Math 201A, Homework 3 (Lebesgue measure)

Problem1. Let μ be Lebesgue measure and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue-measurable subsets of $[0,1]$. Assume the set B consists of those points $x \in [0,1]$ that belong to infinitely many of the A_n.

1. Prove that B is Lebesgue-measurable.
2. Prove that if $\mu(A_n) > \delta > 0$ for every $n \in \mathbb{N}$, then $\mu(B) \geq \delta$.
3. Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(B) = 0$.
4. Give and example where $\sum_{n=1}^{\infty} \mu(A_n) = \infty$, but $\mu(B) = 0$.

Problem2. Prove that if $A \subset \mathbb{R}$ is Lebasgue-measurable with $\mu(A) > 0$, then there is a subset of A that is not Lebesgue-measurable.

Problem3. Let μ be Lebesgue measure on $\mathbb{R}$. Construct a Borel set $A \subset \mathbb{R}$ such that $\mu(A) > 0$ and $\mu(A \cap I) < \mu(I)$ for every non-degenerate interval $I \subset \mathbb{R}$.

Problem4. Let $A \subset \mathbb{R}$ be a Lebesgue-measurable set. Prove that the set

$$B = \cup_{x \in A}[x-1, x+1]$$

is Lebesgue-measurable.