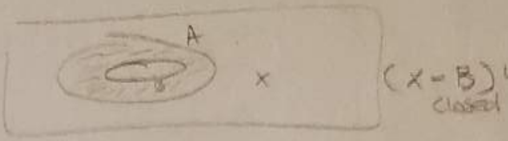
Spring 2018



1. In each case give a proof, or give a counterexample & prove it is a counterexample.

(a) $\text{int}(\text{int}(\mathcal{U})) = \text{int}(\mathcal{U})$

This is true. By definition $\text{int}(\text{int}(\mathcal{U}))$ is the largest open set contained in $\text{int}(\mathcal{U})$. Since $\text{int}(\mathcal{U})$ is open, $\text{int}(\text{int}(\mathcal{U})) = \text{int}(\mathcal{U})$

(b) $\text{cl}(\text{cl}(\mathcal{U})) = \text{cl}(\text{int}(\mathcal{U}))$ $\quad \mathcal{U} =$

This is not true. Let $\mathcal{U} = [0,1] \cup \{2\}$ in $\mathbb{R}$ with the standard topology. Then

$\text{cl}(\text{cl}(\mathcal{U})) = \text{cl}([0,1] \cup \{2\}) = [0,1] \cup \{2\}$

However,

$\text{cl}(\text{int}(\mathcal{U})) = \text{cl}((0,1)) = [0,1]$

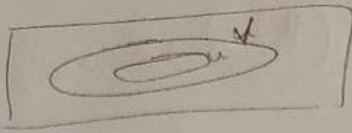
(c) $\text{int}(\text{cl}(\text{int}(\mathcal{U}))) = \text{int}(\mathcal{U})$

This is not true. Let $\Omega = \{a,b,c\}$ & $\mathcal{J}_\Omega = \{\Omega, \emptyset, \{a,b\}\}$ then let $\mathcal{U} = \{a,b\}$. Then $\text{int}(\mathcal{U}) = \{a,b\}$. Then $\text{Cl}(\text{int}(\mathcal{U})) = \Omega$ & $\text{int}(\text{cl}(\text{int}(\mathcal{U}))) = \Omega$

(d) $\text{int}(\mathcal{U} \times V) = \text{int}(\mathcal{U}) \times \text{int}(V)$

This is true. $\text{int}(\mathcal{U}) \times \text{int}(V)$ is an open set in $\mathcal{U} \times V$ so $\text{int}(\mathcal{U}) \times \text{int}(V) \subseteq \text{int}(\mathcal{U} \times V)$. On the other hand, $\forall (x,y) \in \text{int}(\mathcal{U} \times V)$ $\exists$ $W_1 \times W_2$, W_1 open in $\mathcal{U}$, W_2 open in V where $(x,y) \in W_1^x \times W_2^y \subseteq \text{int}(\mathcal{U} \times V) \subseteq \mathcal{U} \times V$. Then $\bigcup_{x \in \mathcal{U}} W_1^x$ is an open set in $\mathcal{U}$ & $\bigcup_{x \in V} W_2$ is an open set in V. Thus,

$\bigcup W_1 \subseteq \text{int}(\mathcal{U})$ & $\bigcup W_2 \subseteq \text{int}(V) \Rightarrow$
$\text{int}(\mathcal{U} \times V) \subseteq \text{int}(\mathcal{U}) \times \text{int}(V)$ $\quad \square$



2 An ultrafilter on a set X is a collection $\mathcal{U}$ of subsets of X with the properties that

(i) If $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$

(ii) If $U \subset X$ then $U \in \mathcal{U}$ or $X \setminus U \in \mathcal{U}$

(iii) $\emptyset \notin \mathcal{U}$

(a) Show that if $V \supset U$ & $U \in \mathcal{U}$ then $V \in \mathcal{U}$

From condition (ii) either $V \in \mathcal{U}$ or $X \setminus V \in \mathcal{U}$. If $X \setminus V \in \mathcal{U}$ then by condition (i) $(X \setminus V) \cap U = \emptyset \in \mathcal{U}$ which contradicts (iii). Thus $X \setminus V \notin \mathcal{U}$ so $V \in \mathcal{U}$.

(b) Show that $\mathcal{T} := \mathcal{U} \cup \{\emptyset\}$ is a topology on X.

First note that $\emptyset \in \mathcal{T}$. By condition (iii) $\emptyset \notin \mathcal{U}$ so by condition (ii) $X \setminus \emptyset = X \in \mathcal{U} \Rightarrow X \in \mathcal{T}$.

