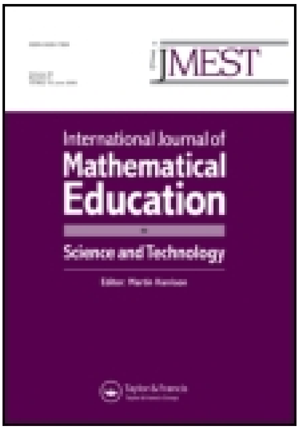

International Journal of Mathematical Education in Science and Technology
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tmes20

The Gibbs phenomenon for series of orthogonal polynomials

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Published online: 30 Nov 2006.

To cite this article: T.H. Fay & P. Hendrik Kloppers (2006) The Gibbs phenomenon for series of orthogonal polynomials, International Journal of Mathematical Education in Science and Technology, 37:8, 973-989, DOI: 10.1080/00207390601083617

To link to this article: http://dx.doi.org/10.1080/00207390601083617

International Journal of Mathematical Education in Science and Technology, Vol. 37, No. 8, 15 December 2006, 973–1000

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Classroom notes

The Gibbs phenomenon for series of orthogonal polynomials

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(*Received 15 January 2003*)

This note considers the four classes of orthogonal polynomials – Chebyshev, Hermite, Laguerre, Legendre – and investigates the Gibbs phenomenon at a jump discontinuity for the corresponding orthogonal polynomial series expansions. The perhaps unexpected thing is that the Gibbs constant that arises for each class of polynomials appears to be the same as that for Fourier series expansions. Each class of polynomials has features which are interesting numerically. Finally a plausibility argument is included showing that this phenomenon for the Gibbs constants should not have been unexpected. These findings suggest further investigations suitable for undergraduate research projects or small group investigations.

1. Introduction

Students in beginning courses on partial differential equations are introduced to the concepts of eigenfunctions and eigenvalues, when employing the method of separation of variables. Sturm–Liouville theory and boundary value problems arise naturally in this context and classes of functions orthogonal with respect to a weight function lead to series expansions of functions, and Fourier series are often introduced for the first time in this way [1].

When dealing with series expansions, one needs to consider convergence and the Fourier series provide an important example for the illustration of difference between uniform and pointwise convergence. Moreover, there is the interesting Gibbs phenomenon that occurs at jump discontinuities, which is easy to display, straightforward to calculate, and provides the motivation for convergence discussions. This has been the thrust of the recent articles [2,3] appearing in this journal.

In the authors' article [4], Fourier–Bessel series are considered and the Gibbs constant is calculated numerically for three examples. This constant appeared to the

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International Journal of Mathematical Education in Science and Technology
ISSN 0020–739X print/ISSN 1464–5211 online © 2006 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/00207390601083617

same as that obtained using Fourier series. Thus it seems natural to investigate other classes of orthogonal functions with regard to the Gibbs phenomenon. In doing so, we are able to observe how quickly (or how slowly) each generalized Fourier series converges and to give some computational shortcuts for finding the series coefficients illustrating that the multitude of relationships that each class of orthogonal polynomials enjoys are useful and not theoretical curiosities.

We give a brief description of each orthogonal polynomial class[1] and study the behaviour of the truncated series representation for a step function having a simple jump discontinuity of height 2. We will see that the Gibbs constant appears to be the same for each class. Finally we will give a plausibility argument to show that we should have expected the Gibbs constant to be exactly the same.

The descriptions of these orthogonal polynomials and their many properties, found in such standard references as Abramowitz and Stegun [5] or Thompson [6], can be overwhelming to the beginning student (see also [7]). By focusing on this geometric problem of the Gibbs phenomenon, students can begin to see the necessity for discovering many of the relationships as computational needs drive much of this theory. The Gibbs phenomenon provides a beautiful and simple application to introduce much of this important material early in the curriculum either by enriching a differential equations class or as a source for undergraduate research projects.

2. The Gibbs phenomenon for Fourier series

Recall that for a periodic, piecewise continuous and piecewise smooth function $f(x)$, there is an effect that occurs near each jump discontinuity of $f(x)$ called the Gibbs effect or Gibbs phenomenon when using truncated Fourier series to approximate $f(x)$. This is most often illustrated with the step function

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 0 & x = -\pi, 0, \pi \\ 1 & 0 < x < \pi \end{cases} \tag{2.1}$$

In figure 1, we plot $f(x)$ and the sum of the first ten terms and the sum of the first 100 terms of the Fourier series for $f(x)$. One can clearly see that although the Fourier series for $f(x)$ converges pointwise to $f(x)$ at each value of x, the 'under- and overshoots' just to the left and right of the jump discontinuities do not disappear as n tends to infinity, but rather tends to a well defined limit which can be shown to be

$$\frac{2}{\pi}\int_0^{\pi} \frac{\sin x}{x}\, dx \approx 1.178979744. \tag{2.2}$$

In this article we shall call this number the Gibbs constant. This effect or phenomenon occurs not just for step functions, but whenever $f(x)$ has a jump discontinuity. We shall investigate a similar Gibbs effect for the truncated sums of series representations employing families of orthogonal polynomials.

