Math 550
Homework 3
Dr. Fuller
Solutions

1. Let A be the matrix whose entry in row r, column c is $dx_{i_r}(e_{j_c})$.

   If $i_1 = j_1, i_2 = j_2, \dots, i_k = j_k$, then $A = I$, and det $A = 1$.

   Otherwise, suppose m is the first index with $i_m \neq j_m$ (so $i_1 = j_1, i_2 = j_2, \dots, i_{m-1} = j_{m-1}$). If $i_m < j_m$, then $i_m \neq j_\ell$ for all $1 \leq \ell \leq k$, so $dx_{i_m}(e_{j_\ell}) = 0$ for all ℓ. Thus the i_m row of A is all zeroes, and det $A = 0$. If $i_m > j_m$, then $j_m \neq i_\ell$ for all $1 \leq \ell \leq k$, so $dx_{i_\ell}(e_{j_m}) = 0$ for all ℓ. Thus the j_m column of A is all zeroes, and det $A = 0$.

2. (a) -20 and -11.

   (b) -3

3. (a) 20

   (b) 1

   (c) $\nu(V)$ describes the function $(x,y,z) \mapsto 2y^2 + 2yz$
   $\omega(V,W)$ describes the function $(x,y,z) \mapsto 2x^2y^2 + xz^2$.

4. (a) $a_1b_2 - a_2b_1\ dx \wedge dy$

   (b) $x^3\ dx \wedge dy \wedge dz$

   (c) $2\ dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$

5. (a) From Proposition 5, we have $\omega \wedge \omega = (-1)^{k^2} \omega \wedge \omega = -\omega \wedge \omega$. This implies $\omega \wedge \omega = 0$.

   (b) See 4 (c)

6. (a) It is routine to confirm that the function $\varphi : \mathbf{R}^n_p \to (\mathbf{R}^n_p)^*$ given by $\varphi(v) = T_v$, where $T_v(w) = \langle v, w\rangle$ is a linear function. If $v \in \ker \varphi$, then $\langle v, w\rangle = 0$ for all $w \in \mathbf{R}^n_p$, which implies $v = 0$. Thus φ is one-to-one. Since both $\mathbf{R}^n_p$ and $(\mathbf{R}^n_p)^*$ have dimension n, φ is an isomorphism.

   (b) $\omega_X = f_1 dx_1 \wedge \cdots \wedge f_n dx_n$