Math 550
Homework 1

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September 4, 2018

1. The set $\Lambda^n(\mathbb{R}^n)$ of all alternating, multilinear functions on $(\mathbb{R}^n)^n$ forms a vector space. (You do not have to prove this.) What is its dimension? Find a basis for this vector space.

   Answer: We know by Thm 1 that there exists only one alternating multilinear function D on $(\mathbb{R}^n)^n$ such that $D(I_n) = 1$ (where I_n denotes the standard basis of $\mathbb{R}^n, \{\vec{e}_1, \ldots, \vec{e}_n\}$). Thus, D is completely and uniquely determined by its behavior on I_n.

   Claim: $\{D\}$ is a basis for $\Lambda^n(\mathbb{R}^n)$, and therefore, the space has dimension 1.

   Proof Let F be some element of $\Lambda^n(\mathbb{R}^n)$, and let $k = F(I_n)$. Also let A be an arbitrary element of $(\mathbb{R}^n)^n$ written as $A = \{\sum_i a_{ij} e_i\}_{j=1}^n$. Then,

$$\begin{aligned}
F(A) &= F\left(\sum_{i_1} a_{i_1 1} e_{i_1},\ \sum_{i_2} a_{i_2 2} e_{i_2},\ \ldots,\ \sum_{i_n} a_{i_n n} e_{i_n}\right) \\
&= \sum_{i_n} \cdots \sum_{i_1} (a_{i_1 1})(\cdots)(a_{i_n n}) F(e_{i_1}, \ldots, e_{i_n}) \\
&= \sum_{\sigma} (a_{\sigma(1)1})(\cdots)(a_{\sigma(n)n}) F(e_{\sigma(1)}, \ldots, e_{\sigma(n)}) \\
&= \sum_{\sigma} (a_{\sigma(1)1})(\cdots)(a_{\sigma(n)n})(\pm k) \\
&= k \sum_{\sigma} (\operatorname{sign}\sigma)(a_{\sigma(1)1})(\cdots)(a_{\sigma(n)n}) \\
&= kD(A)
\end{aligned}$$

   Thus, $F = kD$ since they agree at any arbitrary point, and so $\{D\}$ spans the space. ∎

2. Let V be an n-dimensional vector space with an inner product $\langle\ ,\ \rangle$. Suppose $S \in \Lambda^n(V)$ is an alternating multilinear function on V.

   (a) Let $(\vec{u}_1, \ldots, \vec{u}_n)$ be a basis for V. Suppose $(\vec{v}_1, \ldots, \vec{v}_n)$ is a collection of vectors in V with $\vec{v}_j = \sum_i a_{ij} \vec{u}_i$. Prove that $S(\vec{v}_1, \ldots, \vec{v}_n) = \det[a_{ij}] S(\vec{u}_1, \ldots, \vec{u}_n)$.

   Proof

$$\begin{aligned}
S(\vec{v}_1, \ldots, \vec{v}_n) &= S\left(\sum_{i_1} a_{i_1 1} u_{i_1},\ \sum_{i_2} a_{i_2 2} u_{i_2},\ \ldots,\ \sum_{i_n} a_{i_n n} u_{i_n}\right) \\
&= \sum_{i_n} \cdots \sum_{i_1} (a_{i_1 1})(\cdots)(a_{i_n n}) S(u_{i_1}, \ldots, u_{i_n}) \\
&= \sum_{\sigma} (a_{\sigma(1)1})(\cdots)(a_{\sigma(n)n}) S(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \\
&= \sum_{\sigma} (\operatorname{sign}\sigma)(a_{\sigma(1)1})(\cdots)(a_{\sigma(n)n}) S(u_1, \ldots, u_n) \\
&= \left(\sum_{\sigma} (\operatorname{sign}\sigma)(a_{\sigma(1)1})(\cdots)(a_{\sigma(n)n})\right) S(u_1, \ldots, u_n) \\
&= \det[a_{ij}] S(\vec{u}_1, \ldots, \vec{u}_n)
\end{aligned}$$

∎

(b) Suppose that $(\vec{u}_1, \ldots, \vec{u}_n)$ and $(\vec{v}_1, \ldots, \vec{v}_n)$ are two orthonormal bases for V, with $\vec{v}_j = \sum_i a_{ij}\vec{u}_i$. Let $A = [a_{ij}]$. Prove that $AA^T = I$. (Hint: start by considering $\langle \vec{v}_i, \vec{v}_j \rangle$.)

(c) Prove that $|S(\vec{u}_1, \ldots, \vec{u}_n)| = |S(\vec{v}_1, \ldots, \vec{v}_n)|$ for any two orthonormal bases of V.