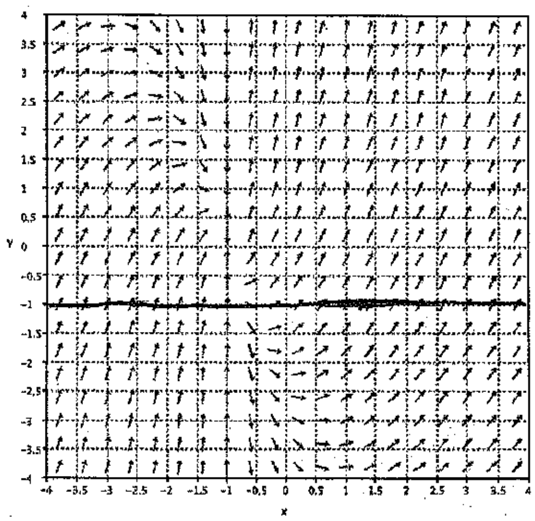
Math 4B Midterm Review Problems
October 30, 2014

These are practice problems to help you prepare for your midterm, you do not need to turn in solutions. You should think of this as a starting point for organizing your study plan. You should also review your DPs, old homework problems and problems from the lecture slides.

Your midterm will cover material from sections 1.1, 1.2, 2.1-2.5, 2.9, 3.1-3.4 and 3.7

1. The slope field below corresponds to a linear DE of the form

$$y' + p(t)y = g(t)$$



(a) Is the corresponding DE is autonomous? How can you tell?

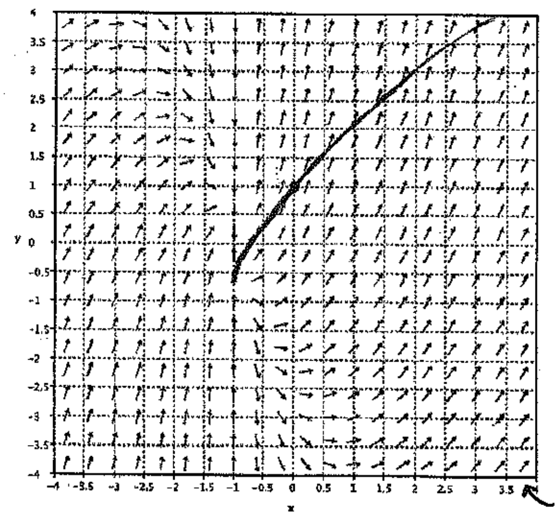
No, the corresponding DE is not autonomous.

The slope field of an autonomous DE should have horizontal isoclines.

Actually, every horizontal line in the slope field of an autonomous DE is a isocline. But the above slope field does not satisfy this.

For example, along the horizontal line $y=-1$ that I indicate in the above slope field, for $x=-1$ and $x=1$ we have differential slopes.

(b) Sketch the solution curve corresponding to the initial condition $y(0) = 1$. What is the largest interval in which this solution of the IVP can exist? How can you tell?

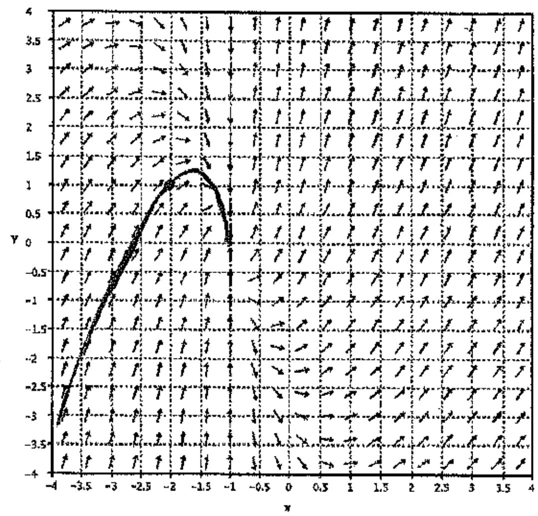


The largest interval is $[-1, +\infty)$.
We have to stop when we reach the vertical line $x=-1$, since when $x=-1$, $y'=\infty$ and a function $y(x)$ can not have a vertical tangent line.

