Math 501
Homework 2

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1. Professor Doofus mistakenly writes the following on the blackboard.

   Theorem. The following are equivalent.

   (1) $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at all $x \in \mathbb{R}^n$ (with the δ-ϵ definition)

   (2) For every open set $U \subset \mathbb{R}^n$, the image $f(U) \subset \mathbb{R}^m$ is open.

   Give an example which shows why Doofus is wrong.

   Example. Suppose $n = m$, and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be defined as

   $$f(x_1, x_2, \ldots, x_n) = (|x_1|, |x_2|, \ldots, |x_n|).$$

   Claim: (1.1) $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at all $x \in \mathbb{R}^n$.

   Proof Let $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$ be given. Now, for any $x \in B(x_o, \delta)$ where $\delta = \epsilon$,

   $$\begin{aligned} \epsilon &> d(x, x_0) \\ &= \sqrt{\sum_{i=1}^{n} (x_i - x_{0i})^2} \\ &= \sqrt{\sum_{i=1}^{n} (|x_i - x_{0i}|)^2} \\ &\geq \sqrt{\sum_{i=1}^{n} (||x_i| - |x_{0i}||)^2} \\ &> d(|x|, |x_0|). \end{aligned}$$

   Thus, if $x \in B(x_o, \delta)$, then $f(x) \in B(f(x_o), \epsilon)$, so f is continuous and (1) holds. ■

Claim: **(1.2)** There exists an open set $U \subset \mathbb{R}^n$, such that the image $f(U) \subset \mathbb{R}^m$ is not open.

Proof Consider the open set $U = B(\vec{0}, 1) \subset \mathbb{R}^n$.

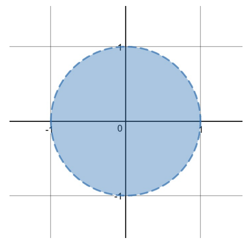

Figure 1: The set U in $\mathbb{R}^2$

Now, under f, every n-tuple in $\mathbb{R}^n$ maps either to itself, or to a corresponding n-tuple in the first orthant (or on its boundary); so the image of U is $f(U) = U \cap I$, where I denotes the closure of the first orthant.

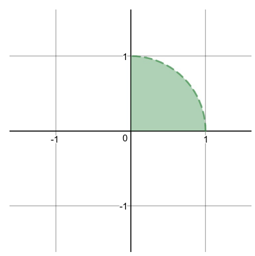

Figure 2: The set $f(U)$ in $\mathbb{R}^2$

The set $f(U)$ is not open; since the origin $\vec{0} \in f(U)$, but every $B(\vec{0}, r)$ contains points in every orthant, so no open ball $B(\vec{0}, r)$ is a subset of $f(U)$.

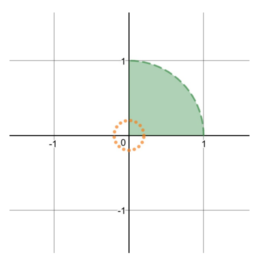

Figure 3: Every $B(\vec{0}, r) \not\subset f(U)$

Therefore, (2) fails. Thus, (1) $\not\Longrightarrow$ (2). ■

2. Let $X = [0, \infty)$, and let $\mathscr{T}$ consist of $\emptyset$, X, and all sets of the form (a, ∞) with $a \geq 0$. Show that $\mathscr{T}$ forms a topology on X.

   To prove that $\mathscr{T}$ forms a topology on X, it suffices to show the following:

   (i) $\emptyset, X \in \mathscr{T}$ (This is true by assumption).

   (ii) $\mathscr{T}$ is closed under arbitrary unions.

   (iii) $\mathscr{T}$ is closed under finite intersections.

Proof (ii) Let S be any subset of $\mathscr{T}$, and let $B = \{x : (x, \infty) \in S\}$. Then, since $\mathscr{T}$ contains all sets of the form (a, ∞) with $a \geq 0$, it suffices to show that

$$\bigcup S = (\inf(B), \infty).$$

B is bounded below by 0, so $\inf(B) \geq 0$ must exist.

$\subset$: For every $x \in B$ such that $x \neq \inf(B)$, $\inf(B) < x < \infty$. Therefore, every set in S is a subset of $(\inf(B), \infty)$, so $\bigcup S \subset (\inf(B), \infty)$.

