

# Welcome To Math 34A!

## Differential Calculus

### Instructor:

Trevor Klar, [trevorklar@math.ucsb.edu](mailto:trevorklar@math.ucsb.edu)

South Hall 6431X (Grad Tower, 6th floor, blue side, first door on the right)

### Office Hours:

MTWR after class 2:00-3:00, and by appointment. Details on Gauchospace.

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# Summary of Logs

$\log(y)$  is how many tens you multiply together to get  $y$ .

$$10^{\log(y)} = y$$

$$\log(10^a) = a$$

$$10^a \times 10^b = 10^{a+b}$$

$$\log(x \times y) = \log(x) + \log(y)$$

$$(10^a)^p = 10^{ap}$$

$$\log(a^p) = p \log(a)$$

Each of these pairs of equalities says one thing!

# General Compound Interest

If the interest rate is  $r\%$ , then each year money multiplies by

$$m = 1 + \frac{r}{100}.$$

If you start with an initial amount  $A$  of money then after  $t$  years you have

$$A \times m^t = A \times \left(1 + \frac{r}{100}\right)^t$$

**5.** If you invest \$1000 at 14% interest, how much will you have 5 years later? (Guess!)

$A \approx \$700$     $B \approx \$1400$     $C \approx \$1500$     $D \approx \$1700$     $E \approx \$2000$  E

After 5 years, you have

$$\$1,000 \times \left(1 + \frac{14}{100}\right)^5 = \$1,000 \times (1.14)^5.$$

How much is this? **Smart way: 14% in 1 year  $\approx$  7% per year for 2.**

## §7.9: Population Growth

Assume each generation of bunnies has 3 times as many bunnies as previous one. Initially have 100 bunnies. How many bunnies after  $n$  generations?

$$\begin{array}{lll} A = 100 \times 3n & B = 100 + 3n & C = 100(1 + 3n) \\ D = 100^{3n} & E = 100 \times 3^n & \end{array}$$

Answer: E

Start with 100

After 1 generation have  $100 \times 3$  bunnies

After 2 generations have  $100 \times 3 \times 3$  bunnies

After 3 generations have  $100 \times 3 \times 3 \times 3$  bunnies

So...after  $n$  generations have

$$100 \times \underbrace{3 \times 3 \times \cdots \times 3}_{n \text{ times}} = 100 \times 3^n \text{ bunnies.}$$

# More Bunnies

We saw that:

- if we start with 100 bunnies, and
- the bunny population triples every generation,

then we have  $100 \times 3^n$  bunnies after  $n$  generations.

**1.** How many generations until there are  $10^7 = 10,000,000$  bunnies?

$$A = \log(5/3) \quad B = 5 - \log(3) \quad C = 5 / \log(3)$$

$$D = 5/3 \quad E = 10^5/3$$

$$A \approx 0.22 \quad B \approx 4.52 \quad C \approx 10.48$$

$$D \approx 1.67 \quad E \approx 3,333 \quad \boxed{C}$$

# Flu Outbreak

- 2.** At the start of an outbreak of H1N1 flu in a large class of students, there were 5 infected individuals. The numbers doubles every 3 days. How many days until there are 80 infected students?

$$A = \log(16)/\log(2) \quad B = \log(16/2) \quad C = 16/\log(2)$$

$$D = 3 \log(16)/\log(2) \quad E = \log(48/2) \quad \boxed{D}$$

# Doubling Time Formula

Suppose something doubles every  $K$  minutes\*. If there is a mass of  $A$  at time  $t = 0$ , how much is there at time  $t$  minutes?

$$\text{mass after } t \text{ minutes} = A \times 2^{(t/K)}$$

Idea:  $t/K$  is number of doubling periods in  $t$  minutes.

- 3.** A disease spreads through a community. On March 1 there were 100 infected people. The number of people doubles in a 3 days. How many infected people are there  $t$  days after March 1?

$$A = 2^t \quad B = 3 \times 2^{t/100} \quad C = 100 \times 2^t \quad D = 100 \times 2^{t/3} \quad \boxed{D}$$

How many days until there are 1,000 infected people?

$$A = \log(10)/\log(2) \quad B = 3 \log(10)/\log(2) \quad C = 3 \log(5) \\ D = 3(\log(10) - \log(2)) \quad E = 3 \log(20) \quad \boxed{B}$$

\*Any time unit will work, not just minutes. Just be consistent!

