Exponential Growth through Pattern Exploration

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Betty Brown

Exponential Growth through Pattern Exploration

Do your students think of arithmetic, algebra, and geometry as separate entities? Do the laws of exponents tend to confuse them? Participants at the summer institute Pattern Exploration: Integrating Math and Science in the Middle Grades developed and used the three activities in this article. The activities have proved to be exceptional tools for providing students in middle school and the early years of high school with models of exponent laws while relating differing fields of mathematics.

THE MIDPOINT ACTIVITY

In this part of the activity, students use iterative geometric constructions to create the Sierpinski triangle, and they complete a chart by using information derived from their constructions. This activity reinforces the concepts of exponents, ratios, and algebraic expressions for the *n*th stage of their investigations. The only materials needed are triangular dot paper, rulers, colored pencils, and **sheet 1,** Finding Ratios in the Sierpinski Triangle.

On the dot paper (see **fig. 1**) students construct an equilateral triangle with sides of sixteen units, using seventeen dots to do so. This initial triangle is Stage 0 on **sheet 1.** With one triangle, students can only complete the first and last columns of **sheet 1.**

Students then mark the midpoint of each side of the triangle and connect the midpoints, forming four congruent triangles—stage 1 (see **fig. 2**). Students use a light color to shade in the triangle that is "pointing down" and complete the stage 1 row of **sheet 1.**

Although this activity does not specifically address area, the teacher can take this opportunity to point out that when the sides of the triangle are cut in half, the area of each resulting triangle is one-fourth the area of the original triangle.

To form stage 2 (see **fig. 3**), students find and connect the midpoints of the unshaded triangles from stage 1. Again, they use the light color to shade the triangles that point down and complete the row for stage 2 on **sheet 1.** The teacher can point out that the ratios should not be reduced. At this stage and at all successive stages, the term *congruent triangles* refers to the triangles of the new size formed by connecting the midpoints.

Students may begin to see a pattern emerging in the numbers in the chart; the total numbers of unshaded triangles are powers of three. Some students may also notice patterns emerging in other

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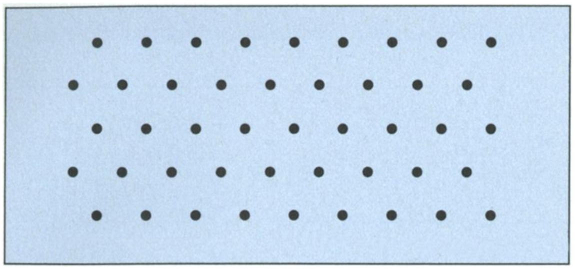

Fig. 1 Dot paper sample

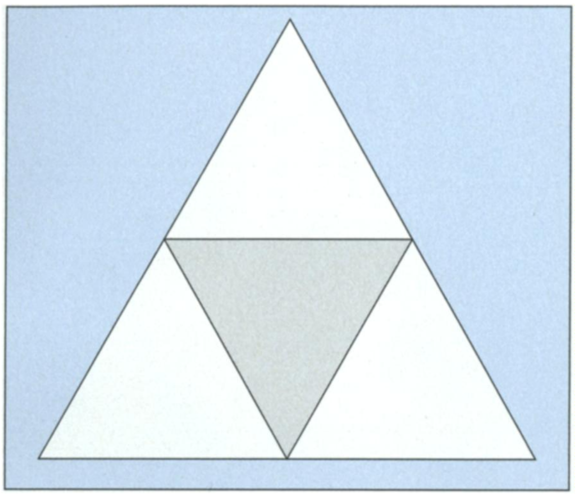

Fig. 2 Sierpinski triangle, stage 1

columns. The teacher can ask students to predict the number of triangles that will be formed at the next stages and complete the next stage to learn whether their predictions are correct.

Students next complete stages 3 and 4 (see **figs. 4** and **5**) and compare the ratios within each column. The teacher can ask them to explain why the ratios are equivalent. The students should answer that the triangles are equivalent because the same iterative process is occurring at each step.

The students next express stage n algebraically. They will need to use exponents with more than one term, such as $n + 1$ or $n - 1$. Prime factorizations of the numbers may make these patterns more evident. The teacher must point out that n is greater than or equal to 1 in the first column.

