MATH 221B
Winter 2020
FINAL

NAME ______________________ ___________
LAST FIRST

Directions: No human contact is allowed. Accessing books, notes, and electronic resources is permitted but all answers should be rewritten in your own words. No plagarism.
Due Thursday March 19, 2020 by 5pm (or earlier)

1. **Carefully define each of the following:**
   (a) **the fundamental group of based space**
   (b) **a based covering map** $p\colon (Y, y_0) \to (X, x_0)$
   (c) **isomorphic based covering maps.**
2. **Carefully prove that** $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.
3. **Use induced maps to prove that if** $f\colon (X, x_0) \to (Y, y_0)$ **is a homotopy equivalence then** $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ **is a group isomorphism.**
4. **Let** $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$.
   (a) **Find** $\pi_1(X)$.
   (b) **Find the universal cover of** X.
   (c) **Find all of its connected** 2**-sheeted covers.**
5. **Let** $X = \mathbb{RP}^2 \times \mathbb{RP}^2$.
   (a) **Find** $\pi_1(X)$.
   (b) **Find the universal cover of** X.
   (c) **Find all of its connected** 2**-sheeted covers.**

6. **Calculate the fundamental group of $S(K^2 \vee T^2)$ where S denotes suspension, K^2 is the klein bottle, and T^2 is the torus.**

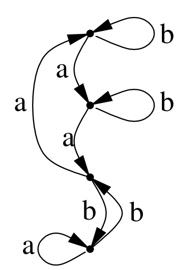
7. **Let $\mathbb{D}$ be the unit disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$, let $\mathbb{S}^1$ be the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$, and let X_n be the space obtained by attaching $\partial\mathbb{D}$ to $\mathbb{S}^1$ using the map $z \mapsto z^n$. In other words, the boundary of the disk is wrapped n times around $\mathbb{S}^1$.**

   (a) **Use the Seifert-van Kampen theorem to calculate $\pi_1(X_n)$.**

   (b) **Find all subgroups of $\pi_1(X_{18})$ and all of the corresponding connected based covering spaces. The covers will be hard to draw but you should be able to describe them fairly precisely.**

   (c) **Draw the subgroup lattice (i.e. illustrate all of the inclusion relations between subgroups in a diagram).**

8. **Consider the cover of $\mathbb{S}^1 \vee \mathbb{S}^1$ drawn below based at its bottom vertex. Is the image of its fundamental group in $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ a normal subgroup? Why or why not?**



9. **(Extra credit) Use the 2-dimensional Brouwer fixed point theorem to prove that every 3×3 matrix with positive real entries has a positive real eigenvalue. [Hint: focus on how the matrix acts on rays through the origin in $\mathbb{R}^3$, particularly those in that point into the first octant.]**