

Final Exam

1. Let M^n be an embedded submanifold of N^m . Show that TM is an embedded submanifold of TN .

Proof Since M is an embedded submanifold of N , then the inclusion map $\iota : M \hookrightarrow N$ is an embedding; that is, a smooth map of rank n which is a homeomorphism onto its image with the subspace topology.

If we consider the global differential $d\iota : TM \rightarrow TN$, we know that $d\iota$ is smooth since ι is smooth (Proposition 3.21). Now observe that

$$\begin{aligned} d\iota(x^1, \dots, x^n, v^1, \dots, v^n) &= \left(\iota^1(x), \dots, \iota^n(x), \frac{\partial \iota^1}{\partial x^i}(x)v^i, \dots, \frac{\partial \iota^n}{\partial x^i}(x)v^i \right) \\ &= (x^1, \dots, x^n, v^1, \dots, v^n), \end{aligned}$$

where the last line comes from the fact that $\iota^i(x)$ just gives the i -th coordinate of x and $\frac{\partial \iota^j}{\partial x^i}(x)$ is constantly 1 if $i = j$ and constantly 0 if $i \neq j$. This means that the rank of $d\iota$ is $2n$, and $d\iota$ is the inclusion map $TM \hookrightarrow TN$. Since ι is a homeomorphism onto its image and T_pM is a linear subspace of T_pN for every $p \in M$, then $d\iota$ is also a homeomorphism onto its image, so $d\iota$ is an embedding, and we are done. ■

2. Let X, Y, Z be vector fields on \mathbb{R}^3 given by

$$X = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \quad Y = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \quad Z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$

Let A be the linear space spanned by X, Y, Z . Show that A is a 3-dimensional Lie algebra with the Lie bracket of $\mathfrak{X}(\mathbb{R}^3)$.

Proof Since X, Y, Z are clearly linearly independent, then their span over coefficients in \mathbb{R} is a 3-dimensional vector space. Now we show that the bracket $[X, Y] = XY - YX$ on A satisfies the desired properties. By the symmetry in their definitions, $[X, Y]$, $[Y, Z]$, and $[Z, X]$ will all have the same properties, and while an arbitrary vector field in A will be of the form $(aX + bY + cZ)$, it suffices to show that the desired properties hold for the basis vectors.

(i) BILINEARITY:

$$[aX + bY, Z]f = (aX + bY)Zf - Z(aX + bY)f$$

$$\begin{aligned} (aX + bY)Zf &= \left(ay \frac{\partial}{\partial z} - az \frac{\partial}{\partial y} + bz \frac{\partial}{\partial x} - bx \frac{\partial}{\partial z} \right) \circ \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) f \\ &= \left(bz \frac{\partial}{\partial x} - az \frac{\partial}{\partial y} + (ay - bx) \frac{\partial}{\partial z} \right) \circ \left(x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x} \right) \\ &= bz \frac{\partial f}{\partial y} + bxz \frac{\partial^2 f}{\partial y \partial x} - bz y \frac{\partial^2 f}{\partial x^2} - axz \frac{\partial^2 f}{\partial y^2} + az \frac{\partial f}{\partial x} + ayz \frac{\partial^2 f}{\partial x \partial y} \\ &\quad + (ay - bx) \left(x \frac{\partial^2 f}{\partial y \partial z} - y \frac{\partial^2 f}{\partial x \partial z} \right) \end{aligned}$$

$$\begin{aligned} -Z(aX + bY)f &= - \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \circ \left(bz \frac{\partial}{\partial x} - az \frac{\partial}{\partial y} + (ay - bx) \frac{\partial}{\partial z} \right) f \\ &= \left(-x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x} \right) \circ \left(bz \frac{\partial f}{\partial x} - az \frac{\partial f}{\partial y} + ay \frac{\partial f}{\partial z} - bx \frac{\partial f}{\partial z} \right) \\ &= -bxz \frac{\partial^2 f}{\partial x \partial y} + axz \frac{\partial^2 f}{\partial y^2} - ax \frac{\partial f}{\partial z} - axy \frac{\partial^2 f}{\partial z \partial y} + bx^2 \frac{\partial^2 f}{\partial z \partial y} \\ &\quad + byz \frac{\partial^2 f}{\partial x^2} - ayz \frac{\partial^2 f}{\partial y \partial x} + ay^2 \frac{\partial^2 f}{\partial z \partial x} - by \frac{\partial f}{\partial z} - bxy \frac{\partial^2 f}{\partial z \partial x} \end{aligned}$$

