Math 6A
Summer A 2020
Final Celebration of Knowledge
2020/7/27 - 2020/8/2
Due: Sunday, August 2, 11:59 PM

Name: ________________________

Perm Number: ________________________

IT IS NOT MY FAULT IF YOU LOSE POINTS BECAUSE YOU DID NOT FOLLOW DIRECTIONS.

This exam is different from the previous ones. Please observe the following.

- Your solutions should be written on paper or an electronic device, or they should be typed using LaTeX. Begin each numbered question on a new page, and clearly label each question. If you write your answers on paper, I recommend a smartphone scanner app to produce a clean PDF. When you are done, upload your exam to GradeScope. I have given you some freedom in which problems you answer. In return, you MUST match problems when you submit to GradeScope. You will lose points if you do not.

- If you do not use complete sentences in your explanations, you will lose points. Each answer should not only perform the required computations, but you must COMMUNICATE what you did and why you did it to the reader. On the timed exam, grading on communication was much more relaxed. Since you have time on this exam and can get answers from your classmates, you MUST clearly communicate that you understand what you're doing to receive full points.
  On a related note, since you have time, the solutions you submit should be better written. Once you are done solving a problem, please write a final draft to be submitted which is clean and nicely organized. It should also not have anything incorrect in it. If you submit two solutions and only one of them is correct, you will not receive credit for it.

- Collaboration on this exam is allowed. However, you MUST turn in your own solutions. Moreover, you must write your own solutions as well. You may talk to each other, but what you turn in should be your own work. Identical solutions will receive zeros.
  - You MUST note all people you collaborated with on the paper you turn in. This is something you should always be doing. It will not affect your grade, and you may receive a zero for plagiarism if you do not do this.

- You MAY consult the class notes, the UCSB Math 6A/6B textbook, as well as any other textbook on multivariable calculus, Khan Academy videos, and Paul's Online Math Notes, though any work you submit should still be your own. If you use resources outside of our class, you should cite them.
  You may also use WolframAlpha or a calculator for difficult computations, but you must include in your explanation a method for performing that computation. For example, if you use WolframAlpha to compute an integral, you must explain how that integral should be computed by hand. If you have a question about a specific resource and whether it is allowed, ask me.

- You MAY NOT use people outside this class or things like StackExchange, MathOverflow, Chegg, or Reddit to look up or post the questions. You have so much freedom on this exam. Please just turn in your own work.
  - If I or a TA catch an individual breaking these rules, they will receive a zero for the exam and an Academic Misconduct report.
  - If I or a TA finds any of these problems newly posted online with a solution, I will consider the problem unusable for the exam and replace it with another problem, which I guarantee will be more difficult.
  - I think many students will let me know of questions being posted online out of integrity, but just to encourage this, the first student to report a problem being posted will receive a 2% boost on their exam (limited to one per student).
  - If this does become an issue, I will invalidate the entire exam, and we will only have the regular 2-hour option.
- Do your best. Some problems may seem more difficult than others. The key is to START EARLY so that your brain has a week to mull over the problems. If you need to turn the exam in and you're still not sure what to do, explain whatever you do know. It is likely worth partial credit.
- Actual footage of every student at the beginning of a class: https://bit.ly/30SYu8x

Your exam WILL NOT BE GRADED if the following statement is not signed. Please copy the statement and sign it on the exam you turn in.

On my honor as a student of the University of California, Santa Barbara, I have neither given nor received aid on this exam except as permitted by the instructor.

X____________________________________

Please answer 5 of the following 9 problems. If you turn in more than 5, your final grade will be based on the LOWEST 5. All questions are worth 20 points.

(Feel free to send me other solutions after turning in your exam if you're actually interested in feedback.)

