Topology Fall 2017 Selected Solutions

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Abstract

Quals are annoying, but collaboration before the quals doesn't have to be. Let me know if you find errors in any of these solutions and I'll correct them.

1 Problem 4

A collection, $\{B_\alpha\}_{\alpha \in I}$ of subsets of a set X is called a **basis for a topology on** X if the following holds:

1. Given $x \in X$, there is a $B\beta$ in our collection that contains x.

2. Given two basis elements B_α, B_β who's intersection is non-empty, for every $y \in B_\alpha \cap B_\beta$, there is a basis element $B_\gamma \ni y$ such that,

$$y \in B_\gamma \subseteq B_\alpha \cap B_\beta$$

Let X be the power set of the naturals. We show that the given collection of subsets of X is a basis. Let $E \in X$. Then $E \in [\emptyset, X]$. Now, suppose $[A, B], [C, D]$ have non-empty intersection and pick $E \in [A, B] \cap [C, D]$. Then $A \subseteq E$ and $C \subseteq E$. Hence, $A \cup C \subseteq E$. Moreover, $E \subseteq B$ and $E \subseteq D$. Since $B \cap D$ has finite compliment and $E \subseteq B \cup D$, we have $[A \cup C, B \cap D]$ is a basis element containing E. If $F \in [A \cup C, B \cap D]$, then $A \subseteq F$ and $C \subseteq F$. Moreover, $F \subseteq B$ while $F \subseteq D$. Thus,

$$E \in [A \cup C, B \cap D] \subseteq [A, B] \cap [C, D]$$

This shows the given collection is a basis.

Now, fix an element $x \in \mathbb{N}$. Then the elements $[\{x\}, X]$ and $[\emptyset, X \setminus \{x\}]$ are disjoint basis elements who's union is X. Thus, X is disconnected. Moreover, if $E, F \in X$ are differing elements, then suppose $y \in E$ and $y \notin F$ (without loss of generality). Then the aforementioned construction inducing a separation on X will produce disjoint open sets containing E and F, respectively. This shows X is Hausdorff.

Finally, let $(E_1, E_2) \in f^{-1}([A, B])$ for some fixed basis element $[A, B]$. Then $E_1 \cap E_2 \in [A, B]$. Thus,

$$(E_1, E_2) \in [A, B \cup E_1] \times [A, B \cup E_2] \tag{1}$$

Moreover, if K_1, K_2 are elements with the property that,

$$(K_1, K_2) \in [A, B \cup E_1] \times [A, B \cup E_2]$$

Then $A \subseteq K_1 \cap K_2$. But also, we have,

$$K_1 \cap K_2 \subseteq (B \cup E_1) \cap (B \cup E_2) = B \cup (E_1 \cap E_2) = B$$

so that $(K_1, K_2) \in f^{-1}([A, B])$. We conclude that the open set on the right hand side of (1) is contained entirely in $f^{-1}([A, B])$. Thus, we have furnished an open neighborhood of (E_1, E_2) contained in $f^{-1}([A, B])$, so f is continuous.

2 Problem 6

We show the proof for part b) here. Suppose f is not surjective. Since M is compact, $f(M)$ is compact and since M is a metric space, it is Hausdorff. Hence, $f(M)$ is closed. Let $y \notin f(M)$. Since $M \setminus f(M)$ is open, let $\epsilon > 0$ have the property that $B_\epsilon(y) \cap f(M) = \emptyset$. Thus, for every $z \in f(M)$, $d(y, z) > \epsilon$. Put $x_1 := y$ and let $x_2 := f(x_1)$. Having defined x_n, let $x_{n+1} := f(x_n)$. Fix $n, m > 0$ and suppose $n > m$. Since f is an isometry, we have,

$$d(x_n, x_m) = d(f(x_{n-1}), f(x_{m-1})) = d(x_{n-1}, x_{m-1}) = ... = d(x_{n-m+1}, x_1)$$

Since,

$$d(x_{n-m+1}, x_1) = d(x_{n-m+1}, y) > \epsilon$$

we conclude that $d(x_n, x_m) > \epsilon$ for any n, m. Thus, no subsequence of $\{x_n\}$ converges. But M is a compact metric space, hence sequentially compact!!

