Math 501
Homework 3

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1. Prove

   Theorem. 1: (Openness Criterion) *Let $(X, \mathscr{T})$ be a topological space. A set $S \subset X$ is open if and only if for every $x \in S$, there exists an open set $U_x \subset S$.*

   Proof
   $\Longrightarrow$: Suppose $S \subset X$ is open. For any $x \in S$, let $U_x = S$. U_x is an open set such that $x \in U_x \subset S$, so we are done. $\square$
   $\Longleftarrow$ Suppose for every $x \in S$, there exists an open set $U_x \subset S$. Consider

   $$\bigcup_{x \in S} U_x.$$

   Now, every element of $\bigcup_{x \in S} U_x$ is in S, and every element of S is in $U_x \subset \bigcup_{x \in S} U_x$, so $\bigcup_{x \in S} U_x = S$. Since any arbitrary union of open sets is open, S is open. $\blacksquare$

2. Prove that in a Hausdorff space, a set consisting of a single point is a closed set.

   Proof Let $(X, \mathscr{T})$ be a Hausdorff space, and let x_0 be any point in X.
   Claim: $S = X - \{x_0\}$ is open, so $\{x_o\}$ is closed.
   Since $(X, \mathscr{T})$ is Hausdorff, for any $x \in X$ which is distinct from x_0, there exist open sets $U_x \in \mathscr{T}$ and $V_x \in \mathscr{T}$ such that

   $$x \in U_x,\ x_0 \in V_x, \text{ and } U_x \cap V_x = \emptyset.$$

   Now consider $\bigcup_{x \in S} U_x \equiv \bigcup U_x$. For every $x \in S$,

   $$x \in U_x \subset \bigcup U_x,$$

   and every $U_x \subset S$ since $x_0 \notin U_x \subset X$, which means that $\bigcup U_x \subset S$. Therefore, $\bigcup U_x = S$.

   Thus, we have shown that S can be written as a union of open sets, and since an arbitrary union of open sets is open, $S = X - \{x_0\}$ is open in X, so $\{x_o\}$ is closed in X. $\blacksquare$

3. Let U be open and C closed subsets of a space X, with $C \subset U$. Prove that $U - C$ is open.

   Proof By definition of set subtraction, $U - C = U \cap (X - C)$, and since C is closed in X, $(X - C)$ is open. Thus, $U - C$ is the intersction of two open sets, so it is open. $\blacksquare$

4. Let A be a subset of a space X. Prove that C is closed in A if and only if $C = A \cap F$, where F is closed in X.

Proof

$\Longrightarrow$: Assume C is closed in A. By definition of closed in A, there exists some U which is open in X such that

$$A - C = A \cap U.$$

Let $F = X - U$. Now, F and U are complements in X, and $A \subset X$. This means that $A \cap U$ and $A \cap F$ are complements in A. Thus, taking the complement in A of both sides, we find that

$$A - C = A \cap U \implies C = A \cap F.$$

This completes the proof. ■

$\Longleftarrow$: Assume $C = A \cap F$, where F is closed in X. Let $U = X - F$. Then, using the same reasoning as above, we can take the complement in A of both sides to obtain

$$C = A \cap F \implies A - C = A \cap U.$$

Thus, C is closed in A by definition. ■

5. Find $\overline{A}$, int A, and A^ℓ for the following sets A in $\mathbb{R}^2$. (Just answers, no proofs.)

(a) $A = \{(x, 0) : 0 \leq x < 1\}$
- $\overline{A} = \{(x, 0) : 0 \leq x \leq 1\}$
- $\text{int}(A) = \emptyset$
- $A^\ell = \overline{A} = \{(x, 0) : 0 \leq x \leq 1\}$

(b) $A = \{(x, y) : x^2 + y^2 \leq 10\}$
- $\overline{A} = A$
- $\text{int}(A) = \{(x, y) : x^2 + y^2 < 10\}$
- $A^\ell = A$

(c) $A = \{(x, y) : x, y \in \mathbb{Q},\ x^2 + y^2 \leq 10\}$
- $\overline{A} = \{(x, y) : x^2 + y^2 \leq 10\}$
- $\text{int}(A) = \emptyset$
- $A^\ell = \overline{A} = \{(x, y) : x^2 + y^2 \leq 10\}$

(d) $A = \{(x, y) : x, y \in \mathbb{Z},\ x^2 + y^2 \leq 10\}$
- $\overline{A} = A$
- $\text{int}(A) = \emptyset$
- $A^\ell = \emptyset$

6. Let $\mathbb{R}^1_f$ denote the real numbers endowed with the finite complement topology. What is the set of limit points of $\mathbb{Z}$ in $\mathbb{R}^1_f$?

