Math 450B
Homework 2 Solutions
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1. Note that $U - C = U \cap (\mathbf{R}^n - C)$. This expresses $U - C$ as the intersection of two open sets, so the result follows from Proposition 3.

2. Give the interior, exterior, and boundary for the following subsets of $\mathbf{R}^n$. No proofs, just give answers.

   (a) Int = $\emptyset$. Ext = $\{(x,y) : xy \neq 0\}$. Boundary = $\{(x,y) : xy = 0\}$.

   (b) Int = $\{(x,y) : xy \neq 0\}$. Ext = $\emptyset$. Boundary = $\{(x,y) : xy = 0\}$.

   (c) Int = $\emptyset$. Ext = $\mathbf{R}^3 - \{(x,y,z) : x^2+y^2 \leq 1 \text{ and } z = 0\}$. Boundary = $\{(x,y,z) : x^2+y^2 \leq 1 \text{ and } z = 0\}$.

   (d) Int = $\{(x,y,z) : x^2+y^2 < 1\}$. Ext = $\{(x,y,z) : x^2+y^2 > 1\}$. Boundary = $\{(x,y,z) : x^2+y^2 = 1\}$.

   (e) Int = Ext = $\emptyset$. Boundary = $\mathbf{R}^n$.

3.

	closed?	bounded?	compact?
(a)	yes	yes	yes
(b)	yes	yes	yes
(c)	yes	yes	yes
(d)	yes	yes	yes
(e)	no	yes	no
(f)	yes	no	no

4. (a) Since $\mathbf{R}^n - A$ is open, there is $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset \mathbf{R}^n - A$. This implies that $\|\mathbf{x} - \mathbf{y}\| > \delta$ for all $y \in A$.

   (b) For each $\mathbf{w} \in C$, we can find $\delta(\mathbf{w}) > 0$ as in part (a). Consider the smaller ball $B(\mathbf{w}, \delta(\mathbf{w})/2)$: if $\mathbf{x} \in B(\mathbf{w}, \delta(\mathbf{w})/2)$ and $\mathbf{y} \in A$, then

$$\delta < \|\mathbf{w} - \mathbf{y}\| \leq \|\mathbf{w} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{y}\| < \frac{\delta(\mathbf{w})}{2} + \|\mathbf{x} - \mathbf{y}\|.$$

   This implies $\|\mathbf{x} - \mathbf{y}\| > \frac{\delta(\mathbf{w})}{2}$ for all $\mathbf{x} \in B(\mathbf{w}, \delta(\mathbf{w})/2)$ and $\mathbf{y} \in A$.
   $\{B(\mathbf{w}, \delta(\mathbf{w})/2)\}_{\mathbf{w} \in C}$ is an open cover of C. Since C is compact, then there is a finite subcover $B(\mathbf{w}_1, \delta(\mathbf{w}_1)/2), \ldots, B(\mathbf{w}_N, \delta(\mathbf{w}_N)/2)$ of C. Let $\delta = \min(\delta(\mathbf{w}_1)/2, \ldots, \delta(\mathbf{w}_N)/2)$. Then for any $\mathbf{x} \in C$ and $\mathbf{y} \in A$, we have $\mathbf{x} \in B(\mathbf{w}_k, \delta(\mathbf{w}_k)/2)$ for some k, and so $\|\mathbf{x} - \mathbf{y}\| > \delta(\mathbf{w}_k)/2 \geq \delta$.

   (c) $\{(x, 0) : x \in \mathbf{R}\}$ and $\{(x, \frac{1}{x}) : x \neq 0\}$