3) \to \mathbb{R}$ is a smooth function and $\operatorname{codim}(\{1\}; \mathbb{R}) = 1$, † then $SO(3) = \det^{-1}\{1\}$ has codimension 1 in $O(3)$, so $\dim(SO(3)) = \dim(T_I SO(3)) \le 2$. Then there exists a surjective linear transformation

$$d\pi_I : T_I SU(2) \longrightarrow T_I SO(3).$$

Let π be a smooth function so that its derivative at I is the above linear transformation.‡ Since $SU(2)$ is a Lie group, multiplication is a diffeomorphism, so

$$\begin{aligned} d(\pi(wv))_I &= d(\pi \circ L_w(v))_I \\ &= d(\pi)_w \circ d(L_w)_I(v). \end{aligned}$$

I don't know why π is a homomorphism.

π has to be surjective because $d\pi_I$ is onto. I've done this problem SO WRONG. ∎

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† I is definitely a regular point for det because we can smoothly vary the entries of a matrix in $O(3)$ to increase or decrease the determinant.

‡ Can I do this? Formally I don't think we've justified this.

③ <u>Def</u>: Given any two ordered bases

$$\beta = \{v_1, \cdots, v_k\} \qquad \beta' = \{v_1', \cdots, v_k'\}$$

on a k-dimensional vector space, we say

$$\beta \sim \beta' \Leftrightarrow \det T > 0$$

where T is the linear transformation that maps each $v_i \mapsto v_i'$.

<u>Def</u>: An **orientation** on a vector space is an equivalence class $[\beta]$ of a particular basis.

<u>Def</u>: The **standard orientation on $\mathbb{R}^n$** is the equivalence class of the standard basis.

<u>Def</u>: An **oriented vector space** is a vector space with a preferred orientation $[\beta]$. Any basis $\beta' \in [\beta]$ is called **positively-oriented**, and a basis $\tilde{\beta} \notin [\beta]$ is called **negatively-oriented.**

<u>Def</u>: Given two oriented vector spaces V and W, a linear isomorphism $T: V \to W$ is **orientation-preserving** if the image of a positively-oriented basis is positively-oriented. That is,

$$T(\beta_V) \sim \beta_W$$

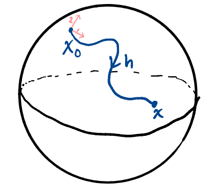
for positively-oriented bases $\beta_V \subset V$ $\beta_W \subset W$.

<u>Def</u>: A manifold M is **orientable** iff $\forall x \in M$ we can orient T_xM such that each chart of M near x is orientation-preserving, that is if $\phi: U \subset \mathbb{R}^k \to M$ with $u \mapsto x$, then

$$d\phi_u \text{ is orientation-preserving.}$$

③ ⓘ Suppose that M^k is simply connected. Prove that M is orientable.

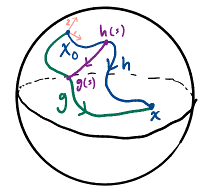
Proof: Pick an arbitrary basepoint $x_0 \in M$. Take a chart $\phi: U \subset \mathbb{R}^k \to V \subset M$ with $\phi(0) = x_0$, and orient $T_{x_0}M$ by choosing the orientation induced by $d\phi_0$, and orient T_vM similarly for all $v \in V$.



Part I: Fix $x_1 \in M$, and $\in M$, Let $h: I \to M$ be some path with $h_0 = x_0$, $h_1 = x_1$. For all $x \in im(h)$, orient T_xM as follows:

The image of h is the smooth image of a compact set, so we can cover $im(h)$ with a finite collection of charts $\{\psi_i\}_{i=1}^n$ with each $\psi_i: U_i \to V_i$ so that $\bigcup_{i=1}^n V_i \supset im(h)$. WLOG assume $h(t)$ encounters the sets V_i in order. Now $x_0 \in V_1$, so $T_{x_0}M$ has a preferred orientation and $d\psi_{1\psi_1^{-1}(x_0)}$ either preserves or reverses orientation, so $\forall v \in V_1$ orient T_vM so that orientation of $d\psi_{1u}$ is consistent with $d\psi_{1\psi_1^{-1}(x_0)}$ $\forall u \in U_1$. Each V_i contains an element $\tilde{v} \in V_{i-1}$ for $i > 1$, so in the same way, for each $v \in V_i$ orient T_vM so that $d\psi_{iu}$ is consistent with $d\psi_{(i-1)\psi_{(i-1)}^{-1}(\tilde{v})}$ $\forall u \in U_i$. Thus for every $x \in im(h)$, we have defined an orientation of T_xM which is consistent with $T_{x_0}M$, and Part I is proved.