Claim: $\mathbb{Z}^\ell$ in $\mathbb{R}^1_f$ is $\mathbb{R}$.

Proof In the finite complement topology on $\mathbb{R}$, every open set has a finite complement; that is, there is a greatest element of $\mathbb{R}$ which the set does not contain. This means every open set is unbounded above. Since $\mathbb{Z}$ is also unbounded above, every set which is open in $\mathbb{R}$ contains elements of $\mathbb{Z}$. Thus, for any real number x, every open set containing x also contains elements of $\mathbb{Z}$ distinct from x, so x is a limit point. ■

7. Let X be a space, A a subset of X. A point $p \in A$ is called an *isolated point* of A if p is not a limit point of A.

   (a) What is the set of all isolated points of $\mathbb{Z}$ in $\mathbb{R}$, where $\mathbb{R}$ has the usual topology?
   Claim: The set of all isolated points of $\mathbb{Z}$ is $\mathbb{Z}$.

   Proof For any integer n, the interval $B = (n - \frac{1}{2}, n + \frac{1}{2})$ is an open set such that $A \cap (B - \{n\}) = \emptyset$, so every integer is an isolated point. ∎

   (b) What is the set of all isolated points of $\mathbb{Z}$ in $\mathbb{R}^1_f$, where $\mathbb{R}^1_f$ denotes the finite complement topology?
   Answer: The set of all isolated points of $\mathbb{Z}$ in $\mathbb{R}^1_f$ is $\emptyset$, since we proved in Exercise 6 that $\mathbb{Z}^\ell$ in $\mathbb{R}^1_f$ is $\mathbb{R}$, and $\mathbb{Z} \subset \mathbb{R}$.

8. Prove that in a Hausdorff space X with subset A, x is a limit point of A if and only if every open set containing x contains infinitely many points of A.

   Proof
   $\Longleftarrow$: If every open set U containing x contains infinitely many points of A, then $A \cap (U - \{x\}) \neq \emptyset$, so x is a limit point and we are done.

   $\Longrightarrow$: Assume that x is a limit point of A, and let U be an arbitrary open set which contains x. Suppose for contradiction that U contains only finitely many elements of A, denoted $a_1, a_2, \ldots, a_n$. Since X is Hausdorff, then U with the subspace topology is also Hausdorff, so there exist $n+1$ disjoint sets which are open in U and contain $a_1, a_2, \ldots, a_n$, and x, respectively. Let U_x denote the last of these sets, which intersects with A only at the point x. Now, by definition, U_x is the intersection of some open set and U, so U_x is also open in X.

   Since x is a limit point of A, then $A \cap (U_x - \{x\}) \neq \emptyset$, so U_x contains at least one element of A distinct from x, which contradicts our construction of U_x. Therefore, every open set U contains infinitely many points of A. ∎

9. Suppose that A is a subset of a space X. Prove that $\overline{A} = A \cup A^\ell$.

10. Let A be a subset of a space X. We say A is *dense* in X if $\overline{A} = X$. (For example, $\mathbb{Q}$ is dense in $\mathbb{R}$.)

    Prove that A is dense in X if and only if for every nonempty open set U in X, we have $U \cap A \neq \emptyset$.

    Proof
    $\Longrightarrow$: Suppose A is dense in X. Then, by definition,

    $$\overline{A} = \bigcap \{F : F \subset X \text{ is closed and } F \supset A\} = X,$$

    so the only closed set which contains A is X. Let U be any nonempty open set in X. Now, $(X - U)$ is closed and not equal to X, so it does not contain A. This means that U, the complement of $(X - U)$, must contain some elements of A. So, $U \cap A \neq \emptyset$ and we are done.

    $\Longleftarrow$: Suppose that for every nonempty open set U in X, it is true that $U \cap A \neq \emptyset$. Let F be any closed set in X which is not equal to X. Now, $(X - F)$ is a nonempty open set in X, so $(X - F) \cap A \neq \emptyset$. This means that $F \not\supset A$. Since it is given that $A \subset X$, X can be the only closed set which contains A, so $\overline{A} = X$. ∎