# A More Complicated Example

$$\text{mass after } t \text{ minutes} = A \times 2^{(t/K)}$$

where

- $K$  is the doubling time, and
- $t/K$  is the number of doubling periods in  $t$  minutes.

4. A colony of mold is growing on a cheeseburger in the back of a dorm refrigerator. When discovered it has a mass of 10 mg. One week later it was found to have a mass of 30 mg. What is the doubling time measured in days?

$$A = \log(2)/\log(3) \quad B = 7 \log(2)/\log(3) \quad C = 7 \log(2/3) \\ D = 7 \log(3/2)$$

**Hint:** We know  $A$  and the mass  $t$  days after discovery (for some  $t$ ).

Solving  $30 = 10 \times 2^{7/K}$  gives B



# More Bunnies

We saw that:

- if we start with 100 bunnies, and
- the bunny population triples every generation,

then we have  $100 \times 3^n$  bunnies after  $n$  generations.

**1.** How many generations until there are  $10^7 = 10,000,000$  bunnies?

$$\begin{array}{lll} A = \log(5/3) & B = 5 - \log(3) & C = 5 / \log(3) \\ D = 5/3 & E = 10^5/3 & \end{array}$$

$$\begin{array}{lll} A \approx 0.22 & B \approx 4.52 & C \approx 10.48 \\ D \approx 1.67 & E \approx 3,333 & \boxed{C} \end{array}$$

# Flu Outbreak

2. At the start of an outbreak of H1N1 flu in a large class of students, there were 5 infected individuals. The numbers doubles every 3 days. How many days until there are 80 infected students?

$$A = \log(16)/\log(2) \quad B = \log(16/2) \quad C = 16/\log(2)$$

$$D = 3 \log(16)/\log(2) \quad E = \log(48/2) \quad \boxed{D}$$

# Doubling Time Formula

Suppose something doubles every  $K$  minutes<sup>†</sup>. If there is a mass of  $A$  at time  $t = 0$ , how much is there at time  $t$  minutes?

$$\text{mass after } t \text{ minutes} = A \times 2^{(t/K)}$$

Idea:  $t/K$  is number of doubling periods in  $t$  minutes.

- 3.** A disease spreads through a community. On March 1 there were 100 infected people. The number of people doubles in a 3 days. How many infected people are there  $t$  days after March 1?

$$A = 2^t \quad B = 3 \times 2^{t/100} \quad C = 100 \times 2^t \quad D = 100 \times 2^{t/3} \quad \boxed{D}$$

How many days until there are 1,000 infected people?

$$A = \log(10)/\log(2) \quad B = 3 \log(10)/\log(2) \quad C = 3 \log(5) \\ D = 3(\log(10) - \log(2)) \quad E = 3 \log(20) \quad \boxed{B}$$

<sup>†</sup>Any time unit will work, not just minutes. Just be consistent!

# A More Complicated Example

$$\text{mass after } t \text{ minutes} = A \times 2^{(t/K)}$$

where

- $K$  is the doubling time, and
- $t/K$  is the number of doubling periods in  $t$  minutes.

4. A colony of mold is growing on a cheeseburger in the back of a dorm refrigerator. When discovered it has a mass of 10 mg. One week later it was found to have a mass of 30 mg. What is the doubling time measured in days?

$$A = \log(2)/\log(3) \quad B = 7 \log(2)/\log(3) \quad C = 7 \log(2/3) \\ D = 7 \log(3/2)$$

**Hint:** We know  $A$  and the mass  $t$  days after discovery (for some  $t$ ).

Solving  $30 = 10 \times 2^{7/K}$  gives B

## §7.11: Half-Life, Doubling Time

The half-life of a radioactive isotope is the time it takes for **half** of the isotope to decay.

**Example:** Isotope W has a half-life of 10 years. How much remains after 20 years? **None?**

$$\frac{1}{2} \times \frac{1}{2} \times (\text{amount you start with})$$

**Idea:** In half-life problems, convert time into **half-lives**.

In this problem, the half-life is 10 years. Therefore, 20 years is two half-lives.

