Math 201A, Homework 4 (Measurable Functions)

Problem1. Let X be a nonempty topological space and let μ be a measure on X. Prove that if the functions $f_n \colon X \to [-\infty, +\infty]$ are μ-measurable for $n = 1, 2, \dots$, then the set

$$A = \{x \in X \ : \ \lim_{n\to\infty} f_n(x) \ \text{ exists}\}$$

is $\mu-$measurable.

Problem2. Prove that any Lebasgue-measurable function $f \colon \mathbb{R} \to \mathbb{R}$ that satisfies the relation

$$f(x+y) = f(x) + f(y) \quad \text{for all} \quad x, y \in \mathbb{R}$$

must be linear.

Problem3. Let $f \colon (0,1) \to \mathbb{R}$ be such that for every $x \in (0,1)$ there exists $\delta > 0$ and a Borel-measurable function $g \colon \mathbb{R} \to \mathbb{R}$ (both dependent on x), such that $f(y) = g(y)$ for all $y \in (x-\delta, x+\delta) \cap (0,1)$. Prove that f is Borel-measurable. (You can assume that $f(x) = 0$ outside the interval $(0,1)$).

Problem4. Give an example of a collection of Lebesgue-measurable nonnegative functions $\{f_\alpha\}_{\alpha\in A}$ ($f_\alpha \colon \mathbb{R} \to \mathbb{R}$) such that the function

$$g(x) = \sup_{\alpha\in A} f_\alpha(x), \quad x \in \mathbb{R}$$

is finite for all $x \in \mathbb{R}$ but g is not Lebesgue-measurable. Here A is a nonempty index set.