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SOME HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS*

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Abstract. Explicit formulas and orthogonality relations are given for some polynomials which include as special or limiting cases the classical polynomials, related polynomials with discrete orthogonalities, some polynomials of Pollaczek and the 6-j symbols of angular momentum.

1. Introduction. This paper contains derivations and discussion of some polynomial orthogonality relations which include as special or limiting cases the orthogonalities for the 6-j symbols of quantum mechanics, the classical polynomials and many related families of orthogonal polynomials. In [7], we give properties of these polynomials which extend the recurrence relations, differential equations and Rodrigues formulas for the classical polynomials.

The polynomials may be expressed as hypergeometric series:

$$\begin{aligned} p_n(t^2) &= p_n(t^2; a, b, c, d) \\ &= (a+b)_n(a+c)_n(a+d)_n \sum_{k=0}^{n} \frac{(-n)_k(a+b+c+d+n-1)_k(a-t)_k(a+t)_k}{(a+b)_k(a+c)_k(a+d)_k k!} \qquad (1.1) \\ &= (a+b)_n(a+c)_n(a+d)_n \, {}_4F_3\left(\begin{matrix} -n, a+b+c+d+n-1, a-t, a+t; \\ a+b, a+c, a+d \end{matrix} \; 1\right), \end{aligned}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ if $k \geqq 1$ and $(a)_0 = 1$. (We use the same ${}_4F_3$ notation for p_n even if one of $a+b$, $a+c$, or $a+d$ is a negative integer $-N$ with $N \geqq n$.) This is a polynomial of degree n in t^2 since $(a-t)_k(a+t)_k = \prod_{j=0}^{k-1}((a+j)^2 - t^2)$. The ${}_4F_3$ is balanced, meaning that it is a finite series and the sum of the denominator parameters equals the sum of the numerator parameters plus one. Such series satisfy a transformation formula (Bailey [1, p. 56]):

$$ {}_4F_3\left(\begin{matrix} -n, b, c, d; 1 \\ e, f, g \end{matrix}\right) = \frac{(f-b)_n(g-b)_n}{(f)_n(g)_n} {}_4F_3\left(\begin{matrix} -n, b, e-c, e-d; 1 \\ e, b-f-n+1, b-g-n+1 \end{matrix}\right), \qquad (1.2)$$

provided $e+f+g = -n+b+c+d+1$. (This formula, when iterated, contains the symmetries of the 6-j symbols.) In terms of the ${}_4F_3$ polynomials, (1.2) says that

$$p_n(x; a, b, c, d) = p_n(x; b, a, c, d), \qquad (1.3)$$

so that p_n is symmetric in all four parameters.

There are various orthogonality relations for $\{p_n\}$ (with respect to positive measures on the real line) corresponding to various conditions on a, b, c and d. These relations are derived in § 3 from the complex orthogonality in § 2.

2. Complex orthogonality. We prove the complex orthogonality relation

$$\frac{1}{2\pi i}\int_C f(z)p_m(z^2)p_n(z^2)\,dz = \delta_{mn}Mh_n \qquad (2.1)$$

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with

$$f(z)=\frac{\Gamma(a+z)\Gamma(a-z)\Gamma(b+z)\Gamma(b-z)\Gamma(c+z)\Gamma(c-z)\Gamma(d+z)\Gamma(d-z)}{\Gamma(2z)\Gamma(-2z)},$$

$$M=\frac{2\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}$$

and

$$h_n=\frac{n!(a+b+c+d+n-1)_n(a+b)_n(a+c)_n(a+d)_n(b+c)_n(b+d)_n(c+d)_n}{(a+b+c+d)_{2n}}.$$

Here a, b, c, and d are complex. The contour C is the imaginary axis deformed so as to separate the increasing sequences of poles of $f(z)$ $\cdot(\{a+k\}_{k=0}^{\infty}, \{b+k\}_{k=0}^{\infty}, \{c+k\}_{k=0}^{\infty}, \{d+k\}_{k=0}^{\infty})$ from the decreasing sequences $(\{-a-k\}_{k=0}^{\infty}, \cdots, \{-d-k\}_{k=0}^{\infty})$. Of course, we need to assume that these two sets of poles are disjoint, i.e.,

$$2a, a+b, a+c, \cdots, c+d, 2d \notin \{0, -1, -2, \cdots\}. \tag{2.2}$$

The case $m=n=0$,

$$\begin{aligned}&\frac{1}{2\pi i}\int_C \frac{\Gamma(a+z)\Gamma(a-z)\Gamma(b+z)\Gamma(b-z)\Gamma(c+z)\Gamma(c-z)\Gamma(d+z)\Gamma(d-z)}{\Gamma(2z)\Gamma(-2z)}\,dz\\ &\qquad=\frac{2\cdot\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}\end{aligned} \tag{2.3}$$

is an integral analog of the ${}_5F_4$ summation theorem (Bailey [1, p. 27]) which may be written

$$\begin{aligned}&{}_5F_4\left(\begin{matrix}2a, a+1, a+b, a+c, a+d; \\ a, a-b+1, a-c+1, a-d+1\end{matrix}\quad 1\right)\\ &\qquad=\frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(-a-b-c-d+1)}{\Gamma(2a+1)\Gamma(-b-c+1)\Gamma(-b-d+1)\Gamma(-c-d+1)}\end{aligned} \tag{2.4}$$

