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Final Exam

1. Let the sequences $\{a_n\}, \{r_n\} \subset \mathbb{R}$ be such that

$$\sum_{i=1}^{\infty} |a_n| < \infty.$$

Prove that the series

$$\sum_{i=1}^{\infty} \frac{a_n}{\sqrt{|x - r_n|}}$$

converges absolutely for almost every $x \in \mathbb{R}$.

Proof Let $\{a_n\}, \{r_n\}$ be given as above, and let $g_n(x) = \frac{1}{\sqrt{|x-r_n|}}$. We will show that the integral

$$\int_{\alpha}^{\alpha+2} \sum_{n=1}^{\infty} a_n \, g_n(x) \, dx$$

is finite over any region of length 2,[†] which means the series is infinite on a set of measure 0. First note that for any fixed n,

$$\int_{\alpha}^{\alpha+2} g_n \, dx = \int_{\alpha}^{\alpha+2} (x - r_n)^{-1/2} \, dx = \frac{2(\alpha + 2 - r_n)}{\sqrt{|\alpha + 2 - r_n|}} - \frac{2(\alpha - r)}{\sqrt{|\alpha - r|}},$$

and by differentiating with respect to α we find that this value is greatest when

$$0 = \frac{1}{\sqrt{|\alpha - r_n - 2|}} - \frac{1}{\sqrt{|\alpha - r_n|}},$$

which is to say that $\alpha - r_n = 1$. Thus we conclude that

$$\int_{\alpha}^{\alpha+2} g_n \, dx \leq \int_{r_n-1}^{r_n+1} g_n \, dx = \int_{-1}^{1} \frac{1}{\sqrt{t}} \, dt = 4$$

for all $\alpha \in \mathbb{R}$ and all $n \in \mathbb{N}$.

To finish the proof, observe that

$$\int_{\alpha}^{\alpha+2} \sum_{n=1}^{\infty} \frac{|a_n|}{\sqrt{|x - r_n|}} \, dx = \sum_{n=1}^{\infty} \int_{\alpha}^{\alpha+2} |a_n| \, g_n(x) \, dx = \sum_{n=1}^{\infty} |a_n| \int_{\alpha}^{\alpha+2} g_n(x) \, dx \leq 4 \sum_{n=1}^{\infty} |a_n|$$

which is finite. Since the series is infinite only on a zero-measure subset of an arbitrary interval of length 2, then the union of all such subsets also has measure zero, and we're done. ■

[†] That is, for any $\alpha \in \mathbb{R}$.

2. Suppose $f \in L^1[0,1]^2$ with respect to two dimensional Lebesgue measure μ. Prove that if

$$\int_{[0,a],[0,b]} f \, d\mu = 0$$

for all $(a,b) \in [0,1]^2$, then $f = 0$ μ-*a.e.* in $[0,1]^2$.

Proof Since we can decompose f as $f = f^+ - f^-$, then

$$f(x) = 0 \iff f^+(x) = f^-(x) = 0$$

so without loss of generality suppose f is nonnegative.

($\int_R = 0$) Observe that the integral over any rectangle

$$\begin{aligned}\int_{[a,b]\times[c,d]} f \, d\mu &= \int_{[0,b]\times[0,d]} f \, d\mu - \int_{[0,a]\times[0,d]} f \, d\mu - \int_{[0,b]\times[0,c]} f \, d\mu + \int_{[0,a]\times[0,c]} f \, d\mu \\ &= 0 - 0 - 0 + 0.\end{aligned}$$

($\int_U = 0$) Let U be open. For each $n \in \mathbb{N}$, define a cover of $[0,1]^2$ by 2^{2n} squares of side length 2^{-n} and denote it $\{Q_n^i\}_{i=1}^{\infty}$. For any given n, there are finitely many $Q_n^i \subseteq U$, and so the union of all such is a countable union of cubes. To see that it covers U, let $(x,y) \in U$. Since U is open, some neighborhood of (x,y) is a subset of U, and certainly some sufficiently large n gives a cube fully contained in that neighborhood which contains n[†]. Then

$$\int_U f \, d\mu = \sum_{i,n: Q_n^i \subseteq U} \int_{Q_n^i} f \, d\mu = 0.$$

($\int_G = 0$) Let G be a G_δ set, so $G = \bigcap_{i=1}^{\infty} U_i$, where each U_i is open. Then

$$\int_G f \, d\mu = 0$$

[I can't figure out how to prove this part.]