(c) Describe the long term behavior of your solution from part (b).

When we start at the point $(0,1)$ and go in the right direction, we will always go up since we can not hit any horizontal slope fields.
And when x goes to $+\infty$, $y(t)$ approaches to $+\infty$.
On other hand, when we go in the left direction, as we explained in part (b), we have to stop when $x=-1$.

(d) Sketch the solution curve corresponding to the initial condition $y(-2) = 1$. What is the largest interval in which this solution of the IVP can exist? How can you tell?



The largest interval is $(-\infty, -1]$.
when have to stop when we reach the vertical line $x=-1$, as we explained in part (b).

(e) Describe the long term behavior of your solution from part (d).

When we start at the point $(-2,1)$ and go in the ~~lft~~ left direction, we will always go down since we can not hit any horizontal slope fields. And when x goes to $-\infty$, $y(x)$ approach to $-\infty$.
On the other hand, when we go in the right direction, as we explained in part (b), we have to stop at $x=-1$.

2. Which of the following differential equations are separable? Circle *all* that apply.

(A) $y' + y^2 \sin x = 0$

(B) $y' = \dfrac{3y^2 - x^2}{2xy}$

(C) $y' = \dfrac{x(x^2+1)}{4y^3}$

(D) $\dfrac{dr}{d\theta} = \dfrac{r^2}{\theta}$

3. Consider the following *autonomous differential equation*:

$$y' = f(y)$$

Which of the following statements are true? Circle *all* that apply and justify your choices.

(A) We can find the values of y that give equilibrium solutions to the DE by solving the equation $f(y) = 0$.

(B) The corresponding IVP with any initial conditions will have unique solutions.

(C) The equation is separable.

(D) The slope field for such an equation will have horizontal isoclines.

(A) Because equilibrium solutions are just constant solutions (i.e. constant functions) where derivatives are zero, and $y' = f(y)$ so we can find the values of y that give equilibrium solutions by solving the equation $f(y) = 0$.

(B) In general, solutions to the corresponding IVP is not unique. For example:

$f(y) = \sqrt{y}$, with initial condition $y(0) = 0$. i.e. IVP : $\begin{cases} y' = \sqrt{y} \\ y(0) = 0. \end{cases}$

It's easy to check both $y(t) \equiv 0$ & $y(t) = \frac{1}{4}t^2$ are solutions to the IVP.

(C) The equation is separable, since we can divide both sides by $f(y)$ to separate variable. After doing that, we can get $\frac{1}{f(y)} y' = 1$.

(d) A horizontal line in ty-plane is given by equation $y \equiv$ constant.

So from the equation $y' = f(y)$, we can see along a horizontal line $y \equiv$ constant

$y' \equiv f(\text{constant})$ is constant. So we have horizontal isoclines.

8. A deposit into a savings account earns interest, which is just a fraction of your deposit added to the total at regular intervals.

(a) Suppose your account earns 8% each year and that interest is compounded once a year, i.e. 8% of the amount is added each year. How much money will you have after 5 years with an initial deposit of \$100? After N years?

(b) Now suppose the interest is compounded monthly. How much will you have in the account after 5 years?

(c) Write down a *difference equation* that describes how the account value is changing. Suppose the annual interest rate r is compounded n times per year. Your difference equation should look something like

$$A_{k+1} - A_k = ??$$

(d) Now suppose the bank makes it's payments more and more often: daily, hourly, every minute, every second... continuously. What will your difference equation look like if interest is compounded continuously? HINT: Let $A(t) = A_k$ and let $\Delta t = \frac{1}{n}$, then find an expression for $\Delta A = A(t + \Delta t) - A(t)$. In the limit as $\Delta t \to 0$, you should get a *differential equation*.

(e) Compare the return after 5 years on two accounts with $A_0 = \$100$ and $r = 8\%$ - one compounded monthly and one compounded continuously. What kind of account do you want to invest in?