or

$$\begin{aligned}&\sum_{z=a,a+1,\cdots}\frac{\Gamma(1-2z)\Gamma(1+2z)}{\Gamma(1-a-z)\Gamma(1-a+z)\cdots\Gamma(1-d-z)\Gamma(1-d+z)}\\ &\qquad=\frac{\Gamma(1-a-b-c-d)}{\Gamma(1-a-b)\Gamma(1-a-c)\Gamma(1-a-d)\Gamma(1-b-c)\Gamma(1-b-d)\Gamma(1-c-d)}\end{aligned}$$

provided $\operatorname{Re}(a+b+c+d)<1$ for convergence. (Extending the sum over $z=\pm a, \pm(a+1), \cdots$ improves the resemblance to the integral formula.) Similar Mellin–Barnes integrals are found in Bailey [1], and in fact it is possible to derive (2.3) from his formula (1) on p. 47.

To prove (2.3), we need an asymptotic estimate for the integrand $f(z)$. We use the reflection formula $\Gamma(z)=\pi/\Gamma(1-z)\sin \pi z$ along with Stirling's formula

$$\Gamma(a+z)=\sqrt{2\pi}\cdot z^{a+z-1/2}\,e^{-z}\left(1+O\left(\frac{1}{z}\right)\right)$$

as $z\to\infty$ in $S_\theta=\{z: |\arg z|<\theta\}$, $0<\theta<\pi$, and the estimates

$$\sin 2\pi z = O(e^{2\pi|\mathcal{I}m z|})$$

in the entire plane, and

$$\sin \pi(a-z) = O(e^{-\pi|\mathcal{I}m z|})$$

in the plane excluding ε-neighborhoods of the poles. The implied constants in these formulas may be chosen independently of a if a takes values in a bounded set. These give

$$f(z) = \frac{-2\pi^3 z \sin 2\pi z \Gamma(a+z)\cdots\Gamma(d+z)}{\sin \pi(a-z)\cdots \sin \pi(d-z)\Gamma(1-a+z)\cdots\Gamma(1-d+z)}$$

$$(2.5) \qquad = z^{2(a+b+c+d)-3} O(e^{-2\pi|\mathcal{I}m z|})$$

$$= O(|z|^{2\,\mathrm{Re}\,(a+b+c+d)-3} e^{-2\pi|\mathcal{I}m z|})$$

as $z \to \infty$ in S_θ excluding ε-neighborhoods of the poles. Since $f(z)$ is an even function, (2.5) holds as $z \to \infty$ in the plane excluding ε-neighborhoods of the poles.

In particular, (2.5) holds as $z \to \infty$ on C, so $\int_C f(z)\,dz$ is convergent. Furthermore, since the implied constant in (2.5) is independent of a (in bounded sets), the integral defines an analytic function of a in $\{a : a, a+1, a+2, \cdots$ are to the right of C and $-a, -a-1, \cdots$ are to the left of $C\}$. We will use Cauchy's theorem to prove (2.3) under the condition $\mathrm{Re}\,(a+b+c+d) < 1$. This condition may then be removed by analytic continuation.

Consider $\int_{C_1+C_2} f(z)\,dz$, where $C_1 = C_1(\omega)$ is the piece of C from $-i\omega$ to $+i\omega$, and $C_2(\omega)$ is the path consisting of the three line segments from $i\omega$ to $\omega+i\omega$ to $\omega-i\omega$ to $-i\omega$. We will let $\omega \to \infty$ through value $\omega_0, \omega_0+1, \omega_0+2, \cdots$, where ω_0 is chosen so that the contours $C_2(\omega_0+k)$ avoid the poles of $f(z)$. Then

$$\left|\int_{C_2} f(z)\,dz\right| \leqq 4\omega \max\{|f(z)| : z \text{ on } C_2\}$$

$$= O(\omega^{2\,\mathrm{Re}\,(a+b+c+d)-2}).$$

So with $\mathrm{Re}\,(a+b+c+d) < 1$, $\int_{C_2} f(z)\,dz$ vanishes as $\omega \to \infty$.

It follows that

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \lim_{\omega\to\infty} \frac{1}{2\pi i}\int_{C_1+C_2} f(z)\,dz,$$

which is minus the sum of the residues at the poles to the right of C. The residue at $z = a+k$ is

$$\frac{\Gamma(2a+k)((-1)^{k+1}/k!)\Gamma(a+b+k)\Gamma(b-a-k) \cdot \Gamma(c+a+k)\Gamma(c-a-k)\Gamma(d+a+k)\Gamma(d-a-k)}{\Gamma(2a+2k)\Gamma(-2a-2k)}$$

$$= \frac{-\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b-a)\Gamma(c-a)\Gamma(d-a)}{\Gamma(-2a)}$$

$$\cdot \frac{(2a)_k(a+1)_k(a+b)_k(a+c)_k(a+d)_k}{(1)_k(a)_k(a-b+1)_k(a-c+1)_k(a-d+1)_k}.$$

Here we are assuming that the poles are simple, but in (2.3) this condition is removable.

