450b HW4 Trevor Klar

1. If $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are differentiable on an open set U, and $\alpha, \beta \in \mathbb{R}$, prove that $\alpha f + \beta g : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and $D(\alpha f + \beta g)(\vec{a}) = \alpha D(f)(\vec{a}) + \beta D(g)(\vec{a})$.

<u>Proof</u>: Let $\vec{a} \in U$. Since f, g are diff'ble on U, then there exist $Df(\vec{a})$ and $Dg(\vec{a})$ s.t.

$$\lim_{\vec{x} \to \vec{a}} \frac{\| f(\vec{x}) - f(\vec{a}) - Df(\vec{a})(\vec{x}-\vec{a}) \|}{\| x - a \|} = 0,$$

and similar for g.

Now consider the function αf.

<u>Claim</u>: αf is differentiable at $\vec{a}$ and

$\alpha Df(\vec{a})(\vec{x})$

is its derivative. Observe:

$$\lim_{\vec{x} \to \vec{a}} \frac{\| \alpha f(\vec{x}) - \alpha f(\vec{a}) - \alpha Df(\vec{a})(\vec{x}-a) \|}{\| \vec{x} - \vec{a} \|}$$

$$= \lim_{\vec{x} \to \vec{a}} \frac{\| \alpha \ (f(\vec{x}) - f(\vec{a}) - Df(\vec{a})(\vec{x}-a) \|}{\| \vec{x} - \vec{a} \|}$$

$$= \| \alpha \| \lim_{\vec{x} \to \vec{a}} \frac{\| f(\vec{x}) - f(\vec{a}) - Df(\vec{a})(\vec{x}-a) \|}{\| \vec{x} - \vec{a} \|}$$

$$= 0.$$

Similarly, βg is diff'ble and $\beta Dg(\vec{a})(\vec{x})$ is the desired linear transformation.

Now Consider $\alpha f + \beta g$.

$$\lim_{\vec{x}\to\vec{a}} \frac{\|[\alpha f+\beta g](\vec{x}) - [\alpha f+\beta g](\vec{a}) - [\alpha Df + \beta Dg](\vec{a})(\vec{x}-\vec{a})\|}{\|\vec{x}-\vec{a}\|}$$

$$= \lim_{\vec{x}\to\vec{a}} \frac{\|[\alpha f(\vec{x}) - \alpha f(\vec{a}) - \alpha Df(\vec{a})(\vec{x}-\vec{a})] + [\beta g(\vec{x}) - \beta g(\vec{a}) - \beta Dg(\vec{a})(\vec{x}-\vec{a})]\|}{\|\vec{x}-\vec{a}\|}$$

$$\leq \lim_{\vec{x}\to\vec{a}} \frac{\|\alpha f(\vec{x}) - \alpha f(\vec{a}) - \alpha Df(\vec{a})(\vec{x}-\vec{a})\|}{\|\vec{x}-\vec{a}\|} + \lim_{\vec{x}\to\vec{a}} \frac{\|\beta g(\vec{x}) - \beta g(\vec{a}) - \beta Dg(\vec{a})(\vec{x}-\vec{a})\|}{\|\vec{x}-\vec{a}\|}$$

$$= \quad 0 \quad + \quad 0 .$$

Thus, since our desired limit is nonegative and ≤ 0, it is 0.

Note, $[\alpha Df + \beta Dg](\vec{a}) = \alpha Df(\vec{a}) + \beta Dg(\vec{a})$ was used as linear transformation showing that $(\alpha f + \beta g)$ is diff'ble, so $D(\alpha f + \beta g)(\vec{a}) = \alpha Df(\vec{a}) + \beta Dg(\vec{a})$, since the derivative is unique. $\blacksquare$

2. If $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a constant f'n, prove that $Df(\vec{a}) = 0$ for all $\vec{a} \in U$.

Proof: let $\vec{a} \in U$. Since f is a constant f'n, $f(\vec{a}) - f(\vec{x}) = 0$ for any $x \in U$. Now observe that

$$\lim_{\vec{x}\to\vec{a}} \frac{\|f(\vec{a}) - f(\vec{x}) - 0(\vec{x}-\vec{a})\|}{\|\vec{x}-\vec{a}\|}$$

$$= \lim_{\vec{x}\to\vec{a}} \frac{\|\vec{0} - \vec{0}\|}{\|\vec{x}-\vec{a}\|}$$

$$= \lim_{\vec{x}\to\vec{a}} 0$$

$$= 0.$$

and we are done. $\blacksquare$

3. Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ be $f(x,y) = 0$, where $A = \{(x,y) : x \in [0,1], y = 0\}$. Prove that the derivative of f is not unique on A.