<u>Part II: Claim:</u> The orientation on T_xM given above is the same for all homotopic paths connecting x_0 to x.



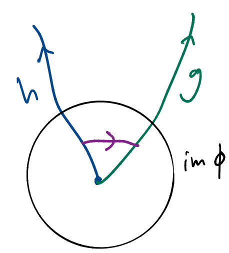
Let $h \simeq g$ be path from x_0 to x_1, and let $F_t(s)$ be a path homotopy from $h(s)$ to $g(s)$, that is, $\forall t \in I$ $F_t(0) = x_0$ and $F_t(1) = x_1$, and also $\forall s \in I$ $F_0(s) = h(s)$ and $F_1(s) = g(s)$.

Orient T_yM $\forall y \in im(g) \cup im(h)$ using the construction from earlier, and denote the orientation of a point $\gamma(s)$ given by a curve γ as $O\gamma(s)$.

Partition I into $\{s_0, \cdots, s_i, \cdots s_n\}$ so that $\forall i = 1 \dots n$, the image $F_t((s_{i-1}, s_i))$ is covered by some chart.†

We will show by induction on i that $Og(s) = Oh(s)$ $\forall s \in I$.

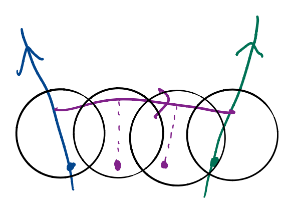
First note that $OF_t(0) = Oh(0) = Og(0)$ $\forall t \in I$, since they are all defined to be the same as $d\phi_0$.



This means that $\forall s \in [0, s_1]$, $OF_t(s)$ is constant $\forall t \in I$.

Next assume that $OF_t(s)$ is constant $\forall s \in [s_{i-1}, s_i]$ $\forall t \in I$.

$\forall t \in I$ $F_t(s_i)$ is covered by some chart, and $im(F_t(s_i))$ is compact so we can take a finite subcover and find that $\forall s \in [s_i, s_{i+1}]$, $OF_t(s)$ is constant.



Thus $OF_t(s)$ is constant $\forall s, t \in I$, so part II is proved.

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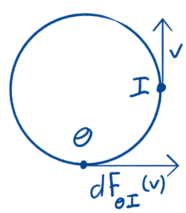
† This can be made rigorous using a Lebesgue number argument.

Part III:

Consider the collection P of all possible paths from x_0 to x_1. Since every connected manifold is path connected,† then $\forall x \in M$, the path $x_0 \sim x \sim x_1$ is in P, so the paths in P cover every point in M. Given any two homotopic curves g, h connecting $x_0 \sim x_1$, Part II gives a consistent orientation of $T_x M$ for all $x \in im(h) \cup im(g)$, and all the paths in P are homotopic since they have common endpoints and M is simply connected, so the family P gives consistent orientations of $T_x M$ $\forall x \in M$ and we're done. $\blacksquare$

③ (ii) **Prove** that a Lie group G is orientable.

Proof: Use a chart to orient $T_I G$ where I denotes the identity. Then $\forall \theta \in G$, the function $F_\theta : G \to G$ given by $F_\theta(x) = \theta x$ is a global diffeomorphism, so if we orient the rest of the manifold so that $dF_{\theta I} : T_I G \to T_\theta G$ is orientation preserving for all $\theta \in G$, then we're done.



$\blacksquare$

† I prove this in the Lemma of problem 5 (ii)

④ <u>**Def:**</u> A polynomial $P(x_1, \cdots, x_n)$ is said to be **homogeneous** if $P(t\vec{x}) = t^m P(\vec{x})$, for some $m \in \mathbb{N}$.

