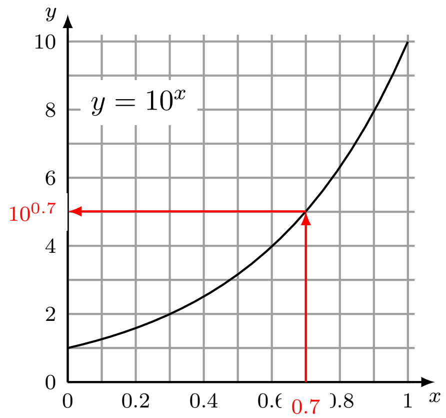
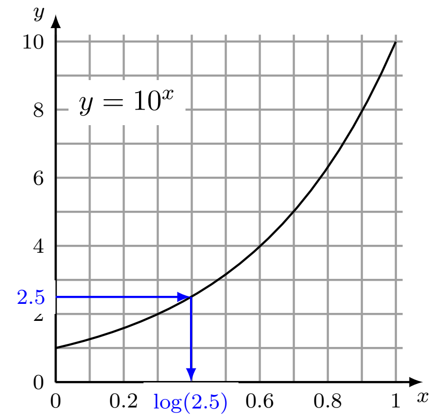
Math 34A Winter 2020
Old Final Exam #3 Solutions

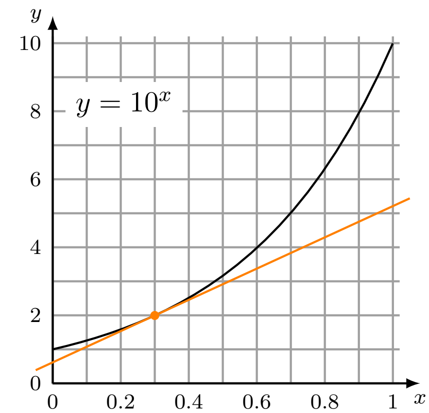
1. Here are the three graphs we'll use in solving these problems:



Part (a)



Parts (a) and (b)



Part (c)

(a) To solve $\log(2x) = 2.7$, we first take the antilog; this equation becomes $2x = 10^{2.7}$. We can write 2.7 as $2 + 0.7$, so $10^{2.7} = 10^{2+0.7} = 10^2 \times 10^{0.7}$. We can find this value directly from the graph; we get $10^{0.7} \approx 5.0$. Thus $2x \approx 10^2 \times 5.0 = 500$, so $\boxed{x \approx 250}$.

(b) It's difficult to compute 2.5^{10} directly, but we can compute $\log(2.5^{10})$, and then $2.5^{10} = \text{antilog}(\log(2.5^{10}))$.

We start with $\log(2.5^{10})$, which by the rules of logs is $10\log(2.5)$. From the middle graph we see that $\log(2.5) \approx 0.40$, so $\log(2.5^{10}) \approx 10(0.40) = 4.0$. Thus $2.5^{10} = \text{antilog}(\log(2.5^{10})) \approx 10^{4.0} = \boxed{10,000}$.

Mathematica tells me that $2.5^{10} \approx 9,536.7$, so we're within 64 of the correct answer (about 2/3 of a percent error).

(c) We've drawn the tangent line at $x = 0.3$ on the third graph, above. We pick two points on this line that are reasonably far apart; we'll take $(x, y) = (0.3, 2)$ and $(1, 5.2)$. Thus the slope of this line is about

$$m = \frac{5.2 - 2}{1 - 0.3} = \frac{3.2}{0.7} \approx \boxed{4.57}.$$

The actual slope of the tangent line to $y = 10^x$ at $x = 0.3$ is $m = 10^{0.3}\ln(10) \approx 4.594\,26\ldots$, so as usual we're pretty close.

2. We write down the answers without much commentary. Notice that $f'(x) = 20x^4 - 7$, $f''(x) = 80x^3$, and $g'(x) = 3x^2$.

(a) $\dfrac{d}{dx}\left(f(x) + 4g(x)\right) = f'(x) + 4g'(x) = 20x^4 - 7 + 4(3x^2) = \boxed{20x^4 + 12x^2 - 7}$.

(b) We've already said that $f''(x) = \boxed{80x^3}$.

(c) $f''(1) - 2g'(1) = 80(1)^3 - 2 \times 3(1)^2 = 80 - 6 = \boxed{74}$.

3. Again we don't say too much in computing these derivatives.

(a) $\dfrac{d}{dx}\left(e^{kx} + x^{3k}\right) = \boxed{ke^{kx} + 3kx^{3k-1}}$.

