Welcome To Math 34A! Differential Calculus

Instructor:

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MTWR after class 2:00-3:00, and by appointment. Details on Gauchospace.

Suppose x and y are related variables. So as one changes, the other changes. We can ask:

How much does y change per unit change in x?

Answer: The derivative of y with respect to x tells us, and it depends on the current value of x!

If we write y as a function of x like this: $y = f(x)$, then the derivative is written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{or} \quad \frac{\mathrm{d}f}{\mathrm{d}x} \quad \text{or} \quad f'(x)$$

It is the limit of "average rate of change" over shorter and shorter Δx:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

also known as "instantaneous rate of change"

Why use h to find the derivative?

Without h: $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

Here is an example without h. For $f(x) = x^2$, if we wanted to find $f'(2)$ it would be the limit of the average rate of change from 2 to a second point x as that second point approaches 2.

$$\lim_{x \to 2} \frac{x^2 - 2^2}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x \to 2} x + 2 = 4$$

Second example: For $g(x) = x^3$, if we wanted to find $g'(5)$ it would be the limit of the average rate of change from 5 to a second point x as that second point approaches 5.

$$\lim_{x \to 5} \frac{x^3 - 5^3}{x-5} = \lim_{x \to 5} \frac{(x-5)(x^2+5x+5^2)}{(x-5)} = \lim_{x \to 5} x^2 + 5x + 5^2 = 75$$

It's often harder to find the derivative this way, so we just make $\Delta x = h$ and let h disappear.

On the other hand...

$$\text{With } h: \quad f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

For $f(x) = x^2$, we can find $f'(2)$ this way.

$$\lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h\to 0} \frac{2^2 + 4h + h^2 - 2^2}{h} = \lim_{h\to 0} \frac{4h + h^2}{h}$$

$$= \lim_{h\to 0} 4 + h = 4$$

For $g(x) = x^3$, we can find $g'(5)$ this way.

$$\lim_{h\to 0} \frac{(5+h)^3 - 5^3}{h} = \lim_{h\to 0} \frac{5^3 + 75h + 15h^2 + h^3 - 5^3}{h} = \lim_{h\to 0} \frac{75h + 15h^2 + h^3}{h}$$

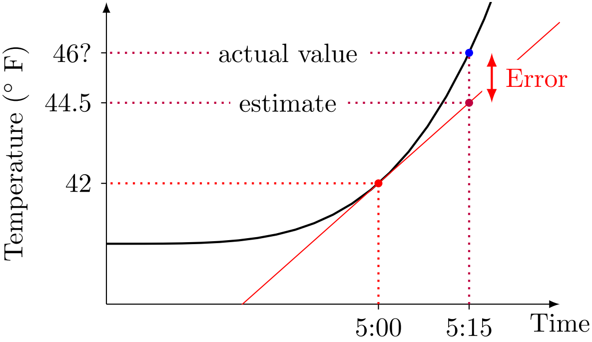
$$= \lim_{h\to 0} 75 + 15h + h^2 = 75$$

§8.6: Tangent Line Approximation

Question: At 5am the temperature is 42° F and increasing at a rate of 10° F per hour. Which of the following do you think is closest to the temperature at 5:15am?

A= 2.5° F B= 52° F C= 43.5° F D= 44.5° F E= 5.15° F

Answer: D



Continuing this example

Same set-up:

- $f(x)$ = temperature at time x hours after midnight
- $f(5) = 42$ (42° F at 5:00am)
- $f'(5) = 2$

(1) Find the equation of tangent line to $y = f(x)$ at $x = 5$.

A $y = 5x + 42$ B $y = 2x + 5$ C $y = 2(x - 5) + 42$
D $y - 5 = 2(x - 42)$ E $y - 42 = 2x - 5$

Answer: C

(2) Use this to predict the approximate temperature at 4am.

A= 40 B= 41 C= 42 D= 43 E= 44 A

(3) The tangent line approximation is used to estimate the temperature at the following times. Which do you think is most accurate?

A 4am B 4:50am C 5:25am D 6am E midnight B

Tangent Line Approximation

To do a tangent line approximation:

(i) Find the equation of the tangent line.

(ii) Plug in the required value(s) into this equation.

Suppose $f(4) = 2$ and $f'(4) = 3$.

(a) The equation of the tangent line to $y = f(x)$ at $x = 4$ is $y =$?

A$= 4x - 14$ B$= 3x - 10$ C$= 2x - 6$
D$= 3x - 4$ E$= 2x - 5$ B

(b) Use this tangent line approximation to estimate $f(4.1)$.

A$= 2.3$ B$= 1.7$ C$= 2.6$ D$= 1.4$ E$= 2$ A

(c) Use the tangent line approximation to estimate the value of x which gives $f(x) = 2.9$.

