Math 550
Homework 8
Dr. Fuller
Solutions

2. To apply Stokes', note that C is the boundary of a disk D in the plane $x+y+z=0$, and we compute $\int_D d(z^3 dx) = -\int_D 3z^2\, dx \wedge dz = \frac{\pi}{2\sqrt{3}}$.

   (To compute the integral, the challenge is to parameterize D. This can be done by finding an explicit orthonormal basis $\{\vec{u}, \vec{v}\}$ for the vector space $x+y+z=0$, and defining $g(r,\theta) = (r\cos\theta)\vec{u} + (r\sin\theta)\vec{v}$ for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.)

3. Direct calculation gives $d\omega = 0$. But $\int_{S^2} \omega|_{S^2} = \pm 4\pi$ (depending on a choice of orientation of S^2), so by the Corollary to Stokes' Theorem, ω is not closed.

4. Examples abound. For instance, if M is the open upper hemisphere of S^2 parameterized and oriented by $g(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$ with $0 < \theta < 2\pi$, and $0 < \varphi < \frac{\pi}{2}$, then $\int_M d(x\,dy) = \int_M dx \wedge dy = \frac{1}{2}$.

   However, $\partial M = \emptyset$, so $\int_{\partial M} x\,dy = 0$.

5. This follows immediately from Stokes' Theorem: $\int_M d\omega = \int_{\partial M} \omega = 0$.

   The counterexample given in the previous problem also works here, with $\omega = x\,dy$.

6. Direct calculation gives $d\omega = 0$.

   If $(0,0) \notin C$, then ω is defined on C, and we may use Stokes' Theorem to get $\int_{\partial C} \omega = \int_C d\omega = 0$.

   If $(0,0) \in C$, then $(0,0) \notin C - B_\varepsilon$, so we may use Stokes' Theorem on $C - B_\varepsilon$ to get

   $$0 = \int_{C-B_\varepsilon} d\omega = \int_{\partial(C-B_\varepsilon)} \omega = \int_{\partial C} \omega + \int_{\partial B_\varepsilon} \omega = \int_{\partial C} \omega - 2\pi.$$

   In the above, $\int_{\partial B_\varepsilon} \omega = -2\pi$ comes from direct calculation, keeping in mind that we must use the boundary orientation on ∂B_ε inherited from the standard orientation on C. This requires a clockwise orientation on ∂B_ε, and the use of a parameterization such as $g(\theta) = (\varepsilon\sin\theta, \varepsilon\cos\theta)$.