Math 501
Homework 12

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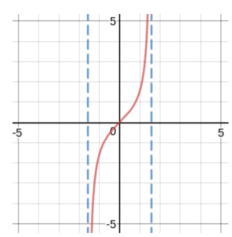
1. Let X be a topological space, with $\mathscr{B}$ a basis for the topology on X. Prove that if every open cover of X by sets in $\mathscr{B}$ has a finite subcover, then X is compact.

   PROOF Let $\{U_\alpha\}_{\alpha\in\Gamma}$ be an arbitrary open cover of X. Since each U_α is open, we know that for each $x \in U_\alpha$, there exists a basic open set B such that $x \in B \subset U_\alpha$. So for each $\alpha \in \Gamma$, and each $x \in U_\alpha$, let $B_{(\alpha,x)}$ denote a basic open set such that $x \in B \subset U_\alpha$. This means that $\bigcup_{\alpha\in\Gamma}\{B_{(\alpha,x)} : \forall x \in U_\alpha\}$ is an open cover of X by sets in $\mathscr{B}$, so it has a finite subcover $\{B_i\}_{i=1}^N$. Each B_i is a subset of some U_α, so for each $i \in \{1,\ldots,N\}$, let α_i be an element of Γ such that $B_i \subset U_{\alpha_i}$. Therefore, $\bigcup_{i=1}^N\{\{U_{\alpha_i}\}\}$ is a finite subcollection of $\{U_\alpha\}_{\alpha\in\Gamma}$ which covers X, so X is compact.[1] ■

2. Let $\{X_\alpha\}_{\alpha\in\Gamma}$ be a collection of spaces.

   (a) Prove that the projection $\pi_\beta : \prod\limits_{\alpha\in\Gamma} X_\alpha \to X_\beta$ is not necessarily closed.

   PROOF Consider the graph of $\tan|_{(-\frac{\pi}{2},\frac{\pi}{2})}$ as a subset of $\mathbb{R}^2_{usual}$; $S = \{(x, tan(x) : -\frac{\pi}{2} < x < \frac{\pi}{2}\}$.



   This set S is closed in $\mathbb{R}^2$, but $\pi_1(S) = (-\frac{\pi}{2}, \frac{\pi}{2})$ is not closed in $\mathbb{R}$. ■

   (b) Prove that $g : Y \to \prod\limits_{\alpha\in\Gamma} X_\alpha$ is continuous if and only if $\pi_\alpha \circ g$ is continuous for each $\alpha \in \Gamma$.

   PROOF ($\implies$) Suppose $g : Y \to \prod\limits_{\alpha\in\Gamma} X_\alpha$ is continuous. To show that $\pi_\alpha \circ g : Y \to X_\alpha$ is continuous for each $\alpha \in \Gamma$; let $\beta \in \Gamma$ be arbitrary, and let U_β be an arbitrary open set in X_β. Now,

$$\pi_\beta^{-1}(U_\beta) = U_\beta \times \prod_{\alpha\in(\Gamma-\beta)} X_\alpha,$$

   Which is open in $\prod\limits_{\alpha\in\Gamma} X_\alpha$. Since g is continuous, then $g^{-1}\left(\pi_\beta^{-1}(U_\beta)\right) = (\pi_\alpha \circ g)^{-1}(U_\beta)$ is open in Y, and we are done. ■

[1] There could be $U_{\alpha_i} = U_{\alpha_j}$ for some $i \neq j$, so this union notation is used to to clarify the fact that this is a collection of sets which does not repeat. It is a union of singletons of sets, not a union of the sets themselves.

Proof ($\Longleftarrow$) Suppose that $\pi_\alpha \circ g : Y \to X_\alpha$ is continuous for each $\alpha \in \Gamma$. Let $U \in \prod\limits_{\alpha\in\Gamma} X_\alpha$ be any basic open set. By definition,

$$U = \prod \begin{cases} U_\alpha & \alpha \in \{\alpha_1, \ldots, \alpha_N\} \\ X_\alpha & \alpha \notin \{\alpha_1, \ldots, \alpha_N\} \end{cases}$$

Now we can see that $g^{-1}(U)$ is open by the following diagram chase, since there are finitely many nontrivial component sets of U:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[rr, bend left, "\pi_\alpha \circ g"] & \prod\limits_{\alpha\in\Gamma} X_\alpha \arrow[r, "\pi_\alpha"] & X_\alpha \end{tikzcd}$$