In general: After  $n$  half-lives,

$$\text{remaining amount} = \left(\frac{1}{2}\right)^n \times (\text{amount started with})$$

- 5.** Start with 120 grams of an isotope with a half-life of 12 years. How many grams remains after 36 years?

A= 0    B= 10    C= 15    D= 20    E= 40    C

# Another Example

**In general:** After  $n$  half-lives,

$$\text{remaining amount} = \left(\frac{1}{2}\right)^n \times (\text{amount started with})$$

**6.** An isotope has a half-life of **5 years**.

(a) If we start with **70 grams**, how many grams will be left after  $t$  years?

$$\begin{aligned} A &= 70 \left(\frac{1}{2}\right)^t & B &= 5 \left(\frac{1}{2}\right)^{70t} & C &= 70 \left(\frac{1}{2}\right)^{5t} \\ D &= 70 \left(\frac{1}{2}\right)^{t/5} & E &= 0 & \boxed{D} \end{aligned}$$

(b) How many years until **10** grams remain?

$$\begin{aligned} A &= 5(\log(7) - \log(2)) & B &= \log(7)/\log(2) & C &= 5 \log(7/2) \\ D &= 5 \log(7)/\log(2) & E &= \log(7)/(5 \log(2)) & \boxed{D} \end{aligned}$$

# Half-Life Formula

Suppose something has a half-life of  $K$  years<sup>‡</sup>. If there is a mass of  $A$  at time  $t = 0$ , how much is there at time  $t$  years?

$$\text{mass after } t \text{ years} = A \times \left(\frac{1}{2}\right)^{(t/K)}$$

Idea:  $t/K$  is number of half-lives in  $t$  years.

7. (Radiocarbon Dating) A bone is found with 2% of the usual amount of carbon-14 in it. The half-life of carbon-14 is 5730 years. How old (in years) is the bone?

$$A = 5730 \log(.01) / \log(2) \quad B = 5730 \log(50) / \log(2)$$

$$C = 5730 \times 50 \quad D = \text{wicked old}$$

Answer: B  $\approx 32,000$  years

<sup>‡</sup>Any time unit will work, not just years. Just be consistent!

## §7.13: Logs in Other Bases

$\log(y)$  is how many tens you multiply together to get  $y$ .

$\log_2(y)$  is how many twos you multiply together to get  $y$ .

So  $2^3 = 8$  means the same thing as  $\log_2(8) = 3$

Examples:

$$\log_2(16) = 4$$

$$\text{because } 2^4 = 16$$

$$\log_2(32) = 5$$

$$\text{because } 2^5 = 32$$

$$\log_2(1/8) = -3$$

$$\text{because } 2^{-3} = 1/8$$

The five laws of logs work for any base  $b$  exactly the same way except...

$$b^{\log_b(y)} = y$$

$$\log_b(b^a) = a$$



# Summary & Examples

## Important bases:

- $\log_2$  is used extensively in computer science
- $\ln = \log_e$  is used everywhere (the **natural log**) ( $e \approx 2.718$ )  
 $\log_e(y) = x$  means  $e^x = y$   
 $\log_e(y)$  is how many  $e$ 's you multiply to get  $y$ .  
 Read as: “log base  $e$  of  $y$  equals  $x$ .”

## Examples:

$$\log_3(81) = \quad A=0 \quad B=1 \quad C=2 \quad D=3 \quad E=4 \quad \boxed{E}$$

$$\log_5(25) = \quad A=0 \quad B=1 \quad C=2 \quad D=3 \quad E=4 \quad \boxed{C}$$

$$\text{Simplify } \ln \left( (e^{3x} \times e^y)^2 \right)$$

$$A = 6x + y \quad B = 2x + 2y \quad C = 3x + 2y \quad D = 6x + 2y \quad E = 6xy \quad \boxed{D}$$

Teaser:  $e$  is special because the derivative of  $e^x$  is  $e^x$  whatever that means.

# Derivatives & Differential Calculus

...are about **how quickly things change**.

- Need to understand PRACTICAL significance in various situations

Spread of infectious disease, population growth, speed, acceleration, marginal rates in economics, global warming

- Calculate (or estimate) rate of change from various sources:

graph  
table of data  
formula

- Applications:

measure change  
predict the future  
**optimization** – find the best, or smallest, or biggest, or most...

