Math 550
Homework 5
Dr. Fuller
Due October 2

1. Calculate the differential of each of the following.
   (a) $\omega = e^{xy}\,dx$
   (b) $\omega = x_1x_2\,dx_3 \wedge dx_4$
   (c) $\omega = f(x,y)\,dx + g(x,y)\,dy$
   (d) $\omega = f(x,y,z)\,dy \wedge dz - g(x,y,z)\,dx \wedge dz + h(x,y,z)\,dx \wedge dy$
2. Determine if the following 2-forms are exact.
   (a) $\omega = x\,dx \wedge dy$
   (b) $\omega = z\,dx \wedge dy$
   (c) $\omega = z\,dx \wedge dy + y\,dx \wedge dz + z\,dy \wedge dz$
3. (a) Let $\alpha \in \Omega^1(\mathbf{R}^3)$ satisfy $\alpha(p) \neq 0$ for all $p \in \mathbf{R}^3$. Prove that $\ker\alpha$ is a 2-dimensional subspace (i.e. a plane) of $\mathbf{R}^3_p$ for all $p \in \mathbf{R}^3$.
   (b) Let $\alpha_1 = dz$. Sketch the planes described in part (a).
   (c) Let $\alpha_2 = x\,dy + dz$. Sketch the planes described in part (a).
   (d) Show that $\alpha_1 \wedge d\alpha_1 = 0$ and $\alpha_2 \wedge d\alpha_2 \neq 0$ (at all $p \in \mathbf{R}^3$).
4. Prove that if $\omega \in \Omega^k(\mathbf{R}^n)$ is exact and $\varphi \in \Omega^\ell(\mathbf{R}^n)$ is closed, then $\omega \wedge \varphi$ is exact.
5. Show that the image of the curve $c(t) = (\cos 2t \cos t, \cos 2t \sin t)$ for $t \in (-\pi/2, \pi/4)$ is not a 1-dimensional manifold.
6. Let $f : \mathbf{R}^3 \to \mathbf{R}$ be defined by $f(x,y,z) = x^2 + y^2 - z^2$.
   (a) For which values of a is $f^{-1}(a)$ a manifold?
   (b) Find two different values a and b so that the manifolds $f^{-1}(a)$ and $f^{-1}(b)$ are not homeomorphic, and prove that they are not homeomorphic.
7. Let S^2 denote the unit sphere in $\mathbf{R}^3$. Give a basis for the tangent space S^2_p at any $p \in S^2$.
8. Prove that the unit sphere S^{n-1} is $\mathbf{R}^n$ cannot be parameterized as a manifold by a single parameterization. Can you generalize your proof into a more general result?
9. Let V be a k-dimensional vector subspace of $\mathbf{R}^n$.
   (a) Prove that V is a k-dimensional manifold in $\mathbf{R}^n$.
   (b) Let V_p denote the tangent space to V at $p \in V$. Prove that $V_p = V$.