Math 450
Homework 2
Dr. Fuller
Due February 6

1. Let U be an open set in $\mathbf{R}^n$ and C be a closed set in $\mathbf{R}^n$, with $C \subset U$. Prove that $U - C$ is open.

2. (☐) Give the interior, exterior, and boundary for the following subsets of $\mathbf{R}^n$. No proofs, just give answers.
   (a) $\{(x,y) : xy = 0\} \subset \mathbf{R}^2$
   (b) $\{(x,y) : xy \neq 0\} \subset \mathbf{R}^2$
   (c) $\{(x,y,z) : x^2+y^2 < 1 \text{ and } z = 0\} \subset \mathbf{R}^3$
   (d) $\{(x,y,z) : x^2+y^2 < 1\} \subset \mathbf{R}^3$
   (e) $\{(x_1,\ldots,x_n) : \text{each } x_i \in \mathbf{Q}\} \subset \mathbf{R}^n$

3. Decide if the following subsets of $\mathbf{R}^n$ are closed, bounded, and compact.
   (a) A finite set of points in $\mathbf{R}^n$
   (b) $\overline{B}(\mathbf{0},2) - B(\mathbf{0},1)$
   (c) $\{(x_1,\ldots,x_n) \in \overline{B}(\mathbf{0},1) : x_n = 0\}$
   (d) $\{(x_1,\ldots,x_n) \in \overline{B}(\mathbf{0},10) : \text{each } x_i \in \mathbf{Z}\} \subset \mathbf{R}^n$
   (e) $\{(x_1,\ldots,x_n) \in \overline{B}(\mathbf{0},10) : \text{each } x_i \in \mathbf{Q}\} \subset \mathbf{R}^n$

4. (a) (☐) Suppose A is a closed subset of $\mathbf{R}^n$, and $\mathbf{x} \notin A$. Prove that there is a $\delta > 0$ such that $\|\mathbf{x}-\mathbf{y}\| \geq \delta$ for all $\mathbf{y} \in A$.
   (b) Suppose that A and C are closed subsets of $\mathbf{R}^n$, with C compact, and $A \cap C = \emptyset$. Prove that there exists $\delta > 0$ such that $\|\mathbf{x}-\mathbf{y}\| \geq \delta$ for all $\mathbf{y} \in A$ and $\mathbf{x} \in C$. (Hint: For each $\mathbf{w} \in C$, find an open ball such that this inequality holds for all $\mathbf{x} \in B(\mathbf{w}, r(\mathbf{w}))$.)
   (c) (☐) Give a counterexample in $\mathbf{R}^2$ to part (b) when A and B are closed, but neither is compact.

5. Determine if the following examples are continuous on the indicated domain. Justify your answers.
   (a) $f : \mathbf{R}^2 - \{\mathbf{0}\} \to \mathbf{R}$ given by $f(x,y) = \dfrac{xy}{x^2+y^2}$
   (b) $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   (c) $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

6. Prove that $f : \mathbf{R}^n \to \mathbf{R}$ given by $f(x) = \|x\|$ is continuous.

7. Suppose that $f : U \subset \mathbf{R}^n \to \mathbf{R}^m$ and $g : U \subset \mathbf{R}^n \to \mathbf{R}^m$ are continuous at $\mathbf{a} \in \mathbf{R}^n$. Prove that $\langle f,g \rangle : U \subset \mathbf{R}^n \to \mathbf{R}$ defined by $(\langle f,g \rangle)(\mathbf{x}) = \langle f(\mathbf{x}), g(\mathbf{x}) \rangle$ is continuous at $\mathbf{a}$.