Homework 3

15. Enumerate all the subcomplexes of S^∞, with the cell structure on S^∞ that has S^n as its n skeleton.

Answer: The cell structure S^∞ that has S^n as its n skeleton is given by building each k-th sphere on top of the $k-$th sphere: so beginning with S^0 (two points), we attach two e^1s by attaching their boundaries to what has already been built.

$$S^0 \qquad S^1 \qquad S^2 \qquad etc.$$

Thus a diagram of the attaching maps can be drawn as follows:
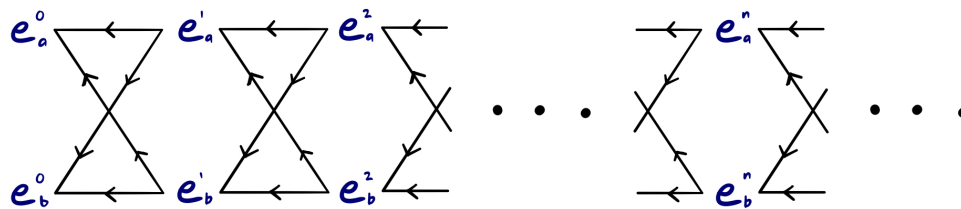
Now we will show that any subcomplex is either a hemisphere or a sphere, and the following list enumerates them:

$$\left\{ \langle e_a^0 \rangle, \ \langle e_b^0 \rangle, \ \left(\langle e_a^0 \rangle \cup \langle e_b^0 \rangle \right), \ \langle e_a^1 \rangle, \ \langle e_b^1 \rangle, \ \left(\langle e_a^1 \rangle \cup \langle e_b^1 \rangle \right), \ \dots \right\}$$

where $\langle e_\delta^i \rangle$ denotes the subcomplex generated by e_δ^i. Observe that for any single cell $e_\delta^n \in S^\infty$, $\langle e_\delta^n \rangle$ is a hemisphere, since each e^i is attached to both e^{i-1} cells. Thus to examine the subcomplex generated by any finite collection of cells E, we need only consider the cells of highest dimension, namely e_a^n and/or e_b^n. If only one of $e_a^n, e_b^n \in E$, then we have an n-hemisphere. Otherwise, we have the n-sphere. Now if E a collection of cells of S^∞ is infinite, then $\langle E \rangle = S^\infty$ since for any $e_\delta^n \in S^\infty$, there is some $e_\gamma^k \in E$ with $k > n$. ∎

16. Show that S^∞ is contractible.

Proof We can actually construct a deformation retraction of S^∞ to e_a^0 as a countable concatenation of homotopies.

(1) Let F_1 be the homotopy which deformation retracts $\langle e_a^1 \rangle$ (the a-th semicircle) to e_a^0. This has the effect of retracting all of S^0 to e_a^0.
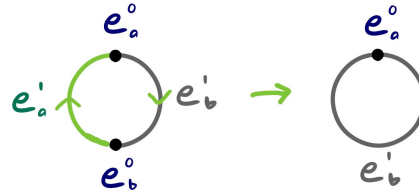
Note that any point in S^∞ except e_a^0 can move to accomplish this, but we constrain points S^n to never leave S^n for all $n > 0$. Here we draw more dimensions to illustrate:


(i) For each $i \geq 2$, let F_i be the homotopy which deformation retracts $\langle e_a^i \rangle$ (the a-th i-hemisphere) to e_a^0. We know we can do this because each hemisphere is a disk which contains e_a^0. This has the effect of retracting all of S^{i-1} to e_a^0.

Now we concatenate all of these homotopies, allotting time $\frac{1}{2^i}$ for each F_i, so that they are all completed as t goes from 0 to 1. Every $S^i \subset S^\infty$ is deformation retracted to the point e_a^0 by the time $t = 1 - 2^{-(i+1)}$, so S^∞ is contractible. ∎

18. Show that $S^1 * S^1 = S^3$, and more generally $S^m * S^n = S^{m+n+1}$.

Proof By definition, $S^m * S^n$ is a shape which has an m-sphere on one end, an n-sphere on the other end, and everywhere in between is shaped like a $S^m \times S^n$. Following is an illustration when $m = n = 1$:

$t = 0$ $t = 0.5$ $t = 1$
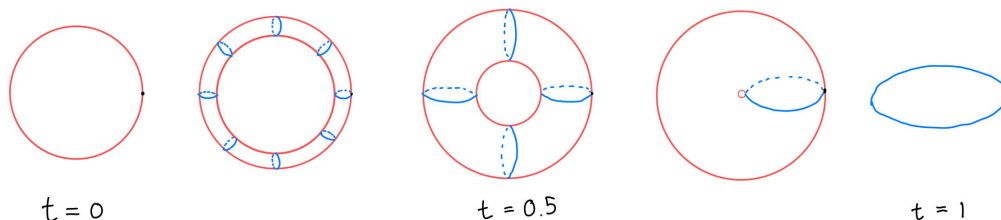
We can consider two halves of this shape, glued together on their boundaries. Since every point in the m-sphere is connected to every point in the n-sphere, we can see this as

$$S^m \times C(S^n) \bigsqcup_{S^m \times S^n} C(S^m) \times S^n$$

which is the same as

$$S^m \times D^{n+1} \bigsqcup_{S^m \times S^n} D^{m+1} \times S^n$$

(there's a really cool picture of interlocking solid tori which doesn't turn out to be helpful for what we want to prove here, so I've left it out).

