Math 201B, Homework 2 (Integration, Differentiation, Density)

Problem1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be increasing. Define the function $\nu\colon 2^{\mathbb{R}} \to [0, \infty]$ as follows:
1. For any open set $U = \cup_{i=1}^{\infty}(a_i, b_i)$ where (a_i, b_i) are disjoint, set

$$\nu(U) = \sum_{i=1}^{\infty}(f(b_i-) - f(a_i+)),$$

where

$$f(x+) = \lim_{y \to x+} f(y) \quad \text{and} \quad f(x-) = \lim_{y \to x-} f(y) \quad \text{for} \quad x \in \mathbb{R}$$

(the two limits obviously exist as f increases).
2. For any $A \subset \mathbb{R}$ define

$$\nu(A) = \inf\{\nu(U) \ : \ A \subset U, U - \mathit{open}\}.$$

Prove that ν is a measure on $\mathbb{R}$.

Problem2. Let m be Lebesgue measure on $\mathbb{R}$.
1. Construct an $m-$integrable function $f\colon \mathbb{R} \to [-\infty, \infty]$ for which there exists a set $A \subset \mathbb{R}$ such that $m(A) > 0$ and for any $x \in A$ the limit

$$\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)dy$$

exists but is different from $f(x)$.
2. Prove that in fact for any $\epsilon > 0$ one can reach $m(\mathbb{R} - A) < \epsilon$ in the first part.

Problem3. Let $\alpha \in (0, 1)$ and let m be Lebesgue measure on $\mathbb{R}$. Construct a Borel set $E \subset [-1, 1]$ such that

$$\lim_{r \to 0} \frac{m(E \cap [-r, r])}{2r} = \alpha.$$

Problem4. For a function $f\colon [a, b] \to \mathbb{R}$ define for every $x \in [a, b)$

$$D^+ f(x) = \limsup_{h \to 0+} \frac{f(x + h) - f(x)}{h}.$$

Prove that if $f\colon [a, b] \to \mathbb{R}$ is continuous and $D^+ f(x) \geq 0$ for all $x \in [a, b)$, then $f(b) \geq f(a)$.

Problem5. Let the function $f\colon [a, b] \to \mathbb{R}$ be differentiable at every point $x \in [a, b]$. Is f necessarily absolutely continuous on $[a, b]$?