This is all about *understanding* the world.

# Philosophical problem

How quickly is something changing at **one moment** in time?

Example: Does a ball **stop** when I throw it straight up?

Example: How fast is the temperature rising at 7am?

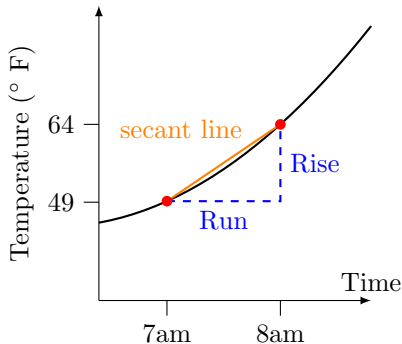
$$\left( \begin{array}{l} \text{change in temp} \\ \text{between 7am \& 8am} \end{array} \right)$$

$$= 64 - 49 = 15^\circ \text{ F}$$

$$\left( \begin{array}{l} \text{average rate of} \\ \text{change in temp} \\ \text{between 7am \& 8am} \end{array} \right)$$

$$= \frac{15^\circ \text{ F}}{1 \text{ hour}} = 15^\circ \text{ F/hour}$$

$$= \text{slope of secant line}$$



# Continuing Example

Similarly,

$$\left( \begin{array}{c} \text{average rate of} \\ \text{change in temp} \\ \text{between 6am \& 8am} \end{array} \right) = \frac{\text{change in temp}}{\text{time taken}}$$

**Question:** Suppose temperature at time  $t$  given by the formula  $f(t) = t^2$ . What is the average rate of change of temperature from 6am to 8am?

A= 1    B= 7    C= 9    D= 14    E= 28    D

# Average Rate of Change

Suppose temperature at time  $t$  given by the formula  $f(t) = t^2$ .

Using a calculator one can find the **average rate of change** over shorter and shorter time spans  $\Delta t$ , starting at 7am:

$\Delta t$	$(f(7 + \Delta t) - f(7))/\Delta t$	ave rate of change °F/hr
1	$(8^2 - 7^2)/1$	15
0.1	$(7.1^2 - 7^2)/0.1$	14.1
0.01	$(7.01^2 - 7^2)/0.01$	14.01
0.001	$(7.001^2 - 7^2)/0.001$	14.001
0.0001	$(7.0001^2 - 7^2)/0.0001$	14.0001
0.00001	$(7.00001^2 - 7^2)/0.00001$	14.00001
0	$(7^2 - 7^2)/0$	0/0      arghhhh

Table: Average rate of change over various time spans

What would you **guess** the **exact** instantaneous rate of change of temperature at precisely 7am is? Yes! 14. But how do we get this? Answer: it is a **limit**!

# Instantaneous Rate of Change

What does the limit

$$\lim_{\Delta t \rightarrow 0} \frac{f(7 + \Delta t) - f(7)}{\Delta t}$$

mean in practice?

Work out the average rate of change over a **very short** time interval.  
That is **very nearly** the correct answer.

The shorter the time interval you use, the more accurate you expect the answer to be.

To get the **exact** answer you would need to take a time interval of zero length.

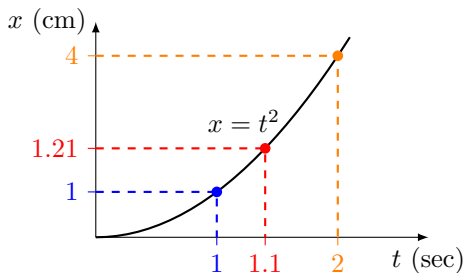
This leads to the nonsense **0/0**. So you can't do this.

That is the **philosophical** problem.

Mathematical solution: **take the limit**.

# An Example

A hamster runs along the  $x$ -axis, so that after  $t$  seconds the hamster is  $t^2$  cm from the origin. Our goal is to find the hamster's speed at time  $t = 1$  sec.



$$\left( \begin{array}{c} \text{average speed from} \\ t = 1 \text{ to } t = 2 \end{array} \right) = \frac{\text{distance gone}}{\text{time taken}} = \frac{2^2 - 1^2}{2 - 1} = 3 \text{ cm/sec}$$

$$\left( \begin{array}{c} \text{average speed from} \\ t = 1 \text{ to } t = 1.1 \end{array} \right) = \frac{\text{distance gone}}{\text{time taken}} = \frac{1.1^2 - 1^2}{1.1 - 1} = 2.1 \text{ cm/sec}$$

# Example Concluded

How do we work out the **exact** speed of the hamster after 1 second?

