Standard

4 Let $X \subset \mathbb{R}^2$ be the subspace $\{(x, (\frac{1}{x})\sin(\frac{1}{x})) : x > 0\}$
$\cup \{(x, y) : x \leq 0\}$

(a) Is X connected?

<u>Lemma</u> If X is connected & $f: X \to Y$ is continuous then $f(X)$ is connected.

Let $U \cup V = f(X)$ be a separation. Then $f^{-1}(U) \cup f^{-1}(V)$ is a ~~sep~~ cover of X & a separation of X since if $x \in f^{-1}(U)$ $f(x) \in U \Rightarrow f(x) \notin V$. $\#$

Now $\{(x, (1/x)\sin(1/x)) : x > 0\}$ is the image of the continuous function $f: (0, \infty) \to (0, \infty) \times (0, \infty)$ where $f(x) = (x, (1/x)\sin(1/x))$

how much proof here? $\to$ So $\{(x, (1/x)\sin(1/x))\}$ is connected. Similarly, $\{(x, y) : x \leq 0\}$ is connected. Further, $\overline{\{(x, (\frac{1}{x})\sin(\frac{1}{x})) : x > 0\}} = \{(x, (\frac{1}{x})\sin(\frac{1}{x})) : x > 0\} \cup \{x = 0\}$ & so $\overline{\{(x, (\frac{1}{x})\sin(\frac{1}{x})) : x > 0\}} \cap \{(x, y) : x \leq 0\} \neq \emptyset$ so their union is connected. $\Rightarrow \{(x, y) : x \leq 0\} \subset \{(x, (\frac{1}{x})\sin(\frac{1}{x})) : x > 0\} \cup \{(x, y) : x \leq 0\}$
$\subset$

$\Rightarrow$ the middle is connected. $\square$

$\to$ (b) Is X path-connected?

No. Suppose $\exists$ a path $f: [0,1] \to \underbrace{\{(x, y) : x \leq 0\}}_{S} \cup \underbrace{\{(x, (\frac{1}{x})\sin(\frac{1}{x})) : x > 0\}}_{T}$ where $f(0) = (0,0)$ & $f(1) = (\frac{1}{2}, \sin(2))$. Then consider $f^{-1}(S)$. Then this is an ordered set in $[0,1] \Rightarrow \exists t \in [0,1)$, $\sup\{t : f(t) \in S\}$. Then if $f(t) \in S$ then $f: [t, 1] \to S \cup T$ maps $f(t)$ to S & everything else to T. Create a sequence t_n, $0 < u < x(\frac{1}{n})$ s.t. $\frac{1}{u}\sin(\frac{1}{u}) = \frac{1}{u}$. Then ?

Use Intermediate Value Theorem

Suppose $\exists$ path $p(0) = (0,0)$ $p(1) = (x, \frac{1}{x}\sin(\frac{1}{x})) \Rightarrow$

$\{p(0,0)\}$ is closed $\quad f(t) = (x(t), y(t)) \quad x(0) = 0$

Given n choose $0 < x < \frac{1}{n}$ s.t. u $\quad$ ~~y~~ $x) = \frac{1}{t}\sin(\frac{1}{t})$