Math 220A - First Homework - Due October 9

1. As noted in class, the matrix presentation of the quaternion group Q shows it is generated by two elements i and j subject to the relations $i^4 = j^4 = e$, $i^2 = j^2 = -1$, $ij = (-1)ji$. Show that the quaternion group has only one element $-1 \in Q$ of order 2, and that it commutes with all elements of Q. Deduce that Q is not isomorphic to D_4, and that every subgroup of Q is normal.

2. Let $V = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \subset S_4$. Prove that V is a normal subgroup of S_4, and then show that $S_4/V \cong S_3$.

3. Consider the elements $a, b \in \mathrm{GL}_2(\mathbb{Z})$

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Show that $a^4 = e$ and $b^3 = e$, but that ab has infinite order, and hence that the group generated by a and b (denoted $\langle a, b\rangle$) is infinite.

4. Let G_1 be the subgroup of $\mathrm{GL}_3(\mathbb{Z}/2\mathbb{Z})$ defined by

$$G_1 := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z}/2\mathbb{Z} \right\}.$$

Show that G_1 is a group of order 8 and identify this group as one of the five possible groups of order 8.

5. Let G_2 be the subgroup of $\mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$ defined by

$$G_2 := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a = 1, 3 \in \mathbb{Z}/4\mathbb{Z}, b \in \mathbb{Z}/4\mathbb{Z} \right\}.$$

Show that G_2 is a group of order 8 and identify this group as one of the five possible groups of order 8.

6. Show that every finite group of even order contains an element of order 2.

7. Let N be a normal subgroup of G of index n. Show that if $g \in G$, then $g^n \in N$. Give an example to show that this may be false when N is not normal.

8. Suppose that G is a finite group with normal subgroups $N_1, N_2, \ldots, N_t$ for which $N_i \cap \prod_{j \neq i} N_j = \{e\} \subset G$ for all i, and for which $|G| = |N_1| \cdot |N_2| \cdots |N_t|$.
(i) Prove that is $n_i \in N_i$ and $i \neq j$ then $n_i \cdot n_j = n_j \cdot n_i$. (Hint: Think about $n_i n_j n_i^{-1} n_j^{-1}$.)
(ii) Prove that G is isomorphic to the direct product $N_1 \times N_2 \times \cdots \times N_t$.