9:04

1. Let X be the set of subsets of $\mathbb{N}$. If A is a finite subset of $\mathbb{N}$ & B is a subset of $\mathbb{N}$ whose complement is finite, define a subset $[A,B]$ of X by
$$[A,B]=\{E\subset\mathbb{N} : A\subset E\subset B\}$$
(a) Show that the sets $[A,B]$ form a base for a topology on X.

Let $E\subseteq\mathbb{N}$, then $[\emptyset,\mathbb{N}]$ satisfies that $\emptyset\subset E\subset\mathbb{N}$. So every element of $2^{\mathbb{N}}$ belongs to some $[A,B]$.

Let $F\in[A,B]\cap[C,D]$. Note that A & C are finite so so is $A\cup B$. Similarly B & D have complement that is finite so so does $B\cap D$. Therefore $[A\cup B, C\cap D]$ is an element of the basis. Further, since $A\subseteq F$ & $C\subseteq F$, $A\cup C\subseteq F$, & since $F\subseteq B$ & $F\subseteq D$, $F\subseteq B\cap D$. Thus $F\in[A\cup C, B\cap D]$. Lastly, let $G\in[A\cup C, B\cap D]$. Then $A\cup C\subseteq G$ so $A\subseteq G$ & $C\subseteq G$. Also $G\subseteq B\cap D$ so $G\subseteq B$ & $G\subseteq D$. Thus $G\in[A,B]\cap[C,D]$. Thus $F\in[A\cup C, B\cap D]\subseteq[A,B]\cap[C,D]$. Thus this collection forms a basis. □ $\boxed{9:14}$

(b) Prove that with this topology X is Hausdorff.

Let E & $F\in 2^{\mathbb{N}}$ & $E\neq F$. Then either $\exists\, n\in\mathbb{N}$ with $n\in E$ & $n\notin F$ or $n\in F$ and $n\notin E$. Without loss of generality let $\exists\, n\in\mathbb{N}$ with $n\in E$ & $n\notin F$. Therefore,
$\{n\}\subseteq E\subseteq\mathbb{N}$ so $E\in[\{n\},\mathbb{N}]$ and
$\emptyset\subseteq F\subseteq\mathbb{N}-\{n\}$ so $F\in[\emptyset,\mathbb{N}\setminus\{n\}]$

Now suppose $G\in[\{n\},\mathbb{N}]\cap[\emptyset,\mathbb{N}\setminus\{n\}]$.
Then $\{n\}\subseteq G$ so $n\in G$ & $G\subseteq\mathbb{N}\setminus\{n\}$

which implies $n \notin G$, a contradiction. Thus $[\emptyset, \mathbb{N}\setminus\{n\}]$ & $[\{n\}, \mathbb{N}]$ are two disjoint open sets containing F & E respectively so our space is Hausdorff. $\square$ $\boxed{9:19}$

(c) Prove that with this topology X is disconnected.

As shown in part (b), $[\{1\}, \mathbb{N}]$ & $[\emptyset, \mathbb{N}\setminus\{1\}]$ are disjoint open sets. Let $E \in 2^{\mathbb{N}}$. If $1 \in E$ then $E \in [\{1\}, \mathbb{N}]$. If $1 \notin E$ then $E \in [\emptyset, \mathbb{N}\setminus\{1\}]$. Further these sets are nonempty because $\{1,2\} \in [\{1\}, \mathbb{N}]$ & $\{2\} \in [\emptyset, \mathbb{N}\setminus\{1\}]$

Thus $2^{\mathbb{N}} = [\{1\}, \mathbb{N}] \amalg [\emptyset, \mathbb{N}\setminus\{1\}]$

So X is disconnected. $\square$ $\boxed{9:22}$

(d) Prove that the function $f: X \to X$ given by $f(E) = \mathbb{N}\setminus E$ is continuous.

Let $[A,B]$ be an arbitrary basis element of X & consider $f^{-1}([A,B])$. Let $E \in f^{-1}([A,B])$. Then since A is finite, $\mathbb{N}\setminus A$ has finite complement & since B has finite complement, $\mathbb{N}\setminus B$ is finite.

Now note that since $E \in f^{-1}([A,B])$, $f(E) = \mathbb{N}\setminus E \in [A,B]$ which implies $A \subseteq \mathbb{N}\setminus E \subseteq B$ so $\mathbb{N}\setminus B \subseteq E \subseteq \mathbb{N}\setminus A$ therefore $E \in [\mathbb{N}\setminus B, \mathbb{N}\setminus A]$. Further, if $F \in [\mathbb{N}\setminus B, \mathbb{N}\setminus A]$ then $\mathbb{N}\setminus B \subseteq F \subseteq \mathbb{N}\setminus A$, so $f(F) = \mathbb{N}\setminus F$ satisfies that $A \subseteq \mathbb{N}\setminus F \subseteq B$ so $f(F) \in [A,B]$. Thus $E \in [\mathbb{N}\setminus B, \mathbb{N}\setminus A] \subseteq f^{-1}([A,B])$ so f is continuous. $\square$ $\boxed{9:29}$

2. Give a proof or counterexample for each of the following:

(a) Every closed subset of a compact space is compact.

