Trevor Klar

Homework 6

8. Does the Borsuk-Ulam Theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x,y) \in S^1 \times S^1$ such that $f(x,y) = f(-x,-y)$?

Answer: No. Consider $f(x,y) = (\cos x, \sin x)$. This is clearly a map since it is constant with respect to y and it is the inclusion map with respect to x, but f never vanishes and is odd, so $f(x,y) \neq f(-x,-y)$ for all (x,y). ■

10. From the isomorphism $\pi_1\big((X \times Y), (x_0, y_0)\big) \approx \pi_1(X, x_0) \times \pi_1(Y, y_0)$ it follows that loops in $X \times \{y_0\}$ and $\{x_0 \times Y\}$ represent commuting elements of $\pi_1\big((X \times Y), (x_0, y_0)\big)$. Construct an explicit homotopy demonstrating this.

Answer: Let

- $\gamma = \big(\widetilde{\gamma}(s), y_0\big)$, where $\widetilde{\gamma}$ is a loop in (X, x_0).
- $\eta = \big(x_0, \widetilde{\eta}(s)\big)$ where $\widetilde{\eta}$ is a loop in (Y, y_0).
- For any loop α in any space, let $\omega_t \cdot \alpha$ denote a reparametrization of α which is constant for $s \in [0,t]$, then does α over the remaining s-values in $[0,1]$. Define $\alpha \cdot \omega_t$ similarly except do α first, then wait.

Observe that

$$\gamma \cdot \eta = (\widetilde{\gamma} \cdot \omega_{1/2}\ ,\ \omega_{1/2} \cdot \widetilde{\eta})$$
$$\eta \cdot \gamma = (\omega_{1/2} \cdot \widetilde{\gamma}\ ,\ \widetilde{\eta} \cdot \omega_{1/2}),$$

so

$$f_t = (\omega_{t/2} \cdot \widetilde{\gamma} \cdot \omega_{1-t/2}\ ,\ \omega_{1-t/2} \cdot \widetilde{\eta} \cdot \omega_{t/2})$$

is the desired homotopy. ■

15. Given a map $f : X \to Y$ and a path $h : I \to X$ from x_0 to x_1, show that $f_* \beta_h = \beta_{fh} f_*$ in the diagram at the right.

$$\begin{tikzcd} \pi_1(X, x_1) \arrow[r, "\beta_h"] \arrow[d, "f_*"'] & \pi_1(X, x_0) \arrow[d, "f_*"] \\ \pi_1(Y, f(x_1)) \arrow[r, "\beta_{fh}"] & \pi_1(Y, f(x_0)) \end{tikzcd}$$

Proof Let γ be a loop based at x_1. Then

$$\gamma \xmapsto{\beta_h} h \cdot \gamma \cdot \overline{h} \xmapsto{f_*} f(h \cdot \gamma \cdot \overline{h}) = fh \cdot f\gamma \cdot f\overline{h}$$

and

$$\gamma \xmapsto{f_*} f\gamma \xmapsto{\beta_{fh}} fh \cdot f\gamma \cdot \overline{fh} = fh \cdot f\gamma \cdot f\overline{h},$$

so the diagram commutes. ■

16. Show that there are no retractions $r : X \to A$ in the following cases:

(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.

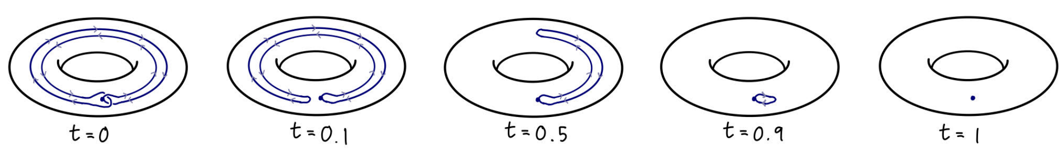
Proof Suppose for contradiction there is a retraction $r : X \to A$, and let γ be any loop in A, where x_0 denotes the basepoint. We know $\mathbb{R}^3$ is contractible, so by composing such a homotopy with γ we can produce f_t, a straight-line homotopy in $\mathbb{R}^3$ from γ to x_0. Then $r \circ f_t$ is a homotopy in A from γ to x_0, contradicting that $\pi_1(A) \neq 0$. □

(b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.

