Math 201A, Homework 5 (Integration of Functions)

Problem1 (Integrability of positive functions). Let X be a nonempty set and let μ be a measure on X. Prove that any nonnegative $\mu-$measurable function $f\colon X \to [0,\infty]$ is μ-integrable on X, i.e., the lower integral equals the upper integral:

$$\int_{*X} f d\mu = \int^{*}_{X} f d\mu.$$

Problem2 (Integrability of the product). Let X be a nonempty set and let μ be a measure on X. Prove that if $\mu-$measurable functions $f, g\colon X \to [-\infty,\infty]$ are such that f is μ-summable on X, and g is bounded on X ($|g(x)| \le M$ for all $x \in X$), then the product fg is $\mu-$summable and

$$\int_X |fg| d\mu \le M \int_X |f| d\mu.$$

Problem3 (Absolute continuity of the integral). Let μ be a Radon measure on $\mathbb{R}^n$ and let the function $f\colon \mathbb{R}^n \to \mathbb{R}$ be μ-summable on $\mathbb{R}^n$, i.e.,

$$\int_{\mathbb{R}^n} |f(x)| d\mu < \infty.$$

Prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that for every $\mu-$measurable set $A \subset \mathbb{R}^n$ with $\mu(A) < \delta$ one has

$$\int_A |f| d\mu < \epsilon.$$

Problem4. Let X be a nonempty set and let μ be a measure on X. Assume $\mu-$summable functions $f, f_n\colon X \to [-\infty,\infty]$ are such that

$$f_n \to f \quad \mu - \text{a.e. in} \quad X,$$

and

$$\int_X |f_n| d\mu \to \int_X |f| d\mu.$$

Prove that

$$\int_X |f_n - f| d\mu \to 0.$$

Problem5. Compute the limit

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n \ln\left(2 + \cos\left(\frac{x}{n}\right)\right) dx.$$