Homework 1

1. Let c_0 be the vector space of sequences limiting to 0 with the $\|\cdot\|_{l^\infty}$-norm. Prove that c_0 is a closed subspace of l^∞ (and hence is a Banach space). Prove that $l^1 \cong c_0^*$ in the following sense. For every $f = (f_n) \in l^1$ define

$$F_f(x) = \sum_{n=1}^{\infty} x_n f_n, \quad x = (x_n) \in c_0.$$

Prove that $F_f \in c_0^*$, $\|F_f\|_* = \|f\|_{l^1}$, and for every $\phi \in c_0^*$ there exists $f \in l^1$ such that $\phi = F_f$.

Proof (a) Let $f^1, f^2, \ldots \in c_0 \subset \ell^\infty$, with $f^i \xrightarrow{i} f$ in the ℓ^∞ norm. Let $\varepsilon > 0$. Then $\exists M$ such that if $i > M$, then $\sup_n |f_n^i - f_n| < \varepsilon$.

Now fix $i > M$. Since $f^i \in c_0$, then $f_n^i \xrightarrow{n} 0$, so $\exists N$ such that if $n > N$, then $|f_n^i| < \varepsilon$. Thus $\forall n > N$,

$$\begin{aligned} |f_n| &\leq \left|f_n - f_n^i\right| + \left|f_n^i\right| \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon, \end{aligned}$$

and we conclude that $f_n \xrightarrow{n} 0$, and $f \in c_0$. □

(b) (i) Note that F_f is obviously linear. For all $f \in \ell^1$, F_f is bounded since $||F_f||_* = \sup\limits_{||x|| \leq 1} |F_f(x)|$ and for all x with $||x||_{c_0} \leq 1$,

$$\begin{aligned} \left|\sum_n x_n f_n\right| &\leq \sum_n |x_n f_n| \\ &\leq \sum_n |f_n| \qquad \text{since } \sup |x_n| = 1 \end{aligned}$$

thus $||F_f||_* \leq ||f||_{\ell^1}$ which is finite since $f \in \ell^1$.
Thus F_f is a bounded linear functional $c_0 \to \mathbb{R}$, so $F_f \in c_0^*$. □

(ii) We have shown already that $||F_f||_* \leq ||f||_{\ell^1}$, so to prove that $||F_f||_* = ||f||_{\ell^1}$, it remains to prove the other direction.

Let $f \in \ell^1$. For each $f_n \in \mathbb{C}$, let $u_n = z_n e_n$, where $e_n = (\overbrace{0, 0, \ldots, 0, 1}^{n}, 0, \ldots)$ and z_n is the complex number such that $f_n z_n = |f_n|$ (note that $|z_n| = 1$). Then $v_j = \sum_{n=1}^{j} u_n$ has norm 1 for all j, and $F_f(v_j) \nearrow ||f||$ as $j \to \infty$. Thus $||F_f||$ cannot be less than $||f||$, so $||F_f||_* \geq ||f||_{\ell^1}$. □

(iii) Let $\varphi \in c_0^*$. Let f be the sequence defined by $f_n = \varphi(e_n)$ for all n.
CLAIM: $\sum_n |f_n| < \infty$. To see this, suppose for contradiction that $\sum_n |f_n| = \infty$. We know that $\left|\left|\sum_{n=1}^{N} e_n\right|\right|_{c_0} = 1$ for every $N \in \mathbb{N}$, so

$$\left\{\sum_{n=1}^{N} e_n : N \in \mathbb{N}\right\} \subset \{x \in c_0 : ||x|| \leq 1\}. \tag{$*$}$$

Thus

$$\begin{aligned}
\infty &= \sum_{n=1}^{\infty} |f_n| \\
&= \sup_N \sum_{n=1}^{N} |f_n \cdot 1| \\
&= \sup_N \sum_{n=1}^{N} |\varphi(e_n)| \\
&\leq \sup_N \left|\varphi\left(\sum_{n=1}^{N} e_n\right)\right| && \text{by } \Delta \text{ ineq. and linearity} \\
&\leq \sup_{||x||=1} |\varphi(x)| && \text{by } (*) \\
&= ||\varphi||_* \\
&< \infty.
\end{aligned}$$

Thus we have shown by contradiction that $\sum_n |f_n| < \infty$. ■

2. Let X be a Banach space with $E \subset X^*$. Suppose for every $x \in X$ the set $\{\varphi(x)\}_{\varphi \in E} \subset \mathbb{R}$ is bounded. Prove that E is strongly bounded in X^*. Explain why your proof collapses if X is not complete.

Proof This follows immediately from the Uniform Boundedness Principle below, which requires X to be Banach. ■

Theorem. (Uniform Boundedness Principle) Let X be a Banach space and let Y be a normed linear space. Let $\mathcal{F}$ be a collection of bounded linear operators from X to Y. If for every $x \in X$ we have that $\sup_{T \in \mathcal{F}} ||T(x)||_Y < \infty$ then $\sup_{T \in \mathcal{F}} ||T||_* < \infty$.

3. (a) Let X be a Banach space and (φ_j) be a sequence in X^*. Suppose that $\langle \varphi_j, x\rangle$ converges for any $x \in X$. Prove that there exists $\varphi \in X^*$ such that $\varphi_j \xrightarrow{w*} \varphi$. (In fancy terminology "X^* is always $w*$ sequentially complete".)