Minus the sum of these residues for $k=0, 1, 2, \cdots$ is given by the ${}_5F_4$ formula (2.4) as

$$\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b-a)\Gamma(c-a)\Gamma(d-a)}{\Gamma(-2a)} \cdot \frac{\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(-a-b-c-d+1)}{\Gamma(2a+1)\Gamma(-b-c+1)\Gamma(-b-d+1)\Gamma(-c-d+1)} = \frac{1}{2} R \cdot S_a,$$

where R is the right-hand side of (2.3), and

$$S_a = \frac{\sin(-2\pi a) \sin \pi(b+c) \sin \pi(b+d) \sin \pi(c+d)}{\sin \pi(b-a) \sin \pi(c-a) \sin \pi(d-a) \sin \pi(a+b+c+d)}.$$

If we define S_b, S_c, S_d symmetrically, then adding the contributions from all the poles $a+k, b+k, c+k, d+k$ for $k=0, 1, 2, \cdots$ gives

$$\frac{1}{2\pi i}\int_C f(z)\,dx = \frac{1}{2} R \cdot (S_a+S_b+S_c+S_d).$$

Finally, a tedious trigonometric computation or a contour integration argument shows $S_a+S_b+S_c+S_d=2$. This proves (2.3).

To prove the orthogonality (2.1), first note that, by the symmetry (1.3),

$$p_m(z^2) = (-1)^m (a+b+c+d+m-1)_m (b-z)_m (b+z)_m + \sum_{j=0}^{m-1} \gamma_j (b-z)_j (b+z)_j.$$

For $j=0, 1, \cdots, n$,

$$\frac{1}{2\pi i}\int_C f(z) p_n(z^2)(b-z)_j(b+z)_j\,dz = (a+b)_n(a+c)_n(a+d)_n \cdot \sum_{k=0}^{n} \frac{(-n)_k(n+a+b+c+d-1)_k}{(a+b)_k(a+c)_k(a+d)_k(1)_k} \cdot \frac{1}{2\pi i}\int_C f(z)(a-z)_k(a+z)_k(b-z)_j(b+z)_j\,dz.$$

The integral here may be written

$$\frac{1}{2\pi i} \cdot \int_C \frac{\Gamma(a+k+z)\Gamma(a+k-z)\Gamma(b+j+z)\Gamma(b+j-z)\Gamma(c+z)\Gamma(c-z)\Gamma(d+z)\Gamma(d-z)\,dz}{\Gamma(2z)\Gamma(-2z)}$$

and, by (2.3), its value is

$$\frac{2\cdot\Gamma(a+b+k+j)\Gamma(a+c+k)\Gamma(a+d+k)\Gamma(b+c+j)\Gamma(b+d+j)\Gamma(c+d)}{\Gamma(a+b+c+d+k+j)}$$
$$= \frac{2\cdot\Gamma(a+b+j)\Gamma(a+c)\Gamma(a+d)\Gamma(b+c+j) \cdot \Gamma(b+d+j)\Gamma(c+d)(a+b+j)_k(a+c)_k(a+d)_k}{\Gamma(a+b+c+d+j)(a+b+c+d+j)_k}.$$

Therefore,

$$\frac{1}{2\pi i}\int_C f(z)p_n(z^2)(b-z)_j(b+z)_j\,dz$$

$$\begin{aligned}&=2\cdot\frac{\Gamma(a+b+j)\Gamma(a+c+n)\Gamma(a+d+n)\Gamma(b+c+j)\Gamma(b+d+j)\Gamma(c+d)(a+b)_n}{\Gamma(a+b+c+d+j)}\\ &\quad\cdot{}_3F_2\left(\begin{matrix}-n,\, n+a+b+c+d-1,\, a+b+j;\\ a+b,\, a+b+c+d+j\end{matrix}\;1\right)\\ &=2\cdot\frac{\Gamma(a+b+j)\Gamma(a+c+n)\Gamma(a+d+n)\Gamma(b+c+j)\Gamma(b+d+j)\Gamma(c+d+n)(-j)_n}{\Gamma(a+b+c+d+j+n)}.\end{aligned}\tag{2.6}$$

We have evaluated the ${}_3F_2$ by the formula of Pfaff and Saalschütz (Bailey [1, p. 9]):

$${}_3F_2\left(\begin{matrix}-n,\, b,\, c;\\ d,\, e\end{matrix}\;1\right)=\frac{(d-c)_n(e-c)_n}{(d)_n(e)_n}$$

provided $d+e=-n+b+c+1$. The result of (2.6) is zero if $j<n$. Therefore

$$\frac{1}{2\pi i}\int_C f(z)p_m(z^2)p_n(z^2)\,dz=0\quad\text{if } m<n,$$

while

$$\begin{aligned}&\frac{1}{2\pi i}\int_C f(z)p_n(z^2)^2\,dz\\ &\quad=(-1)^n(a+b+c+d+n-1)_n\frac{1}{2\pi i}\int_C f(z)p_n(z^2)(b-z)_n(b+z)_n\,dz\\ &\quad=\frac{2\cdot n!(a+b+c+d+n-1)_n\Gamma(a+b+n)\Gamma(a+c+n)\cdots\Gamma(c+d+n)}{\Gamma(a+b+c+d+2n)}\end{aligned}$$

as required.

