1. Let (X, d) be a metric space.

<u>Metric topology on X:</u> The metric topology on X is the topology with the following Basis:

$$\mathcal{B} = \{ B(x, \varepsilon) \mid x \in X, \varepsilon > 0 \}$$

where $B(x, \varepsilon) = \{ y \in X : d(x, y) < \varepsilon \}$

To show that the metric topology is a topology it suffices to show that $\mathcal{B}$ is indeed a basis.

<u>claim</u>: $X = \bigcup_{B(x,\varepsilon) \in \mathcal{B}} B(x, \varepsilon)$.

Let $x \in X$. Then for $\varepsilon > 0$, we know $x \in B(x, \varepsilon)$ and $B(x, \varepsilon) \in \mathcal{B}$. So $x \in \bigcup_{B(x,\varepsilon) \in \mathcal{B}} B(x, \varepsilon)$, which implies

$X \subset \bigcup_{B(x,\varepsilon) \in \mathcal{B}} B(x, \varepsilon)$. Clearly, $\bigcup_{B(x,\varepsilon) \in \mathcal{B}} B(x, \varepsilon) \subset X$.

So, $X = \bigcup_{B(x,\varepsilon) \in \mathcal{B}} B(x, \varepsilon)$.

<u>Claim</u>: Let $y \in B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$

Now, let $\varepsilon = \min(\varepsilon_1 - d(y, x_1), \varepsilon_2 - d(y, x_2))$

Clearly, $B(y, \varepsilon) \in \mathcal{B}$, and clearly $y \in B(y, \varepsilon)$.

We claim $B(y, \varepsilon) \subset B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$

$\rightarrow$

Proof of Klaim: Let $z \in B(y, \varepsilon)$.

Then
$$\begin{aligned} d(z, x_1) &\leq d(z, y) + d(y, x_1) \\ &< \varepsilon + d(y, x_1) \\ &\leq \varepsilon_1 - d(x_1, y) + d(y, x_1) \\ &= \varepsilon_1 - d(y, x_1) + d(y, x_1) \\ &= \varepsilon_1 \end{aligned}$$

So $d(z, x_1) < \varepsilon_1 \Rightarrow z \in B(x_1, \varepsilon_1)$

Similarly,
$$\begin{aligned} d(z, x_2) &\leq d(z, y) + d(y, x_2) \\ &< \varepsilon + d(y, x_2) \\ &\leq \varepsilon_2 - d(x_2, y) + d(y, x_2) \\ &= \varepsilon_2 \end{aligned}$$

So $d(z, x_2) < \varepsilon \Rightarrow z \in B(x_2, \varepsilon_2)$.

So $z \in B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$.

Therefore $B(y, \varepsilon) \subseteq B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$.

So $\mathcal{B}$ is indeed a basis. So the metric topology is a topology.

Prove that if $A \subset X$ and $x \in X$, then x is in the closure of A iff x is the limit of a sequence of points in A.

<u>Proof:</u> Let $A \subset X$.

<u>Claim</u>: $x \in \overline{A}$ if and only if for every open neighborhood U of x, $U \cap A \neq \emptyset$.

<u>Proof of claim</u>: we prove this by the contrapositive.

<u>That is</u>: $x \notin \overline{A}$ iff $\exists$ an open neighborhood U of x such that $U \cap A = \emptyset$.

$\Rightarrow$ Suppose $x \notin \overline{A}$. So there exists a closed set C such that $x \notin C$ and $A \subset C$. (since the closure of A is the intersection of all closed sets containing A)

Therefore $x \in X \setminus C \subseteq X \setminus A$,

~~[illegible]~~ so $X \setminus C \cap A = \emptyset$.

However, $X \setminus C$ is open since C is closed which yields our desired result.

$\Leftarrow$ Now, suppose $\exists$ an open neighborhood of ~~[illegible]~~ x (call it U) such that $U \cap A = \emptyset$. So $A \subseteq X \setminus U$.