Proof We know that $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$, so any loop in A is homotopic to a loop (ω_n, γ_m) for $n, m \in \mathbb{Z}$ which goes around the torus n times in one direction and m times in the the other. Consider $(\omega_n, \gamma_m) \in X$, and note that D^2 is contractible, so there is a homotopy f_t in X between (ω_n, γ_m) and $(\omega_n, 0)$. Assuming a retraction $r : X \to A$ exists, we can compose $r \circ f$ to find that $(\omega_n, \gamma_m) \simeq (\omega_n, x_0)$ in A, contradicting that $\pi_1(S^1 \times S^1) \neq \mathbb{Z} \times 0$. □

(c) $X = S^1 \times D^2$ with A the circle shown in the figure.

Proof We will use the fact that if X retracts to A, then the inclusion map $A \hookrightarrow X$ induces an injection $\pi_1(A) \rightarrowtail \pi_1(X)$ (and in fact, would have saved time by using it on parts (a) and (b) as well). Observe that the loop which traverses A once is homotopic to 0 in X by the following homotopy:



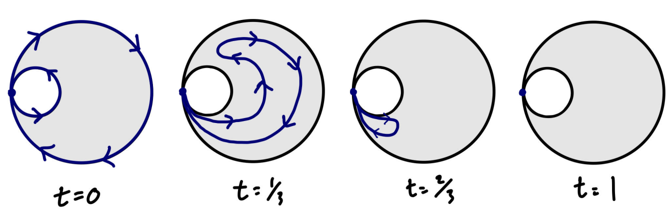
Thus $1 \in \pi_1(A) \mapsto 0 \in \pi_1(X)$, so the kernel is nontrivial, and the homomorphism is not injective. □

(d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.

Proof We know that $\pi_1(X) = 0$ since X is contractible and $\pi_1(A)$ is the free group on two generators, so any homomorphism from $\pi_1(A) \to \pi_1(X)$ fails to be injective since everything maps to zero. ■

(e) X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.

Proof If g, f are the generators of $\pi_1(A)$, then $g\overline{f}$ is a nonzero element in $\pi_1(A)$ which maps to $0 \in \pi_1(X)$ under the homomorphism induced by the inclusion:



so the homomorphism is not injective. □

(f) X the Möbius band and A its boundary circle.

Proof I can't figure this out. X deformation retracts to its central circle, so $\pi_1(X) = \mathbb{Z}$, but also A is a circle, so $\pi_1(A) = \mathbb{Z}$ as well. I can't come up with any reason why ι^* would fail to be injective either, since we can't homotope a nontrivial loop in the boundary to a constant loop in X in any obvious way. In fact, it seems to me that $\iota^*(n) = 2n$, which *is* injective. ■

17. Construct infinitely many nonhomotopic retractions $S^1 \vee S^1 \to S^1$.

Answer: Denote points in $S^1 \vee S^1$ by $\theta \in (0, 4\pi)$, so that $(0, 2\pi]$ represents a point in the first circle, $[2\pi, 4\pi)$ represents a point in the second circle, and 2π is in both circles. Then define

$$r_n(\theta) = \begin{cases} \theta, & \theta \in (0, 2\pi] \\ n\theta, & \theta \in [2\pi, 4\pi) \end{cases}$$

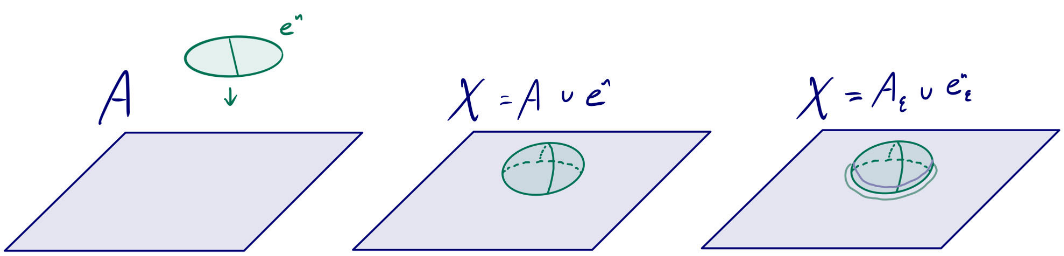
where we make the usual identification in the codomain that $\theta \sim 2n\pi\ \forall n \in \mathbb{N}$. It is clear that these are nonhomotopic maps since they are loops in distinct equivalence classes of $\pi_1(S^1)$.