[1]More and deeper information and further references about each orthogonal class can be found in Eric Weinstein's *world of* MATHEMATICS at www.mathworld.wolfram.com and at the engineering fundamentals website www.efunda.com.

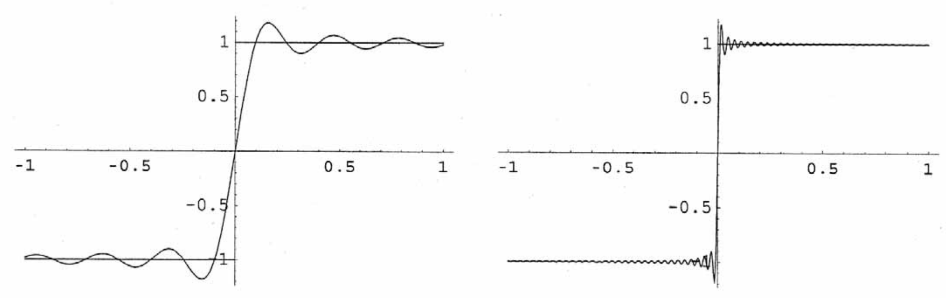

Figure 1. Sums of 10 and 100 terms of the Fourier series.

3. Chebyshev polynomials

The Chebyshev (sometimes written Tchebysheff) polynomials $T_n(x)$ are solutions to Chebyshev's equation

$$(1-x^2)y'' - xy' + \nu^2 y = 0$$

when $\nu = n$, a non-negative integer. They are polynomials of degree n and are odd when n is odd and even when n is even; the first few are listed:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \end{aligned}$$

These polynomials can be generated by the Rodrigues formula

$$T_n(x) = \frac{\sqrt{1-x^2}}{(-1)^n(2n-1)(2n-3)\cdots 1}\frac{d^n}{dx^n}(1-x^2)^{n-1/2}$$

Particular values are $T_n(-1) = (-1)^n$, $T_n(1) = 1$, and $T_n(0) = 0$ if n is odd and $T_n(0) = (-1)^{n/2}$ if n is odd. It is of interest to note that the Chebyshev polynomials $T_n(x)$ arise from the coefficients in the expression $\cos(n\theta)$, if we set $\theta = \arccos(x)$. For example

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= 4\cos^3\theta - 3\cos\theta \end{aligned}$$

It follows that

$$T_n(x) = \cos(n\mathrm{Arc}\cos(x)) \tag{3.1}$$

The Chebyshev polynomials form a complete orthogonal set on the interval $[-1, 1]$ with respect to the weight function $1/\sqrt{1-x^2}$. It can be shown that

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}T_n(x)T_m(x)\,dx = \begin{cases} 0 & \text{for } n \neq m \\ \pi & \text{for } n = m = 0 \\ \pi/2 & \text{for } n = m = 1, 2, \ldots \end{cases}$$

Using this orthogonality, given a piecewise continuous function $f(x)$ defined on $[-1, 1]$, we can form its *Fourier–Chebyshev* series

$$\Omega(x) = \sum_{n=0}^{\infty} c_n T_n(x)$$

where

$$c_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\,dx$$

and for $n > 0$,

$$c_n(x) = \frac{2}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) T_n(x)\,dx \tag{3.2}$$

This series converges to $f(x)$ if f is continuous at the point x, and converges to the average of the left- and right-hand limits if f has a jump discontinuity at x.

3.1. *A simple step function*

We consider the simple step function

$$g(x) = \begin{cases} 1 & -1 \leqslant x < 0 \\ 0 & x = 0 \\ -1 & 0 < x \leqslant 1 \end{cases} \tag{3.3}$$

To investigate the Fourier–Chebyshev series approximations of g, we set

$$\Omega_n(x) = \sum_{k=1}^{n} c_k T_k(x) \tag{3.4}$$

and we plot g and Ω_{50}, and Ω_{500} in figure 2.
A Gibbs phenomenon can be seen quite clearly in these figures.

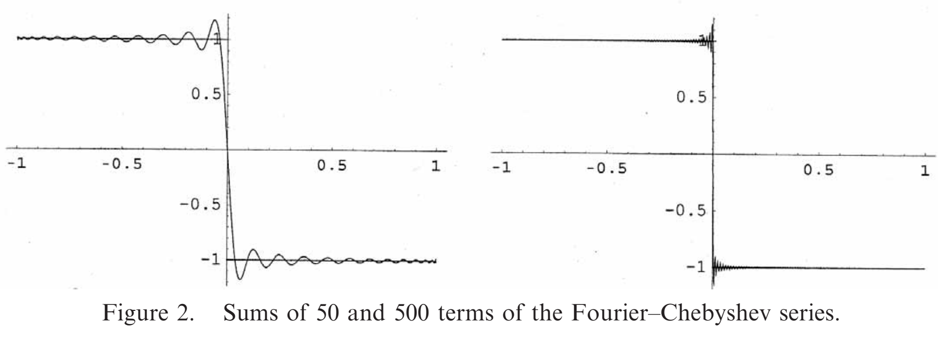

Figure 2. Sums of 50 and 500 terms of the Fourier–Chebyshev series.

