Math 550
Homework 7

Trevor Klar

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1. Let M be a k-dimensional manifold in $\mathbb{R}^N n$. Prove that if there exists a nowhere zero k-form on M, then M is orientable.

   Proof Since M is a manifold, then for every point $x \in M$ there exists a parameterization $\varphi_x : U_x \to V_x$ with $x \in V_x$. Thus the collection $\{\varphi_x\}_{x\in M}$ parametrizes all of M. Let

   $$\begin{array}{rl} G = & \{\varphi_x \,|\, {\varphi_x}^*\omega(\varphi_x^{-1}(x))\,(e_1,\ldots,e_k) > 0\}, \\ B = & \{\varphi_x \,|\, {\varphi_x}^*\omega(\varphi_x^{-1}(x))\,(e_1,\ldots,e_k) < 0\}, \text{ and} \\ \tau_{12}: & \mathbb{R}^k \to \mathbb{R}^k \text{ defined by } \tau_{12}(x_1,x_2,x_3,\ldots,x_k) = (x_2,x_1,x_3,\ldots,x_k). \end{array}$$

   Consider a point $p_g \in M$ such that for some $x_1, x_2 \in M$, we have $p_g \in V_{x_1} \cap V_{x_2}$ and $\varphi_{x_1}, \varphi_{x_2} \in G$. Then by a previous homework problem, φ_{x_1} and φ_{x_2} induce the same orientation on M_{p_g} for every $p_g \in V_{x_1} \cap V_{x_2}$. Thus all the parameterizations in G have compatible orientations. The same argument shows that the parameterizations in B are also compatible. Now, define

   $$\psi_x = \begin{cases} \varphi_x & \text{if } \varphi_x \in G \\ \tau_{12} \circ \varphi_x & \text{if } \varphi_x \in B. \end{cases}$$

   Thus, the parameterizations $\{\psi_x\}_{x\in M}$ are all compatible. To see this, observe that for any $\varphi_x \in B$,

   $$\begin{array}{rcl} {\psi_x}^*\omega(e_1,\ldots,e_k) & = & (\tau_{12}\circ\varphi_x)^*\omega(e_1,e_2,\ldots,e_k) \\ & = & {\varphi_x}^*{\tau_{12}}^*\omega(e_1,e_2,\ldots,e_k) \\ & = & {\varphi_x}^*\omega(e_2,e_1,\ldots,e_k) \\ & = & -{\varphi_x}^*\omega(e_1,e_2,\ldots,e_k) \\ & > & 0 \end{array}$$

   Thus, we have produced a collection of parameterizations $\{\psi_x\}_{x\in M}$ covering M for which all ψ_{x_1}, ψ_{x_2} induce the same orientation on M_x whenever $x \in V_{x_1} \cup V_{x_2}$, so M is orientable. ∎

2. There is a general correspondence between k-forms and $(n-k)$-forms on $\mathbb{R}^n$, for all $1 \le k \le n$. Given $\omega \in \Omega^k(\mathbb{R}^n)$, we define $\star\omega \in \Omega^{n-k}(\mathbb{R}^n)$ using the rule

   $$\star(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \pm dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}},$$

   and extending linearly, where $i_1 < \cdots < i_k, j_1 < \cdots < j_{n-k}$, and $\{i_1,\ldots,i_k,j_1,\ldots,j_{n-k}\} = \{1,\ldots,n\}$. The sign is chosen so that $\omega \wedge \star\omega = dx_1 \wedge \cdots \wedge dx_n$.
   Notation Another way to notate $\star$ is: Let $\Gamma \subset \{1,\ldots n\}$, and define $\star$ by

   $$\star\left(\bigwedge_{i\in\Gamma} dx_i\right) = \pm \bigwedge_{j\in\Gamma^{\complement}} dx_j$$

   and extending linearly. The sign is chosen so that $\omega \wedge \star\omega = \bigwedge_1^n dx_i$.
   Example In $\mathbb{R}^5$, $\star(dx_1 \wedge dx_4) = dx_2 \wedge dx_3 \wedge dx_5$ and $\star(dx_1 \wedge dx_3) = -\,dx_2 \wedge dx_4 \wedge dx_5$.
   Prove that $\star\star\,\omega = (-1)^{k(n-k)}\omega$.

Proof By definition of $\star$, we know that $\omega \wedge \star\omega = \bigwedge_1^n dx_i$ and $\star\omega \wedge \star\star\omega = \bigwedge_1^n dx_i$. So, by properties of wedge products, we can commute and write

$$\begin{aligned} (-1)^{k(n-k)} \star\star\omega \wedge \star\omega &= \bigwedge_1^n dx_i \\ \omega \wedge \star\omega &= \bigwedge_1^n dx_i \end{aligned}$$

Now since $\star\star\omega$ and ω differ by at most a sign[1], then $\mathrm{sign}(\star\star\omega) = (-1)^{k(n-k)}$. Thus, $\star\star\omega = (-1)^{k(n-k)}\omega$ as desired. ■

[1]Since $\star\star\omega$ is a wedge indexed over $\Gamma^{\complement\complement} = \Gamma$, and all that remains is to determine the sign.