A$= 4.9$ B$= 4.1$ C$= 2.9$ D$= 4.1$ E$= 4.3$ E

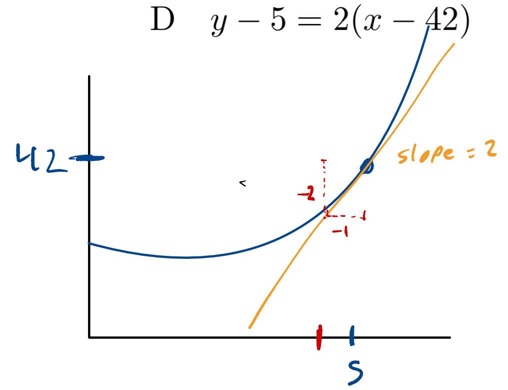
Continuing this example

Same set-up:

- $f(x)$ = temperature at time x hours after midnight
- (5, 42) $f(5) = 42$ (42° F at 5:00am)
- $f'(5) = 2$ slope = 2

(1) Find the equation of tangent line to $y = f(x)$ at $x = 5$.

A $y = 5x + 42$ B $y = 2x + 5$ $\boxed{\text{C } \ y = 2(x-5) + 42}$
D $y - 5 = 2(x - 42)$ E $y - 42 = 2x - 5$



Point-slope form:

$y - y_1 = m(x - x_1)$

$y - 42 = 2(x - 5)$

Tangent Line Approximation

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(b) Use this tangent line approximation to estimate $f(4.1)$.

A= 2.3 B= 1.7 C= 2.6 D= 1.4 E= 2

$3(4.1) - 10$

$$\begin{array}{r} 4.1 \\ \times \quad 3 \\ \hline 12.3 \\ -\,10.0 \\ \hline 2.3 \end{array}$$

Standard Estimation Problem

Question: Approximate $\sqrt{26}$.

A= 0.1 B= 5.01 C= 5.05 D= 5.1 E= 5.2 D

Some tools: For $g(x) = \sqrt{x}$, $g'(25) = 1/10$ and $g(25) = \sqrt{25} = 5$.

Better estimate: $\sqrt{26} \approx 5.09902$, so the error in the tangent line approximation here is

$$\text{error} \approx 5.1 - 5.09902 \approx 0.001$$

This is a percentage error of only 0.02%.

Standard Estimation Problem

Question: Approximate $\sqrt{26}$.

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Some tools: For $g(x) = \sqrt{x}$, $g'(25) = 1/10$ and $g(25) = \sqrt{25} = 5$.

tangent line:

$$y - 5 = \frac{1}{10}(x - 25)$$

$$y = \frac{1}{10}(x - 25) + 5$$

$$y = \frac{1}{10}(1) + 5$$

$$y \approx 5.1$$

Another Example:

- $f(t)$ = number of grams of a chemical reagent after t seconds
- We're told $f(0) = 20$ and $f'(0) = -3$

Question: Roughly how many grams are there after t seconds?

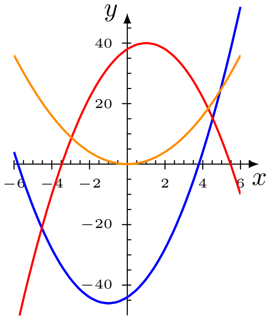
A= $4 - 3t$ B= $20 - 3t$ C= $20 - 4t$ D= $20 + 4t$ E= $32 - 3t$

Answer: B

Sketching some simple graphs

It's useful to be able to sketch…

(1) Quadratics



$y = 2x^2 + 4x - 44$

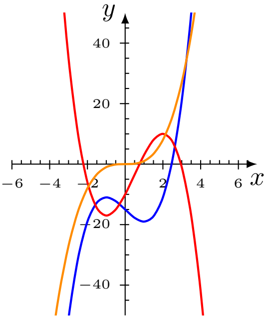
$y = -2x^2 + 4x + 38$

- $y = ax^2 + bx + c$
- Bowl-shaped:
  - ⋆ Opens up if $a > 0$
  - ⋆ Opens down if $a < 0$
- Model curve: $y = x^2$
  Shown here!

Sketching some simple graphs

It's useful to be able to sketch…

(2) Cubics



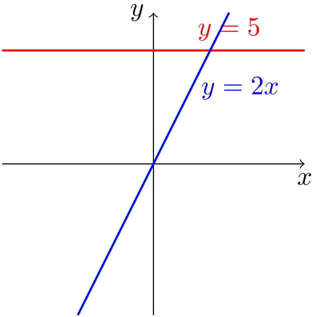
$y = 2x^3 - 6x - 15$

$y = -2x^3 + 3x^2 + 12x - 10$

- $y = ax^3 + bx^2 + cx + d$
- "S"-shaped:
  - ⋆ Goes to $+\infty$ if $a > 0$
  - ⋆ Goes to $-\infty$ if $a < 0$
- Model curve: $y = x^3$
  Shown here!