However, since U is open $X \setminus U$ is closed. So $X \setminus U$ is a closed set containing A that does not contain x.

So $x \notin \overline{A}$ which proves our claim

Now, suppose $x \in \overline{A}$. Then, by our claim we know for every $n \in \mathbb{N}$, $B(x, \frac{1}{n}) \cap A \neq \emptyset$. So choose $x_n \in B(x, \frac{1}{n})$ and form $\{x_n\}$.

<u>Claim:</u> $\{x_n\} \to x$ as $n \to \infty$.

Let $\varepsilon > 0$. Then, $\exists\ N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.

So for all $m \geq N$, $m \in \mathbb{N}$ we have

$$d(x, x_m) \leq \frac{1}{m} \leq \frac{1}{N} < \varepsilon.$$ So $\{x_n\} \to x$.

Conversely, suppose there exists a sequence in A such that $\{x_n\} \to x$. Let U be an open neighborhood. Then ~~choose~~ there exists $\varepsilon > 0$, such that $B(x, \varepsilon) \subseteq U$. Since $\{x_n\} \to x$ $\exists\ N \in \mathbb{N}$ such that for all $m \geq N$, we have $x_m \in B(x, \varepsilon)$. So $x_m \in A$ and $A \cap U \neq \emptyset$. Thus, by our original claim we know $x \in \overline{A}$.

$\blacksquare$

2. Let $d: \mathbb{Z}^+ \times \mathbb{Z}^* \to \mathbb{R}$ be ~~the~~ the function

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{x} + \frac{1}{y} & \text{if } x \neq y \end{cases}$$

Prove $(\mathbb{Z}^+, d)$ is a metric Space but not a complete metric space.

<u>Proof</u>: We check to see if $(\mathbb{Z}^+, d)$ fulfills the three requirements of a metric space.

1. Suppose $x = y$, then $d(x,y) = 0$ by definition.

Suppose $d(x,y) = 0$, and suppose $x \neq y$.

Then $d(x,y) = \frac{1}{x} + \frac{1}{y} = 0$

$$\Rightarrow \quad \frac{1}{x} = -\frac{1}{y}$$

$y = -x$ which is a contradiction as $x, y \in \mathbb{Z}^+$.

So, if $d(x,y) = 0$, then $x = y$.

2. Now, $d(x,y) = \frac{1}{x} + \frac{1}{y} = \frac{1}{y} + \frac{1}{x} = d(y,x)$.
for $x = y$

3. $d(x,y) + d(y,z) = 1$

$d(x,z) = $ ~~then~~ $\frac{1}{x} + \frac{1}{z} \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{y} + \frac{1}{z} = d(x,y) + d(y,z)$

$\therefore (\mathbb{Z}^+, d)$ is a metric space.

2b. (Not a complete metric space).

Let $\{x_n\} = (1, 2, 3, \dots)$ so $x_n = n$; for ~~[illegible]~~ $n \in \mathbb{N}$

~~M~~ we claim $\{x_n\}$ is cauchy but not convergent.

$\boxed{\text{Cauchy}}$: Let $\varepsilon > 0$. Then $\exists$ $K \in \mathbb{N}$ such that

$\frac{1}{K} < \varepsilon$ (by the Archimedean property?). Now

for all ~~M~~ $m, n \geq 2K$ we know
($\Rightarrow \frac{1}{m} \leq \frac{1}{2K}$ and $\frac{1}{n} \leq \frac{1}{2K}$)

$$d(x_n, x_m) = d(n, m) = \frac{1}{n} + \frac{1}{m} \leq \frac{1}{2K} + \frac{1}{2K} = \frac{1}{K} < \varepsilon.$$

Therefore, $\{x_n\}$ is cauchy.