ⓘ **Prove** Euler's identity for homogeneous polynomials

$$\sum_j x_j \frac{\partial P}{\partial x_j} = m P(\vec{x}).$$

<u>**Proof:**</u> Note that

$$\begin{aligned}
P(t\vec{x}) &= P(tx_1, \cdots, tx_n) \\
&= a_1 (tx_1)^{p_{11}} (tx_2)^{p_{12}} \cdots (tx_n)^{p_{1n}} \\
&\quad + a_2 (tx_1)^{p_{21}} (tx_2)^{p_{22}} \cdots (tx_n)^{p_{2n}} \\
&\quad \vdots \\
&\quad + \\
&\quad \vdots \\
&\quad + a_k (x_1)^{p_{k1}} (x_2)^{p_{k2}} \cdots (x_n)^{p_{kn}} \\
&= a_1 t^{p_{11} + \cdots + p_{1n}} (x_1)^{p_{11}} (x_2)^{p_{12}} \cdots (x_n)^{p_{1n}} \\
&\quad + \\
&\quad \vdots \\
&\quad + a_k t^{p_{k1} + \cdots + p_{kn}} (x_1)^{p_{k1}} (x_2)^{p_{k2}} \cdots (x_n)^{p_{kn}} \\
\text{by homogeneity} \quad &= t^m \begin{pmatrix} a_1 (x_1)^{p_{11}} (x_2)^{p_{12}} \cdots (x_n)^{p_{1n}} \\ + \\ \vdots \\ + a_k (x_1)^{p_{k1}} (x_2)^{p_{k2}} \cdots (x_n)^{p_{kn}} \end{pmatrix} \\
&= t^m P(\vec{x})
\end{aligned}$$

So it must be true that the powers $p_{11} + p_{12} + \cdots + p_{1n}$ etc. all sum to m.

Thus

$$\sum_j x_j \frac{\partial P}{\partial x_j} = \sum_j \begin{pmatrix} p_{1j} a_1 (x_1)^{p_{11}} \cdots (x_j)^{p_{1j} - 1 + 1} \cdots (x_n)^{p_{1n}} \\ + \\ \vdots \\ + p_{kj} a_k (x_1)^{p_{k1}} \cdots (x_j)^{p_{kj} - 1 + 1} \cdots (x_n)^{p_{kn}} \end{pmatrix} = \begin{matrix} m a_1 (x_1)^{p_{11}} \cdots (x_n)^{p_{1n}} \\ + \\ \vdots \\ + m a_k (x_1)^{p_{k1}} \cdots (x_n)^{p_{kn}} \end{matrix} = m P(\vec{x})$$

and Euler's identity is proved. ∎

④ **Def:** The **Special Linear Group** is

$$SL(n,\mathbb{R}) = \{A \in M_n\mathbb{R} \mid \det(A) = 1\}$$

Def: Let $f: X^k \xrightarrow{\text{smooth}} Y^\ell$ and let $y \in Y$. We say y is **regular** for f if $\forall x \in f^{-1}(y)$, df_x is onto.

Thm: Let $f: X^k \xrightarrow{\text{smooth}} Y^\ell$ and denote $f^{-1}(y) = Z$. If y is a regular point of f, then Z is a smooth manifold of codimension ℓ in X, and $\forall z \in Z$, $T_z Z = \ker(df_z)$.

Def: A **Lie Group** is a smooth manifold with a group structure such that multiplcation and inversion are smooth maps $X \times X \to X$ and $X \to X$, respectively.

ⓘⓘ **Deduce** that $SL(n,\mathbb{R})$ is a Lie group. Compute $T_I(SL)$.

Proof: $SL(n,\mathbb{R}) = \det^{-1}(1)$ where $\det: M_n\mathbb{R} \to \mathbb{R}$ is a homogeneous polynomial of degree n.† So if 1 is a regular value, then $SL(n,\mathbb{R})$ is a smooth manifold. Let $A \in \det^{-1}(1)$. Since $\det$ maps into $\mathbb{R}$ which is 1-dimensional, the derivative $d(\det)_A$ is surjective as long as it is not the zero map. Note that $A = 0 \Rightarrow \det A = 0$, so since $\det A = 1$, $A \neq 0$. This means that if $d(\det)_A(A) \neq 0$, then $d(\det)_A$ is not the zero map, so it is surjective. Since $M_n\mathbb{R} \cong \mathbb{R}^{n^2}$

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† Inductively, this isn't too hard to see. It's clear for $n=2,3$ by the basket-weaving methods for 2×2 and 3×3 matrices, and $n\times n$ is computed via expansion by minors, with n terms of a matrix entry times an $(n-1)\times(n-1)$ determinant.