$\supset$: Let $x \in (\inf(B), \infty)$; that is, $0 < \inf(B) < x$. If $\inf(B) \in B$, then $x \in (\inf(B), \infty) \subset \bigcup S$ and we are done. However, if $\inf(B) \notin B$, then $\inf(B)$ must be a limit point of B, so every neighborhood of $\inf(B)$ contains an element of B distinct from $\inf(B)$. Let $\delta = \frac{x - \inf(B)}{2}$, and consider the neighborhood $(\inf(B) - \delta, \inf(B) + \delta)$. Call b the element of B distinct from $\inf(B)$, and we find that $b < x$, so $x \in (b, \infty) \subset \bigcup S$. ■

Proof (iii) Let $U, V \in \mathscr{T}$. In the trivial case where either set is empty, then $U \cap V = \emptyset$, and we are done. There is also another trivial case where $U = X$ and $V = X$. In this case, $U \cap V = X$, and we are done. So suppose U and V are nonempty sets, at least one of which is distinct from X; and let $u = \inf(U)$ and $v = \inf(V)$. Then,

$$U \cap V = (\max(u, v), \infty),$$

which is a set of the form (a, ∞), so $(\max(u, v), \infty) \in \mathscr{T}$. ■

3. Let $X = [-1, 1]$, and let $\mathscr{T}$ consist of all sets which either do not contain 0, or do contain the interval $(-1, 1)$. Show that $\mathscr{T}$ forms a topology on X.

   Proof Let $\mathscr{T}_U$ be the collection of all subsets of X which do not contain 0, and let $\mathscr{T}_V$ be the collection of all subsets of X which contain the interval (-1,1).

   (i) $0 \notin \emptyset$ and $(-1, 1) \in X$, so $\emptyset, X \in \mathscr{T}$.

   (ii) Let $\{S_\alpha\}$ be an arbitrary collection of subsets of $\mathscr{T}$. If any of the sets in $\{S_\alpha\}$ contain $(-1, 1)$, then $\bigcup_\alpha S_\alpha$ contains $(-1, 1)$ as well. Otherwise, it consists only of sets which do not contain 0 (since they are all in $\mathscr{T}$), so $\bigcup_\alpha S_\alpha$ does not contain 0 either. Thererfore, $\mathscr{T}$ is closed under arbitrary unions.

   (iii) Let S_1, S_2 be two sets in $\mathscr{T}$. If either set does not contain 0, then their intersection does not either. If neither set contains 0, then they both must contain $(-1, 1)$, since they are both in $\mathscr{T}$. So, their intersection also contains $(-1, 1)$. Therefore, $\mathscr{T}$ is closed under finite intersections.

   ■

4. Let X be a nonempty set, with $p \in X$. Which of the following collections $\mathscr{T}$ form a topology on X? If it does, is it Hausdorff?

   (a) $\mathscr{T}$ consists of all subsets of X which contain p, together with $\emptyset$.
   Claim: $\mathscr{T}$ is a topology on X.

   Proof

   (i) $\emptyset, X \in \mathscr{T}$

   (ii) Let $\{S_\alpha\}$ be an arbitrary collection of subsets of $\mathscr{T}$. If they all are empty, then their union is empty. Otherwise, at least one is nonempty, and so must contain p, so their union either contains p, or is empty. Thus, $\mathscr{T}$ is closed under arbitrary unions.

   (iii) Let S_1, S_2 be two sets in $\mathscr{T}$. If either one is empty, then their intersection is empty. Otherwise, they both contain p, so their union either contains p, or is empty. Thus, $\mathscr{T}$ is closed under finite intersections.

   ■

Claim: $(X, \mathscr{T})$ is not Hausdorff. Following is a counterexample:

Example. Consider $x, p \in X$. Since every set in $\mathscr{T}$ which is nonempty contains p, then there does not exist an open set which contains x and not p. ■

(b) $\mathscr{T}$ consists of all subsets of X which do not contain p, together with X.
Claim: $\mathscr{T}$ is a topology on X.

Proof

(i) $\emptyset, X \in \mathscr{T}$

(ii) Let $\{S_\alpha\}$ be an arbitrary collection of subsets of $\mathscr{T}$. If any of them are the set X, then their union is X. Otherwise, none of them contain p, so their union either does not contain p, or is X. Thus, $\mathscr{T}$ is closed under arbitrary unions.

(iii) Let S_1, S_2 be two sets in $\mathscr{T}$. If both $S_1 = X$ and $S_2 = X$, then their intersection is X. Otherwise, at least one of them does not contain p, so their union either does not contain p, or is empty. Thus, $\mathscr{T}$ is closed under finite intersections.

■

Claim: $(X, \mathscr{T})$ is not Hausdorff. Following is a counterexample:

Example. Consider $x, p \in X$. Since the only set in $\mathscr{T}$ which contains p is X, then there do not exist any open set which are disjoint from the one containing p. ■

5. Let $(X, \mathscr{T})$ be a topological space, with $q \notin X$. Is $\mathscr{T}^* = \{U \cup \{q\} : U \in \mathscr{T}\} \cup \{0\}$ a topology on $X \cup \{q\}$? If so, is it Hausdorff?
Claim: $\mathscr{T}$ is a topology on X, but is not Hausdorff.

Proof First, note that $\mathscr{T}^*$ consists only of subsets of $X \cup \{q\}$ which contain q, together with $\emptyset$. This topological space has the same form as that of problem 4(a). So, by the same reasons given in problem 4(a), $\mathscr{T}$ is a topology on X, but is not Hausdorff. ■