Trevor Klar

Homework 5

1. Show that composition of paths satisfies the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$, then $f_0 \simeq f_1$.

Definition. (Concatenation of Path Homotopies $F \cdot G$) Given homotopic paths $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$ such that $f_s \cdot g_s$ is defined, then

$$F \cdot G := \begin{cases} F(2s,t) & s \in [0, \frac{1}{2}] \\ G(2s-1,t) & s \in [\frac{1}{2}, 1] \end{cases}$$

or in words, apply F in the first region, and G in the second.

Proof Since $f_0 \cdot g_0 \simeq f_1 \cdot g_1$, then call the path homotopy relating them Φ, and the homotopy relating g_0 and g_1 G. We can (making a minor abuse of notation) consider G to be a homotopy between the inverses $\bar{g}_0$ and $\bar{g}_1$ as well. Using the waiting homotopy we have discussed in class, we know that $f \simeq f \cdot (g \cdot \bar{g})$ whenever the concatenation is defined. Then

$$\begin{aligned} f_0 &\simeq f_0 \cdot (g_0 \cdot \bar{g}_0) && \text{by the waiting homotopy} \\ &\simeq (f_0 \cdot g_0) \cdot \bar{g}_0 && \text{reparametrizing} \\ &\simeq (f_1 \cdot g_1) \cdot \bar{g}_1 && \text{by } \Phi \cdot G \\ &\simeq f_1 \cdot (g_1 \cdot \bar{g}_1) && \text{reparametrizing} \\ &\simeq f_1 && \text{by the waiting homotopy} \end{aligned}$$

and we're done. ■

2. Show that the change-of-basepoint homomorphism β_h depends only on the homotopy class of h.

Proof Suppose $h_0 \overset{H}{\simeq} h_1$. Then for any loop f,

$$\begin{aligned} h_0 \cdot (f \cdot \bar{h}_0) &\simeq h_1 \cdot (f \cdot \bar{h}_0) && \text{by } H \cdot \mathbb{1} \\ &\simeq (h_1 \cdot f) \cdot \bar{h}_0 && \text{reparametrizing} \\ &\simeq (h_1 \cdot f) \cdot \bar{h}_1 && \text{by } \mathbb{1} \cdot H \end{aligned}$$

Thus $\beta_{h_0}[f] = [h_0 \cdot f \cdot \bar{h}_0] = [h_1 \cdot f \cdot \bar{h}_1] = \beta_{h_1}[f]$ and we're done. ■

3. For a path-connected space X, show that $\pi_1(X)$ is abelian if and only if all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.

Proof For $\pi_1(X)$ to be abelian means that for all loops f, g with the same basepoint, $f \cdot g \simeq g \cdot f$. For the change-of-basepoint homomorphism β_h to depend only on the endpoints of the path h means that if h, h' have the same endpoints, then $h \cdot f \cdot \bar{h} \simeq h' \cdot f \cdot \bar{h'}$. We will show that these are equivalent.

$(\implies)$ Suppose $\pi_1(X)$ is abelian, let f be a loop, and let h, h' have the same endpoints. Then

$$\begin{aligned}
h \cdot f \cdot \bar{h} &\simeq (h' \cdot \bar{h}') \cdot h \cdot f \cdot \bar{h} \cdot (h' \cdot \bar{h}') && \text{by the waiting homotopy} \\
&\simeq h' \cdot (\bar{h}' \cdot h) \cdot f \cdot (\bar{h} \cdot h') \cdot \bar{h}' && \text{reparametrizing} \\
&\simeq h' \cdot f \cdot (\bar{h}' \cdot h) \cdot (\bar{h} \cdot h') \cdot \bar{h}' && \pi_1(X) \text{ is abelian} \\
&\simeq h' \cdot f \cdot (\bar{h}' \cdot (h \cdot \bar{h}) \cdot h') \cdot h' && \text{reparametrizing} \\
&\simeq h' \cdot f \cdot (\bar{h}' \cdot h') \cdot h' && \text{by the waiting homotopy} \\
&\simeq h' \cdot f \cdot h' && \text{by the waiting homotopy}
\end{aligned}$$

$(\impliedby)$ Let f, g be loops with the same basepoint, and suppose β_h depends only on the endpoints of h. Then

$$\begin{aligned}
f \cdot g &\simeq f \cdot g \cdot \bar{f} \cdot f && \text{by the waiting homotopy} \\
&\simeq g \cdot g \cdot \bar{g} \cdot f && f, g \text{ have the same endpoints} \\
&\simeq g \cdot f && \text{by the waiting homotopy}
\end{aligned}$$

∎

5. Show that for a space X, the following conditions are equivalent:

(a) Every map $S^1 \to X$ is homotopic to a constant map.