If students have already studied the laws of exponents, they can use those laws to simplify the ratios for the nth terms. They will see that the expressions simplify to the results in stage 1, as follows:

$$\frac{3^n}{3^{n-1}} = 3^{n-(n-1)} = 3^1 = 3 = \frac{3}{1};$$
$$\frac{3^{n-1}}{4(3^{n-1})} = \frac{3^{(n-1)-(n-1)}}{4} = \frac{3^{(n-1-n+1)}}{4} = \frac{3^0}{4} = \frac{1}{4};$$
$$\frac{3^n}{4(3^{n-1})} = \frac{3^{n-(n-1)}}{4} = \frac{3^{n-n+1}}{4} = \frac{3^1}{4} = \frac{3}{4}$$

Students can predict what will happen if the geometric iterative process in the midpoint activity is

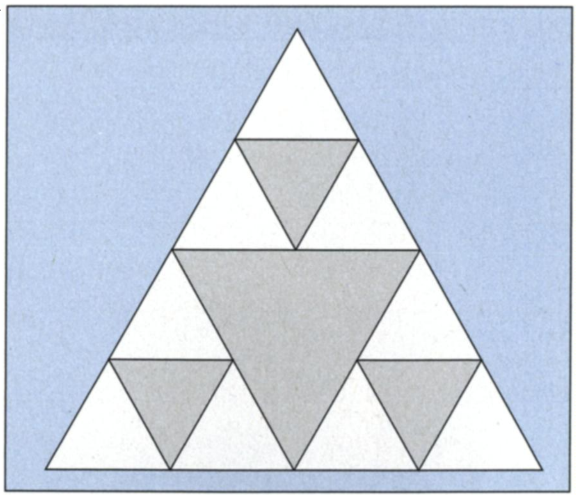

Fig. 3 Sierpinski triangle, stage 2

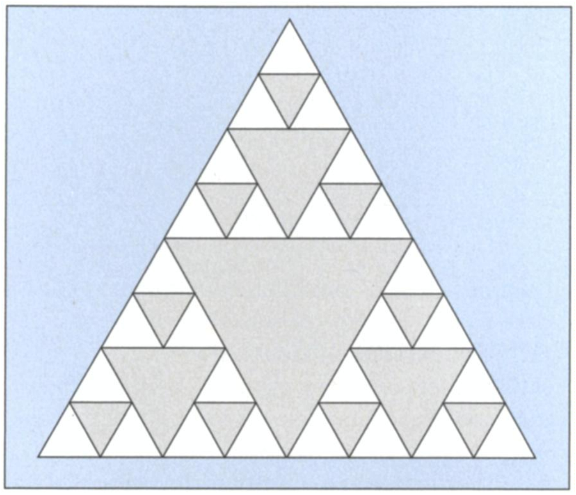

Fig. 4 Sierpinski triangle, stage 3

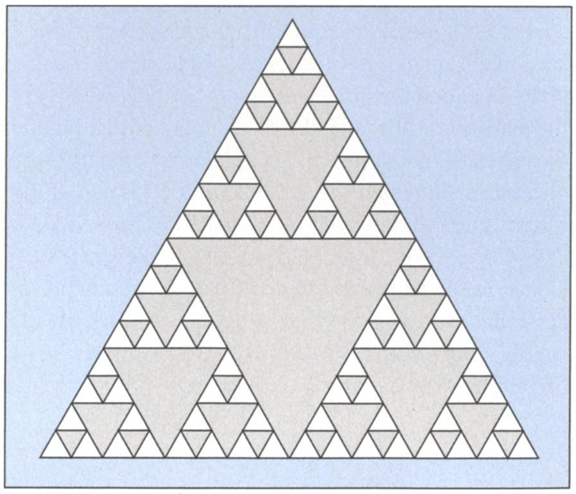

Fig. 5 Sierpinski triangle, stage 4

continued infinitely. (The triangles are reduced to points, so the limit figure has an infinite perimeter but no area.) **Figure 5** shows the stage 4 Sierpinski triangle that the students have created. **Table 1** shows the completed chart for the midpoint activity.