Solution: (a) $A_0 = 100$

$A_1 = 100 + 100 \times 0.08 = 100(1+0.08)$

$A_2 = 100(1+0.08)^2$

$A_3 = 100(1+0.08)^2 + 100(1+0.08)^2 \times 0.08 = 100(1+0.08)^3$

$A_4 = 100(1+0.08)^3 + 100(1+0.08)^3 \times 0.08 = 100(1+0.08)^4$

$A_5 = 100(1+0.08)^4 + 100(1+0.08)^4 \times 0.08 = 100(1+0.08)^5$

After 5 years, we will have \$ $100(1+0.08)^5$.

And similarly, after N years, we will have $\boxed{\$\ 100(1+0.08)^N}$.

(b) Monthly interest rate is $r = \frac{0.08}{12}$.

So if the interest is compounded monthly, after 5 years, we will have $\boxed{\$100\left(1+\frac{0.08}{12}\right)^{60}}$, since 5 years = 60 months.

(c). Annual interest rate $= r$, then interest rate for each $\frac{1}{n}$ year is $\frac{r}{n}$. So we have:

$$\boxed{A_{k+1} - A_k = \frac{r}{n}\cdot A_k} \quad \text{----} \quad (*)$$

And this difference equation is what we want.

(d). Let $t = \frac{k}{n}$, $\Delta t = \frac{1}{n}$, then: $t+\Delta t = \frac{k+1}{n}$ and,

$A(t) = A_k$, $A(t+\Delta t) = A_{k+1}$, $\frac{r}{n} = r\cdot\Delta t$, $\{n\to+\infty\} \Leftrightarrow \{\Delta t = \frac{1}{n}\to 0\}$

Then from the difference equation (*), we have:

$$A(t+\Delta t) - A(t) = r\cdot\Delta t\cdot A(t)$$

$$\xrightarrow[\text{both sides by } \Delta t]{\text{divide}} \quad \frac{A(t+\Delta t) - A(t)}{\Delta t} = rA(t) \quad \text{....} \quad (**)$$

By the definition of derivative, $\lim\limits_{\Delta t\to 0} \frac{A(t+\Delta t) - A(t)}{\Delta t} = A'(t)$.

So by taking the limit for equation (**) as $\Delta t \to 0$, we get:

$$\boxed{A'(t) = rA(t)} \quad \text{----} \quad (***)$$

Thus if the bank makes it's payments continuously, the difference equation (*) will became the differential equation (***).

(e). We've already gotten in part (b) that if the interest is ~~pr~~ compounded monthly, after 5 years we will have:

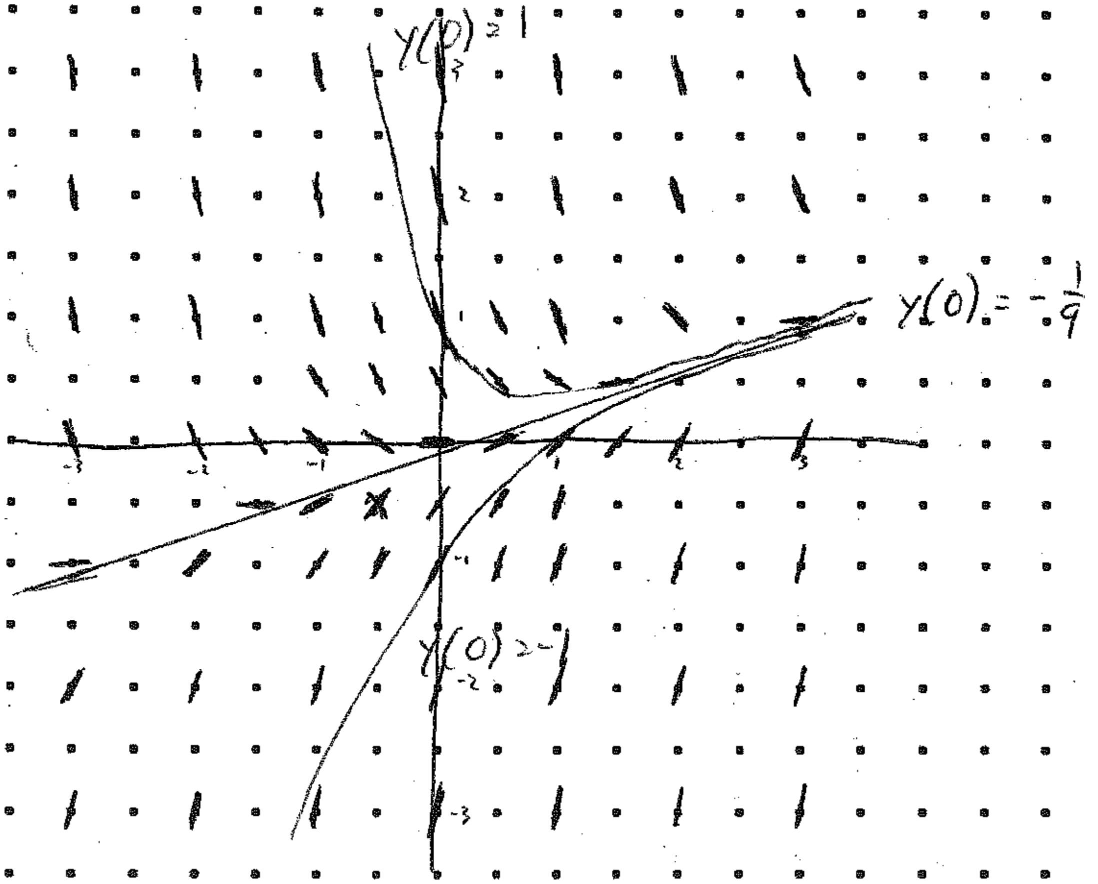
$$\boxed{\$100\left(1+\frac{0.08}{12}\right)^{60} \approx \$148.98}$$