3. Real orthogonalities. We wish to obtain from (2.1) orthogonality relations with respect to positive measures on the real line. Note that, when the real parts of a, b, c, and d are positive, C may be taken to be the imaginary axis. If, furthermore, a, b, c, and d are real except for conjugate pairs, then for imaginary z,

$$f(z)=\left|\frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c+z)\Gamma(d+z)}{\Gamma(2z)}\right|^2.$$

Letting $z=it$ in the integral gives the orthogonality relation

$$\begin{aligned}&\frac{1}{2\pi}\int_0^\infty\left|\frac{\Gamma(a+it)\Gamma(b+it)\Gamma(c+it)\Gamma(d+it)}{\Gamma(2it)}\right|^2 p_n(-t^2)p_m(-t^2)\,dt\\ &\quad=\delta_{m,n}n!(n+a+b+c+d-1)_n\frac{\Gamma(a+b+n)\Gamma(a+c+n)\cdots\Gamma(c+d+n)}{\Gamma(a+b+c+d+2n)}\end{aligned}\tag{3.1}$$

(a, b, c, and d positive real except for complex conjugate pairs with positive real parts). With these conditions on the parameters, the polynomials $p_n(t^2)$ are real for real values of t^2. This is clear from the definition (1.1) in the case where a and b are real (and either c and d are real or $c=\bar{d}$). If $a=\bar{b}$ and $c=\bar{d}$, then it is clear that $p_n(t^2;\, b, a, c, d)=\overline{p_n(t^2;\, a, b, c, d)}$. But then the symmetry (1.3) shows that $p_n(t^2;\, a, b, c, d)$ is real.

Return now to (2.1) and consider the case where $a<0$ and $a+b$, $a+c$, $a+d$ are positive except possibly for one pair of complex conjugates with positive real parts. These conditions will yield an orthogonality (3.3) with respect to a positive weight function consisting of a continuous function plus some point masses. By Cauchy's theorem

$$\frac{1}{2\pi i}\int_C f(z)p_n(z^2)p_m(z^2)\,dz$$
$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f(z)p_n(z^2)p_m(z^2)\,dz$$
$$-(\text{sum of residues of integrand at } z=a+k, \text{ with } a\leqq a+k<0)$$
$$+(\text{sum of residues at } z=-(a+k), \text{ with } a\leqq a+k<0) \tag{3.2}$$
$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f(z)p_n(z^2)p_m(z^2)\,dz-2\ (\text{sum of residues at } z=a+k,\ a\leqq a+k<0)$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(it)p_n(-t^2)p_m(-t^2)\,dt$$
$$+2\,\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b-a)\Gamma(c-a)\Gamma(d-a)}{\Gamma(-2a)}$$
$$\cdot\sum_{\substack{k=0,1,\cdots\\ a+k<0}}\frac{(2a)_k(a+1)_k(a+b)_k(a+c)_k(a+d)_k}{(1)_k(a)_k(a-b+1)_k(a-c+1)_k(a-d+1)_k}\,p_n((a+k)^2)p_m((a+k)^2).$$

Therefore, (2.1) becomes

$$\frac{1}{2\pi}\int_0^\infty\left|\frac{\Gamma(a+it)\cdots\Gamma(d+it)}{\Gamma(2it)}\right|^2 p_n(-t^2)p_m(-t^2)\,dt$$
$$+\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b-a)\Gamma(c-a)\Gamma(d-a)}{\Gamma(-2a)}$$
$$\sum_{\substack{k=0,1,\cdots\\ a+k<0}}\frac{(2a)_k(a+1)_k(a+b)_k(a+c)_k(a+d)_k}{(1)_k(a)_k(a-b+1)_k(a-c+1)_k(a-d+1)_k}\,p_n((a+k)^2)p_m((a+k)^2) \tag{3.3}$$
$$=\delta_{m,n}n!(n+a+b+c+d-1)_n\frac{\Gamma(a+b+n)\cdots\Gamma(c+d+n)}{\Gamma(a+b+c+d+2n)}$$

if $a<0$, and $a+b$, $a+c$, $a+d>0$ except possibly for a pair of conjugates with positive real parts. Condition (2.2) requires that $2a\not\in\{0,-1,-2,\cdots\}$, but here this condition is removable.

Formula (2.1) also yields some purely discrete orthogonality relations for $p_n(t^2)$. Take $a+b=-N+\varepsilon$, N a positive integer. (Condition (2.2) requires $\varepsilon\neq 0$.) Use Cauchy's theorem as in (3.2) to replace the contour C by the imginary axis and add some residues. Then divide the equation by $\Gamma(a+b)=\Gamma(-N+\varepsilon)$. As $\varepsilon\to 0$, the integral term vanishes because $1/\Gamma(-N+\varepsilon)\to 0$, and the result may be written

$$\frac{(a-c+1)_N(a-d+1)_N}{(2a+1)_N(1-c-d)_N}\sum_{k=0}^{N}\frac{(2a)_k(a+1)_k(a+b)_k(a+c)_k(a+d)_k}{(1)_k(a)_k(a-b+1)_k(a-c+1)_k(a-d+1)_k}\cdot p_n((a+k)^2)p_m((a+k)^2)$$

$$= \delta_{m,n}\frac{n!(n+a+b+c+d-1)_n(a+b)_n(a+c)_n(a+d)_n(b+c)_n(b+d)_n(c+d)_n}{(a+b+c+d)_{2n}} \tag{3.4}$$

when $a+b=-N$. Interchanging a and b here is equivalent to summing in the reverse order. The case $m=n=0$ is the terminating series version of (2.4):

$$ {}_5F_4\left(\begin{matrix}2a,\ a+1,\ a+b,\ a+c,\ a+d;\\ a,\ a-b+1,\ a-c+1,\ a-d+1\end{matrix}\ 1\right)=\frac{(2a+1)_N(1-c-d)_N}{(a-c+1)_N(a-d+1)_N}$$

when $a+b=-N$. Formula (3.4) can also be proven directly from this ${}_5F_4$ theorem just as the complex orthogonality (2.1) was proven from the integral formula (2.3).

