Harmonic Analysis - PCMI, 2018

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Contents

Note: If you find any typos in these notes, please let me know at trevor.klar.834@my.csun.edu. If you could include the page number, that would be helpful.

These notes were taken during the lectures of the Park City Mathematics Institute, Undergraduate Summer School 2018. Lectures given by Eyvindur Palsson at The Prospector in Park City, Utah.

1 Introduction (Palsson)

1.1 Combinatorics

1. The Erdös distinct distance problem

Question (Erdös, 1947): What is the least number of distinct distances determined by n points in a plane?

Example (1.1). asdfgasdf

There is clearly an upper bound here, $\binom{N}{2} = \frac{N(N-1)}{2} \sim N^2$, where $\sim$ denotes "is of the order".

Example (1.2). sdamsdf

The problem here, is that there are some elements of this list that are not possible. i.e., there are no integers such that $a^2 + b^2 = 3$. So Ramanujan showed that

$$\lim_{N\to\infty} (\text{Number of distinct distances}) \sim \frac{N}{\sqrt{log(N)}}$$

Erdös conjectured that even for random placements of points,

$$\lim_{N\to\infty} (\text{Number of distinct distances}) \sim \frac{N}{\sqrt{log(N)}}.$$

He was only able to prove in 1946 that

$$\lim_{N\to\infty} (\text{Number of distinct distances}) \sim \text{at least } \sqrt{N}$$

In 2015, Guth and Katz showed that it holds for

$$\sim \frac{N}{\log(N)}.$$

1.2 Crescent Configurations

Consider N points in the plane. Suppose some distance d_1 appears once, d_2 appears 2 times, $\ldots, d_{N-1}$ appears $N-1$ times.

What does this look like?

Answer Equidistant points on a line.

Okay, so suppose you require *general position.* This means no more than 2 points are collinear, and no more than 3 points are concyclic. Call this a *crescent configuration.*

Do these exist?

Example (Palasti, 1989). asdsokmoim

Conjecture (Erdös): Eventually they don't exist.

Question: Find many (all) crescent configurations for some N.

- $N = 4$, all are known. (add image)
- $N = 5$, many but not all are known.

2 Introduction (Saenz)

Motivation: Diffusion

- $u(x, y, t) =$ temperature at time t.
- $H(t) = \int\int_S u(x, y, t) dx\, dy$
- $\frac{dH}{dt}(t) = \int\int_S \frac{\partial u}{\partial t}(x, y, t) dx\, dy$
- heat equation
$$\sigma \frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$
- at equilibrium: $\frac{\partial u}{\partial t} = 0$:
  Laplace Equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$
  that is $\Delta u = 0$. We say Δu is the Laplacian of u.
- Dirichlet problem
$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f & \text{given } f \end{cases}$$

Definition. A **harmonic function** is a function such that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

3 Day 2 (Palsson)

Today's Perspective: Linear Algebra

- Let V be a set of vectors, F scalars
- you know how + scalar $\cdot$
- the usual axioms of a v. space.

Example. $C[a, b]$ is the space of continuous functions defined on $[a, b]$ (this is a vector space.)

Complex Numbers sidebar

- $\mathrm{Re}(x+iy) = x$
- $\mathrm{Im}(x+iy) = y$
- $\overline{x+iy} = x - iy$
- $z\overline{z} = \left|z^2\right|$
- Euler's Formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

From now on, think of V as a function space, and F as either $\mathbb{R}$ or $\mathbb{C}$.

Definition. An **inner product** is a map

$$\langle \cdot\,, \cdot \rangle : V \times V \to F$$

such that for all $f, g, h \in V$ and $\alpha, \beta \in F$,

- **(Positive definite)** $\langle f, f \rangle \geq 0$ and if $\langle f, f \rangle = 0$, then $f = 0$.
- **Conjugate commutation** $\langle f, g \rangle = \overline{\langle g, f \rangle}$
- **Left distribution** $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$

Example.