Plan:

- Find the **average speed** over a short time interval  $\Delta t$ , then
- Take the **limit** as  $\Delta t \rightarrow 0$ .

$$\begin{aligned}
 \left( \begin{array}{l} \text{average speed from} \\ t = 1 \text{ to } t = 1 + \Delta t \end{array} \right) &= \frac{\text{distance gone}}{\text{time taken}} \\
 &= \frac{(1 + \Delta t)^2 - 1^2}{(1 + \Delta t) - 1} \\
 &= \frac{(1 + 2\Delta t + (\Delta t)^2) - 1}{\Delta t} \\
 &= \frac{2\Delta t + (\Delta t)^2}{\Delta t} \\
 &= 2 + \Delta t
 \end{aligned}$$

The **limit** of this as  $\Delta t \rightarrow 0$  is 2.

Conclusion: At  $t = 1$  sec, the **exact** speed of the hamster is 2 cm/sec.



# Hamster Summary

Soon we will calculate that...

the **exact speed** of the hamster after  $t$  seconds is  $2t$  cm/sec.

Summary:

$f(t) = t^2$  = **distance** in cm of hamster from origin after  $t$  seconds  
(a function that gives the distance the hamster has traveled at time  $t$ )

$f'(t) = 2t$  = **speed** of hamster in cm/sec after  $t$  seconds  
(called the **derivative** of  $t^2$  because it can be **derived** or **obtained** from the function  $t^2$ )

**Question:** How many cm had the hamster run by the time its **speed** was 8 cm/sec?

A= 4    B= 8    C= 16    D= 32    E= 64    C

# Exact Hamster Speed

Now we calculate that...

the **exact speed** of the hamster after  $t$  seconds is  $2t$  cm/sec.

Do this as before: working out the **average speed** over a short time interval  $\Delta t$  and taking the **limit** as  $\Delta t \rightarrow 0$

$$\begin{aligned}
 \left( \begin{array}{c} \text{average speed from} \\ t \text{ to } t + \Delta t \end{array} \right) &= \frac{\text{distance gone}}{\text{time taken}} \\
 &= \frac{(t + \Delta t)^2 - t^2}{(t + \Delta t) - t} \\
 &= \frac{(t^2 + 2t\Delta t + (\Delta t)^2) - t^2}{\Delta t} \\
 &= \frac{2t\Delta t + (\Delta t)^2}{\Delta t} \\
 &= 2t + \Delta t
 \end{aligned}$$

The **limit** of this as  $\Delta t \rightarrow 0$  is  $2t$ .

# Hamster Questions!

After  $t$  seconds, the hamster is  $f(t) = t^2$  cm from origin.

(1) What is the **exact** speed (in cm/sec) of the hamster at  $t = 2$ ?

A = 1   B = 2   C = 4   D = 6   E = 8   ☐ C

(2) What is the **exact** speed (in cm/sec) of the hamster at  $t = 4$ ?

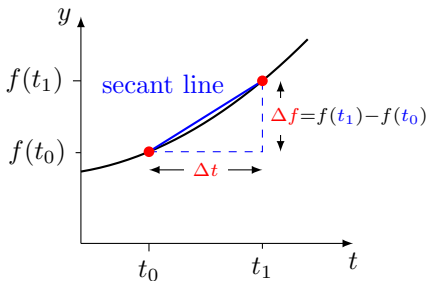
A = 1   B = 2   C = 4   D = 6   E = 8   ☐ E

(3) What is the **average** speed (in cm/sec) of the hamster from  $t = 2$  to  $t = 4$  seconds?

A = 1   B = 2   C = 4   D = 6   E = 8   ☐ D

Does this make sense?

