**Math 550**
**Homework 1**
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Due September 4, 2018

1. The set $\Lambda^n(\mathbf{R}^n)$ of all alternating, multilinear functions on $(\mathbf{R}^n)^n$ forms a vector space. (You do not have to prove this.) What is its dimension? Find a basis for this vector space.

2. Let $V$ be an $n$-dimensional vector space with an inner product $\langle\ ,\ \rangle$. Suppose $S \in \Lambda^n(V)$ is an alternating, multilinear function on $V$.

   (a) Let $(u_1,\dots,u_n)$ be a basis for $V$. Suppose $(v_1,\dots,v_n)$ is a collection of vectors in $V$ with $v_j = \sum_i a_{ij}u_i$. Prove that $S(v_1,\dots,v_n) = \det[a_{ij}]S(u_1,\dots,u_n)$.

   (b) Suppose that $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$ are two orthonormal bases for $V$, with $v_j = \sum_i a_{ij}u_i$. Let $A = [a_{ij}]$. Prove that $AA^T = I$. (Hint: start by considering $\langle v_i, v_j\rangle$.)

   (c) Prove that $|S(u_1,\dots,u_n)| = |S(v_1,\dots,v_n)|$ for any two orthonormal bases of $V$.

3. Give a counterexample to show that the change of variables formula does not hold if $g$ is not one-to-one, even if $\det Dg(x) \neq 0$ for all $x \in \Omega$. (Hint: Take $f = 1$ and $g(x,y) = (e^x\cos y, e^x \sin y)$ for a suitable region $\Omega$.)

4. (a) Calculate $\int_{B_r} e^{-x^2-y^2}\,dx\,dy$, where $B_r = \{(x,y) : x^2+y^2 \le r^2\}$.

   (b) Show that $\int_{C_r} e^{-x^2-y^2}\,dx\,dy = (\int_{-r}^{r} e^{-x^2}\,dx)^2$, where $C_r = [-r,r]\times[-r,r]$.

   (c) Show that
   $$\lim_{r\to\infty}\int_{B_r} e^{-x^2-y^2}\,dx\,dy = \lim_{r\to\infty}\int_{C_r} e^{-x^2-y^2}\,dx\,dy.$$

   (d) Show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

5. (a.) Let $D$ be the unit ball in $\mathbf{R}^3$, and let $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$. Calculate $\int_D f$ using a change of variables.

   (b.) Let $E$ be the ellipsoid $\{(x,y,z) \in \mathbf{R}^3 : (x^2/a^2)+(y^2/b^2)+(z^2/c^2) \le 1\}$, where $a, b$, and $c$ are positive constants. Compute the volume of $E$ using a change of variables.

6. Let $\langle e_1,\dots,e_n\rangle$ denote the standard basis for $\mathbf{R}^n$, and let $T$ denote the linear operator on $\mathbf{R}^n$ defined by $T(e_1) = (1,1,1,1,\dots,1), T(e_2) = (1,2,1,1,\dots,1), T(e_3) = (1,2,3,1,\dots,1),\dots,T(e_n) = (1,2,3,4,\dots,n)$.
   Suppose that $f : \Omega \to \mathbf{R}$ is integrable, and $\int_\Omega f = 1$. Compute $\int_{T^{-1}(\Omega)} f \circ T$.