3. Recall that $GL_2(\mathbb{R})$ is the group of invertible 2×2 matrices, $SL_2(\mathbb{R})$ is its subgroup of all invertible 2×2 matrices with determinant equal to one.

   (a) Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R}, a, c, \neq 0 \right\}$. **Prove** H is a subgroup of $GL_2(\mathbb{R})$.

   Proof We must show that (1) every element of H is in $GL_2(\mathbb{R})$, and (2) H is closed under matrix multiplication. To see that (1) holds, observe that for any $h = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in H$, we have that $\det(h) = ac$, and since $h \in H$, then $a, c \neq 0$, so $ac \neq 0$. Thus h is invertible, and $h \in GL_2(\mathbb{R})$. Now we will prove (2). Let $h_1, h_2 \in H$, with

$$h_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}, \text{ and } h_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}.$$

   Then,

$$\begin{aligned} h_1 h_2 &= \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1 a_2 & (a_1 b_2 + b_1 c_2) \\ 0 & c_1 c_2 \end{bmatrix} \end{aligned}$$

   and, since we know that $a_1, a_2, c_1, c_2 \neq 0$, then $a_1 a_2, c_1 c_2 \neq 0$ as well. Thus, $h_1 h_2 \in H$, and H is closed. ■

   (b) Consider the following two matrices in $GL_2(\mathbb{R})$: $x = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, y = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$.

   Explain why $SL_2(\mathbb{R})x = SL_2(\mathbb{R})y$ (that is, why are these two right cosets equal? Think of the equivalence relation).

   Proof (Or how I expected the proof to go)
   It suffices to show that $x \sim_R y$, because since equivalence relations are transitive and symmetric, any matrix which is in one coset will also be in the other. To see that $x \sim_R y$, observe that

$$\begin{aligned} xy^{-1} &= \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{4-2} \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & -1 \\ -1/2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -4 \\ -1 & 4 \end{bmatrix} \end{aligned}$$

   and $\det(xy^{-1}) = 1$, so $xy^{-1} \in SL_2(\mathbb{R})$. **EXCEPT:** $\det(xy^{-1})$ is not 1, it's 4. So is this my mistake? Or should the numbers in the problem be slightly different?

   ■