Math 462 - Advanced Linear Algebra
Homework 2

Trevor Klar

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Exercises:

5. Let G_0 and G_1 be groups. Consider the set

$$G_0 \times G_1 = \{(s_0, s_1) : s_0 \in G_0, s_1 \in G_1\}$$

This is just the Cartesian product of the two sets G_0 and G_1. Define an operation on $G_0 \times G_1$ as follows:

$$(s_0, s_1)(t_0, t_1) = (s_0 t_0, s_1 t_1) \quad \forall s_0, t_0 \in G_0, s_1, t_1 \in G_1$$

Show that $G_0 \times G_1$ is a group with respect to this operation.

Proof To show that $G_0 \times G_1$ is a group with respect to this operation, it suffices to show that the associative, identity, and inverse properties hold.

Associative property: Consider the product $(s_0, s_1)(t_0, t_1)(u_0, u_1)$.

$$\begin{aligned} ((s_0, s_1)(t_0, t_1))(u_0, u_1) &= (s_0 t_0, s_1 t_1)(u_0, u_1) \\ &= ((s_0 t_0) u_0, (s_1 t_1) u_1) \\ &= (s_0 (t_0 u_0), (s_1 (t_1 u_1)) \\ &= (s_0, s_1)(t_0 u_0, t_1 u_1) \\ &= (s_0, s_1)((t_0, t_1)(u_0, u_1)) \end{aligned}$$

Identity property: If e_0 and e_1 are the identities of G_0 and G_1, respectively, then (e_0, e_1) is the identity of $G_0 \times G_1$ as shown below:

$$\begin{aligned} (s_0, s_1)(e_0, e_1) &= (s_0 e_0, s_1 e_1) \\ &= (s_0, s_1) \end{aligned}$$

Inverse property: Since G_0 and G_1 are groups, then the inverse of any (s_0, s_1) can be found using the inverses of its components; that is, (s_0^{-1}, s_1^{-1}). Proof follows:

$$\begin{aligned} (s_0, s_1)(s_0^{-1}, s_1^{-1}) &= (s_0 s_0^{-1}, s_1 s_1^{-1}) \\ &= (e_0, e_1) \end{aligned}$$

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6. Show that $G_0 \times G_1 \cong G_1 \times G_0$. (Explicitly construct the isomorphism. This is easy.)

Example. Let $f(s_0, s_1) = (s_1, s_0)$. First we will show that f is a homomorphism:

$$\begin{aligned} f((s_0, s_1)(t_0, t_1)) &= f(s_0 t_0, s_1 t_1) \\ &= (s_1 t_1, s_0 t_0) \\ &= (s_1, s_0)(t_1, t_0) \\ &= f(s_0, s_1) f(t_0, t_1) \end{aligned}$$

Next, we mention that f is clearly bejective. (s_0, s_1) can clearly be the only element which maps to (s_1, s_0), so f is 1-1. Similarly, given any (s_1, s_0), there is always a corresponding (s_0, s_1) such that $f(s_0, s_1) = (s_1, s_0)$. ■

7. Continuing in this same context, consider the functions

$$\begin{array}{rcl} \rho_0 : G_0 \times G_1 & \to & G_0 \\ (s_0, s_1) & \mapsto & s_0 \end{array}$$

$$\begin{array}{rcl} \rho_1 : G_0 \times G_1 & \to & G_1 \\ (s_0, s_1) & \mapsto & s_1 \end{array}$$

These are called *projection maps*. Show that both maps are surjective homomorphisms and compute the kernel of each.

