1. ***Heine-Borel Theorem.*** *A set $E \subset \mathbb{R}$ is compact iff E is closed and bounded.*

Proof (converse direction) Suppose E is closed and bounded. We will show that E is compact; that is, every open cover of E has a finite subcover. Let $\{G_\lambda\}_{\lambda\in A}$ be an arbitrary open cover of E. To show that $\{G_\lambda\}_{\lambda\in A}$ has a finite subcover of of E, we will assume that $\{G_\lambda\}_{\lambda\in A}$ does not have a finite subcover of of E; and show that this assumption is self-contradictory.

Since E is bounded, there is a closed interval $[\alpha, \beta]$ which covers E. Let γ_0 be the midpoint of $[\alpha, \beta]$. Since E cannot be covered by a finite subfamily of $\{G_\lambda\}_{\lambda\in A}$, then either

$$[\alpha, \gamma_0] \cap E \text{ or } [\gamma_0, \beta] \cap E$$

cannot be covered by a finite subfamily of $\{G_\lambda\}_{\lambda\in A}$. Choose one and call it $[\alpha_1, \beta_1]$, and call γ_1 the midpoint of $[\alpha_1, \beta_1]$. Now again, either $[\alpha_1, \gamma_1] \cap E$ or $[\gamma_1, \beta_1] \cap E$ cannot be covered by a finite subfamily of $\{G_\lambda\}_{\lambda\in A}$. Choose one and call it $[\alpha_2, \beta_2]$. Continuing in this fashion, we obtain a sequence of closed intervals $[\alpha_n, \beta_n]$ with the following properties:

1. $\beta_n - \alpha_n = \frac{1}{2^n}(\beta - \alpha)$. (The length of each interval is half the length of the previous interval)
2. $[\alpha_{n+1}, \beta_{n+1}] \subset [\alpha_n, \beta_n]$ for all n. (This is a sequence of nested intervals)
3. Every set $[\alpha_n, \beta_n] \cap E$ cannot be covered by a finite subfamily of $\{G_\lambda\}_{\lambda\in A}$.

By (3), $[\alpha_n, \beta_n] \cap E$ is nonempty for each $n = 1, 2, \dots$; so we may choose an element of each of these sets. Consider the set

$$P = \{x_n : x_n \in [\alpha_n, \beta_n] \cap E\}.$$

There are only two possibilities: either P is finite or it is infinite. We will consider each case seperately.

Case I (P is finite): If P is finite then by (2), there is an $x_{n_0} \in P$ such that, for every $[\alpha_n, \beta_n]$,

$$x_{n_0} \in [\alpha_n, \beta_n] \cap E.$$

Since $\{G_\lambda\}_{\lambda\in A}$ is an open cover of E, there is some $\lambda_0 \in A$ such that

$$x_{n_0} \in G_{\lambda_0}.$$

Also, since G_{λ_0} is open, there is $\epsilon > 0$ such that

$$(x_{n_0} - \epsilon, x_{n_0} + \epsilon) \subset G_{\lambda_0}.$$

We now have a single element of $\{G_\lambda\}_{\lambda \in A}$ which covers a neighborhood of x_{n_0}. We will proceed to show that this neighborhood covers one of the "uncoverable" sets $[\alpha_n, \beta_n] \cap E$ from (3).
Since $\beta_n - \alpha_n = \frac{1}{2n}(\beta - \alpha)$ for all n by (1), choose N large enough that

$$\beta_N - \alpha_N = \frac{1}{2^N}(\beta - \alpha) < \epsilon.$$

So, we have that the length of $[\alpha_N, \beta_N]$ is less than ϵ, and $x_{n_0} \in [\alpha_N, \beta_N]$. So,

$$x_{n_0} \leq \beta_N < x_{n_0} + \epsilon$$

and

$$x_{n_0} - \epsilon < \alpha_N \leq x_{n_0}.$$

Therefore, $(x_{n_0} - \epsilon, x_{n_0} + \epsilon)$ covers $[\alpha_N, \beta_N]$, which contradicts (3).