Now observe that the unit sphere $S^{m+n+1} \in \mathbb{R}^{m+n+2}$ is the locus of points such that

$$\overbrace{x_1^2 + \cdots + x_{m+1}^2}^{m+1} + \overbrace{x_{m+2}^2 + \cdots + x_{m+n+2}^2}^{n+1} = 1$$

or equivalently,

$$x_1^2 + \cdots + x_{m+1}^2 = t \quad \text{and} \quad x_{m+2}^2 + \cdots + x_{m+n+2}^2 = (1-t) \quad \text{for} \quad t \in [0,1].$$

which is the same as

$$x_1^2 + \cdots + x_{m+1}^2 = t \quad \text{and} \quad x_{m+2}^2 + \cdots + x_{m+n+2}^2 = (1-t) \quad \text{for} \quad t \in [0, \tfrac{1}{2}]. \qquad (1)$$

$$\text{together with}$$

$$x_1^2 + \cdots + x_{m+1}^2 = (1-s) \quad \text{and} \quad x_{m+2}^2 + \cdots + x_{m+n+2}^2 = s \quad \text{for} \quad s \in [0, \tfrac{1}{2}]. \qquad (2)$$

where we attach (1) and (2) along the set where $s = t = \frac{1}{2}$ (note how the equations agree in that case).

What regions do these equations describe? First consider (1). In the left hand equation, t determines the radius of an $m-$sphere, so letting t vary gives an $m+1$ disk. Using t in the

right-hand equation gives an n-sphere whose radius is a function of t. Thus, for every point in an $m+1$ disk of radius $\sqrt{\frac{1}{2}}$, we can choose any point in an n-sphere to satisfy (1). Thus (1) determines a region which is the shape of $D^{m+1} \times S^n$.

Using similar reasoning, we can see that (2) determines a region which is the shape of $S^m \times D^{n+1}$. Finally, we observe that for $t = s = \frac{1}{2}$, both equations determine the same set of points $S^m \times S^n$. Thus an $m+n+1$-sphere is the same homotopy type as

$$S^m \times D^{n+1} \bigsqcup_{S^m \times S^n} D^{m+1} \times S^n,$$

which is $S^m * S^n$.   ■

19. Show that the space obtained from S^2 by attaching n 2–cells along any collection of n circles in S^2 is homotopy equivalent to the wedge sum of $n+1$ 2–spheres.

Proof Any circle in S^2 divides S^2 into two 2-disks which are contractible. For every one of these circles, call part of S^2 the inside of this circle and contract it to a point. Then move the attaching maps so all the spheres are attached to each other at the same point, and we're done.
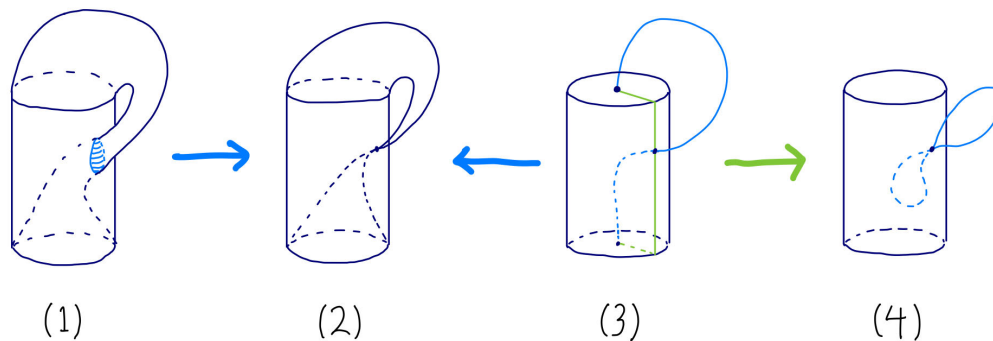

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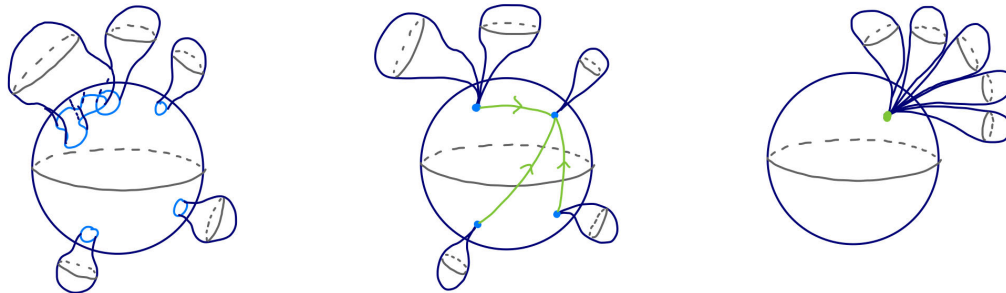
20. Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle, as shown in the figure, is homotopy equivalent to $S^1 \vee S^1 \vee S^2$.
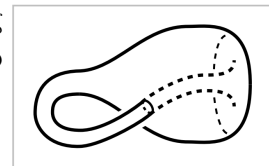
Proof See the following figure.

Observe that contracting the blue disk in (1) to a point gives us (2), and contracting both blue 1-cells in (3) to a point also gives (2), so those three figures are homotopy equivalent. The green lines show how the attaching maps for the top and bottom points are homotopic to attaching maps which attach the blue 1-cells' endpoints all to the same point, so (3) $\simeq$ (4). ■

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15.

16.

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