Math 450b
Homework 8

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1. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and suppose there is a constant M such that $||f(\mathbf{x})|| \leq M\,||\mathbf{x}||^2$ for all $\mathbf{x} \in \mathbb{R}^n$. Let $g(\mathbf{x}) = T(\mathbf{x}) + f(\mathbf{x})$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation. Prove that g is locally invertible near $\mathbf{0}$.

   Proof To prove that g is locally invertible near $\mathbf{0}$, we will show that $\det(Dg(\mathbf{0})) \neq 0$. Since

   $$g(\mathbf{x}) = T(\mathbf{x}) + f(\mathbf{x}),$$

   Then by linearity of derivatives,

   $$\begin{aligned} Dg(\mathbf{0})(\mathbf{x}) &= DT(\mathbf{0})(\mathbf{x}) + Df(\mathbf{0})(\mathbf{x}) \\ &= T(\mathbf{x}) + Df(\mathbf{0})(\mathbf{x}) \end{aligned}$$

   By an earlier homework problem, since $||f(\mathbf{x})|| \leq M\,||\mathbf{x}||^2$ for all $\mathbf{x} \in \mathbb{R}^n$, then $Df(\mathbf{0}) \equiv \mathbf{0}$. Thus,

   $$Dg(\mathbf{0})(\mathbf{x}) = T(\mathbf{x}),$$

   ans since T is invertible, $\det(Dg(\mathbf{0})) = \det(T) \neq 0$. ■

2. Determine whether the system

   $$\begin{aligned} u &= x + xyz \\ v &= y + xy \\ w &= z + 2x + 3z^2 \end{aligned}$$

   can be solved for x, y, z in terms of u, v, w near $(0, 0, 0)$.

   Proof Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $f(x, y, z) = (u, v, w)$. Note that $f(\mathbf{0}) = \mathbf{0}$. We seek some $f^{-1}(u, v, w) = (x, y, z)$ near $\mathbf{0}$. If $\det(Df(\mathbf{0})) \neq 0$ (We can already can see that f is continuous), then the Inverse Function Theorem guarantees the desired function.

   $$Df = \begin{bmatrix} 1+yz & xz & xy \\ y & 1+x & 0 \\ z & 0 & 1+6z \end{bmatrix} \quad Df(\mathbf{0}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

   $Df(\mathbf{0})$ is lower-triangular, so $\det Df(\mathbf{0}) = (1)(1)(1) \neq 0$. Thus, f is locally invertible near $\mathbf{0}$ and we are done. ■

3. Suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and one-to-one, with $Df(\mathbf{a}) \neq 0$ for all $\mathbf{a} \in U$. Prove that $f(U)$ is an open set.

   Proof Let $\mathbf{y} \in f(U)$ with $f(\mathbf{x}) = \mathbf{y}$ for some $\mathbf{x} \in U$. Since $\det(Df(\mathbf{x})) \neq 0$ by assumption, the Inverse Function Thm gives open sets V, W such that

   $$\mathbf{x} \in V \subset U, \quad \mathbf{y} \in W \subset f(U),$$

   thus $f(U)$ is open by the openness criterion. ■

4. Determine whether the system

$$\begin{aligned} 3x + 2y + z^2 + u + v^2 &= 0 \\ 4x + 3y + z + u^2 + v + w + 2 &= 0 \\ x + z + w + u^2 + 2 &= 0 \end{aligned}$$

can be solved for u, v, w in terms of x, y, z near $x = y = z = u = v = 0, w = -2$.

