Labutin, Spring 2020
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Homework 4

Chapter 2

2. (a) Prove 2.1(6): When $f \in L^\infty(\Omega) \cap L^q(\Omega)$ for some q, then $f \in L^p(\Omega)$ for all $p > q$ and

$$||f||_\infty = \lim_{p\to\infty} ||f||_p .$$

(b) Prove that when $\infty \geq r \geq q \geq 1$,

$$f \in L^r(\Omega) \cap L^q(\Omega) \implies f \in L^p(\Omega)$$

for all $r \geq p \geq q$.

Proof (a) Let $f \in L^q$, and $p > q$ with $||f||_\infty < \infty$. We need to show that $|f|^p$ is summable, that is, $\int |f|^p < \infty$.[†]

$$\begin{aligned}\int |f|^p &= \int |f|^q \, |f|^{p-q} \\ &\leq \int |f|^q \cdot ||f||_\infty^{p-q} \\ &= ||f||_\infty^{p-q} \int |f|^q \\ &< \infty.\end{aligned}$$

Thus $f \in L^p(\Omega)$.

Now observe that for all $p > q$, $||f||_p \leq ||f||_\infty$.

$$\begin{aligned}||f||_p &= \left(\int |f|^p\right)^{\frac{1}{p}} \\ &\leq \left(\int ||f||_\infty^p\right)^{\frac{1}{p}} \\ &= \left(||f||_\infty^p \, \mu(\Omega)\right)^{\frac{1}{p}} \\ &= ||f||_\infty \cdot \left(\mu(\Omega)\right)^{\frac{1}{p}}\end{aligned}$$

and since $\mu(\Omega)$ is finite, then

$$\lim_{p\to\infty} \left(||f||_\infty \cdot \left(\mu(\Omega)\right)^{\frac{1}{p}}\right) = ||f||_\infty .$$

Thus part (a) is proved. $\square$

[†] To suppress notation, we will omit the region of integration to denote that the region is all of Ω. We will also omit the measure unless it is needed.

(b) Let p such that $\infty \geq r \geq p \geq q \geq 1$, and $f \in L^r(\Omega) \cap L^q(\Omega)$. Denote

$$\{|f| \leq 1\} = \{x \in \Omega : |f(x)| \leq 1\}, \text{ and}$$
$$\{|f| > 1\} = \{x \in \Omega : |f(x)| > 1\}.$$

then observe that $|f|^p$ is summable:

$$\begin{aligned} \int |f|^p &= \int_{\{|f|\leq 1\}} |f|^p + \int_{\{|f|>1\}} |f|^p \\ &= \int_{\{|f|\leq 1\}} |f|^q + \int_{\{|f|>1\}} |f|^r \\ &= \int |f|^q + \int |f|^r \\ &< \infty. \end{aligned}$$

■

9. In Sect. 2.9 three ways are shown for which an $L^p(\mathbb{R}^n)$ sequence f^k can converge weakly to zero but f^k does not converge to anything strongly. Verify this for the three examples given in 2.9 (page 56):

(i) f_k 'oscillates to death': An example is $f_k(x) = \sin kx$ for $0 \leq x \leq 1$ and zero otherwise.

Proof As we did in HW1 Problem 5, we use integration by parts and find that $\forall\, g \in C(\mathbb{R})$,

$$\begin{aligned} \int_0^1 g(x)\sin(kx)\,dx &= -g(x)\frac{1}{k}\cos(kx) - \int_0^1 g'(x)\frac{1}{k}\cos(kx)\,dx \\ &= \frac{1}{k}\left[-g(x)\cos(kx) - \int_0^1 g'(x)\cos(kx)\,dx\right], \end{aligned}$$

and in the limit as $n \to \infty$, everything goes to 0.
For arbitrary $g \in L^2(\mathbb{R})$, g' may not so exist, so we use Weierstrauss Approximation Theorem, for every $\varepsilon > 0$, there exists a polynomial h such that $\sup_I |g - h| \leq \varepsilon$. Thus,

