Baire Category & Nowhere Dense Sets

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The following theorem expresses a very useful property of **complete metric spaces**. It was first proved by W. F. Osgood in 1897 for the real line $\mathbb{R}$, and (independently) by R. Baire in 1899 for $\mathbb{R}^n$.

Theorem 1 (Baire Category Theorem) *If a (nonempty) complete metric space E is the union of a countable family $(F_k)_{k\in\mathbb{N}}$ of closed subsets, then at least one of these closed subsets contains a nonempty open set.*

Proof. Suppose no F_k contains a nonempty open set. Then, in particular, no F_k equals E. In particular $F_1 \neq E$, so $\complement F_1$ is a nonempty open set which must therefore contain an open ball $B_1 = B(x_1;\epsilon_1)$ with $0 < \epsilon_1 < \frac{1}{2}$. The set F_2 does not contain the open ball $B(x_1;\epsilon_1/2)$. Hence the nonempty open set $\complement F_2 \cap B(x_1;\epsilon_1/2)$ contains an open ball $B_2 = B(x_2;\epsilon_2)$ with $0 < \epsilon_2 < \frac{1}{4}$. By the **principle of inductive definition** we obtain a sequence $B_k = B(x_k;\epsilon_k)$ of open balls such that, for all integers $k \geq 1$, $0 < \epsilon_k < \frac{1}{2^k}$, $B_{k+1} \subset B(x_k;\epsilon_k/2)$, and $B_k \cap F_k = \emptyset$. In particular, the family $(F_k)_{k\in\mathbb{N}}$ *must be infinite*. (That is, in the finite case the proof is complete.) Since, for $n < m$,

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) < \sum_{k=n}^{m-1} \frac{1}{2^{k+1}} < \frac{1}{2^n},$$

the centers x_k of these balls form a Cauchy sequence; and so converge to a point x^* of E. Since for all $m > n$

$$d(x_n, x^*) \leq d(x_n, x_m) + d(x_m, x^*) < \frac{\epsilon_n}{2} + d(x_m, x^*) \quad \text{and} \quad \lim_{m\to\infty} d(x_m, x^*) = 0,$$

it follows that $d(x_n, x^*) \leq \frac{\epsilon_n}{2}$ which means that $x^* \in B_n$ for every $n \geq 1$. Therefore x^* is in none of the sets F_n, and so is not in their union which is E. But this contradicts the hypothesis of the theorem. We can only conclude that at least one of the closed sets F_k does contain a nonempty open set. This completes the proof of Theorem 1. □

Definition 1 *A subset N of a topological space E is said to be* **nowhere dense (n. d.)** *if its closure has no interior points.*

Definition 2 *The* **exterior** *of a set $A \subset E$, denoted* $\mathrm{Ext}(A)$*, is defined by* $\mathrm{Ext}(A) := \complement\bar{A}$*, which is the same as* $\mathrm{Int}(\complement A)$*, the* **interior** *of* $(\complement A)$.

Exercise 1 (a) *Show that* $\complement\bar{A} = \mathrm{Int}(\complement A)$. (b) *Also show that* $\complement\,\mathrm{Int}(A) = \overline{\complement A}$.

Proposition 1 *A subset N of a topological space E is nowhere dense if and only if it satisfies any one, and therefore all, of the following equivalent conditions.*

(N1) Its closure has no interior points.

(N2) $\mathrm{Ext}(N)$ *is dense (in E).*

(N3) Every nonempty open set U contains a nonempty open set V not intersecting N.

(N4) Its closure $\overline{N}$ is nowhere dense.

Exercise 2 *Prove the equivalence of the four conditions stated in Proposition 1.*

Proposition 2 *Some properties of nowhere dense sets.*

1. *A closed set is nowhere dense iff it has no interior point.*
2. *Every subset of a nowhere dense set is nowhere dense.*
3. *A nowhere dense subset N of a subspace $S \subset E$ is also nowhere dense in E.*
4. *If G is open and dense in E, then $\complement G$ is nowhere dense.*
5. *If F is closed and nowhere dense, then $\complement F$ is dense.*

Exercise 3 *Prove the five properties stated in Proposition 2.*

The Baire Category Theorem can be stated a second way as follows.

Theorem 2 (Baire Category Theorem) *A complete metric space cannot be expressed as a countable union of nowhere dense subsets.*

Definition 3 *A subset M of a topological space is said to be* **meagre** *(or* **first category***) if it can be expressed as the countable union of nowhere dense subsets. It is said to be* **nonmeagre** *(or* **second category***) if it is not meagre.*

The Baire Category Theorem can be restated a third way as follows.