(b) Evey map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi_1(X) = 0$ for all $x_0 \in X$.

Proof $(a \implies b)$ Let $f : S^1 \to X$ be a loop, and let $F : S^1 \times I \to X$ be a homotopy with $f_1 = f$ and $f_0 \equiv x_0$ for some $x_0 \in X$. For each point $(\theta, r) \in D^2$, we can consider θ to be in S^1 and r to be in I. Then F is a map $D^2 \to X$, and it is well defined since if $r = 0$ then $(\theta_1, r) \sim (\theta_2, r)$, and F is constant when $r = 0$. ∎

Proof $(b \implies c)$ Let $x_0 \in X$, and let f be a loop with basepoint x_0. Then extend f to $F : D^2 \to X$. We know that D^2 deformation retracts to any point in D^2, so let $G : D^2 \times I \to D^2$ be a deformation retraction to $f_0^{-1}(x_0)$. If we restrict G to $S^1 \times I$, then G is a homotopy of loops, and

$$F(G|_{S^1 \times I}(\theta, t)) \text{ is a path homotopy from } f \text{ to } x_0,$$

since $g_0 = \mathbb{1}$ and $g_1 \equiv F^{-1}(x_0)$. ∎

Proof $(c \implies a)$ By definition of trivial fundamental group, any two loops in X are homotopic, including any loop based at x_0 and the constant loop x_0. ∎

6. We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Let $[S^1, X]$ be the set of homotopy classes of maps $S^1 \to X$ with no conditions on basepoints. Thus there is a natural map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints.

Show that

(i) Φ is onto if X is path-connected, and

(ii) $\Phi([f]) = \Phi([g])$ iff $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

(iii) Hence Φ induces a one-to-one correspondence between $[S^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

Proof (i) Let x_0 be the basepoint of X, and let g be a loop with arbitrary start and end point x_1. Since X is path-connected, there exists a path p from x_0 to x_1. Now observe that $p \cdot g \cdot \bar{p}$ is a loop with basepoint x_0, and $\Phi([p \cdot g \cdot \bar{p}]) = \Phi([g])$ since we can produce a homotopy of maps (not of loops) that shows $p \cdot g \cdot \bar{p} \simeq g \cdot \bar{p} \cdot p$ as follows:

$$\begin{aligned} h_0(s) &= p \cdot g \cdot \bar{p}, && \text{parametrized in 3 equal parts} \\ h_t(s) &= h_0\left(s + \tfrac{t}{3}\right) && \text{where we identify } s \sim (s-1) \text{ if } s > 1. \end{aligned}$$

and of course $g \cdot \bar{p} \cdot p \simeq g$ by the waiting homotopy. ■

Proof (ii)($\Longleftarrow$) Since $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$, then there exists a loop p such that $f \simeq p \cdot g \cdot \bar{p}$, and $p \cdot g \cdot \bar{p} \simeq g$ by the same reasoning as in (i). ■

Proof (ii)($\Longrightarrow$) Since $\Phi([f]) = \Phi([g])$, then

$$f(s) \overset{H(t,s)}{\simeq} g(s)$$

where H is a homotopy of maps. Thus the common basepoint of f and g (denoted x_0), is not fixed over time t. Define a path p by $p(t) = H(t, 0)$, the image of the basepoint in the homotopy (we will hereafter write p as a function of s). Since f and g have the same basepoint, p is a loop with endpoint x_0. Define a homotopy of maps $p_t(s) := p(ts)$ so that

$$\begin{aligned} p_0 &\equiv x_0 \\ p_1 &= p \\ p_t(1) &= H(t, 0), \end{aligned}$$

and also define a homotopy $\bar{p}_t$ that gives the inverse path of p_t for each time t. Now observe that for all t, $p_t(0) = x_0, p_t(1) = H(t, 0), H(t, 1) = \bar{p}_t(0), \bar{p}_t(1) = x_0$, so we can concatenate

$$P \cdot H \cdot \bar{P}$$

to obtain an actual homotopy of loops showing $f \simeq p \cdot g \cdot \bar{p}$. To see that we have done this, observe that for all time the endpoints are fixed at x_0, at $t = 0$ we have $\{x_0\} \cdot f \cdot \{x_0\} \simeq f$, and at $t = 1$ we have $p \cdot g \cdot \bar{p}$. ■

Proof (iii) Consider the map induced by Φ,

$$\widetilde{\Phi} : \{\text{conjugacy classes of } \pi_1(X)\} \to [S^1, X].$$

- Since Φ is onto, then $\widetilde{\Phi}$ is onto.
- Since $\Phi([f]) = \Phi([g])$ implies $[f]$ and $[g]$ are conjugate, then $\widetilde{\Phi}$ is one-to-one.
- Since $[f]$ and $[g]$ are conjugate implies $\Phi([f]) = \Phi([g])$, then $\widetilde{\Phi}$ is well-defined. ■

11. If X_0 is the path-component of a space X containing the basepoint x_0, show that the inclusion $X_0 \hookrightarrow X$ induces an isomorphism $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$.

