Math 550
Homework 4
Dr. Fuller
Solutions

1. (a) $\sin^2 u\, du \wedge dv$

   (b) $d\theta$

2. We need to show that $g^*(dx_1 \wedge \cdots \wedge dx_n)_p(e_1,\ldots,e_n)_p = \det Dg(p)(dx_1 \wedge \cdots \wedge dx_n)_p(e_1,\ldots,e_n)_p$ for all $p \in \mathbf{R}^n$.

   Here are two ways to show that.

   Solution 1.

$$\begin{aligned}
g^*(dx_1 \wedge \cdots \wedge dx_n)_p(e_1,\ldots,e_n)_p &= (dx_1 \wedge \cdots \wedge dx_n)_{g(p)}(Dg(p)(e_1),\ldots,Dg(p)(e_n))_{g(p)} \\
&= \det \begin{pmatrix} | & | & & | \\ Dg(p)(e_1) & Dg(p)(e_2) & \cdots & Dg(p)(e_n) \\ | & | & & | \end{pmatrix} \\
&= \det Dg(p) \\
&= \det Dg(p)(dx_1 \wedge \cdots \wedge dx_n)_p(e_1,\ldots,e_n)_p
\end{aligned}$$

   The next-to-last equality follows from recognizing the matrix representation of $Dg(p)$ with respect to the standard basis.

   Solution 2.

$$\begin{aligned}
g^*(dx_1 \wedge \cdots \wedge dx_n)_p(e_1,\ldots,e_n)_p &= (dx_1 \wedge \cdots \wedge dx_n)_{g(p)}(Dg(p)(e_1),\ldots,Dg(p)(e_n))_{g(p)} \\
&= (dx_1 \wedge \cdots \wedge dx_n)_{g(p)}(\textstyle\sum_i \frac{\partial g_i}{\partial x_1}(p)e_i,\ldots,\sum_i \frac{\partial g_i}{\partial x_n}(p)e_i)_{g(p)} \\
&= (\textstyle\sum_\sigma (-1)^{\text{sign }\sigma} \frac{\partial g_{\sigma(1)}}{\partial x_1}(p)\cdots\frac{\partial g_{\sigma(n)}}{\partial x_n}(p))\ (dx_1 \wedge \cdots \wedge dx_n)_{g(p)}(e_1,\ldots,e_n)_{g(p)} \\
&= \det\left[\frac{\partial g_i}{\partial x_j}(p)\right]\ (dx_1 \wedge \cdots \wedge dx_n)_{g(p)}(e_1,\ldots,e_n)_{g(p)} \\
&= \det Dg(p)(dx_1 \wedge \cdots \wedge dx_n)_p(e_1,\ldots,e_n)_p
\end{aligned}$$

3. $-\frac{\pi}{2}$

4. $\frac{1}{3}$