Theorem 3 (Baire Category Theorem) *A complete metric space E is second category in itself.*

Definition 4 *A subset R of a topological space E is said to be* **residual** *(or* **comeagre***) in E if $\complement R$ is first category (meagre) in E.*

Definition 5 *A topological space E is called a* **Baire space** *if it satisfies any one, and therefore all, of the following equivalent properties:*

(B1) Every countable intersection of dense open sets in E is dense in E.

(B2) Every countable union of nowhere dense closed sets has no interior point.

(B3) Every meagre set in E has empty interior.

(B4) Every nonempty open set in E is nonmeagre.

(B5) Every residual set in E is dense in E.

Proof that (B1) $\Rightarrow$ (B2): Let $F_\sigma = \bigcup_n F_n$ be a countable union of closed nowhere dense subsets of E. Then $G_\delta = \complement F_\sigma = \bigcap_n G_n$ (where $G_n = \complement F_n$) is a countable intersection of open dense sets in E. Therefore G_δ is dense in E by (B1). But if F_σ contains an interior point, then there exists an open set G such that $G \subset F_\sigma$ or $G \cap \complement F_\sigma = G \cap G_\delta = \emptyset$,which is impossible if G_δ is dense.

Proof that (B2) $\Rightarrow$ (B3): Let $M = \bigcup_n N_n$ be a meagre subset of E. Since each N_n is nowhere dense, so is $\overline{N}_n$ for each n. Define $\widehat{M} = \bigcup_n \overline{N}_n$. Then (B2) implies $\mathrm{Int}(\widehat{M}) = \emptyset$. But since $M \subset \widehat{M}$, it follows that $\mathrm{Int}(M) = \emptyset$.

Proof that (B3) $\Rightarrow$ (B4): This is obvious.

Proof that (B3) $\Rightarrow$ (B5): To say that the set R is residual means that $M = \complement R$ is meagre. Then (B3) implies that $\mathrm{Int}(M) = \mathrm{Int}(\complement R) = \emptyset$. But $\mathrm{Int}(\complement R) = \complement \overline{R}$. Therefore $\complement \overline{R} = \emptyset$ or $\overline{R} = E$.

Proof that (B5) $\Rightarrow$ (B1): Let $G_\delta = \bigcap_n G_n$ be a countable intersection of open dense sets G_n. Then $F_\sigma = \complement G_\delta = \bigcup_n F_n$, where $F_n = \complement G_n$ is nowhere dense. Thus F_σ is meagre so that G_δ is residual. It now follows from (B5) that G_δ is dense.

Theorem 4 *Every complete metric space is a Baire space.*

Proof. We use Theorem 1 (Baire Category Theorem) to establish property (B1) of Definition 4. Let $G_\delta = \bigcap_n G_n$ be a countable intersection of open sets G_n each dense in E. If G_δ is not dense, then there exists a nonempty open set $G \subset E$ such that

$$\emptyset = G \cap G_\delta = \bigcap\nolimits_n (G \cap G_n). \tag{1}$$

Let $H_n = G \cap G_n$. Then each H_n is an open and dense subset of $\overline{G}$ so that $\overline{G} \setminus H_n$ is a closed and nowhere dense subset of $\overline{G}$. Applying a De Morgan Formula to (1) we obtain

$$\overline{G} = \overline{G} \setminus \emptyset = \bigcup\nolimits_n (\overline{G} \setminus H_n). \tag{2}$$

But $\overline{G}$ is a closed (and hence complete) subspace of E, so that it cannot be expressed as a countable union of closed nowhere dense sets (by Theorem 1) as it is in (2). Thus $G \cap G_\delta \neq \emptyset$ and so G_δ must be dense. □

Remark 1 *Another class of Baire spaces is provided by the class of locally compact spaces, metrizable or not. However, there are Baire spaces which are neither locally compact nor metrizable; there are also metrizable Baire spaces which are not complete with respect to any equivalent metric. Every nonempty open subspace of a Baire space is a Baire space. If every point of a topological space E has a neighborhood which is a Baire space, then E itself is a Baire space. In a Baire space E, the complement of a meagre set is a Baire space. See N. Bourbaki,* Elements of Mathematics*,* **General Topology***,* Part 2, Chapter IX, § 5, pages 190–195. *Addison-Wesley Pub. Co. 1966.*

