Math 450b
Homework 1

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1. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Prove that $|\langle \mathbf{x}, \mathbf{y} \rangle| = ||\mathbf{x}||\,||\mathbf{y}||$ if and only if $\mathbf{y} = r\mathbf{x}$ for some $r \in \mathbb{R}$.

Proof Both directions of this proof will rely on the fact that $\mathbf{x} \neq \vec{0}$, so before we begin we will address that possibility. Suppose $\mathbf{x} = \vec{0}$. Then, $|\langle \mathbf{x}, \mathbf{y} \rangle| = |\langle \mathbf{0}, \mathbf{y} \rangle| = |\sum_{i=1}^{n} 0 y_i| = 0$ and $||x||\,||y|| = 0\,||y|| = 0$. Thus, $|\langle \mathbf{x}, \mathbf{y} \rangle| = 0 = ||\mathbf{x}||\,||\mathbf{y}||$, so the converse direction holds (since the conclusion is always true). However, if $\mathbf{x} = \vec{0}$ and $\mathbf{y} \neq \vec{0}$, then there is no such $r \in \mathbb{R}$ such that $\mathbf{y} = r\mathbf{x}$, so the forward direction actually does not hold in this case (the hypothesis is always true, but the conclusion is always false).

Since the theorem does not always hold when $\mathbf{x} = \vec{0}$, we will assume that $\mathbf{x} \neq \vec{0}$ in the rest of this proof. ■

Proof ($\Longleftarrow$) Suppose that $\mathbf{y} = r\mathbf{x}$ for some $r \in \mathbb{R}$. Then we have the following:

$$\begin{aligned} 0 &= ||\mathbf{y} - r\mathbf{x}||^2 \\ 0 &= \langle \mathbf{y} - r\mathbf{x}, \mathbf{y} - r\mathbf{x} \rangle \\ 0 &= ||\mathbf{y}||^2 - 2r\langle \mathbf{x}, \mathbf{y} \rangle + r^2 ||\mathbf{x}||^2 \end{aligned}$$

Before we proceed further, we can use the fact that $\mathbf{y} = r\mathbf{x}$ to obtain a value for r:

$$\begin{aligned} \langle \mathbf{x}, r\mathbf{x} \rangle &= \langle \mathbf{x}, r\mathbf{x} \rangle \\ r\langle \mathbf{x}, \mathbf{x} \rangle &= \langle \mathbf{x}, \mathbf{y} \rangle \\ r\,||\mathbf{x}||^2 &= \langle \mathbf{x}, \mathbf{y} \rangle \\ r &= \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||^2} \end{aligned}$$

Now we plug this in for r in our previous equation and simplify:

$$\begin{aligned} 0 &= ||\mathbf{y}||^2 - 2\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||^2}\right)\langle \mathbf{x}, \mathbf{y} \rangle + \left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||^2}\right)^2 ||\mathbf{x}||^2 \\ 0 &= ||\mathbf{y}||^2 - 2\frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{||\mathbf{x}||^2} + \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{||\mathbf{x}||^2} \\ 0 &= ||\mathbf{y}||^2 - \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{||\mathbf{x}||^2} \end{aligned}$$

From this, we can rearrange to find that $\langle \mathbf{x}, \mathbf{y} \rangle^2 = ||\mathbf{x}||^2\,||\mathbf{y}||^2$ and take square roots, yielding $|\langle \mathbf{x}, \mathbf{y} \rangle| = ||\mathbf{x}||\,||\mathbf{y}||$ and we are done. ■

Proof ($\Longrightarrow$) Suppose that $|\langle \mathbf{x}, \mathbf{y} \rangle| = ||\mathbf{x}||\,||\mathbf{y}||$.

