201 Real Analysis
Assignment 2

1. Prove that both weak and weak* topologies are Hausdorff.

2. Find the $\sigma(X, X^*)$-closure (the weak closure) of the unit sphere $S = \{x\colon \|x\| = 1\}$. (Think X is Banach.)

3. Give the detailed proof of the statement from the lectures: sets $W(\phi; x_1, \ldots, x_n)$ with $\phi \in X^*$, $x_j \in X$, $n < \infty$, form a basis of a topology on X^*. Convergence of a sequence (ϕ_n) in this topology is equivalent to the weak* convergence as defined earlier.

4. Let (x_n) be a sequence in l^1 such that $x_n \xrightarrow{w} x$ and $\|x_n\|_{l^1} \to \|x\|_{l^1}$. Prove that (x_n) converges to x strongly.

5. Prove that the closed unit ball $\bar{B}(X)$ in a Banach space X is $\sigma(X, X^*)$-closed. Prove that the closed unit ball $\bar{B}(X^*)$ is $\sigma(X^*, X)$-closed.

6. Prove the statement from the lectures: let X be a separable Banach space with the dense set $\{x_n\}_{n\in\mathbf{N}}$, and let $\bar{B}^*$ be the closed unit ball in X^*. Then the weak* topology restricted to $\bar{B}^*$, $\sigma(X^*, X)|_{\bar{B}^*}$, coincides with the topology of the metric

$$d(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \frac{|(\phi - \psi)(x_n)|}{1 + |(\phi - \psi)(x_n)|}.$$

The problems below will not be graded and are not obligatory. However, if you are thinking of choosing "analysis" for your research subject, then it's a good idea to attempt to solve them.

7. Prove that for a linear operator $A\colon X \to X$ in a Banach space X the following are equivalent: (i) A is continuous: if $x_n \to 0$ then $Ax_n \to 0$; (ii) if $x_n \stackrel{w}{\to} 0$ then $Ax_n \stackrel{w}{\to} 0$; (iii) if $x_n \to 0$ then $Ax_n \stackrel{w}{\to} 0$.

8. Prove in full details that the weak topology on X is the weakest topology in which all $\phi \in X^*$ are continuous.

9. Prove that a sequence in the space l^1 converges weakly if and only if it converges strongly. However, the weak topology on l^1 is different from the strong topology.