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Homework 2

Problem1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be increasing. Define the function $\nu\colon 2^{\mathbb{R}} \to [0,\infty]$ as follows:
1. For any open set $U = \cup_{i=1}^{\infty}(a_i, b_i)$ where (a_i, b_i) are disjoint, set

$$\nu(U) = \sum_{i=1}^{\infty}(f(b_i-) - f(a_i+)),$$

where

$$f(x+) = \lim_{y\to x+} f(y) \quad \text{and} \quad f(x-) = \lim_{y\to x-} f(y) \quad \text{for} \quad x \in \mathbb{R}$$

(the two limits obviously exist as f increases).
2. For any $A \subset \mathbb{R}$ define

$$\nu(A) = \inf\{\nu(U) \ : \ A \subset U, U - \mathit{open}\}.$$

Prove that ν is a measure on $\mathbb{R}$.

Proof

- (Monotonicity) Let $A \subseteq B$, and denote $\mathcal{O}_A$ the collection of all open sets containing A, and $\mathcal{O}_B$ the collection of all open sets containing B.

$$\begin{aligned} \nu(B) &= \inf\{\nu(U) : U \in \mathcal{O}_B\} \\ &\geq \inf\{\nu(U) : U \in \mathcal{O}_A\} \qquad \text{since } \mathcal{O}_B \subseteq \mathcal{O}_A, \\ &= \nu(A). \end{aligned}$$

- $\nu(\emptyset) = 0$[†], since

$$\begin{aligned} \lim_{n\to\infty} \nu\left(1, 1+\tfrac{1}{n}\right) &= \lim_{n\to\infty}\left(\lim_{-} f\left(1+\frac{1}{n}\right) - \lim_{+} f(1)\right) \\ &= \lim_{+} f(1) - \lim_{+} f(1) \\ &= 0 \end{aligned}$$

  Thus for any $\epsilon > 0$, there exists n such that $\nu\left(1, 1+\frac{1}{n}\right) < \epsilon$, and by monotonicity, $\nu(\emptyset) \leq \nu\left(1, 1+\frac{1}{n}\right) < \epsilon$.

[†] Note, we have to require that our definition of ν on open sets only applies to nonempty open sets, since otherwise $\nu(\emptyset) = \nu(a,a) = f(a-) - f(a+) \neq 0$ if a is a point of discontinuity.

- (Subadditivity for intervals)[†] Let $\{I_i\}$ be any countable collection of nonempty open intervals. Without loss of generality we can assume that f is continuous on each I_i, since we could always partition an interval by splitting at (at most countable many) discontinuities. Then

$$\sum_{i=1}^{\infty} \nu(I_i) = \sum_{i=1}^{\infty} f(b_i) - f(a_i)$$

and to consider the union $\bigcup_{i=1}^{\infty} I_i$, we just consider the list of all endpoints $\Gamma = \{(a_j, b_j)\}_{j=1}^{\infty}$, where we omit any endpoint which is contained in another interval. i.e. if $(a_1, b_1) \cap (a_2, b_2) \neq \emptyset$, then we omit b_1 and a_2 to obtain $\{(a_1, b_2)\}$. Thus

$$\begin{aligned}
\sum_{i=1}^{\infty} f(b_i) - f(a_i) &\geq \sum_{(a_j,b_j)\in\Gamma} f(b_j) - f(a_j) \\
&= \sum_{(a_j,b_j)\in\Gamma} \nu(a_j, b_j) \\
&= \nu\left(\bigcup_{i=1}^{\infty} I_i\right)
\end{aligned}$$

- (Subadditivity for open sets) Every open set U_i is a union of disjoint open intervals, so

$$\begin{aligned}
\nu \bigcup_{i\in\mathbb{N}} U_i &= \nu \bigcup_{i\in\mathbb{N}} \left(\bigcup_{j\in\mathbb{N}} I_{ij}\right) \\
&= \nu \bigcup_{i,j\in\mathbb{N}} I_{ij} \\
&\leq \sum_{i,j\in\mathbb{N}} \nu(I_{ij}) \\
&= \sum_{i\in\mathbb{N}} \left(\sum_{j\in\mathbb{N}} \nu(I_{ij})\right) \\
&= \sum_{i\in\mathbb{N}} \nu(U_i)
\end{aligned}$$

[†] In order for our definitions of $\nu(U)$ and $\nu(A)$ to agree when $U = A$ where U is an interval containing a point of jump discontinuity, we need to restrict our definition of $\nu(U)$ to nonempty open sets *on which f is continuous*. We can only have jump discontinuities, and only countably many, since f is increasing.