This is true. Let X be a compact space & let $C \subseteq X$ be closed. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Delta}$ be an open cover of C. Then

$$\mathcal{V} = \{X \setminus C, U_\alpha\}_{\alpha \in \Delta}$$

is an open cover of X, so $\exists$ a finite subcover

$$\bar{\mathcal{V}} = \{X \setminus C, U_i\}_{i=1}^{n}$$

of X. $X \setminus C \cap C = \emptyset$ so $\{U_i\}_{i=1}^{n}$ must cover C. Thus C is compact. $\square$ $\boxed{9{:}33}$

(b) The product of any two connected spaces is connected.

This is true. Let X & Y be connected. Suppose $\exists$ $f: X \times Y \to \{0,1\}$ where $\{0,1\}$ is given the discrete topology & f is continuous. Without loss of generality suppose that $\exists$ $(\bar{x}, \bar{y}) \in X \times Y$ such that $f(\bar{x}, \bar{y}) = 0$. Then since X is connected so is $X \times \{\bar{y}\}$. Therefore by the lemma, $f|_{X \times \{\bar{y}\}}$ must be the constant function 0. Similarly, $\{\bar{x}\} \times Y$ is connected, so $f|_{\{\bar{x}\} \times Y}$ is the constant function 0. Therefore,

f is the constant function 0 & $X \times Y$ is connected.

<u>Lemma</u> X is connected if & only if every function $f: X \to \{0,1\}$ where $\{0,1\}$ is given the discrete topology is constant

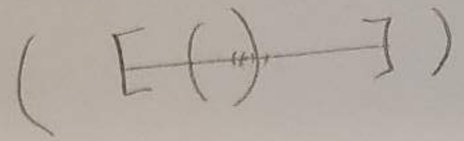
Proof of Lemma: Let $X = U \amalg V$ be a separation.
Then define $f: X \to \{0,1\}$

$$x \mapsto \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in V \end{cases}$$

f is continuous since $f^{-1}(\{0,1\}) = X$, $f^{-1}(\emptyset) = \emptyset$ & $f^{-1}(0) = U$ & $f^{-1}(1) = V$.

On the other hand if $f: X \to \{0,1\}$ is nonconstant & continuous then $f^{-1}(0)$ & $f^{-1}(1)$ are nonempty open sets in X & $X = f^{-1}(0) \amalg f^{-1}(1)$ so X is not connected. □

9:41



③ Prove that a metric space is compact if & only if it's sequentially compact.

Let (X,d) be a compact metric space & let (x_n) be a sequence. Suppose for contradiction that (x_n) has no convergent subsequence. Then every element $x \in X$ has an open set $\mathcal{U}_x$ of X such that $\mathcal{U}_x$ contains finitely many elements of (x_n). Then $\{\mathcal{U}_x\}_{x \in X}$ is an open cover of X & so $\exists$ finite cover $\{\mathcal{U}_{x_i}\}_{i=1}^{n}$. However each $\mathcal{U}_{x_i}$ only contains finitely many elements of the infinite sequence (x_n) which is a contradiction. Thus X must be sequentially compact.

Conversely, let (X,d) be sequentially compact. Let $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in \Delta}$ be an open cover of X.

Since X is sequentially compact $\exists$ Lebesgue number $\delta > 0$ such that $\forall x \in X$ $\exists \alpha \in \Delta$ with $B_\delta(x) \subseteq \mathcal{U}_\alpha$. Further X is totally bounded, so for this δ $\exists$ $x_1, \dots, x_n \in X$ where $\{B_\delta(x_i)\}_{i=1}^{n}$ is an open cover of X. Therefore, we can pick the associated $\mathcal{U}_i \in \mathcal{U}$ where $B_\delta(x_i) \subseteq \mathcal{U}_i$ & $\{\mathcal{U}_i\}_{i=1}^{n}$ will be a finite subcover of $\mathcal{U}$. □

$\boxed{16.12}$

④ A topological space is regular if for every closed subset C of X & every point $p \in X \setminus C$ $\exists$ disjoint open sets $\mathcal{U}, V \subset X$ with $C \subset \mathcal{U}$ & $p \in V$. Prove that every compact Hausdorff space is regular.

Let X be a compact Hausdorff space, & let $C \subseteq X$ be closed & $p \in X \setminus C$. As proved in problem 2, C is compact. Since X is Hausdorff $\forall$ $c \in C$ $\exists$ disjoint open sets $\mathcal{U}_p^c$ of p & V_c^p of C.

