Math 550
Homework 9
Dr. Fuller
Due November 6

1. Professor Doofus gives the following "rule" for canceling differential forms $\omega \in \Omega^k(\mathbf{R}^n)$ and $\alpha, \beta \in \Omega^\ell(\mathbf{R}^n)$:
$$\text{If } \omega \wedge \alpha = \omega \wedge \beta, \text{ then } \alpha = \beta.$$
Give an example which shows that Doofus is mistaken.

2. Compute the volume of the unit ball in $\mathbf{R}^3$ by integrating an appropriate 2-form over the unit sphere in $\mathbf{R}^3$.

3. Suppose C is a 1-dimensional manifold in $\mathbf{R}^n$, oriented by a parameterization $c : [a,b] \to M$. Prove that $\int_{[a,b]} c^* ds = \int_a^b [\sum_{i=1}^n (c_i'(t))^2]^{\frac{1}{2}} \, dt$.

4. Let X be a vector field on $\mathbf{R}^3$, and let ω_X^1 and ω_X^2 denote the associated 1- and 2-forms, respectively. Let $f : \mathbf{R}^3 \to \mathbf{R}$ be a 0-form.
   (a) Show that: $df = \omega^1_{\text{grad } f}$, $d(\omega_X^1) = \omega^2_{\text{curl } X}$, and $d(\omega_X^2) = \text{div } X \; dx \wedge dy \wedge dz$.
   (b) Prove that curl grad $f = 0$ and div curl $X = 0$.

5. For a vector field $X = (f_x, f_y)$ on $\mathbf{R}^2$, we may define an associated 1-form, different from the one in class, by
$$\star\omega_X^1 = -f_y \, dx + f_x \, dy.$$
We may also define
$$\text{div } X = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}.$$
   (a) Let M be a compact 2-dimensional manifold with boundary in $\mathbf{R}^2$. Show that for all points $p \in \partial M$, the equation $\star\omega_X^1 = X \cdot n \, ds$ holds.
   (b) Prove the following *Divergence form of Green's Theorem:* Let M be a 2-dimensional manifold-with-boundary in $\mathbf{R}^2$, and let X be a vector field on M. Then
$$\int_M \text{div } X \, dA = \int_{\partial M} X \cdot n \, ds.$$

6. Let M be a compact 3-dimensional manifold-with-boundary in $\mathbf{R}^3$, with $(0,0,0) \in M - \partial M$. Consider the vector field $X(p) = \frac{p}{\|p\|^3}$ defined on $\mathbf{R}^3 - \{(0,0,0)\}$. Prove that
$$\int_{\partial M} X \cdot N \, dA = 4\pi.$$