Proof Denote the point

$$x = y = z = u = v = 0, w = -2$$

as $-2\mathbf{e}_w$. Let $F : \mathbb{R}^6 \to \mathbb{R}^3$ be defined as

$$F(x, y, z, u, v, w) = \begin{bmatrix} 3x + 2y + z^2 + u + v^2 \\ 4x + 3y + z + u^2 + v + w + 2 \\ x + z + w + u^2 + 2 \end{bmatrix}$$

Observe that $F(-2\mathbf{e}_w) = \mathbf{0}$. If

$$\det\left(\left[\frac{\partial F_i}{\partial j}(-2\mathbf{e}_w)\right]_{\substack{i\in\{1,2,3\}\\ j\in\{u,v,w\}}}\right) \neq 0,$$

then the Implicit Function Theorem gives open sets $V_1 \in \mathbb{R}^3, V_2 \in \mathbb{R}^3$ with $\mathbf{0} \in V_1$ and $(0, 0, -2) \in V_2$, and some $f : V_1 \to V_2$ such that $F(x, y, z, f(x, y, z)) = \mathbf{0}$ for all $(x, y) \in V_1$. Thus, we calculate the above determinant.

$$\det\left(\left[\frac{\partial F_i}{\partial j}(-2\mathbf{e}_w)\right]_{\substack{i\in\{1,2,3\}\\ j\in\{u,v,w\}}}\right) = \det\left(\left.\begin{bmatrix} 1 & 2v & 0 \\ 2u & 1 & 1 \\ 2u & 0 & 1 \end{bmatrix}\right|_{-2\mathbf{e}_w}\right) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

as desired, and we are done. ■

5. Show that the equations

$$\begin{aligned} x^2 - y^2 - u^3 + v^2 + 4 &= 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 &= 0 \end{aligned}$$

determine functions $u(x, y), v(x, y)$ near $x = 2, y = -1$ such that $u(2, -1) = 2$, $v(2, -1) = 1$. Compute $\frac{\partial u}{\partial x}$.

Proof Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ be defined as

$$F(x, y, u, v) = \begin{bmatrix} x^2 - y^2 - u^3 + v^2 + 4 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 \end{bmatrix}.$$

Observe that $F(2, -1, 2, 1) = \mathbf{0}$. If

$$\det\left(\left[\frac{\partial F_i}{\partial j}(2, -1, 2, 1)\right]_{\substack{i\in\{1,2\}\\ j\in\{u,v\}}}\right) \neq 0,$$

then the Implicit Function Theorem gives open sets $V_1 \in \mathbb{R}^2, V_2 \in \mathbb{R}^2$ with $(2, -1) \in V_1$ and $(2, 1) \in V_2$, and some $f : V_1 \to V_2$ such that $F(x, y, f(x, y)) = F(x, y, u(x, y), v(x, y)) = \mathbf{0}$ for all $(x, y) \in V_1$. Thus, we calculate the above determinant.

$$\det\left(\left[\frac{\partial F_i}{\partial j}(2, -1, 2, 1)\right]_{\substack{i\in\{1,2\}\\ j\in\{u,v\}}}\right) = \det\left(\left.\begin{bmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{bmatrix}\right|_{(2,-1,2,1)}\right) = \begin{vmatrix} -12 & 2 \\ -8 & 12 \end{vmatrix} = -144 + 16 \neq 0$$

as desired, and we are done. ■

Solution Now we calculate $\frac{\partial u}{\partial x}$ by differentiating $F(x,y,u,v) = \mathbf{0}$ implicitly.

$$\frac{\partial}{\partial x}\left[\begin{array}{c} x^2 - y^2 - u^3 + v^2 + 4 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 \end{array}\right] = \frac{\partial}{\partial x}\left[\begin{array}{c} 0 \\ 0 \end{array}\right]$$

This gives the following system of equations, if we consider u and v to be functions of x but y to be constant with respect to x:

$$\begin{array}{cccc} 2x & -3u^2\frac{\partial u}{\partial x} & +2v\frac{\partial v}{\partial x} & = & 0 \\ 2y & -4u\frac{\partial u}{\partial x} & +12v^3\frac{\partial v}{\partial x} & = & 0 \end{array}$$

Solving this first equation for $\frac{\partial v}{\partial x}$ gives

$$\frac{\partial v}{\partial x} = \frac{3u^2}{2v}\frac{\partial u}{\partial x} - \frac{x}{v},$$

and we can substitute this into the second equation to find

$$\frac{\partial u}{\partial x} = \frac{6v^2x - y}{9v^2u^2 - 2u}$$

and we are done. ■