$\mathbb{R}^n$) $\quad \langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y}$

$\mathbb{C}^n$) $\quad \langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \overline{\vec{y}}$

$\mathcal{C}[a,b]$) $\quad \langle f, g \rangle = \frac{1}{b-a} \int_a^b f(x)\overline{g(x)}dx$

Definition. A **norm** is a function $||\cdot|| : V \to \mathbb{R}$ such that for all $f, g \in V$ and $\alpha \in F$,

- **(Positive definite)** $||f|| \geq 0$
- **(Scaling)** $||\alpha f|| = |\alpha|\, ||f||$
- **(Triangle inequality)** $||f + g|| \leq ||f|| + ||g||$

Note: In a vector space with inner product $\langle \cdot\,, \cdot \rangle$, then

$$||f|| := \sqrt{\langle f, f \rangle}$$

is a norm.

Theorem (Cauchy-Schwartz Inequality).

$$|\langle f, g\rangle| \leq ||f||\,||g||$$

Theorem. One can use the CS inequality to prove the triangle inequality:

Proof

$$\begin{aligned}
||f+g||^2 &= \langle f+g, f+g\rangle \\
&= ||f||^2 + \langle f, g\rangle + \overline{\langle f, g\rangle} + ||g||^2 \\
\text{(CS)} \quad &\leq ||f||^2 + 2\,||f\,||g|||| + ||g||^2 \\
&= \big(\,||f|| + ||g||\,\big)^2
\end{aligned}$$

■

Example (of a norm). On $\mathcal{C}[a, b]$,

$$||f||_2 = \left(\frac{1}{b-a}\int_a^b |f(x)|^2\,dx\right)^{1/2}$$

Definition. We say that $f, g \in V$ are **orthogonal** if iff

$$\langle f, g\rangle = 0.$$

We say that $\phi_1, \ldots, \phi_n$ are **orthogonal** iff

$$\langle \phi_j, \phi_k\rangle = 0 \text{ if } j \neq k$$

$$\text{and } \phi_i \neq 0\ \forall i.$$

If $\phi_1, \ldots, \phi_n$ are orthogonal and of unit length, then we say they are **orthonormal**.

Corollary. Pythagorean Theorem If $\phi_1, \ldots, \phi_n$ are orthogonal, then

$$\left|\left|\sum \phi_i\right|\right|^2 = \sum ||\phi_i||^2$$

Definition (Projection). let $\phi \in V$ with $||\phi|| = 1$.

$$\mathrm{proj}_\phi(f) = \langle f, phi\rangle\,\phi.$$

Definition. Let W_n be a subspace of V with an orthonormal basis $\{\phi_1, \ldots, \phi_n\}$. The **projection of f onto W_n** is

$$\mathrm{proj}_{W_n}(f) = \langle f, \phi_1 \rangle \phi_1 + \cdots + \langle f, \phi_n \rangle \phi_n = \sum_{j=1}^{n} \langle f, \phi_j \rangle \phi_j.$$

Theorem. $\mathrm{proj}_{W_n}(f) = f$ if and only if $f \in W_n$.

Theorem. $f - \mathrm{proj}_{W_n}(f)$ is orthogonal to every $g \in W_n$.

4 Day 3 (Palsson)

Theorem. Let $f \in V$, and let W_n be a subspace of V with an orthonormal basis $\{\phi_1, \ldots, \phi_n\}$. Then $w \in W_n$ that minimizes $||f - w||$ is

$$w = \mathrm{proj}_{W_n}(f).$$

Proof Since $w \in W_n$, we can write it as $w = \sum_{i=1}^{n} \beta_i \phi_i$. Let $\alpha_i = \langle f, \phi_i \rangle$. Now,

