Welcome To Math 34A!
Differential Calculus

Instructor:

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South Hall 6431X (Grad Tower, 6th floor, blue side, first door on the right)

Office Hours:

MTWR after class 2:00-3:00, and by appointment. Details on Gauchospace.

Warm-up

- $\log(x) = 5$ $\quad x = \boxed{10^5} = \boxed{100,000}$
- $10^x = 1,000,000$ $\quad x = \boxed{\log(10^6)} = \boxed{6}$
- $\log(\log(x)) = 2$ $\quad x = \boxed{10^{10^2}} = \boxed{10^{100}}$
- $10^x = 2$ $\quad x = \boxed{\log(2)} \approx \boxed{.3}$
- $10^x = 8700$ $\quad x = \boxed{\log(8700)} \approx ?$

  $8700 = 8.7 \cdot 10^3$ $\quad \log(8700) = \log(8.7) + 3$
- $10^{4x-5} = 7$ $\quad x = \boxed{(\log(7) + 5)/4} \approx \boxed{?}$ just leave it that way

Logarithm Strategy

- $4^{2x+1} = 3$ $\qquad x = \boxed{(\log_4(3) - 1)/2} = (\frac{\log(3)}{\log(4)} - 1)/2$

In general,

$$\boxed{\log_b(x) = \frac{\log(x)}{\log(b)}}$$

Midterm 2: Wednesday

Bring:

- A pen or sharp pencil.
- A 3" $\times$ 5" card with your notes.
- Student ID.

Don't bring:

- A calculator

No bluebook or scratch paper necessary, just the above materials and hopefully a fresh, well-practiced you! Scratch paper will be provided.

Midterm 2 Topics

- All topics from Midterm 1
- Sums (like the example below, more examples on Gauchospace)

$$\sum_{n=1}^{4} 2^n - 1$$

- Advanced Logarithm Methods (the full chapter on logarithms in the book)
- Change and Average Rate of Change for a function or graph.
- Limits with h (used to find exact speed, examples on the old midterm and extra problems)

If you struggled on Midterm 1 with algebra or word problems, you need to improve these skills immediately. **They are essential for success in this course**.

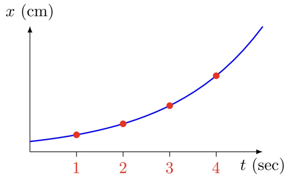
Midterm 2 Topics

To refresh your memory, let's do this example right now:

$$\sum_{n=1}^{4} 2^n - 1$$

It would be a great idea to look back over Lecture 6 to prepare for the midterm.

Speed=Slope=Derivative



The graph shows the distance from the origin in cm after t seconds of a hamster. Which of the numbers below is the largest?

Hint: Speed is a slope!

A = speed of the hamster at $t = 1$
B = speed of the hamster at $t = 2$
C = speed of the hamster at $t = 3$
D = average speed of the hamster between $t = 2$ and $t = 3$
E = average speed of the hamster between $t = 3$ and $t = 4$

- Sums (like the example below, more examples on Gauchospace)

end

$$\sum_{n=1}^{4} 2^n - 1$$

Formula for each term

start

$$= (2^1 - 1) + (2^2 - 1) + (2^3 - 1) + (2^4 - 1)$$
$$= 2 - 1 + 4 - 1 + 8 - 1 + 16 - 1$$
$$= 1 + 3 + 7 + 15$$
$$= \boxed{26}$$

Context: Population

Suppose $f(t)$ = the population of the ancient city of Lyrad in year t. We are told that $f(1550) = 1820$ and $f'(1650) = 1100$. Which of the following is true?

