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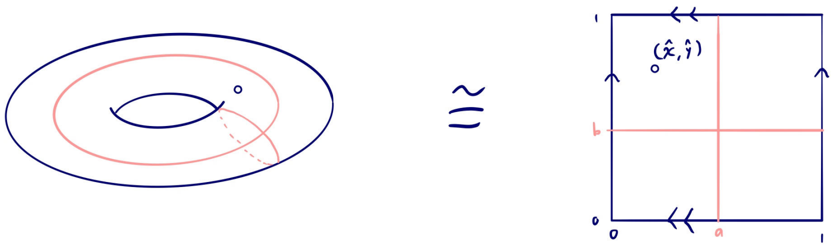
Homework 1

1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

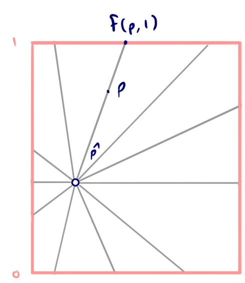
Answer: Let T be the torus given by $\mathbb{R}^2/\mathbb{Z}^2$ with one point $\hat{p} = (\hat{x}, \hat{y})$ removed, and let two longitude and meridian circles be the lines $y = b$ and $x = a$, respectively. Denote the union of these two circles C.



We can choose our coordinate system however we like, so without loss of generality suppose $a = b = 0$. Then let $F : T \times [0, 1] \to T$ be given by

$$F(p,t) = \big(1-t\big)p + t\big(\hat{p} + \lambda(p - \hat{p})\big),$$

where λ is the scalar such that $F(p, 1)$ lies on C:

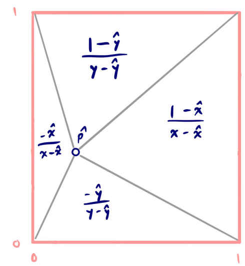


We can find λ explicitly as a function of $p = (x, y)$ by constraining λ to be the minimum positive defined quantity from the following list: $\left(\frac{1-\hat{x}}{x-\hat{x}}, \frac{1-\hat{y}}{y-\hat{y}}, \frac{-\hat{x}}{x-\hat{x}}, \frac{-\hat{y}}{y-\hat{y}}\right)$.

Then

- $F(p, 0) = p$,
- $F(p, 1) \in C$ for all $p \in T$,
- for all $p \in C$ one can check that $\lambda = 1$, so $F(p,t) = p$ for all time t,
- F is continuous.

This last point is perhaps not trivial, since we must show that λ is a continuous function of p. To see this, partition the torus into 4 regions by connecting the corners to $\hat{p}$:



Each region has a corresponding λ-value from the above list, and they agree on their boundaries (that is, for any p on a boundary line, the expressions on each side of the line give the same values for λ). In the interior of each region, λ is determined by its equation for that region, which is continuous. Thus $\lambda(p)$ is continuous, and so is F. ■

2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

Answer: Let $F(p,t) = p(1-t) + \frac{p}{||p||}(t)$. Then

- $F(p, 0) = p$,
- $F(p, 1) \in S^{n-1}$ for all $p \in T$, since $\frac{p}{||p||}$ has norm 1,
- for all $p \in S^{n-1}$, we have that $F(p,t) = p(1-t) + p(t) = p$, so F fixes every point in S^{n-1},
- F is continuous since $p \neq 0$ so the denominator in F never vanishes.

■

3. (a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

Proof By assumption, there exist homotopy equivalence maps $f, \bar{f}, g, \bar{g}$ such that $\bar{f}f \simeq \mathbb{1}_X$ and $\bar{g}g \simeq \mathbb{1}_Y$. This means that $gf : X \to Z$ is a homotopy equivalence map, since

$$gf\bar{f}\bar{g} \simeq g\mathbb{1}_Y\bar{g} = g\bar{g} \simeq \mathbb{1}_Z$$

and

$$\bar{f}\bar{g}gf \simeq f\mathbb{1}_Y\bar{f} = f\bar{f} \simeq \mathbb{1}_X.$$

thus

- $X \simeq X$, since $\mathbb{1}_X$ is a homotopy equivalence map,
- $X \simeq Y \implies Y \simeq X$ by definition of a homotopy equivalence map, and
- $X \simeq Y$ and $Y \simeq Z \implies X \simeq Z$ as shown above. $\square$

(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

Proof Let $f, g, h : X \to Y$ be maps.