and det is a homogenous polynomial, let's write $p(x_1, \dots, x_{n^2}) = \det A$ and apply Euler's Identity:

$$dp_{\vec{x}}(\vec{x}) = \sum_j x_j \frac{\partial p}{\partial x_j} = m\, p(\vec{x}) = m.$$

Thus we conclude that $SL(n, \mathbb{R})$ is a smooth submanifold of the Lie group $M_n(\mathbb{R})$. Note that $\forall A, B \in SL(n, \mathbb{R})$,

$$\det(AB) = \det(A)\det(B) = 1$$

and

$$\det(A^{-1}) = \frac{1}{\det(A)} = 1,$$

so $SL(n, \mathbb{R})$ is closed under its group operations. Since these operations are smooth for $M_n(\mathbb{R})$, then they are for $SL(n, \mathbb{R})$ as well, so $SL(n, \mathbb{R})$ is a Lie group.

Now let's compute the tangent space $T_I SL(n, \mathbb{R})$. By the theorem on the previous page, $T_I SL(n, \mathbb{R}) = \ker(d(\det)_I)$ so let A be in the kernel and observe:

$$\begin{aligned} 0 &= d(\det)_I(A) \\ &= \sum_{ij} a_{ij} \frac{\partial \det}{\partial x_{ij}}(I) \end{aligned}$$

We can compute this partial derivative, with some patience. In the 3×3 case, consider $\frac{\partial \det}{\partial x}$ in the following matrix:

$$\frac{\partial}{\partial x}\begin{vmatrix} a & z_1 & b \\ z_2 & x & z_3 \\ c & z_4 & d \end{vmatrix} = \frac{\partial}{\partial x}\left(x\begin{vmatrix} a & b \\ c & d \end{vmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

using cofactor expansion we can see that in general

$\frac{\partial \det}{\partial x_{ij}} = C_{ij}$ so in our computation

$$\begin{aligned} 0 &= \sum_{ij} a_{ij} \frac{\partial \det}{\partial x_{ij}}(I) \\ &= \sum_{ij} a_{ij}\, C_{ij}(I) \\ &= \sum_{i=j} a_{ij}(1) + \sum_{i\neq j} a_{ij}(0) \\ &= \operatorname{tr}(A) \end{aligned}$$

So $T_I SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \operatorname{tr}(A) = 0\}$. $\blacksquare$

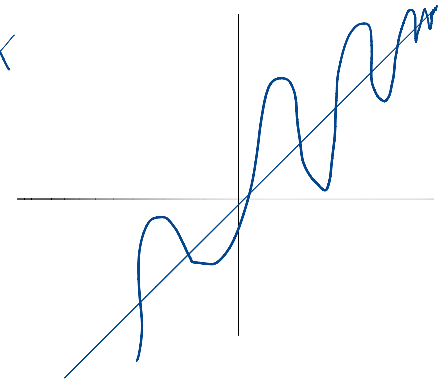
⑤ Suppose that $f: X \to X$ fixes a pt x^*. Define this to be a **Lefschetz fixed pt** if df_{x^*} has no eigenvalue $=1$.

ⓘ **Prove** that if all the fixed pts of f are Lefschetz, then f has only finitely many fixed pts.

Def: Define two submanifolds of $X \times X$

① $\Delta = \{(x,x) \mid x \in X\}$ "the diagonal"

② $G_f = \{(x, f(x)) \mid x \in X\}$ "the graph"



Proposition: We showed in lecture that $G_f \pitchfork \Delta$ † exactly when every point $(x,x) \subset G_f \cap \Delta$ is such that x is a Lefschetz fixed point of f.

Thm: If $f: X \to Y$ with $Z \underset{\text{manif.}}{\subseteq} Y$ and $f \pitchfork Z$, then $f^{-1}(Z) \underset{\text{manif.}}{\subseteq} X$ with $\text{codim}(f^{-1}(Z); X) = \text{codim}(Z; Y)$. (Proved in lecture)

Proof of ⑤ⓘ: Suppose that all the fixed pts of f are Lefschetz, then $G_f \pitchfork \Delta$. Then we can find that $i^{-1}(\Delta)$ is a submanifold of codimension k in G_f. Since $G_f \cong X$, it is a compact submanifold of dimension k, so $i^{-1}(\Delta)$ is a compact manifold of dimension 0, that is, a finite set of points. This set is the set of all the (Lefschetz) fixed points of f, so we're done. ∎

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† I'm making an abuse of the notation here, technically the inclusion map $i: G_f \to X^2$ is transverse to the submanifold Δ.