A In 1550, the population was 1820 and rising at a rate of 1100 people per year

B In 1650, the population was 1100 more than in 1550

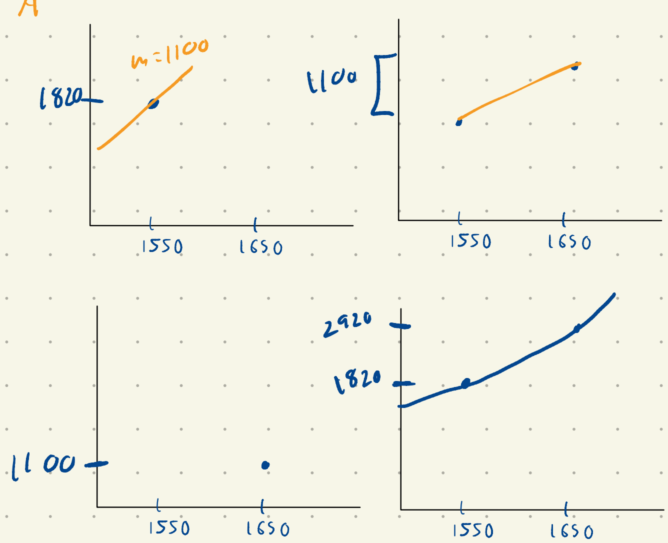
C In 1650, Lyrad contained 1100 people

D In 1550, there were 1820 people in Lyrad, and by 1650 this had increased to 2920

E None of above

In the year 1550, Lyrad had a population of 1820 people, and in the year 1650, the population was increasing at a rate of 1100 people per year.

A



Derivatives & Differential Calculus

. . . are about how quickly things change.

- Need to understand PRACTICAL significance in various situations

  Spread of infectious disease, population growth, speed, acceleration, marginal rates in economics, global warming

- Calculate (or estimate) rate of change from various sources:

  graph
  table of data
  formula

- Applications:

  measure change
  predict the future
  optimization – find the best, or smallest, or biggest, or most. . .

This is all about *understanding* the world.

Philosophical problem

How quickly is something changing at one moment in time?

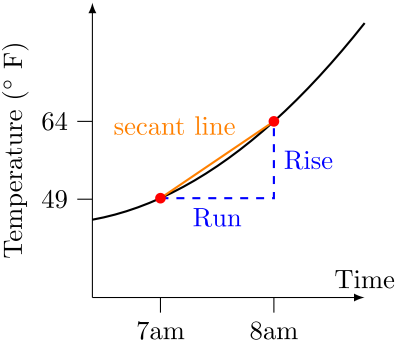
Example: Does a ball stop when I throw it straight up?

Example: How fast is the temperature rising at 7am?

$$\begin{pmatrix} \text{change in temp} \\ \text{between 7am \& 8am} \end{pmatrix} = 64 - 49 = 15^\circ \text{ F}$$

$$\begin{pmatrix} \text{average rate of} \\ \text{change in temp} \\ \text{between 7am \& 8am} \end{pmatrix} = \frac{15^\circ \text{ F}}{1 \text{ hour}} = 15^\circ \text{ F/hour}$$

= slope of secant line



Continuing Example

Similarly,

$$\begin{pmatrix} \text{average rate of} \\ \text{change in temp} \\ \text{between 6am \& 8am} \end{pmatrix} = \frac{\text{change in temp}}{\text{time taken}}$$

Question: Suppose temperature at time t given by the formula $f(t) = t^2$. What is the average rate of change of temperature from 6am to 8am?

A= 1 B= 7 C= 9 D= 14 E= 28 $\boxed{\text{D}}$

Average Rate of Change

Suppose temperature at time t given by the formula $f(t) = t^2$. Using a calculator one can find the average rate of change over shorter and shorter time spans Δt, starting at 7am:

Δt	$(f(7+\Delta t) - f(7))/\Delta t$	ave rate of change ^{o}F/hr	
1	$(8^2 - 7^2)/1$	15	
0.1	$(7.1^2 - 7^2)/0.1$	14.1	
0.01	$(7.01^2 - 7^2)/0.01$	14.01	
0.001	$(7.001^2 - 7^2)/0.001$	14.001	
0.0001	$(7.0001^2 - 7^2)/0.0001$	14.0001	
0.00001	$(7.00001^2 - 7^2)/0.00001$	14.00001	
0	$(7^2 - 7^2)/0$	0/0	arghhhh

Table: Average rate of change over various time spans

What would you guess the exact instantaneous rate of change of temperature at precisely 7am is? Yes! $\boxed{14}$. But how do we get this? Answer: it is a limit!

Instantaneous Rate of Change

What does the limit

$$\lim_{\Delta t \to 0} \frac{f(7+\Delta t) - f(7)}{\Delta t}$$

mean in practice?

