Definition

CW-complex

Anything that can be constructed with the following type of construction:

- Start with any set of points X^0 with the discrete topology.
- Form $X^n = D^n_\alpha \sqcup_{\varphi_\alpha} X^{n-1}$ by attaching n-cells to the $(n-1)$-skeleton.
- If you go infinitely, use the weak topology; where $A \underset{\text{open}}{\subset} X$ if $A \underset{\text{open}}{\subset} X^n$ for all n.

Definition

g Homotopic to h rel A

$g \simeq h \operatorname{rel} A$ if $\exists$ a homotopy F s.t.

- $f_0 = g$
- $f_1 = h$
- $f_{t_1}(a) = f_{t_2}(a) \quad \forall a \in A$

Definition

Homotopy Equivalent rel A

$\exists f : X \to Y, g : Y \to X$ such that

- $gf \simeq \mathbb{1} \operatorname{rel} A$
- $fg \simeq \mathbb{1} \operatorname{rel} A$

Definition

Homotopy Extension Property

The following are equivalent:

- $\forall F : A \times I \to Y$ and
  $f : X \to Y$ s.t. f extends F_0,
  $\exists \bar{F} : X \times I$ which extends F and f.
- $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Definition

Smash Product

$$X \wedge Y = {}^{X \times Y}/_{X \vee Y},$$

where we wedge X and Y at their respective base points x_0, y_0, usually given.

Definition

Concatenation of Paths $f \cdot g$

(Product Path)

Given two paths $f, g : I \to X$ such that $f(1) = g(0)$, then

$$f \cdot g(s) = \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ g(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

or in words, do f then do g, but go twice as fast.

Definition

Concatenation of Path Homotopies $F \cdot G$

(Not defined in Hatcher)

Given homotopic paths $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$ such that $f_s \cdot g_s$ is defined, then

$$F \cdot G := \begin{cases} F(2s, t) & s \in [0, \frac{1}{2}] \\ G(2s-1, t) & s \in [\frac{1}{2}, 1] \end{cases}$$

or in words, apply F in the first region, and G in the second.

Definition

Composition of Path Homotopies $F \square F'$

(Not defined in Hatcher)

Given homotopic paths $f_0 \overset{F}{\simeq} f_1 \overset{F'}{\simeq} f_2$, we can compose the homotopies by

$$F \cdot F'(s,t) = \begin{cases} F(s, 2t) & t \in [0, \frac{1}{2}] \\ F'(s, 2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

That is, smoothly change f_1 into f_2 via F', then change f_2 into f_3 via F'.

Definition

Simply Connected Space

- X is path-connected
- $\pi_1(X) = 0$, that is, the fundamental group is the trivial group.

Theorem

If X is path-connected, then $\pi_1(X)$...

is independent of basepoint, since the change-of-basepoint homomorphism is an isomorphism.

Theorem

If $\varphi : X \to Y$ is a homotopy equivalence map, what can we say about π_1?

φ_* is an isomorphism, so $\pi_1(X) \cong \pi_1(Y)$.

Theorem

If X retracts to A, what can we say about π_1? What if it deformation retracts?

ι_* is injective, so $\pi_1(A) \underset{\text{group}}{\subset} \pi_1(X)$ up to isomorphism.
$X \simeq A$, so $\pi_1(X) \cong \pi_1(A)$.