As in the converse direction (with steps reversed), we can square both sides and rearrange to find that

$$0 = ||\mathbf{y}||^2 - 2\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||^2}\right)\langle \mathbf{x}, \mathbf{y} \rangle + \left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||^2}\right)^2 ||\mathbf{x}||^2.$$

Now since we have assumed that $\mathbf{x} \neq \vec{0}$, we know that $\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||^2}$ is a real number. So let $r = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||^2}$ and substitute to obtain

$$0 = ||\mathbf{y}||^2 - 2r\langle \mathbf{x}, \mathbf{y} \rangle + r^2 ||\mathbf{x}||^2.$$

Again as we did in the converse direction, we can rearrange to find that $0 = ||\mathbf{y} - r\mathbf{x}||^2$. This means that $\mathbf{y} = r\mathbf{x}$, and we are done. ■

2. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ be nonzero. Prove that $||\mathbf{x}+\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Proof ($\Longleftarrow$) Suppose $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. Then

$$\begin{aligned} ||\mathbf{x}+\mathbf{y}||^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle \\ &= ||\mathbf{x}||^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + ||\mathbf{y}||^2 \end{aligned}$$

and, since $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, so

$$\begin{aligned} & ||\mathbf{x}||^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + ||\mathbf{y}||^2 \\ = \ & ||\mathbf{x}||^2 + ||\mathbf{y}||^2 \end{aligned}$$

and we are done. ■

Proof ($\Longrightarrow$) Suppose that $||\mathbf{x}+\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$. Then,

$$\begin{aligned} ||\mathbf{x}+\mathbf{y}||^2 &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 \\ ||\mathbf{x}||^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + ||\mathbf{y}||^2 &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 \\ 2\langle \mathbf{x}, \mathbf{y} \rangle &= 0 \end{aligned}$$

Thus, $\mathbf{x}$ and $\mathbf{y}$ are orthogonal by definition. ■

3. Let $\mathbf{x} = (1, 1, \ldots, 1)$ and $\mathbf{y} = (1, 2, \ldots, n)$ in $\mathbb{R}^n$. Let θ_n be the angle between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$. Find $\lim_{n\to\infty} \theta_n$.

We know that

$$\cos\theta_n = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||\,||\mathbf{y}||},$$

So we will compute each of the parts.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$||\mathbf{x}|| = \sqrt{n}$$

$$||\mathbf{y}|| = \sqrt{\sum_{i=1}^{n} i^2} = \sqrt{\frac{n(n+1)(2n+1)}{6}}$$

Plugging these terms in and canceling, we find that

$$\cos\theta_n = \sqrt{\frac{3n+3}{4n+2}}$$

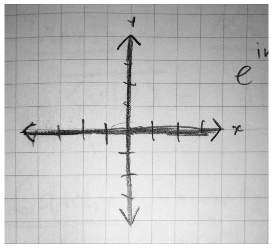
So, to find $\lim_{n\to\infty} \theta_n$, we find

$$\lim_{n\to\infty} \left(\cos^{-1} \sqrt{\frac{3n+3}{4n+2}} \right) = \cos^{-1}\left(\frac{\sqrt{3}}{2} \right) = \frac{\pi}{6}$$

and we are done. ■

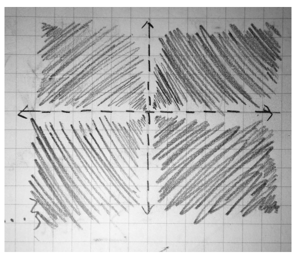
4. (□) Decide if the following subsets of R^n are open and/or closed. (Draw pictures, and give answers. No proofs necessary.)

(a) $\{(x, y) : xy = 0\} \subset \mathbb{R}^2$
Answer: Closed and not open.



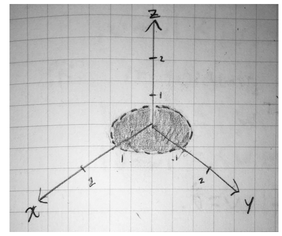
□

(b) $\{(x, y) : xy \neq 0\} \subset \mathbb{R}^2$
Answer: Open and not closed.