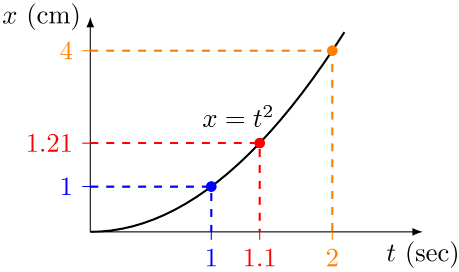
Work out the average rate of change over a very short time interval.
That is very nearly the correct answer.
The shorter the time interval you use, the more accurate you expect the answer to be.

To get the exact answer you would need to take a time interval of zero length.
This leads to the nonsense 0/0. So you can't do this.
That is the philosophical problem.

Mathematical solution: take the limit.

An Example

A hamster runs along the x-axis, so that after t seconds the hamster is t^2 cm from the origin. Our goal is to find the hamster's speed at time $t = 1$ sec.



$$\left(\begin{array}{c}\text{average speed from}\\ t = 1 \text{ to } t = 2\end{array}\right) = \frac{\text{distance gone}}{\text{time taken}} = \frac{2^2 - 1^2}{2 - 1} = 3 \ \text{ cm/sec}$$

$$\left(\begin{array}{c}\text{average speed from}\\ t = 1 \text{ to } t = 1.1\end{array}\right) = \frac{\text{distance gone}}{\text{time taken}} = \frac{1.1^2 - 1^2}{1.1 - 1} = 2.1 \text{ cm/sec}$$

Example Concluded

How do we work out the exact speed of the hamster after 1 second?

Plan:

- Find the average speed over a short time interval Δt, then
- Take the limit as $\Delta t \to 0$.

$$\begin{aligned}
\begin{pmatrix} \text{average speed from} \\ t = 1 \text{ to } t = 1 + \Delta t \end{pmatrix} &= \frac{\text{distance gone}}{\text{time taken}} \\
&= \frac{(1+\Delta t)^2 \; - \; 1^2}{(1+\Delta t) - 1} \\
&= \frac{(1 + 2\Delta t + (\Delta t)^2) \; - \; 1}{\Delta t} \\
&= \frac{2\Delta t + (\Delta t)^2}{\Delta t} \\
&= 2 + \Delta t
\end{aligned}$$

The limit of this as $\Delta t \to 0$ is 2.

Conclusion: At $t = 1$ sec, the exact speed of the hamster is 2 cm/sec.

Hamster Summary

Soon we will calculate that...

> the exact speed of the hamster after t seconds is $2t$ cm/sec.

Summary:

$f(t) = t^2$ = distance in cm of hamster from origin after t seconds

(a function that gives the distance the hamster has traveled at time t)

$f'(t) = 2t$ = speed of hamster in cm/sec after t seconds

(called the derivative of t^2 because it can be derived or obtained from the function t^2)

Question: How many cm had the hamster run by the time its speed was 8 cm/sec?

A= 4 B= 8 C= 16 D= 32 E= 64 $\boxed{\text{C}}$

Hamster Summary

Soon we will calculate that...

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$\boxed{f(t) = t^2}$ = distance in cm of hamster from origin after t seconds
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Question: How many cm had the hamster run by the time its speed was 8 cm/sec?

A= 4 B= 8 C= 16 D= 32 E= 64

$$f'(t) = 8 = 2t$$

$$t = 4$$

$$f(4) = 4^2$$

$$= \boxed{16 \text{ cm}}$$

$$\frac{2\Delta t - (\Delta t)^2}{\Delta t}$$

$$\frac{\cancel{\Delta t}(2 - \Delta t)}{\cancel{\Delta t}}$$

Exact Hamster Speed

Now we calculate that...

> the exact speed of the hamster after t seconds is $2t$ cm/sec.