Necessary and sufficient conditions on a, b, c, d for the positivity of the weights in (3.4) are quite messy, but some sufficient conditions are

$$a+b=-N,\quad b<-\tfrac{1}{2}<a,\quad -a<c<a+1\quad\text{and}\quad\text{either } d>-b \text{ or } d<b+1. \tag{3.5}$$

Of course, interchanging a and b in (3.5) also gives sufficient conditions.

4. Limiting cases. We now describe how, as claimed in § 1, many orthogonality relations for previously known polynomials are included in the ${}_4F_3$ orthogonalities as limiting cases. The appropriate limit processes can usually be determined by comparing the hypergeometric series representations of the polynomials. It sometimes helps to write the ${}_4F_3$ polynomials, with a change of variable and parameters, as

$$r_n(\lambda(x);\alpha,\beta,\gamma,\delta)={}_4F_3\left(\begin{matrix}-n,\ n+\alpha+\beta+1,\ -x,\ x+\gamma+\delta+1;\\ \alpha+1,\ \beta+\delta+1,\ \gamma+1\end{matrix}\ 1\right) \tag{4.1}$$

with $\lambda(x)=x(x+\gamma+\delta+1)$.

Then (3.4) becomes

$$\sum_{k=0}^{N}\frac{(\gamma+\delta+1)_k((\gamma+\delta+3)/2)_k(\alpha+1)_k(\beta+\delta+1)_k(\gamma+1)_k}{(1)_k((\gamma+\delta+1)/2)_k(\gamma+\delta-\alpha+1)_k(\gamma-\beta+1)_k(\delta+1)_k}\cdot r_n(\lambda(k))r_m(\lambda(k))$$

$$=\delta_{m,n}M\cdot\frac{n!(n+\alpha+\beta+1)_n(\beta+1)_n(\alpha-\delta+1)_n(\alpha+\beta-\gamma+1)_n}{(\alpha+\beta+2)_{2n}(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n} \tag{4.2}$$

if $\alpha+1, \beta+\delta+1$, or $\gamma+1=-N$, with

$$M=\begin{cases}\dfrac{(\gamma+\delta+2)_N(-\beta)_N}{(\gamma-\beta+1)_N(\delta+1)_N} & \text{if } \alpha+1=-N,\\[2ex] \dfrac{(\gamma+\delta+2)_N(\delta-\alpha)_N}{(\gamma+\delta-\alpha+1)_N(\delta+1)_N} & \text{if } \beta+\delta+1=-N,\\[2ex] \dfrac{(-\delta)_N(\alpha+\beta+2)_N}{(\alpha-\delta+1)_N(\beta+1)_N} & \text{if } \gamma+1=-N.\end{cases}$$

Letting $\delta\to\infty$ with $\gamma+1=-N$ gives the Hahn polynomial orthogonality:

$$\sum_{x=0}^{N}\frac{(\alpha+1)_x(-N)_x}{(-N-\beta)_x x!}Q_n(x;\alpha,\beta,N)Q_m(x;\alpha,\beta,N)=0,\qquad m\neq n,$$
$$Q_n(x;\alpha,\beta,N)={}_3F_2\left(\begin{matrix}-n,\,n+\alpha+\beta+1,\,-x;\\ \alpha+1,\,-N\end{matrix}\;1\right),\qquad 0\leqq n\leqq N. \tag{4.3}$$

Letting $\beta\to\infty$ in (4.2) with $\alpha+1=-N$ gives the dual Hahn orthogonality (Karlin and McGregor [3])

$$\sum_{x=0}^{N}\frac{(\gamma+\delta+1)_x((\gamma+\delta+3)/2)_x(\gamma+1)_x(-N)_x(-1)^x}{x!((\gamma+\delta+1)/2)_x(\delta+1)_x(N+\gamma+\delta+2)_x}$$
$$\cdot R_n(\lambda(x);\gamma,\delta,N)R_m(\lambda(x);\gamma,\delta,N)=0,\qquad m\neq n,$$
$$R_n(\lambda(x);\gamma,\delta,N)={}_3F_2\left(\begin{matrix}-n,\,-x,\,x+\gamma+\delta+1;\\ -N,\,\gamma+1\end{matrix}\;1\right),\qquad 0\leqq n\leqq N.$$