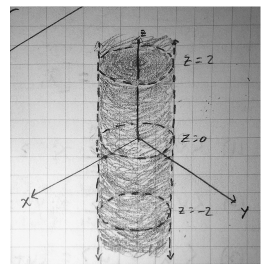
□

(c) $\{(x, y, z) : x^2 + y^2 < 1 \text{ and } z = 0\} \subset \mathbb{R}^3$
Answer: Not open and not closed.



□

(d) $\{(x, y, z) : x^2 + y^2 < 1\} \subset \mathbb{R}^3$
Answer: Open and not closed.

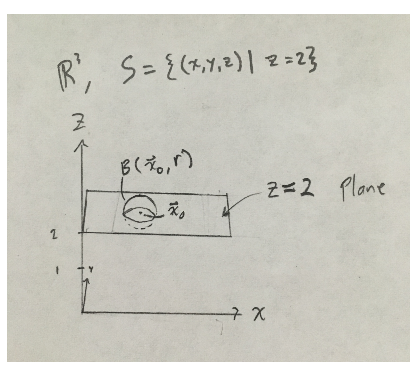


□

(a) $\{(x_1, \ldots, x_n) : \text{each } x_i \in \mathbb{Q}\} \subset \mathbb{R}^n$
Answer: Not open and not closed.
This set is impossible to draw. I imagine it something like a dense infinite point grid, like a field of stars in space. Each element has infinitely many other elements surrounding it in every direction, as well as elements not in the set surrounding it in a similar way. ■

5. (□) Let S be an $(n-1)$-dimensional vector subspace of $\mathbb{R}^n$. Prove that S is not an open set.

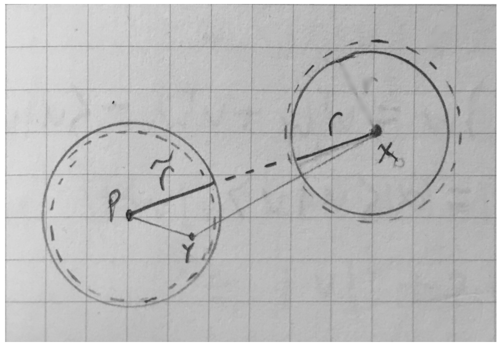
Proof Since every vector space has a basis, let $B = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{n-1}}\}$ be a basis for S. Now, since B has only $(n-1)$ elements, it cannot span $\mathbb{R}^n$, and thus can be extended to a spanning set by including another vector, $\mathbf{u}$. Now, to see that S is not open, observe that for every $\mathbf{x} \in S$, and every $B(\mathbf{x}, r)$ where $r \in \mathbb{R}^+$, the point $\mathbf{x} + \frac{r\mathbf{u}}{2||\mathbf{u}||}$ is an element of $B(\mathbf{x}, r)$, but not an element of S. The following image illustrates this for $\mathbb{R}^3$ and $S = \{(x, y, z) : z = 2\}$:



■

6. ($\square$) Let $\mathbf{x} \in \mathbb{R}^n, r \geq 0$, and define $\overline{B}(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R} : ||\mathbf{x} - \mathbf{y}|| \leq r\}$. Prove that $\overline{B}(\mathbf{x}, r)$ is closed.

Proof To show that $\overline{B}(\mathbf{x}, r)$ is closed, we will show that its complement is open. Let $\mathbf{p}$ be in $\mathbb{R}^n$ such that $\mathbf{p} \notin \overline{B}(\mathbf{x}, r)$. Let $\tilde{r} = \frac{||\mathbf{p}-\mathbf{x}||-r}{2}$.
Claim: $B(\mathbf{p}, \tilde{r}) \subset (\mathbb{R}^n - \overline{B}(\mathbf{x}, r))$.



