A Problem from Erdös About Products of 2 or 3 Primes

PUMP Research Symposium

Trevor Klar & Eli Moore
California State University, Northridge

April 28th, 2017

CSUNlogo.png

Motivation

Take two prime numbers, say 2 and 3, and make a list of all the natural numbers which can be formed using only 2 and 3 as factors.

$$\left.\begin{array}{cccccccc} 2 & 3 & 4 & 6 & 8 & 9 & 12 & \dots \\ 2^1 & 3^1 & 2^2 & 2^1 3^1 & 2^3 & 3^2 & 2^2 3^1 & \dots \end{array}\right.$$

There is an interesting pattern here: there keep ocurring pairs of numbers which have only 2 or only 3 as a factor.

$$\begin{array}{cccccccc} \dots & 16 & 18 & 24 & 27 & 32 & 36 & \dots \\ \dots & 2^4 & 2^1 3^2 & 2^3 3^1 & 3^3 & 2^5 & 2^2 3^2 & \dots \end{array}$$

Motivation

Does this keep happening?

...	*1536*	*1728*	*1944*	*2048*	*2187*	*2304*	...
...	$2^9 3^1$	$2^6 3^3$	$2^3 3^5$	2^{11}	3^7	$2^8 3^2$	...

Does it still happen with any primes?

59	*61*	*3481*	*3599*	*3721*	*205379*	*212341*	...
59^1	61^1	59^2	$59^1 61^1$	61^2	59^3	$59^2 61^1$	...

... $59^3 61^2$ $59^2 61^3$ $59^1 61^4$ 61^5 59^6 $59^5 61^1$ ...

Motivation

What if you do it with *three* primes (i.e. 2, 3, and 5)?

...	*18*	*20*	*24*	*25*	*27*	*30*	...
...	$2^1 3^2$	$2^2 5^1$	$2^3 3^1$	5^2	3^3	$2^1 3^1 5^1$	...

Question!

Paul Erdös asked the following question about three distinct primes: If we construct a sequence of all the products of their powers, with the sequence arranged in increasing order, is it true infinitely often that consecutive terms in this sequence are both prime-powers?

Definitions

Let p, q be distinct primes, and let $m, n \in \mathbb{Z}^+$.

Definition

A pure power of p is an integer of the form p^m.

Definition

A mixed power of p and q is an integer of the form $p^m q^n$.

Definition

A critical pair of p and q is a pair of pure powers of p and q which do not have a mixed power between them.

Developing Intuition

Lemma 1

If $a_k = q^n$, then $a_{k+1} \neq q^{n+1}$.

Ex.

$$\ldots \quad 2^4 \quad 2^1 3^2 \quad 2^3 3^1 \quad 3^3 \quad \ldots$$

Here, 3^4 can't come next, because $3^3 < 3^3 2^1 < 3^4$.

Early Stages

Lemma 2

There exist at most finitely many $a_k = p^m$ such that $a_{k+1} = p^{m+1}$.

Even in a dramatic example like $p = 2$, $q = 509$, the sequence starts out all powers of 2,

$$2^1 \quad 2^2 \quad 2^3 \quad 2^4 \quad 2^5 \quad \dots$$

but this can't continue forever.

$$2^7 \quad 2^8 \quad 509^1 \quad 2^9 \quad 2^1 509^1 \quad \dots$$

Early Stages

Lemma 3

If $a_k = p^m$ and $a_{k+1} = q^n$, then m and n are relatively prime.

Sketch of Proof. Suppose for contradiction that $\gcd(m, n) = d$. Then $m = m'd$ and $n = n'd$ for some $m', n' \in \mathbb{N}$. Since $p^m = p^{m'd} < q^{n'd} = q^n$, we have $p^{m'} < q^{n'}$. Thus:

$$\begin{aligned}
p^m &= p^{m'd} \\
&= p^{m'd-m'+m'} \\
&= p^{m'(d-1)}p^{m'} \\
&< \underline{p^{m'(d-1)}q^{n'}} \\
&< q^{n'(d-1)}q^{n'} \\
&= q^{n'd} \\
&= q^n.
\end{aligned}$$

The "Tail" of the Sequence of Exponents

slope_plot4.png

2	3	4	6	8	9	12	...
2^1	3^1	2^2	$2^1 3^1$	2^3	3^2	$2^2 3^1$	...

The "Tail" of the Sequence of Exponents

slope_plot4.png

Notice how the lines seem to approach a constant slope!
As $k \to \infty$, we see that $\frac{n_k}{m_k} \to -\frac{\log p}{\log q}$.