For each $\alpha_i \in \{\alpha_1, \ldots, \alpha_N\}$, denote the preimage

$$(\pi_{\alpha_i} \circ g)^{-1}(U_{\alpha_i}) = V_{\alpha_i}.$$

Note that V_{α_i} may differ from $g^{-1}(U)$, since $\pi_{\alpha_i}^{-1}(\pi_{\alpha_i}(U))$ may differ from U. However, $g^{-1}(U) \subset V_{\alpha_i}$. We know that V_{α_i} is open, since $\pi_{\alpha_i} \circ g$ is continuous. Now we are done, since $\bigcap V_{\alpha_i} = g^{-1}(U)$ is a finite intersection of open sets, and thus is open. To see that this equality holds, let $p \in \bigcap V_{\alpha_i}$. So, for every $\alpha \in \Gamma$,

$$(\pi_\alpha \circ g)(p) \in \begin{cases} U_\alpha & \alpha \in \{\alpha_1, \ldots, \alpha_N\} \\ X_\alpha & \alpha \notin \{\alpha_1, \ldots, \alpha_N\} \end{cases}$$

so $g(p) \in U$. Thus, $\bigcap V_{\alpha_i} \subset g^{-1}(U)$. Now we show that $\bigcap V_{\alpha_i} \supset g^{-1}(U)$. Let $p \in g^{-1}(U)$. So $g(p) \in U$, and

$$(\pi_\alpha \circ g)(p) \in \begin{cases} U_\alpha & \alpha \in \{\alpha_1, \ldots, \alpha_N\} \\ X_\alpha & \alpha \notin \{\alpha_1, \ldots, \alpha_N\} \end{cases}$$

thus, $p \in \bigcap V_{\alpha_i}$ by definition of V_{α_i}. Thus, we have shown that for every basic open set U, $g^{-1}(U)$ is open, therefore g is continuous. ∎

3. Describe the box topology on $\prod\limits_{x\in X} \{0,1\}_X$. Show that the box topology on $\prod\limits_{x\in X} \{0,1\}_X$ is not necessarily compact.

   Answer: The space itself is the set of all functions $f : X \to \{0,1\}$ such that for all $x \in X$, $f(x) = 0$ or $f(x) = 1$. The topology is the discrete topology, since a set U is open in $\prod\limits_{x\in X} \{0,1\}_X$ if $\pi_x(U)$ is open for all $x \in X$, and each $\{0,1\}$ has the discrete topology.

   Proof To show that the box topology on $\prod\limits_{x\in X} \{0,1\}_X$ is not necessarily compact, consider $\{0,1\}^{\mathbb{N}}$ where $\mathbb{N} = \{1, 2, \ldots\}$. For each $i \in \mathbb{N}$, let $S_i = \{f : \mathbb{N} \to \{0,1\} | f(i) = 1\}$, and let S_0 be the singleton set $\{f \equiv 0\}$.
   Now $\{S_i\}_{i=0}^{\infty}$ is an open cover of $\{0,1\}^{\mathbb{N}}$, since if $f \not\equiv 0$ then there is some $i \in \mathbb{N}$ such that $f(i) = 1$. Also, this open cover has no finite subcover, since removing any element S_j of the cover results in a collection which does not contain $f : \mathbb{N} \to \{0,1\}$ such that $f(j) = 1$ and $f(i) = 0$ for all $i \neq j$. ∎

5. Prove the converse to the Tychonoff Theorem: If the product topology $\prod\limits_{\alpha\in\Gamma} X_\alpha$ is compact, then each X_α is compact.

   Proof Suppose $X = \prod\limits_{\alpha\in\Gamma} X_\alpha$ is compact. Let $\alpha_0 \in \Gamma$ be arbitrary, and let $\{U_{\beta_{\alpha_0}}\}_{\beta\in\Delta}$ be an arbitrary open cover of $X\alpha_0$. Let

$$\{U_\beta\}_{\beta\in\Delta} = \left\{ U_{\beta_{\alpha_0}} \times \prod_{\alpha} U_{\beta_\alpha} : \beta \in \Delta,\, \alpha \in (\Gamma - \alpha_0) \right\}$$

be an arbitrary open cover of X. Since X is compact, $\{U_\beta\}$ has a finite subcover

$$\{U_{\beta^i}\}_{i=1}^N = \left\{\prod_{\alpha\in\Gamma} U_{\beta^i_\alpha}\right\}_{i=1}^N$$

where each $\beta^i \in \Delta$. Now consider the image of this collection in the projection π_{α_0}:

$$\{\pi_{\alpha_0}\left(U_{\beta^i}\right)\}_{i=1}^N = \{U_{\beta^i_{\alpha_0}}\}_{i=1}^N$$

Observe that $\{U_{\beta^i_{\alpha_0}}\}_{i=1}^N \subset \{U_{\beta_{\alpha_0}}\}_{\beta\in\Delta}$, and it also covers all of X_{α_0} (since $\{U_{\beta^i}\}_{i=1}^N$ covers X). Thus, we have produced a finite subcover of any open cover of any X_α, so we are done. ∎