To show this, we will prove that $||\mathbf{y} - \mathbf{x}|| > r$.
Let $\mathbf{y} \in B(\mathbf{p}, \tilde{r})$. Then by the triangle inequality,

$$||\mathbf{p} - \mathbf{x}|| \leq ||\mathbf{p} - \mathbf{y}|| + ||\mathbf{y} - \mathbf{x}||$$

and subtracting $||\mathbf{p} - \mathbf{y}||$, we find that

$$||\mathbf{p} - \mathbf{x}|| - ||\mathbf{p} - \mathbf{y}|| \leq ||\mathbf{y} - \mathbf{x}|| .$$

Now, $||\mathbf{p} - \mathbf{x}|| = r + 2\tilde{r}$ by definition, and $-\tilde{r} < -||\mathbf{p} - \mathbf{y}||$ as well, so

$$r + \tilde{r} = (r + 2\tilde{r}) - \tilde{r} < ||\mathbf{p} - \mathbf{x}|| - ||\mathbf{p} - \mathbf{y}|| \leq ||\mathbf{y} - \mathbf{x}|| ,$$

Thus $r < ||\mathbf{y} - \mathbf{x}||$ and we are done. ■

7.

(a) Prove that $\mathbb{R}^n$ is an open set.

Proof Let $\mathbf{x} \in \mathbb{R}^n$, and let $r > 0$. Observe that $B(\mathbf{x}, r) \subset \mathbb{R}^n$, so R^n is open. ■

(b) Let $\{U_\alpha\}_{\alpha \in \Gamma}$ be a collection of an arbitrary number of open sets in $\mathbb{R}^n$. Prove that $\bigcup_{\alpha \in \Gamma} U_\alpha$ is an open set.

Proof Let $\mathbf{x} \in \bigcup_{\alpha \in \Gamma} U_\alpha$. By definition, $\mathbf{x} \in U_\beta$ for some $\beta \in \Gamma$. Since U_β is open, there exists some $r > 0$ such that $B(\mathbf{x}, r) \subset U_\beta$. Thus, $B(\mathbf{x}, r) \subset \bigcup_{\alpha \in \Gamma} U_\alpha$, so it is open. ■

(c) Let U_1 and U_2 be open sets in $\mathbb{R}^n$. Prove that $U_1 \cap U_2$ is an open set.

Proof Let $\mathbf{x} \in U_1 \cap U_2$. Since U_1 and U_2 are open sets, there exist $r_1, r_2 > 0$ such that $B(\mathbf{x}, r_1) \subset U_1$ and $B(\mathbf{x}, r_2) \subset U_2$. Let $r = \min(r_1, r_2)$. Then, $B(\mathbf{x}, r) \subset B(\mathbf{x}, r_1) \subset U_1$ and $B(\mathbf{x}, r) \subset B(\mathbf{x}, r_2) \subset U_2$; so

$$B(\mathbf{x}, r) \subset U_1 \cap U_2$$

and we are done. ■

8. Let $\{C_\alpha\}_{\alpha\in\Gamma}$ be an arbitrary collection of closed sets in $\mathbb{R}^n$.

   (a) Prove that $\bigcap\limits_{\alpha\in\Gamma} C_\alpha$ is a closed set.

   Proof To prove that $\bigcap\limits_{\alpha\in\Gamma} C_\alpha$ is closed, we will prove that its complement is open; that is, $\bigcup\limits_{\alpha\in\Gamma} C_\alpha^{\complement}$ is open. Since each C is closed, then each $C^{\complement}$ is open. Then, by problem 7(b), $\bigcup\limits_{\alpha\in\Gamma} C_\alpha^{\complement}$ is also open, and we are done.

   ■

   (b) Professor Doofus writes that in addition $\bigcup\limits_{\alpha\in\Gamma} C_\alpha$ is a closed set. Give an example which shows that Doofus is wrong.
   Answer: Let $\{C_n\}_{n=1}^{\infty}$ be the collection of all $C_n = \overline{B}(\mathbf{0}, 1-{}^1\!/n)$. So since $sup\left\{\left(1-\frac{1}{n}\right) : n\in\mathbb{N}\right\} = 1$, then $\bigcup\limits_{n=1}^{\infty} C_n = B(\mathbf{0},1)$. And we already know that open balls are not closed.