Actually, the dual Hahn polynomials have continuous, discrete, and mixed orthogonality relations (with positive weight functions) which come from the orthogonalities for the ${}_4F_3$'s as $d\to\infty$. For example, (3.1) becomes

$$\frac{1}{2\pi}\int_0^\infty\left|\frac{\Gamma(a+it)\Gamma(b+it)\Gamma(c+it)}{\Gamma(2it)}\right|^2p_n(-t^2)p_m(-t^2)\,dt$$
$$=\delta_{m,n}n!\Gamma(a+b+n)\Gamma(a+c+n)\Gamma(b+c+n), \tag{4.4}$$

where

$$p_n(z^2)=(a+b)_n(a+c)_n\,{}_3F_2\left(\begin{matrix}-n,\,a-z,\,a+z;\\ a+b,\,a+c\end{matrix}\;1\right)$$

and a, b, and c are all positive except possibly for a pair of complex conjugates with positive real parts. The complex orthogonality (2.1) becomes

$$\frac{1}{2\pi i}\int_C\frac{\Gamma(a+z)\Gamma(a-z)\Gamma(b+z)\Gamma(b-z)\Gamma(c+z)\Gamma(c-z)}{\Gamma(2z)\Gamma(-2z)}p_n(z^2)p_m(z^2)\,dz$$
$$=\delta_{m,n}2\cdot n!\Gamma(a+b+n)\Gamma(a+c+n)\Gamma(b+c+n)$$

with $p_n(z^2)$ as above, and C separating the increasing and decreasing sequences of poles.

It is known that, by taking limits of (4.3), we can obtain the discrete orthogonalities for the Meixner, Krawtchouk and Charlier polynomials, as well as the orthogonalities for the classical polynomials of Jacobi, Laguerre, and Hermite.

It is also interesting that the classical polynomial orthogonalities can be realized in a different way as limits of the continuous orthogonality relations (3.1) and (4.4). In (3.1), let $a=b=(\alpha+1)/2$ and $c=\bar d=(\beta+1)/2+i\omega$, and change variable $t=\omega s$ to get:

$$\int_0^\infty\frac{1}{2\pi}\left|\frac{\Gamma((\alpha+1)/2+i\omega s)^2\Gamma((\beta+1)/2+i\omega(s+1))\Gamma((\beta+1)/2+i\omega(s-1))}{\Gamma(2i\omega s)\Gamma((\alpha+\beta+2)/2+i\omega)^2}\right|^2$$
$$\cdot\,{}_4F_3\left(\begin{matrix}-n,\,n+\alpha+\beta+1,\,(\alpha+1)/2+i\omega s,\,(\alpha+1)/2-i\omega s;\\ \alpha+1,\,(\alpha+\beta+2)/2+i\omega,\,(\alpha+\beta+2)/2-i\omega\end{matrix}\;1\right){}_4F_3\left(\begin{matrix}-m,\cdots\\ \cdots\end{matrix}\right)\omega\,ds$$
$$=\delta_{mn}\frac{n!(n+\alpha+\beta+1)_n\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2n)}\frac{(\beta+1)_n}{(\alpha+1)_n}.$$

As $\omega \to +\infty$, the weight function is asymptotic to $2 \cdot s^{2\alpha+1}|1-s^2|^\beta e^{-\pi\omega(|s-1|+s-1)}$ by Stirling's formula, and therefore has limit $2s^{2\alpha+1}(1-s^2)^\beta$ if $0<s<1$, 0 if $s>1$. At least formally, this gives

$$\int_0^1 s^{2\alpha}(1-s^2)^\beta {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1; \\ \alpha+1\end{matrix}\; s^2\right) {}_2F_1\left(\begin{matrix}-m, m+\alpha+\beta+1; \\ \alpha+1\end{matrix}\; s^2\right) d(s^2)$$
$$= \delta_{mn} \frac{n!(n+\alpha+\beta+1)_n \Gamma(\alpha+1)\Gamma(\beta+1)(\beta+1)_n}{\Gamma(\alpha+\beta+2n)(\alpha+1)_n},$$

which is the Jacobi orthogonality with a change of variable.

In (4.4), let $a=b=(\alpha+1)/2$ and $c=\omega^2$, and change variable $t=\omega s$. As $\omega \to +\infty$, a messy application of Stirling's formula gives

$$\int_0^\infty e^{-s^2}(s^2)^\alpha {}_1F_1\left(\begin{matrix}-n; s^2 \\ \alpha+1\end{matrix}\right) {}_1F_1\left(\begin{matrix}-m; s^2 \\ \alpha+1\end{matrix}\right) d(s^2) = \delta_{mn} \frac{\Gamma(\alpha+1)n!}{(\alpha+1)_n}, \tag{4.5}$$

which is the Laguerre polynomial orthogonality. The cases of formula (4.5) with $\alpha=-\frac{1}{2}$ and $\alpha=\frac{1}{2}$ are the orthogonalities for the Hermite polynomials of even and odd degrees, respectively. Of course, the Laguerre and Hermite orthogonalities are also limiting cases of the Jacobi polynomial orthogonality.