For a polynomial, the highest power of x dominates when x is big

The Derivatives of Simple Functions



The derivative of a constant is…? zero because:

- derivative = rate of change
- constants don't change

- derivative = slope
- slope = 0

So $\dfrac{d}{dx}(5) = 0$

The derivative of a straight line is…? its slope because

- derivative = slope

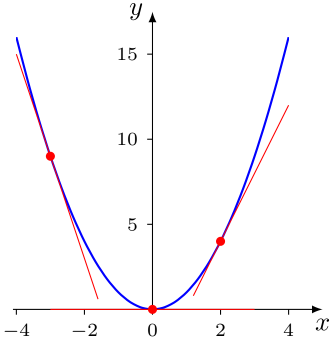
So $\dfrac{d}{dx}(2x) = 2$

Meaning of Derivatives

$$\frac{d}{dx}\left(x^2\right) = 2x$$

What this means

The slope of the graph of $y = x^2$ at $x = a$ is $2a$



at $x = -3$, slope is $2(-3) = -6$

at $x = 0$, slope is $2(0) = 0$

at $x = 2$, slope is $2(2) = 4$

derivative = rate of change = slope of graph = slope of tangent line

General Rule:

$$\frac{d}{dx}\left(x^{2}\right) = 2x$$
$$\frac{d}{dx}\left(x^{3}\right) = 3x^{2}$$
$$\frac{d}{dx}\left(x^{4}\right) = 4x^{3}$$

$$\boxed{\frac{d}{dx}\left(x^{n}\right) = nx^{n-1}}$$

The exponent comes out front. Then subtract one from exponent.

Examples:

1. $\frac{d}{dx}\left(x^{7}\right) =$

A$= 7x^7$ B$= 6x^6$ C$= 6x^7$ D$= 7x^6$ E$= 0$ D

2. $\frac{d}{dx}\left(x^{-3}\right) =$

A$= 3x^{-2}$ B$= -3x^{-2}$ C$= -2x^{-4}$ D$= -3x^{-4}$ D

More Examples

$$\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}}$$

3. $\frac{d}{dx}\left(x^{1/2}\right) =$

A= $\frac{1}{2}x^{1/2}$ B= $-\frac{1}{2}x^{-1/2}$ C= $\frac{1}{2}x^{-1/2}$ C

Rule: ALWAYS rewrite the thing you want derivative of as x^n

4. $\frac{d}{dx}\left(\frac{1}{x^3}\right) =$

A= $\frac{1}{3x^2}$ B= $-3x^{-2}$ C= $-3x^{-4}$ C

5. $\frac{d}{dx}\left(\sqrt{x}\right) =$

A= $-\frac{1}{2}\sqrt{x}$ B= $\frac{1}{2}x^{-1/2}$ C= $-\frac{1}{2}x^{-1/2}$ B

Polynomials

$$\frac{d}{dx}\left(4x^5 + 7x^2 - 5x + 7\right) = 4(5)x^4 + 7(2)x^1 - 5 + 0$$

Special cases

- $\frac{d}{dx}\left(-5x\right) = -5$
- $\frac{d}{dx}\left(7\right) = 0$

6. $\frac{d}{dx}\left(3x^4 + 9x^3 + 7\right) = ?$

A= I have an answer B= I am working on it C= Help!

Fun Trick

Imagine you are asked to find the vertex (highest/lowest point) of the parabola

$$f(x) = x^2 + 3x + 1.$$

Problem: Who remembers that formula?!
What is the slope of $f(x)$ at the highest/lowest point? It's zero!

$$f'(x) = 2x + 3$$

When is this 0?

$$2x + 3 = 0 \text{ when } x = -\frac{3}{2}$$

Bingo!

The Meanings of Derivatives

The derivative of $f(x) = x^2 + 3x + 1$ is $f'(x) = \frac{df}{dx} = 2x + 3$. This means:

- This is the slope of the graph $y = x^2 + 3x + 1$ at the point x
- It is the instantaneous rate of change of $f(x)$ at x.

That $f'(2) = 7$ means:

- The slope of the graph $y = f(x)$ at $x = 2$ is 7.
- The slope of the tangent line to the graph at $x = 2$ is 7.
- The instantaneous rate of change of $f(x)$ at $x = 2$ is 7.
- At $x = 2$ the output (value of $f(x)$) changes 7 times as fast as the input (value of (x)).
- $\Delta f \approx 7\Delta x$ near $x = 2$.
- $f(2 + \Delta x) \approx f(2) + 7\Delta x$.

Applications

7. What is the slope of the graph $y = 3x^2 - 7x + 5$ at $x = 1$?

A$= -2$ B$= -1$ C$= 0$ D$= 1$ E$= 2$ B

8. What is the instantaneous rate of change of $f(x) = x^3 - 2x + 3$ at $x = 1$?

A$= -2$ B$= -1$ C$= 0$ D$= 1$ E$= 2$ D

9. After t seconds a hamster on a skate board is $4t^2 + 2t$ cm from the origin on the x-axis. What is the exact speed of the hamster (in cm/sec) after 2 seconds?

A$= 10$ B$= 16$ C$= 18$ D$= 20$ E$= 14$ C

That's it. Thanks for being here.