Math 450b
Homework 2

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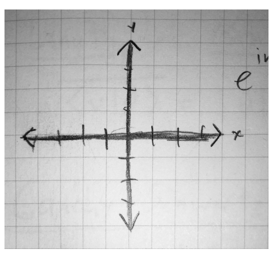
1. Let U be an open set in $\mathbb{R}^n$ and C be a closed set in $\mathbb{R}^n$, with $C \subset U$. Prove that $U - C$ is open.

   Proof By definition of set subtraction, $U - C = U \cap C^{\complement}$, with U and $C^{\complement}$ open. Thus, $U - C$ is open. ■

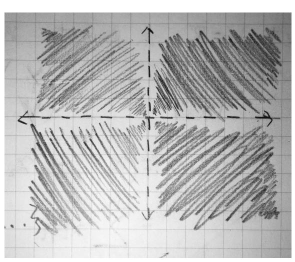
2. (□) Give the interior, exterior, and boundary for the following subsets of $\mathbb{R}^n$. No proofs, just give answers.

   (a) $S = \{(x, y) : xy = 0\} \subset \mathbb{R}^2$
   Answer: $\text{int}(S) = \emptyset$,
   $\text{bdy}(S) = S$,
   $\text{ext}(S) = S^{\complement}$



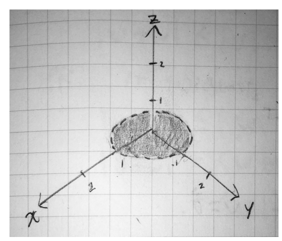
□

   (b) $S = \{(x, y) : xy \neq 0\} \subset \mathbb{R}^2$
   Answer: $\text{int}(S) = S$,
   $\text{bdy}(S) = S^{\complement}$,
   $\text{ext}(S) = \emptyset$



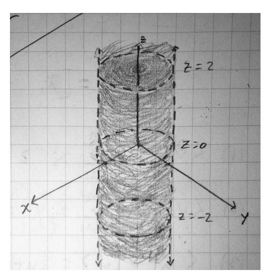
□

   (c) $S = \{(x, y, z) : x^2 + y^2 < 1 \text{ and } z = 0\} \subset \mathbb{R}^3$
   Answer: $\text{int}(S) = \emptyset$,
   $\text{bdy}(S) = \{(x, y, z) : x^2 + y^2 \leq 1 \text{ and } z = 0\}$,
   $\text{ext}(S) = \{(x, y, z) : x^2 + y^2 > 1 \text{ or } z \neq 0\}$



□

   (d) $S = \{(x, y, z) : x^2 + y^2 < 1\} \subset \mathbb{R}^3$
   Answer: $\text{int}(S) = S$,
   $\text{bdy}(S) = \{(x, y, z) : x^2 + y^2 = 1\}$,
   $\text{ext}(S) = \{(x, y, z) : x^2 + y^2 > 1\}$



□

   (e) $\{(x_1, \ldots, x_n) : \text{each } x_i \in \mathbb{Q}\} \subset \mathbb{R}^n$
   Answer: $\text{int}(S) = \emptyset$,
   $\text{bdy}(S) = \mathbb{R}^n$,
   $\text{ext}(S) = \emptyset$
   This set is impossible to draw. I imagine it something like a dense infinite point grid, like a field of stars in space. Each element has infinitely many other elements surrounding it in every direction, as well as elements not in the set surrounding it in a similar way. ■

3. Decide if the following subsets of R^n are closed, bounded, and compact.

(a) A finite set of points in $\mathbb{R}^n$.

Answer: Let S be such a set. S is bounded. To see this, let $r = \max\{||\mathbf{y}|| : \mathbf{y} \in S\}$. Since $S \subset \overline{B}(\mathbf{0}, r)$, we are done. S is also closed. To see this, let $\mathbf{x} \in S^{\mathsf{C}}$, and let $r = \min(||\mathbf{x} - \mathbf{y}|| : \mathbf{y} \in S)$. Since $B(\mathbf{x}, r) \subset S^{\mathsf{C}}$, we are done. Since S is closed and bounded, then it is compact by Heine-Borel.

(b) $\overline{B}(\mathbf{0}, 2) - B(\mathbf{0}, 1)$

Answer: $\overline{B}(\mathbf{0}, 2) - B(\mathbf{0}, 1) = \overline{B}(\mathbf{0}, 2) \cap B(\mathbf{0}, 1)^{\mathsf{C}}$ and so is closed. Also, $\overline{B}(\mathbf{0}, 2) - B(\mathbf{0}, 1) \subset \overline{B}(\mathbf{0}, 2)$, and so is bounded. Thus, the set is closed and bounded, and so it is compact by Heine-Borel.

(c) $\{(x_1, \ldots, x_n) \in \overline{B}(\mathbf{0}, 1) : x_n = 0\}$

Answer: This is the n-dimensional version of problem 2(c). This set is closed, and is bounded by $\overline{B}(\mathbf{0}, 1)$. Thus, it is compact by Heine-Borel.

(d) $\{(x_1, \ldots, x_n) \in \overline{B}(\mathbf{0}, 10) : \text{each } x_i \in \mathbb{Z}\} \subset \mathbb{R}^n$

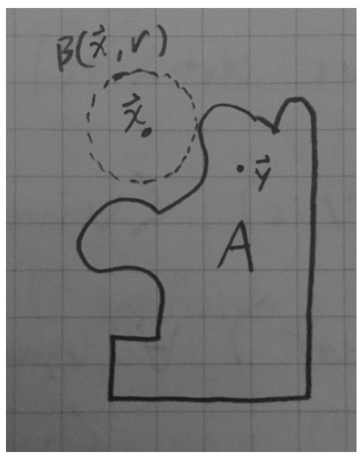
Answer: This is a finite set, so it is closed, bounded, and compact by 3(a).