Another limiting case of (3.1) is Pollaczek's orthogonality relation [4]:

$$\int_{-\infty}^\infty e^{(-\pi+2\phi)x}|\Gamma(\lambda+ix)|^2 P_n^{(\lambda)}(x;\phi) P_m^{(\lambda)}(x;\phi)\, dx = 0, \qquad m \neq n,$$

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n e^{in\phi}}{n!} {}_2F_1\left(\begin{matrix}-n, \lambda+ix; \\ 2\lambda\end{matrix}\; 1-e^{-2i\phi}\right), \qquad n \geqq 0,$$

$\lambda>0$, $0<\phi<\pi$. We extend (3.1) to a symmetric integral on $(-\infty, \infty)$. Then we take $a=\lambda+i\omega$, $b=\lambda-i\omega$, and $c=d=\omega \cot(\phi/2)$, and substitute $t=x-\omega$:

$$\frac{1}{2\pi}\int_{-\infty}^\infty \left|\frac{\Gamma(\lambda+ix)\Gamma(\lambda+ix-2i\omega)\Gamma(\omega\cot(\phi/2)+i(x-\omega))^2}{\Gamma(2i(x-\omega))}\right|^2$$
$$\cdot {}_4F_3\left(\begin{matrix}-n, n+2\lambda+2\omega\cot(\phi/2)-1, \lambda+ix, \lambda-ix+2i\omega; \\ 2\lambda, \lambda+\omega(\cot(\phi/2)+i), \lambda+\omega(\cot(\phi/2)+i)\end{matrix}\; 1\right) {}_4F_3\left(\begin{matrix}-m, \cdots \\ \cdots\end{matrix}\right) dx$$
$$= 2\delta_{mn} \frac{n!(n+2\lambda+2\omega\cot(\phi/2)-1)_n \Gamma(2\lambda+n) \cdot \Gamma(2\omega\cot(\phi/2)+n)|\Gamma(\lambda+\omega(\cot(\phi/2)+i)+n)|^2}{\Gamma(2\lambda+2\omega\cot(\phi/2)+2n)}.$$

As $\omega \to +\infty$, we get Pollaczek's orthogonality by another application of Stirling's formula.

5. The 6-j symbols. The 6-j symbols $\bar{W}\begin{pmatrix} a, b\ c \\ d, e, f\end{pmatrix}$, important in quantum mechanics in the coupling of angular momenta, satisfy an orthogonality relation which we will show to be equivalent to certain cases of (3.4). It appears that this orthogonality was recognized as a polynomial orthogonality only in very special cases (Biedenharn et al. [2, p. 253]). $\bar{W}\begin{pmatrix} a, b, c \\ d, e, f\end{pmatrix}$ is defined for half-integers a, b, c, d, e, f which are nonnegative and satisfy certain triangle conditions: $a+b+c$ is an integer, $a \leqq b+c$, $b \leqq a+c$, and $c \leqq a+b$, so that a, b, and c are the sides of some triangle; each of the triples (a, e, f), (d, b, f), and (d, e, c) satisfies similar conditions. An explicit formula for the 6-j symbol is

$$\bar{W}\begin{pmatrix} a, b, c \\ d, e, f \end{pmatrix} = \Delta(a, b, c)\Delta(a, e, f)\Delta(d, b, f)\Delta(d, e, c) \tag{5.1}$$
$$\cdot \sum_k \frac{(-1)^k (k+1)!}{(k-a-b-c)!(k-a-e-f)!(k-d-b-f)!(k-d-e-c)! \cdot (a+b+d+e-k)!(a+c+d+f-k)!(b+c+e+f-k)!}$$

where

$$\Delta(a, b, c) = \left[\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}\right]^{1/2}$$

and the sum is over all integers $k \geqq 0$. Only finitely many terms of the sum are nonzero. There are symmetries here which are easier to understand if we consider the tetrahedron with edges a, b, c, d, e and f and faces (a, b, c), (a, e, f), (d, b, f), and (d, e, c). The value of $\bar{W}\begin{pmatrix} a, b, c \\ d, e, f \end{pmatrix}$ is preserved under any permutation of the parameters which preserves the tetrahedron.

Racah's orthogonality [5] is

$$\sum_c (2c+1)\bar{W}\begin{pmatrix} a, b, c \\ d, e, f \end{pmatrix} \bar{W}\begin{pmatrix} a, b, c \\ d, e, f' \end{pmatrix} = \frac{\delta_{f,f'}}{2f+1}. \tag{5.2}$$

The permissible values for f and f' and the values of the summation variable are determined by the triangle conditions. The inequalities involved are

$$|a-e|, |b-d| \leqq f, f' \leqq a+e, b+d \tag{5.3}$$

and

$$|a-b|, |d-e| \leqq c \leqq a+b, d+e. \tag{5.4}$$

By the tetrahedron symmetries, there is no loss of generality in assuming that

$$\max(|a-b|, |d-e|) = |a-b| = a-b. \tag{5.5}$$

We need to consider two cases, distinguished by the upper limit on c.

Consider first the case where

$$d+e \leqq a+b. \tag{5.6}$$

With conditions (5.5) and (5.6) we must have $\max(|a-e|, |b-d|) = a-e$ and $b+d \leqq a+e$, so (5.3) and (5.4) reduce to $a-e \leqq f$, $f' \leqq b+d$, and $a-b \leqq c \leqq d+e$. Let $N = b+d+e-a$ and replace the variable c by $d+e-x$, so the orthogonality is on the $N+1$ points where $x = 0, 1, \cdots, N$. Replace f by $b+d-n$, so the orthogonal functions are indexed by n, $0 \leqq n \leqq N$. We claim that

$$\bar{W}\begin{pmatrix} a, b, d+e-x \\ d, e, b+d-n \end{pmatrix} = C\frac{\Delta(a, b, d+e-x)\Delta(d, e, d+e-x)}{x!(N-x)!(2d-x)!(a+b-d-e+x)!} \tag{5.7}$$
$$\cdot {}_4F_3\left(\begin{matrix} -n, n-2b-2d-1, -x, x-2d-2e-1; \\ -a-b-d-e-1, -2d, -N \end{matrix}\; 1\right),$$