Now let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a collection of elements of $\mathcal{T}$. Consider $\bigcup_{\alpha \in \Lambda} U_\alpha$. If $\bigcup_{\alpha \in \Lambda} U_\alpha = \emptyset$ then all $U_\alpha = \emptyset$ & $\bigcup_\alpha U_\alpha \in \mathcal{T}$. Now assume $U_\alpha \neq \emptyset \ \forall \alpha \in \Lambda$. So $U_\alpha \in \mathcal{U} \ \forall \alpha \in \Lambda$.

? If $\bigcup_\alpha U_\alpha \notin \mathcal{U}$ then $X \setminus \bigcup_\alpha U_\alpha \in \mathcal{U}$ & by condition (i) $(X \setminus \bigcup_\alpha U_\alpha) \cap U_\alpha = \emptyset \in \mathcal{U}$ which violates (iii). So $X \setminus \bigcup_\alpha U_\alpha \notin \mathcal{U}$ so $\bigcup_\alpha U_\alpha \in \mathcal{U}$

$\bigcup_{\alpha \in \Lambda} U_\alpha \supset U_\beta$ for $\beta \in \Lambda$

Now I prove that if $U_1, \ldots, U_n \in \mathcal{U}$, then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$ by induction on n. The case $n=2$ is given by condition (i). Now assume it holds up to $n-1$. Then consider $U_1 \cap \cdots \cap U_n$. $U_1 \cap \cdots \cap U_{n-1} \in \mathcal{U}$ by induction & $U_n \in \mathcal{U}$ so condition 1 gives $U_1 \cap \cdots \cap U_n \in \mathcal{U} \subseteq \mathcal{T}$.

If any $U_i = \emptyset$, then $U_1 \cap \cdots \cap U_n = \emptyset \in \mathcal{T}$. Therefore $\mathcal{T}$ is a topology on X.

(c) If X is infinite show that the topology is connected, but not compact & not Hausdorff.

First I show X is not Hausdorff. Let $x, y \in X$. Then if $\exists\, \mathcal{U}_x \ni x$ & $V_y \ni y$ where $\mathcal{U}_x, V_x \in \mathcal{T}$ (which really means $\mathcal{U}_x, V_x \in \mathcal{U}$ since they are nonempty) & $\mathcal{U}_x \cap V_y = \emptyset$ then this implies $\emptyset \in \mathcal{U}$ which violates (iii). So X is not Hausdorff.

Now I show X is connected. If $\exists$ disjoint, nonempty $\mathcal{U}, V \in \mathcal{T}$ (which really means $\mathcal{U}$ since nonempty) such that $\mathcal{U} \cup V = X$. Then $\mathcal{U} \cap V = \emptyset \in \mathcal{U}$ which violates (iii).

Finally I show X is not compact. Let $x \in X$. Then either $\{x\} \in \mathcal{U}$ or $X \setminus x \in \mathcal{U}$. First consider $\{x\} \in \mathcal{U}$. Then by part (a), $\{x, y\} \in \mathcal{U}$ $\forall\, y \in X$. Thus $\{\{x,y\}\}_{y \in X}$ is an open cover of X & since X is infinite, it has no finite subcover.

The above trick works for any $x \in X$. So it is left to show that if $\forall\, x \in X$, $X \setminus \{x\} \in \mathcal{U}$. Therefore for any $x, y \in X$, $X \setminus \{x\} \cap X \setminus \{y\} = X \setminus \{x, y\} \in \mathcal{U}$. We can then use induction to show that $X \setminus \{\text{any finite set}\} \in \mathcal{U}$.

Where to go from here?

Take $\Sigma \subset X$ so $|\Sigma| = \infty$ & $|X \setminus \Sigma| = \infty$

One of Σ or $X \setminus \Sigma$ is in $\mathcal{T}$. Then add points one at a time

4 Let $X \subset \mathbb{R}^2$ be the Subspace ~~$\{(x, (\frac{1}{x})\sin(\frac{1}{x})) : x>0\}$~~
$\{(x, (\frac{1}{x})\sin(\frac{1}{x})) : x>0\} \cup \{(x,y) : x \leq 0\}$
(a) Is X connected?