and all of the second-order derivatives cancel[†], the sum is given by

$$[aX + bY, Z]f = az \frac{\partial f}{\partial x} + bz \frac{\partial f}{\partial y} - (ax + by) \frac{\partial f}{\partial z}.$$

Now we check that this is equal to $a[X, Z]f + b[Y, Z]f$:

$$\begin{aligned} a[X, Z]f &= a(XZf - ZXf) \\ &= a \left[\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \circ \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) f - \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \circ \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) f \right] \\ &= a \left[\left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \circ \left(x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x} \right) - \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \circ \left(y \frac{\partial f}{\partial z} - z \frac{\partial f}{\partial y} \right) \right] \\ &= a \left[-x \frac{\partial f}{\partial z} + z \frac{\partial f}{\partial x} \right] \\ &= -ax \frac{\partial f}{\partial z} + az \frac{\partial f}{\partial x} \end{aligned}$$

[†]Since $f \in C^\infty(\mathbb{R})$, mixed partials are equal.

$$\begin{aligned}
b[Y, Z]f &= b(YZf - ZYf) \\
&= b \left[\left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \circ \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) f - \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \circ \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) f \right] \\
&= b \left[z \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial z} \right] \\
&= bz \frac{\partial f}{\partial y} - by \frac{\partial f}{\partial z}
\end{aligned}$$

So,

$$\begin{aligned}
[aX + bY, Z]f &= az \frac{\partial f}{\partial x} + bz \frac{\partial f}{\partial y} - (ax + by) \frac{\partial f}{\partial z} \\
&= -ax \frac{\partial f}{\partial z} + az \frac{\partial f}{\partial x} + bz \frac{\partial f}{\partial y} - by \frac{\partial f}{\partial z} \\
&= a[X, Z]f + b[Y, Z]f
\end{aligned}$$

and a similar proof will show that $[Z, aX + bY] = a[Z, X] + b[Z, Y]$. Thus, the bilinearity property is shown.

(ii) ANTISYMMETRY:

$$\begin{aligned}
[X, Y]f &= XYf - YXf \\
&= -(-XYf + YXf) \\
&= -[Y, X]f
\end{aligned}$$

(iii) JACOBI IDENTITY:

By examining some of the computations in the bilinearity part of this proof, we can see that

$$\begin{aligned}
[Y, X] &= Z, \\
[X, Z] &= Y, \text{ and} \\
[Z, Y] &= X.
\end{aligned}$$

Thus

$$\begin{aligned}
[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] &= [X, -X] + [Y, -Y] + [Z, -Z] \\
&= 0 + 0 + 0
\end{aligned}$$

Therefore, A is a 3-dimensional vector space with a bracket operation having the bilinearity, antisymmetry, and Jacobi identity properties, so A is a 3-dimensional Lie algebra. ■

3. a) Compute the flow of the vector field X on \mathbb{R}^2 :

$$X = y \frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

Answer: The vector field X corresponds to the ODE system

$$\begin{aligned}\dot{y} &= -1 \\ \dot{x} &= y\end{aligned}$$

which can be solved one at a time to find the solutions $y(t) = -t + y_0$ and $x(t) = -\frac{t^2}{2} + y_0 t + x_0$. This corresponds to the flow