$$\begin{aligned}
||f - w||^2 &= \langle f - w, f - w \rangle \\
&= ||f||^2 - \langle f, \sum \beta_i \phi_i \rangle - \overline{\langle f, \sum \beta_i \phi_i \rangle} + \langle \sum \beta_i \phi_i, \sum \beta_j \phi_j \rangle \\
&= ||f||^2 - \sum \overline{\beta_i} \alpha_i - \sum \beta_i \overline{\alpha_i} + \sum ||\beta_i||^2 \\
&= ||f||^2 + \left(\sum ||\alpha_i||^2 - \sum \overline{\beta_i} \alpha_i - \sum \beta_i \overline{\alpha_i} + \sum ||\beta_i||^2 \right) - \sum ||\alpha_i||^2 \\
&= ||f||^2 + \left(\sum (\beta_i - \alpha_i) \overline{(\beta_i - \alpha_i)} \right) - \sum ||\alpha_i||^2 \\
&= ||f||^2 + \sum |\beta_i - \alpha_i|^2 - \sum |\langle f, \phi_i \rangle|^2
\end{aligned}$$

Now, since f, ϕ_i, α_i are all given, we can see that the above expression is minimized when we choose $\beta_i = \alpha_i$. ■

Corollary.

$$\sum_{i=1}^{n} |\langle f, \phi_i \rangle|^2 \leq ||f||^2$$

Theorem (Bessel's Inequality). The above holds for infinite sums, as long as $||f||$ is finite.

Theorem (Riemann-Lebesgue Lemma). As long as the above hold,

$$\lim_{i \to \infty} \langle f, \phi_i \rangle = 0.$$

Motivation.

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(nx)\cos(mx)dx = \begin{cases} 0 & n \neq m \\ \frac{1}{2} & n = m \neq 0 \\ 1 & n = m = 0 \end{cases}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(nx)\sin(mx)dx = \begin{cases} \frac{1}{2} & n = m \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(nx)\cos(mx)dx = \begin{cases} 0 & \text{always} \end{cases}$$

So, these vectors must be scaled to be ortho**normal**.

$$\left\{1, \sqrt{2}\cos(nx), \sqrt{2}\sin(mx) : n, m \in \mathbb{Z}\right\} \text{ is orthonormal.}$$

Now we have the best approximation for a function on $[0, 2\pi)$ is

$$a_0 + a_1\sqrt{2}\cos(x) + b_1\sqrt{2}\sin(x) + \dots$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi}\langle f(x), 1\rangle\, dx \\ a_j &= \frac{1}{2\pi}\int_0^{2\pi}\left\langle f(x), \sqrt{2}\cos(jx)\right\rangle dx \\ b_k &= \frac{1}{2\pi}\int_0^{2\pi}\left\langle f(x), \sqrt{2}\sin(kx)\right\rangle dx \end{aligned}$$

Fourier Trigonometric Series.

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\Big(A_n\cos(nx) + B_n\sin(nx)\Big)$$

where

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx \quad \text{(including n=0)} \\ B_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx \end{aligned}$$

Simplification (?). We could have started with the orthonormal sequence.

$$e^{inx}, \quad n \in \mathbb{Z} \text{ on } [0, 2\pi)$$

Let's verify normality.

$$\begin{aligned} \left\langle e^{inx}, e^{imx}\right\rangle &= \frac{1}{2\pi}\int_0^{2\pi} e^{inx}\overline{e^{imx}}dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)x}dx \\ &= \frac{1}{2\pi}\left[\frac{1}{i(n-m)}e^{i(n-m)x}\right]_0^{2\pi} \\ &= \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases} \end{aligned}$$

Fourier Exponential Series. This gives us the series

$$\sum_{n\in\mathbb{Z}} C_n e^{inx}$$

where

$$\begin{aligned} C_n &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{e^{inx}}dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx \end{aligned}$$

Note.

$$\begin{aligned} C_n &= \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} f(x)[\cos(nx) - i\sin(nx)]dx \\ &\quad \text{(Note that the - has vanished and moved, since cos and sin are even and odd, respectively.)} \\ &= \begin{cases} \frac{1}{2}(A_n - iB_n) & n > 0 \\ \frac{1}{2}A_0 & n = 0 \\ \frac{1}{2}(A_{|n|} + iB_{|n|}) & n < 0 \end{cases} \end{aligned}$$

So, for all $n > 0$,

$$C_n e^{inx} + C_{(-n)}e^{i(-n)x} = A_n\cos(nx) + B_n\sin(nx)$$

Intervals.