Yes X is connected. Call $S = \{(x, \frac{1}{x}\sin(\frac{1}{x})) : x>0\}$ & $T = \{(x,y) : x \leq 0\}$. Note that S & T are both path-connected ; & thus are connected. Note that $\bar{S} = S \cup \{(0,y) : y \in \mathbb{R}\}$ Since $\bar{S} \cap T \neq \emptyset$, this means $\forall (0,y) \in T$, $\forall$ open sets of $(0,y)$, $\mathcal{U}$, $\mathcal{U} \cap S \neq \emptyset$. Thus $\{(0,y) : y \in \mathbb{R}\}$ must belong to the connected component of S. Since T is connected $\Rightarrow$ all T belongs to the connected component of $S \Rightarrow S \cup T$ is connected.

(b) Is X path-connected?

No $S \cup T$ is not path-connected. Assume it were. Then there would exist a path, $p : [0,1] \to S \cup T$ where $p(0) = (0,0)$ & $p(1) = (1, \sin(1))$. Note that T is closed so $p^{-1}(T)$ is closed in $[0,1]$. Therefore $\exists$ max $t \in p^{-1}(T)$.

Had to look up from here

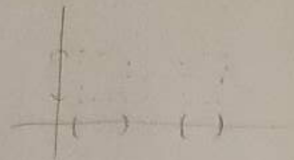
Thus $p|_{[t,1]}$ maps $p(t) \in T$ but $p((t,1]) \subseteq S$. Rescale p so that $t = 0$ & call the new map $\bar{p}$, i.e. $\bar{p}(s) = p((1-s)t + s)$ Now $\bar{p}(s) = (x(s), y(s))$. Then $x(0) = 0$ & $x(s) > 0$ if $s \neq 0$. & $y(s) = \frac{1}{s}\sin(\frac{1}{s})$ for $s \neq 0$.

We now create a sequence s_n: given n, chose u where $0 < u < x(1/n)$ s.t. $\frac{1}{u}\sin(\frac{1}{u}) = (-1)^n$. Then by IVT $\exists$ s_n with $0 < s_n < 1/n$ s.t. $x(s_n) = u$.

Now we have $s_n \to 0$ & $y(s_n) = (-1)^n$ However y is continuous so $y(s_n)$ should converge which is a contradiction.

Learn Latin please

Thus there is no path from $(0,0)$ to $(1, \sin(1))$ so X is not path-connected. $\square$



5. Prove that the product of two compact Hausdorff spaces is compact & Hausdorff.

Let X & Y be two compact Hausdorff spaces. First I show $X \times Y$ is Hausdorff. Let $(x,y) \neq (w,z) \in X \times Y$. Then either $x \neq w$ or $y \neq z$. WLOG let $x \neq w$. Since X is Hausdorff $\exists$ disjoint open sets $\mathcal{U}_x \ni x$ & $\mathcal{U}_w \ni w$. Then $(x,y) \in \mathcal{U}_x \times Y$ & $(w,z) \in \mathcal{U}_w \times Y$. Further, $(\mathcal{U}_x \times Y) \cap (\mathcal{U}_y \times Y) = (\mathcal{U}_x \cap \mathcal{U}_y) \times Y = \emptyset$. So $\mathcal{U}_x \times Y$ & $\mathcal{U}_y \times Y$ are two disjoint open sets containing (x,y) & (w,z) respectively. Thus, $X \times Y$ is Hausdorff.

Now let $\mathcal{V} = \{\mathcal{U}_\alpha\}_{\alpha \in \Delta}$ be an open cover of $X \times Y$. Note that since Y is compact, so is $\{x\} \times Y$ for any $x \in X$. Thus $\{\mathcal{U}_\alpha\}$ is an open cover for $\{x\} \times Y$ so there exists a finite subcover $\mathcal{U}_x = \{\mathcal{U}_i\}_{i=1}^{n_x}$. Similarly $X \times \{y\}$ is compact $\forall y \in Y$, so $\{\mathcal{U}_x\}_{x \in X}$ is an open cover of $X \times \{y\}$. Thus there exists a finite subcover $\{\mathcal{U}_{x_i}\}_{i=1}^{m_y}$. Thus $\bigcup_{i=1}^{m_y} \mathcal{U}_{x_i} \supseteq X \times \{y\}$ & $\mathcal{U}_{x_i} \supseteq \{x_i\} \times Y$. Therefore

$\bigcup_{i=1}^{m_y} \mathcal{U}_{x_i} \supseteq X \times Y$. Thus $\{\mathcal{U}_\alpha \in \mathcal{V} : \mathcal{U}_\alpha \in \mathcal{U}_{x_i},\ 1 \le i \le m_y\}$ is an open cover of $X \times Y$ & is finite since there are finitely many $\mathcal{U}_{x_i}$ & each $\mathcal{U}_{x_i}$ has finitely many elements. So $X \times Y$ is compact. $\square$

7. Let (X,d) be a metric space. Recall that a metric completion of X is a complete metric space (X', d') & a map $f: X \to X'$ s.t. f is an isometry onto $f(X)$ & $f(X)$ is dense in X'

(a) Carefully define a metric space (X', d') & map $f: X \to X'$ that forms a completion of X. State, without proof, everything that must be checked to confirm that your definition is a metric completion of X.