$$\theta_t(x, y) = \left(-\frac{t^2}{2} + ty + x, -t + y\right),$$

where integral curves are parabolas. ■

- b) Let $M = M_n(\mathbb{R})$ be the space of all $n \times n$ matrices. For $A \in M$ let V_A be the vector field on M so that $V_A(X) = AX$, where $X \in M$ (we have used the identification $T_X M = \mathbb{R}^{n^2} = M$). Compute the flow θ_t generated by V_A . **Answer:** The vector field AX corresponds to the linear system of ODEs $X' = AX$, and if we write X as a concatenation of column vectors, we obtain

$$\begin{bmatrix} \begin{array}{c} | \\ (\mathbf{x}^1)' \\ | \end{array} & \cdots & \begin{array}{c} | \\ (\mathbf{x}^n)' \\ | \end{array} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \begin{array}{c} | \\ \mathbf{x}^1 \\ | \end{array} & \cdots & \begin{array}{c} | \\ \mathbf{x}^n \\ | \end{array} \end{bmatrix},$$

and this system can be solved column by column (they will all have the same family of solutions, differing only in their initial values) as

$$(\mathbf{x}^i)' = \mathbf{A}\mathbf{x}^i, \text{ for } i \in 1, \dots, n$$

according to the usual method of finding eigenvalues and eigenvectors, finding coefficients, and determining constants assuming that $\mathbf{x}^i(0) = \mathbf{x}_0^i$ for each i . For example,

$$\mathbf{x}^1(t) = \varphi_1(t, \mathbf{x}_0^1) + \cdots + \varphi_n(t, \mathbf{x}_0^1),$$

Where each φ_j is \mathbb{R}^n -valued. Now each $\mathbf{x}^i(t)$ is given by the same set of functions with different initial points, so

$$\mathbf{x}^i(t) = \varphi_1(t, \mathbf{x}_0^i) + \cdots + \varphi_n(t, \mathbf{x}_0^i) \quad \text{for each } i.$$

To write the flow, let $\tau_t(\mathbf{x}) = \varphi_1(t, \mathbf{x}) + \cdots + \varphi_n(t, \mathbf{x})$, then

$$\theta_t(\mathbf{X}) = \begin{bmatrix} \begin{array}{c} | \\ \tau_t(\mathbf{x}^1) \\ | \end{array} & \cdots & \begin{array}{c} | \\ \tau_t(\mathbf{x}^n) \\ | \end{array} \end{bmatrix}$$

■

4. a) Give an example of complete vector field and an example of incomplete vector field. Explain why it is complete or incomplete.

Answer: We will discuss examples from the text, as they are readily available. Let $M = \mathbb{R}^2$, $V = \frac{\partial}{\partial x}$. Then the flow generated by V is

$$\tau_t(x, y) = (x + t, y),$$

and this is a global flow, since given any $(x, y) \in M$, we can see that $\tau_t(x, y)$ determines an integral curve which is defined for all $t \in \mathbb{R}$. Thus V is complete.

For an incomplete example, let $M = \mathbb{R}^2$, $V = x^2 \frac{\partial}{\partial x}$. This corresponds to the ODE system

$$\frac{dx}{dt} = x^2 \qquad \frac{dy}{dt} \equiv 0$$

which has solution[†]

$$x(t) = \frac{1}{\frac{1}{x_0} - t} \qquad y(t) = y_0$$

so the flow is given by

$$\theta_t(x, y) = \left(\frac{1}{\frac{1}{x} - t}, y \right).$$

Consider $\theta^{(1,0)}(t)$. Since

$$\theta^{(1,0)}(t) = \left(\frac{1}{1-t}, 0 \right),$$

then the integral curve cannot be continuously extended past $t = 1$, since $x \rightarrow \infty$ as $t \nearrow 1$. ■

- b) Let X be a vector field on a manifold M , and $\gamma(t)$ an integral curve of X starting at $p \in M$. If $f \in C^\infty(M)$, $f > 0$, find the integral curve of fX starting at p .

