Theorem

Fatou's Lemma

If f_n are all μ-measurable, then

$$\int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n \, d\mu.$$

Definition

A measure

- $\mu(\emptyset) = 0$, positive
- monotonicity
- subadditivity

Definition

Borel measure

Borel sets are measurable. (The measure splits)

Definition

Regular measure

"Every set is μ-almost a measurable set."

Definition

Borel-regular measure

"Every set is μ-almost a Borel set, and μ is Borel."

Definition

Radon measure

"Compact sets have finite μ-measure, and μ is Borel-regular."

Theorem

Radon-Nikodym Theorem

Let μ, ν be Radon measures on $\mathbb{R}^n$, with $\nu << \mu$. Then

$$\nu(A) = \int_A D_\mu \nu \, d\mu$$

for all μ-measurable $A \subseteq \mathbb{R}^n$.

Theorem

Young's Inequality

If $a, b \geq 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem

Holder Inequality

If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ where p, q conjugate exponents, then

$$||fg||_{L^1} \leq ||f||_{L^p} \, ||g||_{L^q} \,, \quad \text{that is,}$$

$$\int_\Omega |f(x)g(x)| \, dx \leq \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q \, dx \right)^{\frac{1}{q}}$$

This can be thought of a "sort of" a Cauchy-Schwarz for Banach Spaces.

Definition

Lipschitz Continuity

A function $f : X \to Y$ is *Lipschitz continuous* if there exists $M > 0$ such that for all $x_1, x_2 \in X$,

$$||f(x_1) - f(x_2)||_Y \leq M \, ||x_1 - x_2||_X \,.$$

If $f : X \to X$ and $M < 1$, then f is in particular a *contraction*.

Definition

Baire First Category

A set A is ___________ if it is a countable union of nowhere dense sets.

Definition

Nowhere dense set

A set A is ___________ if $\overline{A}$ has empty interior.

Definition

Sequentially compact space

A topological space X is ____________ if every sequence in X has a convergent subsequence (to a point in X).