Do this as before: working out the average speed over a short time interval Δt and taking the limit as $\Delta t \to 0$

$$\begin{aligned}\begin{pmatrix}\text{average speed from}\\ t \text{ to } t+\Delta t\end{pmatrix} &= \frac{\text{distance gone}}{\text{time taken}}\\ &= \frac{(t+\Delta t)^2 \ - \ t^2}{(t+\Delta t)-t}\\ &= \frac{(t^2+2t\Delta t+(\Delta t)^2) \ - \ t^2}{\Delta t}\\ &= \frac{2t\Delta t+(\Delta t)^2}{\Delta t}\\ &= 2t+\Delta t\end{aligned}$$

The limit of this as $\Delta t \to 0$ is $2t$.

$$f(t) = t^2$$

$$f'(t) = 2t$$

take limit as $\Delta t \to 0$ of slope between t and $t + \Delta t$:

$$\lim_{\Delta t \to 0} \frac{f_2 - f_1}{t_2 - t_1} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{(t + \Delta t) - t}$$

Exact Hamster Speed

Now we calculate that...

the exact speed of the hamster after t seconds is $2t$ cm/sec.

Do this as before: working out the average speed over a short time interval Δt and taking the limit as $\Delta t \to 0$

$$\begin{aligned}
\begin{pmatrix} \text{average speed from} \\ t \text{ to } t + \Delta t \end{pmatrix} &= \frac{\text{distance gone}}{\text{time taken}} \\
&= \frac{(t+\Delta t)^2 \ - \ t^2}{(t+\Delta t) - t} \\
&= \frac{(t^2 + 2t\Delta t + (\Delta t)^2) \ - \ t^2}{\Delta t} \\
&= \frac{2t\Delta t + (\Delta t)^2}{\Delta t} \\
&= 2t + \Delta t
\end{aligned}$$

The limit of this as $\Delta t \to 0$ is $2t$.

Hamster Questions!

After t seconds, the hamster is $f(t) = t^2$ cm from origin.
(1) What is the exact speed (in cm/sec) of the hamster at $t = 2$?

A= 1 B= 2 C= 4 D= 6 E= 8 $\boxed{\text{C}}$

(2) What is the exact speed (in cm/sec) of the hamster at $t = 4$?

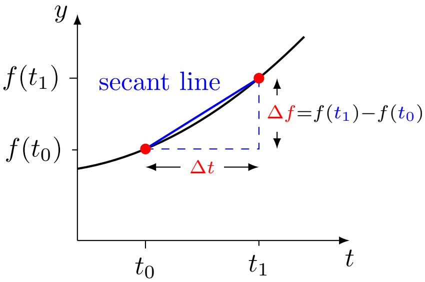
A= 1 B= 2 C= 4 D= 6 E= 8 $\boxed{\text{E}}$

(3) What is the average speed (in cm/sec) of the hamster from $t = 2$ to $t = 4$ seconds?

A= 1 B= 2 C= 4 D= 6 E= 8 $\boxed{\text{D}}$

Does this make sense?

Derivatives: Graphical Approach



Δf = change in f

Δt = change in t

Many ways to say same thing:

$$\left(\begin{array}{c}\text{average rate of}\\ \text{change of } f\end{array}\right) = \frac{\text{change in } f}{\text{change in } t}$$

$$= \frac{\Delta f}{\Delta t}$$

$$= \text{slope of secant line} = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

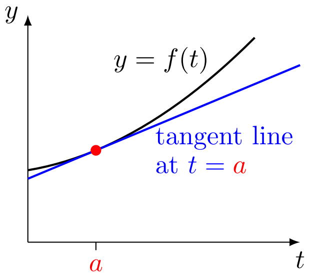
The derivative is defined to be

$$\lim_{\Delta t \to 0} \left(\frac{\Delta f}{\Delta t}\right) = \frac{df}{dt}$$

Idea: As t_1 moves closer to t_0 the secant line approaches the tangent line at t_0. This is the line with the same slope as the graph at t_0.

