Math 550
Homework 11
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Solutions

1. Let $\vec{x} = g(u,v)$. We know that $(Dg(u,v)(e_1), Dg(u,v)(e_2))$ is a positively oriented basis of the tangent space $g(U)_{\vec{x}}$. Thus the unit outward normal to $g(U)$ at $\vec{x}$ is $N_{\vec{x}} = \frac{Dg(u,v)(e_1)\times Dg(u,v)(e_2)}{\|Dg(u,v)(e_1)\times Dg(u,v)(e_2)\|}$. Then for all $(u,v)\in U$

$$\begin{aligned}
g^*dA(u,v)(e_1,e_2) &= dA(\vec{x})(Dg(u,v)(e_1), Dg(u,v)(e_2)) \\
&= \det\begin{pmatrix} | & | & | \\ N_{\vec{x}} & Dg(u,v)(e_1) & Dg(u,v)(e_2) \\ | & | & | \end{pmatrix} \\
&= \det\begin{pmatrix} | & | & | \\ \frac{Dg(u,v)(e_1)\times Dg(u,v)(e_2)}{\|Dg(u,v)(e_1)\times Dg(u,v)(e_2)\|} & Dg(u,v)(e_1) & Dg(u,v)(e_2) \\ | & | & | \end{pmatrix} \\
&= \|Dg(u,v)(e_1)\times Dg(u,v)(e_2)\| \\
&= \|Dg(u,v)(e_1)\times Dg(u,v)(e_2)\|\, du\wedge dv(e_1,e_2).
\end{aligned}$$

   This shows that $g^*dA = \|Dg(u,v)(e_1)\times Dg(u,v)(e_2)\|\, du\wedge dv$, and the integral formula follows.

2. For every $\vec{x}\in S^{2k}$, define $X(\vec{x}) = P(f(\vec{x}))$, where $P : \mathbf{R}^{2k+1}_{\vec{x}} \to S^{2k}_{\vec{x}}$ is the projection of $\mathbf{R}^{2k+1}_{\vec{x}}$ onto the subspace $S^{2k}_{\vec{x}}$. (In more detail, if we write $f(\vec{x}) = \vec{w} + \lambda N_{\vec{x}}$, for unique $\vec{w}\in S^{2k}_{\vec{x}}$ and $\lambda\in\mathbf{R}$, then $P(f(\vec{x})) = \vec{w}$.) X defines a vector field on S^{2k}. By Theorem 28, $0 = X(\vec{x}) = P(f(\vec{x}))$ for some $\vec{x}$, and at that point we have $f(\vec{x}) = \pm N_{\vec{x}} = \pm\vec{x}$.

3. Suppose $f(\vec{x})\neq 0$ for all $\vec{x}\in D^n$. Then we can define $\frac{f}{\|f\|} : D^n \to S^{n-1}$, and also consider its restriction $\frac{f}{\|f\|} : S^{n-1}\to S^{n-1}$. Since $\|f(\vec{x})-\vec{x}\| < 1$ for all $\vec{x}\in S^{n-1}$, we can define a homotopy

$$H(x,t) = \frac{t\vec{x} + (1-t)\frac{f}{\|f\|}(\vec{x})}{\|t\vec{x} + (1-t)\frac{f}{\|f\|}(\vec{x})\|}$$

   between $\frac{f}{\|f\|} : S^{n-1}\to S^{n-1}$ and the identity $S^{n-1}\to S^{n-1}$.

   Then if ν is the volume form on S^{n-1}, we have

$$\int_{S^{n-1}} \nu = \int_{S^{n-1}} (f/\|f\|)^*\nu = \int_{D^n} d(f/\|f\|)^*\nu = \int_{D^n} (f/\|f\|)^* d\nu = 0.$$

   This contradicts that $\int_{S^{n-1}}\nu > 0$.

4. Let $x_0\in M$. Since M is contractible, there exists a homotopy $H : M\times[0,1]\to M$ with $H(x,0) = x$ and $H(x,1) = x_0$, for all $x\in M$.

   To verify M is path connected, let $x_1, x_2\in M$. Define $\gamma_1 : [0,1]\to M$ by $\gamma_1(t) = H(x_1,t)$; this is a path from x_1 to x_0. Similarly, $\overline{\gamma_2}(t) = H(x_2, 1-t)$ gives a path from x_0 to x_2. We may concatenate the two paths to get a path from x_1 to x_2.

   Let $\gamma : S^1\to M$ be any closed curve. Then $G : S^1\times[0,1]\to M$ given by $G(s,t) = H(\gamma(s),t)$ gives a homotopy between γ and the constant curve at x_0.

5. It was shown in class that S^k is simply connected for $k \geq 2$. But each such S^k is a compact oriented manifold without boundary, and therefore not contractible by Homework 10, addendum problem 2.

6. (a) Since ω_1 and ω_2 are cohomologous, we have $\omega_1 = \omega_2 + d\eta$. So

$$\int_M \omega_1 = \int_M \omega_2 + \int_M d\eta = \int_M \omega_2 + \int_{\partial M = \emptyset} \eta = \int_M \omega_2.$$

   (b) Part (a) shows that $\int_M : H^k(M) \to \mathbf{R}$ given by $\int_M([\omega]) = \int_M \omega$ is well-defined. The linearity of $\int_M$ follows immediately from the linearity of the integral.

   (c) If $M = \partial W$, then by Stokes' Theorem we have $\int_M([\omega]) = \int_M \omega = \int_W d\omega = 0$.

7. (a) Since $H^n(S^n)$ has dimension 1 and is generated by the class $[\nu]$ of the volume form, we can write $\omega = r\nu + d\eta$, for some $r \in \mathbf{R}$ and some $(n-1)$-form η. Then

$$\int_{S^n} \omega = r\int_{S^n} \nu + \int_{S^n} d\eta = r\int_{S^n} \nu + \int_{D^{n+1}} d^2\eta = r\int_{S^n} \nu.$$

   This implies $\int_{S^n} \omega = 0$ if and only if $r = 0$ if and only if $\omega = d\eta$.

   (b) Part (a) shows that $\ker \int_{S^n} = \{[0]\}$, so $\int_{S^n}$ is one-to-one. Then since $H^n(S^n)$ and $\mathbf{R}$ are both 1 dimensional, $\int_{S^n}$ is an isomorphism.