Math 550
Homework 10
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Solutions

1. (a) We may write $X(p) = w + (X_p \cdot n_p)n_p$, where $w \in \partial M_p$ and n_p is the unit outward normal at p. (Recall that in this case, the two different outward normal vectors n_p and N_p coincide.) Let $u = (u_1, u_2) \in \partial M_p$. Then

$$\begin{aligned}
(\star\omega_X^1)_p(u) &= (-f_y(p)\,dx + f_x(p)\,dy)(u_1,u_2) \\
&= \det\begin{pmatrix} f_x(p) & u_1 \\ f_y(p) & u_2 \end{pmatrix} \\
&= \det\begin{pmatrix} | & | \\ X_p & u \\ | & | \end{pmatrix} \\
&= \det\begin{pmatrix} | & | \\ w + (X_p \cdot n_p)n_p & u \\ | & | \end{pmatrix} \\
&= X_p \cdot n_p \det\begin{pmatrix} | & | \\ n_p & u \\ | & | \end{pmatrix} = X_p \cdot n_p\, ds(u).
\end{aligned}$$

(b) Direct calculation gives $d(\star\omega_X^1) = \text{div } X\, dA$. So

$$\int_M \text{div } X\, dA = \int_M d(\star\omega_X^1) = \int_{\partial M} \star\omega_X^1 = \int_{\partial M} X \cdot n\, ds.$$

2. Direct calculation gives div $X = 0$.

Since X is defined on M^3, we apply the Divergence Theorem to $M - B_\varepsilon^3$, where B_ε^3 is a small open ball of radius ε centered at $(0,0,0)$. Then

$$0 = \int_{M-B_\varepsilon^3} \text{div } X\, dV = \int_{\partial(M-B_\varepsilon^3)} X \cdot N\, dA = \int_{\partial M} X \cdot N\, dA + \int_{S_\varepsilon^2} X \cdot N\, dA = \int_{\partial M} X \cdot N\, dA - 4\pi.$$

To see that $\int_{S_\varepsilon^2} X \cdot N\, dA = -4\pi$, note that the boundary orientation induced on S_ε^2 from $M - B_\varepsilon^3$ is opposite the orientation induced from the standard orientation on B_ε^3. Also, if $p = (x,y,z) \in S_\varepsilon^2$, then $X(p) = \frac{1}{\varepsilon^3}(x,y,z)$ and $N(p) = \frac{1}{\varepsilon}(x,y,z)$, so $X(p) \cdot N(p) = \frac{1}{\varepsilon^4}(x,y,z) \cdot (x,y,z) = \frac{1}{\varepsilon^4}\varepsilon^2 = \frac{1}{\varepsilon^2}$. Thus

$$\int_{S_\varepsilon^2} X \cdot N\, dA = -\frac{1}{\varepsilon^2}\int_{S_\varepsilon^2} dA = -\frac{1}{\varepsilon^2}4\pi\varepsilon^2 = -4\pi.$$

3. (a) $d(\omega_X^1) = \omega_{\text{curl } X}^2 = 0$. Then ω_X^1 is exact by the Poincare Lemma, so there exists f with $\omega_X^1 = df = \omega_{\text{grad } f}^1$. Thus $X = \text{grad } f$.

(b) $d(\omega_X^2) = \text{div } X = 0$. Then ω_X^2 is exact by the Poincare Lemma, so there exists a 1-form η with $\omega_X^2 = d\eta$. Note that if we write $\eta = f_x\, dx + f_y\, dy + f_z\, dz$, then $\eta = \omega_Y^1$ for the vector field $Y = (f_x, f_y, f_z)$. Then we have $\omega_X^2 = d\eta = d(\omega_Y^1) = \omega_{\text{curl } Y}^2$. Thus $X = \text{curl } Y$.

4. Suppose that ω extends to a 1-form $\widetilde{\omega}$ on $\mathbf{R}^2$. Then at $p \neq (0,0,0)$, we have $d\widetilde{\omega}(p) = d\omega(p) = 0$. Thus the coefficient functions of $d\widetilde{\omega}$ are identically zero on $\mathbf{R}^2 - \{(0,0)\}$, and by continuity on $\mathbf{R}^2$ as well. So $d\widetilde{\omega}((0,0,0)) = 0$. Thus $\widetilde{\omega}$ is closed. By the Poincare Lemma, $\widetilde{\omega}$ is exact. But this would mean that its restriction ω to $\mathbf{R}^2 - \{(0,0)\}$ is exact, a contradiction to the fact that ω has been shown to be otherwise.

Addendum

1. Let $g : U \to \mathbf{R}^n$ be a local parameterization of M which induces the given orientation. Then we can write $g^*\nu = f\ dx_1 \wedge \cdots \wedge dx_k$, and since $g^*\nu(u)(e_1, \ldots, e_k) > 0$ for all $u \in U$, we have

$$f(u) = f(u)\ dx_1 \wedge \cdots \wedge dx_k(e_1, \ldots, e_k) = g^*\nu(u)(e_1, \ldots, e_k) > 0$$

   for all $u \in U$. Therefore,

$$\int_{g(U)} \nu = \int_U g^*\nu = \int_U f dx_1 \cdots dx_k > 0.$$

   Finally, taking $\{\varphi\}$ to be a partition of unity subordinate to the cover of M by $\{g_\alpha(U_\alpha)\}$, this implies $\int_M \nu = \sum_\varphi \int_M \varphi\nu > 0$.

2. If M were contractible, then the identity $i : M \to M$ would be homotopic to a constant function $c : M \to M$. Then if ν is the volume form on M, we have $\int_M \nu = \int_M i^*\nu = \int_M c^*\nu = 0$. This contradicts the previous problem.