TOPOLOGY QUALIFYING EXAM - DRAFT

May 2017

INSTRUCTIONS:

•This exam has 8 questions. You should answer exactly 6 of them.

•Use a separate sheet of paper for each problem.

Problem 1. Prove or give a counterexample for each of the following.

(a) Any quotient of a Hausdorff space is Hausdorff.

(b) Any metric space is normal.

(c) If X is a topological space and $A \subset B \subset X$ and $\overline{A}$ is the closure of A then $\overline{A} \cap B$ is the closure of A with respect to the subspace topology on B.

Problem 2. Suppose X is a compact topological space, Y is a topological space, and $\mathcal{C}$ is an open cover of $X \times Y$. Prove that for all $y \in Y$ there exists an open neighborhood U of y such that $X \times U$ is contained in the union of finitely many sets from $\mathcal{C}$.

Problem 3. Suppose X is a locally compact Hausdorff space, and $A \subset X$. Prove that if $A \cap C$ is closed for every compact subset C of X then A is closed.

Problem 4. Prove that, in any topological space, the intersection of two open dense sets is open and dense. Prove that, in any complete metric space, the intersection of countably many open dense sets is nonempty. (Do not use the Baire category theorem, which would make the question too easy.)

Problem 5. Prove that $\mathbb{R}$ is connected. (Use whatever other properties of $\mathbb{R}$ you want.) Prove that if a topological space X has a connected dense subset then X is connected.

Problem 6. State and prove the contraction mapping theorem. Give an example of a complete metric space X and a function $f\colon X \to X$ such that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ but f has no fixed point.

Problem 7. Prove that any finite sheeted covering space of a compact metric space is compact.

Problem 8. Let M^6 be the 6-manifold $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{S}^2$. Calculate $\pi_1(M^6)$. How many covers does M^6 have? Roughly describe each cover and the subgroup with which it corresponds.