TABLE 1

Solution to Sheet 1, Finding Ratios in the Sierpinski Triangle

Stage	Total Number of Congruent Triangles	Ratio of $\frac{\text{Unshaded} \cong \text{Triangles}}{\text{Shaded} \cong \text{Triangles}}$	Ratio of $\frac{\text{Shaded} \cong \text{Triangles}}{\text{Total} \cong \text{Triangles}}$	Ratio of $\frac{\text{Unshaded} \cong \text{Triangles}}{\text{Total} \cong \text{Triangles}}$	Total Number of Unshaded Congruent Triangles
0	1				1
1	4	$\frac{3}{1}$	$\frac{1}{4}$	$\frac{3}{4}$	3
2	12	$\frac{9}{3}$	$\frac{3}{12}$	$\frac{9}{12}$	9
3	36	$\frac{27}{9}$	$\frac{9}{36}$	$\frac{27}{36}$	27
4	108	$\frac{81}{27}$	$\frac{27}{108}$	$\frac{81}{108}$	81
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$4(3^{n-1})$, $n \geq 1$	$\frac{3^n}{3^{n-1}}$	$\frac{3^{n-1}}{4(3^{n-1})}$	$\frac{3^n}{4(3^{n-1})}$	3^n

THE TRISECTION VARIATION

For this part of the activity, students use the trisection variation dot paper (see **fig. 6**) to make a triangle. They create sides of twenty-seven units and then trisect the sides. Following the same procedures that they used in in the previous part of the activity, they lightly shade the triangles that point down at each stage and complete **sheet 2,** Number Patterns in the Trisection Variation. This trisection variation produces results that are similar to those obtained in the midpoint activity. Although the mathematical computations are more difficult than those for the midpoint activity, the terms of the nth stage once again simplify to the stage 1 ratios. This part of the activity gives students many opportunities to use the laws of exponents. It provides another opportunity to look at the scaling factor. Teachers can help students understand that, when the sides are cut into thirds, the resulting figures each have areas that are one-ninth the original area. Students can then see that

$$\left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

If students shade the remaining triangles in a darker color at the end of the trisection variation, the results of this activity also correspond to the figure obtained when the numbers of Pascal's triangle that are divisible by 3 are shaded.

Figure 7 shows the results that students should obtain, and **table 2** shows the completed chart for the trisection variation.

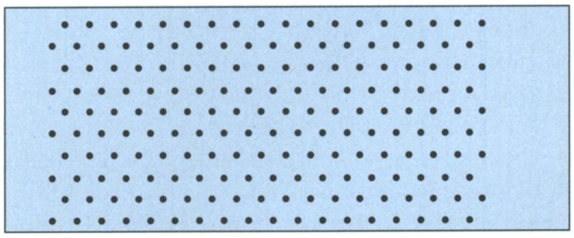

Fig. 6 Trisection variation dot paper sample

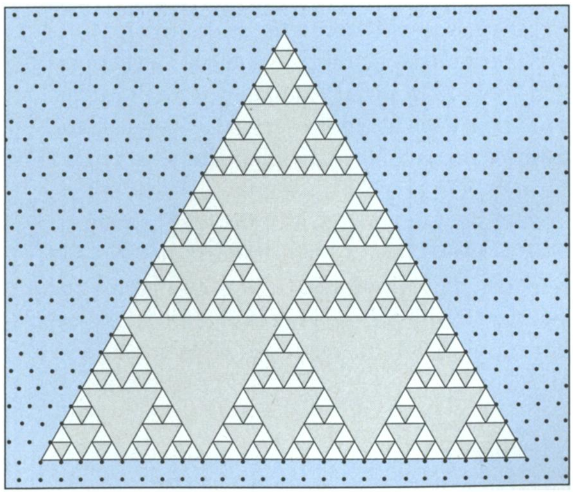

Fig. 7 The trisection variation key

BUILDING THE SIERPINSKI TRIANGLE

This hands-on part of the activity allows students to explore the concepts of similarity, self-similarity, area, and perimeter while reinforcing exponent and

