# Math 550
# Homework 8

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2. Let $C$ be the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$, oriented counterclockwise as viewed from above the $xy$-plane. Use Stokes' Theorem to evaluate

$$\int_C z^3\,dx.$$

**Answer** Let $\tilde{C}$ be the part of the given plane which is bounded by $C$. Then since $C = \partial\tilde{C}$,

$$\int_C z^3\,dx = \int_{\tilde{C}} d(z^3\,dx) = \int_{\tilde{C}} 3z^2\,dz \wedge dx = \int_{\tilde{C}} -3z^2\,dx \wedge dz$$

Now we parameterize the disc $\tilde{C}$. Let

$$g(r,\theta) = \begin{pmatrix} ar\cos\theta + abr\sin\theta, \\ ar\cos\theta - abr\sin\theta, \\ -2ar\cos\theta \end{pmatrix}, \text{ where } a = \frac{1}{\sqrt{2}},\ b = \frac{1}{\sqrt{3}}$$

To see that $g$ parameterizes $\tilde{C}$[1], first note that the boundary $C$ is can be found by solving the system of equations, and one will find the solution set is given by $2x^2+2xy+2y^2 = 1$, which is an ellipse oriented diagonally. Now Observe that $g_1$ and $g_2$ are given by rotation $\rho_{\pi/4}$ composed with the parameterization in polar coordinates for an ellipse with half-width 1 and half-height $1/\sqrt{3}$:

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} r\cos\theta \\ \frac{r}{\sqrt{3}}\sin\theta \end{bmatrix}$$

and $g_3$ is given by $-g_1 - g_2$. Now that we have a parameterization of $\tilde{C}$, we calculate the pullback.

$$\begin{array}{rcl} \int_C z^3\,dx &=& \int_{\tilde{C}} -3z^2\,dx \wedge dz \\ &=& \int_{g^{-1}(\tilde{C})} g^*(-3z^2)\,dx \wedge dz \end{array}$$

To compute the pullback, we calculate

$$\begin{array}{rcl} g^*(-3z^2) &=& -6r^2\cos^2\theta \\ g^*dx &=& (a\cos\theta + ab\sin\theta)\,dr + (-ar\sin\theta + abr\cos\theta)\,d\theta \\ g^*dz &=& (a\cos\theta - ab\sin\theta)\,dr + (-ar\sin\theta - abr\cos\theta)\,d\theta \end{array}$$

Thus after quite some simplifying we find that

$$g^*(-3z^2)\,dx \wedge dz = 4\sqrt{3}r^3\cos^2\theta\,dr \wedge d\theta.$$

Then we integrate and obtain $\int_0^{2\pi}\int_0^1 4\sqrt{3}r^3\cos^2\theta\,dr\,d\theta = \sqrt{3}\pi$. ■

[1] In checking my work afterwards, I realized that my parameterization for $g_2$ has the wrong sign. This changes everything, so the rest of the work is based on a faulty parameterization. Have mercy on my soul!

3. Show that

$$\omega = \frac{x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

is closed, but not exact.

**Proof** (Closed) To reduce notation, let $\rho^2 = x^2+y^2+z^2$. Then

$$\begin{aligned}
d\omega &= d(x\rho^{-3}\,dy\wedge dz) - d(y\rho^{-3}\,dx\wedge dz) + d(z\rho^{-3}\,dx\wedge dy)\\
&= \phantom{+}(-2x^2+z^2+y^2)\rho^{-5}\;dx\wedge dy\wedge dz\\
&\quad +(+2y^2-x^2-z^2)\rho^{-5}\;dy\wedge dx\wedge dz\\
&\quad +(-2z^2+y^2+x^2)\rho^{-5}\;dz\wedge dx\wedge dy\\
&= \phantom{+}(-2x^2+z^2+y^2)\rho^{-5}\;dx\wedge dy\wedge dz\\
&\quad -(+2y^2-x^2-z^2)\rho^{-5}\;dx\wedge dy\wedge dz\\
&\quad +(-2z^2+y^2+x^2)\rho^{-5}\;dx\wedge dy\wedge dz\\
&= 0
\end{aligned}$$

