Math 550
Homework 5
Dr. Fuller
Solutions

1. (a) $-xe^{xy}\, dx \wedge dy$
   (b) $x_2\, dx_1 \wedge dx_3 \wedge dx_4 + x_1\, dx_2 \wedge dx_3 \wedge dx_4$
   (c) $(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x})\, dx \wedge dy$
   (d) $(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z})\, dx \wedge dy \wedge dz$

2. (a) Yes. For instance, $\omega = d(\frac{x^2}{2} dy)$.
   (b) No, as ω is not closed.
   (c) Yes. For instance, $\omega = d(-yz\, dx - \frac{z^2}{2}\, dy)$.

3. (a) For all p, $\alpha(p)$ is a linear transformation $\mathbf{R}^3_p \to \mathbf{R}$. Since $\alpha(p) \neq 0$, we have $\dim\ \mathrm{im}\alpha(p) = 1$. The rank-nullity theorem implies $\dim\ \ker\alpha(p) = 2$.
   (b) $\ker\alpha_1(x,y,z)$ has (e_1,e_2) as a basis. The kernel at each point is simply the (xy)-plane.
   (c) $\ker\alpha_2(x,y,z)$ has $((1,0,0),(0,1,-x))$ as a basis. How to sketch these planes will be discussed in class.
   (d) $\alpha_1 \wedge d\alpha_1 = dz \wedge d(dz) = dz \wedge 0 = 0$. $\alpha_2 \wedge d\alpha_2 = dx \wedge dy \wedge dz \neq 0$.

4. Since ω is exact, there is $\eta \in \Omega^{k-1}(\mathbf{R}^n)$ with $d\eta = \omega$. Then

$$d(\eta \wedge \varphi) = d\eta \wedge \varphi + (-1)^{k-1}\eta \wedge d\varphi = \omega \wedge \varphi + (-1)^{k-1}\eta \wedge 0 = \omega \wedge \varphi.$$

5. Suppose $M = c(((-\pi/2, \pi/4))$, and let $g : (a,b) \to \mathbf{R}^2$ be a local parameterization with $(0,0) \in g((a,b))$. Consider the open set $V = M \cap B((0,0), \frac{1}{4})$ in M; it is connected, and so $g^{-1}(V)$ is an interval in (a,b). But then $V - \{(0,0)\}$ has three connected components, while $g^{-1}(V - \{(0,0)\}$ will have only two. This contradicts that g^{-1} is continuous.

6. (a) When $a \neq 0$, $f^{-1}(a)$ is a hyperboloid, which is a manifold. When $a = 0$, it is not a manifold. In this case, one may argue using connectedness as in the previous problem that there can be no local parameterization around $(0,0,0) \in f^{-1}(0)$.
   (b) When $a > 0$, $f^{-1}(a)$ is a hyperboloid of one sheet, which is connected, but when $a < 0$, $f^{-1}(a)$ is a hyperboloid of two sheets, which is not.

8. Suppose $g : U \subset \mathbf{R}^{n-1} \to \mathbf{R}^n$ is a parameterization with $g(U) = S^{n-1}$. Since S^{n-1} is compact, we have that $g^{-1}(S^{n-1})$ is a non-empty compact open subset of $\mathbf{R}^{n-1}$. Since $\mathbf{R}^{n-1}$ is connected, this is impossible.

   The argument shows that any compact k-dimensional manifold in $\mathbf{R}^n$ cannot be parameterized by a single parameterization.