$\boxed{\text{not convergent:}}$ Clearly $(1, 2, 3, \dots)$ is not convergent as $x_n \to \infty$ as $n \to \infty$. $\square$

4. Prove that a metric space is compact iff it is sequentially compact.

Proof: $\Rightarrow$ Suppose (X, d) is a compact metric space.

Now, if there exists $x \in X$ such that $\forall \varepsilon > 0$,

$B_\varepsilon(x)$ contains infinitely many points of $\{x_n\}$

then we <u>claim</u> $\{x_n\}$ has a convergent subsequence.

<u>proof of claim:</u> Let $\varepsilon = 1$. Then choose $x_{n_1} \in B_1(x) \cap \{x_n\}$.

Now let $\varepsilon = \frac{1}{2}$,

then since $B_{\frac{1}{2}}(x)$ has infinitely many points

of $\{x_n\}$ we know $\exists\, m > n_1$ such that

$x_m \in B_{\frac{1}{2}}(x)$. Let $x_{n_2} = x_m$. We can continue

this process and we can choose $x_{n_i} \in B_{\frac{1}{i}}(x)$ for each $i \in \mathbb{N}$.

Now, it is clear that $\{x_{n_i}\} \to x$ so $\{x_n\}$

has a convergent subsequence.

Now, on the other hand, suppose $\exists\, \varepsilon > 0$ s.t. every $x \in X$, then

$B_\varepsilon(x)$ contains only finitely many points

of X. Then $\{B_\varepsilon(x)\}_{x \in X}$ form an

open cover of X. So, we can extract a

finite subcover (since X is compact). So finitely

many balls containing finitely many points of $\{x_n\}$

cover X. Therefore $\{x_n\}$ must only have

finitely many distinct points. So, it must

have a convergent subsequence.

$\Leftarrow$ Suppose X is sequentially compact.
Then we know X has a Lebsegue # and
X is totally bounded. (need to prove this?)

Let $\{U_\alpha\}$ be an open cover for X. ~~Given~~ And
let δ be the ~~c~~ lebsegue #. Then for all
$x \in X$, $B_\delta(x) \subset U_\beta$ for some β.
Further, since X is totally bounded,
$$X = \bigcup_{i=1}^{n} B_\delta(x_i) \text{ where } x_i \in X.$$
Therefore since $B_\delta(x_i) \subseteq U_{\alpha_i}$ where $U_{\alpha_i} \in \{U_\alpha\}$
we know $X = \bigcup_{i=1}^{n} U_{\alpha_i}$ which is
a finite subcover. $\square$

5. Suppose X is second countable. prove that A is closed in X iff $A \cap K$ is closed in K for all compact $K \subseteq X$

Proof: First we prove a small claim.

Claim: If X is second countable, then $x \in \overline{A}$ iff there exists a sequence in ~~A~~: $\{x_n\}$ such that $\{x_n\} \to x$.

Proof of claim: $\Rightarrow$ Let $x \in \overline{A}$. Since X is 2^{nd} countable, it is first countable. Let $B = \{B_n\}$ be the local ~~basis~~ countable basis for x. Now, define $U_n = \bigcap_{i=1}^{n} B_i$. Since U_n is open, and contains x we know $U_n \cap A \neq \emptyset$. Therefore pick $x_n \in U_n \cap A$.

We claim $\{x_n\} \to x$. Let V be an open neighborhood of x. Then $\exists\ B_n \in B$ such that $B_n \subset V$. Now, for all $m \geq n$, we know $x_m \in \bigcap_{i=1}^{m} B_i$ so $x_m \in B_n$, which implies $x_m \in V$. Therefore, $\{x_n\} \to x$.

$\Leftarrow$ Now, suppose there exists a sequence in X such that $\{x_n\} \to x$. Let U be an open neighborhood of x. Then $\exists\ \cancel{M}\ N \in \mathbb{N}$ s.t for all $m \geq N$ $x_m \in U$. So $U \cap A \neq \emptyset$. Therefore $x \in \overline{A}$, which proves our claim.