Proof: Observe that for any $\vec{a} \in A$, $Df(\vec{a})(x,y) = y$ satisfies the definition of a Derivative:

$$\lim_{\vec{x} \to \vec{a}} \frac{\| f(\vec{x}) - f(\vec{a}) - Df(\vec{a})(\vec{x} - \vec{a}) \|}{\| \vec{x} - \vec{a} \|}$$

$$= \lim_{\vec{x} \to \vec{a}} \frac{\| 0 - 0 - Df(\vec{a})(\vec{x} - \vec{a}) \|}{\| \vec{x} - \vec{a} \|}$$

now, since $\vec{x}, \vec{a} \in A$, $\vec{x}_y = 0$ and $\vec{a}_y = 0$, so $(\vec{x} - \vec{a})_y = 0$.

thus, our limit becomes

$$= \lim_{\vec{x} \to \vec{a}} \frac{0}{(\vec{x} - \vec{a})} = 0 .$$

now, we have already shown that $Df(\vec{a}) \equiv \vec{0}$ satisfies the definition of a derivative for any constant f'n, so we are done. $\blacksquare$

4. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be $f(x,y) = \sqrt{|xy|}$ Prove that f is not diff'ble at $\vec{0}$. Let Df be some linear transformation.

$$\lim_{\vec{x} \to \vec{0}} \frac{\| f(\vec{x}) - f(\vec{0}) - Df(\vec{0})(\vec{x} - \vec{0}) \|}{\| \vec{x} - \vec{0} \|}$$

and, since Df is a L.T., there exists some $M > 0$ st. $Df(\vec{x}) \leq M \|\vec{x}\|$ $\forall \vec{x}$.

$$\geq \lim_{\vec{x} \to \vec{0}} \frac{\left| \sqrt{|xy|} - 0 - M \|\vec{x}\| \right|}{\|\vec{x}\|}$$

Claim: for some $\varepsilon > 0$, we have that for every $\delta > 0$, There exists some $\vec{x}$ such that $\|\vec{x}\| < \delta$ and

$$\left| \sqrt{|xy|} - M \|\vec{x}\| \right| \geq \varepsilon \|\vec{x}\|.$$

Let $\varepsilon = \frac{M}{2}$, and let $\delta > 0$ be given. Now,

$$\left| \sqrt{|xy|} - M\|\vec{x}\| \right| = \|\vec{x}\| \frac{\left| \sqrt{|xy|} - M\|\vec{x}\| \right|}{\|\vec{x}\|}$$

$$= \|\vec{x}\| \left| \frac{\sqrt{|xy|}}{\|\vec{x}\|} - M \right|$$

$\square$

now if we choose $\vec{x} = (0, \frac{\delta}{2})$, $\|\vec{x}\| < \delta$ and

$$\|\vec{x}\| \left| \frac{\sqrt{|xy|}}{\|\vec{x}\|} - M \right| = \|\vec{x}\| \left| \frac{0}{\|\vec{x}\|} - M \right|$$

$$= \|\vec{x}\| M$$

$$\geq \|\vec{x}\| \frac{M}{2}$$

$$= \|\vec{x}\| \varepsilon.$$

Thus, the negation of the definition of differentiability holds, so we are done. $\blacksquare$

5. Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and suppose there is a constant M such that $\|f(\vec{x})\| \le M\|\vec{x}\|^2$ for all $\vec{x} \in \mathbb{R}^n$. Prove that f is diff'ble at $\vec{0}$ and $Df(\vec{0}) = 0$.

__Proof:__ First, note that since $\|f(\vec{x})\| \le M\|\vec{x}\|^2$, then $f(\vec{0}) = \vec{0}$. Now, consider the definition of differentiability for $Df(\vec{0}) = 0$.

$$\lim_{\vec{x} \to \vec{0}} \frac{\|f(\vec{x}) - f(\vec{0}) - Df(\vec{0})(\vec{x})\|}{\|\vec{x} - \vec{0}\|}$$

$$= \lim_{\vec{x} \to \vec{0}} \frac{\|f(\vec{x}) - \vec{0} - \vec{0}\|}{\|\vec{x}\|}$$

$$\le \lim_{\vec{x} \to \vec{0}} \frac{M\|\vec{x}\|^2}{\|\vec{x}\|}$$

$$= \lim_{\vec{x} \to \vec{0}} M\|\vec{x}\|$$

$$= 0.$$

6. Suppose f is as defined in (5), and let $g(\vec{x}) = T(\vec{x}) + f(\vec{x})$, where $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Prove that g is diff'ble at $\vec{0}$ and $Dg(\vec{0}) = T$.

__Proof:__ we have already shown that f is diff'ble, and T is diff'ble by prop 11, so by problem (1), $g = T + f$ is diff'ble and $Dg = DT + Df = T + \vec{0} = T$. ■

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$$= \lim_{\vec{x} \to \vec{0}} \frac{\|f(\vec{x}) - \vec{0} - \vec{0}\|}{\|\vec{x}\|}$$

$$\le \lim_{\vec{x} \to \vec{0}} \frac{M\|\vec{x}\|^2}{\|\vec{x}\|}$$

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