References

William Fogg Osgood [1864–1943] René Baire [1874–1932]

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Appendix I
Some theorems which can be proved by means of the Baire Category Theorem

1. If a metric space has no isolated points, then it is uncountable.
   Corollary: $\mathbb{R}$ and the Cantor Ternary Set are uncountable. [Boas]

2. If $f : [0,1] \to \mathbb{R}$ has derivatives of all orders and if f is not a polynomial, then there exists a point $x \in [0,1]$ at which no derivative $f^{(n)}(x)$ of f vanishes. [Boas]

3. There exist many continuous functions $f : \mathbb{R} \to \mathbb{R}$ which are nowhere differentiable. Indeed, they form a set of second category in the space $\mathcal{C}$ of continuous functions. [Boas]

4. A closed interval in $\mathbb{R}$ cannot be expressed as the union of a denumerable family of disjoint nonempty closed sets.

5. If $(E, \|x\|)$ is a complete normed linear space then the its Hamel base dimension cannot be $\aleph_0$ (read "aleph-naught"), nor any cardinal which is the limit of a denumerable number of smaller cardinals. [Goffman]

6. Let E be a vector space (over $\mathbb{R}$ or $\mathbb{C}$) which has an infinite Hamel basis H. Define the **norm** $\|x\|_H$ on E by the formula $\|x\|_H = \sum_k |\xi_k|$, where $x = \sum_k \xi_k \alpha_k$, $\alpha_k \in H$. Then the normed linear space $(E, \|x\|_H)$ is not complete.

7. The **Banach-Steinhaus Principle of Uniform Boundedness**.

8. The **Interior Mapping Principle**.

9. A homeomorphism f of a separable complete metric space X is transitive iff it is transitive at some point; i.e., iff some point $x \in X$ has a dense orbit in X. A homeomorphism $f : X \to X$ of a topological space X onto itself is called **transitive** if for every pair of nonempty open sets U and V there is an integer $n \in \mathbb{Z}$ such that $f^{[n]}(U) \cap V \neq \emptyset$. Theorem: If $f : X \to X$ is transitive on a complete metric space, then the set T of all transitive points of f is an invariant residual G_δ.

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Appendix II
SOLUTIONS TO THE EXERCISES

Solution to Exercise 1 (a): We want to show that $\complement\overline{A} = \mathrm{Int}(\complement A)$.

$\complement\overline{A}$ is an open set contained in $\complement A$. (Because $S \subset \overline{S}$ & $S \subset T \Rightarrow \complement T \subset \complement S$.)

Therefore $\complement\overline{A} \subset \mathrm{Int}(\complement A)$. (1)

$\forall x \in \mathrm{Int}(\complement A)$, $x \notin \overline{A}$. (Because $x \in \overline{A}$ means every neighborhood of x intersects A.)

So $\forall x \in \mathrm{Int}(\complement A)$, $x \in \complement\overline{A}$. That is, $\mathrm{Int}(\complement A) \subset \complement\overline{A}$. (2)

Combining (1) and (2) we have $\complement\overline{A} = \mathrm{Int}(\complement A)$. □

Solution to Exercise 1 (b): We want to show that $\complement\,\mathrm{Int}(A) = \overline{\complement A}$.

$\overline{\complement A} \supset \complement A$. (Because $\overline{S} \supset S$.)

Therefore $\complement\overline{\complement A} \subset \complement\complement A = A$. So $\complement\overline{\complement A}$ is an open set contained in A.

Therefore $\complement\overline{\complement A} \subset \mathrm{Int}(A)$. (Because $\mathrm{Int}(A)$ is the *largest* open set contained in A.)

Thus $\overline{\complement A} \supset \complement\,\mathrm{Int}(A)$. (3)

But $\mathrm{Int}(A) \subset A$, so $\complement\,\mathrm{Int}(A) \supset \complement A$; and so with (3) we obtain $\overline{\complement A} \supset \complement\,\mathrm{Int}(A) \supset \complement A$.

But $\overline{\complement A}$ is the *smallest* closed set containing $\complement A$, so $\overline{\complement A} = \complement\,\mathrm{Int}(A)$. □

Solution to Exercise 2:

Proof that (N1) ⇒ (N2):

Suppose *(N2)* is false: I.e., suppose $\complement\overline{N}$ ($= \mathrm{Ext}N$) is *not dense* in E.

