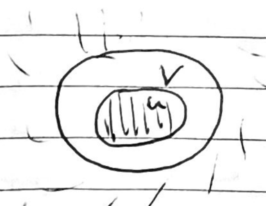
Topology Spring 2018

2. An ultrafilter on a set $X$ is a collection $\mathcal{U}$ of subsets of $X$ with the properties that



(i) If $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$

(ii) If $U \subset X$ then $U \in \mathcal{U}$ or $X \setminus U \in \mathcal{U}$

(iii) $\emptyset \notin \mathcal{U}$

(a) Show that if $V \supset U$ & $U \in \mathcal{U}$ then $V \in \mathcal{U}$

If $V \notin \mathcal{U}$ then $X \setminus V \in \mathcal{U}$. Then $X \setminus V \cap U \in \mathcal{U}$ but $X \setminus V \cap U = \emptyset$ ※ so $V \in \mathcal{U}$.

(b) Show that $\mathcal{T} := \mathcal{U} \cap \{\emptyset\}$ is a topology on $X$.

Since $\emptyset \notin \mathcal{U} \Rightarrow X - \emptyset = X \in \mathcal{U}$ so $X$ & $\emptyset \in \mathcal{T}$. We will prove that if $U_1, \ldots, U_n \in \mathcal{T}$ then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$ by induction. Case $n=2$ is condition (i). Assume the result holds up to $n-1$. Then consider $\bigcap_{i=1}^{n} U_i = \bigcap_{i=1}^{n-1} U_i \cap U_n$. $U_n \in \mathcal{U}$ & $\bigcap_{i=1}^{n-1} U_i \in \mathcal{U}$ by induction so $\bigcap_{i=1}^{n} U_i \in \mathcal{U}$. Now consider $\{U_\alpha\}_{\alpha \in A}$ $U_\alpha \in \mathcal{T}$. If $\bigcup U_\alpha \notin \mathcal{T}$ then $\bigcup U_\alpha \neq \emptyset$ & $\bigcup U_\alpha \notin \mathcal{U} \Rightarrow$ $X \setminus \bigcup U_\alpha \in \mathcal{U} \Rightarrow (X \setminus \bigcup U_\alpha) \cap U_\alpha = \emptyset \in \mathcal{U}$ ※

→ (c) If $X$ is infinite show that the topology is connected, but not compact & not Hausdorff.

Suppose $\exists\, U, V \in \mathcal{T} \setminus \{\emptyset\}$ s.t. $U \cap V = \emptyset \Rightarrow \emptyset \in \mathcal{U}$ ※. So $X$ is connected. Let $U_1 \in \mathcal{U}$ & create an open cover of $X$ by $U_1 \subset U_1 \cup \{x_1\} = U_2$ where $x_1 \notin X$. Start with $U_1$ so that $X - U_1$ is not finite. Then $\{U_i\}$ is an open cover of $X$ with no finite subcover. Otherwise, there would be a maximal & this would not work.

Finite Intersection Property