Math 501
Homework 8

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1. Prove that $[0,1) \times [0,1)$ is homeomorphic to $[0,1] \times [0,1)$, but not to $[0,1] \times [0,1]$. (Assume that all factors are topologized as subspaces of R with the usual topology.)

   Proof Let S be the unit circle minus the set $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, y \geq 0\}$, as pictured. Consider a function f which maps points in S to points in $[0,1) \times [0,1)$ as follows:

   $$f(x,y) = \left(rd, \theta - \tfrac{\pi}{4}\right),$$

   where (r,θ) are the usual polar coordinates for (x,y) and d is the distance from $\left(\frac{1}{2}, \frac{1}{2}\right)$ in the direction of $\theta - \frac{\pi}{4}$ to the boundary of $[0,1) \times [0,1)$. Also, $f(0,0) = (\frac{1}{2}, \frac{1}{2})$.

   Now, f is a homeomorphism because conversion to polar coordinates is a homeomorphism, and the final mapping is a composition of homeomorphisms (multiplying and adding). Note, d actually depends on x and y, but comes from trig functions which are all continuous. In case you're not convinced that this is a bijection, note that every point is S has exactly 1 corresponding point in the target set, and vice-versa. ■

2. Given $f : X \to Y$ between spaces, we define the graph of f as

   $$G_f = \{(x,y) \in X \times Y : y = f(x)\}.$$

   (a) Suppose that X and Y are Hausdorff. Prove that if f is continuous, then G_f is closed.

   Proof First, note that since X and Y are Hausdorff, then $X \times Y$ is Hausdorff. Suppose that $f : X \to Y$ is continuous. To show that G_f is closed, we will show that $(X \times Y) - G_f$ is open. Let $(x,y) \in (X \times Y) - G_f$, that is, $f(x) \neq y$. Since Y is Hausdorff, there exists disjoint open subsets of Y, V_y and $V_{f(x)}$, which contain y and $f(x)$ respectively. Since f is continuous, the preimage $U = f^{-1}\left(V_{f(x)}\right)$ is open, and $x \in U$. Now, since $f(U) \subset V_{f(x)}$, and $V_y \cap V_{f(x)} = \emptyset$, then $V_y \cap f(U) = \emptyset$. Thus, $U \times V_y \cap G_f = \emptyset$, so $U \times V_y \subset ((X \times Y) - G_f)$. Therefore, by the openness criterion, $((X \times Y) - G_f)$ is open and G_f is closed. ■

   (b) Suppose that X and Y are compact and Hausdorff. Prove that if G_f is closed, then f is continuous. (Hint: If C is any subset of Y, then $f^{-1}(C)$ can be expressed in terms of π_X.)

   Proof Suppose G_f is closed. Let B, C be open sets such that $B \times C \subset X \times Y$. Now,

   $$f^{-1}(C) = \pi_x(G_f \cap B \times C) \subset \pi_x(B \times C) = B.$$

   I can't figure out what to do next. ■

(c) Show by example that the statement in part (b) is false if X and Y are not compact. Consider $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} 0 & x \in \{\frac{\pi}{2}, \frac{3\pi}{2}, \ldots\} \\ \tan(x) & \text{otherwise} \end{cases}$$

Now, G_f is closed, but the function is clearly not continuous at any multiple of $\frac{\pi}{2}$.