Converging Ratios

As $k \to \infty$, we see that $\frac{n_k}{m_k} \to -\frac{\log p}{\log q}$.

critical_slopes3.png

Two Primes

After developing an understanding for the problem, we began to analyze the details that Erdös glossed over. This lead us to a proof for the following theorem:

Theorem 1

For any two distinct prime numbers p, and q, there exist infinitely many critical pairs.

Key Lemmas

Lemma 4

Consider the pure powers p^a, q^b with $p^a < q^b$ and $a, b \in \mathbb{Z}^+$. If, for all critical pairs p^s, q^t with $s < a$ and $t < b$,

$$1 < \frac{q^b}{p^a} < \frac{q^t}{p^s}, \quad s, t \in \mathbb{Z}^+$$

then p^a, q^b is a critical pair.

A sketch of the proof follows.

Assume that for all critical pairs p^s, q^t with $s < a$ and $t < b$,

$$1 < \frac{q^b}{p^a} < \frac{q^t}{p^s},$$

and suppose for contradiction that p^a, q^b is not a critical pair. Since p^a, q^b is not a critical pair, then there exists an intermediate mixed power of the form

$$p^a < q^{b-\tilde{b}} p^{\tilde{a}} < q^b$$

So, since $p^a < q^{b-\tilde{b}} p^{\tilde{a}}$,

$$\frac{q^b}{p^a} > \frac{q^b}{q^{b-\tilde{b}} p^{\tilde{a}}} = \frac{q^{\tilde{b}}}{p^{\tilde{a}}}.$$

If $p^{\tilde{a}}$ and $q^{\tilde{b}}$ are a critical pair, we have a contradiction. If not, repeat.

Key Lemmas

Dirichlet's Approximation Theorem

Let α be a real number. Given any $\epsilon > 0$, there exist M, N $\in \mathbb{Z}$ such that

$$\left|\alpha - \frac{M}{N}\right| < \epsilon.$$

The "Tail" of the Sequence of Exponents

slope_plot4.png

By the way, the fact that as $k \to \infty$, $\frac{n_k}{m_k} \to -\frac{\log p}{\log q}$.
follows from Dirichlet's Approximation Theorem and Lemma 4.

Key Lemmas

Dirichlet's Approximation Theorem

Let α be a real number. Given any $\epsilon > 0$, there exist M, N $\in \mathbb{Z}$ such that

$$\left|\alpha - \frac{M}{N}\right| < \epsilon.$$

Lemma 5

Let α be an irrational number. Given any $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $n\alpha - \lfloor n\alpha \rfloor < \epsilon$.

Proof of the Result

Let p and q be distinct primes. Suppose for contradiction that there exist finitely many critical pairs. Of these critical pairs, consider the critical pair with $p^k < q^\ell$ such that $\frac{q^\ell}{p^k}$ is smallest. This means that

$$1 < \frac{q^\ell}{p^k}.$$

Choose some $\epsilon \in \mathbb{R}$ such that

$$1 < p^\epsilon < \frac{q^\ell}{p^k}.$$

Now, consider the irrational number $\log_p q$. By Lemma 5, there exists some $\Omega \in \mathbb{N}$ such that

$$\Omega \log_p q - \lfloor \Omega \log_p q \rfloor < \epsilon.$$

Proof of the Result

To simplify the notation, let $a = \lfloor \Omega \log_p q \rfloor$. Thus, with a little algebra,

$$\begin{aligned} \Omega \log_p q - a &< \epsilon \\ \Omega \log_p q &< a + \epsilon \\ q^\Omega &< p^a p^\epsilon \\ \frac{q^\Omega}{p^a} &< \quad p^\epsilon \quad < \tfrac{q^\ell}{p^k} \end{aligned}$$

Hence,

$$1 < \frac{q^\Omega}{p^a} < \frac{q^\ell}{p^k}.$$

Therefore, by Lemma 4, (q^Ω, p^a) is a critical pair with ratio closer to 1, which is a contradiction.

Redeveloped Proof

Problem

Everything depends on

$$1 < \frac{q^{\Omega}}{p^{a}} < \frac{q^{\ell}}{p^{k}}.$$

Issues with 3 Primes

We have attempted the problem with various techniques: Mean Value Theorem/Taylor Polynomial bounds, Geometric Arguments, Linear Programming, Special cases (59,61,3601, (p,q,r $=$ pq $+2$)), etc.

Moving Forward

Currently, we are continuing to look into our geometric argument in three dimensions.

Acknowledgments

Thank you to our advisor Dr. Werner Horn for overseeing this project, as well as Dr. Helena Noronha and the rest of the PUMP program for providing us with this opportunity.

Thank you to the NSF for providing funding through the PUMP program award, NSF Grant DMS-1247679.

NSFLogo.png CSUNlogo.png