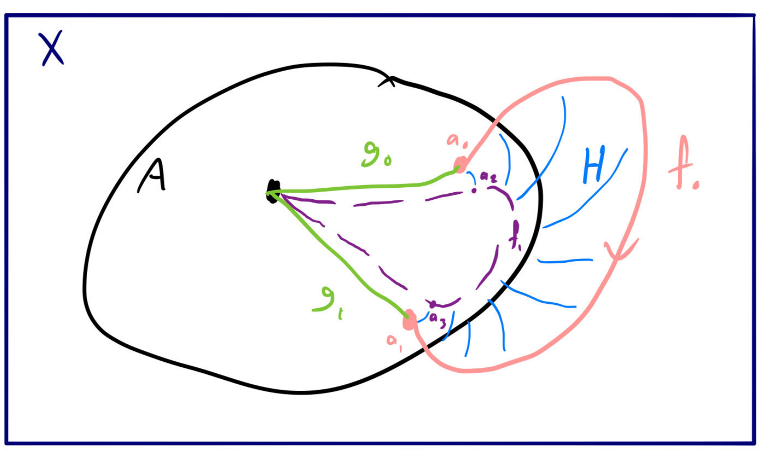
Proof

- Any loop in X with basepoint x_0 must be in the path-component of x_0, since the loop itself connects every point in it to x_0. Thus the induced map $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$ is onto.
- It is obviously true that $f \simeq g$ iff $f \simeq g$, so the induced map $\pi_1(X_0, x_0) \to \pi_1(X, x_0)$ is one-to-one and well-defined. ■

13. Given a space X and a path-connected subspace A containing the basepoint x_0, show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in X with endpoints in A is homotopic to a path in A.

Proof ($\Longleftarrow$) Every loop f in X with basepoint x_0 is a path with basepoints in A, so it is homotopic to a loop a in A. Thus $[a] = [f]$, and the map is surjective. ■

Proof ($\Longrightarrow$) Suppose the map is onto. Let f_0 be a path with endpoints $a_0, a_1 \in A$. Since A is path-connected, let g_0, g_1 be paths $x_0 \to a_0$ and $a_1 \to x_0$ respectively. Then $[g_0 \cdot f_0 \cdot g_1] \in \pi_1(X, x_0)$, so there exists $g_0 \cdot f_0 \cdot g_1 \overset{H(s,t)}{\simeq} \widetilde{f}$ such that $[\widetilde{f}] \in \pi_1(A, x_0)$. Here we are parameterizing $g_0 \cdot f_0 \cdot g_1$ in three equal parts, so $H(\frac{1}{3}, 0) = a_0$ and $H(\frac{2}{3}, 0) = a_1$. Call $a_2 = H(\frac{1}{3}, 1)$ and $a_3 = H(\frac{2}{3}, 1)$, and note that $H(\frac{1}{3}, t)$ and $H(\frac{2}{3}, t)$ are paths connecting a_0 to a_2 and a_1 to a_3 respectively.



To finish up, we concatenate three homotopies of maps to form a homotopy of paths, similar to how we did in Problem 6(ii).

$$\begin{aligned} \text{Let } P^0(s,t) &= H\left(\tfrac{1}{3}, st\right), \\ \text{let } P^1(s,t) &= H\left(\tfrac{2}{3}, st\right), \text{ and} \\ \text{let } \widetilde{H}(s,t) &= H\Big|_{\left[\frac{1}{3},\frac{2}{3}\right]\times I}, \text{ rescaled so that } \widetilde{H} : I \times I \to X. \end{aligned}$$

Then for all t,

- $P^0(0,t) = a_0$,
- $P^0(1,t) = H\left(\frac{1}{3}, t\right) = \widetilde{H}(0,t)$
- $P^1(1,t) = H\left(\frac{2}{3}, t\right) = \widetilde{H}(1,t)$
- $P^1(0,t) = a_1$,

so we can concatenate $P^0 \cdot \widetilde{H} \cdot \bar{P}^1$ to produce a path homotopy between f_0 and $p_1^0 \cdot \widetilde{h}_1 \cdot \bar{p}_1^1$, the latter which path is completely in A. ■

Collaborators:

1. Zach Wagner

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3.

5. Zach Wagner

6. Sandy Schoettler

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13.