TOPOLOGY QUALIFYING EXAM
MAY 2015

There are seven questions. Answer exactly six.
If you answer more, we will count only the *lowest* six.
Any theorems that you use should be stated precisely.

(1) Let X be a topological space.
  (a) Define what it means for X to be connected.
  (b) Define what it means for X to be path-connected.
  (c) Using, for example, the least upper bound property of $\mathbb{R}^1$, show that the interval $I = [0,1] \subset \mathbb{R}^1$ is both connected and path-connected.

(2) (a) Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a family of connected subsets of a topological space X. Assume there is a connected $A \subset X$ so that for each $\lambda \in \Lambda$, $A_\lambda \cap A \neq \emptyset$. Show that $A \cup (\cup_\lambda A_\lambda)$ is connected.
  (b) Show that a topological space X is path-connected if and only if
   - X is connected and
   - each $x \in X$ has a path-connected neighborhood.

  Note: If you have not done the previous problem, be sure to state the appropriate definitions.

(3) (a) For topological spaces X and Y give a careful definition of the product topology on $X \times Y$.
  (b) Show that a topological space X is Hausdorff if and only if the diagonal of $X \times X$ is closed in the product topology.
  (c) Let X be a Hausdorff topological space and suppose $f : X \to X$ is continuous. Show that the set of fixed points for f is a closed subset of X.
  (d) Prove or disprove the following claim: any infinite Hausdorff space contains an infinite isolated set (i.e. an infinite set with the discrete topology).

(4) Let X be a compact topological space and $\mathcal{F}$ be a set of continuous functions $X \to \mathbb{R}$ with these two properties:
- If $f, g \in \mathcal{F}$ then so is their pointwise multiplication $f \cdot g$ (defined as $(f \cdot g)(x) := f(x) \cdot g(x)$).
- For each $x \in X$ there is a neighborhood $U(x)$ of x in X and a function $f \in \mathcal{F}$ so that $f^{-1}(0) = U(x)$.

Prove that $\mathcal{F}$ contains the function $f \equiv 0$.

(5) For (M, d) a metric space and $\emptyset \neq A \subset M$, define the diameter of A to be $\delta(A) = \sup\{d(x, y) \mid x, y \in A\}$ if the supremum exists, and ∞ if it does not. Show that for any $A, B \subset M$:
(a) $\delta(A) = 0 \iff A$ contains at most one point.
(b) If $A \subset B$ then $\delta(A) \leq \delta(B)$.
(c) $\delta(closure(A)) = \delta(A)$.
(d) If $A \cap B \neq \emptyset$ then $\delta(A \cup B) \leq \delta(A) + \delta(B)$.

(6) Show the following:
(a) A closed subspace of a compact space is compact.
(b) A compact subspace of a Hausdorff space is closed.
(c) If $f : X \to Y$ is a continuous bijection, X is compact and Y is Hausdorff, then f is a homeomorphism.

(7) Let X be a topological space.
(a) Define what it means for X to be simply connected.
(b) Define what it means for X to be locally path connected.
(c) Suppose X is both simply connected and locally path connected. Using covering space theory, show that any continuous function from X to the torus $S^1 \times S^1$ is null-homotopic.
(d) Is the same true if the torus is replaced by S^2? (Prove it, or give a convincing counterexample.)