$$\begin{aligned} \int g(x)\sin(kx)\,dx &= \int g(x)\sin(kx)\,dx - \int h(x)\sin(kx)\,dx + \int h(x)\sin(kx)\,dx \\ &= \int (g(x) - h(x))\sin(kx)\,dx + \int g(x)\sin(kx)\,dx, \end{aligned}$$

and this integral is bounded above and below by

$$\int (\pm\varepsilon + h(x))\sin(kx)\,dx$$

respectively, which integrands are themselves polynomials, so they vanish in the limit. Therefore $\lim\limits_{n\to\infty} \int g(x)\sin(kx)\,dx = 0$ by the squeeze theorem, and we conclude that $f_k \xrightarrow{w} 0$.
Finally, note that if we fix x and let $k \to \infty$, then $\sin(x)$ takes values all over $[0, 1]$, so it doesn't converge pointwise. Therefore, f_k doesn't converge strongly to anything. □

(ii) f_k 'goes up the spout': An example is $f_k(x) = k^{1/p} g(kx)$ where g is any fixed function in $L^p(\mathbb{R}^1)$. This sequence becomes very large near $x = 0$.

Proof $f_k \xrightarrow{w} 0$ because for any $h \in L^{p'}(\mathbb{R})$,

$$\begin{aligned}
\int f_k(x)h(x)\,dx &= k^{1/p} \int g(kx)h(x)\,dx \\
&= k^{1/p} \int g(kx)h(x)\,dx \\
&\leq k^{1/p} \left(\int |g(kx)|^p \right)^{\frac{1}{p}} ||h||_{p'} \\
&\leq \frac{k^{1/p}}{k^p} ||g||_p \, ||h||_{p'} && \text{by change in variables}
\end{aligned}$$

and as $k \to \infty$, this all goes to zero.

Observe that f_k converges pointwise to zero, so if it does converge strongly, it must converge to 0: Indeed, since $\chi_{B_1(x_0)}$ is an $L^{p'}(\mathbb{R})$ function for any fixed x_0, and since $f_k \xrightarrow{w} 0$, then

$$\int f_k \cdot \chi_{B_1(x_0)} \to 0,$$

so $f_k(x) \xrightarrow{k} 0$. However, $||f_k||_p$ is constant and nonzero:

$$\begin{aligned}
||f_k||_p &= \left(\int \left| k^{1/p} g(kx) \right|^p \right)^{\frac{1}{p}} \\
&= k^{1/p} \left(\int |g(kx)|^p \right)^{\frac{1}{p}} \\
&= k^{1/p} \left(\frac{1}{k} \int |g(t)|^p \right)^{\frac{1}{p}} && \text{by change in variables} \\
&= \frac{k^{1/p}}{k^{1/p}} \left(\int |g(t)|^p \right)^{\frac{1}{p}} \\
&= ||g||_p
\end{aligned}$$

Thus f_k cannot converge strongly to zero, and it cannot converge strongly to anything else. $\square$

(iii) f_k 'wanders off to infinity': An example is $f_k(x) = g(x+k)$ for some fixed function g in $L^p(\mathbb{R}^1)$.

Proof Assuming we can prove that $f_k \xrightarrow{w} 0$, then f_k converges pointwise to zero as well for the same reasons as in (ii). Thus if the sequence converges strongly, then it converges to zero. However f_k clearly cannot converge strongly to zero, since Lebesgue measure is translation-invariant, so $||f_k||_p = ||g||_p$ for all k.

As regards weak convergence, I think we can approximate with compact supported functions and use Dominated Convergence to yield the result. Intuitively, $g(x+k)$ aught to get small as $k \to \infty$, and multiplying by the fixed number $h(x)$ won't stop the small-enizing. As to the details, I ran out of time. This homework was too long. ■

11. With the usual $j_\varepsilon \in C_c^\infty$, show that if f is continuous on $\mathbb{R}$, then $j_\varepsilon * f(x)$ converges to $f(x)$ for all x, and it does so uniformly on each compact subset of $\mathbb{R}^n$.

