Some Problems About Consecutive Products of Primes

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1 Introduction

Suppose p and q are both prime numbers with $p < q$. Consider all integers of the form $p^\alpha q^\beta$ with $\alpha, \beta \in \mathbb{N}$ and let $\{a_n\}$ be the sequence of these integers in increasing order.

Definition. If two integers p^m, q^n are elements of $\{a_k\}$ such that $p^m = a_i$ and $q^n = a_{i+1}$, we say that (p^m, q^n) is a *critical pair*. Note that this notation means that $p^m, < q^n$. It is also possible that (q^n, p^m) is a critical pair, so that $q^n < p^m$.

Lemma 1.1. *If $a_k = q^n$, then $a_{k+1} \neq q^{n+1}$.*

Proof *(By Contradiction)* Assume that $a_k = q^n$, and suppose for contradiction that $a_{k+1} = q^{n+1}$. Since $1 < p < q$, then $q^n < pq^n < q^{n+1}$. However, this contradicts our assumption that $a_k = q^n$ and $a_{k+1} = q^{n+1}$, as pq^n must be a term of the sequence which falls between a_k and a_{k+1}. ■

Lemma 1.2. *There exist at most finitely many $a_k = p^n$ such that $a_{k+1} = p^{n+1}$.*

Proof Since $p < q$, let n be the largest $n \in \mathbb{N}$ such that $p^n < q$. Then it follows that $p^n < q < p^{n+1}$. This means that $\{a_n\}$ begins as

$$\{a_n\} = \{1, p, p^2, ..., p^n, q, p^{n+1}, ...\}.$$

Claim: $\forall m \in \mathbb{N}, \quad p^{n+m} < p^m q < p^{n+m+1}$. Let $T(m)$ denote this statement. We now prove this claim by induction on m. We already know that $p^n < q < p^{n+1}$, so $p^{n+1} < pq < p^{n+2}$. Thus, $T(1)$ holds. We now assume $T(m)$ and show $T(m+1)$ holds:

$$p^{n+m} < p^m q < p^{n+m+1} \implies p^{n+m+1} < p^{m+1} q < p^{n+m+2}$$

As such, every integer $p^{n+m} > p^n$ is followed by the term $p^m q$ before p^{n+m+1} in the sequence a_n. Thus, $a_k = p^n$ and $a_{k+1} = p^{n+1}$ can only occur at the beginning of the sequence (finitely many times) as shown above. ■

Lemma 1.3. *If $a_i = p^m$ and $a_{i+1} = q^n$, then m and n are relatively prime.*

Proof *(By Contradiction)* Assume $a_i = p^m$ and $a_{i+1} = q^n$ and suppose for contradiction that $\gcd(m, n) = d$. Then $m = m'd$ and $n = n'd$ for some $m', n' \in \mathbb{N}$. Since $p^m = p^{m'd} < q^{n'd} = q^n$, we have $p^{m'} < q^{n'}$. Consider the following inequality:

$$\begin{aligned} p^m &= p^{m'd} \\ &= p^{m'd-m'+m'} \\ &= p^{m'(d-1)}p^{m'} \\ &< p^{m'(d-1)}q^{n'} \\ &< q^{n'(d-1)}q^{n'} \\ &= q^{n'd} \\ &= q^n. \end{aligned}$$

Since $p^{m'(d-1)}q^{n'}$ must come between p^m and q^n, p^m and q^n cannot be consecutive terms in a_k. Thus we have reached a contradiction. Notice, a similar argument holds for when $a_i = q^n$ and $a_{i+1} = p^m$.

■

Lemma 1.4. *For any two consecutive p^m, $q^n \in \{a_k\}$,*

$$\lim_{k\to\infty} \frac{m}{n} = \frac{\ln(q)}{\ln(p)}.$$

Proof Since $p < q$, we know already that

$$\begin{aligned} m\ln p - n\ln q &< \min(\ln p, \ln q) \\ &= \ln p, \end{aligned}$$

So we can divide by $n \ln p$ to find that

$$\begin{aligned} \frac{m}{n} - \frac{\ln q}{\ln p} &< \frac{\ln p}{n \ln p} \\ &= {}^1\!/_n \end{aligned}$$

■

2 The (flawed) Proof

Definition. Let p, q be distinct primes, and let $a, b \in \mathbb{Z}^+$.

