Real Analysis - Horn, 2019

Trevor Klar

February 10, 2019

Contents

Chapter 1

Measure Theory

1.1 Preliminaries

Definition. We say $\alpha = \inf S$ iff:

- $\alpha \leq s$ for all $s \in S$ and
- for any $\epsilon > 0$, there exists $s \in S$ such that $s < \alpha + \epsilon$.

Definition. A (closed) **rectangle** R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d],$$

where $a_j \leq b_j$ are real numbers.

Definition. The **measure** of a rectangle R is defined to be

$$|R| = \prod_{i=1}^{d} (b_i - a_i) = (b_1 - a_1)(b_2 - a_2) \cdots (b_d - a_d).$$

Definition. A union of rectangles is said to be **almost disjoint** if the interiors of the rectangles are disjoint. (We pretty much only use closed rectangles when we say they are almost disjoint).

Lemma 1.1. *Let R be a rectangle which is the almost disjoint union of finitely many other rectangles, that is, $R = \bigcup_{k=1}^{N} R_k$. Then,*

$$|R| = \sum_{k=1}^{N} |R_k|.$$

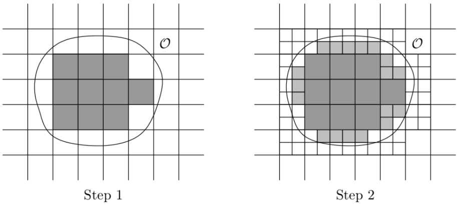
Lemma 1.2. *Let $R, R_1, \ldots, R_N$ be rectangles, with $R \subseteq \bigcup_{k=1}^{N} R_k$. Then,*

$$|R| \leq \sum_{k=1}^{N} |R_k| \,.$$

Theorem 1.3. *Every open subset $O \subseteq \mathbb{R}^1$ can be written uniquely as a countable union of disjoint open intervals.*[1]

Theorem 1.4. *Every open subset $O \subseteq \mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes.*

Proof Basically, do this:



■

[1] I have deliberately changed the notation slightly here. I will continue to use script letters i.e. $\mathcal{ABCQO}$ to denote collections of sets, and ordinary capitals i.e. $ABCQO$ to denote sets.

1.2 The exterior measure

Definition. Let $E \subseteq \mathbb{R}^d$. Let $\mathcal{Q} = \{Q_j\}_1^\infty$ denote a countable collection of *closed* cubes which cover E, and let Γ denote the collection of all possible countable covers of E. That is, for all $\mathcal{Q} \in \Gamma$, $E \subset \bigcup_1^\infty Q_j$ where each $Q_j \in \mathcal{Q}$.

The **exterior measure** of E is defined as

$$m_*(E) = \inf_{\mathcal{Q} \in \Gamma} \sum_{j=1}^\infty |Q_j|$$

Remark. Since the exterior measure is defined with an infimum, then $m_*(E) \leq \sum |Q_i|$ for any cover $\{Q_i\}$ of E.

Remark. Note that $|\cdot|$ is only defined for rectangles. For any other sets, we have only the exterior measure, m_*.

Observations about exterior measure.

1. **(Monotonicity)**
   If $E_1 \subseteq E_2$, then $m_*(E_1) \subseteq m_*(E_1)$.
2. **(Countable sub-additivity)**
   If $E = \bigcup_{j=1}^\infty E_j$, then $m_*(E) \leq \sum_{j=1}^\infty m_*(E_j)$.
3. Let $E \subseteq \mathbb{R}^d$, and let $\mathcal{O} = \{\text{open sets } O : E \subseteq O\}$.
   Then $m_*(E) = \inf_{O \in \mathcal{O}} m_*(O)$.

   (**Corollary to 3.**) If $m_*(E) < \infty$, then $\exists$ open $O \supset E$ such that $m_*(O) < m_*(E) + \epsilon$ for any $\epsilon > 0$.
4. If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$,
   then $m_*(E) = m_*(E_1) + m_*(E_2)$.
5. If a set E is the countable union of almost disjoint closed cubes $E = \bigcup_{j=1}^\infty Q_j$, then $m_*(E) = \sum_{j=1}^\infty |Q_j|$.

1.3 Measurable sets and the Lebesgue measure

Definition. We say that $E \subseteq R^d$ is **measurable** if

for any $\epsilon > 0$, there exists an open set $O \supseteq E$ with $m_*(O - E) < \epsilon$.

If the distinction is important, we can be more specific and say the set is **Lebesgue measurable**.

Definition. We define the Lebesgue **measure** of a measurable set E by its exterior measure,

$$m(E) = m_*(E).$$

Now we give some propositions about the Lebesgue Measure.

Property 1. Every open set in R^d is measurable.

Proof Let $E \subseteq \mathbb{R}^d$ be open. Then E is an open set containing E where $m_*(E - E) = 0 < \epsilon$ for any $\epsilon > 0$, so E is measurable. ■

Property 2. If $m_*(E) = 0$, then E is measurable. In particular, if $F \subseteq E$ and $m_*(E) = 0$, then F is measurable.

Proof Let $F \subset E \subseteq \mathbb{R}^d$ with $m_*(E) = 0$, and let $\epsilon > 0$ be given. By Observation 3 about exterior measure, there exists on open set $O \supset E$ such that $m_*(O) < \epsilon$. Thus by monotonicity, $m_*(O - E) < m_*(O - F) < m_*(O) < \epsilon$ and we are done. ■

Property 3. A countable union of measurable sets is measurable.

Proof (idea) Choose open sets so that each one is $\frac{1}{\epsilon^j}$, and they all sum to $< \epsilon$. ■

Property 4. Closed sets are measurable.