For this step function g, the Fourier–Chebyshev coefficients c_n are easy to calculate. Computing,

$$\begin{aligned} c_0 &= \frac{1}{\pi}\left(\int_{-1}^{0}\frac{1}{\sqrt{1-x^2}}T_0(x)\,dx - \int_0^1 \frac{1}{\sqrt{1-x^2}}T_0(x)\,dx\right) \\ &= \frac{1}{\pi}\left(\int_{-1}^{0}\frac{1}{\sqrt{1-x^2}}\,dx - \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx\right) \\ &= \frac{1}{\pi}\left(\frac{\pi}{2}-\frac{\pi}{2}\right) \\ &= 0 \end{aligned} \tag{3.5}$$

In fact, since $g(x)$ is an odd function and $T_{2k}(x)$ is even, each $c_{2k}=0$. More generally, for $n>0$,

$$\begin{aligned} c_n &= \frac{2}{\pi}\left(\int_{-1}^{0}\frac{1}{\sqrt{1-x^2}}T_n(x)\,dx - \int_0^1 \frac{1}{\sqrt{1-x^2}}T_n(x)\,dx\right) \\ &= \frac{2}{\pi}\left(\int_{-1}^{0}\frac{1}{\sqrt{1-x^2}}\cos(n\cos^{-1}x)\,dx - \int_0^1 \frac{1}{\sqrt{1-x^2}}\cos(n\cos^{-1}x)\,dx\right) \\ &= \frac{2}{\pi}\left(\frac{\sin(n\pi)-\sin(n\pi/2)}{n} - \frac{\sin(n\pi)}{n}\right) \\ &= \frac{2\sin(n\pi)-4\sin(n\pi/2)}{n\pi} \\ &= -\frac{4\sin(n\pi/2)}{n\pi} \end{aligned} \tag{3.6}$$

Consequently

$$\begin{aligned} c_n &= \begin{cases} 0, & \text{if } n \text{ is even} \\ -4\sin(n\pi/2)/n\pi, & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} 0, & \text{if } n \text{ is even} \\ -4/n\pi, & \text{if } n = 1,\ 5,\ 9,\ \ldots \\ 4/n\pi, & \text{if } n = 3, 7, 11,\ \ldots \end{cases} \end{aligned} \tag{3.7}$$

Hence

$$c_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ 4/\pi\dfrac{(-1)^k}{2k-1}, & \text{if } n = 2k-1 \text{ is odd} \end{cases} \tag{3.8}$$

The Fourier–Chebyshev series for g is given by

$$\Omega(x) = \sum_{k=1}^{\infty} c_{2k-1}T_{2k-1}(x) \tag{3.9}$$

and the nth partial sum is denoted by

$$\Omega_n(x) = \sum_{k=1}^{n} c_{2k-1} T_{2k-1}(x) \\ = \frac{4}{\pi}\sum_{k=1}^{n} \frac{(-1)^k}{2k-1} T_{2k-1}(x) \tag{3.10}$$

Incidentally, to avoid calculation errors in *Mathematica* when plotting, instead of using the Chebyshev polynomial $T_{2k-1}(x)$ in the summation (3.10), we use the equivalent form $\cos((2k-1)\mathrm{Arc}\cos(x))$ which is more computationally friendly. In this way, it is only time consuming to generate a plot of $\Omega_{500}(x)$, for example.

In order to estimate the Gibbs constant and because of the obvious symmetries, we will investigate the value $\Omega_n(x_n) - 1$ where $(x_n, \Omega_n(x_n))$ is the first turning point (local maximum) to the left of the y-axis. Of course x_n depends very much on the value of n. Hence we need to calculate the derivative of $\Omega_n(x)$ and find its largest negative root. It is straightforward to calculate

$$\Omega_n'(x) = \frac{d}{dx}\left\{\sum_{k=1}^{n} c_{2k-1}\cos((2k-1)\mathrm{Arc}\cos(x))\right\} \\ = \frac{d}{dx}\left\{\frac{4}{\pi}\sum_{k=1}^{n}\frac{(-1)^k}{2k-1}\cos((2k-1)\mathrm{Arc}\cos(x))\right\} \\ = \frac{4}{\pi}\sum_{k=1}^{n}\frac{(-1)^k}{\sqrt{1-x^2}}\sin((2k-1)\mathrm{Arc}\cos(x)) \tag{3.11}$$

Mathematica's **FindRoot** command is useful in determining x_n. Using the statement

$$\textbf{FindRoot}\left[\frac{4}{\pi}\sum_{k=1}^{n}\frac{(-1)^k}{\sqrt{1-x^2}}\sin((2k-1)\mathrm{Arc}\cos(x)) == 0, \{x, m\}\right]$$

where m is a graphical estimate of x_n called a starting value. *Mathematica* uses the secant method to calculate the desired zero or root. Note the double equal signs is *Mathematica* syntax. Other computer algebra systems will have similar routines for Newton's method and the secant method for finding such roots (see [8] for more details).