(b) First we re-write this as $4/\sqrt{x} = 4/x^{1/2} = 4x^{-1/2}$, so

$$\frac{d}{dx}(4/\sqrt{x}) = 4\left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right) = \boxed{-2x^{-3/2}}.$$

(c) First we multiply out the product, then we can differentiate. Multiplying out, we get $(x^2+3)(x^3+5) = x^5+3x^3+5x^2+15$. Thus

$$\frac{d}{dx}\big((x^2+3)(x^3+5)\big) = \frac{d}{dx}\big(x^5+3x^3+5x^2+15\big) = \boxed{5x^4+9x^2+10x}.$$

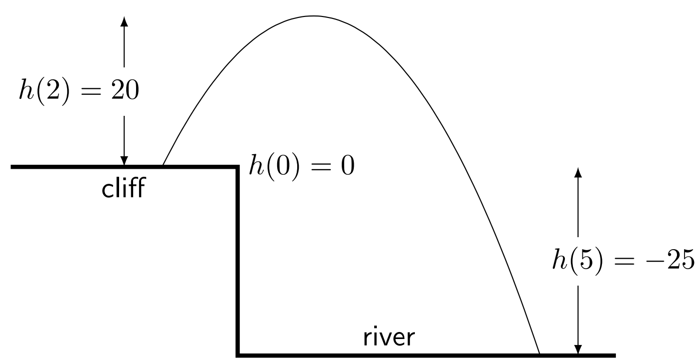
4. Since $y = 3x^2 - 7x + 2$, we have $y' = 6x - 7$.

(a) The slope of the graph is 1 when $y' = 1$. Since $y' = 6x - 7$, this happens when $6x - 7 = 1$. This means $6x = 8$, or $x = 8/6 = \boxed{4/3}$.

(b) This function is a minimum when $y' = 0$. This happens when $6x - 7 = 0$, or when $\boxed{x = 7/6}$.

(c) At $x = 2$, the slope of the tangent line is $m = y'(2) = 6(2) - 7 = 5$. Since $y(2) = 3(2)^2 - 7(2) + 2 = 0$, the equation of the tangent line is $\boxed{y - 0 = 5(x-2)}$ or $\boxed{y = 5x - 10}$.

5. Here's a picture of the situation:



(a) Remember that the velocity of the ball when it hits the river is $h'(5)$. Since $h'(t) = 20 - 10t$ m/s, we find that the velocity is $h'(5) = -30$ m/s. Thus the speed (which is always positive) is $\boxed{30 \text{ m/s}}$.

(b) The height of the cliff is the height at time zero: $h(0) = 0$. The height of the cliff is the height at time $t = 5$: $h(5) = -25$ meters. Thus the cliff is $\boxed{\text{25 meters above the river}}$.

(c) The ball's maximum height occurs when $h'(t) = 0$. This means $20 - 10t = 0$, which happens when $t = 20/10 = 2$ seconds. Thus the maximum height is $h(2) = \boxed{\text{20 meters}}$.

(d) Since the ball's maximum height is 20 meters above the cliff, and the cliff is 25 meters above the river, the maximum height of the ball above the river is $\boxed{\text{45 meters}}$.

6. (a) There are $4 + 9 = 13$ liters of paint in total, 4 of which are red. Thus the percentage of red paint is

$$\frac{\text{amount of red paint}}{\text{total amount of paint}} \times 100\% = \frac{4 \text{ liters}}{13 \text{ liters}} \times 100\% = \frac{400}{13}\ \%.$$

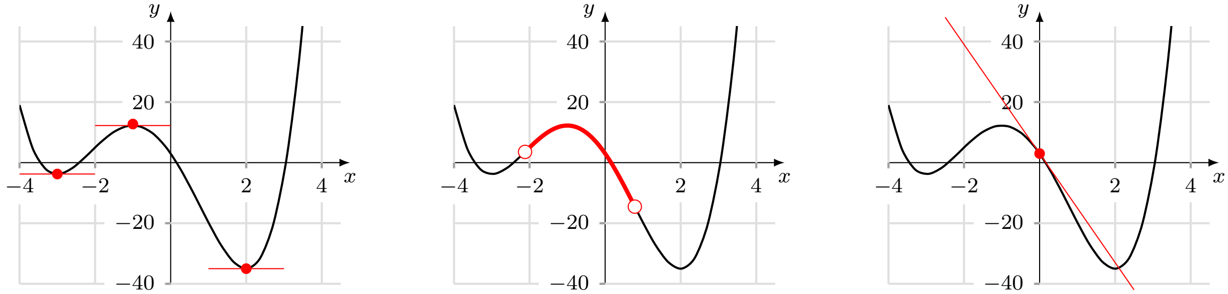
(b) Let R be the amount (in liters) of red paint added to 9 liters of blue paint. Thus the total paint is $9 + R$ liters. Then, as in the previous part, the percentage of red paint is

$$\frac{\text{amount of red paint}}{\text{total amount of paint}} \times 100\% = \frac{R \text{ liters}}{9+R \text{ liters}} \times 100\% = \frac{100R}{(9+R)}\ \%.$$

This is supposed to be 30%, so we need to solve $\frac{100R}{(9+R)} = 30$. Multiplying through by $9+R$ gives us the equation $100R = 30(9+R)$, which simplifies to $100R - 30R = 270$. Thus $R = 270/70 = \boxed{27/7 \text{ liters}}$.

