Homework 9

1. Find *all* distributional solutions y to the following equations:

(a) $xy = 0$ in $\mathcal{D}'(\mathbb{R})$
[Hint: Represent any $\phi \in \mathcal{D}$ as $\phi(x) \equiv \phi(0)\eta(x) + x\psi(x)$ with η independent of ϕ.

Proof We can write

$$\phi = \phi_0\eta + x\psi,$$

where $\eta \in \mathcal{D}$ is any test function with $\eta(0) = 1$ and $\psi := \frac{1}{x}(\phi - \phi_0\eta)$. Suppose T is a solution, so xT is the zero distribution. Then

$$\begin{aligned} T\phi &= T(\phi_0\eta + x\psi) \\ &= \phi_0 T\eta + T(x\psi) && \text{by linearity} \\ &= \phi_0 T\eta + xT(\psi) && \text{by def. of mult. by } C^\infty \text{ f'ns} \\ &= \phi_0 T\eta && \text{by assumption} \\ &= (T\eta)\delta_0(\phi) \end{aligned}$$

and since $T\eta$ is an arbitrary constant with respect to ϕ, then any multiple of the Dirac delta δ_0 is a solution to (a). ■

(b) $y' = 0$ in $\mathcal{D}'((a, b))$

Proof Suppose T is a solution, so the distributional derivative $DT = 0$. Since

$$DT = 0 \in C^0(a, b),$$

then by Theorem 6.10

$$T = f \in C^1(a, b)$$

for some f, and $0 = DT$ is the classical derivative f'. Since $f' = 0$, then it is a constant, so $T = C$. Thus the solutions to (b) are constant distributions. ■

2. Consider the function $f(x) = |x|^{-1}$ on $\mathbb{R}$. Although this function is not in L^1_{loc}, it is defined as a distribution for test functions on $\mathbb{R}$ that vanish at the origin, by

$$T_f(\phi) = \int_{\mathbb{R}} |x|^{-1}\phi(x)\, dx.$$

(a) Show that there is a distribution $T \in \mathcal{D}'(\mathbb{R})$ that agrees with T_f for functions that vanish at the origin. Give an explicit formula for one such T.

Proof Note that for all $\phi \in \mathcal{D}$, $(\phi - \phi_0) \in \mathcal{D}$ and vanishes at zero, so $T_f(\phi - \phi_0)$ is a distribution. If ϕ vanishes at 0, then

$$T_f(\phi - \phi_0) = T_f(\phi - 0) = T_f(\phi).$$

■

(b) Characterize all such T's. Theorem 6.14 may be helpful here.

3. Compute the following limit in $\mathcal{D}'(\mathbb{R})$.

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\pi x^2} \sin^2\left(\frac{x}{\varepsilon}\right)$$

Answer: The answer is the Dirac delta δ_0. To see this, observe that $\int_{\mathbb{R}} \frac{1}{\pi x^2} \sin^2(x)\, dx = 1$, so

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \left(\int_{\mathbb{R}} \frac{\varepsilon}{\pi x^2} \sin^2\left(\tfrac{x}{\varepsilon}\right) \phi(x)\, dx - \phi(0)\right)\\
=&\lim_{\varepsilon\to 0} \left(\int_{\mathbb{R}} \frac{1}{\pi u^2} \sin^2(u)\phi(\varepsilon u)\, du - \phi(0)\right)\\
=&\lim_{\varepsilon\to 0} \left(\int_{\mathbb{R}} \frac{1}{\pi u^2} \sin^2(u)\phi(\varepsilon u)\, du - \phi(0)\int_{\mathbb{R}} \frac{1}{\pi u^2} \sin^2(u)\, du\right)\\
=&\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \frac{1}{\pi u^2} \sin^2(u)\Big(\phi(\varepsilon u) - \phi(0)\Big)\, du
\end{aligned}$$

and note that $\phi \in \mathcal{D}$ and in particular, ϕ is continuous with compact support, so $||\phi(\varepsilon u) - \phi(0)||_\infty \leq C < \infty$, so the sequence of integrands above is dominated by $\frac{C}{\pi u^2}\sin^2(u)$, so by the dominated convergence theorem we can move the limit inside the integral and find that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \frac{1}{\pi u^2} \sin^2(u)\Big(\phi(\varepsilon u) - \phi(0)\Big)\, du\\
=&\int_{\mathbb{R}} \frac{1}{\pi u^2} \sin^2(u)\Big(\phi(0) - \phi(0)\Big)\, du\\
=&0.
\end{aligned}$$

Therefore the action of the limit is exactly that of δ_0. ■