Math 360
Section 1.2 Exercises

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1. The three things we must check in order to confirm that a function $\phi : S \to S'$ is an isomorphism are the following:
   - ϕ is one-to-one
   - ϕ is onto
   - For all $a, b \in S$, we have $\phi(a * b) = \phi(a) *' \phi(b)$.

Determine whether the given map ϕ is an isomorphism of the first binary structure with the second. If not, why not?

2. $\langle \mathbb{Z}, + \rangle$ with $\langle \mathbb{Z}, + \rangle$ where $\phi(n) = -n$ for $n \in \mathbb{Z}$

   Answer: Yes, because ϕ is a bijection, and for all $a, b \in \mathbb{Z}$, $-(a + b) = -(a) + -(b)$.

3. $\langle \mathbb{Z}, + \rangle$ with $\langle \mathbb{Z}, + \rangle$ where $\phi(n) = 2n$ for $n \in \mathbb{Z}$

   Answer: No, because ϕ is not onto; there is no $n \in \mathbb{Z}$ such that $2n = 3$, for example.

5. $\langle \mathbb{Q}, \cdot \rangle$ with $\langle \mathbb{Q}, \cdot \rangle$ where $\phi(x) = \frac{x}{2}$ for $x \in \mathbb{Q}$

   Answer: No, because for any nonzero $a, b \in \mathbb{Q}$, we have $\frac{a \cdot b}{2} \neq \frac{a}{2} \cdot \frac{b}{2}$

7. $\langle \mathbb{R}, \cdot \rangle$ with $\langle \mathbb{R}, \cdot \rangle$ where $\phi(x) = x^3$ for $x \in \mathbb{R}$.

   Answer: Yes. It is clear from its graph that ϕ is a bijection. Also, for all $a, b \in \mathbb{R}$, we have $(a \cdot b)^3 = a^3 \cdot b^3$.

8. $\langle M_2(\mathbb{R}), \cdot \rangle$ with $\langle \mathbb{R}, \cdot \rangle$ where $\phi(A)$ is the determinant of matrix A.

   Answer: No, since ϕ is not one-to-one. To see this, observe that for $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$, we have $A \neq B$ but $\phi(A) = 0 = \phi(B)$.

9. $\langle M_1(\mathbb{R}), \cdot \rangle$ with $\langle \mathbb{R}, \cdot \rangle$ where $\phi(A)$ is the determinant of matrix A.

   Answer: Yes. Since any $A \in M_1(\mathbb{R})$ is a matrix of the form $A = [a]$ where $a \in \mathbb{R}$, then $\det(A) = a$. So clearly, ϕ is a bijection. Now we apply the definition of matrix multipliaction. For any $A, B \in M_1(\mathbb{R})$, we have

   $$\det(AB) = \det([a][b]) = \det[ab] = a \cdot b = \det([a]) \cdot \det([b]) = \det(A) \cdot \det(B)$$

   and we are done.

Same instructions. Let $F = \{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 0, f \in C^\infty\}$.

11. $\langle F, + \rangle$ with $\langle F, + \rangle$ where $\phi(f) = f'$.

    Answer: Yes, though verifying this does require a bit of thought. For any $f, g \in F$, we have that $(f + g)' = f' + g'$ by linearity. We can also see that ϕ is onto, since every element of F is smooth and therefore integrable. Thus, for any $g \in F$, we have that $f(x) = \int_0^x g(t)dt$ is an element of F such that $\phi(f) = g$. Now to see that ϕ is one-to-one, we point out that $f(x) = \int_0^x g(t)dt$ is the *only* element of F which maps to g under ϕ. Let $h : \mathbb{R} \to \mathbb{R}$ be a function such that $\phi(h) = g$. Then, since f and h have the same derivative, they differ only by a constant. Thus, if $h(0) = 0$, then $h = f$; and otherwise, $h \notin F$; and we are done.

12. $\langle F, +\rangle$ with $\langle \mathbb{R}, +\rangle$ where $\phi(f) = f'(0)$.

   Answer: Let $f, g \in F$. We can see that ϕ commutes with the operations, since

   $$\phi(f+g) = (f+g)'(0) = f'(0) + g'(0) = \phi(f) + \phi(g).$$

   However, ϕ is not one-to-one, since given some $x \in \mathbb{R}$, there are many functions in F whose derivative at zero is x. For example, consider $x = 0$. The identically zero function and x^2 are both in F, and $\phi(0) = \phi(x^2) = 0$.