TABLE 2

Solution to Sheet 2, Number Patterns in the Trisection Variation

Stage	Total Number of Congruent Triangles	Ratio of $\frac{\text{Unshaded} \cong \text{Triangles}}{\text{Shaded} \cong \text{Triangles}}$	Ratio of $\frac{\text{Shaded} \cong \text{Triangles}}{\text{Total} \cong \text{Triangles}}$	Ratio of $\frac{\text{Unshaded} \cong \text{Triangles}}{\text{Total} \cong \text{Triangles}}$	Total Number of Unshaded Congruent Triangles
0	1				1
1	9	$\frac{6}{3}$	$\frac{3}{9}$	$\frac{6}{9}$	6
2	54	$\frac{36}{18}$	$\frac{18}{54}$	$\frac{36}{54}$	36
3	324	$\frac{216}{108}$	$\frac{108}{324}$	$\frac{216}{324}$	216
⋮	⋮	⋮	⋮	⋮	⋮
n	$(3^{n+1})(2^{n-1})$, $n \geq 1$	$\frac{2^n \cdot 3^n}{(2^{n-1})(3^n)}$	$\frac{(2^{n-1})(3^n)}{(2^{n-1})(3^{n+1})}$	$\frac{2^n \cdot 3^n}{(2^{n-1})(3^{n+1})}$	6^n

rational number operations. The teacher needs to give each student copies of the triangles from **sheet 3** (enlarged 200 percent) as indicated below:

- One 8" shaded triangle
- Three 8" white triangles
- One sheet of 4" shaded triangles
- One sheet of 1" and 2" shaded triangles

In addition, students need the following materials:

- Scissors
- Glue sticks
- **Sheet 4,** Finding Patterns in Sierpinski Triangle

Teachers also need to make transparencies of **sheet 4** and **figures 3–5**.

After the students receive all the template sheets, they cut out the four 8-inch triangles. They should keep the shaded triangle on their desks. Refer to it as stage 0. The teacher displays the stage 1 triangle on the overhead for approximately ten seconds and then asks students to build Stage 1 with their remaining pieces. To do so, they should cut out the 4-inch triangles and glue three of them onto a white 8-inch triangle. They should also keep the stage 1 triangle on their desks for reference. The teacher then displays the stage 2 triangle on the overhead for approximately twenty seconds and asks students to construct the figure that they have just seen by using the materials they have. They cut out nine of the 2-inch shaded triangles and glue them onto an 8-inch white triangle.

To explore the concepts of similarity and self-similarity, students compare the three figures that are on their desks. They should understand that the shaded triangles within each stage are the same size, but they are different in that each figure has smaller triangles, with three times as many as at the previous stage. The parts of each successive stage are identical to those of the previous stage, but they are smaller, so the figures are self-similar.

The teacher uses a transparency made from **figure 3** to display stage 1 of Perimeter in the Sierpinski Triangle. She explains that each stage yields a figure with an area and a perimeter. The white triangles are not considered part of the figure when finding perimeter or area. The perimeter at each stage is the sum of the sides of all the shaded triangles, and the area is a fraction of the previous area. The teacher shows transparencies made from **figures 4** and **5** when discussing stages 2 and 3.

After distributing **sheet 4**, Finding Patterns in the Sierpinski Triangle, the teacher explains that the first 8-inch shaded triangle is stage 0, with side s, perimeter $3s$, and area A. Stages 1 and 2 were constructed next. Together with the teacher, the students complete the chart for stages 0, 1, and 2. Stage 1 triangles have sides of length $(1/2)s$. With three triangles and a total of nine sides, the total length of the sides is $3 \cdot 3(1/2)s$, or $9(1/2)s$. Four stage 1 triangles make the stage 0 triangle, so each of the stage 1 triangles has an area of $(1/4)A$. The total area of the shaded triangles at stage 1 is $3 \cdot (1/4)A$. At stage 2, the sides of each triangle are one-half the length of the sides of the triangles of stage 1, or $(1/2)(1/2)s$. The total

TABLE 3

Solution to Sheet 4, Finding Patterns in the Sierpinski Triangle

Stage	Number of Shaded Triangles	Length of Side of One Shaded Triangle	Total Length of All Sides of All Shaded Triangles (perimeter)	Area of One Shaded Triangle	Total Area of All Shaded Triangles
0	1	s	$3 - s$	A	A
1	3	$\left(\frac{1}{2}\right)s$	$9\left(\frac{1}{2}\right)s$	$\left(\frac{1}{4}\right)A$	$3\left(\frac{1}{4}\right)A$
2	9	$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$27\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$	$9\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$
3	27	$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$81\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$	$27\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$
4	81	$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$243\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)s$	$\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$	$81\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)A$
⋮	⋮	⋮	⋮	⋮	⋮
n	3^n	$\left(\frac{1}{2}\right)^n s$	$3 \cdot 3^n\left(\frac{1}{2}\right)^n s$	$\left(\frac{1}{4}\right)^n A$	$3^n\left(\frac{1}{4}\right)^n A$

perimeter is 3 • 9 • (1/2)(1/2)*s*, or 27 • (1/2)(1/2)*s*. Each of the nine triangles of stage 2 has an area that is one-fourth the area of a stage 1 triangle, or (1/4)(1/4)*A*; and the total area is 9 • (1/4)(1/4)*A*.

