Math 462 - Advanced Linear Algebra
Group Homework 1

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Exercises:

2. Give an example of a group G and a nonempty subset H of G which is closed under the operation defined on G, but is not a subgroup of G. (Hint: G must be infinite for this to occur.)

   Example. Consider $G = (\mathbb{Z}, +)$ and $H = (\mathbb{Z}_1^\infty, +)$, where $+$ is the usual addition of integers, and $\mathbb{Z}_1^\infty$ denotes $\{n \in \mathbb{Z} : 1 \leq n\}$. Note that G is clearly a group as it has the appropriate properties:

   - Closure: $\mathbb{Z}$ is closed under addition
   - Identity: For any integer n, $0 + n = n$.
   - Inverse: For any integer n, the integer $-n$ is its inverse, since $n + (-n) = 0$.
   - Associative: Addition of integers is associative, since $(a + b) + c = a + (b + c)$ for any integers a, b, c.

   Now consider the subset $\mathbb{Z}_1^\infty \subset \mathbb{Z}$. The magma $H = (\mathbb{Z}_1^\infty, +)$ is closed under $+$, as the sum of two positive integers is a positive integer. However, since there is no identity in H, it is not a group. Thus, H is not a subgroup of G. ■

22. Show that a field F can never have zero-divisors, i.e.

$$\forall a, b \in F, \quad ab = 0 \implies a = 0 \text{ or } b = 0.$$

   Proof by contradiction. Suppose $a \neq 0$ and $b \neq 0$. Note that that for any ring (and thus, any field) we have already shown that $x0 = 0$ for any element x of the ring.

   Since F is a field and $a \neq 0$, there exists $a^{-1} \in F$. Now,

$$ab = 0 \implies a^{-1}ab = a^{-1}0 \implies b = 0,$$

   which contradicts our assumption that $b \neq 0$.

   We can also observe that, since F is a field and $b \neq 0$, there exists $b^{-1} \in F$. In this case,

$$ab = 0 \implies abb^{-1} = 0b^{-1} \implies a = 0,$$

   which contradicts our assumption that $a \neq 0$.

   Therefore, $ab = 0 \implies a = 0$ or $b = 0$. ■