Fall 2018 Topology Qual Solution Sketches

Seth Althauser

April 2, 2020

Note–these solutions were typed very quickly, so please let me know if you spot any egregious mistakes.

1 Problem 1

The metric topology on (X, d) is the topology having as basis all open sets of the form $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. It is an easy exercise to show that this indeed forms a basis for a topology on X.

Now let $A \subseteq X$ and $x \in X$. We will show that x is in the closure of A if and only if x is the limit of a sequence of points in A. To do so, we first prove the following characterization of elements in the closure of some set.

Lemma 1.1. *Let $A \subseteq X$ and X an arbitrary topological space. Then $x \in \overline{A}$ if and only if every open neighborhood U of x intersects A.*

Proof of Lemma: We prove this characterization by the contrapositive, namely $x \notin \overline{A}$ iff there exists an open neighborhood U of x such that $A \cap U = \emptyset$. First suppose $x \notin \overline{A}$. Then there exists some closed set C such that $A \subseteq C$ and $x \notin C$. Then $x \in (X \setminus C) \subseteq (X \setminus A)$, meaning that $(X \setminus C) \cap A = \emptyset$ and $X \setminus C$ is open since C is closed, hence this is the desired open neighborhood of x. Conversely, suppose there exists an open neighborhood U of x such that $A \cap U = \emptyset$. Then $X \setminus U$ is a closed set which contains A and not x, so $x \notin \overline{A}$. □

Now, let (X, d) be a metric space as above and suppose that $x \in \overline{A}$. Then we construct a sequence of balls $B_n = B(x, \frac{1}{n})$, and choose some x_n in each $B_n \cap A$. Note that this intersection is nonempty by the above characterization of the closure. It is easy to prove that this sequence converges to x using the definition of the metric topology and the fact that our sequence of open balls is nested.

Now suppose conversely that there is sequence (x_n) of points in A which converges to x. Then for every open neighborhood U of x, we have that the sequence (x_n) is eventually in U. Because this sequence is comprised of points in A, this means that $U \cap A \neq \emptyset$ and so $x \in \overline{A}$ by the above closure characterization.

2 Problem 2

Proving that with the function $d : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{1}{x} + \frac{1}{y} & \text{if } x \neq y, \end{cases}$$

$(\mathbb{Z}^+, d)$ is a metric space is simple, and just involves a careful description of cases.

To see that $(\mathbb{Z}^+, d)$ is not a complete metric space we exhibit a Cauchy sequence which doesn't converge to a point in $(\mathbb{Z}^+, d)$. The sequence (x_n) defined by $x_n = n$ will work. To see the sequence is Cauchy, note that for $x_n = n$ and $x_m = m$, either $n = m$ and $d(n, m) = 0$ or $d(n, m) = \frac{1}{n} + \frac{1}{m} \leq 2 \max\{\frac{1}{n}, \frac{1}{m}\}$. So given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. We can prove that this sequence doesn't converge to any point in $\mathbb{Z}^+$, as for any $x \in \mathbb{Z}^+$ for points $y \neq x$ we have that $d(x, y) \geq \frac{1}{x} > 0$.

3 Problem 3

A basis for $\mathbb{R}^\omega$ with the product topology is given by all sets of the form $\prod_n U_n$ where the U_i are all open subsets of $\mathbb{R}$ and all but finitely many of them are equal to $\mathbb{R}$. Take an arbitrary point $x = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$. An arbitrary basic open neighborhood U of x is of the form described above, so for the finitely many $U_i \neq \mathbb{R}$ take an arbitrary point $y_i \in U_i$. For the remaining $U_i = \mathbb{R}$, choose $y_i = 0$. Then $y \in U$ and all but finitely many terms in the sequence $y = (y_1, y_2, \ldots)$ are zero, so $y \in A$. Thus $A \cap U \neq \emptyset$ and because U was arbitrary this shows that $x \in \overline{A}$. Because x was an arbitrary point in $\mathbb{R}^\omega$, this proves that $\overline{A} = \mathbb{R}^\omega$, in other words, A is dense in $\mathbb{R}^\omega$ with the product topology.

