A Problem from Erdös About Products of 2 or 3 Primes

Joint Mathematics Meeting

Eli Moore & Trevor Klar

California State University, Northridge

Research Information:

This research was funded by the PUMP-URG Program under the NSF grant DMS-1244679.

Advisor: Dr. Werner Horn

werner.horn@csun.edu

CALIFORNIA STATE UNIVERSITY NORTHRIDGE

Abstract

Paul Erdös asked the following question about three distinct primes: If we construct a sequence of all the products of their powers, with the sequence arranged in increasing order, is it true infinitely often that consecutive terms in this sequence are both prime-powers?[1] The way the question is stated indicates that the situation is known for two primes. However, we were not able to find a reference for this, and we suspect that he may have found this problem to be too easy to be worth publishing. Erdös' assumed opinion notwithstanding, our team did not find the solution to be trivial. We explore this question with two distinct primes, and give the proof that it does happen infinitely often. We also further explore the question with three distinct primes.

Definitions and Introductory Lemmas

Let p, q be distinct primes, and let $a, b \in \mathbb{Z}^+$.

Definition. A *pure power* of p is an integer of the form p^a.

Definition. A *mixed power* of p and q is an integer of the form $p^a q^b$.

Definition. A *critical pair* of p and q is a pair of pure powers of p and q which do not have an intermediate mixed power.

Example:

$$\begin{array}{cccccccccccccc} 2 & 3 & 4 & 6 & 8 & 9 & 12 & 16 & 18 & 24 & 27 & 32 & 36 & \ldots \\ 2^1 & 3^1 & 2^2 & 2^1 3^1 & 2^3 & 3^2 & 2^2 3^1 & 2^4 & 2^1 3^2 & 2^3 3^1 & 3^3 & 2^5 & 2^2 3^2 & \ldots \end{array}$$

To develop some intuition about the problem, we first proved a few introductory results.

For example, powers of the larger of the two primes cannot fall consecutively in the sequence.

Lemma 1. If $a_k = q^n$, then $a_{k+1} \neq q^{n+1}$.

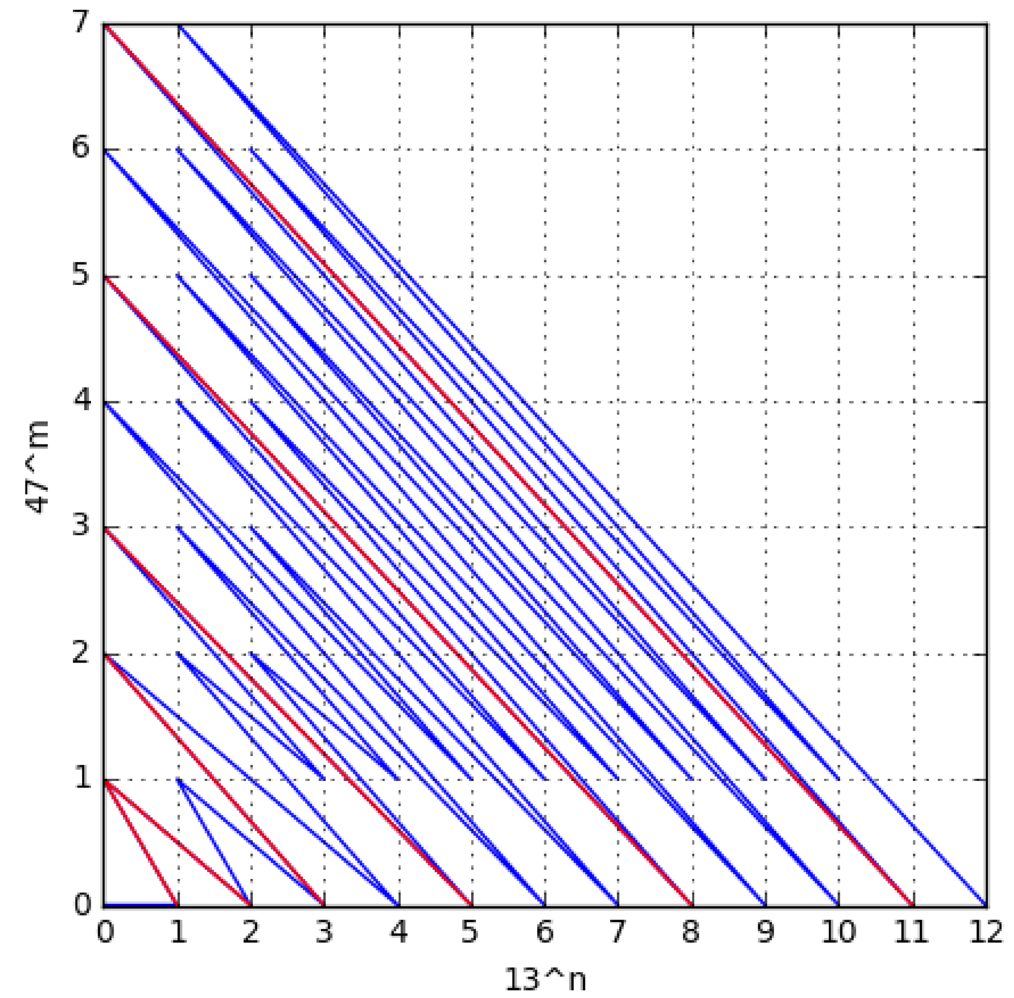
Powers of the smaller primes can happen consecutively, but only finitely many times, and then never again.

Lemma 2. There exist at most finitely many $a_k = p^n$ such that $a_{k+1} = p^{n+1}$.