Now if the interest is compounded continuously, we solve the differential equation (***) with the initial condition $A(0)=\$100$ & $r=0.08=8\%$.

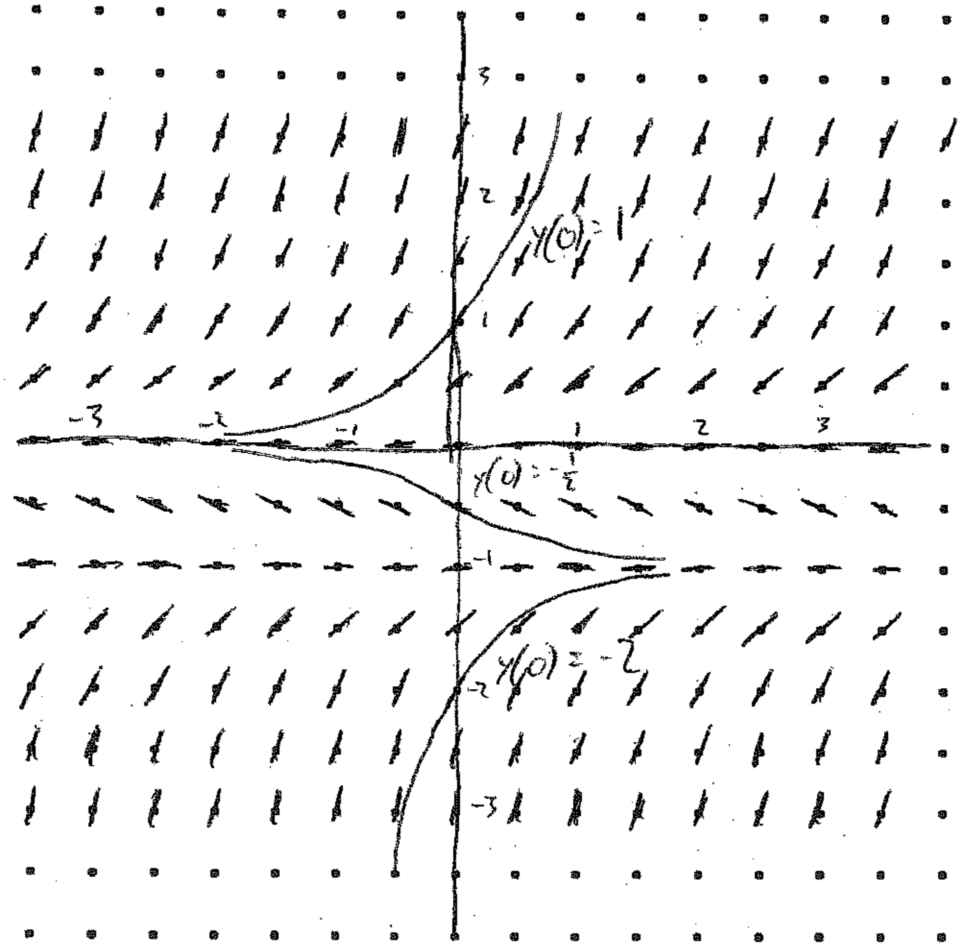
$$A(t)=100e^{0.08t}. \Rightarrow A(5)=100e^{0.08\times 5}$$