Understanding Derivatives

There are many ways to think about derivatives. We need to understand how derivatives apply to problems.



slope of graph at a

= slope of tangent line

= instantaneous rate of change of f at a

$$= \left(\begin{array}{c} \text{limit of average rate of change} \\ \text{of } f \text{ over shorter and shorter} \\ \text{time intervals starting at } a \end{array} \right)$$

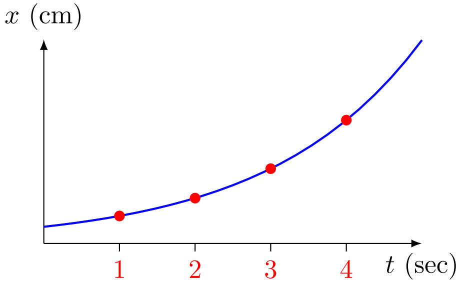
= limit of slopes of secant lines

$$= f'(a) \; = \; \left.\frac{df}{dt}\right|_{t=a}$$

Summary

- How fast something changes = rate of change
- Instantaneous rate of change is the limit of the average rate of change over shorter and shorter time spans. This gets around the changing speed problem, and works a whole lot better that getting frustrated and trying $0/0$.
- speed = rate of change of distance traveled.

Speed=Slope=Derivative



The graph shows the distance from the origin in cm after t seconds of a hamster. Which of the numbers below is the largest?

Hint: Speed is a slope!

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C = speed of the hamster at $t = 3$

D = average speed of the hamster between $t = 2$ and $t = 3$

E = average speed of the hamster between $t = 3$ and $t = 4$

Answer: $\boxed{\text{E}}$

Practical Meaning

Our goal is that you understand the practical meaning of the derivative in various situations.

$f(t) =$ temperature in $^\circ$ F at t hours after midnight

$f(7) = 48$ means the temperature at 7am was 48° F

$f'(7) = 3$ means at 7am the temperature was rising at a rate of 3° F/hr

$f'(9) = -5$ means at 9am the temperature was falling at a rate of 5° F/hr or rising at a rate of -5° F/hr

$g(t) =$ distance from origin in cm of hamster on x-axis after t seconds

$g(7) = 3$ means after 7 seconds hamster was 3 cm from origin

$g'(9) = -5$ means after 9 seconds our furry friend was running towards the origin at a speed of 5 cm/sec

Another Context

Suppose $f(t)$ = temperature of oven in °C after t minutes.

What do $f(3) = 20$ and $f'(3) = 15$ mean?

- A After 20 minutes the oven was at 3° C and heating up at a rate of 15° C/min
- B After 3 minutes oven temperature was 15° C and cooling down at a rate to 20° C/min
- C The oven was heating up at rate of 3° C/min after 15 minutes and also after 20 minutes
- D After 3 minutes the oven was at 20° C and heating up at a rate of 15° C/min
- E None of the above

Answer: $\boxed{\text{D}}$

Context: Population

Suppose $f(t)$ = the population of the ancient city of Lyrad in year t. We are told that $f(1550) = 1820$ and $f'(1650) = 1100$. Which of the following is true?

- A In 1550, the population was 1820 and rising at a rate of 1100 people per year
- B In 1650, the population was 1100 more than in 1550
- C In 1650, Lyrad contained 1100 people
- D In 1550, there were 1820 people in Lyrad, and by 1650 this had increased to 2920
- E None of above

Answer: $\boxed{\text{E}}$

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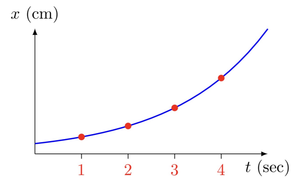
Speed=Slope=Derivative



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- Sums (like the example below, more examples on Gauchospace)

end

$$\sum_{n=1}^{4} 2^n - 1$$

Formula for each term

start

$$= (2^1 - 1) + (2^2 - 1) + (2^3 - 1) + (2^4 - 1)$$
$$= 2 - 1 + 4 - 1 + 8 - 1 + 16 - 1$$
$$= 1 + 3 + 7 + 15$$
$$= \boxed{26}$$

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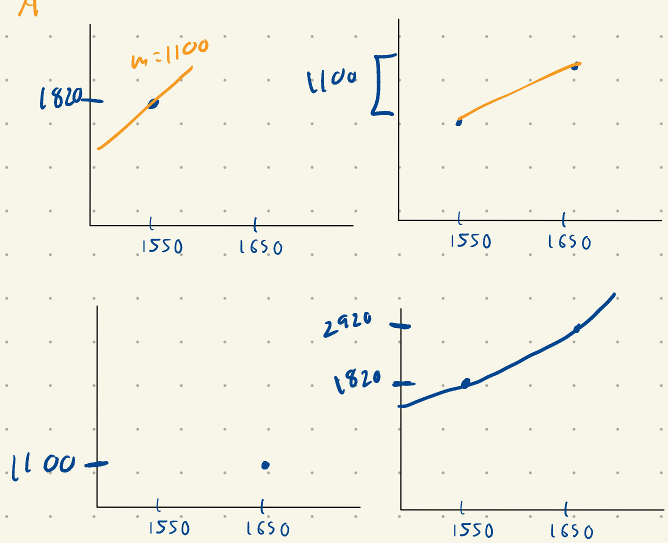
C In 1650, Lyrad contained 1100 people

