Final Exam

1. Let X be a nonempty topological space and let $\{\mu_n\}_{n=1}^\infty$ be a sequence of Borel-regular measures on X. Assume for any $A \subset X$ the sequence $\mu_n(A) \searrow \mu(A)$. Prove that if $\mu_1(X) < \infty$, then μ is a measure on X.

Proof (Null empty set) Observe that

$$\mu(\emptyset) = \lim_{n\to\infty} \mu_n(\emptyset) = \lim_{n\to\infty} 0 = 0.$$

(Monotonicity and subadditivity) Suppose that $A \subset \bigcup_{i=1}^\infty A_i$. Letting

$$\begin{aligned} B_1 &= A_1 \\ B_n &= A_n \setminus B_{n-1} \end{aligned}$$

gives $A \subset \bigcup_{i=1}^\infty A_i = \coprod_{i=1}^\infty B_i$. Thus since every $B_i \subseteq A_i$, then $\mu_n(B_i) \le \mu_n(A_i)$ for all n, so $\mu(B_i) \le \mu(A_i)$, which means it suffices to show that $\mu(A) \le \sum_{i=1}^\infty \mu(B_i)$.

Observe that

$$\begin{aligned} \mu(A) &= \lim_{n\to\infty} \mu_n(A) \\ &\le \lim_{n\to\infty} \sum_{i=1}^\infty \mu_n(B_i) \\ &= \limsup_{n\to\infty} \sum_{i=1}^\infty \mu_n(B_i) \\ &\left.\begin{aligned} &= \limsup_{n\to\infty} \int_{x=1}^\infty \mu_n(B_{\lfloor x \rfloor})\, dx \\ &\le \int_{x=1}^\infty \limsup_{n\to\infty} \mu_n(B_{\lfloor x \rfloor})\, dx \end{aligned}\right\} \text{By reverse Fatou's lemma} \\ &= \int_{x=1}^\infty \mu(B_{\lfloor x \rfloor})\, dx \\ &= \sum_{i=1}^\infty \mu(B_i). \end{aligned}$$

To finish the proof, we will justify the above use of reverse Fatou's lemma.

Observe that $\mu_n(B_{\lfloor x\rfloor})$ is a countable sequence of real-valued measurable functions of x, and $\mu_1(B_{\lfloor x\rfloor})$ dominates the sequence since $\mu_n(B) \leq \mu_1(B)$ for all sets B, and

$$\begin{aligned}
\int_1^\infty \mu_1(B_{\lfloor x\rfloor})\,dx &= \sum_{i=1}^\infty \mu_1(B_i) \\
&\left.\begin{aligned} &= \sum_{i=1}^\infty \mu_1(B_i) \\ &= \mu_1\left(\bigcup_{i=1}^\infty B_i\right) \end{aligned}\right\}\text{Since } B_i \text{ sets are disjoint} \\
&\leq \mu_1(X) \\
&< \infty.
\end{aligned}$$

Thus $\mu_1(B_{\lfloor x\rfloor})$ is dx-summable, and the conditions of reverse Fatou's lemma are satisfied. ■

2. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue-measurable. Prove that there exists a Borel-measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ Lebesgue-*a.e.* in $\mathbb{R}$.

Proof Approximate f by simple functions as follows. Let $A_k^n = [\frac{n}{2^k}, \frac{n+1}{2^k}]$ ■