With the materials provided, students can construct their prediction for stage 3 without gluing the pieces down. Students should be given time to explore—and perhaps struggle—with this task. If they need assistance, the teacher can remind them that stage 2 is made of three smaller stage 1 figures and stage 3 is made of three smaller stage 2 figures. When students have arranged their shaded triangles, they can explain how they arrived at their configuration. Before they glue the triangles, students should have placed twenty-seven of the 1-inch triangles in the correct position on the white triangle. They can then compare this figure with the previous stages. At this point, it is important for teachers to summarize and emphasize the following points:

- Each figure was constructed by using a step-by-step iterative process.
- Each figure consists of three identical parts.
- Each stage looks like the previous stage but is smaller.

After students complete **sheet 4** for stage 4, they can discuss their answers to verify that they have filled in the row correctly and that they have determined the pattern. Together with the students, teachers complete the table for stage *n* by examining the patterns of change that occur in each column. Then they can reinforce the exponent law

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}.$$

Teachers can discuss with their students the amazing fact that the perimeter of this figure increases while the area decreases. This experience will foster a more comprehensive understanding of the relationship between area and perimeter.

The three parts of this activity are excellent ways to engage students and reinforce several mathematical concepts, including the laws of exponents. ∞

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Finding Ratios in the Sierpinski Triangle

Sheet 1

Stage 0: On dot paper, construct an equilateral triangle with sides of sixteen units. Complete the first and last columns of the Stage 0 row of the chart.

Stage 1: Mark the midpoint of each side of your triangle. Connect the midpoints and lightly shade the interior triangle formed (pointing down). Complete the Stage 1 row of the chart.

Stage 2: Mark the midpoints of the sides of the unshaded triangles. Connect the new midpoints and shade the new interior triangles (pointing down). Complete the new Stage 2 row of the chart.

Continue forming new interior triangles to complete the rest of the chart.

Stage	Total Number of Congruent Triangles	Ratio of $\frac{\text{Unshaded} \cong \text{Triangles}}{\text{Shaded} \cong \text{Triangles}}$	Ratio of $\frac{\text{Shaded} \cong \text{Triangles}}{\text{Total} \cong \text{Triangles}}$	Ratio of $\frac{\text{Unshaded} \cong \text{Triangles}}{\text{Total} \cong \text{Triangles}}$	Total Number of Unshaded Congruent Triangles
0					
1					
2					
3					
4					
⋮	⋮	⋮	⋮	⋮	⋮
n					

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Number Patterns in the Trisection Variation

Sheet 2

Stage 0: On the trisection variation dot paper, construct an equilateral triangle with sides of twenty-seven units. Complete the first and last columns of the Stage 0 row of the chart.

Stage 1: Use points to trisect or divide each side of your triangle into three equal parts. Connect the points and lightly shade the interior triangles formed (pointing down). Complete the Stage 1 row of the chart.

Stage 2: Use points to trisect the sides of the unshaded triangles. Connect the new points and shade the new interior triangles (pointing down). Complete the new Stage 2 row of the chart.

Continue forming new interior triangles to complete the rest of the chart.

Stage	Total Number of Congruent Triangles	Ratio of $\frac{\text{Unshaded} \cong \text{Triangles}}{\text{Shaded} \cong \text{Triangles}}$	Ratio of $\frac{\text{Shaded} \cong \text{Triangles}}{\text{Total} \cong \text{Triangles}}$	Ratio of $\frac{\text{Unshaded} \cong \text{Triangles}}{\text{Total} \cong \text{Triangles}}$	Total Number of Unshaded Congruent Triangles
0					
1					
2					
3					
⋮	⋮	⋮	⋮	⋮	⋮
n					