D In 1550, there were 1820 people in Lyrad, and by 1650 this had increased to 2920

E None of above

In the year 1550, Lyrad had a population of 1820 people, and in the year 1650, the population was increasing at a rate of 1100 people per year.

A



Context: Mathematics

Suppose $f(0) = 50$ and $f(10) = 70$. Which of the following is true?

- A For all t between 0 and 10, the derivative is $f'(t) = 2$
- B $f'(0) = 2$
- C It is possible that $f'(0) = -8$
- D It is impossible that $f'(0) = -8$
- E None of above

Answer: $\boxed{\text{C}}$

We'll see later that, for example, that $f(x) = x^2 - 8x + 50$ has $f(0) = 50$, $f(10) = 70$, and $f'(0) = -8$.

Context: Mathematics

Average rate of change $= \frac{f_2 - f_1}{x_2 - x_1} = \frac{70-50}{10-0} = \frac{20}{10} = 2$

Suppose $f(0) = 50$ and $f(10) = 70$. Which of the following is true?

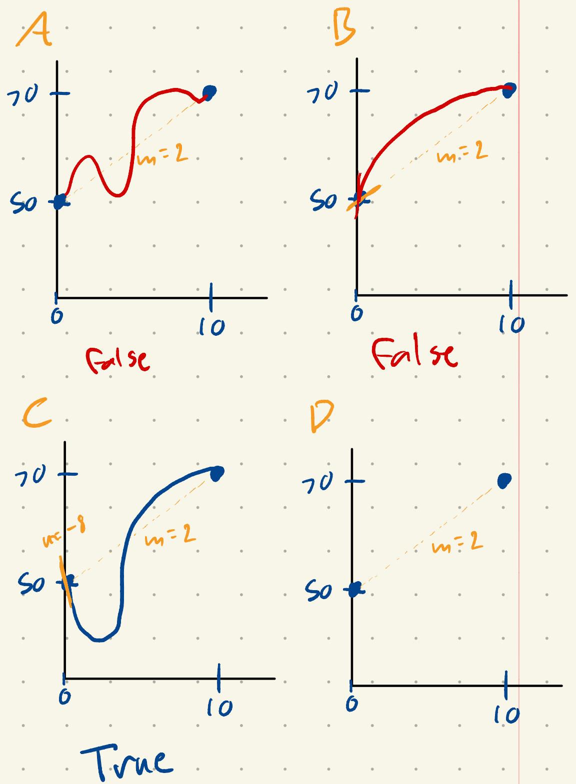
A For all t between 0 and 10, the derivative is $f'(t) = 2$

B $f'(0) = 2$

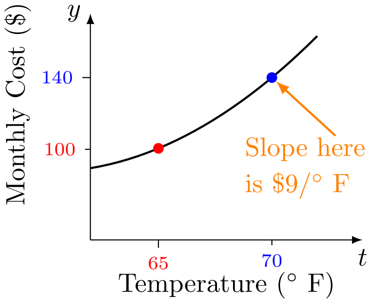
C It is possible that $f'(0) = -8$

D It is impossible that $f'(0) = -8$

E None of above



It doesn't have to be about time!



$f(x)$ = monthly cost of heating house to $x°$ F

$f(70) = 140$ means it costs $140 to heat the house for one month to a temperature of 70°F.

$f'(70) = 9$ means rate at which cost increases as temperature changes is $9 for each extra ° F.

In practical terms this means you pay an extra $9 during each month for each extra 1^oF . If you turn it up two degrees you pay an extra $18 each month. Each extra degree of warmth costs an extra $9 each month. In economics this is called a marginal cost or marginal rate

This is not exactly true:

average rate of change versus instantaneous rate of change.

