Labutin, Spring 2020
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Midterm Exam

1. Let $\varepsilon > 0$. Prove that there exists a sequence (x_n) of real numbers such that

$$\sum_{n=1}^{\infty} x_n^2 < \infty \text{ but } n^\varepsilon x_n \not\to 0.$$

Proof Consider the sequence $\{n^\varepsilon\}$. For all $k \in \mathbb{N}$, there exists some $n_k \in \mathbb{N}$ such that

$$n_k^\varepsilon > k^2$$

so let $\{n_k^\varepsilon\}$ be that subsequence. For the same indeces n_k, Let

$$x_{n_k} = k^{-1}$$

and for all other n, let

$$x_n = n^{-1}.$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty} x_n^2 &= \sum_{k=1}^{\infty} x_{n_k}^2 + \sum_{n \notin \{n_k\}} x_n^2 \\ &< \sum_{k=1}^{\infty} k^{-2} + \sum_{n \in \mathbb{N}} n^{-2} \\ &< \infty,\end{aligned}$$

but $n^\varepsilon x_n \not\to 0$, since the subsequence indexed by n_k is

$$\begin{aligned}n_k^\varepsilon x_{n_k} &= n_k^\varepsilon k^{-1} \\ &> k^2 k^{-1} \\ &= k\end{aligned}$$

and this goes to infinity as $k \to \infty$. ■

2. (i) Let X be a normed. Assume X^* is separable. Prove that X is separable.

Proof Since X^* is separable, then so is

$$S(X^*) = \{\varphi \in X^* \ : \ ||\varphi|| = 1\},$$

so let $\{\varphi_n\}$ be a countable dense subset of $S(X^*)$. Since for each n we have that

$$\sup_{||x||=1} |\langle \varphi_n, x\rangle| = ||\varphi_n|| = 1,$$

then for each φ_n we can choose some $x_n \in X$ with $||x_n|| = 1$ such that

$$|\langle \varphi_n, x_n\rangle| > \frac{1}{2}. \tag{†}$$

Let

$$\begin{aligned} D &= \operatorname{span}\{x_n\} \\ &= \left\{ \sum_{j=1}^{n} r_j x_j \ : \ r_j \in \mathbb{R}, n \in \mathbb{N} \right\} \end{aligned}$$

and denote A as the set of all finite linear combinations of $\{x_n\}$ with rational coefficients, that is,

$$A = \left\{ \sum_{j=1}^{n} q_j x_j \ : \ q_j \in \mathbb{Q}, n \in \mathbb{N} \right\}.$$

Then A is countable since $\mathbb{Q} \times \mathbb{N}$ is countable, and (as we will show presently) it is dense in D. Let $\sum_{j=1}^{n} r_j x_j \in D$, and let $\varepsilon > 0$. For each j, we can find some $q_j \in \mathbb{Q}$ such that $|r_j - q_j| < \frac{\varepsilon}{n||x_j||}$,

$$||r_j x_j - q_j x_j|| = |r_j - q_j| \, ||x_j|| < \frac{\varepsilon}{n},$$

so by triangle inequality,

$$\left|\left| \sum_j^n r_j x_j - \sum_j^n q_j x_j \right|\right| = \left|\left| \sum_j^n (r_j x_j - q_j x_j) \right|\right| \leq \sum_j^n ||r_j x_j - q_j x_j|| < \varepsilon.$$

Now we will show that D is dense in X.
Suppose for contradiction that $\overline{D} \neq X$. Since $\overline{D}$ is the span of vectors in X, then it is a linear subspace of X, and so by Hahn-Banach we can find some $\psi \in X^*$ such that $\psi|_{\overline{D}} = 0$. Since $\{\varphi_n\}$ is dense in $S(X^*)$, then we can find a particular φ_n such that

$$||\psi - \varphi_n||_* < \frac{1}{4}.$$

Now since every $x_n \in \overline{D}$ with $||x_n|| = 1$, we have that $\langle \psi, x_n\rangle = 0$, so by applying (†) and the equation above,

$$\frac{1}{2} < |\langle \varphi_n, x_n\rangle| = |\langle \varphi_n, x_n\rangle - \langle \psi, x_n\rangle| \leq ||\varphi_n - \psi||_* \, ||x_n|| < \frac{1}{4}$$

which is a contradiction. Thus A is a countable set, and A is dense in D which is dense in X, so A is dense in X, and we're done. $\square$

(ii) Give an example of a separable X_0 such that X_0^* is not separable. Prove the separability and non-separability of your example.