where the factor C is independent of x. In fact, this is just the ${}_4F_3$ transformation (1.2), but since a little care is needed to avoid zero denominators in the computation, we give details. By applying (5.1) to $\bar{W}\begin{pmatrix} a, b, d+e-x \\ d, e, b+d-n \end{pmatrix}$ and substituting $a+b+d+e-j$ for k, we get

$$\begin{aligned}
&\bar{W}\begin{pmatrix} a, b, d+e-x \\ d, e, b+d-n \end{pmatrix} \\
&\quad = C\cdot \Delta(a, b, d+e-x)\Delta(d, e, d+e-x) \\
&\qquad \cdot \sum_{j=0}^{n} \frac{(-1)^{a+b+d+e-j}(a+b+d+e-j+1)!}{(x-j)!(n-j)!(n+a-b-d+e-j)!(x+a+b-d-e-j)! \cdot j!(2d-n-x+j)!(N-n-x+j)!} \\
&\quad = C\cdot \lim_{\varepsilon\to 0} \Delta(a, b, d+e-x)\Delta(d, e, d+e-x) \\
&\qquad \cdot \sum_{j=0}^{n} \frac{(-1)^{a+b+d+e-j}(a+b+d+e-j+1)!}{\Gamma(x+\varepsilon-j+1)(n-j)!(n+a-b-d+e-j)!} \\
&\qquad \cdot \frac{1}{\Gamma(x+\varepsilon+a+b-d-j+1)j!\,\Gamma(2d-n-x-\varepsilon+j+1)\Gamma(N-n-x-\varepsilon+j+1)} \\
&\quad = C\cdot \lim_{\varepsilon\to 0} \frac{\Delta(a, b, d+e-x)\Delta(d, e, d+e-x)}{\Gamma(x+\varepsilon+1)\Gamma(x+\varepsilon+a+b-d-e+1) \cdot \Gamma(2d-n-x-\varepsilon+1)\Gamma(N-n-x-\varepsilon+1)} \\
&\qquad \cdot {}_4F_3\begin{pmatrix} -n, -n-a+b+d-e, -x-\varepsilon, -x-\varepsilon-a-b+d+e; \\ -a-b-d-e-1, 2d-n-x-\varepsilon+1, N-n-x-\varepsilon+1 \end{pmatrix}1\Bigr).
\end{aligned}$$

By transformation (1.2), this is

$$\begin{aligned}
&C\cdot \lim_{\varepsilon\to 0} \frac{\Delta(a, b, d+e-x)\Delta(d, e, d+e-x)}{\Gamma(x+\varepsilon+1)\Gamma(x+\varepsilon+a+b-d-e+1)\Gamma(2d-x-\varepsilon+1)\Gamma(n-x-\varepsilon+1)} \\
&\qquad \cdot {}_4F_3\begin{pmatrix} -n, n-2b-2d-1, -x-\varepsilon, x+\varepsilon-2d-2e-1; & \\ -a-b-d-e-1, -2d, -N & 1 \end{pmatrix} \\
&\quad = \frac{C\cdot \Delta(a, b, d+e-x)\Delta(d, e, d+e-x)}{x!(x+a+b-d-e)!(2d-x)!(N-x)!} \\
&\qquad \cdot {}_4F_3\begin{pmatrix} -n, n-2b-2d-1, -x, x-2d-2e-1; & \\ -a-b-d-e-1, -2d, -N & 1 \end{pmatrix}.
\end{aligned}$$

This establishes (5.7). If $a' = -d-e-\frac{1}{2}$, $b' = a-b+\frac{1}{2}$, $c' = e-d+\frac{1}{2}$, and $d' = -a-b-\frac{1}{2}$, then the ${}_4F_3$ we have is $C\cdot p_n((a'+x)^2; a', b', c', d')$. It is now an easy matter to compare the weight functions (or apply a general theorem) to see that (3.4) and (5.2) represent the same orthogonality.

The case where $a+b \leqq d+e$ is dealt with similarly. The limits on f and c become $d-b \leqq f, f' \leqq b+d$ and $a-b \leqq c \leqq a+b$. Let $N = 2b$; replace f by $b+d-n$; and replace c by $a+b-x$. Then corresponding to (5.7) is

$$\begin{aligned}
\bar{W}\begin{pmatrix} a, b, a+b-x \\ d, e, b+d-n \end{pmatrix} &= \frac{C\cdot \Delta(a, b, a+b-x)\Delta(d, e, a+b-x)}{x!(N-x)!(d+e-a-b+x)!(a+b+d-e-x)!} \\
&\qquad \cdot {}_4F_3\begin{pmatrix} -n, n-2b-2d-1, -x, x-2a-2b-1; & \\ -a-b-d-e-1, e-a-b-d, -2b & 1 \end{pmatrix}.
\end{aligned}$$

The ${}_4F_3$ here is $Cp_n((a'+x)^2; a', b', c', d')$ with $a' = -a-b-\frac{1}{2}$, $b' = a-b+\frac{1}{2}$, $c' = e-d+\frac{1}{2}$, and $d' = -d-e-\frac{1}{2}$.

In both cases, the weight functions satisfy the positivity conditions (3.5).

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