5 (ii) **Def:** Given $f: X \to X$ and y a Lefschetz fixed pt of f, define

$$\text{sign}(y) = \begin{cases} +1 & \text{if } \prod_i (\lambda_i - 1) > 0 \\ -1 & \text{if } \prod_i (\lambda_i - 1) < 0 \end{cases}$$

where $\{\lambda_i\}_{i=1}^{k}$ are the eigenvalues of f.

Def: Given $f: X \to X$, define the **Lefschetz number of f**

$$\mathcal{L}(f) = \sum_{y \in L(f)} \text{sign}(y)$$

where $L(f)$ [distinct from $\mathcal{L}(f)$] denotes the set of all Lefschetz points of f.

Thm: $\mathcal{L}(f)$ is invariant under smooth homotopy.

Def: Define the **Euler characterisic** of a closed smooth manifold to be

$$\chi_X = \mathcal{L}(\mathbb{1}),$$

where $\mathbb{1}: X \to X$ is the identity map. In practice we choose some $f \simeq \mathbb{1}$ and compute $\mathcal{L}(f)$ to find χ_X.

(ii) **Prove** that a compact connected Lie group X has Euler characteristic zero.

Lemma: Every connected manifold is path-connected.

Proof: Suppose a manifold X is connected and choose $x_0 \in X$. Let

$$P = \{ y \in X \mid x_0 \sim y \}.^{\dagger}$$

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† In this proof, I write $x \sim y$ to mean "$\exists$ a path connecting x to y."

Observe $P \neq \emptyset$ since $x \sim x$ by the constant path. For any $y \in P$, we can take a chart $\phi: U^{\subset \mathbb{R}^k} \to V^{\subset X}$ and find that since $V \ni y$ is diffeomorphic to a subset of $\mathbb{R}^k$, then V is open and path-connected. Thus $\forall v \in V$ we have $x_0 \sim y \sim v$, so $V \subset P$ which means P is open. Now for any $z \in P^c$, we can use a chart ψ of X near z to obtain an open set $\tilde{V}$ s.t. $\forall v \in \tilde{V}$, $v \sim z$. This means $\tilde{V} \subset P^c$, since if not then some $v \in \tilde{V}$ is such that $x_0 \sim v \sim z$, contradicting $z \in P^c$. Thus P open nonempty, P^c open, $P \cup P^c = X$ connected, so P^c empty and X path-connected. $\square$

__**Proof 5 ii:**__ Let X be a compact connected Lie group. By the Lemma, X is path-connected. Let $e \in X$ be the identity element, let $\psi \in X$ be any $\psi \neq e$. Note that multiplication in X is a smooth map, and $f_\psi : X \to X$ given by $x \mapsto \psi x$ has no fixed points.† Let $\gamma : I \to X$ be a smooth path $e \sim \psi$, then

$$F_t(x) := \gamma(t) x$$

is a smooth homotopy from $X \times I \to X$ with $F_0 = \mathbb{1}$ and $F_1 = f_\psi$, so $\chi_X = \mathcal{L}(\mathbb{1}) = \mathcal{L}(f_\psi)$ which is an empty sum, so $\chi_X = 0$. $\blacksquare$

__**Afterword:**__ I am really disappointed that I didn't get a chance to do HW 2 for this class, not only because

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† This is a common result about Lie groups which ultimately boils down to the fact that $\forall \psi \in X$, f_ψ is a diffeomorphism so $f_\psi(e) = \psi$, so $f_\psi(\psi) \neq \psi$ unless $\psi = e$.

those problems were on this exam, but also because I genuinely enjoy topology and there was some cool stuff in these questions that I'd like to learn more about (and I may still do).

I'm not exaggerating when I say that the students of Denis Labutin's 201 C class were veritably opressed by the mounds of reading and mandatory HW we had to do, and Topology got only whatever time I had left as a result.

Hopefully I passed 201C and I can spend my time in future quarters studying topics I love!

Thanks for everything this quarter, looking forward to in-person classes again,

- Trevor