Proof Let

$$j_\varepsilon = \varepsilon \chi_{[0,1/\varepsilon]}.$$

Then

$$\begin{aligned} j_\varepsilon * f(x) &= \int f(x-t)\varepsilon\chi_{[0,1/\varepsilon]}(t)\,dt \\ &= \varepsilon \int_0^{1/\varepsilon} f(x-t)\,dt \\ &= \text{average value of } f(t) \text{ on } [x-1/\varepsilon, x] \end{aligned}$$

and since f is continuous, taking $\lim\limits_{\varepsilon\to\infty}$[†], we obtain $\lim\limits_{t\to x^-} f(x) = f(x)$. ■

18. Prove that every convex function f has a support plane at every x in the interior of its domain $D \subset \mathbb{R}$, as claimed in Sect. 2.1. See also Exercise 3.1.

Proof We know that every convex function is continuous on an open set (such as $\text{int}D$), but and if f is also differentiable, then clearly it has a tangent plane which is in fact a support plane.

In this spirit, we can find a support plane for a general convex function by using right derivatives, which always exist. The support plane is

$$\text{span}(1, f'_+(x)),$$

To see this, observe that since f is convex, then

$$\frac{f(x+t) - f(x)}{t}$$

[†] I'm *very* uncomfortable with sending a variable called ε to ∞. Maybe what I'm doing is fine, but it's late, my brain hurts, and this homework was way too long.

decreases as $t \to 0^+$, so since

$$f'_+(x) = \lim_{t\to 0^+} \frac{f(x+t)-f(x)}{t},$$

then $f(x) + tf'_+(x) \le f(x+t)$ for all nonnegative t.

Now considering negative values of t; since f is convex, then

$$f'_- \le f'_+,$$

and f'_- increases as $t \to 0^-$, so $f(x) + tf'_+(x) \le f(x) + tf'_-(x) \le f(x+t)$ for $t < 0$, so

$$f(x) + tf'_+(x) \le f(x+t)$$

always, and we're done. ■

23. Find a sequence of functions with the property that

(i) $f_n \xrightarrow{w} 0$ in $L^2(\Omega)$, and

(ii) $f_n \to 0$ strongly in $L^{\frac{3}{2}}(\Omega)$, but

(iii) $f_n \not\to 0$ strongly in $L^2(\Omega)$.

Answer: Let f_n be the following sequence in $g \in L^2(\mathbb{R})$ which "goes up the spout":

$$f_n = \sqrt{n}\,\chi_{[0,1/n]}$$

Proof (i) For any $g \in L^2(\mathbb{R})$,

$$\begin{aligned}\left|\int f_n g\,d\mu\right| &= \left|\int_0^{1/n} \sqrt{n} g\right| \\ &\le \sqrt{n}\int_0^{1/n} |g| && \text{by Holder} \\ &\le \sqrt{n}\int_0^{1/n} |g|^2 && \text{since } x > x^2 \text{ in } [0,1/n]\end{aligned}$$

And since $\int_{\mathbb{R}} |g|^2$ is finite, then $\lim_{n\to\infty} \int_0^{1/n} |g|^2 = 0$. Thus $f_n \xrightarrow{w} 0$. □

(ii) Observe,

$$\begin{aligned}||f_n||_{\frac{3}{2}} &= \left|\int_0^{1/n} \sqrt{n}^{\frac{3}{2}}\right|^{\frac{2}{3}} \\ &= n^{-5/6} \\ &\xrightarrow{n} 0.\end{aligned}$$

Thus $f_n \to 0$ strongly in $L^{\frac{3}{2}}(\mathbb{R})$. □

(iii) However, $||f_n||_2$ is constantly 1, since

$$\begin{aligned}
||f_n||_2 &= \left(\int_0^n \sqrt{n}^2\right)^{\frac{1}{2}} \\
&= \sqrt{1} \\
&= 1,
\end{aligned}$$

so $f_n \not\to 0$ in $L^2(\mathbb{R})$. $\blacksquare$