We conclude that y is surjective.

3 Problem 7

A metric space is **complete** if every Cauchy sequence in the metric space is convergent. The contraction mapping principle states that, if (X, d) is a complete metric space and $f : X \to X$ is a contraction mapping, then f has a unique fixed point. That is, if there is an $\alpha < 1$ so that,

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for every $x, y \in X$, then there is a unique $z \in X$ so that $f(z) = z$. We will prove this principle.

Pick $x_0 \in X$ and define $x_n := f(x_{n-1})$ for $n \geq 1$. Observe,

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \leq ... \leq \alpha^n d(x_1, x_0)$$

Moreover, for $m > n \geq 0$,

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n)$$

$$\leq \alpha^{m-1} d(x_1, x_0) + ... + \alpha^n d(x_1, x_0) = d(x_1, x_0)\alpha^n \sum_{j=0}^{m-n-1} \alpha^j \tag{2}$$

The latter sum in (2) is bounded by $\frac{1}{1-\alpha}$ since $\alpha < 1$, so $d(x_m, x_n) \to 0$ as $n \to \infty$. Thus, $\{x_n\}$ is Cauchy, so since X is complete, $x_n \to z$ for some z. f is continuous (being Lipschitz) so,

$$f(z) = \lim_{n\to\infty} f(x_{n-1}) = \lim_{n\to\infty} x_n = z$$

Uniqueness is immediate: if w, z are two fixed points, then,

$$d(w,z) = d(f(w), f(z)) \leq \alpha d(w,z) < d(w,z)$$

which is impossible. Thus, $d(w,z) = 0$, so $w = z$.
For the second part of this exercise, introduce the following function,

$$g : \mathbb{R}^n \to \mathbb{R}^n$$

$$g(x_1, ..., x_n) := \left(\sum_{j=1}^{n} a_{1j} f(x_j) + b_1, ..., \sum_{j=1}^{n} a_{nj} f(x_j) + b_n \right)$$

Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we examine the following:

$$|g(\mathbf{x}) - g(\mathbf{y})|^2$$

$$= \left| \left(\sum_{j=1}^{n} a_{1j}(f(x_j) - f(y_j)), ..., \sum_{j=1}^{n} a_{nj}(f(x_j) - f(y_j)) \right) \right|^2$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{kj}(f(x_j) - f(y_j)) \right)^2 \tag{3}$$

Applying mean value theorem gives, for each j, a c_j so that,

$$f(x_j) - f(y_j) = f'(c_j)(x_j - y_j) < M(x_j - y_j) \tag{4}$$

Combining (3) and (4) yields,

$$|g(\mathbf{x}) - g(\mathbf{y})|^2 < \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{kj} M(x_j - y_j) \right)^2 \tag{5}$$

By Cauchy Schwartz,

$$\sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{kj} M(x_j - y_j) \right)^2 \leq \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{kj}^2 \sum_{j=1}^{n} M^2 (x_j - y_j)^2 \right) \tag{6}$$

$$= M^2 |\mathbf{x} - \mathbf{y}|^2 \sum_{k,j} a_{kj}^2 < \alpha |\mathbf{x} - \mathbf{y}|^2 \tag{7}$$

where α is chosen so that,

$$\sum_{kj} a_{kj}^2 < \frac{\alpha}{M^2} < \frac{1}{M^2}$$

Thus, combining (5), (6), and (7) yields,

$$|g(\mathbf{x}) - g(\mathbf{y})| < \alpha |\mathbf{x} - \mathbf{y}|$$

Thus, g is an α contraction map on a complete metric space. As such, a fixed point of g exists. But this is exactly the desired point due to how g was defined.