Proof We can of course define a functional $\varphi(x) = \lim_j \langle \varphi_j, x\rangle$ and note that it is linear, but we need to show that this φ is bounded. Since the sequence $\langle \varphi_j, x\rangle$ is convergent and thus bounded, then by problem 2 the set $\{\varphi_j\}_j$ is bounded in X^*, call this bound M. Thus for all x with $||x|| \leq 1$,

$$\begin{aligned} |\langle \varphi, x\rangle| &= |\lim_j \langle \varphi_j, x\rangle| \\ &= \lim_j |\langle \varphi_j, x\rangle| \\ &\leq M \end{aligned}$$

and we're done. ■

(b) Formulate the analogous statement for the w-convergence for a sequence $(x_n) \in X$. Try to extend your proof to this situation. when does the proof collapse?

Question Let X be a Banach space and (x_n) be a sequence in X. Suppose that $\langle \varphi, x_n\rangle$ converges for any $\varphi \in X^*$. Does there exist $x \in X$ such that $x_n \xrightarrow{w} x$?

Answer: We can follow the strategy from (a) and define a *functional* $\hat{x} \in X^{**}$ so that $\langle \varphi, x_n\rangle \to \langle \varphi, \hat{x}\rangle$, but we are only guaranteed that a corresponding $x \in X$ exists exactly when X is reflexive. ■

4. Let X be Banach. Prove that a sequence (φ_j) in X^* converges $w*$ if and only if it is strongly bounded and there exists a dense set E with $\overline{E} = X$, such that the number sequence $\langle \varphi_j, u\rangle$ converges for all $u \in E$.

Proof The forward direction is straightforward; the sequence is strongly bounded by problem 2, and X is of course dense in itself and has the desired property.

For the converse direction, suppose $\sup_j ||\varphi_j|| = C$ and E exists as above. It suffices to show that $\langle \varphi_j, x\rangle$ also converges if $x \in E^{\complement}$, since problem 3 completes the proof. Let $x \in X$. Since $x \in \overline{E}$ there exists a sequence $(u_n) \in E$ which converges to x.

$$\begin{aligned} \lim_j \langle \varphi_j, x\rangle &= \lim_j \left\langle \varphi_j, \lim_n u_n \right\rangle \\ &= \lim_j \lim_n \langle \varphi_j, u_n\rangle \\ &= \lim_n \lim_j \langle \varphi_j, u_n\rangle \end{aligned}$$

Claim: The sequence $\left(\lim_j \langle \varphi_j, u_n \rangle\right)_{n=1}^{\infty}$ is a real Cauchy sequence, and thus converges.

Proof of claim: Let $\varepsilon > 0$. Since $u_n \to x$, then it is also a Cauchy sequence, so there exists $N > 0$ such that $\forall n, m > N$,

$$|u_n - u_m| < \varepsilon$$
$$\implies \forall j \;\; |\varphi_j(u_n) - \varphi_j(u_m)| = |\varphi_j(u_n - u_m)| \leq ||\varphi_j|| \, \varepsilon$$
$$\implies \left|\lim_j \varphi_j(u_n) - \lim_j \varphi_j(u_m)\right| \leq C\varepsilon$$

so $\lim_n \lim_j \langle \varphi_j, u_n \rangle = \lim_j \langle \varphi_j, x \rangle$ converges for all $x \in X$, and we can define $\varphi(x) = \lim_j \langle \varphi_j, x \rangle$. Problem 3 assures us that $\varphi \in X^*$, and we are done. ■

5. Let $I = [0,1]$. Let $C^1(I)$ denote the space of continuously differentiable functions,[†] and let $d\phi_n = \cos(\pi n x) \, d\lambda^1(x)$.

(a) Prove that

$$\int_I g \, d\phi_n \xrightarrow{n} 0 \qquad \forall g \in C^1(I).$$

(b) Prove that $d\phi_n \xrightarrow{w*} 0$ as measures in $C(I)^*$.

Proof (a) Using integration by parts, we find that

$$\begin{aligned} \int_I g(x) \cos(\pi n x) \, dx &= g(x) \frac{1}{\pi n} \sin(\pi n x) - \int_I g'(x) \frac{1}{\pi n} \sin(\pi n x) \, dx \\ &= \frac{1}{\pi n} \left[g(x) \sin(\pi n x) - \int_I g'(x) \sin(\pi n x) \, dx \right], \end{aligned}$$

and in the limit as $n \to \infty$, everything goes to 0. □

(b) For arbitrary $f \in C(I)$, f' may not so exist, so we can't use integration by parts. However, by the Weierstrauss Approximation Theorem, for every $\varepsilon > 0$, there exists a polynomial g such that $\sup_I |f - g| \leq \varepsilon$. Thus,

$$\begin{aligned} \int f \, d\phi_n &= \int f \, d\phi_n - \int g \, d\phi_n + \int g \, d\phi_n \\ &= \int (f - g) \, d\phi_n + \int g \, d\phi_n, \end{aligned}$$

and this integral is bounded above and below by

$$\int (\pm\varepsilon + g) \, d\phi_n$$

respectively, which integrands are themselves polynomials, so they vanish in the limit. Therefore $\lim_{n\to\infty} \int f \, d\phi_n = 0$ by the squeeze theorem. ■

[†]That is, $g, g' \in C(I)$. For example, a polynomial.