Now suppose that $\mathbb{R}^\omega$ is given the box topology. We claim in this case that $\overline{A} = A$, in other words A is closed in the box topology. To prove this, we will prove that $\mathbb{R}^\omega \setminus A$ is open. Let $x \in \mathbb{R}^\omega \setminus A$. Then $x = (x_1, x_2, \ldots)$ and $x_i \neq 0$ for infinitely many i. For each such i take an open interval U_i around x_i which doesn't contain zero. Let the remaining $U_i = \mathbb{R}$. Then any point in $U := \prod_n U_n$ must have infinitely many non-zero terms, since infinitely many of the U_i above do not contain zero. So U is an open neighborhood of x contained in $\mathbb{R}^\omega \setminus A$ which shows that $\mathbb{R}^\omega \setminus A$ is open, and therefore A is closed in the box topology.

4 Problem 4

Proving that a metric space is compact if and only if it is sequentially compact is a standard (but tedious) proof. As is almost always the case, one direction is easier.

Let (X, d) be a compact metric space. Suppose for contradiction that X is not sequentially compact. Then there exists a sequence (x_n) with no convergent subsequence. This sequence must have infintely many distinct points, otherwise there is a constant (hence convergent) subsequence. Now for each $x \in X$, there exists an epsilon ball $B(x, \varepsilon_x)$ such that this ball contains no points of the sequence (x_n) except perhaps for x itself. The collection of all such balls forms an open cover of X with no finite subcover, as any finite subcollection only contains finitely many points of the sequence (x_n) and therefore cannot cover all of X. This is a contradiction, so X is sequentially compact.

The other direction is more involved and I don't really want to type it up. One standard proof involves proving first the Lebesgue number lemma for sequentially compact metric spaces, then proving that a sequentially compact metric space is totally bounded, and finally using these results to show compactness. You can find a proof in Munkres as part of Theorem 28.2 for example.

5 Problem 5

(In fact, it should be sufficient that X is a first countable space for this claim to hold.)

We will first need the following lemma.

Lemma 5.1. *If X is first countable and $A \subseteq X$, then $x \in \overline{A}$ if and only if there exists a sequence (x_n) of points in A which converges to x.*

Proof of Lemma 5.1. First let (x_n) be a sequence of points in A which converges to x. Then for all open neighborhoods U of x, this sequence is eventually in U, hence every open neighborhood has non-trivial intersection with A and so $x \in \overline{A}$.

Conversely, suppose $x \in \overline{A}$. Let $\mathcal{B} = \{B_n\}$ be a countable local base for x. Define $U_n := \bigcap_{i=1}^n B_i$, so each U_n is an open neighborhood of A, and hence $U_n \cap A \neq \emptyset$ for all n by characterization of the closure. For each n, choose some $x_n \in U_n \cap A$. This is the desired sequence of points in A which converges to x. $\square$

We now return to the original claim. One direction is trivial: if A is closed in X, then $A \cap K$ is closed in K by definition of the subspace topology.

Suppose conversely that $A \cap K$ is closed in K for all compact subspaces $K \subseteq X$. We will show that A is closed in X.

Let $x \in \overline{A}$. By Lemma 5.1 there exists a sequence (x_n) in A which converges to x. Define the set

$$K := \{x\} \cup \{x_n \mid n \in \mathbb{N}\}.$$

We claim that K is compact. Let $\mathcal{U}$ be an open cover of K by sets open in X. Suppose that $U \in \mathcal{U}$ is such that $x \in U$. Then because the sequence (x_n) converges to x, all but finitely many points in the sequence are contained in U. For each of the remaining points x_i in the sequence such that $x_i \notin U$, take some open neighborhood V_i of x_i from the open cover $\mathcal{U}$. Then $U \cup \{V_i \mid x_i \notin U\}$ is our finite subcover of K, and K is compact.

Since K is compact, by assumption we have that $A \cap K$ is closed in K. Since (x_n) converges to x in X, for every open neighborhood U of x in X, the sequence (x_n) is eventually in U and therefore $U \cap (A \cap K) \neq \emptyset$. Therefore, for every open neighborhood $U \cap K$ of x in K, we have that $(U \cap K) \cap (A \cap K) = U \cap (A \cap K) \neq \emptyset$, and so $x \in \overline{A \cap K}^K$, where the notation here means the closure in the subpsace topology on K. But $A \cap K$ is closed in K, so $\overline{A \cap K}^K = A \cap K$, hence $x \in A \cap K$ and in particular $x \in A$. We have shown that an arbitrary $x \in \overline{A}$ is also in A and so A is closed as claimed.

6 Problem 6

It is a general result that the closure of a connected set is connected.

