Topology Exam: Fall 2016.
Answer SIX of the NINE questions

1. Let X be the set of subsets of $\mathbb{N}$ (the set of positive integers). If A is a finite subset of $\mathbb{N}$, and B is a subset of $\mathbb{N}$ whose complement is finite, define a subset $[A, B]$ of X by

$$[A, B] = \{E \subset \mathbb{N} \mid A \subset E \subset B\}$$

   (a) Show that the sets $[A, B]$ form a base for a topology on X.
   (b) Prove that with this topology X is Hausdorff.
   (c) Prove that with this topology X is disconnected.
   (d) Prove that the function $f : X \to X$ given by $f(E) = \mathbb{N} \setminus E$ is continuous.

2. Give a proof or counterexample for each of the following.
   (a) Every closed subset of a compact space is compact.
   (b) The product of any two connected spaces is connected.

3. Prove that a metric space is compact if and only if it is sequentially compact.

4. A topological space X is *regular* if for every closed subset C of X and point $p \in X \setminus C$ there are disjoint open sets $U, V \subset X$ with $C \subset U$ and $p \in V$. Prove that every compact Hausdorff space is regular.
(It's also normal, but you don't have to prove that.)

5. For each of the following either give a proof, or give a counterexample and prove it is a counterexample.
   (a) Suppose A and B are non-empty topological spaces and $A \times B$ has the product topology.
   Let $\sim$ be the equivalence relation on $A \times B$ defined by $(a, b) \sim (a', b')$ if and only if $b = b'$.
   Is $A \times B/\sim$ homeomorphic to A?
   (b) Suppose B and C are subspaces of a topological space A.
   If B is homeomorphic to C, does it follow that A/B is homeomorphic to A/C?

6. Give an example of a space that is connected but not path connected. Prove that your example works.

7. State the contraction mapping theorem. Prove there is a unique continuous $f : [0, 1] \to [0, 1]$ which satisfies

$$\forall x \in [0, 1] \qquad f(x) = (f(\sin x) + \cos x)/2$$

[Hint: You might (or might not) wish to consider the metric space consisting of all continuous functions $f : [0, 1] \to [0, 1]$ with the metric $d(f, g) = \sup |f(x) - g(x)|$.]

8. Let X be the square $\{(x, y) \mid 0 \leq x, y \leq 1\}$ with the subspace topology from $\mathbb{R}^2$, and let $Y = X \setminus \{(0, 0)\}$.
Let $\sim$ be the equivalence relation $(x, y) \sim (x', y')$ if and only if either $(x, y) = (x', y')$ or $x = x' = 0$.
Are $X/\sim$ and $Y/\sim$ homeomorphic? Prove your answer is correct.
[hint: are $X/\sim$ and $Y/\sim$ compact ?]

9. In this question S^1 denotes the circle given by $\{(x, y) \mid x^2 + y^2 = 1\}$ as a subspace of $\mathbb{R}^2$. Suppose $p : X \to S^1$ is a connected covering space and $\pi : \mathbb{R} \to S^1$ is the covering space given by $\pi(t) = (\cos t, \sin t)$.
Prove there is a covering space $q : \mathbb{R} \to X$ with $\pi = p \circ q$.