Case II (P is infinite): Suppose P is infinite. Since $P \subset E$ and E is bounded, P is an infinite bounded set, which means it has an accumulation point. Call this x_0. Since x_0 is an accumulation point of P, $P \subset E$, and E is closed; then x_0 is an accumulation point of E and $x_0 \in E$. Since $\{G_\lambda\}_{\lambda \in A}$ covers E, there is a $\lambda_1 \in A$ such that $x_0 \in G_{\lambda_1}$; and since G_{λ_1} is open, there is $\epsilon > 0$ such that

$$x_0 \in (x_0 - \epsilon, x_0 + \epsilon) \subset G_{\lambda_1}.$$

We will now show that this neighborhood of x_0 covers one of the "uncoverable" intervals. By the same reasoning in Case I, choose N large enough that

$$\beta_N - \alpha_N < \frac{\epsilon}{2}.$$

Since x_0 is an accumulation point of P, there are infinitely many elements of P in each neighborhood of x_0, so we can find an $M > N$ such that

$$|x_0 - x_M| < \frac{\epsilon}{2}.$$

Now, since $M > N$, then $[\alpha_M, \beta_M] \subset [\alpha_N, \beta_N]$, so

$$\beta_M - \alpha_M < \beta_N - \alpha_N < \frac{\epsilon}{2}.$$

This means that since $x_M \in [\alpha_M, \beta_M]$, we now have that for any $x \in [\alpha_M, \beta_M]$,

$$|x_M - x| < \frac{\epsilon}{2}.$$

Recall that by (3), no finite subfamily of $\{G_\lambda\}_{\lambda\in A}$ covers $[\alpha_M, \beta_M]$. However, for any $x \in [\alpha_M, \beta_M]$,

$$\begin{aligned} |x_0 - x| &= |x_0 - x_M + x_M - x| \\ &\leq |x_0 - x_M| + |x_M - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

Therefore, $(x_0 - \epsilon, x_0 + \epsilon) \subset G_{\lambda_1}$ covers $[\alpha_M, \beta_M]$, which is a contradiction.

Thus, we have shown that $\{G_\lambda\}_{\lambda\in A}$, an arbitrary open cover of E, must necessarily have a finite subcover of E, so E is compact. ■

Proof (forward direction) Suppose E is compact. We will show that E is closed and bounded by contrapositive; that is, if E is either not closed or not bounded, then E is not compact.

Case I (E is not bounded): Assume E is not bounded. To show that E is not compact, we will produce an open cover of E which has no finite subcover. Let $\{G_n\}_{n=1}^{\infty}$ be the collection of all open intervals

$$G_n = (-n, n).$$

Now, $\{G_n\}_{n=1}^{\infty}$ is an open cover of $\mathbb{R}$, so it certainly covers E. To specify an arbitrary subcover of $\{G_n\}_{n=1}^{\infty}$; let $S \subset \mathbb{N}$ be some finite set of positive integers. Since, for any $n < m$, $G_n \subset G_m$, then

$$\bigcup_{n\in S} G_n = G_{max(S)}.$$

However, since E is unbounded, $E \not\subset G_{max(S)}$. Thus, $\{G_n\}_{n=1}^{\infty}$ is an open cover of E with no finite subcover, so E is not compact.

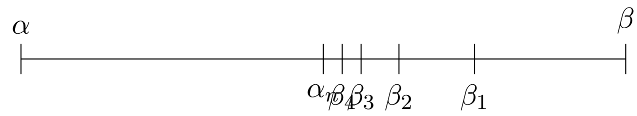
Case II (E is not closed): Assume E is not closed. Then there is an accumulation point of E (call it x_0) such that $x_0 \notin E$. For each positive integer n, define

$$G_n = \left(-\infty, x_0 - \tfrac{1}{n}\right) \cup \left(x_0 + \tfrac{1}{n}, \infty\right).$$

Note that $\{G_n\}_{n=1}^{\infty}$ covers $\mathbb{R} \setminus \{x_0\}$, so it also covers E. We will again specify an arbitrary subcover of $\{G_n\}_{n=1}^{\infty}$ by letting $S \subset \mathbb{N}$ be some finite set of positive integers; and noting again that, for any $n < m$, $G_n \subset G_m$, so

$$\bigcup_{n\in S} G_n = G_{max(S)}.$$

Let $N = max(S)$. Then, $\bigcup_{n\in S} G_n = G_N = (-\infty, x_0 - {}^1\!/\!{}_N) \cup (x_0 + {}^1\!/\!{}_N, \infty)$. However, since x_0 is an accumulation point of E, $(x_0 - {}^1\!/\!{}_N, x_0 + {}^1\!/\!{}_N)$ contains infinitely many elements of E, none of which are in G_N. Therefore, $\{G_n\}_{n=1}^{\infty}$ has no finite subcover of E, so E is not compact. ■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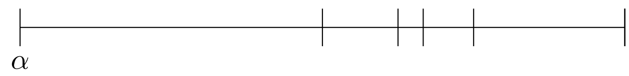
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