The values of x_n can be computed with starting values $m = 1/n$. In the following table we record x_n, $\Omega_n(x_n)$, and the over-shoot of $\Omega_n(x_n)$ above the Gibbs constant 1.1789879 for various values of n.

n	x_n	$\Omega_n(x_n)$	over-shoot
10	0.309017	1.182328	0.0033485
50	0.062791	1.179113	0.0001333
100	0.031411	1.178013	0.0000333
200	0.015707	1.178988	0.0000083
500	0.006283	1.178981	0.0000013
1000	0.003142	1.178980	0.0000003

In this table, we see that as n increases, the over-shoot decreases, and that the values of $\Omega_n(x_n)$ appear to be converging to the number 1.178979 which is the Gibbs constant rounded to 6 decimal places.

4. Hermite polynomials

The Hermite polynomials are polynomial solutions to Hermite's differential equation

$$y'' - 2xy' + 2\nu y = 0 \tag{4.1}$$

for $\nu = n$, a non-negative integer, and are often denoted by $H_n(x)$. The first few are

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x$$

These polynomials can be generated by the Rodrigues' formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \text{ where } n = 0,\ 1,\ 2,\ \ldots$$

The Hermite polynomials enjoy a symmetry condition

$$H_n(-x) = (-1)^n H_n(x) \tag{4.2}$$

Each $H_n(x)$ is a polynomial of degree n and is an odd function if n is odd and an even function if n is even.

There are a number of identities relating the H_n and their derivatives, again too many to list here, but we note that

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{4.3}$$

and

$$H_n'(x) = 2nH_{n-1}(x) \tag{4.4}$$

so that

$$\int_0^x H_n(t)dt = \frac{1}{2(n+1)}(H_{n+1}(x) - H_{n+1}(0)) \tag{4.5}$$

and finally

$$\frac{d}{dx}\left\{e^{-x^2} H_n(x)\right\} = -2xe^{-x^2} H_n(x) + 2ne^{-x^2} H_{n-1}(x) \tag{4.6}$$

so that

$$\int_0^x e^{-t^2} H_n(t)\, dt = H_{n-1}(0) - e^{-x^2} H_{n-1}(x) \tag{4.7}$$

Because e^{-x^2} is an even function and the parity of $H_n(x)$ is either odd or even, it follows that

$$\int_{-x}^0 e^{-t^2} H_n(t)\, dt = (-1)^n (H_{n-1}(0) - e^{-x^2} H_{n-1}(x)) \tag{4.8}$$

These polynomials are orthogonal with respect to the weight function $w(x) = e^{-x^2}$, and can be shown to form a complete orthogonal system over the interval $(-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \begin{cases} 0 & \text{for } n \neq m \\ 2^n \cdot n! \sqrt{\pi} & \text{for } n = m \end{cases} \tag{4.9}$$

Thus these polynomials can be used to produce a generalized Fourier series called *Fourier–Hermite* series. Given a function $f(x)$ defined over $(-\infty, \infty)$, the Fourier–Hermite series for $f(x)$ is

$$\Psi(x) = \sum_{n=0}^{\infty} c_n H_n(x) \tag{4.10}$$

where

$$c_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) f(x)\, dx \tag{4.11}$$

4.1. *A simple step function*

To investigate the Gibbs phenomenon for Fourier–Hermite series, we consider the step function

$$f(x) = \begin{cases} 0 & \text{for } |x| > 1 \\ -1 & \text{for } -1 < x < 0 \\ 1 & \text{for } 0 < x < 1 \end{cases} \tag{4.12}$$

In addition, the values $f(0) = 0$ and $f(\pm 1) = \pm 1/2$ are chosen so that they are the average of the left- and right-hand limits. We chose to define $f(x) = 0$ for $|x| > 1$ in order that the Fourier–Hermite series coefficients are simple to compute numerically in a timely fashion. The coefficients c_n of the Fourier–Hermite series for $f(x)$ are given by

$$\begin{aligned} c_n &= \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) f(x)\, dx \\ &= \frac{1}{2^n n! \sqrt{\pi}} \left\{ \int_0^1 e^{-x^2} H_n(x)\, dx - \int_{-1}^0 e^{-x^2} H_n(x)\, dx \right\} \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-2}{2^n n! \sqrt{\pi}} (H_{n-1}(0) - e^{-1} H_{n-1}(1)) & \text{if } n \text{ is odd} \end{cases} \end{aligned} \tag{4.13}$$

The integrals within the curly braces are not difficult to evaluate using equation (4.8). In figure 3, we show $f(x)$ and the sum of the first 50 and first 200 (non-zero) terms of its Fourier–Hermite series.

