Math 550
Homework 11
Dr. Fuller
Due December 4, 2018

1. Suppose $g : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ is a parameterization of an oriented surface $g(U)$ in $\mathbf{R}^3$. Prove
$$\int_U g^* dA = \int_U \|Dg(u,v)(e_1) \times Dg(u,v)(e_2)\| \, du\, dv.$$
(Remark: This problem is a companion to Homework 9, problem 3. Together, they show that the definitions of line integrals and surface integrals from elementary vector calculus can be viewed as integrals of volume forms on 1- and 2-dimensional manifolds, repectively.)

2. Let $f : S^{2k} \to S^{2k}$ be a C^∞ function. Prove that there exists $\vec{x} \in S^{2k}$ with either $f(\vec{x}) = \vec{x}$ or $f(\vec{x}) = -\vec{x}$.

3. Let $n \geq 2$, and suppose $f : D^n \to \mathbf{R}^n$ is C^∞, with $\|f(\vec{x}) - \vec{x}\| < 1$ for all $\vec{x} \in S^{n-1}$. Prove that there exists $\vec{x} \in D^n$ such that $f(\vec{x}) = 0$.

4. Prove that if M contractible, then M is simply connected.

5. Show that the converse of Exercise 4 is false.

6. (a) Suppose that ω_1 and ω_2 are cohomologous k-forms on a compact oriented k-dimensional manifold M. Prove that
$$\int_M \omega_1 = \int_M \omega_2.$$
   (b) Show that integration over M defines a linear functional
$$\int_M : H^k(M) \to \mathbf{R}.$$
   (c) Suppose that M bounds; that is, suppose M is the boundary of some compact oriented $(k+1)$-dimensional manifold. Show that $\int_M$ is zero.

7. (a) Prove that a closed n-form ω on S^n is exact if and only if $\int_{S^n} \omega = 0$.
   (b) Prove that the linear function $\int_{S^n} : H^n(S^n) \to \mathbf{R}$ is an isomorphism.

8. For $\ell > 0$, prove
$$H^\ell(\mathbf{R}^k - \{(-1,0,\ldots,0),(1,0,\ldots,0)\} \cong \begin{cases} \mathbf{R}^2 & \text{if } \ell = k-1, \\ 0 & \text{if } \ell \neq k-1. \end{cases}$$