In the following examples we will ignore this subtlety.

Get Pumped!

Adrenaline cause the heart to speed up.
x = number of mg (milligrams) of adrenaline in the blood.
$f(x)$ = number of beats per minute (bpm) of the heart with x mg of adrenaline in the blood.

What does $f'(5) = 2$ mean? Answer: $\boxed{\text{E}}$

A When there are 5 mg of adrenaline the heart beats at 2 pbm

B When the amount of adrenaline is increased by 2 mg the heart speeds up by 5 bpm

C When the heart beats at 5 bpm the adrenaline is increased by 2 mg

D When there are 5 mg of adrenaline the heart speeds up by 2bpm

E When there are 5 mg of adrenaline in the blood the heart speeds up by 2 bpm for each extra mg of adrenaline.

Hint: The units of $f'(5)$ are bpm per milligram of adrenaline

Summary of Derivatives

One quantity, y, depends on another quantity x.
In other words y is a function of x so $y = f(x)$. Example: $y = 7x$

If you change x, then y changes.
Question: How quickly does y change as x changes?

Answer: The derivative tells you.

In our example, the derivative is 7. This tells you:

the output $= y$ of the function changes
7 times as fast
as the input $= x$ to the function.

If x is changed by 0.1 how much does y change by?

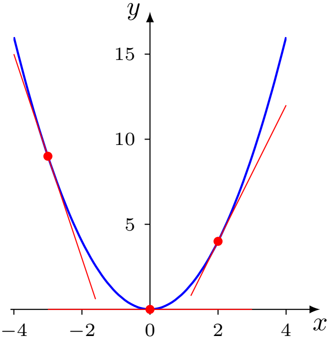
A= 7 B= 7.1 C= 0.7 D= 0.1/7 E= other $\boxed{\text{C}}$

Graphical Meaning

$$\frac{d}{dx}\left(x^2\right) = 2x$$

What this means

The slope of the graph of $y = x^2$ at $x = a$ is $2a$



at $x = -3$, slope is $2(-3) = -6$

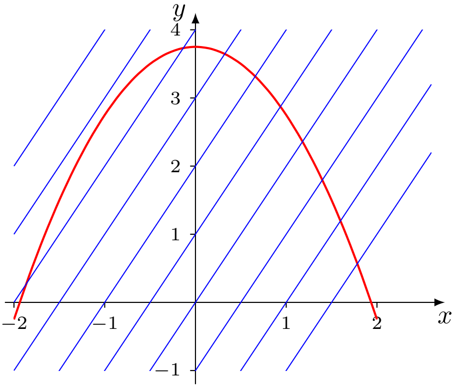
at $x = 0$, slope is $2(0) = 0$

at $x = 2$, slope is $2(2) = 4$

derivative = rate of change = slope of graph = slope of tangent line

Slope Question

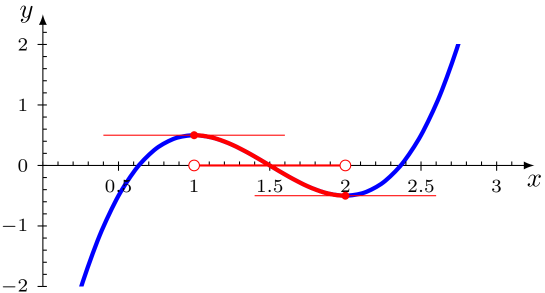
This graph shows $y = f(x)$ and lines parallel to $y = 2x$



Question: For which values of x is $f'(x) > 2$?

A $x < 1.2$ B $x < 0$ C $x < -1.5$ D $x < -1$ E $x < -0.5$ $\boxed{\text{D}}$

More Slope Questions



(1) For which values of x is $f'(x) = 0$?

A= none B= $\{0.63, \ 1.5, 2.38\}$ C= 1 D= $\{1, 2\}$ E= 2 $\boxed{\text{D}}$

(2) For which values of x is $f'(x) < 0$?

A $x < 0.63$ B $x < 1$ C $1 < x < 2$ D $1.5 < x < 2.38$ E none $\boxed{\text{C}}$

That's it. Thanks for being here.