In this case, after 5 years, we will have $\boxed{\$100e^{0.08\times 5}\approx \$149.18}$.

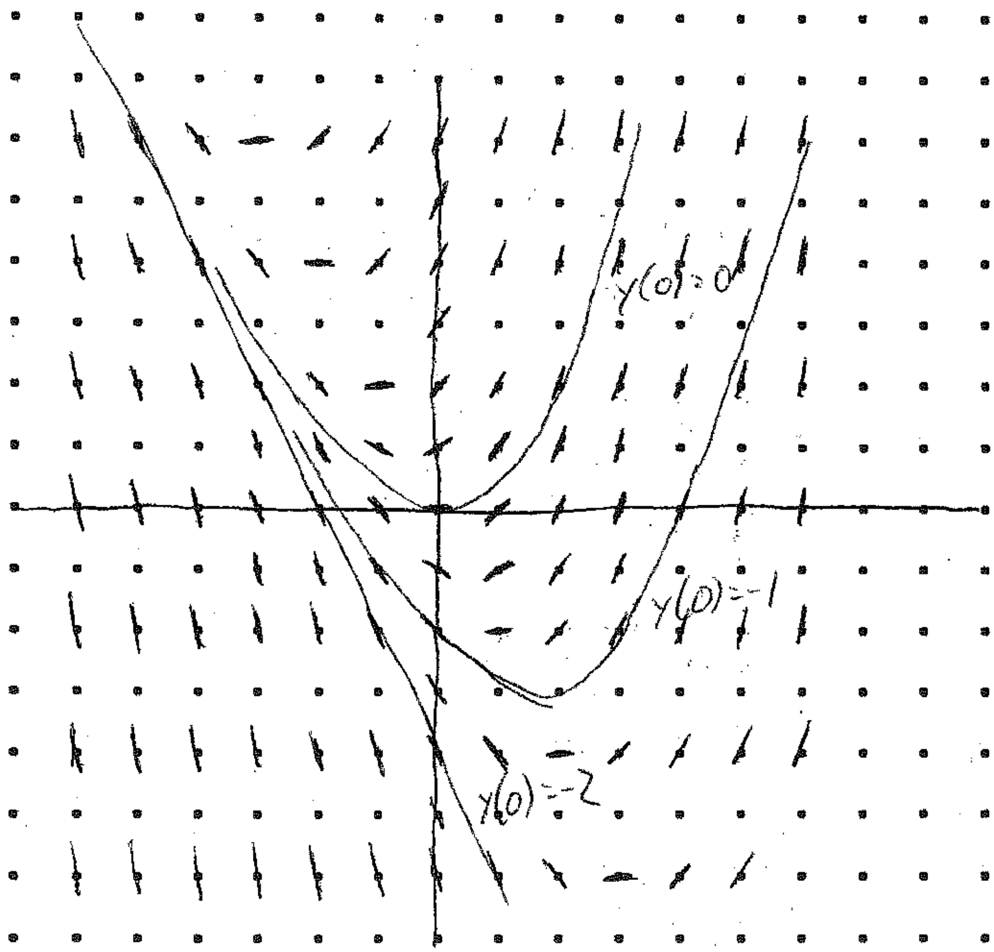
So we should choose continuously compounded interest account.