18. Using Lemma 1.15, show that if a space X is obtained from a path-connected subspace A by attaching a cell e^n with $n \geq 2$, then the inclusion $A \hookrightarrow X$ induces a surjection on π_1. Use this to show:

(i) The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.

(ii) For a path-connected CW complex X the inclusion map $X^1 \hookrightarrow X$ of its 1-skeleton induces a surjection $\pi_1(X^1) \to \pi_1(X)$. [For the case that X has infinitely many cels, see Proposition A.1 in the appendix.]

Proof Consider X as the union of A_ϵ and e^n_ϵ, which are ϵ-neighborhoods of A and e^n respectively in X. Since A, e^n are deformation retractions of A_ϵ, e^n_ϵ, it suffices to show that the property holds for those subspaces.



For path-connected spaces, we know that π_1 is independent of base point, and A and e^n are both path-connected, which means X is as well. Thus without loss of generality we can suppose the basepoint $x_0 \in (A_\epsilon \cap e^n_\epsilon)$. Given any loop f in X, we can use Lemma 1.15 to write it as $f = \left(\blacksquare_{i=1}^{n} \gamma_i\right)$, where each γ_i is in A_ϵ or e^n_ϵ. Let

$$E = \{i : \gamma_i \in e^n_\epsilon\}.$$

Since e^n_ϵ is contractible, then for each γ_i such that $i \in E$, $\gamma_i \simeq \mathbb{1}$, so

$$f = \left(\mathop{\blacksquare}_{i=1}^{n} \gamma_i \right) \simeq \left(\mathop{\blacksquare}_{i \in E^{\complement}} \gamma_i \right)$$

which is a loop in A_ϵ.

Thus for every loop in X, there is a homotopy equivalent loop in A, which means $\pi_1(X)$ is a subgroup of $\pi_1(A)$ (up to isomorphism). □

Proof (i) It suffices to show that the inclusion $S^1 \hookrightarrow S^1 \vee S^2$ induces an injective homomorphism $\pi_1(S^1) \to \pi_1(S^1 \vee S^2)$, since it is already surjective by above. For any loop $\gamma \in S^1$ such that $\gamma \not\simeq c$ where c is the constant loop, consider $\gamma \in (S^1 \vee S^2)$. Call the intersection point of the wedge x_0, and without loss of generality suppose x_0 is the basepoint. Suppose there is a homotopy f_t of loops in $(S^1 \vee S^2)$ such that $f_0 = \gamma$ and $f_1 = c$. Since any loop with points in $S^1 - \{x_0\}$ and $S^2 - \{x_0\}$ must pass through $\{x_0\}$, then at every time t, f_t is a concatenation of loops in S^1 with loops in S^2. Then we can write F as a concatenation of homotopies

$$f_t = \left(\mathop{\blacksquare}_{i=1}^{n} f_{i,t} \right).$$

Since $f_1 = c$, then $f_{i,1} = c$ for all i, which means that if $f_{j,0}$ is any of the loops in S^1, then $f_{j,t}$ is a homotopy between that loop and c ion S^1, contradicting that $\gamma \not\simeq c$. □

Proof (ii) Inductively, we can see that

$$\pi_1(X^1) \twoheadrightarrow \pi_1(X^2)$$

since X^2 is exactly X^1 with 2-cells attached, and

$$\pi_1(X^1) \twoheadrightarrow \pi_1(X^n) \implies \pi_1(X^1) \twoheadrightarrow \pi_1(X^{n+1})$$

since X^n is exactly X^{n+1} with $(n+1)$-cells attached. Thus the map is surjective for finite-dimensional CW complexes.

Proposition A.1 shows that a compact subspace of a CW complex is contained in a finite subcomplex. This means that for an arbitrary curve $\gamma \in X^\infty$, the image of the curve is compact since I is compact, so it is contained in X^n for some n. Thus there is some $\gamma' \in X^n$ with $\gamma' \simeq \gamma$ and we are done. ■

Collaborators:

None for this homework.