Math 550
Homework 6
Dr. Fuller
Due October 9

1. Let S^2 denote the unit sphere in $\mathbf{R}^3$. Give a basis for the tangent space S^2_p at any $p \in S^2$.

2. Let V be a k-dimensional vector subspace of $\mathbf{R}^n$.

   (a) Prove that V is a k-dimensional manifold in $\mathbf{R}^n$.

   (b) Let V_p denote the tangent space to V at $p \in V$. Prove that $V_p = V$.

3. Suppose that M is a k-dimensional manifold in $\mathbf{R}^n$. Prove that the *tangent bundle*

$$TM = \{(p,v) \in M \times \mathbf{R}^n : v \in M_p\}$$

   is a $2k$-dimensional manifold in $\mathbf{R}^{2n}$.

4. Let M be a k-dimensional manifold-with-boundary. Prove that ∂M is a $(k-1)$-dimensional manifold.

5. Let $f : U \to f(U)$ and $g : V \to g(V)$ be two parameterizations of $S = f(U) = g(V)$ in $\mathbf{R}^n$, where $U, V \subset \mathbf{R}^k$. Let $\omega \in \Omega^k(S)$ be any k-form which is non-zero at $x \in S$. Prove that f and g induce the same orientation on S_x if and only if $f^*\omega(e_1, \ldots, e_k)$ and $g^*\omega(e_1, \ldots, e_k)$ have the same sign. (Hint: Recall Problem 2 from Homework 4.)

6. Let $f(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ with $0 < \theta < 2\pi$ and $0 < \varphi < \pi$. Let $g(u,v) = (u, \sqrt{1-u^2-v^2}, v)$ for $\{(u,v) : u^2+v^2 < 1\}$. Do f and g induce the same orientation on $\{(x,y,z) \in S^2 : y > 0\}$? (Hint: Regard the previous problem as a criterion to compare orientations. You pick the form ω and the point of evaluation.)

7. The manifold $\partial\mathbf{H}^k$ can be oriented as the boundary of $\mathbf{H}^k$ with the usual orientation. It can also be oriented using the usual orientation of $\mathbf{R}^{k-1}$ (using the obvious identification of $\partial\mathbf{H}^k$ with $\mathbf{R}^{k-1}$). Prove that these orientations agree if and only if k is even.