Suppose for contradiction that $\overline{A}$ is not connected. Then there exist sets U, V open in X such that $(U \cap \overline{A}) \cup (V \cap \overline{A}) = \overline{A}$ and $U \cap \overline{A} \neq \emptyset$ and $V \cap \overline{A} \neq \emptyset$ but $U \cap V \cap \overline{A} = \emptyset$. We claim that $U \cap A$ and $V \cap A$ form a separation of A. First note that since $U \cap \overline{A} \neq \emptyset$, there exists some x in this intersection. Since x is in the closure of A, every open neighborhood of x has nonempty intersection with A, so in particular $U \cap A \neq \emptyset$ and likewise for $V \cap A$.

Also,

$$A = A \cap \overline{A} = A \cap [(U \cap \overline{A}) \cup (V \cap \overline{A})] = (A \cap U) \cup (A \cap V).$$

Finally note that $(U \cap A) \cap (V \cap A) = U \cap V \cap A \subseteq U \cap V \cap \overline{A} = \emptyset$, so this indeed forms a separation of A which is a contradiction. Thus, the closure of a connected set is connected.

For the second part of the problem, one nice way to prove this is using the characterization that a space Y is connected if and only if every continuous function $f : Y \to \{0, 1\}$ is constant. You might be able to take this as the definition of connectedness (you should ask), but it is also a very simple proof to just write out as a lemma.

So suppose that X and Y are both connected spaces and consider a map $f : X \times Y \to \{0, 1\}$. Note that the restriction of a continuous map is continuous, so $f|_{X \times \{y_0\}}$ and $f|_{\{x_0\} \times Y}$ are continuous for all $x_0 \in X$ and $y_0 \in Y$. Note that $X \times \{y_0\}$ is homeomorphic to X, so is connected (easy enough to write out this explicitly if you have time), so the restricted map is constant.

Therefore, for any two points $(x_0, y_0), (x_1, y_1) \in X \times Y$ we have that $f(x_0, y_0) = f(x_0, y_1) = f(x_1, y_1)$ and so f is constant.

Alternatively, we could have shown that the union of connected spaces with a point in common is connected and then gone through the same argument with the union of slices.

7 Problem 7

Let $f : X \to Y$ be a quotient map. Suppose for contradiction that $X = A \cup B$ is a separation of X. The key idea here is that a map f is a quotient map if and only if f is continuous and maps saturated open sets of X to open sets of Y. Note a set is saturated if it contains any fiber with which it has nonempty intersection. Here A and B are in fact saturated open sets exactly because each $f^{-1}(\{y\})$ is connected, and so a separation of X pushes forward to a separation of Y which is a contradiction.

Formally, since f is surjective we know that $Y = f(X) = f(A) \cup f(B)$. We also have that $f(A) \cap f(B) = \emptyset$, otherwise there exists some $a \in A, b \in B$ and $y \in Y$ such that $f(a) = y = f(b)$ and then A and B form a separation of $f^{-1}(y)$ which is a contradiction.

Finally, we show that $f(A)$ and $f(B)$ are open by showing that WLOG $A = f^{-1}(f(A))$. We trivially have that $A \subseteq f^{-1}(f(A))$. Now if $x \in f^{-1}(f(A))$ and $x \in B$, then $f(x) \in f(f^{-1}(f(A))) \subseteq f(A)$ and also $f(x) \in f(B)$, but this contradicts that $f(A) \cap f(B) = \emptyset$, therefore $f^{-1}(f(A)) = A$. So since f was a quotient map, $f^{-1}(f(A))$ is open if and only if $f(A)$ is open, so because A is open, this implies that $f(A)$ is also open. Similarly for $f(B)$ and therefore this would form a separation of Y.

8 Problem 8

Note: I believe that there is a typo in this question and should include the assumption that the space is locally path-connected.

Let (X, x_0) be a path-connected, locally path-connected, and semilocally simply-connected topological space. Then there is a bijective correspondence between the set of all basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$ obtained by associating the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ to the covering space $(\tilde{X}, \tilde{x_0})$.

The canonical example of a space with fundamental group $\langle a, b \rangle$ is the wedge of two circles, $S^1 \vee S^1$. This can be proven using Van Kampen's theorem.

Covering spaces corresponding to the listed groups can be found using, for example, the fact that the fundamental group of a connected graph X (i.e. a 1-dimensional CW complex) is free with basis in one-to-one correspondence with the edges in $X - T$, where T is a maximal tree. The details of this are in the appendix to chapter 1 in Hatcher.