A *pure power* of p is an integer of the form p^a.

This is as opposed to a *mixed power* of p and q, which is an integer of the form $p^a q^b$.

Definition. Let p, q be distinct primes, and let $a, b, \alpha, \beta \in \mathbb{Z}^+$.

We say that $p^\alpha q^\beta$ is an *intermediate mixed power* of p^a and q^b if $p^\alpha q^\beta$ is between p^a and q^b.

Definition. Let p, q be distinct primes, and let $a, b \in \mathbb{Z}^+$.

A *critical pair* of p and q is a pair of pure powers of p and q which do not have an intermediate mixed power.

Lemma 2.1. *If p, q are distinct prime integers, then there exists at least one critical pair of p and q.*

Proof Without loss of generality, suppose that $p < q$. Then, let n be the largest $n \in \mathbb{N}$ such that $p^n < q$. Now, $p^n < q^1$ is a critical pair and we are done. ∎

Algorithm 2.2. Let p, q be distinct primes, and let $a, b \in \mathbb{Z}^+$. Suppose $q^a \approx p^b$ and, without loss of generality, suppose that $q^a > p^b$. That is,

$$1 < \frac{q^a}{p^b} < 1 + \epsilon, \quad \text{where } \epsilon << .$$

If an intermediate mixed power exists, it is of the form

$$p^b < q^{a-k} p^{b+\ell} < q^a \tag{1}$$

where $k, \ell \in \mathbb{Z}^+$. So, since $q^{a-k} p^{b+\ell} > p^b$,

$$\begin{aligned} 1 + \epsilon &> \tfrac{q^a}{p^b} \\ &> \tfrac{q^a}{q^{a-k} p^{b+\ell}} \\ &= \tfrac{q^k}{p^{b+\ell}}. \end{aligned}$$

Now, let $k = \tilde{a}$, and let $b + \ell = \tilde{b}$. If an intermediate mixed power exists, apply Algorithm 2.2 until one no longer exists. Note, since $a > k \geq 1$ and $b < b + \ell$, this process cannot continue indefinitely.

Thus, we can always apply this algorithm to find a critical pair between any two pure powers of p and q.

Claim: There exist infinitely many critical pairs of any two distinct primes p and q.

PROOF by Induction. Let p, q be distinct primes. Let $P(n)$ be the statement "There exist n distinct critical pairs of p and q." We will prove that there are infinitely many critical pairs of p and q by induction on n.

By Lemma 2.1, there must exist at least one critical pair $p^{b_0} < q^{a_0}$. Thus, $P(1)$ holds.

Now, assume that $P(n)$ holds. Let $p^b = p^{b_n}$, and choose some q^a such that

$$1 < \frac{q^a}{p^b} < 1 + \epsilon.$$

Apply Algorithm 2.2 to obtain $p^{b_{n+1}}, q^{a_{n+1}}$ such that $p^{b_n} < p^{b_{n+1}}$, and $p^{b_{n+1}} < q^{a_{n+1}}$ are a critical pair. Thus, we have a critical pair such that $p^{b_{n+1}} > p^{b_n} > \ldots > p^{b_1}$, so we have $n+1$ distinct critical pairs. Therefore, P(n+1) holds. ■

Issue: There is a critical problem with this proof. The statement given in Equation 1 is false. It is actually true that if an intermediate power exists, it is of the form

$$p^b < q^{a-k} p^{0+\ell} < q^a$$

or

$$p^b < q^{0+k} p^{b-\ell} < q^a.$$

3 Working proof

Definition. Let p, q be distinct primes, and let $a, b \in \mathbb{Z}^+$.

A *pure power* of p is an integer of the form p^a.

This is as opposed to a *mixed power* of p and q, which is an integer of the form $p^a q^b$.

Definition. Let p, q be distinct primes, and let $a, b, \alpha, \beta \in \mathbb{Z}^+$.

We say that $p^\alpha q^\beta$ is an *intermediate mixed power* of p^a and q^b if $p^\alpha q^\beta$ is between p^a and q^b. (That is, either $p^a < p^\alpha q^\beta < q^b$ or $q^b < p^\alpha q^\beta < p^a$)

Definition. Let p, q be distinct primes, and let $a, b \in \mathbb{Z}^+$.