Lecture 11/4

Choose $x_0 \in X$. Define a map $i: X \to \mathcal{B}(X, \mathbb{R})$ where $i(x) = f_x$ & $f_x(\xi) = d(x, \xi) - d(\xi, x_0)$

<u>Claim 1</u>: $f_x \in \mathcal{B}(X, \mathbb{R})$

Proof of claim: $\forall \xi$, $d(x, \xi) \leq d(x, x_0) + d(\xi, x_0)$

$\Rightarrow d(x, \xi) - d(\xi, x_0) \leq d(x, x_0)$

$\Rightarrow |f_x| \leq d(x, x_0)$ □

<u>Claim 2</u>: i is an isometry

Proof: $d(i(x), i(y)) = \sup_{\xi \in X} |f_x(\xi) - f_y(\xi)|$

$= \sup_{\xi \in X} |d(x,\xi) - \cancel{d(x_0,\xi)} - d(y,\xi) + \cancel{d(x_0,\xi)}|$

$= \sup_{\xi \in X} |d(x,\xi) - d(y,\xi)|$

$\leq d(x,y)$

On the other hand, if $\xi = x$, then $d(i(x), i(y)) \geq d(x,y)$

Thus $d(i(x), i(y)) = d(x,y)$ □

<u>Claim</u>: $\mathcal{B}(X, \mathbb{R})$ is complete

Proof: Let (f_n) be a Cauchy sequence. Fix $x \in X$. Then $(f_n(x))$ is a sequence of real numbers which is Cauchy & since $\mathbb{R}$ is complete, $f_n(x) \to f(x)$.

Claim: $f \in B(X, \mathbb{R})$

Proof: Pick $\epsilon = 1$, $\exists\ k$ so $d(f_n, f_m) \leq 1$ $\forall n, m \geq k$. In particular, $d(f_k, f_m) < 1\ \forall m \geq k$.

$\Rightarrow |f_m(x)| \leq |f_k(x)| + 1$

Now let $m \to \infty$, so $|f(x)| \leq |f_k(x)| + 1$

so $f \in B(X, \mathbb{R})$.

Claim: $f_n \to f$ in $B(X, \mathbb{R})$

Proof: Let $\epsilon > 0$. Then $\exists\ K \in \mathbb{N}$ s.t. $\forall n, m > K$ $d(f_n, f_m) < \epsilon/2$. Let $m \to \infty$. Then $d(f_n, f) < \epsilon/2$

So $f_n \to f$. $\square$

Thus $B(X, \mathbb{R})$ is complete.

Therefore, i is the map & $B(X, \mathbb{R})$ with sup is the completion.

Everything that must be checked for (X', d') to be a completion of (X, d) with $f: X \to X'$:

1. (X', d') is a complete metric space
2. f is an isometry
3. $\overline{X'} = X$.

$\square$

OK with (c) now?

Long bookwork...

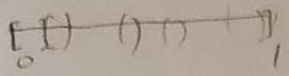
8. Let $X = \mathbb{R}^n \setminus \{0\}$ with the subspace topology from $\mathbb{R}^n$. Let $\sim$ be the equivalence relation on X given by $\vec{x} \sim \vec{y}$ iff $\vec{x} = t\vec{y}$ for some $t > 0$. Prove that $X/\sim$ is homeomorphic to $S^{n-1} := \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = 1\}$ with the subspace topology from $\mathbb{R}^n$.

Pick some $x \in X$. Then denote E_x as the equivalence class for x in $X/\sim$. Let $\|\vec{x}\| = d$. Then $\vec{y} = \frac{\vec{x}}{\|\vec{x}\|}$ is an element of X & $\|\vec{y}\| = 1$. Further, $\vec{x} = d\vec{y}$ so $\vec{y} \in E_x$. Thus every equivalence class has an element of length 1. Further, if $y, y' \in E_x$ & $\|y\| = \|y'\| = 1$, then $\vec{y} = t\vec{y}'$ & $\|\vec{y}\| = t\|\vec{y}'\|$ so $t = 1$ & $y = y'$. Thus every equivalence class has precisely 1 element of norm 1.