We let

$$\Psi_n(x) = \sum_{k=1}^{n} c_{2k-1} H_{2k-1}(x)$$

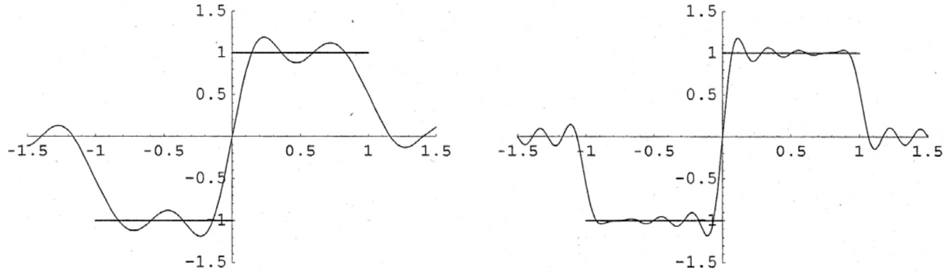

Figure 3. Sums of 50 and 200 terms of the Fourier–Hermite series.

and this time let x_n denote the abscissa of the first turning point to the left of zero for the truncated Fourier–Hermite series. In the following table, we compute the difference between the over-shoot $\Psi_n(x_n)$ and the Gibbs constant 1.178979.

n	x_n	$\Psi_n(x_n)$	difference
50	−0.305234	1.177162	−0.001818
100	−0.233224	1.187841	0.008861
500	−0.099751	1.181217	0.002237
1000	−0.070905	1.180046	0.001066

Clearly the over-shoots $\Psi_n(x_n)$ appear to be converging to 1.178979 but note the slowness of the convergence.

5. Laguerre polynomials

Polynomial solutions to Laguerre's differential equation

$$xy'' + (1-x)y' + \nu y = 0 \tag{5.1}$$

for $\nu = n,$ a non-negative integer, and are often denoted by $L_n(x)$. The first few are

$$L_0(x) = 1$$

$$L_1(x) = 1 - x$$

$$L_2(x) = \frac{1}{2}(2 - 4x + x^2)$$

$$L_3(x) = \frac{1}{6}(6 - 18x + 9x^2 - x^3)$$

These polynomials can be generated by the Rodrigues' formula

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}) \text{ where } n = 0, 1, 2, \ldots \tag{5.2}$$

Each $L_n(x)$ is a polynomial of degree n and $L_n(0) = 1$ for all n.

There are a number of identities relating the L_n and their derivatives. It will be useful to note that the Laguerre polynomials satisfy a recursion relation

$$(n+1)L_{n+1}(x) = (2n + 1 - x)L_n(x) - nL_{n-1}(x) \tag{5.3}$$

and

$$xL_n'(x) = nL_n(x) - nL_{n-1}(x) \tag{5.4}$$

and

$$L_n'(x) = L_{n-1}'(x) - L_{n-1}(x) \qquad (5.5)^2$$

From these equations it follows that

$$\int_0^x L_n(t)\,dt = L_n(x) - L_{n+1}(x) \tag{5.6}$$

Integrating by parts the integral

$$\int e^{-x} L_n(x)\,dx$$

it follows that

$$\int e^{-x} L_{n+1}(x)\,dx = e^{-x}(L_n(x) - L_{n+1}(x))$$

so that

$$\int_0^a e^{-x} L_n(x)\,dx = e^{-a}(L_{n-1}(a) - L_n(a)) \tag{5.7}$$

These polynomials are orthogonal with respect to the weight function $w(x) = e^{-x}$:

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\,dx = \begin{cases} 0 & \text{for } n \neq m \\ 1 & \text{for } n = m \end{cases} \tag{5.8}$$

and can be shown to form a complete orthogonal system over the interval $[0, \infty)$. Thus a generalized Fourier series called the *Fourier–Laguerre* series for a function $f(x)$ defined over $[0, \infty)$ is given by

$$\Phi(x) = \sum_{n=0}^{\infty} c_n L_n(x) \tag{5.9}$$

where

$$c_n = \int_0^\infty e^{-x} L_n(x) f(x)\,dx \tag{5.10}$$

5.1. *A simple step function*

We study the step function

$$f(t) = \begin{cases} 1 & \text{for } 0 < t < 1 \\ -1 & \text{for } 1 < t < 2 \\ 0 & \text{for } 2 < t \end{cases}$$

[2]The formula given on the website www.efunda.com/math/Laguerre is incorrect.

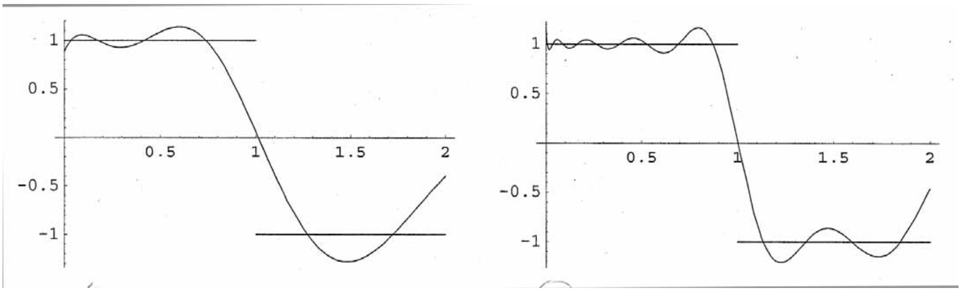

Figure 4. Sums of 50 and 200 terms of the Fourier–Laguerre series.