- (Subadditivity for general sets) Let $\{A_i\}$ be a countable collection of sets in $\mathbb{R}^n$, and for each i, denote $\mathcal{O}_{A_i}$ the collection of all open sets containing A_i.

$$\begin{aligned}
\sum_{i=1}^{\infty} \nu(A_i) &= \sum_{i=1}^{\infty} \inf\{\nu(U) : U \in \mathcal{O}_{A_i}\} \\
&= \inf\left\{\sum_{i=1}^{\infty} \nu(U_i) : U_i \in \mathcal{O}_{A_i} \quad \forall i \in \mathbb{N}\right\} \qquad (\dagger) \\
&\geq \inf\left\{\nu\left(\bigcup_{i=1}^{\infty} U_i\right) : U_i \in \mathcal{O}_{A_i} \quad \forall i \in \mathbb{N}\right\} \\
&= \nu\left(\bigcup_{i=1}^{\infty} A_i\right)
\end{aligned}$$

Thus we have shown that $\nu(\emptyset) = 0$, and that ν has the monotonicity and subadditivity properties, therefore it is a measure. ■

Problem2. Let m be Lebesgue measure on $\mathbb{R}$.
1. Construct an $m-$integrable function $f : \mathbb{R} \to [-\infty, \infty]$ for which there exists a set $A \subset \mathbb{R}$ such that $m(A) > 0$ and for any $x \in A$ the limit

$$\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)dy$$

exists but is different from $f(x)$.
2. Prove that in fact for any $\epsilon > 0$ one can reach $m(\mathbb{R} - A) < \epsilon$ in the first part.

Proof Let $\{q_i\}$ be an enumeration of the rationals, and let

$$A^{\complement} = \bigcup_{i=1}^{\infty} B_{2^{-i}}(q_i), \text{ with}$$

$$f(x) = \begin{cases} 0, & x \in A \\ \infty, & x \in A^{\complement} \end{cases}$$

Observe that $m(A) = \infty$, and $m(A^{\complement}) \leq 2$, but in fact we could scale the radii of the balls comprising A to make the sum arbitrarily small (thus if we prove (1), then (2) follows).

[†] In case this equality is not obvious, observe that for any ϵ we can choose a particular collection of U_i such that $\inf \sum \leq \sum \leq \sum(\inf + \frac{\epsilon}{2^i})$, and conversely we can choose a collection of U_i such that $\sum \inf \leq \sum \leq (\inf \sum) + \epsilon$.

Now let $x \notin A$, and observe that $f(x) = 0$. Also x is irrational, but any ball containing x contains rationals, and thus has nonempty intersection with $B_{2^{-i}}(q_i)$ for some q_i. Thus for any $r > 0$,

$$\begin{aligned} \frac{1}{m(B_r(x))} \int_{B_r(x)} f\, dm &= \frac{1}{m(B_r(x))} \left(\int_{B_r(x) \setminus B_{2^{-1}}(q_i)} f\, dm + \int_{B_r(x) \cap B_{2^{-1}}(q_i)} f\, dm \right) \\ &= \frac{1}{m(B_r(x))} \left(\int_{B_r(x) \setminus B_{2^{-1}}(q_i)} f\, dm + \infty \right) \\ &= \infty \end{aligned}$$

and we are done. ■

Problem3. Let $\alpha \in (0, 1)$ and let m be Lebesgue measure on $\mathbb{R}$. Construct a Borel set $E \subset [-1, 1]$ such that