Thus, we can choose to represent $X/\sim$ by precisely the elements in $\mathbb{R}^n$ with norm 1.

Let $q: X \to X/\sim$ be the quotient map, which we know is a continuous surjection. Then $q|_{S^{n-1}}$ will map ~~each element of~~ S^{n-1} to a unique element of $X/\sim$ by the above argument. Thus, $q|_{S^{n-1}}$ is a homeomorphism. □

Needs a bit more



(9) Suppose $p: Y \to X$ is a covering space.
(a) Prove, using only the definition of covering space, that if $f: [0,1] \to X$ is continuous, & $t_0 = 0$, & $y \in Y$ with $p(y) = f(t_0)$ then there is a continuous map $F: [0,1] \to Y$ such that $p \circ F = f$ & $F(t_0) = y$.

Since $p: Y \to X$ is a covering space, $\exists$ an open cover $\mathcal{U} = \{X_\alpha\}$ s.t. $p^{-1}(X_\alpha)$ is a disjoint union of open sets in Y that each map homeomorphically onto X_α by p.

Are you good now?

Needed help on this:

Pick $t \in [0,1]$. Then $f(t_0) \in X$ so $\exists X_{\alpha_t} \in \mathcal{U}$ s.t. $f(t) \in X_{\alpha_t}$ Since f is continuous, $f^{-1}(X_{\alpha_t})$ is open in $[0,1]$, so there exists $(a_t, b_t) \subseteq [0,1]$ s.t. $f(a_t, b_t) \subseteq X_\alpha$ where $t \in (a_t, b_t)$. We can repeat this for all $t \in [0,1]$ & thus $\{f^{-1}(X_{\alpha_t})\}_{t \in [0,1]}$ is an open cover of $[0,1]$, so there exists a finite subcover $\{f^{-1}(X_{t_i})\}_{i=1}^{n}$ of $[0,1]$.

Now we can create a partition of $[0,1]$, say $s_0 = 0 < s_1 < \cdots < s_m = 1$ where $f([s_j, s_{j+1}])$ lies entirely in one element of our finite subcover, say $([s_j, s_{j+1}]) \subseteq f^{-1}(X_i^j)$

Note that we already know $F(0) = y$. Assume inductively that F has been defined on $[0, s_j]$. Now we'll define F on $[s_j, s_{j+1}]$. Note that $f([s_j, s_{j+1}]) \subseteq X_i^j$. By construction of $\mathcal{U}$ $p^{-1}(X_i^j)$ is a disjoint union of sets homeomorphic to X_i^j under p. Since $[s_j, s_{j+1}]$ is connected & $[s_j, s_{j+1}] \subseteq p^{-1}(X_i^j)$, $[s_j, s_{j+1}]$ belongs to precisely one of these sets, say V_i^j. Then

$p^{-1}\big|_{f([s_j, s_{j+1}])} \circ f\big|_{[s_j, s_{j+1}]} : [s_j, s_{j+1}] \to Y$ is a continuous function

I don't think I need this part for this version of path-lifting. Is that correct?

& define F this way on $[s_j, s_{j+1}]$. Thus by induction, we can extend & get F defined on $[0,1]$ □

$\boxed{\text{Claim This lift is unique}}$ This wasn't required

Proof: Suppose F & G are two lifts such that $p\circ F = f$, $p \circ G = f$ & $F(0) = G(0) = y$. As done previous, choose a partition $0 = t_0 < \dots < t_n = 1$ of $[0,1]$ s.t. $f([t_i, t_{i+1}]) \subseteq X_i \in \mathcal{U}$. Since F & G agree on t_0. Assume by induction they agree up to t_i. Then $p^{-1}(X_i)$ is a disjoint union of open sets in Y, so $F([t_i, t_{i+1}])$ & $G([t_i, t_{i+1}])$ & both connected so lie in exactly one of these sets. Since F & G agree on $t_i \Rightarrow$ they lie in same open set, say V_i. Since $pF = pG$ & p is a homeomorphism on V_i, we get $F = G$ on $[t_i, t_{i+1}]$. Thus by induction the lift is unique □

Claim F is well-defined.

Since the lift is unique,

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ S^1 \arrow[ur] \arrow[r, "id"] & S^1 \end{tikzcd}$$