Topology Fall 2017

① For X & Y topological spaces define what it means for a function $f: X \to Y$ to be continuous. Give the ϵ-δ definition of continuity for metric spaces. Prove that your definitions are equivalent for metric spaces.

Let X & Y be topological spaces. We say a function $f: X \to Y$ is continuous if for all open sets U of Y, $f^{-1}(U)$ is open in X.

Let X be a metric space with metric d_X & Y be a metric space with metric d_Y. Then a function $f: X \to Y$ is continuous at $x_0 \in X$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in X$ with $d_X(x, x_0) < \delta$ we have $d_Y(f(x), f(x_0)) < \epsilon$. We say f is continuous if it is continuous at x for all $x \in X$.

Let $f: X \to Y$ be continuous in the first sense where (X, d_X) & (Y, d_Y) are metric spaces. Let $\epsilon > 0$, & $x_0 \in X$. Then $f^{-1}(B_\epsilon(f(x_0)))$ is open in (X, d_X) & contains x_0. Thus there exists a basis element $B_\delta(x_0) \subset f^{-1}(B_\epsilon(f(x_0)))$ which contains x_0. Therefore if $d_X(x, x_0) < \delta$ then $x \in B_\delta(x_0)$ so $f(x) \in B_\epsilon(f(x_0))$ so $d_Y(f(x), f(x_0)) < \epsilon$ which means f is continuous at x_0, $\forall x_0 \in X$, so f is continuous in the 2nd sense.

Now let f be continuous in the 2nd sense. Let U be open in Y. Then $U = \bigcup_{\substack{\epsilon>0 \\ y \in Y}} B_\epsilon(y)$. Then $f^{-1}(U) = f^{-1}\left(\bigcup_{\substack{\epsilon>0 \\ y \in Y}} B_\epsilon(y)\right) = \bigcup_{\substack{\epsilon>0 \\ y \in Y}} f^{-1}(B_\epsilon(y))$.

In order for $f^{-1}(U)$ to be open we just need to show $\forall \epsilon > 0$ & $y \in Y$ $f^{-1}(B_\epsilon(y))$ is open in X. Let $x_0 \in f^{-1}(B_\epsilon(y))$. Let $\bar{d} = \{d_Y(f(x_0), y)$

Then since f is continuous in the 2nd sense at x_0 $\exists \delta > 0$ s.t. $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon - \bar{d}$. Therefore

$$d_Y(f(x), y) \leq d(f(x), f(x_0)) + d(f(x_0), y)$$
$$< \epsilon - \bar{d} + \bar{d} < \epsilon$$

Therefore $B_\delta(x_0) \subset f^{-1}(B_\epsilon(y))$ so $f^{-1}(B_\epsilon(y))$ is open & f is continuous in the 1st sense. □