Math 450
Homework 1
Dr. Fuller
Solutions

1. Let $\mathbf{x}, \mathbf{y}$ be non-zero vectors in $\mathbf{R}^n$.

   Assume that $|\langle \mathbf{x}, \mathbf{y} \rangle| = \|\mathbf{x}\|\|\mathbf{y}\|$. Let $r_0 = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{y}\|^2}$. A calculation as in the proof of Cauchy-Schwarz given in class shows

   $$0 \leq \|\mathbf{x} - r_0\mathbf{y}\|^2 = \langle \mathbf{x} - r_0\mathbf{y}, \mathbf{x} - r_0\mathbf{y} \rangle = \|\mathbf{x}\|^2 - 2r_0\langle \mathbf{x}, \mathbf{y} \rangle + r_0^2\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\|\mathbf{y}\|^2} = 0.$$

   (The last "$= 0$" follows from the assumption.) This implies $\mathbf{x} - r_0\mathbf{y} = 0$ or $\mathbf{x} = r_0\mathbf{y}$.

   Conversely, assume $\mathbf{y} = r\mathbf{x}$. Then $|\langle \mathbf{x}, \mathbf{y} \rangle| = |\langle \mathbf{x}, r\mathbf{x} \rangle| = |r||\langle \mathbf{x}, \mathbf{x} \rangle| = |r|\|\mathbf{x}\|\|\mathbf{x}\| = \|\mathbf{x}\|\|\mathbf{y}\|$.

2. In general we have: $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2$. Thus $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ if and only if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

3. $\frac{\pi}{6}$

4. (a) $\{(x,y) : xy = 0\} \subset \mathbf{R}^2$ is not open and is closed.
   (b) $\{(x,y) : xy \neq 0\} \subset \mathbf{R}^2$ is open and is not closed.
   (c) $\{(x,y,z) : x^2 + y^2 < 1 \text{ and } z = 0\} \subset \mathbf{R}^3$ is not open and not closed.
   (d) $\{(x,y,z) : x^2 + y^2 < 1\} \subset \mathbf{R}^3$ is open and not closed.
   (e) $\{(x_1, \ldots, x_n) : \text{each } x_i \in \mathbf{Q}\} \subset \mathbf{R}^n$ is not open and not closed.

6. If $\mathbf{y} \in \mathbf{R}^n - \overline{B}(\mathbf{x}, r)$, let $s = \|\mathbf{y} - \mathbf{x}\| - r$. We wish to show $B(\mathbf{y}, s) \subset \mathbf{R}^n - \overline{B}(\mathbf{x}, r)$. To do this, let $\mathbf{w} \in B(\mathbf{y}, s)$. Then

   $$s + r = \|\mathbf{y} - \mathbf{x}\| \leq \|\mathbf{y} - \mathbf{w}\| + \|\mathbf{w} - \mathbf{x}\| < s + \|\mathbf{w} - \mathbf{x}\|.$$

   This implies $r < \|\mathbf{w} - \mathbf{x}\|$, so $\mathbf{w} \in \mathbf{R}^n - \overline{B}(\mathbf{x}, r)$.

7. (a) For any $x \in \mathbf{R}^n$, let $r = 1$. Then certainly $x \in B(x, 1) \subset \mathbf{R}^n$
   (b) Let $x \in \bigcup_{\alpha \in \Gamma} U_\alpha$, so $x \in U_{\alpha_0}$ for some $\alpha_0 \in \Gamma$. Since U_{α_0} is open, there exists $r > 0$ with $x \in B(x, r) \subset U_{\alpha_0} \subset \bigcup_{\alpha \in \Gamma} U_\alpha$.
   (c) Let $x \in U_1 \cap U_2$. Then there exists r_1, r_2 with $B(x, r_1) \subset U_1$ and $B(x, r_2) \subset U_2$. Then $x \in B(x, \min\{r_1, r_2\}) \subset U_1 \cap U_2$.

8. (a) $\mathbf{R}^n - \bigcap_{\alpha \in \Gamma} C_\alpha = \bigcup_{\alpha \in \Gamma} \mathbf{R}^n - C_\alpha$. By problem 7(b) this is an open set, showing that $\bigcap_{\alpha \in \Gamma} C_\alpha$ is closed.
   (b) For each $0 < r < 1$, we have that $\overline{B}(\mathbf{0}, r)$ is closed, but $\bigcup_{r \in (0,1)} \overline{B}(\mathbf{0}, r) = B(\mathbf{0}, 1)$ is not closed.