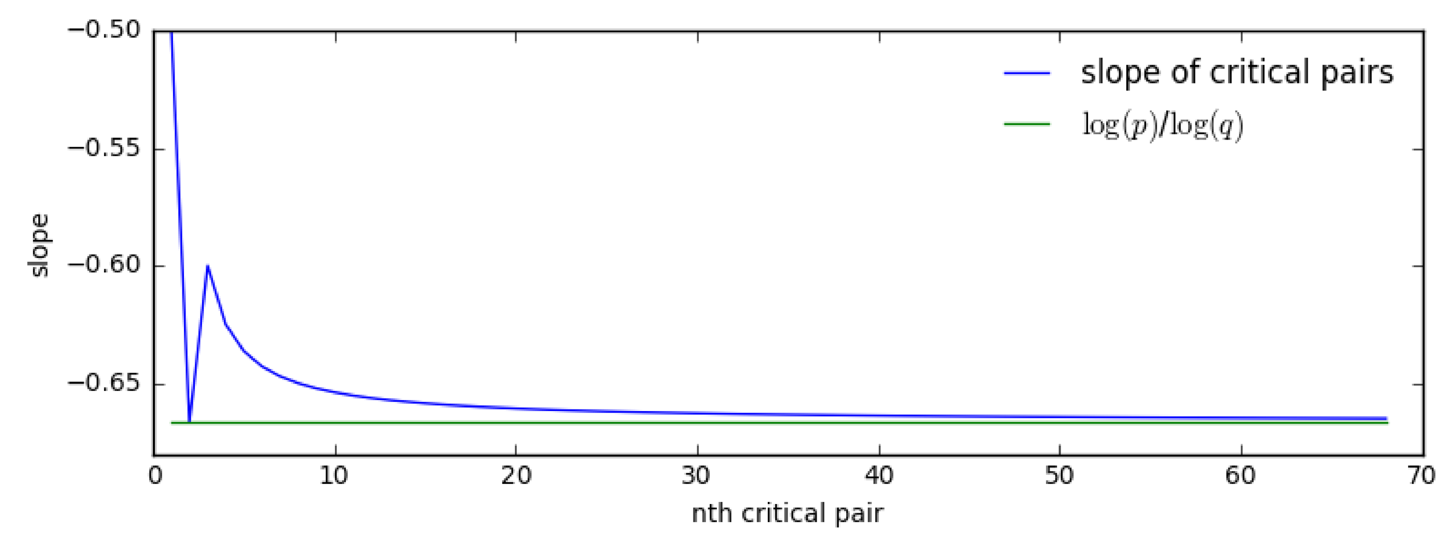
Lemma 3. If $a_i = p^m$ and $a_{i+1} = q^n$, then m and n are relatively prime.

Numerical Exploration

We used Python to plot the sequence, with $p = 13$ and $q = 43$. The blue line shows the order they fall in the sequence, and red segments denote critical pairs.



Notice how the lines seem to approach a constant slope! We found that as the sequence progresses, the ratio $\frac{a_k}{a_{k+1}}$ approaches $\frac{\log p}{\log q}$.



The Goal

After developing an understanding for the problem, we began to analyze the details that Erdös glossed over. This lead us to a proof for the following theorem:

Theorem 4. *For any two distinct prime numbers p, and q, there exist infinitely many critical pairs.*

By approaching this theorem via contradiction, we find that by assuming there exist finitely many critical pairs, we can always develop another critical pair in the sequence.

Key Lemmas (Used to Prove Theorem 1)

Lemma 5. Consider the pure powers p^a, q^b with $p^a < q^b$ and $a, b \in \mathbb{Z}^+$. If, for all critical pairs p^s, q^t with $s < a$ and $t < b$,

$$1 < \frac{q^b}{p^a} < \frac{q^t}{p^s}, \quad s, t \in \mathbb{Z}^+$$

then p^a, q^b is a critical pair.

We proved the above lemma by contradiction. The intuition for this lemma comes from the following fact: if the ratio of the pure powers in the pair (p^a, q^b) are be closer to 1 than any other pair of pure powers (many of which are critical pairs on their own), then we have (p^a, q^b) as a critical pair as well. This will allow us to approach Theorem 1 by contradiction.

Lemma 6. Let α be an irrational number. Given any $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $n\alpha - \lfloor n\alpha \rfloor < \varepsilon$.

The above is actually a close relative of Dirichlet's Approximation Theorem. Through this approximation theorem, we are able to construct a critical pair that has ratio closer to 1 than any other critical pair, which is how we reach our contradiction.

Moving Forward: Difficulties with Three Primes

When considering the problem with three primes, the issue becomes much more complicated. We refer to the primes as p, q,and r, We can use the result already proved to obtain inifitely many critical pairs of p and q, but for any such pair, some $p^m q^n r^\ell$ could potentially fall between them. We numerically explored the spacing of critical pairs to try to tackle this issue, but found the spacing to be too irregular to be useful in a proof.

We also approached the problem geometrically, visualizing the space of all possible mixed and pure powers as a 3-dimensional lattice,

Acknowledgments

We would like to thank Dr. Werner Horn for his guidance through the semester and project. Thank you to Dr. Helena Noronha and the rest of the PUMP program for organizing this program and giving students such as ourselves the opportunity to create undergraduate research.[2]

[1] Guy, R. K., "A Series and a Sequence Involving Primes." In *Unsolved Problems in Number Theory, 2nd ed.* New York: Springer-Verlag, p. 203, 1994.

[2] The authors can be contacted at werner.horn@csun.edu, eli.moore.768@my.csun.edu, and trevor.klar.834@my.csun.edu respectively.