# Graphical Approach



$\Delta f$  = change in  $f$

$\Delta t$  = change in  $t$

Many ways to say same thing:

$$\left( \begin{array}{c} \text{average rate of} \\ \text{change of } f \end{array} \right) = \frac{\text{change in } f}{\text{change in } t}$$

$$= \frac{\Delta f}{\Delta t}$$

$$= \text{slope of secant line} = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

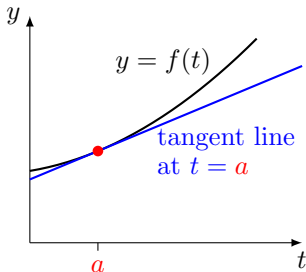
The derivative is defined to be

$$\lim_{\Delta t \rightarrow 0} \left( \frac{\Delta f}{\Delta t} \right) = \frac{df}{dt}$$

Idea: As  $t_1$  moves closer to  $t_0$  the secant line approaches the **tangent line** at  $t_0$ . This is the line with the **same slope** as the graph at  $t_0$ .

# Understanding Derivatives

There are many ways to **think** about derivatives. You **need** to understand these to apply to problems.



slope of **graph** at **a**  
 = slope of **tangent line**  
 = **instantaneous rate of change** of  $f$  at **a**

=  $\left( \begin{array}{l} \text{limit of average rate of change} \\ \text{of } f \text{ over shorter and shorter} \\ \text{time intervals starting at } a \end{array} \right)$

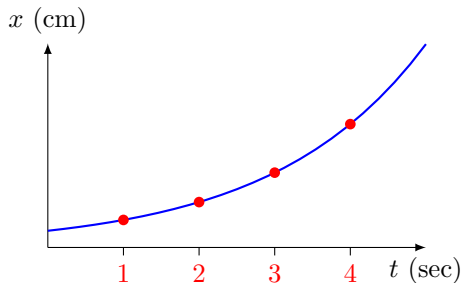
= limit of slopes of secant lines

$$= f'(a) = \left. \frac{df}{dt} \right|_{t=a}$$

# Summary

- How fast something changes = **rate of change**
- **Instantaneous rate of change** is the **limit** of the average rate of change over shorter and shorter time spans. This gets around the **0/0** problem.
- **speed** = rate of change of distance traveled.

# Speed=Slope=Derivative



The graph shows the distance from the origin in cm after  $t$  seconds of a hamster. Which of the numbers below is the largest?

**Hint:** Speed is a slope!

A = speed of the hamster at  $t = 1$

B = speed of the hamster at  $t = 2$

C = speed of the hamster at  $t = 3$

D = average speed of the hamster between  $t = 2$  and  $t = 3$

E = average speed of the hamster between  $t = 3$  and  $t = 4$

Answer: E

# Practical Meaning

Our goal is that you understand the **practical meaning** of the derivative in various situations.

$f(t)$  = temperature in  $^{\circ}$  F at  $t$  hours after midnight

$f(7)$  = 48 means the temperature at 7am was 48 $^{\circ}$  F

$f'(7)$  = 3 means at 7am the temperature was rising at a rate of 3 $^{\circ}$  F/hr

$f'(9)$  = -5 means at 9am the temperature was **falling** at a rate of 5 $^{\circ}$  F/hr  
or **rising** at a rate of -5 $^{\circ}$  F/hr

$g(t)$  = distance from origin in cm of hamster on  $x$ -axis after  $t$  seconds

$g(7)$  = 3 means after 7 seconds hamster was 3 cm from origin

$g'(9)$  = -5 means after 9 seconds our furry friend was running **towards**  
the origin at a speed of 5 cm/sec



# Another Context

Suppose  $f(t)$  = temperature of oven in  $^{\circ}\text{C}$  after  $t$  minutes.

What do  $f(3) = 20$  and  $f'(3) = 15$  mean?

- A After 20 minutes the oven was at  $3^{\circ}\text{C}$  and heating up at a rate of  $15^{\circ}\text{C/min}$
- B After 3 minutes oven temperature was  $15^{\circ}\text{C}$  and cooling down at a rate to  $20^{\circ}\text{C/min}$
- C The oven was heating up at rate of  $3^{\circ}\text{C/min}$  after 15 minutes and also after 20 minutes
- D After 3 minutes the oven was at  $20^{\circ}\text{C}$  and heating up at a rate of  $15^{\circ}\text{C/min}$
- E None of the above

Answer: D

That's it. Thanks for being here.