5a



5b



5c

~~The plots are other pages, but one thing to notice is that (b) is not dependent upon t.~~

6. 7. Consider the separable equation:

$$y' = \frac{x^2}{1-y^2}. \tag{1}$$

Explain how to use separation of variables to solve this DE. Be sure to explain all steps carefully–including how you jump from integrating the left hand side 'dx' to integrating 'dy'.

We start with the given differential equation:

$$y' = \frac{x^2}{1-y^2}. \tag{2}$$

First, we need to get all of the "y's" on the left hand side and the "x's" to the right hand side. To do this, we multiply both sides by $1-y^2$ to obtain

$$(1-y^2)y' = x^2. \tag{3}$$

We now antidifferentiate both sides with respect to x:

$$\int \left(1-y^2\right) y' dx = \int x^2 dx \tag{4}$$

On the left hand side, we make the simple substitution $u = y$, thereby giving us $du = y'dx$. We then have

$$\int (1-u^2) du = \int x^2 dx, \tag{5}$$

and integrating boths sides we get

$$u - \frac{1}{3}u^3 = \frac{1}{3}x^3 + C. \tag{6}$$

Substituting back $u = y$ gives us

$$y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C. \tag{7}$$

This function implicitly defines solutions to the separable differential equation.

7. 8. Newton's Law of Cooling states that an object will cool down (or heat up) at a rate proportional to the difference between the temperature of the object and the ambient temperature.

(a) Write a differential equation that models the rate a cup of coffee will cool down. Let k be the constant of proportionality, $y(t)$ the temperature of the coffee, and T the ambient temperature.

(b) Now suppose the cup of coffee is made with boiling hot water and set in a room where the temperature is 20°C and the coffee cools to 90°C in 2 minutes. How long will it take the coffee to cool to 60°C?

(a) We are told that Newton's Law of Cooling states that an object cools down (or warms up) at a rate [y'] proportional to [$= k$] the difference between the temperature of the object and the ambient temperature [$y - T$]. Thus, we see that

$$y' = k\,(y - T) \tag{8}$$

should be a differential equation that describes the temperature of the coffee at time t.

Note: We could have very well written $T - y$ instead of $y - T$ in the differential equation. This will only result in a sign change of k but will not actually change the final equation at the end.

(b) In order to keep things as general as possible, we will substitute $T = 20°\text{C}$ only at the very end, as with the initial temperature $y(0) = y_0 = 90°\text{C}$. Now, this is a first order linear differential equation, so we we first distribute k and then subtract ky from both sides to obtain

$$y' - ky = -kT. \tag{9}$$

We see that this is a great opportunity for us the use the method of integrating factors. We recall that for the differential equation

$$\frac{dx}{dt} + g(t)x = h(t) \tag{10}$$

the integrating factor μ is given by

$$\mu = \exp\left(\int g(t)dt\right), \tag{11}$$

where $\int g(t)dt$ is *any* antiderivative of $g(t)$. So, in this case, we see that we have

$$\mu = \exp\left(\int (-k)dt\right) = \exp(-kt) = e^{-kt}. \tag{12}$$

We now multiply both sides of Eq. (9) to obtain

$$e^{-kt}y' - ke^{-kt}y = -kTe^{-kt}. \tag{13}$$

From here, we should see that the left hand side of the previous equation factors to

$$\frac{d}{dt}\left(e^{-kt}y\right) = -kTe^{-kt} \tag{14}$$

If we antidifferentiate both sides with respect to t, we see

$$\begin{aligned}\int \frac{d}{dt}\left(e^{-t}y\right)dt &= -\int kTe^{-t}dt\\ e^{-kt}y &= Te^{-kt} + C\end{aligned} \tag{15}$$

Multiplying both sides by e^{kt} will give us the general solution to Eq. (9):

$$y(t) = T + Ce^{kt} \tag{16}$$

If we plug in the fact that $y(0) = y_0$, we see

$$y_0 = T + C, \tag{17}$$

implying that $C = y_0 - T$. Thus, the particular solution to the original differential equation is

$$y(t) = T + (y_0 - T)\,e^{kt}. \tag{18}$$

If we plug in the fact that $T = 20°\text{C}$ and $y_0 = 100°\text{C}$, along with the knowledge that $y(2) = 90°\text{C}$, we see that we have

$$90 = 20 + (100 - 20)\,e^{2k}. \tag{19}$$

If we solve for k, we obtain $\frac{1}{2}\ln\left(\frac{7}{8}\right) \approx -0.0668\ \text{min}^{-1}$. Knowing this, we would now like to know how long it will take for the coffee to cool down to 60°C, so we set $y(t) = 60$ and solve for t. When we do so, we obtain

$$t = 2\frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{7}{8}\right)} \approx 10.4 \text{ minutes}. \tag{20}$$

