Math 550
Homework 8
Dr. Fuller
Due October 30

1. Let $M \subset \mathbf{R}^3$ be the manifold bounded above by the sphere $x^2+y^2+z^2=a^2$ and below by $z=0$. Let

$$\omega = xz\, dy \wedge dz + yz\, dz \wedge dx + (x^2+y^2+z^2)\, dx \wedge dy.$$

   Compute $\int_{\partial M} \omega$ both directly and using Stokes' Theorem. (Answer: πa^4.)

2. Let C be the intersection of the sphere $x^2+y^2+z^2=1$ and the plane $x+y+z=0$, oriented counterclockwise as viewed from above the xy-plane. Use Stokes' Theorem to evaluate

$$\int_C z^3\, dx.$$

3. Show that

$$\omega = \frac{x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

   on $\mathbf{R}^3 - \{(0,0,0)\}$ is closed but not exact.

4. Show that Stokes' Theorem is false if M is not compact.

5. Let M be a compact k-manifold without boundary. Show that $\int_M d\omega = 0$ for all $\omega \in \Omega^{k-1}(M)$. Give a counterexample if M is not compact.

6. Suppose that C is a compact 2-dimensional manifold-with-boundary in $\mathbf{R}^2$, and assume $(0,0) \notin \partial C$. Let $\omega = \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$. Prove that

$$\int_{\partial C} \omega = \begin{cases} 0 & \text{if } (0,0) \notin C, \\ 2\pi & \text{if } (0,0) \in C. \end{cases}$$

   (Hint: If $(0,0) \in C$, then ω is not defined on C, so consider $C - B_\varepsilon$, where B_ε is an open ball centered at the origin with $B_\varepsilon \subset C$.)