201 Real Analysis
Assignment 1

1. Let c_0 be the vector space of sequences limiting to 0 with the $\|\cdot\|_{l^\infty}$-norm. Prove that c_0 is a closed subspace of l^∞ (and hence is a Banach space). Prove that $l^1 \cong c_0^*$ in the following sense. For every $f = (f_n) \in l^1$ define
$$F_f(x) = \sum_{n=1}^{\infty} x_n f_n, \quad x = (x_n) \in c_0.$$
Prove that $F_f \in c_0^*$, $\|F_f\|_* = \|f\|_{l^1}$, and for every $\phi \in c_0^*$ there exists $f \in l^1$ such that $\phi = F_f$.

2. Let X be a *Banach* space, $E \subset X^*$. Suppose for every $x \in X$ the set $\{\phi(x)|\ \phi \in E\} \subset \mathbf{R}$ is bounded.
Prove that E is strongly bounded in X^*.
Explain why your proof collapses if X is not complete.

3. Let X be a Banach space and (ϕ_j) be a sequence in X^*. Suppose that $\langle \phi_j, x\rangle$ converges for any $x \in X$. Prove that there exists $\phi \in X^*$ such that $\phi_j \xrightarrow{w*} \phi$. (In fancy terminology "X^* is always $w*$ sequentially complete".)
Formulate the analogous statement for the w-convergence for a sequence (x_j) in X. Try to extend your proof to this situation. When does the proof collapse? (The statement actually does not hold. Some assumptions are needed for X to be w sequentially complete.)

4. Let X be Banach. Prove that a sequence (ϕ_j) in X^* converges $w*$ if and only if it is strongly bounded and there exists a dense set E, $\overline{E} = X$, such that the number sequence $\langle \phi_j, u\rangle$ converges for all $u \in E$.

5. Let $I = [0,1]$. Let $C^1(I)$ denote the space of continuously differentiable functions g, so $g, g' \in C(I)$. (For example f is a polynomial.) Let $d\phi_n = \cos(\pi n x)\, d\lambda^1(x)$. Prove that
$$\int_I g\, d\phi_n \to 0, n \to \infty, \quad \forall g \in C^1(I).$$
Prove that $\phi_n \to 0$ weakly* as measures in $C(I)^*$. (Hint: for g integrate by parts. For the weak* convergence use Wejerstrass approximation theorem.)

The problems below will not be graded and are not obligatory. However, if you are thinking of choosing "analysis" for your research subject, then it's a good idea to attempt to solve them.

6. Let $1 \le p < \infty$, and let (x_n) be a sequence in l^p, $x_n = (x_{n1}, x_{n2}, \ldots) \in l^p$. Prove that
$$x_n \xrightarrow{w} 0 \Leftrightarrow (x_n) \text{ is strongly bounded and } \forall i \ \lim_{n\to\infty} x_{ni} = 0.$$