Answer: ℓ_1 is separable, and $\ell_1^* = \ell_\infty$ is not separable.

Proof (ℓ_1 is separable) let A be the set of all finite rational sequences, that is,

$$A = \left\{ \sum_{j=1}^{n} q_j e_j \ : \ q_j \in \mathbb{Q}, n \in \mathbb{N} \right\}.$$

Then A is countable since $\mathbb{Q} \times \mathbb{N}$ is countable, and it is dense in ℓ_1 since we can use a similar strategy as in the previous problem to find, for any $\varepsilon > 0$, $q_j \in \mathbb{Q}$ such that

$$||q_j e_j - re_j|| < \varepsilon 2^{-j}$$

for all elements of the form re_j in ℓ_1, and so

$$\left|\left| \sum_{j=1}^{\infty} q_j e_j - \sum_{j=1}^{\infty} r_j e_j \right|\right| < \varepsilon.$$

by triangle inequality for all $\sum_{j=1}^{\infty} r_j e_j \in \ell_1$. □

Proof (ℓ_∞ is not separable) Thinking of elements of ℓ_∞ as functions $\mathbb{N} \to \mathbb{R}$, consider the power set $\mathcal{P}(\mathbb{N})$. We can define a subset $\mathcal{G} \subset \ell_\infty$ given by

$$\mathcal{G} = \{\chi_G(n) : G \in \mathcal{P}(\mathbb{N})\},$$

and we see that this set is uncountable, and no two sequences in $\mathcal{G}$ are closer than 1, since they all differ by 1 at some n. Any countable subset of ℓ_∞ can only be within distance 1/2 of at most countable many elements of $\mathcal{G}$, so it cannot be dense. ■

3. Let X, Y be normed spaces, $A : X \to Y$ be (algebraically) linear. Assume that for any sequence (x_n) in X such that $x_n \to 0$ weakly the corresponding sequence $Ax_n \to 0$ weakly in Y. Prove that A is a bounded operator.

Proof Let $x_n \to 0$ strongly. Then $x_n \xrightarrow{w} 0$, so $Ax_n \xrightarrow{w} 0$. We can write the sequence (x_n) as $(\lambda_n u_n)$, where the scalars $|\lambda_n| \to 0$ and every vector $||u_n|| = 1$. Then since $Ax_n \xrightarrow{w} 0$, then for all $\psi \in Y^*$ we have $\psi(Ax_n) \to 0$, so

$$\begin{aligned}
\psi(Ax_n) &= \psi \circ A(\lambda_n u_n) \\
&= \lambda_n \psi(Au_n) && \text{by linearity} \\
&\xrightarrow{n} 0.
\end{aligned}$$

Suppose for contradiction that there is some (u_n) such that $|\psi(Au_n)| \to \infty$. Then $\frac{1}{|\psi(Au_n)|} \to 0$, and we know that $\lambda_n \psi(Au_n) \xrightarrow{n} 0$ for every sequence $(\lambda_n) \to 0$. But $\frac{|\psi(Au_n)|}{|\psi(Au_n)|} \equiv 1$, which is a contradiction.

Thus $|\psi(Au_n)| \to B < \infty$ for any (u_n) with every $||u_n|| = 1$, so we're done. To see this, note that the sequence $|\psi(Au_n)|$ is bounded for any $\psi \in Y^*$ and any sequence in the unit ball $(u_n) \subset \overline{B}(X)$, which means Au_n is also bounded. Therefore $\sup\limits_{||u||=1} ||Au|| < \infty$. ■

4. Prove that X^* "separates points" of X (a Banach space). That is, prove that for all $x, y \in X$ such that $x \neq y$, there exists $\varphi \in X^*$ such that $\varphi(x) \neq \varphi(y)$.

Proof Fix $x \neq y \in X$, and on the linear subspace

$$\operatorname{span}(y - x),$$

define a linear functional φ by

$$\varphi(y - x) = 1 \text{ and extending linearly.}$$

Then by Hahn-Banach, we can extend φ to a functional on all of X. Then $\varphi(y) = \varphi(x) + 1$ and we're done. ■