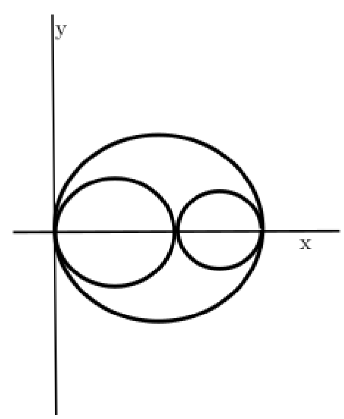
1. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be differentiable with differentiable inverse. Give a proof that $D(f^{-1}) = (Df)^{-1}$.
   Hint: What is the derivative of the identity map id: $\mathbb{R}^n \to \mathbb{R}^n$*?*

2. Suppose a triangle is inscribed in the unit circle centered at the origin in $\mathbb{R}^2$ so that one vertex is at the point $(1, 0)$, and the other two vertices are specified by angles θ_1 and θ_2 in polar coordinates.
   (a) Find a formula for the area in terms of the angles. What are the critical points of the area, and what shape triangle do they correspond to? Classify the critical points.
   (b) What is the average area of these triangles?

3. Let $f(x, y) = \frac{x^2-y^2}{(x^2+y^2)^2}$. Integrate this function over the unit square $[0, 1] \times [0, 1]$, first in the order $dxdy$, then in the order $dydx$. Why aren't these the same? Usually, we can integrate functions over rectangles in either order. What breaks down here?

4. Consider three circles of radius r_1, r_2, and $r_3 = r_1 + r_2$ happily living in $\mathbb{R}^2$. Arrange them so that the circle of radius r_1 and r_3 pass through the origin and are tangent to the y-axis, and the circle of radius r_2 passes through both $(2r_1, 0)$ and $(2r_3, 0)$, as in the picture below. Let R denote the region inside the circle of radius r_3 and outside the two smaller circles.
   (a) Find
   $$\int_0^{2r} \frac{dx}{\sqrt{r^2 - (x-r)^2}}.$$
   Hint: Try a change of variable to get rid of $x - r$*.*
   (b) Calculate the integral
   $$\iint_R \frac{1}{y^2}\, dA.$$

   Is this surprising? Interesting?



5. Consider the region R between the surfaces $z = x^2 + y^2$, $z = 2(x^2 + y^2)$, and $z = 1$.
   (a) Parametrize each of these surfaces on R, and give a sum of integrals which represents the total flux IN through the boundary of R for the vector field $F(x, y, z) = \langle 2x^2, y, -2xz - z\rangle$. Do not evaluate these integrals.
   (b) Consider the integral $\iiint_R 2x\,dV$. Express this integral as an iterated integral in cylindrical coordinates. Do not evaluate it.

   What is the relationship between these two quantities you've found?

6. Calculate the scalar curl of $F_n(x, y) = \left\langle x\sqrt{x^2 + y^2}^{\,n}, y\sqrt{x^2 + y^2}^{\,n}\right\rangle$. For what values of n is F a gradient vector field? When possible, find a function g_n so that $F_n = \nabla g_n$. Are your findings consistent with results we talked about in this class? Explain.

7. Consider the vector field $F = \langle x^2y + y^3 - y, 3x + 2y^2x + e^y\rangle$. For which simple closed curve in the plane does the line integral over this vector field have the maximal value? What is this value? Should we have expected the line integral over all simple closed curves to be zero?

8. Let $F = \langle xy, y^2\rangle$, let c be the clockwise-oriented unit circle centered at the origin, and consider $\int_c F \cdot ds$.
   (a) Which portions of c contribute positively to this integral? Calculate the integral in two ways, first directly and then by using Green's theorem.
   (b) Find the flux of F through c (i.e. the component of F going across c rather than parallel to it).

9. Let c be the curve given by intersecting the surface $z = 42 + x^2 - y^2$ with the cylinder $x^2 + y^2 = 16$, oriented positively. Let

$$F(x, y, z) = (-y + yze^{xy} + 2xyz + y^2z)\hat{\imath} + (x + xze^{xy} + x^2z + 2xyz)\hat{\jmath} + (e^{xy} + x^2y + xy^2)\hat{k}.$$

   Calculate $\int_c F \cdot ds$.
   Hint: This problem does not require a complicated integral.

You made it to the end of the class! Probably not perfectly, but at least having made significant progress from where you started! https://bit.ly/2ZZQUcM

Take a moment to celebrate your achievement! ☺