8 ~~9.~~ A ball is projected vertically upwards in viscious fluid. The force due to the viscosity of the fluid acts to slow the motion of the object at a rate proportional to its (let k be the constant of proportionality). Assume that the gravitational force acting on the ball is mg where $m = 1$ kg is the mass of the ball and $g \approx 10\ \text{m}\,\text{s}^{-2}$ is a good-enough estimate of acceleration due to gravity.

(a) Let $v(t)$ be the velocity of the ball at time t. Use Newton's Law to write a differential equation that models the situation.

(b) Find the velocity $v(t)$ of the ball at any time t if the ball's initial velocity is $v_0 = 5\ \text{m}\,\text{s}^{-1}$.

(c) Describe the behavior of $v(t)$ as $t \to \infty$. What does this indicate about the motion of the ball?

(a) From Newton's Laws of Motion, we know that the sum of the forces acting on an object is equal to the mass times acceleration; symbolically, we have

$$\sum_i F_i = ma. \tag{21}$$

The force of gravity acts with a magnitude of mg and pulls the object down, so we have $F_g = -mg$. The viscious fluid slows down the object proportional to the speed, so we have $F_v = -kv$. We have the negative sign because the force acts in the direction opposite the motion. Thus, we see that

$$ma = \sum_i F_i = F_g + F_v = -mg - kv \tag{22}$$

We know that acceleration is the rate that velocity changes with time ($a = \frac{dv}{dt}$), so we have

$$m\frac{dv}{dt} = -mg - kv. \tag{23}$$

This is a differential equation that could be used to model the motion of the ball.

(b) We need to solve this differential equation. We see that the above equation can be rearranged to

$$\frac{dv}{dt} + \frac{k}{m}v = -g, \tag{24}$$

a form that can solved via the method of integrating factors. We see

$$\mu = \exp\left(\int \frac{k}{m}dt\right) = \exp\left(\frac{k}{m}t\right), \tag{25}$$

and multiplying the equation by this factor gives

$$e^{\frac{k}{m}t}\frac{dv}{dt} + \frac{k}{m}e^{\frac{k}{m}t}v = -ge^{\frac{k}{m}t}. \tag{26}$$

The left hand side factors into

$$\frac{d}{dt}\left(e^{\frac{k}{m}t}v\right) = -ge^{\frac{k}{m}t} \tag{27}$$

and antidifferentiating with respect to t gives us

$$e^{\frac{k}{m}t}v = -\frac{mg}{k}e^{\frac{k}{m}t} + C \tag{28}$$

and from here we see that by solving for v we have

$$v(t) = -\frac{mg}{k} + Ce^{-\frac{k}{m}t}. \tag{29}$$

For full generality, let us assume that it has an initial velocity of v_0; that is, $v(0) = v_0$. Using this information we find

$$v_0 = -\frac{mg}{k} + C, \tag{30}$$

implying that $C = v_0 + \frac{mg}{k}$ and the resulting particular solution of

$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right) e^{-\frac{k}{m}t}. \tag{31}$$

We have $v(0) = v_0 = 5\text{m s}^{-1}$, $m = 1\text{kg}$, and $g \approx 10 \text{ m s}^{-2}$, so

$$v(t) = -\frac{10}{k} + \left(5 + \frac{10}{k}\right) e^{-\frac{k}{m}t}. \tag{32}$$

(c) In order to see the long term behavior, let us look at the limit of $v(t)$ as $t \to \infty$. When we do, we see

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \left[-\frac{10}{k} + \left(5 + \frac{10}{k}\right) e^{-\frac{k}{m}t}\right] = -\frac{10}{k} + \left(5 + \frac{10}{k}\right) \left[\lim_{t\to\infty} e^{-\frac{k}{m}t}\right]. \tag{33}$$

As $t \to \infty$, we see that $e^{-\frac{k}{m}t} \to 0$. Thus, we have $\lim_{t\to\infty} v(t) = -\frac{10}{k}$, which we could call the terminal velocity of the object.