16. The map $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(n) = n + 1$ is a bijection. Give the definition of a binary operation $*$ on $\mathbb{Z}$ such that ϕ is an isomorphism of

   a. $\langle \mathbb{Z}, +\rangle$ with $\langle \mathbb{Z}, *\rangle$
   b. $\langle \mathbb{Z}, *\rangle$ with $\langle \mathbb{Z}, +\rangle$

   In each case, give the identity for $*$ on $\mathbb{Z}$.

   Answer: (a) Let $* : (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z}$ be defined as

   $$a * b = a + b - 1$$

   for all $a, b \in \mathbb{Z}$. To see that ϕ commutes with $+$ and $*$, let $n, m \in \mathbb{Z}$. Now,

   $$\phi(n+m) = n + m + 1 = (n+1) + (m+1) - 1 = \phi(n) * \phi(m).$$

   Note that the identity of $*$ is 1, since $n * 1 = 1 * n = n + 1 - 1 = n$.

   Answer: (b) Let $* : (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{Z}$ be defined as

   $$a * b = a + b + 1$$

   for all $a, b \in \mathbb{Z}$. To see that ϕ commutes with $*$ and $+$, let $n, m \in \mathbb{Z}$. Now,

   $$\phi(n * m) = (n * m) + 1 = (n + m + 1) + 1 = (n+1) + (m+1) = \phi(n) + \phi(m).$$

   Note that the identity of $*$ is -1, since $n * (-1) = (-1) * n = n + (-1) + 1 = n$.

20. The displayed condition for an isomorphism ϕ in Definition 1.2.7 is sometimes summarized by saying "ϕ must commute with the binary operation(s)".[1] Explain how that condition can be viewed in this manner.

   Answer: If we think of $*$ and $*'$ as functions and use function notation (as opposed to binary operation notation), this commutative relationship is more clear. Suppose $\langle A, *\rangle$ and $\langle B, *'\rangle$ are two isomorphic binary structures, and $\phi : A \to B$ is an isomorphism relating them. We would usually write that for all $a_1, a_2 \in A$, $\phi(a_1 * a_2) = \phi(a_1) *' \phi(a_2)$. However using function notation, we see the two functions commuting:

   $$\phi(*(a_1, a_2)) = *'(\phi(a_1), \phi(a_2))$$

23. An identity for a binary operation $*$ as described by Definition 1.2.12 is sometimes referred to as "a two-sided identity." Give analogous definitions for

   a. a *left identity* e_L for $*$, and **b.** a *right identity* e_R for $*$.

   Answer: Let $* : (S \times S) \to S$ be binary operation on a set S. A *left identity* for $*$ is some $e_L \in S$ such that for any $x \in S$, we have $e_L * x = x$. Similarly, a *right identity* for $*$ is some $e_R \in S$ such that for any $x \in S$, we have $x * e_R = x$. $\square$

   (Problem continued) Theorem 1.2.13 shows that if a two-sided identity for $*$ exists, then it is unique. Is the same true for a one-sided identity you just defined? Prove or give a counterexample $\langle S, *\rangle$ for a finite set S and find the first place where the proof of Theorem 1.2.13 breaks down.

[1] I have been saying this in answers to previous problems, having already noticed this question.

Answer: The proof of Theorem 1.2.13 uses one identity as a left identity, and the other as a right identity. We can't do this here, and given two (WLOG) left identities e and e', we have no reason to believe that $e * e' = e' * e$, so while $e * e' = e'$ and $e' * e = e$, we cannot conclude that $e = e'$. Following is a table that give a counterexample:

$*$	e	e'	a	b
e	e	e'	a	b
e'	e	e'	a	b
a	e'	a	b	e
b	a	b	e	e'

Note that $a * e = e'$ while $a * e' = a$, so since we cannot violate the substitution property of equality, we can see that $e \neq e'$. ■

24. Can a binary structure have a left identity and a right identity which are distinct from each other?

    Answer: This is impossible. If there exist left and right identities e and e' respectively, we can apply the proof of Theorem 1.2.13: $e * e' = e$ (right identity), and $e * e' = e'$ (left identity). Thus $e = e'$ and there is one identity which is two-sided. ■

25. Prove that if $\phi : S \to S'$ is an isomorphism of $\langle S, * \rangle$ with $\langle S', *' \rangle$, then the inverse ϕ^{-1} is an isomorphism of $\langle S', *' \rangle$ with $\langle S, * \rangle$.