$$\lim_{r \to 0} \frac{m(E \cap [-r, r])}{2r} = \alpha.$$

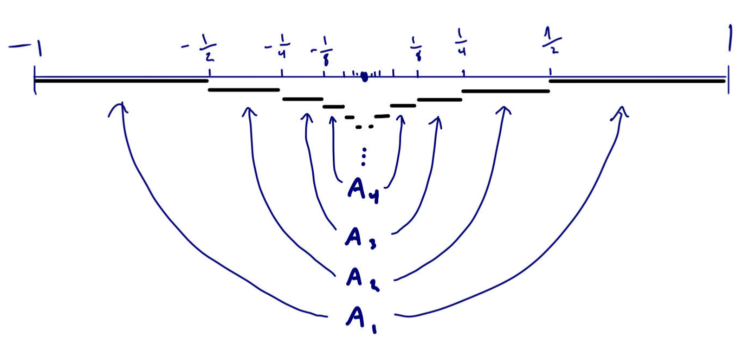
Remark. I think it's worth pointing out that this problem is also a counterexample to the Lebesgue-Besicovitch Differentiation Theorem, as was number 2. In 2, our function wasn't L^1_{loc}, so the function did not equal the limit anywhere in A. In this problem, all the conditions are satisfied since

$$\lim_{r \to 0} \frac{m(E \cap [-r, r])}{2r} = \lim_{r \to 0} \fint_{B_r(0)} \chi_E\, dm,$$

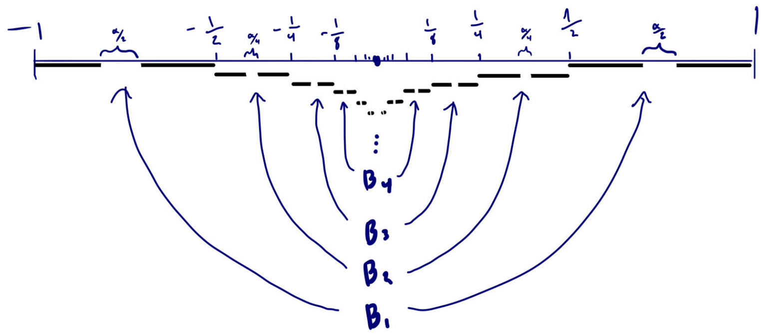
but critically, the theorem only holds μ-*a.e.*, and we are only considering the limit at the single point $\{0\}$.

Proof We will construct E by starting with

$$E_1 = \bigcup_{i=1}^{\infty} A_i,$$
$$\text{where each } A_i = \left[-\frac{2}{2^i}, -\frac{1}{2^i} \right] \cup \left[\frac{1}{2^i}, \frac{2}{2^i} \right]$$



To construct B_i, from each component of A_i remove the central interval whose length is $\frac{\alpha}{2^{-i}}$. Then let $E_2 = \bigcup_{i-1}^{\infty} B_i$.

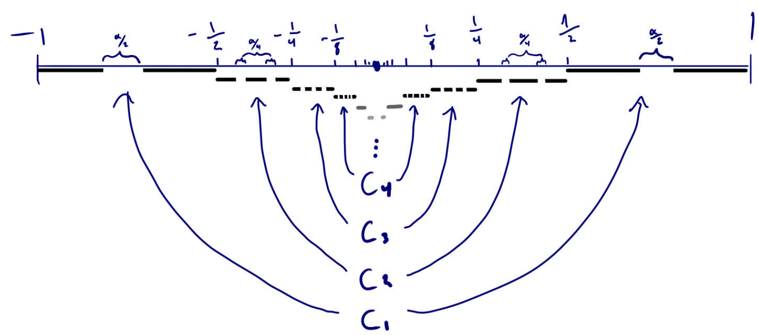


Observe that whenever $r = 2^{-n}$ for some $n \in \mathbb{N}$, we have $⨍_{B_r(0)} \chi_{E_2}\, dm = \alpha$, since the ratio of the removed amount for each component is exactly α by construction. However,

- As r decreases from 2^{-n-1} to $2^{-n-1} - \left(\frac{\alpha}{2}\right) 2^{-n-1}$, the average value decreases,
- As r decreases from $2^{-n-1} - \left(\frac{\alpha}{2}\right) 2^{-n-1}$ to $2^{-n} + \left(\frac{\alpha}{2}\right) 2^{-n-1}$, the average value increases,
- If $r = 2^{-n}, 2^{-n-1}, \text{avg}(2^{-n}, 2^{-n-1})$, then the average value is α.

Furthermore, the maximum and minimum values of $⨍_{B_r(0)} \chi_{E_2}\, dm = \alpha$ do not get any smaller as $r \to 0$, so the limit does not exist.

To fix this, for each B_i define C_i by modifying B_i so that the removed amount is not one central interval, but i equally spaced intervals whose total length is $\alpha 2^{-i}$. Let $E = \bigcup_{i=1}^{\infty} C_i$.



