# Math 201B, Homework 3 (Integration, Differentiation, Density)

**Problem1.** Let $f\colon [0,1] \to \mathbb{R}$ be continuous. Prove that for any $\epsilon > 0$ there exists a continuous function $g_\epsilon\colon [0,1] \to \mathbb{R}$ such that $g'_\epsilon(x)$ exists and equals zero a.e. (w.r.t. Lebesgue measure) in $[0,1]$ and

$$\max_{x\in[0,1]} |f(x) - g_\epsilon(x)| < \epsilon.$$

**Problem2.** Let $A \subset [0,1]$ be a null set (a set that has zero Lebesgue measure). Find an increasing and absolutely continuous function $f\colon [0,1] \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = +\infty$$

for all $x \in A$.

**Problem3.** Let $\mu$ be a Borel measure on $[0,1]$ such that for any $f \in C^1([0,1], \mathbb{R})$ one has the inequality

$$\left| \int_0^1 f'(x) d\mu \right| \le \left( \int_0^1 f^2(x) dx \right)^{1/2}.$$

1. Prove that $\mu$ is in fact a Radon measure that is absolutely continuous with respect to Lebesgue measure on $[0,1]$.
2. If $g$ is the Radon-Nikodym derivative of $\mu$ w.r.t. Lebesgue measure, then there exists a constant $C > 0$ such that

$$|g(x) - g(y)| \le C\sqrt{|x-y|}$$

for a.e. $x, y \in [0,1]$.

**Problem4.** Let $p \ge 1$ and let $f, g \in L^p(\mathbb{R})$. Prove that the function

$$\varphi(t) = \int_{\mathbb{R}} |f(x) + tg(x)|^p dx$$

is differentiable a.e. in $\mathbb{R}$.

Hint: Use Young's inequality: If $p, q > 0$ such that $1/p + 1/q = 1$, then for all $a, b \ge 0$ one has the estimate

$$ab \le \frac{a^p}{p} + \frac{a^q}{q}.$$