Now, we return to the main statement.

$\Rightarrow$ Suppose A is closed in X. Then, by definition of the subspace topology, we know $A \cap K$ is closed for all compact $K \subseteq X$.

$\Leftarrow$ Now suppose $A \cap K$ is closed in K for all compact $K \subseteq X$. Let $x \in \overline{A}$. Then by our claim, $\exists \{x_n\} \to x$ such that $x_n \in A$.

Now, define
$$K = \{x\} \cup \{x_n \mid n \in \mathbb{N}\}.$$

We claim K is compact.

<u>Proof of claim:</u> Let $\{U_\alpha\}$ be an open cover of K. Then, let $U \in \{U_\alpha\}$ such that $x \in U$. Then since U is open and $\{x_n\} \to x$ $\exists N \in \mathbb{N}$ such that for all $m \geq N$ $x_m \in U$. Now for each $1 \leq i \leq N$, there exists $U_i \in \{U_\alpha\}$ such that $x_i \in U_i$. Therefore $\{U_i\}_{i=1}^{N}$ form a finite subcover of K and K is compact.

Therefore, by our assumption $A \cap K$ is closed in K. Let U be an open neighborhood ~~open set~~ of x. Then we know $\{x_n\}$ is eventually in U. So by definition of K, $U \cap (A \cap K) = (U \cap K) \cap (A \cap K) \neq \emptyset$. So $U \cap (A \cap K) = \emptyset$. So and thus every open neighborhood of x in K has non-trivial intersection with $A \cap K$. Thus, $x \in \overline{A \cap K} = A \cap K$. So $x \in A$.

in K

$\blacksquare$

6. Prove that if a topological space X has dense connected subset, then X is connected.

<u>Proof:</u> By way of contradiction, suppose X is not connected. Then there exists seperated sets U and V such that $X = U \cup V$ and $U \cap V = \emptyset$.

We claim $U \cap E$ and $V \cap E$ form a seperation for E.

Since $U \cap X \neq \emptyset$ and E is dense, every open set in X must contain a point of E. So $U \cap E \neq \emptyset$. Similarly $V \cap E \neq \emptyset$. By definition of subspace topology $U \cap E$ and $V \cap E$ are open.

Now, we know

$$E = E \cap X = E \cap [U \cup V] = (E \cap U) \cup (E \cap V)$$

Further,

$$(E \cap U) \cap (E \cap V) = (U \cap V) \cap E \subseteq (U \cap V) \cap X = \emptyset$$

So $(E \cap U) \cap (E \cap V) = \emptyset$. Thus, $E \cap U$ and $E \cap V$ form a seperation of E, which is a contradiction as E is connected.

Prove that the product of two connected spaces is connected.

Proof: Suppose X is connected and Y is connected.

~~By way of contradiction, suppose $X \times Y$ is disconnected. So there exists $f: X \times Y \to \{0,1\}$ continuous and non-constant.~~

~~Therefore~~ let $f: X \times Y \to \{0,1\}$. Then, we know for all $x \in X$, $\{x\} \times Y$ is homeomorphic to Y, and therefore connected.

So $f|_{\{x\} \times Y}$ must be constant. Similarly $X \times \{y\}$ is homeomorphic to X so $f|_{X \times \{y\}}$ must also be constant. Therefore for any two points $(x_0, y_0), (x_1, y_1) \in X \times Y$ we know $f(x_0, y_0) = f(x_0, y_1) = f(x_1, y_1)$. So f must be constant. Therefore, $X \times Y$ is connected. (Do I need to prove if $f: X \times Y \to \{0,1\}$ for all continuous f is constant then $X \times Y$ is connected?)

7. Let $f: X \to Y$ be a quotient map. Prove that if Y is connected and each $f^{-1}(\{y\})$ is connected, then X is connected.

Proof: By way of contradiction suppose X is not connected. So $\exists$ $A, B \subseteq X$ such that $X = A \amalg B$ and A, B are open. We claim $f(A)$ and $f(B)$ form a seperation of Y.

