Math 550
Homework 7
Dr. Fuller
Solutions

1. Let ω be the given non-zero k-form on M. Suppose that $g_\alpha : U_\alpha \to M$ is a collection of local parameterizations covering M. Let $i : \mathbf{R}^k \to \mathbf{R}^k$ be $i(x_1, x_2, x_3, \ldots, x_n) = (x_2, x_1, x_3, \ldots, x_n)$. Define a collection of local parameterizations $h_\alpha : V_\alpha \to M$ by the rule

$$h_\alpha = \begin{cases} g_\alpha & \text{if } g_\alpha^* \omega(u)(e_1, \ldots, e_k) > 0 \text{ for } u \in U_\alpha \\ g_\alpha \circ i & \text{if } g_\alpha^* \omega(u)(e_1, \ldots, e_k) < 0 \text{ for } u \in U_\alpha; \end{cases}$$

   We also set $V_\alpha = U_\alpha$ in the upper case, and $V_\alpha = i(U_\alpha)$ in the lower. Then for all α, we have $h_\alpha^* \omega(v)(e_1, \ldots, e_k) > 0$ for $v \in V_\alpha$. This shows that M is orientable.

2. It suffices to verify the formula for $\omega = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. From the definition, it is clear that $\star\star\,\omega = \varepsilon\omega$, for $\varepsilon \in \{-1, 1\}$. We must verify that $\varepsilon = (-1)^{k(n-k)}$.

   We have that

$$\omega \wedge \star\omega = \star\omega \wedge \star\star\,\omega = (-1)^{k(n-k)} \star\star\omega \wedge \star\omega.$$

   The first equality follows because both are equal to $dx_1 \wedge \cdots \wedge dx_n$. Then

$$0 = (\omega - (-1)^{k(n-k)} \star\star\omega) \wedge \star\omega = (1 - (-1)^{k(n-k)}\varepsilon)\ \omega \wedge \star\omega = (1 - (-1)^{k(n-k)}\varepsilon)\ dx_1 \wedge \cdots \wedge dx_n.$$

   Thus $1 - (-1)^{k(n-k)}\varepsilon = 0$, which implies $\varepsilon = (-1)^{k(n-k)}$.