A *critical pair* of p and q is a pair of pure powers of p and q which do not have an intermediate mixed power.

Theorem 3.1. *Consider the pure powers p^a, q^b with $p^a < q^b$ and $a, b \in \mathbb{Z}^+$. If, for all critical pairs p^s, q^t with $s < a$ and $t < b$,*

$$1 < \frac{q^b}{p^a} < \frac{q^t}{p^s}, \qquad s, t \in \mathbb{Z}^+$$

then p^a, q^b is a critical pair.

Proof by contradiction Assume that for all critical pairs p^s, q^t with $s < a$ and $t < b$,

$$1 < \frac{q^b}{p^a} < \frac{q^t}{p^s},$$

and suppose for contradiction that p^a, q^b is not a critical pair. Since p^a, q^b is not a critical pair, then there exists an intermediate mixed power of the form

$$p^a < q^{b-\ell} p^{a-k} < q^b$$

where $1 \leq k < a, 1 \leq \ell < b$. So, since $q^{b-\ell} p^{a-k} > p^a$,

$$\frac{q^b}{p^a} > \frac{q^b}{q^{b-\ell} p^{a-k}} = \frac{q^\ell}{p^{a-k}}.$$

Now, let $a - k = \tilde{a}$, and let $\ell = \tilde{b}$. If $p^{\tilde{a}}$ and $q^{\tilde{b}}$ are a critical pair, we have a contradiction. If they are not, then we can repeat the preceding process in this proof. Note, since $a > \tilde{a} \geq 1$ and $b > \tilde{b} \geq 1$, the process can be repeated at most $\min(a, b)$ times. At the end of this process, we are guaranteed to find at least one critical pair. ■

(this is basically Dirichlet's Lemma, we just need to connect the dots.)

Lemma 3.2. *Let α be an irrational number. Given any $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $n\alpha - \lfloor n\alpha \rfloor < \epsilon$.*

Theorem 3.3. *For any two distinct prime numbers p, and q, there exist infinitely many critical pairs.*

Proof Let p and q be distinct primes. Suppose for contradiction that there exist finitely many critical pairs, and denote the set of these as $S = \{(p^{k_1}, q^{\ell_1}), \ldots, (p^{k_N}, q^{\ell_N})\}$. Of these critical pairs, consider the subset $C = \{(p^{k_i}, q^{\ell_i}) : p^{k_i} < q^{\ell_i}\}$, where $i \in \mathbb{N}$ such that $1 \leq i \leq N$. This means that

$$1 < \frac{q^{\ell_i}}{p^{k_i}}, \quad \forall (p^{k_i}, q^{\ell_i}) \in C.$$

Choose some $\epsilon \in \mathbb{R}$ such that

$$1 < p^{\epsilon} < \min\left(\frac{q^{\ell_i}}{p^{k_i}}\right).$$

Now, consider the irrational number $\log_p q$. By the Lemma, there exists some $\Omega \in \mathbb{N}$ such that

$$\Omega \log_p q - \lfloor \Omega \log_p q \rfloor < \epsilon.$$

To simplify the notation, let $a = \lfloor \Omega \log_p q \rfloor$. Thus, with a little algebra,

$$\begin{aligned} \Omega \log_p q &< a + \epsilon \\ q^{\Omega} &< p^a p^{\epsilon} \\ \frac{q^{\Omega}}{p^a} &< p^{\epsilon} \\ &< \min\left(\tfrac{q^{\ell_i}}{p^{k_i}}\right) \end{aligned}$$

and we find that for all $(q^{\ell_i}, p^{k_i}) \in C$,

$$1 < \frac{q^{\Omega}}{p^a} < \frac{q^{\ell_i}}{p^{k_i}}.$$

Therefore, by Proposition 3.2, (q^{Ω}, p^a) is a critical pair with $p^a < q^{\Omega}$. But, since $\frac{q^{\Omega}}{p^a} < \frac{q^{\ell_i}}{p^{k_i}}$ for all $(q^{\ell_i}, p^{k_i}) \in C$, then $(q^{\Omega}, p^a) \notin C$, which is a contradiction.

Therefore, we have shown that C cannot be finite, and since $C \subset S$, then S cannot be finite either. ■