where we assume for simplicity, $f(1) = 0$ and $f(2) = -1/2$, so that $f(t)$ is defined over the interval $(0, \infty)$ and equals its Fourier–Laguerre series (the convergence is pointwise). The coefficients of the Fourier–Laguerre series for $f(x)$ are easy to compute. It follows from equation (5.7) that

$$\begin{aligned} c_n &= \int_0^\infty e^{-x} f(x) L_n(x)\, dx \\ &= \int_0^1 e^{-x} L_n(x)\, dx - \int_1^2 e^{-x} L_n(x)\, dx \\ &= \frac{2}{e}(L_{n-1}(1) - L_n(1)) - \frac{1}{e^2}(L_{n-1}(2) - L_n(2)) \end{aligned}$$

The nth partial sum of the Fourier–Laguerre series is denoted by $\Phi_n(x)$. The plots shown in figure 4 of $\Phi_{50}(x)$ and $\Phi_{200}(x)$ overlaid with a plot of $f(x)$ indicates the convergence to be relatively slow.

In a similar manner as above, we let x_n denote the abscissa of the local maximum occurring just to the left of the vertical line $x = 1$. These values are computed as before by using the **FindRoot** command of *Mathematica* applied to

$$\begin{aligned} \Phi'_n(x) &= \sum_{k=0}^{n} c_k L'_k(x) \\ &= \sum_{k=0}^{n} c_k \frac{k}{x}(L_k(x) - L_{k-1}(x)) \end{aligned}$$

We record the values of x_n, $\Phi_n(x_n)$, and the difference between $\Phi_n(x_n)$ and the Gibbs constant 1.178979, for various values of n.

n	x_n	$\Phi_n(x_n)$	difference
50	0.600498	1.139897	−0.039083
100	0.699299	1.190787	0.011807
200	0.796707	1.165550	−0.013430
500	0.865052	1.179966	0.000986

6. Legendre polynomials

Legendre polynomials arise from *Legendre's differential equation*

$$(1 - x^2)y'' - 2xy' + \nu(\nu + 1)y = 0 \tag{6.1}$$

where ν is a real number. If $\nu = n$, a non-negative integer, then certain solutions, denoted $P_n(x)$, turn out to be polynomials of degree n and are called *Legendre polynomials*. The first four Legendre polynomials are:

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2} \\
P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x
\end{aligned}
$$

These polynomials can be generated by the formula of Rodrigues:

$$P_0(x) = 1, \;\; P_n(x) = \frac{1}{2^n n!}\frac{d}{dx}(x^2 - 1)^n$$

where $n = 1, 2, \ldots$. Each $P_n(x)$ is a polynomial of degree n and is an odd function if n is odd and an even function if n is even. Moreover, $P_n(1) = 1$ and $P_n(-1) = (-1)^n$ for all $n \geq 0$.

There are a number of identities relating the P_n and their derivatives, we note

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \tag{6.2}$$

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \tag{6.3}$$

$$(x^2 - 1)P'_n(x) = nxP_n(x) - nP_{n-1}(x) \tag{6.4}$$

so that

$$\int P_n(x)\,dx = \frac{1}{2n+1}\left\{P_{n+1}(x) - P_{n-1}(x)\right\} \tag{6.5}$$

and finally,

$$P_n(0) = \begin{cases} 0 & \text{for } n \text{ odd} \\ (-1)^{n/2} \cdot \dfrac{1 \cdot 3 \cdot 5 \cdots (n-1)}{2 \cdot 4 \cdot 6 \cdots n} & \text{for } n \text{ even} \end{cases} \tag{6.6}$$

These polynomials are orthogonal with respect to the weight function $w(x) = 1$:

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0 & \text{for } n \neq m \\ \dfrac{2}{2n+1} & \text{for } n = m \end{cases} \tag{6.7}$$

and can be shown to form a complete orthogonal system over the interval $[-1, 1]$. For more details, see [7, Chapter 7]. Thus these polynomials can be used to produce a generalized Fourier series called a *Fourier–Legendre* series. Given a function $f(x)$ defined over $[-1, 1]$, the Fourier–Legendre series for a function $f(x)$ is

$$\Lambda(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

where

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} P_n(x)f(x)\,dx \tag{6.8}$$

It is of interest to note that there is another way to think about the Legendre polynomials. Given a sequence of functions $g_n(x)$, having no finite number linearly dependent, the Gram–Schmidt orthogonalization process can be used to produce an orthogonal system $\phi_n(x)$ where $\phi_1(x) = g_1(x)$, $\phi_2(x) = c_1 g_1(x) + c_2 g_2(x), \ldots,$ where the constants $c_1, c_2, \ldots$ are determined by the process. The choice of $g_n(x) = x^n$ in this process produces the Legendre polynomials (up to constant multiples).

6.1. *A simple step function*

We consider the simple example of a step function

$$g(x) = \begin{cases} 1 & -1 \leqslant x < 0 \\ 0 & x = 0 \\ -1 & 0 < x \leqslant 1 \end{cases} \tag{6.9}$$

To investigate the Fourier–Legendre series approximations of g, we set

$$\Lambda_n(x) = \sum_{k=1}^{n} c_k P_k(x) \tag{6.10}$$

and we plot g and Λ_{10}, and Λ_{100} in Figure 5.