This causes the maximum and minimum values of $⨍_{B_r(0)} \chi_E\, dm$ to be closer to α by a factor of n, for $r \in (2^{-n}, 2^{-n-1})$. ∎

Problem4. For a function $f\colon [a,b] \to \mathbb{R}$ define for every $x \in [a,b)$

$$D^+f(x) = \limsup_{h\to 0+} \frac{f(x+h) - f(x)}{h}.$$

Prove that if $f\colon [a,b] \to \mathbb{R}$ is continuous and $D^+f(x) \geq 0$ for all $x \in [a,b)$, then $f(b) \geq f(a)$.

We will follow that standard proof that any continuously differentiable function with nonnegative derivative is increasing. We will show that an analogue of Rolle's Theorem implies an analogue of the Mean Value Theorem, which implies the result.

Lemma 1. (Rolle's Theorem analogue) Let $f : [a,b] \to \mathbb{R}$ be continuous, with $f(a) = f(b)$. Then there exists some $c \in (a,b)$ such that

$$\begin{aligned} D^+f(c) &= \limsup_{h\to 0^+} \frac{f(c+h) - f(c)}{h} \\ &\leq 0. \end{aligned}$$

Proof Since f is continuous on a compact set, it attains its max and min on $[a,b]$. If $\max f = \min f = f(a) = f(b)$, then f is constant, so $D^+f = 0$ everywhere on $[a,b]$ and we're done. Otherwise, suppose $f(c)$ is a maximum. Since $f(a) = f(b)$, then $c < b$. Then $f(c+h) \leq f(c)$ for all $h > 0$, so the difference quotient is negative for all h-values. Thus

$$\begin{aligned} D^+f(c) &= \limsup_{h\to 0^+} \frac{f(c+h) - f(c)}{h} \\ &\leq 0. \end{aligned}$$

■

Lemma 2. (Mean Value Theorem analogue) Let $f : [a,b] \to \mathbb{R}$ be continuous. For all $\alpha, \beta \in [a,b]$, there exists $c \in [a,b]$ such that $D^+f(c) \leq \frac{f(\beta)-f(\alpha)}{\beta-\alpha}$.

Proof Write f as $f(x) = g(x) + r(x-\alpha)$, where $r = \frac{f(\beta)-f(\alpha)}{\beta-\alpha}$. Then $g(\alpha) = g(\beta)$, so by Lemma 1, there exists $c \in [\alpha,\beta]$ such that $D^+g(c) \leq 0$, so

$$\begin{aligned} D^+f(c) &= D^+g(c) + D^+(rx)(c) \\ &= D^+g(c) + r \\ &\leq r \\ &= \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \end{aligned}$$

and we're done. ■

Proof (Main result) Suppose $f : [a,b] \to \mathbb{R}$ is continuous, and $D^+f(x) \geq 0$ for each $x \in [a,b)$. By Lemma 2, there exists some c in (a,b) such that $D^+f(c) \leq \frac{f(b)-f(a)}{b-a}$. Now $D^+f(x) \geq 0$ everywhere, so the difference quotient is positive. This means that since $b - a > 0$, then $f(b) - f(a) > 0$, and we're done. ■

Problem5. Let the function $f\colon [a,b] \to \mathbb{R}$ be differentiable at every point $x \in [a,b]$. Is f necessarily absolutely continuous on $[a,b]$?

Yes.

Proof Let $f : [a,b] \to \mathbb{R}$ be differentiable. Then f' is continuous on $[a,b]$, so f' attains its max and min. Let $\epsilon > 0$, we will show that f is absolutely continuous. Let $\alpha = \max(|\max(f')|, |\min(f')|)$. For any $x, y \in [a,b]$, the difference quotient is bounded by α:

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq \alpha$$

since by MVT we can find some c with $f'(c) = \frac{f(y)-f(x)}{y-x}$. Thus

$$|f(y) - f(x)| \leq \alpha |y - x| .$$

This means we can let $\delta = \frac{\epsilon}{\alpha}$ and find that if $\sum_{i=1}^{n} |y_i - x_i| < \delta$, then

$$\begin{aligned} \sum_{i=1}^{n} |f(y_i) - f(x_i)| &\leq \sum_{i=1}^{n} \alpha |y_i - x_i| \\ &< \alpha\delta \\ &= \epsilon \end{aligned}$$

and we're done. ■