Proof

- ρ_0 is surjective because given any $s_0 \in G_0$, we can choose an arbitrary $s_1 \in G_1$ and find that $\rho_0(s_0, s_1) = s_0$.
- ρ_0 is a homomorphism because

$$\begin{array}{rcl} \rho_0((s_0, s_1)(t_0, t_1)) & = & \rho_0(s_0 t_0, s_1 t_1) \\ & = & s_0 t_0 \\ & = & (s_0)(t_0) \\ & = & \rho_0(s_0, s_1)\rho_0(t_0, t_1) \end{array}$$

- $\ker(\rho_0) = \{(e_0, s_1) : \forall s_1 \in G_1\}$, where e_0 is the identity of G_0.
- ρ_1 is surjective because given any $s_1 \in G_1$, we can choose an arbitrary $s_0 \in G_0$ and find that $\rho_1(s_0, s_1) = s_1$.
- ρ_0 is a homomorphism because

$$\begin{array}{rcl} \rho_1((s_0, s_1)(t_0, t_1)) & = & \rho_1(s_0 t_0, s_1 t_1) \\ & = & s_1 t_1 \\ & = & (s_1)(t_1) \\ & = & \rho_1(s_0, s_1)\rho_1(t_0, t_1) \end{array}$$

- $\ker(\rho_1) = \{(s_0, e_1) : \forall s_0 \in G_0\}$, where e_1 is the identity of G_1.

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8. Consider the special case of the direct product $G \times G$ of a group G with itself. Define a subset D of $G \times G$ by

$$D = \{(s, s) : s \in G\}$$

That is, D consists of all elements with both coordinates equal. Show that D is a subgroup of $G \times G$. This is called the *diagonal subgroup*. Do you see why?

Proof To show that D is a subgoup of $G \times G$, it suffices to show that $D \subset G \times G$ and D is closed under the operation. It is given that $D \subset G \times G$.

Let $s, t \in G$ such that $st = u$. Then, $(s, s), (t, t) \in D$.

$$\begin{array}{rcl} (s, s)(t, t) & = & (st, st) \\ & = & (u, u) \\ & \in & D \end{array}$$

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I can see that this is called the diagonal subgroup because a table or graph of $G \times G$ will list the elements of D on its main diagonal.

9. Consider the direct product $\mathbb{R} \times \mathbb{R}$ of the additive group of real numbers with itself and the function $j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $j(x, y) = 2x - y$. Show that j is a homomorphism of groups; describe its kernel and image.

$$\begin{array}{rcl} j : (\mathbb{R}^2, +) & \to & (\mathbb{R}, +) \\ (x, y) & \mapsto & 2x + y \end{array}$$

Claim: j is a homomorphism.

Proof

$$\begin{array}{rcl} j : ((x_1, y_1) + (x_2, y_2)) & = & j : (x_1 + x_2, y_1 + y_2) \\ & = & 2(x_1 + x_2) - (y_1 + y_2) \\ & = & 2x_1 + 2x_2 - y_1 - y_2 \\ & = & (2x_1 - y_1) + (2x_2 - y_2) \\ & = & j : (x_1, y_1) + j(x_2, y_2) \end{array}$$

$\blacksquare$

Kernel: $\ker(j) = \{(x, y) : y = 2x\}$
Image: $\mathrm{Im}(j) = \mathbb{R}$.

3. (From the Problem Set) Given any ring R, the set of all two by two matrices with entries in R, denoted $M_2(R)$, forms a ring under matrix addition and matrix multiplication (when adding and multiplying these matrices together, you would be using the addition and multiplication in R).

   Suppose $R = \mathbb{R}[x]$, the polynomial ring in x over R. Write down the identity under addition in $M_2(R)$, the identity under multiplication in $M_2(R)$, and write down an element (which would be a two by two matrix) in $M_2(R)$ that has an inverse under multiplication, such that at least one of the entries is a non-constant polynomial in x. Such matrices are important in studying the stability of systems in control theory, a branch of systems engineering.

Example.

$$0_{M_2(R)} = \begin{bmatrix} f(x) = 0 & f(x) = 0 \\ f(x) = 0 & f(x) = 0 \end{bmatrix}$$

$$1_{M_2(R)} = \begin{bmatrix} f(x) = 1 & f(x) = 0 \\ f(x) = 0 & f(x) = 1 \end{bmatrix}$$

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$$A = \begin{bmatrix} x & 2x \\ -x & 2x \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2}x & -\frac{1}{2}x \\ \frac{1}{4}x & \frac{1}{4}x \end{bmatrix}$$