    Proof Since ϕ is a bijection, then so is ϕ^{-1}, so all that remains is to show that ϕ^{-1} commutes with $*'$ and $*$. For any $a', b' \in S'$, there exists $a, b \in S$ such that $\phi(a) = a'$ and $\phi(b) = b'$. Since ϕ is an isomorphism, we know that

$$\phi(a * b) = \phi(a) *' \phi(b) = a' *' b'.$$

Now,

$$\begin{aligned} \phi^{-1}(a' *' b') &= \phi^{-1}(\phi(a * b)) \\ &= a * b \\ &= \phi^{-1}(\phi(a)) * \phi^{-1}(\phi(b)) \\ &= \phi^{-1}(a') * \phi^{-1}(b') \end{aligned}$$

and we are done. ■

For 28 through 31, prove that the indicated property of the binary structure $\langle S, * \rangle$ is indeed a structural property.

28. The operation $*$ is commutative.

    Proof To prove that commutativity is a structural property of $\langle S, * \rangle$, we will show that any binary structure which is isometric to $\langle S, * \rangle$ must also have that property. Let ϕ be an isomorphism of $\langle S, * \rangle$ with $\langle S', *' \rangle$, where $\langle S', *' \rangle$ is another structure. Since $*$ is commutative, we know that for all $a, b \in S$, $a * b = b * a$. Now we show that $*'$ is commutative as well. Let $c', d' \in S'$ be given. Then there exist unique $c, d \in S$ such that $\phi(c) = c', \phi(d) = d'$. Now,

$$\begin{aligned} c' *' d' &= \phi(c) *' \phi(d) \\ &= \phi(c * d) \\ &= \phi(d * c) \\ &= \phi(d) *' \phi(c) \\ &= d' *' c' \end{aligned}$$

and we are done. ■

29. The operation $*$ is associative.

Proof Suppose $*$ is asociative, and define $a, b, c \in S$ and $*$ and $a', b', c' \in S'$ and ϕ similarly as above. Then,

$$\begin{aligned}(a' *' b') *' c' &= (\phi(a) *' \phi(b)) *' c' \\ &= \phi(a * b) *' \phi(c) \\ &= \phi([a * b] * c) \\ &= \phi(a * [b * c]) \\ &= \phi(a) *' \phi(b * c) \\ &= a' *' (\phi(b) *' \phi(c)) \\ &= a' *' (b' *' c')\end{aligned}$$

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31. There exists an element $b \in S$ such that $b * b = b$.

Proof Using notation as expected; if $b * b = b$, then $\phi(b * b) = \phi(b)$, so $\phi(b) *' \phi(b) = \phi(b)$, thus $b' *' b' = b'$. Therefore we have found an element $b' \in S'$ with the desired property, so we are done. ■

32. Let H be the subset of $M_2(\mathbb{R})$ consisting of all matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{R}$.

   a. Show that $\langle \mathbb{C}, + \rangle$ is isomorphic to $\langle H, + \rangle$.

   b. Show that $\langle \mathbb{C}, \cdot \rangle$ is isomorphic to $\langle H, \cdot \rangle$.

(We say that H is a *matrix representation* of the complex numbers $\mathbb{C}$.)

Proof Let $\phi : \mathbb{C} \to H$ be defined as $\phi(a + bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. It should be clear that ϕ is one-to-one and onto, since this definition holds for all real numbers a and b. Now we show that ϕ commutes with the operations in parts (a) and (b).

a.

$$\begin{aligned}\phi\big((a + bi) + (c + di)\big) &= \phi\big((a + c) + (b + d)i\big) \\ &= \begin{bmatrix} (a + c) & -(b + d) \\ (b + d) & (a + c) \end{bmatrix} \\ &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\ &= \phi(a + bi) + \phi(c + di)\end{aligned}$$

□

b.

$$\begin{aligned}\phi\big((a + bi)(c + di)\big) &= \phi\big((ac - bd) + (ad + bc)i\big) \\ &= \begin{bmatrix} (ac - bd) & -(ad + bc) \\ (ad + bc) & (ac - bd) \end{bmatrix} \\ &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\ &= \phi(a + bi) \cdot \phi(c + di)\end{aligned}$$

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