- e^{inx} works on any interval of length 2π.
- $e^{2\pi inx/L}$ works on any interval of length L.

Fourier Series. If f is integrable on the interval $[a, b]$ of length L, then the nth **Fourier coefficient** of f is

$$\hat{f}(n) = \frac{1}{L}\int_a^b f(x)e^{-2\pi inx/L}dx, \quad n \in \mathbb{Z}$$

and the **Fourier series** of f is given by

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx/L}$$

5 Day 4 (Palsson)

Question: Does $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx/L}$ converge to $f(x)$?

Example. $f(x) = xon[0, 2\pi)$

$$\hat{f}(x) = \frac{1}{2\pi}\int_0^2 \pi x e^{-inx} dx = \text{int. by parts} = \frac{i}{n}, n \neq 0$$

Uniqueness.

Theorem. Suppose f is integrable, and bounded on an interval with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$.

Then $f(x_0) = 0$ whenever f is continuous at x_0.

Theorem. Suppose f is continuous on the circle and the Fourier series of f is absolutely convergent

$$\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right| < \infty.$$

Then,

$$\lim_{N\to\infty} S_n(f)(x) = f(x)\text{uniformly in } x.$$

Proof Since $\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right| < \infty$, then $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ converges absolutely and in fact, uniformly. If $g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$, then g must be continuous.

Since $f - g$ is continuous and $\widehat{f - g}(n) = 0$ for all $n \in \mathbb{Z}$ then we conclude by uniqueness theorem that $f = g$.

$$\begin{aligned} \hat{g}(n) &= \tfrac{1}{2\pi}\int_{n=0}^{2\pi} g(x)e^{-inx}dx \\ &= \tfrac{1}{2\pi}\int_{n=0}^{2\pi}(\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx})e^{-inx}dx \end{aligned}$$

(more proof in picture) ∎

Morse's Lemma and more

Theorem. Let $G_T(x) = e^{-\pi i\langle Tx,x\rangle}$ where T is an invertible, real symmetric matrix with signature $\sigma = k_+ - k_-$ where the ks are the numbers of positive and negative eigenvalues of the matrix (counting multiplicity).

Then,

$$\hat{G}_T = e^{-\pi i\frac{\sigma}{4}} \left|\det T\right|^{1/2} G_{-T^{-1}}.$$

Theorem. Let T be a invertible, real symmetric matrix with signature σ, $a \in C_c^\infty$ and define

$$I(\lambda) = \int e^{-\pi i\lambda\langle Tx,x\rangle}a(x)dx.$$

Then for any N,

$$I(\lambda) = e^{-\pi i \frac{\sigma}{4}} \left|\det T\right|^{1/2} \lambda^{\frac{-n}{2}} \left(a(0) + \sum_{j=1}^{N} \lambda^{-j} D_j a(0) + O(\lambda^{-(N+1)}) \right).$$

Where D_j are explicit homogeneous constant coefficient differential operators of order $2j$.