For this step function g, the Fourier–Legendre coefficients c_n are also easy to calculate. First note that since $g(x)$ is an odd function and $P_{2k}(x)$ is even, each $c_{2k} = 0$. It follows from equations (6.2) and (6.6) that

$$\begin{aligned} c_{2k-1} &= \frac{4k-1}{2}\left\{\int_{-1}^{0} P_{2k-1}(x)\,dx - \int_{0}^{1} P_{2k-1}(x)\,dx\right\} \\ &= \frac{4k-1}{2}\left\{\frac{P_{2k-2}(-1) - 2P_{2k-2}(0) - P_{2k}(-1) + 2P_{2k}(0)}{4k-1}\right\} \\ &= P_{2k}(0) - P_{2k-2}(0) \end{aligned} \tag{6.11}$$

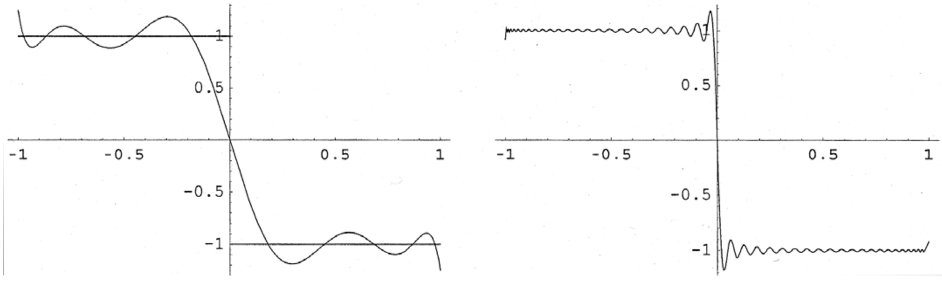

Figure 5. Sums of 10 and 100 terms of the Fourier–Lagendre series.

We can simplify this a bit by taking advantage of the formula (6.6),

$$\begin{aligned}
P_{2k}(0) - P_{2k-2}(0) &= (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!} - (-1)^{k-1}\frac{1\cdot 3\cdot 5\cdots(2k-3)}{2^{k-1}(k-1)!} \\
&= (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-3)\cdot(2k-1)}{2^k k!} \\
&\quad + (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-3)\cdot 2k}{2^{k-1}(k-1)!\cdot 2k} \\
&= (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-3)}{2^k k!}\{2k-1+2k\} \\
&= (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-3)}{2^{k-1}(k-1)!}\left\{\frac{4k-1}{2k}\right\}
\end{aligned}$$

Hence it follows that

$$P_{2k}(0) - P_{2k-2}(0) = -\frac{4k-1}{2k}P_{2k-2}(0) \tag{6.12}$$

so

$$\begin{aligned}
c_{2k-1} &= -\frac{4k-1}{2k}P_{2k-2}(0) \\
&= \left\{\frac{1}{2k}-2\right\}P_{2k-2}(0)
\end{aligned} \tag{6.13}$$

The Fourier–Legendre series for g is given by

$$\Lambda(x) = \sum_{k=1}^{\infty} c_{2k-1}P_{2k-1}(x) \tag{6.14}$$

and the nth partial sum is

$$\Lambda_n(x) = \sum_{k=1}^{n}\left\{\frac{1}{2k}-2\right\}P_{2k-2}(0)P_{2k-1}(x) \tag{6.15}$$

The value of x to the immediate left of 0 for which $\Lambda_n(x)$ is a maximum is denoted x_n. We investigate the over-shoot $\Lambda_n(x_n)$.

n	x_n	$\Lambda_n(x_n)$	difference
10	−0.295758	1.186753	0.007733
50	−0.062179	1.179307	0.000327
100	−0.031256	1.179062	0.000083
200	−0.015668	1.179000	0.000021

Once again the over-shoots appear to be converging to the Gibbs constant.

7. A plausibility argument

Let us suppose we are given a countable family of integrable functions $\{\phi_n(x)\}$ defined over a closed interval $I=[a,b]$ which are orthogonal with respect to an integrable weight function $w(x)$ and so that given an integrable function $f(x)$ defined on I, we have a generalized Fourier series expansion

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \tag{7.1}$$

where each c_n is given by

$$c_n = \int_a^b f(x) w(x) \phi_n(x)\, dx \tag{7.2}$$

and the convergence is uniform over any subinterval of I not containing point at which f has a jump discontinuity. We assume that the Fourier series for f is given by

$$f(x) = \sum_{n=0}^{\infty} \{a_n \cos(nx) + b_n \sin(nx)\} \tag{7.3}$$

as usual (here we have identified f with its periodic extension so that f is defined over the entire real line). In this case the Fourier coefficients of f are given by

$$a_n = \frac{1}{b-a} \int_a^b f(x) \cos(nx)\, dx \quad \text{for } n \geq 1$$

$$b_n = \frac{1}{b-a} \int_a^b f(x) \sin(nx)\, dx \quad \text{for } n \geq 1$$

and

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\, dx$$

Each of the ϕ_n has a Fourier series representation

$$\phi_n(x) = \sum_{k=0}^{\infty} \{a_k^n \cos(kx) + b_k^n \sin(kx)\} \tag{7.4}$$