■

**Proof** (Not exact) Suppose for contradiction that $\omega$ is exact, and write $\omega = d\eta$. Let $M$ be any compact manifold with $\partial M = \emptyset$. Then since Stokes' Thm gives $\int_M d\eta = \int_{\partial M}\eta$, then

$$\int_M \omega = 0.$$

Now, $S^2$ is such a manifold, but we will show that $\int_{S^2}\omega \neq 0$. Observe that

$$g(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

is a parameterization of $S^2$, and also that on $S^2$, $\omega$ is equivalent to $(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy)$. So,

$$\begin{aligned}
\textstyle\int_{S^2}\omega &= \int\limits_{g^{-1}(S^2)} g^*\omega\\
&= \textstyle\int g^*(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy)
\end{aligned}$$

To compute this, we first calculate

$$\begin{aligned}
g^*dx &= -\sin\theta\sin\phi\,d\theta + \cos\theta\cos\phi\,d\phi\\
g^*dy &= \phantom{-}\cos\theta\sin\phi\,d\theta + \sin\theta\cos\phi\,d\phi\\
g^*dz &= -\sin\phi\,d\phi
\end{aligned}$$

Thus,

$$\begin{aligned}
\textstyle\int_{S^2}\omega &= \textstyle\int g^*(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy)\\
&= \textstyle\iint \big[-\cos^2\theta\sin^3\phi - \sin^2\theta\sin^3\phi + \cos\phi(-\cos^2\theta\sin\phi\cos\phi - \sin^2\theta\sin\phi\cos\phi)\big]\,d\theta\,d\phi\\
&\quad \text{(Pythagorean identity 3 times)}\\
&= \int\limits_0^{2\pi} d\theta \int\limits_0^{\pi} -\sin\phi\,d\phi\\
&= 4\pi
\end{aligned}$$

This contradicts our assumption that $\omega$ is exact, so we are done. ■

4. Show that Stokes' Theorem is false if $M$ is not compact.

**Proof** Let $M = \mathbb{R}^2$ and $\omega = x\,dy$, so $\partial M = \emptyset$ and $d\omega = dx\wedge dy$. Then Stokes' Theorem should say that

$$\int\limits_{\mathbb{R}^2} dx\,dy = \int\limits_{\emptyset} x\,dy,$$

but $\int\limits_{\mathbb{R}^2} dx\,dy = \infty$ (that is, the integral diverges) and $\int\limits_{\emptyset} x\,dy = 0$. ■

5. Let $M$ be a compact $k$-manifold without boundary. Show that $\int_M d\omega = 0$ for all $\omega \in \Omega^{k-1}(M)$. Give a counterexample if $M$ is not compact.

**Proof** Since $M$ is compact, it can be parameterized as a $k$-manifold with boundary. To see this, let $\{g_\alpha\}_{\alpha\in\Gamma}$ be a parameterization of $M$. Since $M$ is compact, there is a finite subcollection $\{g_i\}_{i\in 1,\ldots,N}$ which parameterizes $M$. Thus, there is a least element in the set $\{\inf\{x_k : (x_1, \ldots, x_k) \in U_i\} : \forall i\}$ where each $g_i : U_i \to M$. Call this number $\beta$. Then compose each $g_i$ with the translation $\tau_\beta(x_1, \ldots, x_k) = (x_1, \ldots, x_k + |\beta|)$. Now we have a parameterization where all $U_i \subseteq H^k$, so $M$ is a manifold with boundary.

Thus $M$ and $\omega$ satisfy all the criteria for Stokes' Theorem, so

$$\int_M d\omega = \int_{\partial M} \omega = \int_\emptyset \omega = 0.$$

See problem 4 for the requested counterexample. ■

6. Suppose that $C$ is a compact 2-dimensional manifold with boundary in $\mathbb{R}^2$, and assume $(0,0) \notin \partial C$. Let $\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$. Prove that

$$\int_{\partial C} \omega = \begin{cases} 0 & \text{if } (0,0) \in C, \\ 2\pi & \text{if } (0,0) \notin C, \end{cases}$$