Then $\exists$ a nonempty open set G such that $G \cap \complement\overline{N} = \emptyset$.

Therefore $G \subset \overline{N}$, so $\mathrm{Int}\overline{N} \neq \emptyset$, contrary to the hypothesis *(N1)*. □

Proof that (N2) ⇒ (N1):

Suppose *(N2)* is true; that is, suppose $\mathrm{Ext}(N) := \complement\overline{N}$ is dense in E.

Then $\forall$ nonempty open set G, $G \cap \complement\overline{N} \neq \emptyset$, so $G \not\subset \overline{N}$.

Therefore $\mathrm{Int}\overline{N} = \emptyset$, which is *(N1)*. □

Proof that (N1) ⇒ (N3):

Suppose *(N3)* is false: Then there is a nonempty open subset U of E every nonempty open subset of which intersects N. This means N is dense in U so that $\overline{N} \supset U \neq \emptyset$.

Thus, $\overline{N}$ contains interior points contrary to the hypothesis *(N1)*. □

Proof that (N3) ⇒ (N1):

Suppose *(N1)* is false: Then there is a nonempty open set $U \subset \overline{N}$.

By *(N3)*, $\exists$ a nonempty open set $V \subset U$ such that $V \cap N = \emptyset$. (4)

But then we have $\emptyset \neq V \subset U \subset \overline{N}$, so that $V \subset \overline{N} \setminus N$.

This means each point of the nonempty open set V is a limit point of N.

Therefore, $V \cap N \neq \emptyset$ which contradicts (4). □

Proof that (N1) ⇔ (N4):

N is nowhere dense *iff* $\mathrm{Int}\overline{N} = \emptyset$ *iff* $(\mathrm{Int}\overline{\overline{N}} = \emptyset)$ *iff* $\overline{N}$ is nowhere dense. □

Appendix II (continued)
SOLUTIONS TO THE EXERCISES

Solution to Exercise 3:

Proof of Proposition 2.1: We want to prove that
each closed set is n.d. iff it has no interior points.
Let F be a closed set. This means $\overline{F} = F$. So F is n.d. *iff* $\mathrm{Int}\overline{F} = \emptyset$ *iff* $\mathrm{Int}F = \emptyset$. □

Proof of Proposition 2.2: We want to show that
each subset of a nowhere dense set is itself nowhere dense.
Let $N \subset E$ be n.d. in a topological space E: I.e., $\mathrm{Int}\overline{N} = \emptyset$.
Let $S \subset N$. Then $\overline{S} \subset \overline{N}$; so $\mathrm{Int}\overline{S} \subset \mathrm{Int}\overline{N}$. Since $\mathrm{Int}\overline{N} = \emptyset$, $\mathrm{Int}\overline{S} = \emptyset$ too. □

Proof of Proposition 2.3: We want to show that
each n.d. subset N of a subspace S of a topological space E is also n.d. in E.
If not, there $\exists$ a nonempty E-open subset $U \subset \overline{N}^E$; and so $U \cap S \subset \overline{N}^E \cap S = \overline{N}^S$.
(The notation $\overline{N}^S$ denotes **the closure of N in the subspace S**.)
Note that, $\forall\, x \in U \subset \overline{N}^E$, $x \in \overline{N}^E$ and U is an E-neighborhood of x, so $U \cap N \neq \emptyset$.
Then we see that $U \cap S \supset U \cap N \neq \emptyset$, so $U \cap S$ is a nonempty S-open subset of $\overline{N}^S$,
which contradicts our hypothesis that N is n.d. in S. □

Proof of Proposition 2.4: We want to show that
the complement of each open dense set is nowhere dense.
Let G be open and dense in E. Then $\overline{G} = E$.
Therefore $\complement\overline{G} = \complement E = \emptyset$; and $\complement\overline{G} = \mathrm{Int}(\complement G)$ by Exercise 1(a); so $\mathrm{Int}(\complement G) = \emptyset$.
Since $\complement G$ is closed, the latter means that it is n.d. □

Proof of Proposition 2.5: We want to show that
the complement of each closed nowhere dense set is dense.
We are given (1) $\overline{F} = F$, and (2) $\mathrm{Int}\overline{F} = \mathrm{Int}F = \emptyset$.
By Exercise 1(b), we also have $\overline{\complement F} = \complement\,\mathrm{Int}F$.
So $\overline{\complement F} = \complement\,\mathrm{Int}F = \complement\emptyset = E$, which means that $\complement F$ is dense in E. □