TOPOLOGY QUALIFYING EXAM
MAY 2014

There are nine questions. Answer exactly six.
If you answer more, we will count only the *lowest* six.

(1) Let X and Y be sets and let $2^X, 2^Y$ denote their power sets. For $f : X \to Y$ any function define
- $f^* : 2^Y \to 2^X$ to be the function $f^*(B) = f^{-1}(B)$
- $f' : Y \to 2^X$ be the function $f'(y) = f^*(\{y\})$.

(a) Show that if $f'(y_0) \cap f'(y_1) \neq \emptyset$ then $y_0 = y_1$.
(b) Show that if f is surjective then f' is injective.

Further suppose that X and Y are topological spaces, with $\mathcal{T}_X \subset 2^X$, $\mathcal{T}_Y \subset 2^Y$ their respective topologies.

(c) Define continuity of f in terms of f^*, $\mathcal{T}_X$, and $\mathcal{T}_Y$.

(2) Let X be a Hausdorff topological space.
(a) Define what it means for a subspace $C \subset X$ to be compact.
(b) Show that a compact subspace of X is closed.
(c) Show that if $C \subset X$ and $D \subset X$ are compact then $C \cap D$ is also compact.

(3) Let X and Y be topological spaces.
(a) Define the product topology on $X \times Y$.
(b) Define what it means for a space X to be connected.
(c) Show that X and Y are connected if and only if $X \times Y$ is connected.

Recall: A Hausdorff space X is *normal* if, whenever C_1 and C_2 are disjoint closed subsets of X, there are disjoint open sets U_1, U_2 so that $C_i \subset U_i, i = 1, 2$.

(4) Suppose X is a normal topological space and C is a closed set in X. Define an equivalence relation on X by $x \sim y$ whenever both x and y belong to C; otherwise no two distinct points are equivalent. Prove that the quotient space $X/\sim$ is Hausdorff.

(5) Let (M, d) be a metric space.
  (a) Define what this means.
  (b) Show that M is normal.
  (c) Let x_0 be a point in M and define $\rho : M \to \mathbb{R}$ by $\rho(x) = d(x, x_0)$. Show that ρ is a continuous function.

(6) Let X be a metric space.
  (a) Define what it means for a subspace $C \subset X$ to be *complete*.
  (b) Suppose $C \subset X$ and $D \subset X$ are complete subspaces, show that $C \cup D$ is complete.
  (c) Suppose $\{C_\lambda\}$ is any family of complete subspaces. Show that $\cap_\lambda \{C_\lambda\}$ is either empty or a complete subspace.
  (d) Suppose $f : [0, 1] \to X$ is a continuous function and $f(0) \neq f(1)$. Show that X has uncountably many points.

(7) (a) Define covering space and universal covering space.
  (b) State the basic classification theorem for covering spaces
  (c) Let $C_n \subset \mathbb{R}^2$ denote the circle with center $(1/n, 0)$ and with radius $1/n$. Denote $X = \cup_n C_n$. Prove that X has no universal cover.

(8) (a) Suppose X is a simply-connected locally path-connected space. Show that any continuous function $f : X \to S^1 \times S^1$ is null-homotopic.
  (b) Is (a) true if X is replaced by the projective plane $\mathbb{RP}^2$? Either prove it or exhibit a counterexample and prove that it is one.

(9) Suppose G is a *topological group* which means that the group operations of multiplication $\mu : G \times G \longrightarrow G$ and inverse $\iota : G \longrightarrow G$ are continuous. Suppose X is a topological space.
  (a) L is the set of all continuous maps $\alpha : X \longrightarrow G$. Given $\alpha, \beta \in L$ define $\alpha.\beta$ by $(\alpha.\beta)(x) = \mu(\alpha(x), \beta(x))$. Show that this makes L into a group.
  (b) Let K be the subset of L consisting of those maps that are homotopic to the constant map $X \to e$ where $e \in G$ is the identity. Show that K is a normal subgroup of L.
  (c) If X is contractible and G is path connected show that $K = L$.
  (d) Does the conclusion of (c) hold if X is not contractible ? Proof or counterexample.