Math 552
Final Review

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Ch 2 Corr 3.8 Suppose $f(x)$ is a non-negative function on $\mathbb{R}^d$, and let $\mathcal{A} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x)\}$. Prove:

(i) f is measurable on $\mathbb{R}^d$ if and only if $\mathcal{A}$ is measurable in $\mathbb{R}^{d+1}$.

(ii) If the conditions in (i) hold, then $\displaystyle\int_{\mathbb{R}^d} f(x)\, dx = m(\mathcal{A})$.

ex 2.19 Suppose f is integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)|\, dx = \int_0^\infty m(E_\alpha)\, d\alpha.$$

Def: Write the definition of a function of bounded variation.

Thm 3.3 Prove that a real-valued function F on $[a, b]$ is of bounded variation if and only if F is the difference of two increasing bounded functions.

Def: Write the definition of a point of Lebesgue density.

Cor 1.5 Suppose E is a measurable subset of $\mathbb{R}^d$.

Prove:

(a) Almost every $x \in E$ is a point of density of E.

(b) Almost every $x \notin E$ is not a point of density of E.

Ex 3.3 Suppose 0 is a point of (Lebesgue) density of the set $E \subset \mathbb{R}$. Show that for each of the individual conditions below there is an infinite sequence of points $x_n \in E$, with $x_n \neq 0$, and $x_n \to 0$ as $n \to \infty$.

(a) The sequence also satisfies $-x_n \in E$ for all n.

Ex 3.6 In one dimension there is a version of the basic inequality (1) for the maximal function in the form of an identity. We define the "one-sided" maximal function

$$f_+^* = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)|\, dy.$$

Prove that if $E_\alpha^+ = \{x \in \mathbb{R} : f_+^*(x) > \alpha\}$, then

$$m(E_\alpha^+) = \frac{1}{\alpha} \int_{E_\alpha^+} |f(y)|\, dy.$$

Ex 3.7 Use Corollary 1.5 to prove that if a measurable subset E of [0,1] satisfies $m(E \cap I) \geq \alpha m(I)$ for some $\alpha > 0$ and all intervals in $[0,1]$, then E has measure 1.

Ex 3.14 (a) Suppose F is continuous on $[a, b]$. Show that

$$D^+(F)(x) = \limsup_{\substack{h \to 0 \\ h > 0}} \frac{F(x+h) - F(x)}{h}$$

is measurable.

Ex 3.14 (b) Let $F : [a,b] \to \mathbb{R}$ be increasing and bounded, and let $J(x) = \sum_{n=1}^{\infty} \alpha_n j_n(x)$ be the jump function associated with F. Show that

$$\limsup_{h \to 0} \frac{J(x+h) - J(x)}{h}$$

is measurable.

Ex 3.20 This exercise deals with functions F that are absolutely continuous on $[a, b]$ and are increasing. Let $A = F(a)$ and $B = F(b)$.

(a) There exists such an F that is in addition strictly increasing, but such that $F'(x) = 0$ on a set of positive measure.

Ex 3.20 This exercise deals with functions F that are absolutely continuous on $[a, b]$ and are increasing. Let $A = F(a)$ and $B = F(b)$.

(b) Show that $F(x) = \int_a^x \chi_K(t)dt$ has a measurable subset $E \subset [A, B]$ with $m(E) = 0$ and $F^{-1}(E)$ nonmeasurable.

Ex 3.20 This exercise deals with functions F that are absolutely continuous on $[a, b]$ and are increasing. Let $A = F(a)$ and $B = F(b)$.

(c) Modified version solved in class: Prove that for any increasing absolutely continuous F and O, an open subset of $[A, B]$, $m(O) = \int_F^{-1}(0)F'(x)dx$.

Def: State the definition of Caratheodory measurability.

Ex 6.3 Consider the exterior Lebesgue measure m_* introduced in Chapter 1. Prove that a set E in $\mathbb{R}^d$ is Caratheodory measurable if and only if E is Lebesgue measurable in the sense of Chapter 1.