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Homework 1

Definition. Let Σ be a subset of (X,d). We say Σ is **bounded** if there exists $x_0 \in X$ and $0 < r < \infty$ such that $\Sigma \subset B_r(x_0)$.

1. Prove that Σ is bounded if and only if there exists $L > 0$ such that

$$d(x,x') \leq L$$

for any $x, x' \in \Sigma$.

Proof ($\implies$) Suppose Σ is bounded by $B_r(x_0)$. Then for any $x, x' \in \Sigma$, we know $d(x,x_0) < r$ and $d(x',x_0) < r$ since $\Sigma \subset B_r(x_0)$. Thus, by triangle inequality, $d(x,x') < 2r$, so we let $L = 2r$ and we are done. $\square$

($\impliedby$) Suppose there exists $L > 0$ such that $d(x,x') \leq L$ for any $x, x' \in \Sigma$. Then fix any $x_0 \in \Sigma$, and Σ is bounded by $B_L(x_0)$, since for any $x \in \Sigma$, we have $d(x,x_0) \leq L$. $\blacksquare$

2. Suppose Σ is bounded and $A \subset \Sigma$.

(a) Prove that A is bounded.

Proof Obvious.† $\blacksquare$

Definition. Define $\operatorname{diam}(\Sigma) = \sup\limits_{\delta,\delta' \in \Sigma} d(\delta,\delta')$.

(b) Prove that $\operatorname{diam}(A) \leq \operatorname{diam}(\Sigma)$.

Proof For any $x, x' \in A$, we also know $x, x' \in \Sigma$, so $d(x,x') \leq \operatorname{diam}(\Sigma)$. Since $\operatorname{diam}(\Sigma)$ is an upper bound for $d|_{A\times A}$, then $\operatorname{diam}(A) = \sup\limits_{\delta,\delta' \in A} d(\delta,\delta') \leq \operatorname{diam}(\Sigma)$. $\blacksquare$

3. Let (X,d) be a metric space, and let

$$d_p\big((x,y),(x',y')\big) = d(x,x') + d(y,y').$$

(a) Show that d_p is a metric on X^2.

Proof

- Since $(x,y) = (x',y')$ iff both $x = x'$ and $y = y'$, and since d is a metric, then

$$\begin{aligned} d_p\big((x,y),(x',y')\big) = 0 &\iff d(x,x') = 0 \text{ and } d(y,y') = 0 \\ &\iff x = x' \text{ and } y = y', \end{aligned}$$

  so d_p is positive-definite.

†I can't write "obvious" on a homework problem? All right. Observe that $A \subset \Sigma \subset B_r(x_0)$, so $A \subset B_r(x_0)$.

- To see that d_p is symmetric, observe that d is symmetric, so

$$\begin{aligned} d_p\big((x,y),(x',y')\big) &= d(x,x') + d(y,y') \\ &= d(x',x) + d(y',y) \\ &= d_p\big((x',y'),(x,y)\big). \end{aligned}$$

- Now we show that the triangle inequality holds.

$$\begin{aligned} d_p\big((x_1,x_2),(y_1,y_2)\big) + d_p\big((y_1,y_2),(z_1,z_2)\big) &= d(x_1,y_1) + d(x_2,y_2) + d(y_1,z_1) + d(y_2,z_2) \\ &= d(x_1,y_1) + d(y_1,z_1) + d(x_2,y_2) + d(y_2,z_2) \\ &\geq d(x_1,z_1) + d(x_2,z_2) \end{aligned}$$

Thus d_p is positive-definite, symmetric, and has the triangle inequality, so it is a metric on X^2. ■

(b) Prove that $d : X \times X \to (\mathbb{R}, \mathrm{MKM})$ is continuous.

Proof Let $r \in \mathbb{R}$, be given. Then choose $x, y \in X$ such that $d(x,y) \leq r$. Let $\epsilon > 0$. Observe that for any $(x',y') \in B_{\frac{\epsilon}{2}}\big((x,y)\big)$,

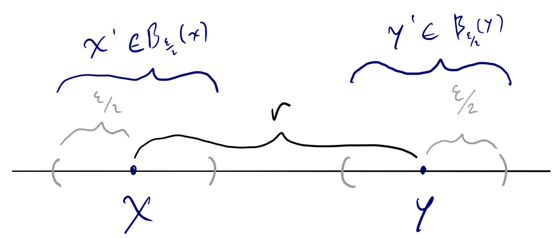
$$d(x,x') + d(y,y') = d_p\big((x',y'),(x,y)\big) < \frac{\epsilon}{2},$$

so $d(x,x') < \frac{\epsilon}{2}$ and $d(y,y') < \frac{\epsilon}{2}$. Now by the triangle inequality,

$$d(x',y') \leq d(x',x) + d(x,y) + d(y,y') = r + \epsilon$$

and

$$d(x',y') \geq d(x,y) - d(x',x) - d(y,y') = r - \epsilon,$$



so $d(x',y') \in B_\epsilon(r)$, which means that d is continuous by the δ-ϵ definition. ■

4. Give examples to show that if $B_r(x) = B_s(y)$, it need not be true that $r = s$ or $x = y$.

Proof Consider $(\mathbb{R}, d_\epsilon)$, where $d_\epsilon(x,y) = \min(|x-y|, 1)$. Then $B_0(100) = B_1(50) = \mathbb{R}$. ■