- $f \simeq f$ since $F(x,t) = f(x)\ \forall t$ is a homotopy.
- If $f \simeq g$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$, then $F(x,(1-t))$ is a homotopy so $g \simeq f$.
- If ϕ_1 is a homotopy relating f and g and ϕ_2 is a homotopy relating g and h, then

$$F(x,t) = \begin{cases} \phi_1(x, 2t) & t \in [0, \frac{1}{2}] \\ \phi_2(x, 2(t - \frac{1}{2})) & t \in (\frac{1}{2}, 1] \end{cases}$$

is a homotopy relating f and h. $\square$

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Proof Let $f, g, \bar{g} : X \to Y$ such that $g \circ \bar{g} \simeq \mathbb{1}$ and $f \simeq g$ with $\begin{matrix} F(x,0) = f(x) \\ F(x,1) = g(x) \end{matrix}$.

Then considering the map $F(\bar{g}(x), t)$[†], we find that $\begin{matrix} F(\bar{g}(x),0) = f \circ \bar{g}(x) \\ F(\bar{g}(x),1) = g \circ \bar{g}(x) \end{matrix}$ so $f\bar{g} \simeq g\bar{g} \simeq \mathbb{1}$, and similarly $\begin{matrix} \bar{g}(F(x,0)) = \bar{g} \circ f(x) \\ \bar{g}(F(x,1)) = \bar{g} \circ g(x) \end{matrix}$ so $\bar{g}f \simeq \bar{g}g \simeq \mathbb{1}$. ■

[†]We know that compositions and restrictions of maps are maps, so we won't get bogged down in mentioning that detail in this or further proofs.

Definition. A *deformation retraction in the weak sense* of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t.

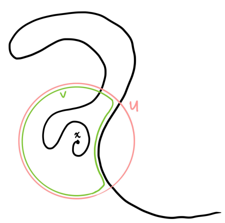
4. Show that if X deformation retracts to A in the weak sense, then the inclusion $\iota : A \hookrightarrow X$ is a homotopy equivalence.

Proof Denote $f_1(x)$ by $g(x)$, a map $X \to A$. Then $\iota \circ g = g \simeq \mathbb{1}_X$ by assumption, and since $g \circ \iota = g|_A$ and $f_t|_A$ is a homotopy between $g|_A$ and $\mathbb{1}_X|_A = \mathbb{1}_A$, then $g \circ \iota = g|_A \simeq \mathbb{1}_A$. ■

5. Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.

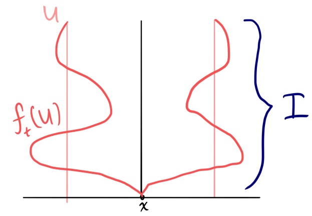
Proof

Let $F : X \times I \to X$ be the above deformation retraction. We want to produce an open set of points which are in U, and which remain in U for all time t.

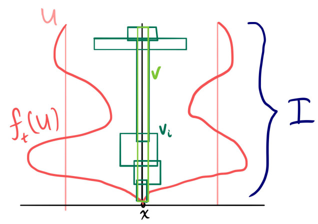


We should exclude the points in U which leave U as $t \to 1$.

Thus we take $(U \times I) \cap F^{-1}(U)$, which gives us the set of all (x, t) such that the point begins in U, and is in U at some time t.