(c) This is the same question as the previous one, except that the "30" has been replaced by "x". Thus our solution is the same, up to when we bring in the 30. We need to solve $\frac{100R}{(9+R)} = x$, which simplifies to $100R = x(9+R) = 9x + xR$ after multiplying through by $9 + R$. Solving, we get $(100 - x)R = 9x$, or $R = \boxed{9x/(100-x) \text{ liters}}$.

7. Here are three views of the same graph, with various markings on them for the three parts of the problem.



(a) (b) (c)

(a) We can see that the tangent line is horizontal (that is, the slope of the graph is zero) at three points: the x values are $\boxed{x = -3,\ x = -1, \text{ and } x = 2}$.

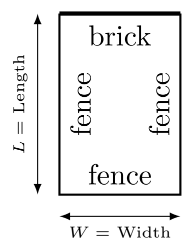
(b) The values of x where $f''(x)$ is negative is exactly the set of x values where $f(x)$ is concave down. We've drawn on the graph where it is concave down; this is roughly $\boxed{-2.1 < x < 0.8}$.

(c) To do this we ask you to draw the tangent line at $x = 0$ and measure the slope as carefully as you can. We're going to estimate that this tangent line passes through the points $(x, y) = (-2, 39)$ and $(2, -33)$. Thus the slope of the tangent line – the value of the derivative at $x = 0$ – is

$$f'(0) = m \approx \frac{-33 - 39}{2 - (-2)} = \frac{-72}{4} = -18.$$

That is, we've estimated that $f'(0) \approx \boxed{-18}$.

8. Here is a reproduction of the picture:



(a) The total cost C of the boundary of the field is

$$\begin{aligned} C &= \$2/\text{meter} \times \Big(\text{length of fence} \Big) + \$5/\text{meter} \times \Big(\text{length of brick wall} \Big) \\ &= 2(2L + W) + 5(W) \\ &= 4L + 7W. \end{aligned}$$

That is, the specified fence and wall will cost $\boxed{4L + 7W \text{ dollars}}$.

(b) Since the area $A = LW$ is 200 square meters, we have $LW = 200$. Thus $\boxed{L = 200/W}$.

(c) Plugging $L = 200/W$ into the cost function from part (a), we get that the cost of the required fence and wall will be $\boxed{800/W + 7W \text{ dollars}}$.

(d) We're trying to find W so that the cost $C = 800/W + 7W$ will be a minimum. This means we compute $C' = \frac{dC}{dW}$ and set it equal to zero. It's easier to differentiate if we first write it as $C = 800W^{-1} + 7W$. Then the derivative is $C' = -800W^{-2} + 7$. Setting this equal to zero gives us $800/W^2 = 7$, or $W^2 = 800/7$. Thus $W = \boxed{\sqrt{800/7} \text{ meters}} \approx 10.69$ meters is the width that will minimize our costs.

9. Marie drives for 3 hours at M km/hr, so she travels (3 hrs)(M km/hr) $= 3M$ km. Similarly, Jason drives 2 hours at J km/hr then 1 hour at $2J$ km/hr, so he travels (2 hrs)(J km/hr) + (1 hr)($2J$ km/hr) $= 4J$ km. Thus they travel $\boxed{3M + 4J = 1000}$ km in total. In the last hour, Marie travels M km and Jason $2J$ km, so together they travel $\boxed{M + 2J = 410}$ km. That is, the two equations we want are

$$\begin{aligned} 3M + 4J &= 1000 \\ M + 2J &= 410 \end{aligned}$$

(although other – equivalent – equations are possible).

We can solve for M by writing $J = (410 - M)/2$ in the second equation, then plugging this into the first equation:

$$3M + 4(410 - M)/2 = 1000 \qquad \text{or} \qquad M + 820 = 1000.$$

Solving, we get $\boxed{M = 180 \text{ km/hr}}$.

10. (a) Notice that, if the price is $\$(2 + x)$, then x is the number of one-dollar price increases. This means that Ermila will sell $\boxed{200 - 10x \text{ burgers}}$.

(b) The total revenue from selling burgers at a price of $\$(2 + x)$ is

$$R = \text{revenue} = \text{price per burger} \times \text{quantity} = (2 + x)(200 - 10x) = \boxed{\$(400 + 180x - 10x^2)}.$$

(c) We maximize $R = 400 + 180x - 10x^2$ by setting $R'(x) = 0$ and solving for x. Since $R'(x) = 180 - 20x$, we get that $R'(x) = 0$ when $x = 9$. That is, the price per burger should be $\$(2+x) = \$(2+9) = \boxed{\$11}$.

(d) When burgers are sold for \$11 (that is, when $x = 9$), Ermila will sell $200 - 10x = 200 - 10(9) = \boxed{110 \text{ burgers}}$.