Then $\{V_c^p\}_{c \in C}$ is an open cover of C so $\exists$ a finite sub cover $\{V_{c_i}^p\}_{i=1}^n$ of C. Thus the associating open sets $\mathcal{U}_{c_i}^p$ satisfy that $\mathcal{U} = \bigcap_{i=1}^n \mathcal{U}_{c_i}^p$ is a nonempty open set containing p, & since $\mathcal{U}_{c_i}^p \cap V_{c_i}^p = \emptyset$ $\Rightarrow$

$$\mathcal{U} \cap \bigcup_{i=1}^n V_{c_i}^p = \emptyset$$

Thus $V = \bigcup_{i=1}^n V_{c_i}^p$ & $\mathcal{U} = \bigcap_{i=1}^n \mathcal{U}_p^{c_i}$ are disjoint open sets containing C & p respectively so X is regular. □

$\boxed{10 : 19}$

6. Give an example of a space that is connected but not path-connected. Prove that this example works.

Let $X = S \cup T$ where $S = \{(x, \sin(\frac{1}{x})) : 0 < x \leq 1\}$ & $T = \{(0,y) : -1 \leq y \leq 1\}$. First I show X is connected.
Note that S & T are both path-connected $\because$ (S is literally an image of a path & y is a segment of $\mathbb{R}$), so S & T are individually connected.
further $\bar{S} = X$ & so $S \subseteq X \subseteq \bar{S}$ which gives
→ that X is connected by the lemma.

Now suppose X is path-connected, thus $\exists$ a path $p : [0,1] \to X$ such that $p(0) = (0,0)$ & $p(1) = (1, \sin(1))$. Consider

$$t = \sup\{s \in [0,1] : p(s) \in T\}$$

Note that $0 \in \{s \in [0,1] : p(s) \in T\}$ so this set is nonempty. Further $[0,1]$ is closed & bounded so
→ this t must exist. Lastly T is closed, so $p^{-1}(T)$ is closed so $p(t) \in T$. Note that $t < 1$.

Now define $\bar{p} : [0,1] \to X$ where $\bar{p}(s) = p((1-s)t + s)$ so $\bar{p}$ is continuous. We can parametrize $\bar{p}$ as $\bar{p}(s) = (x(s), y(s))$. Note that $x(0) = 0$ & $x(s) > 0$ if $s \neq 0$ & $y(s) = \sin(1/x(s))$ if $s \neq 0$. Now we can create a sequence s_n by finding a u for each n where

$$0 < u < x(1/n)$$

& $y(u) = (-1)^n$. Then by the Intermediate Value Theorem $\exists$ $0 < s_n < 1/n$ where $x(s_n) = u$. Thus $s_n \to 0$ as $n \to \infty$ but $y(s_n) = \sin(1/x(s_n)) = \sin(1/u) = (-1)^n$ which does not converge. This contradicts continuity of $\bar{p}$ (& thus p) so X is not path-connected. $\square$

<u>Lemma</u> If A is connected & $A \subset B \subset \bar{A}$ then B is connected.

Proof: Let $f: B \to \{0,1\}$ be a continuous function where $\{0,1\}$ is given the discrete topology. Then since A is connected, problem 2 gives that $f|_A$ is constant. Say $f(A) = 0$ without loss of generality. Then since $B \subset \bar{A}$ if $\exists$ $b \in B$ with $b \in f^{-1}(1)$ $\Rightarrow$ $f^{-1}(1) \cap A = \emptyset$ since $f^{-1}(1)$ is open. This contradicts $f(A) = 0$. Thus $f(B) = 0$ & B is connected by problem 2. $\square$ 10:47

7. State the contraction mapping theorem.

Let X be a complete metric space. Then any $f: X \to X$ that is a contraction has a unique fixed point.

Prove that there is a unique continuous $f: [0,1] \to [0,1]$ whuch satisfies $\forall x \in [0,1]$ $f(x) = (f(\sin(x)) + \cos(x))/2$

Let $\mathcal{C}[0,1]$ be the complete metric space of continuous functions on $[0,1]$ with the sup norm. Let $g: \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ be defined as

$$g(h(x)) = \frac{h(\sin(x)) + \cos(x)}{2}$$

I first prove g is a contraction. Note that for any $h, k \in \mathcal{C}[0,1]$,

$$d(g(h(x)), g(k(x))) = \sup_{x \in [0,1]} |g(h(x)) - g(k(x))|$$

$$= \sup_{x \in [0,1]} \left| \frac{h(\sin(x)) + \cos(x)}{2} - \left(\frac{k(\sin(x)) + \cos(x)}{2}\right)\right|$$

$$= \sup_{x \in [0,1]} \left| \frac{h(\sin(x)) - k(\sin(x))}{2} \right|$$

$$\leq \frac{1}{2} \sup_{x \in [0,1]} |h(x) - k(x)|$$

$$= \frac{1}{2} d(h(x), k(x))$$

So g is indeed a contradiction. Thus the existence of a unique $f \in \mathcal{C}[0,1]$ follows directly from the contraction mapping theorem. $\square$