($\int_B = 0$) Let B be any Borel set in $[0,1]^2$. Since μ is Radon, then for each $n \in \mathbb{N}$ there exists an open set U_n such that $B \subseteq U_n$ and $\mu(U_n \setminus B) < \frac{1}{n}$. Since U_n is a decreasing sequence[‡] of sets with $\mu(U_1) \leq \mu[0,1]^2 = 1 < \infty$, then $\mu(U) = \lim_n(\mu(U - n))$ where $U = \bigcap_n U_n$. Thus

$$\begin{gathered} B \subset U, \\ \text{and } \mu(U \setminus B) < \frac{1}{n} \ \forall n, \\ \text{so } \mu(U \setminus B) = 0. \end{gathered}$$

[†] This can be made more rigorous, but this proof is getting absurdly long.

[‡] Although U_{n+1} is not necessarily a subset of U_n by default, we know that they both contain B, and intersecting them yields $\widetilde{U}_{n+1} = U_n \cap U_{n+1}$ so that $B \subset \widetilde{U}_{n+1} \subset U_n$ and $\mu(\widetilde{U}_{n+1} \setminus B) \leq \mu(U_{n+1} \setminus B) < \frac{1}{n}$. Starting with $\widetilde{U}_1 = U_1$ and constructing the rest inductively yields the desired $\{\widetilde{U}_n\}_{n=1}^{\infty}$. We drop the $\sim$ notation above.

Therefore

$$\begin{aligned}\int_B f\,d\mu &= \int_U f\,d\mu - \int_{U\setminus B} f\,d\mu \\ &= \int_U f\,d\mu \\ &= 0 \text{ since } U \text{ is } G_\delta.\end{aligned}$$

($\int_A = 0$) Let $A = \{(x,y) \in [0,1]^2 : f(x,y) > 0\}$, and suppose for contradiction that $\mu(A) > 0$. This gives us that

$$\int_A f\,d\mu > 0.$$

Since μ is Borel-regular, there exists a Borel set $B \supset A$ such that $\mu(B) = \mu(A)$. This means that $\mu(B \setminus A) = 0$. Therefore

$$\begin{aligned}\int_A f\,d\mu &= \int_B f\,d\mu - \int_{B\setminus A} f\,d\mu \\ &= \int_B f\,d\mu \\ &= 0 \qquad\qquad \text{since } B \text{ is Borel.}\end{aligned}$$

This contradicts that $\int_A f\,d\mu > 0$, and so we conclude that $\mu(A) = 0$. ■

3. Let $A \subset X$ be a closed subspace of of a Banach space X and let $k \in X$ be fixed. Is the distance

$$\operatorname{dist}(k, A) = \inf\{||k - a|| \ : \ a \in A\}$$

attained?

Answer: No. As a counterexample, let $X = \ell_\infty$, the space of all bounded sequences of real numbers, with norm $||x|| = \sup_n |x_n|$. Let

$$\begin{array}{rcllll} k &=& 2, & 2, & 2, & 2, \dots \\ a_1 &=& 0.9, & 2, & 2, & 2, \dots \\ a_2 &=& 2, & 0.99, & 2, & 2, \dots \\ a_3 &=& 2, & 2, & 0.999, & 2, \dots \\ & \vdots & & & & \end{array}$$

Observe that $A = \{a_n : n \in \mathbb{N}\}$ is closed, since it has no accumulation points: For any $n, m \in \mathbb{N}$,

$$\begin{aligned} ||a_n - a_m|| &= \Big|\Big|\big(0, 0, \dots, 0, (1 + 10^{-n}), 0, \dots, 0, (1 + 10^{-m}), 0, 0, \dots\big)\Big|\Big| \\ &= \max\{(1 + 10^{-n}), (1 + 10^{-m})\} \\ &\geq 1. \end{aligned}$$

Next note that for every a_n the distance $||k - a_n|| = 1 + 10^{-n}$, but

$$\begin{aligned} \operatorname{dist}(k, A) &= \inf_n ||k - a_n|| \\ &= \inf_n \{1 + 10^{-n}\} \\ &= 1, \end{aligned}$$

so $\operatorname{dist}(k, A)$ is never attained. ∎