Lemma 1. Suppose ϕ is smooth with $\nabla\phi(p) = 0$ and G is a smooth diffeomorphism $G(0) = p$. Then,

$$H_{\phi\circ G}(0) = DG(0)^T H_\phi(p) DG(0).$$

Thus $H_\phi(p)$ and $H_{\phi\circ G}(0)$ have the same signature and

$$\det(H_{\phi\circ G}(0)) = \det(DG(0))^2 \det(H_\phi(p)).$$

6 measure theory

Definition. A **measure** of a set E is a function $\phi : \mathscr{P}(E) \to \mathbb{R}$ with these properties:

- Non-negativity
- Null empty set
- Countable additivity

Theorem. For an elementary function

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

with disjoint sets A_i define

$$\int f d\mu = \int f(x) d\mu(x) := \sum_{i=1}^{n} \alpha_i \mu(A_i),$$

and we can extend this to more general functions by

$$\int f d\mu = \sup \left\{ \int g(x) d\mu(x) : g \leq f, g \text{ is elementary} \right\}.$$

In particular,

$$\mu(A) = \int_A d\mu.$$

Example. Fourier transform of a measure:

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \xi} d\mu(x)$$

Definition (Hausdorff Dimension). Let $E \subseteq \mathbb{R}^d$. for $\alpha, \epsilon > 0$ define

$$H_\alpha^\epsilon(E) = \inf \left(\sum_{j=1}^{\infty} r_j^\alpha \right)$$

where E is covered by balls of radius $r_j < \epsilon$. We also define the Hausdorff measure as

$$H_\alpha(E) = \lim_{\epsilon \to 0^+} H_\alpha^\epsilon(E).$$

Definition. We say $\dim_H(E) = \alpha_0$ if

- $H_\alpha(E) = \infty$ for $\alpha < \alpha_0$ and
- $H_\alpha(E) = 0$ for $\alpha\, \alpha_0$.

Lemma 2 (Frostman's Lemma). for $A \subseteq D$ compact, $H_s(A) > 0$ if and only if there is a probability measure μ supported on A and a constant $C > 0$ such that

$$\mu(B(x,r)) \leq Cr^s \text{ for all } x \in \mathbb{R}^d, r > 0 \qquad (*)$$

In particular,

$$\dim_H(A) = \sup\{s : \text{there is a measure } \mu \text{ such that } (*) \text{ holds.}\}$$

Such a measure which satisfies $(*)$ is often called a Frostman measure.

Idea: if you have μ satisfying $(*)$ and B_j are balls covering A of radius r_j,

$$\sum_j r_j^s \geq \frac{1}{C}\sum_j \mu(B_j) \geq \frac{1}{C}\mu(A) = \frac{1}{C} > 0$$

whhich implies the forward direction. the other way is *hard.*

Definition (Energy integrals). the s-energy, $s > 0$ of a measure μ

$$I_s(\mu) = \int_{\mathbb{R}^{2d}}\int |x-y|^{-s}\, d\mu(x)\, d\mu(y)$$

Idea:

$$\begin{aligned} I_s(\mu) &= \int\int |x-y|^{-s}\, d\mu(x)\, d\mu(y) \\ &= \int\int \widehat{|x-y|^{-s}}(\xi) e^{2\pi i(x-y)\cdot\xi} d\xi\, d\mu(x)\, d\mu(y) \\ &= \int \gamma(s,d)\, |\xi|^{s-d} \int e^{2\pi i x\xi} d\mu(x) \int e^{-2\pi i y\xi} d\mu(y) d\xi \\ &= \gamma(s,d) \int |\widehat{\mu}(\xi)|^2\, |\xi|^{s-d}\, d\xi \end{aligned}$$

Theorem. For $A \subseteq \mathbb{R}^d$ compact,

$$\dim_H(A) = \sup\{s : \exists \text{ a measure } \mu \text{ supported on } A \text{ such that } I_s(\mu) < \infty\}$$

Definition (Erdös disctict distance problem). let

- $E = \{x_\alpha\}_{\alpha \in \Gamma} \in \mathbb{R}^2$
- $\Delta(E) = \{|x_i - x_j| : 1 \leq i < j \leq N\}$
- $\#(\Delta(E)) \geq ?$
- Guth-katz $\#(\Delta(E)) \gtrsim \frac{N}{\log(N)}$ as $N \to \infty$

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