Now, since the quotient map is surjective; we know $Y = f(A) \cup f(B)$.

Claim: $f(A) \cap f(B) = \emptyset$.

Suppose $y \in f(A) \cap f(B)$. Then, there exists $a \in A$ and $b \in B$ such that $f(a) = y = f(b)$.

However, $f^{-1}(\{y\})$ is connected, so it must lie entirely in A or in B. Thus, $f(A) \cap f(B) = \emptyset$.

Claim $f(A)$ and $f(B)$ are open.

First we show that $A = f^{-1}(f(A))$.

We know $A \subseteq f^{-1}(f(A))$. So let $x \in f^{-1}(f(A))$. Suppose $x \in B$. Then $f(x) \in f(f^{-1}(f(A))) \subseteq f(A)$ and $f(x) \in f(B)$ which is a contradiction since $f(A) \cap f(B) = \emptyset$.

So $f^{-1}(f(A)) = A$. Since f is the quotient map, we know $f^{-1}(f(A))$ is open iff $f(A)$ is open. Therefore ~~A~~ since A is open $f(A)$ is open. Similarly $f(B)$ is open. Therefore $f(A)$ and $f(B)$ form a seperation for Y which is a contradiction. $\boxtimes$

8. <u>Galois correspondence:</u> Let (X, x_0) be a path-connected, locally path-connected, semi-locally simply connected topological space. Then there is a bijective correspondence between

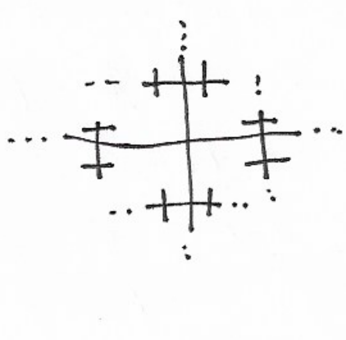
$$\left\{ \begin{array}{l} \text{all base point} \\ \text{perserving isomorphism} \\ \text{classes of path-connected} \\ \text{covering spaces} \\ p: (\tilde{X}, \tilde{x_0}) \to (X, x_0) \end{array} \right\} \xleftrightarrow{1 \text{ to } 1} \left\{ \begin{array}{l} \text{Set of subgroups} \\ \text{of } \pi_1(X, x_0) \text{ obtained} \\ \text{by associating the subgroup} \\ p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \text{ to the} \\ \text{covering space } (\tilde{X}, \tilde{x}_0) \end{array} \right\}$$

Example of a space with fundamental group $\langle a, b \rangle$ is $S^1 \vee S^1$ (wedge of two circles)

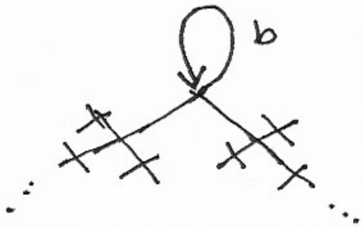
[illegible]

<u>Proof:</u> Consider $S^1 \vee S^1$. Then $S^1 \vee S^1$ is clearly a cell complex. Now $S^1 \vee S^1 = S^1 \cup S^1$. ~~[illegible]~~ (both which contain the base point (pick base point to be in the intersection). Then, by Van Kampen's theorem we know $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} / \{0\} = \mathbb{Z} * \mathbb{Z}$ (since intersection is just a point). which is the free group of two generators.

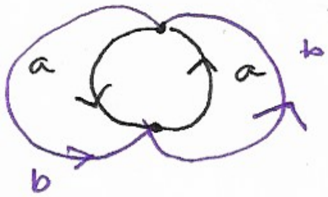
a) trivial group: corresponds to the covering space which is the (Cayley?) graph shown below:



b) $\langle b \rangle$



c) $\langle a^2, b^2, ab \rangle$



(how do I justify these)