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Homework 8

1. For a covering space $p : \widetilde{X} \to X$ and a subspace $A \subset X$, let $\widetilde{A} = p^{-1}(A)$. Show that the restriction $p : \widetilde{A} \to A$ is a covering space.

Proof Let $a \in A$. Since $a \in X$, there exists $U \ni a$ open in X which is evenly covered by p. So $p^{-1}(U) = \coprod_\alpha \widetilde{U}_\alpha$, with each $\widetilde{U}_\alpha$ open in $\widetilde{X}$ and homeomorphic to U. Intersecting with $\widetilde{A}$, we get $\widetilde{V}_\alpha = \widetilde{U}_\alpha \cap \widetilde{A}$ with each $\widetilde{V}_\alpha$ open in the subspace topology on $\widetilde{A}$ and homeomorphic to $\widetilde{A}$, and the collection of sets is disjoint. ■

2. Show that if $p : \widetilde{X} \to X$ and $\rho : \widetilde{Y} \to Y$ are covering spaces, so is their product $p \times \rho : \widetilde{X} \times \widetilde{Y} \to X \times Y$.

Proof Note that I have renamed the spaces in the statement of the problem to simplify notation. Let $(x, y) \in X \times Y$. Using p and ρ, there exist neighborhoods $U \ni x$, $V \ni y$ which are evenly covered by their respective spaces. Observe that the preimage

$$\begin{aligned}
(p \times \rho)^{-1}(U \times V) &= p^{-1}(U) \times \rho^{-1}(V) \\
&= \coprod_\alpha \widetilde{U}_\alpha \times \coprod_\beta \widetilde{V}_\beta \\
&= \coprod_{\alpha,\beta} (\widetilde{U}_\alpha \times \widetilde{V}_\beta)
\end{aligned}$$

is a collection of open rectangles. They are disjoint, since any distinct rectangles differ in either their first coordinate or their second, and $\{\widetilde{U}_\alpha\}$ and $\{\widetilde{V}_\beta\}$ are disjoint collections. For any particular values of α, β, $(\widetilde{U}_\alpha \times \widetilde{V}_\beta)$ is homeomorphic to $(U \times V)$ since $\widetilde{U}_\alpha \cong U$ and $\widetilde{V}_\beta \cong V$. Thus $(U \times V)$ is an open neighborhood of (x, y) which is evenly covered by $(p \times \rho)$. ■

3. Let $p : \widetilde{X} \to X$ be a covering space with $p^{-1}(x)$ finite and nonempty for all $x \in X$. Show that $\widetilde{X}$ is compact Hausdorff iff X is compact Hausdorff.

Proof (Compact $\implies$) Suppose $\widetilde{X}$ is compact, and let $\{U_\alpha\}$ be any open cover of X. Taking preimages we obtain $\{\widetilde{U}_\alpha\}$ which is an open cover of $\widetilde{X}$, since each $\widetilde{U}_\alpha = p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$ and is thus itself an open set. Since $\widetilde{X}$ is compact, there exists a finite subcover $\{\widetilde{U}_i\}$. Since $p^{-1}(x)$ is nonempty for all $x \in X$, then p is onto, so taking images in our cover yields $\{U_i\}$ which covers X. Since each U_i is the image of the preimage of an open set in $\{U_\alpha\}$ that has been reindexed, then each U_i is open. Thus $\{U_i\}$ is a finite subcover of $\{U_\alpha\}$, so X is compact. □

Proof (Compact $\impliedby$) Suppose X is compact Hausdorff, and let $\{\widetilde{U}_\alpha\}_{\alpha\in\Gamma}$ be any open cover of $\widetilde{X}$. We need evenly-coveredness, so for each $x \in X$, let V_x be a neighborhood of x which is evenly covered. Since X is compact, we can take a finite subcover $\{V_i\}$ of $\{V_x\}$. Since p is finite-sheeted, then for each i then $p^{-1}(V_i) = \coprod\limits_j \widetilde{V}_{i,j}$.

Since there are finitely many $\widetilde{V}_{i,j}$, if we can show that each a $\widetilde{V}_{i,j}$ is covered by a finite subcollection of $\{\widetilde{U}_\alpha\}$, then we are done.

