6. Prove that if a topological space X has a dense connected subset then X is connected. Prove that a product of two connected spaces is connected.

Let X have a dense connected subset A. Let $f: X \to \{0,1\}$ be a continuous function. Then $f|_A: A \to \{0,1\}$ is continuous, & so f must be the constant function by the below lemma. Without loss of generality, let $f(A)=0$. If there exists an $x \in X$ such that $f(x)=1$, then $f^{-1}(1)$ is a nonempty open set of X. By density of A, $\exists\ a \in f^{-1}(1)$, which is a contradiction. So f must be the constant function, so X is connected.

<u>Lemma</u> If X is a topological space & $\{0,1\}$ is given the discrete topology, then X is connected iff every continuous function $f: X \to \{0,1\}$ is constant.

Proof: Let X be connected & $f: X \to \{0,1\}$ a continuous function. Then $f^{-1}(0)$ & $f^{-1}(1)$ are open, disjoint sets of X that cover X. If f is nonconstant then $f^{-1}(0) \cup f^{-1}(1)$ is a separation of X which contradicts X being connected.

If there is a separation of X, say $U \cup V = X$ where $U \cap V = \emptyset$ & U & V are nonempty open sets then $f: X \to \{0,1\}$ where $f(x) = \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in V \end{cases}$

is a nonconstant continuous function. □

Let X & Y be connected & let $f: X \times Y \to \{0,1\}$ be a continuous map. Then $f|_{X \times \{y\}}: X \times \{y\} \to \{0,1\}$ is a continuous map. Since $X \times \{y\}$ is homeomorphic to X, $X \times \{y\}$ is connected by the below lemma. So without loss of generality, assume $f(X \times \{y\}) = 0$. Similarly, $f(\{x\} \times Y)$ is a constant. If $\exists\ x \in X$