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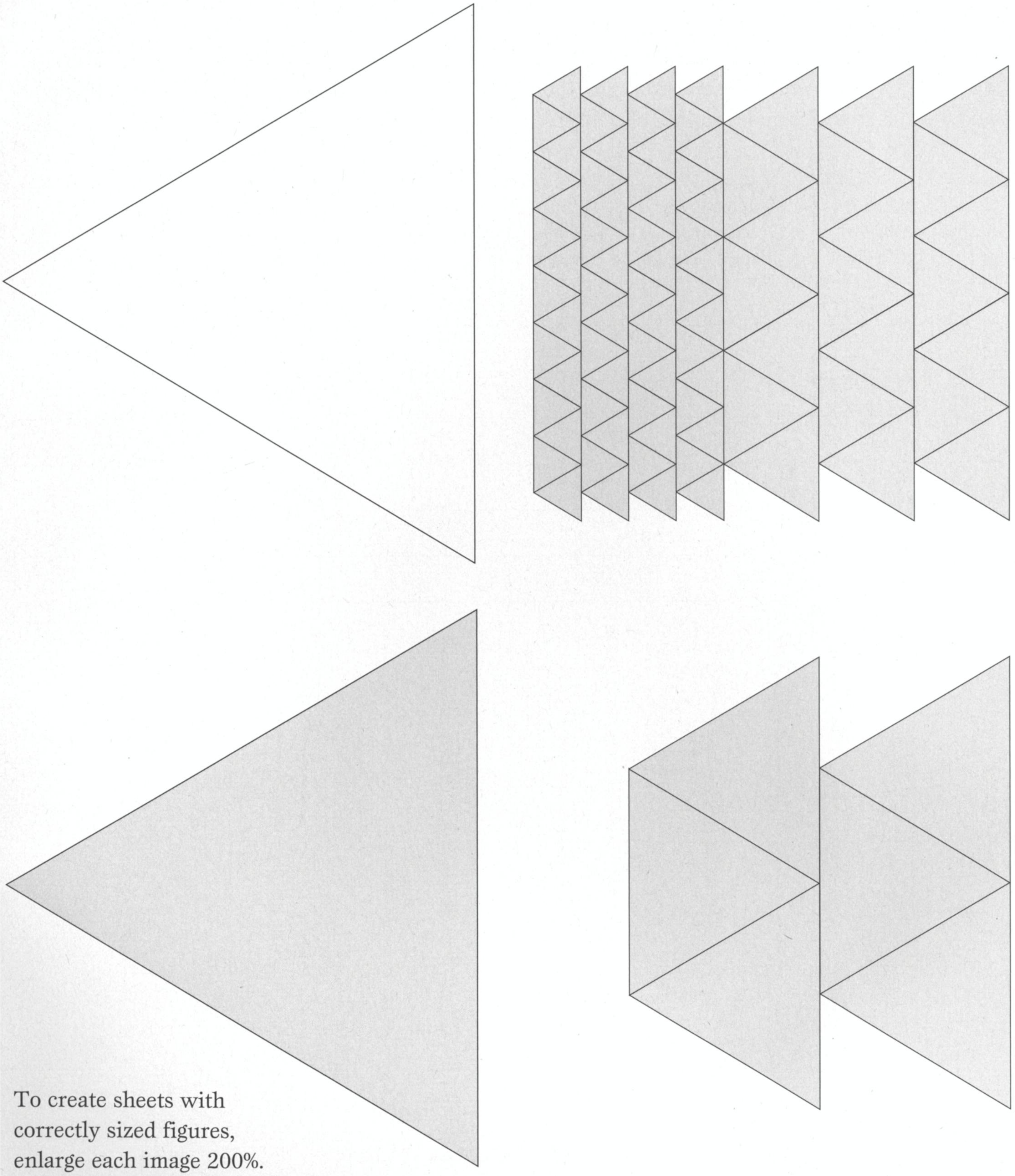
Triangle Templates

Sheet 3



To create sheets with correctly sized figures, enlarge each image 200%.

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Finding Patterns in the Sierpinski Triangle

Sheet 4

Using the templates given, cut out the four 8-inch triangles. Keep the shaded triangle on your desk as Stage 0. With the teacher's instructions and the remaining pieces and templates, you will build Stage 1 and Stage 2 triangles and complete the chart.

Stage	Number of Shaded Triangles	Length of Side of One Shaded Triangle	Total Length of All Sides of All Shaded Triangles (perimeter)	Area of One Shaded Triangle	Total Area of All Shaded Triangles
0					
1					
2					
3					
4					
⋮	⋮	⋮	⋮	⋮	⋮
n					

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