Answer: Assume that fX denotes multiplication, so $(f \cdot X)|_p = f|_p \cdot X|_p$ [‡] for any point $p \in M$. Since $\gamma' = X|_\gamma$, then

$$(f \cdot X)|_\gamma = f|_\gamma \cdot X|_\gamma = f(\gamma) \cdot \gamma'.$$

We seek a function $\Gamma : \mathbb{R} \rightarrow M$ such that $\Gamma' = (f \cdot X)|_\Gamma$, and $f(\gamma) \cdot \gamma'$ looks like a chain rule derivative:

$$\begin{aligned} (f \circ \gamma)(t) \cdot \gamma'(t) &= f(\gamma^1(t), \dots, \gamma^n(t)) \cdot \left(\frac{\partial \gamma^1}{\partial t} + \dots + \frac{\partial \gamma^n}{\partial t} \right) \\ &= \sum_{i=1}^n \frac{\partial \gamma^i}{\partial t} \cdot f(\gamma^1(t), \dots, \gamma^n(t)) \end{aligned}$$

[†]Assuming $x_0 \neq 0$. Otherwise if $x_0 = 0$ then $x = 0$ for all time.

[‡]This is one of those times when using juxtaposition for multiplication, composition, and evaluation makes me a go a liiiiiiiiittle bit crazy.

Thus if there exists a function F such that[†]

$$\frac{\partial F}{\partial x^i} = \frac{\partial F}{\partial x^j} = f \quad \text{for all } 1 \leq i, j \leq n,$$

then we can let $\Gamma = F \circ \gamma$ and we're done. Fortunately, since f is smooth, we can just take indefinite integrals with respect to each x^i and combine the antiderivatives to obtain F :

$$\begin{aligned} F(x^1, \dots, x^n) &= \int f(x^1, \dots, x^n) dx^1 + g(x^2, \dots, x^n) + C_1 \\ &\vdots \\ &= \int f(x^1, \dots, x^n) dx^i + g(x^1, \dots, \hat{x}^i, \dots, x^n) + C_i \\ &\vdots \\ &= \int f(x^1, \dots, x^n) dx^n + g(x^1, \dots, x^{n-1}) + C_{n-1} \end{aligned}$$

Combining these together yields a function $F \in C^\infty(M)$ such that

$$(F(\gamma))' = f(\gamma) \cdot \gamma' = fX|_\gamma,$$

So $\Gamma = F \circ \gamma$ is the desired integral curve. ■

- c) Give an explanation why the following statement could be true: for any manifold M and any vector field X on M , there is a $f \in C^\infty(M)$, $f > 0$, such that fX is complete.

Answer: Anywhere that an integral curve shoots off to infinity in finite time, you can multiply the vector field by a function which goes to zero faster than the integral curve would go to infinity. Anywhere an integral curve goes to a point not in the manifold, you can similarly scale down the vector field so that it takes infinite time to get there.

5. Let M^n be a compact manifold which carries n vector fields X_1, \dots, X_n such that $[X_i, X_j] = 0$ for all $i, j = 1, \dots, n$ and X_1, \dots, X_n are pointwise linearly independent. Let $\theta_t^1, \dots, \theta_t^n$ be the flow generated by X_1, \dots, X_n respectively.

- (i) Show that $F : \mathbb{R}^n \rightarrow M$ defined by $F(x_1, \dots, x_n) = \theta_{x_1}^1 \circ \dots \circ \theta_{x_n}^n(p)$ for some fixed $p \in M$ is well-defined and a submersion. Conclude that F is a local diffeomorphism.

Proof Since $[X_i, X_j] = 0$ for all $i, j = 1, \dots, n$, then the vector fields all commute, so they are invariant under each other's flows. This means that regardless of which vector field we call X_1, X_2 , etc, we are always referring to the same function when we say F is the composition of all n flows starting at p .

F is a submersion because $[X_i, X_j] = 0$ for all $i, j = 1, \dots, n$ and X_1, \dots, X_n are pointwise linearly independent, so each flow is locally moving in a linearly independent direction, which means F has rank n . Since F is a map of constant rank between two n -dimensional manifolds, then it is a local diffeomorphism. ■

[†]Here we are using $x^1 \dots x^n$ to denote the coordinate functions on M .