Let $\Delta = \{\alpha \in \Gamma \mid \widetilde{U}_\alpha \cap \widetilde{V}_{i,j} \neq \emptyset\}$, and observe that $\{\widetilde{U}_\delta\}_{\delta\in\Delta}$ covers the closure $\mathrm{cl}(\widetilde{V}_{i,j})$.[†] This means that taking images in p yields an open[‡] cover $\{U_\delta\}$ of $\overline{V_i} = \mathrm{cl}(p(\widetilde{V}_i))$. Since $\overline{V_i}$ is closed and X is compact, then there exists a finite subcover $\{U_i\}$ of $\{U_\delta\}$ which covers $\overline{V_i}$. Thus the corresponding sets $\{\widetilde{U}_i\}$ cover $\widetilde{V}_{i,j}$, and we are done. $\square$

Proof (Hausdorff $\Longleftarrow$) Suppose X is Hausdorff, and let $\widetilde{x} \neq \widetilde{y} \in \widetilde{X}$. Consider $p(\widetilde{x}) = x$ and $p(\widetilde{y}) = y$. Since X is Hausdorff, there exist disjoint sets $U \ni x, V \ni y$ which are open in X. Then the corresponding preimages $p^{-1}(U) \ni \widetilde{x}$ and $p^{-1}(V) \ni \widetilde{y}$ are disjoint, and they are open because p is a covering space, so $\widetilde{X}$ is Hausdorff. $\square$

Proof (Hausdorff $\Longrightarrow$) Suppose $\widetilde{X}$ is Hausdorff. Let $x \neq y \in X$, and denote $p^{-1}(x) = \{\widetilde{x}_i\}_{i=1}^N$ and $p^{-1}(y) = \{\widetilde{y}_i\}_{i=1}^N$. Since we can separate any two points in $\widetilde{X}$ using the Hausdorff property, we can do it with finitely many points. So let

$$\{\widetilde{U}_i', \widetilde{V}_i'\}_{i=1}^N$$

be a collection of sets open in $\widetilde{X}$ such that $\widetilde{U}_i' \ni \widetilde{x}_i$ and $\widetilde{V}_i' \ni \widetilde{y}_i$ for all $i = 1 \ldots N$ and every pair of sets in the collection is disjoint. Next let E be an evenly covered neighborhood of x, and F an evenly covered neighborhood of y. Then

$$\{\widetilde{E}_i, \widetilde{F}_i\}_{i=1}^N$$

is a collection of sets open in $\widetilde{X}$ such that $\widetilde{E}_i \ni \widetilde{x}_i$ and $\widetilde{F}_i \ni \widetilde{y}_i$ for all $i = 1 \ldots N$ and every $\widetilde{E}_i, \widetilde{F}_i$ is homeomorphic to E, F, respectively. Taking $\widetilde{U}_i = \widetilde{U}_i' \cap \widetilde{E}_i$ and $\widetilde{V}_i = \widetilde{V}_i' \cap \widetilde{F}_i$, we obtain

$$\{\widetilde{U}_i, \widetilde{V}_i\}_{i=1}^N$$

which are open, disjoint, contain $\widetilde{x}_i, \widetilde{y}_i$ as appropriate, and are homeomorphic to U_i, V_i as appropriate, where we denote

$$U_i = p(\widetilde{U}_i) \cap E$$
$$V_i = p(\widetilde{V}_i) \cap F.$$

To finish the proof, we let

$$U = \bigcap_{i=1}^N U_i$$
$$V = \bigcap_{i=1}^N V_i$$

[†] To see this, apply the definition of boundary points and see that for any $b \in \partial(\widetilde{V}_{i,j})$ any open $\widetilde{U}_\delta \ni b$ intersects $\widetilde{V}_{i,j}$, so $\delta \in \Delta$.

[‡] Each U_δ is open because every covering space is an open map, a fact which I choose not to prove here since this problem is insanely long already.

and observe that $x \in U$, $y \in V$ since every $\widetilde{U}_i, \widetilde{V}_i$ contains a point which maps to x, y respectively. To see that U, V are disjoint, suppose $d \in U \cap V$. Observe that for all i, $d \in U \subset U_i \subset p(\widetilde{U}_i)$ so the preimage

$$p|^{-1}(d) \subset \bigcup_{i=1}^{N} \widetilde{U}_i,$$

and similarly $d \in V \subset V_i \subset p(\widetilde{V}_i)$ so

$$p|^{-1}(d) \subset \bigcup_{i=1}^{N} \widetilde{V}_i,$$

but $\bigcup_{i=1}^{N} \widetilde{U}_i$ and $\bigcup_{i=1}^{N} \widetilde{V}_i$ are disjoint. ■

4. Construct a a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is the union of a sphere and a circle intersecting it in two points.

8. Let $\widetilde{X}$ and $\widetilde{Y}$ be simply-connected covering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$ then $\widetilde{X} \simeq \widetilde{Y}$. [Exercise 11 in Chapter 0 may be helpful.]

9. Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic. [Use the covering space $R \to S^1$.]

10. Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphism of covering spaces without basepoints.

Collaborators:

None for this homework.