9 10. Solve the IVPs and then determine where the solution is defined.

(a)

$$\frac{dr}{d\theta} = \frac{r^2}{\theta}, \quad r(1) = 2 \tag{34}$$

(b)

$$y' = (1 - 2x)y^2, \quad y(0) = -\frac{1}{6} \tag{35}$$

(c)

$$y' = \frac{x(x^2 + 1)}{4y^3}, \quad y(0) = -\frac{1}{\sqrt{2}} \tag{36}$$

(a) If we move r^2 to the left hand side and integrate with respect to θ, we have

$$\int \frac{1}{r^2} \frac{dr}{d\theta} d\theta = \int \frac{1}{\theta} d\theta. \tag{37}$$

After making the substitution $u = r$ and $du = r' d\theta$, we have

$$\int \frac{du}{u^2} = \int \frac{d\theta}{\theta}. \tag{38}$$

By integrating both sides of the previous equation and substituting back $u = r$, we obtain

$$-\frac{1}{r} = \ln\theta + C. \tag{39}$$

Using the fact $r(1) = 2$, we find $C = -\frac{1}{2}$. Solving for r gives us

$$r(\theta) = \frac{-1}{\ln\theta - \frac{1}{2}}. \tag{40}$$

We know that we cannot divide by 0, so this tells us that $y(e^{1/2})$ is not defined, as that is when $\ln\theta - \frac{1}{2} = 0$. We recall that the natural logarithm $\ln(x)$ has a vertical asymptote at $x = 0$, so we see that $r(\theta)$ is only defined on $\theta \in (0, e^{1/2})$. This makes sense given that from the original differential equation we have the equation does not make sense for $\theta = 0$ because we cannot divide by 0.

(b) Dividing both sides by y^2 allows us to separate variables so we have

$$\frac{y'}{y^2} = 1 - 2x. \tag{41}$$

If we integrate both sides by x and make the substitution $u = y$, we have

$$\int \frac{du}{u^2} = \int (1 - 2x)\, dx. \tag{42}$$

When we antidifferentiate and substitute $u = y$, we obtain

$$-y^{-1} = x - x^2 + C. \tag{43}$$

Given that $y(0) = -\frac{1}{6}$, we see $C = 6$. Plugging this in and solving for y gives us

$$y(x) = \frac{1}{x^2 - x - 6}. \tag{44}$$

We see that -2 and 3 are roots to the polynomial in the denominator, so this tells us that $y(x)$ is undefined at those points. Therefore, we see that y is defined for $x \in (-2, 3)$.

(c) If we multiply both sides by $4y^3$ and integrate by x, we will have

$$\int 4y^3 y'\, dx = \int \left(x^3 + x^2\right) dx. \tag{45}$$

We make the simple substitution $u = y$ with $du = y'dx$ to obtain

$$\int 4u^3 du = \int \left(x^3 + x^2\right) dx \tag{46}$$

and integrating with substituting back for $u = y$ gives us

$$y^4 = \frac{1}{4}x^4 + \frac{1}{3}x^3 + C. \tag{47}$$

The initial condition $y(0) = \frac{1}{\sqrt{2}}$ gives us that $C = \frac{1}{4}$. Taking the fourth root of both sides gives us

$$y(x) = \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}\right]^{-1/4} \tag{48}$$

Now, we should note that from the previous equation, we have $y(0) = \frac{1}{2}$. Thus, the actual solution to the differential equation with the initial condition is

$$y(x) = -\left[\frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}\right]^{-1/4} \tag{49}$$

From the original differential equation, by setting it equal to 0 we see that we must have $x = 0$. Thus, this is a local maximum and so there is no concern about the $y(x)$ ever being undefined, so y is defined for every real number.