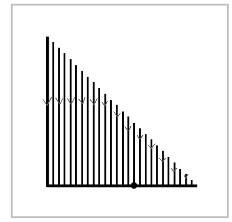
Now $(U \times I) \cap F^{-1}(U)$ is open in the product topology on $X \times I$, so for every point $(x, t) \in \{x\} \times I$, we choose an open rectangle $V_t \subset \{x\} \times I$ containing (x, t), and observe that $\{V_t\}_{t \in I}$ is an open cover of $\{x\} \times I$, a compact space. Thus we can obtain a finite subcover $\{V_i\}$.



(In this figure V should be labeled $V \times I$, and U should be $U \times I$.)

Let $V = \bigcap_i \text{proj}_X(V_i)$. Then for any $p \in V$, $(p, t) \in \bigcup_i V_i \subset (U \times I) \cap F^{-1}(U)$ for all t, so that point remains in V for all time. To finish the proof, observe that $F|_V$ is a homotopy between the inclusion map $V \hookrightarrow U$ and the constant map x. ■

6. (a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point. [See the preceding problem.]



Proof We can deformation retract all the tines down to x-axis, then retract in to the desired point, as shown in the figures:

This fails however if we try to retract to a point on one of the tines, because there are other tines arbitrarily close to the one in question, and they cannot be retracted continuously. In general, no deformation retraction to a point on the tines can exist, because every neighborhood of such a point contains disconnected pieces of other tines, and the identity map on such a neighborhood cannot be nullhomotopic. ■

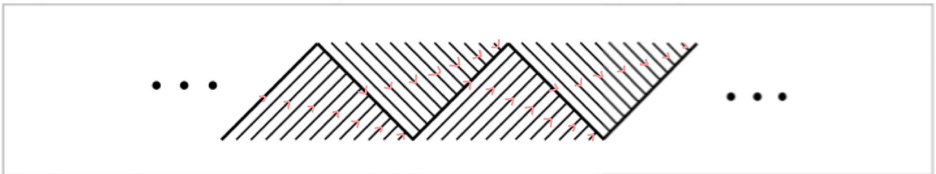
(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.

(c) Let Z be the zigzag subspace of Y homeomorphic to $\mathbb{R}$ indicated by the heavier line. Show there is a deformation retraction in the weak sense (see Exercise 4) of Y onto Z , but no true deformation retraction.

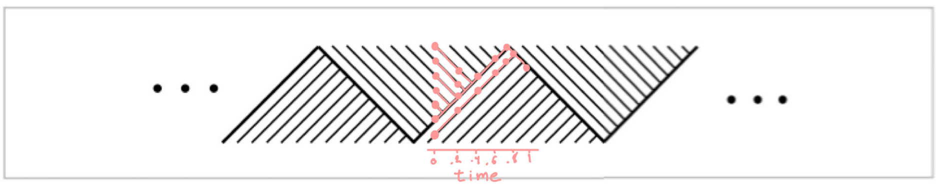
Proof To see that Y does not deformation retract onto any point (b), note that it cannot retract to a point on the tines for the same reason as in part (a), and every segment of Y is a tine for some part of the figure.

To see that Y is contractible (b), we will produce a deformation retraction in the weak sense of Y onto Z (c), and then observe that Z is homeomorphic to $\mathbb{R}$ which is homotopy equivalent to a point.

Each point has an obvious way to move continuously along Y in a rightwards direction; moving down the tines until it is on Z, and then zig-zagging on Z forever.



Moving all the points at a constant speed keeps all points moving rightwards together, so that only their vertical spacing changes:



and after moving for 1 second, every point lies on Z. Thus the homotopy described above is such that $F(x, 0) = x$, $F(x, 1) \in Z$, and $f(z, t) \in Z$ for all $z \in Z$.

To complete the proof, we can choose any point p in X and contract Z in the same way we would $\mathbb{R}$, all in to the point p. The composition of these two homotopies gives a homotopy between $\mathbb{1}_Y$ and the constant map p. ■

Collaborators:

1. Zach Wagner, Nick Bragman, Kyle Hansen

2.

3.

4.

5. Leslie Mavrakis

6. Zach Wagner, Christian Hong (not in this class, but helped me find a key mistake).