Math 450b
Homework 12

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1. Let $f(x,y) = y^2$, and let C denote the upper half semi-circle of the unit circle in $\mathbb{R}^2$.

   Professor Doofus computes: $\int_C f = \int_C y^2 = \int_{-1}^{1} (\sqrt{1-x^2})^2 dx = \frac{4}{5}$.

   Professor Egregious computes: $\int_C f = \int_C y^2 = \int_0^x (\sin t)^2 dt = \frac{x}{2}$

   Compute the path integral correctly, and decide if either of them got it right by accident.

   Answer: Let $\mathbf{c} : (0,\pi) \to \mathbb{R}^2$ be defined by $\mathbf{c}(t) = (\cos t, \sin t)$. Then

   $$\int_C f = \int_C y^2 = \int_0^\pi \sin^2 ||\mathbf{c}'(t)|| \, dt,$$

   so we compute $||\mathbf{c}'(t)||$.

   $$\begin{aligned} \mathbf{c}(t) &= (\cos t, \sin t) \\ \mathbf{c}'(t) &= (-\sin t, \cos t) \\ ||\mathbf{c}'(t)|| &= ||(-\sin t, \cos t)|| \\ &= \sqrt{\sin^2 t + \cos^2 t} \\ &= 1 \end{aligned}$$

   Thus,

   $$\int_C f = \int_0^\pi \sin^2 dt = \int_0^\pi \frac{1}{2}(1 - \cos(2t))dt = \left[\frac{1}{2}(t - \frac{1}{2}\sin(2t))\right]_0^\pi = \frac{\pi}{2}.$$

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