(e) $\{(x_1, \ldots, x_n) \in \overline{B}(\mathbf{0}, 10) : \text{each } x_i \in \mathbb{Q}\} \subset \mathbb{R}^n$

Answer: This set is bounded but not closed, and since Heine-Borel is a biconditional, the set is not compact.

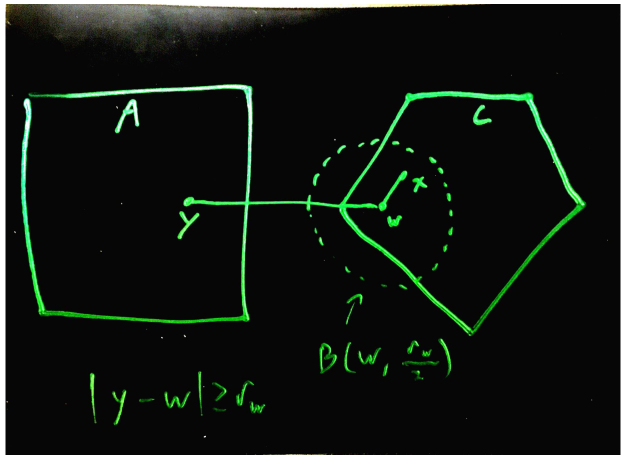
4. (a) ($\square$) Suppose A is a closed subset of $\mathbb{R}^n$, and $\mathbf{x} \notin A$. Prove that there is a $\delta > 0$ such that $||\mathbf{x} - \mathbf{y}|| \geq \delta$ for all $y \in A$.

Proof Let $\mathbf{y} \in A$. Since A^{C} is open and $\mathbf{x} \in A^{\mathsf{C}}$, there is some $\delta > 0$ such that $B(\mathbf{x}, r) \subset A^{\mathsf{C}}$.



Let $\delta = r$. Since $\mathbf{y} \in A = (A^{\mathsf{C}})^{\mathsf{C}}$, then $\mathbf{y} \notin B(\mathbf{x}, \delta) = \{\mathbf{z} \in \mathbb{R}^n : ||\mathbf{z} - \mathbf{x}|| < \delta\}$, so $\mathbf{y} \in \{\mathbf{z} \in \mathbb{R}^n : ||\mathbf{z} - \mathbf{x}|| \geq \delta\}$, and we are done. ■

(b) Suppose that A and C are closed subsets of $\mathbb{R}^n$, with C compact, and $A \cap C = \emptyset$. Prove that there exists $\delta > 0$ such that $||\mathbf{x} - \mathbf{y}|| \geq \delta$ for all $\mathbf{y} \in A$ and $\mathbf{x} \in C$. (Hint: For each $\mathbf{w} \in C$, find an open ball such that this inequality holds for all $\mathbf{x} \in B(\mathbf{w}, r(\mathbf{w}))$.)



Proof Since A is closed, then by 4(a), for each $\mathbf{w} \in C \subset A^{\complement}$, we can find some $r_{\mathbf{w}} > 0$ such that

$$||\mathbf{y} - \mathbf{w}|| \geq r_{\mathbf{w}}$$

for any $\mathbf{y} \in A$. Thus,

$$B\left(\mathbf{w}, \frac{r_{\mathbf{w}}}{2}\right) \subset B(\mathbf{w}, r_{\mathbf{w}}) \subset A^{\complement}.$$

Now, the collection $\{B\left(\mathbf{w}, \frac{r_{\mathbf{w}}}{2}\right) : \mathbf{w} \in C\}$ is an open cover of C, and since C is compact, there exists some finite subcollection $\{B\left(\mathbf{w}_i, \frac{r_{\mathbf{w}_i}}{2}\right)\}_{i=1}^N$ which also covers C. Consider $B\left(\mathbf{w}_k, \frac{r_{\mathbf{w}_k}}{2}\right)$ for some $k \in \{1, \ldots, N\}$. Now, for any $x \in B\left(\mathbf{w}_k, \frac{r_{\mathbf{w}_k}}{2}\right)$, and any $y \in A$,

$$||\mathbf{w}_k - \mathbf{x}|| < \frac{r_{\mathbf{w}_k}}{2}.$$

And, by the Triangle Inequality,

$$\begin{aligned} ||\mathbf{y} - \mathbf{x}|| &\geq \underbrace{||\mathbf{y} - \mathbf{w}_k||}_{> r_{w_k}} - \underbrace{||\mathbf{w}_k - \mathbf{x}||}_{< \frac{r_{\mathbf{w}_k}}{2}} \\ &> \frac{r_{\mathbf{w}_k}}{2}. \end{aligned}$$

Now, to obtain a lower bound that applies to any $\mathbf{x} \in C$, let

$$\delta = \min\left\{\frac{r_{\mathbf{w}_i}}{2} : i \in \{1, \ldots, N\}\right\}.$$

Since every $\mathbf{x} \in C$ is an element of some $B\left(\mathbf{w}_i, \frac{r_{\mathbf{w}_i}}{2}\right)$, then for any $\mathbf{y} \in A, \mathbf{x} \in C$,

$$||\mathbf{y} - \mathbf{x}|| \geq \delta > 0$$

and we are done. ■

(c) (□) Give a counterexample in $\mathbb{R}^2$ to part (b) when both A and B are closed, but neither is compact. **Answer:** Let $A = \{(x, y) : y = \sinh(x)\}$, and let $B = \{(x, y) : y = \cosh(x)\}$.