Math 501
Homework 1

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1. For a set A, let $\{0,1\}^A$ be the set of all functions from A to the two point set $\{0,1\}$. Prove that $|\{0,1\}^A| = |\mathscr{P}(A)|$.

   Proof Let $\Omega : \{0,1\}^A \to \mathscr{P}(A)$ be the following mapping:

   $$\Omega(f) = \{x \in A : f(x) = 1\}$$

   This is to say that, for any function $f : A \to \{0,1\}$, where the set $S \subset A$ is the set such that $f(S) = \{1\}$, then $\Omega(f) = S$.

   Claim: Ω is a bijection, so $|\{0,1\}^A| = |\mathscr{P}(A)|$.

   First, we will show that Ω is 1-1. Let $f, g : A \to \{0,1\}$ be two distinct functions. Since $f \neq g$, then there exists some $x \in A$ which maps to 1 under one function, and 0 under the other. Therefore, the set $\Omega(f) \neq \Omega(g)$, since x is an element of one set, and not the other. Thus, Ω is 1-1.

   Now, we will show that Ω is onto. Let S be an arbitrary subset of A. Since the domain of Ω is $\{0,1\}^A$ (which is the set of *all* functions from A to $\{0,1\}$), then there exists a function $f \in \mathscr{D}(\Omega)$ such that $f(S) = \{1\}$, and $f(A - S) = \{0\}$. This means that $\Omega(f) = S$, and therefore, Ω is onto. ∎

2. Prove that there is no function from a set A onto $\{0,1\}^A$.

   Proof Let Φ be a function from A to $\{0,1\}^A$, and let a be an arbitrary element of A. Then, there exist functions $f_{a_0}, f_{a_1} \in \{0,1\}^A$ such that

   $$f_{a_0}(x) = \begin{cases} 0 & x = a \\ 1 & x \neq a \end{cases} \qquad f_{a_1}(x) = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$$

   This means that for every $x \in A$; $\Phi(x)$ can be either f_{x_0}, or f_{x_1}, (or neither), but not both. Therefore, whenever Φ maps an element of A to a function in $\{0,1\}^A$, it always leaves at least one other function behind. Thus, Φ is not onto. ∎

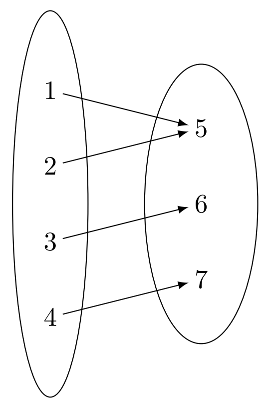
3. Let $f : X \to Y$ be a function bewteen sets X and Y.

   (a) Suppose $A \subset X$. Prove that $A \subset f^{-1}(f(A))$.

   Proof Let x be an arbitrary element of A. Since $x \in \mathscr{D}(f)$, there exists some $y \in Y$ such that $f(x) = y$. Consider the preimage of y; $f^{-1}(\{y\})$. Since $f(x) = y$, then $x \in f^{-1}(\{y\})$, by definition. Also, since $x \in A$ and $f(x) = y$, then $\{y\} \subset f(A)$. Therefore, $x \in f^{-1}(\{y\}) \subset f^{-1}(f(A))$. ∎

(b) Give an example to show $f^{-1}(f(A)) \not\subset A$.

Example. Consider the following function, $f : \{1,2,3,4\} \to \{5,6,7\}$, whose definition is given by the following figure:



Let $A = \{2,3,4\}$. According to the figure above, $f(A) = \{5,6,7\}$, and $f^{-1}(f(A)) = \{1,2,3,4\}$. So, $f^{-1}(f(A)) \not\subset A$. ■

(c) Suppose $B \subset Y$. How are the sets B and $f(f^{-1}(B))$ related? Give a proof and/or example(s) to justify your answer.

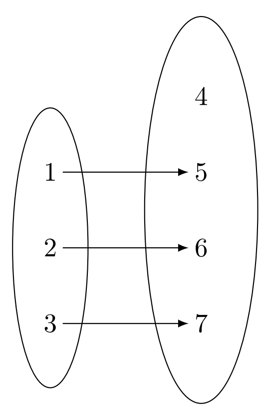
Claim: $f(f^{-1}(B)) \subset B$.

Proof In the trivial case where $f^{-1}(B) = \emptyset$, $f(f^{-1}(B))$ must also be $\emptyset$, so clearly $f(f^{-1}(B)) \subset B$ and we are done.

Now, suppose that $f^{-1}(B) \neq \emptyset$, and let $y \in f(f^{-1}(B))$. Since y is in the image of $f^{-1}(B)$, there must be an $x \in f^{-1}(B)$ such that $f(x) = y$. Since x is in the preimage of B, $f(x)$ must be in B. Therefore, $f(x) = y \in B$. Thus, $f(f^{-1}(B)) \subset B$. ■

Claim: $B \not\subset f(f^{-1}(B))$.

Example. Consider the following function, $f : \{1,2,3\} \to \{4,5,6,7\}$, whose definition is given by the following figure:



Let $B = \{4,5,6,7\}$. According to the figure above, $f^{-1}(B) = \{1,2,3\}$, and $f(f^{-1}(B)) = \{5,6,7\}$. So, $B \not\subset f(f^{-1}(B))$. ■

4. Give an example to show that an arbitrary (i.e. not necessarily finite) intersection of open sets in $\mathbb{R}^n$ need not be open.

   Example. Consider the following set of open intervals:

   $$S = \{\left(0, \tfrac{n+1}{n}\right) : n \in \mathbb{N}\}.$$

   Let I_n denote the nth element of this set, that is, $I_n = (0, \frac{n+1}{n})$. Since the infimum of the set of upper bounds for these intervals is 1,

   $$\bigcap_{n=1}^{\infty} I_n = (0, 1].$$

   The set $(0, 1]$ is not open because every open ball centered at 1 contains real numbers greater than 1, so cannot be a subset of $(0, 1]$. ∎

5. Professor Doofus mistakenly writes the following on the blackboard.

   Theorem. The following are equivalent.

   (1) $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at all $x \in \mathbb{R}^n$ (with the δ-ϵ definition)

   (2) For every open set $U \subset \mathbb{R}^n$, the image $f(U) \subset \mathbb{R}^m$ is open.

   Give an example which shows why Doofus is wrong.

   Example. Let $y \in \mathbb{R}^m$. Consider a constant function,

   $$f(x) = y, \quad \forall x \in \mathbb{R}^n.$$

   The function f is continuous because given any $\epsilon > 0$, we can let $\delta = 1$, and if $x \in B(x_0, \delta)$, then $f(x) \in B(f(x_0), \epsilon)$ because $f(x) = f(x_0) = y$. So, (1) is true in this case.

   However, for any set $A \subset \mathbb{R}^n$, the image $f(A) = \{y\}$ is *not* open. This is because for every $r > 0$, the ball $B(y, r)$ contains points distinct from y, so it cannot be a subset of $\{y\}$. Therefore, (2) is false, so (1) and (2) are not equivalent statements. ∎