Truncating equation (7.1), we have

$$f(x) \approx \sum_{n=1}^{m} c_n \phi_n(x) \tag{7.5}$$

and using a similar truncation in equation (7.4) we have

$$\begin{aligned} f(x) &\approx \sum_{n=1}^{m} c_n \sum_{k=0}^{m} \{a_k^n \cos(kx) + b_k^n \sin(kx)\} \\ &= \sum_{n=1}^{m} \sum_{k=0}^{m} \{c_n a_k^n \cos(kx) + c_n b_k^n \sin(kx)\} \end{aligned}$$

Since both sums are finite, we can commute them and obtain

$$\begin{aligned} f(x) &\approx \sum_{k=0}^{m} \sum_{n=1}^{m} \{c_n a_k^n \cos(kx) + c_n b_k^n \sin(kx)\} \\ &= \sum_{k=0}^{m} \left\{ \left(\sum_{n=1}^{m} c_n a_k^n \right) \cos(kx) + \left(\sum_{n=1}^{m} c_n b_k^n \right) \sin(kx) \right\} \end{aligned} \tag{7.6}$$

Equation (7.6) appears to be 'almost' a partial sum of the Fourier series for f. Thus it is 'plausible' that

$$f(x) = \lim_{m\to\infty} \sum_{k=0}^{m} \left\{ \left(\sum_{n=1}^{m} c_n a_k^n \right) \cos(kx) + \left(\sum_{n=1}^{m} c_n b_k^n \right) \sin(kx) \right\} \tag{7.7}$$

$$= \sum_{k=0}^{\infty} \left\{ \left(\sum_{n=1}^{\infty} c_n a_k^n \right) \cos(kx) + \left(\sum_{n=1}^{\infty} c_n b_k^n \right) \sin(kx) \right\}$$

By the uniqueness to the Fourier series representation, we have

$$a_k = \left(\sum_{n=1}^{\infty} c_n a_k^n \right) \text{ and } b_k = \left(\sum_{n=1}^{\infty} c_n b_k^n \right) \tag{7.8}$$

Hence the truncated generalized Fourier series representations for f are 'almost' truncated Fourier series for f and hence the Gibbs constants should be the same.

If we assume uniform convergence of both the generalized Fourier series in equation (7.1) and the Fourier series for f in equation (7.4) (which occurs if f is continuous), then for $k > 0$ we have

$$\begin{aligned} a_k &= \frac{1}{b-a}\int_a^b f(x)\cos(kx)\,dx = \frac{1}{b-a}\int_a^b \sum_{n=1}^{\infty} c_n\phi_n(x)\cos(kx)\,dx \\ &= \frac{1}{b-a}\sum_{n=1}^{\infty}\int_a^b c_n\phi_n(x)\cos(kx)\,dx \\ &= \sum_{n=1}^{\infty}\frac{c_n}{b-a}\int_a^b \phi_n(x)\cos(kx)\,dx \\ &= \sum_{n=1}^{\infty} c_n a_k^n \end{aligned}$$

Replacing a_k by $a_0/2$ in the above calculation shows that

$$\frac{a_0}{2} = \sum_{n=1}^{\infty} c_n a_0^n$$

A similar calculation for the b_k has

$$b_k = \sum_{n=1}^{\infty} c_n b_k^n$$

This shows that indeed, under the uniform convergence assumption, equation (7.7) is the Fourier series for f.

8. Conclusions

While our numerical investigations are not absolutely conclusive, we believe that the Gibbs constant for a jump discontinuity for Fourier, Fourier–Bessel, Fourier–Chebyshev, Fourier–Hermite, Fourier–Laguerre and Fourier–Legendre are all the same. We believe there is a fundamental principle involved here and think that the plausibility argument can be made rigorous. We hope that these numerical investigations provide the impetus for further student investigations including the following:

- Provide a theoretical proof using ε's for the plausibility argument showing that the Gibbs constants are the same.
- Explain exactly where and why the hypothesis of integrability is needed.
- Investigate the slowness of the convergence of each of the generalized Fourier series representations as compared to the Fourier series representation of each of the step functions used above.
- Which of the generalized Fourier series representations offer computational problems? How long does it take to compute (and plot) these approximations say for $n = 1000$, $n = 2000$, etc. What is the trade-off between computing time and desired accuracy?
- There are other collections of orthogonal polynomials, such as the Jacobi polynomials, forming a complete orthogonal system over an appropriate interval, which can produce generalized Fourier series. And of course there are general Sturm–Liouville series. What is the situation for these series with respect to a Gibbs phenomenon? Are there numerical shortcuts for the rapid calculation of the generalized Fourier coefficients?

We believe that numerical investigations, such as we have outlined above, can be effective in illustrating the need for theoretical work and that the one can go hand-in-hand with the other. The Gibbs phenomenon provides a beautiful visual problem to motivate an investigation into much of the general theory in beginning analysis and approximation theory in particular.

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