Topology Exam: Fall 2017. Four hours.
Answer six out of the seven questions.

1. For X and Y topological spaces define what is means for a function $f : X \to Y$ to be *continuous*. Give the ϵ, δ definition of continuity for metric spaces. Prove that your definitions are equivalent for metric spaces.

2. Are the following statements true or false? Give a proof or counter-example as appropriate.
(a) A closed bounded subset of a topological space is compact.
(b) The image of a closed subset under a continuous map is closed.
(c) If $f : X \to Y$ is a continuous surjection and Y is Hausdorff then so is X.
(d) If $f : X \to Y$ is a continuous surjection and X is Hausdorff then so is Y.
(e) If a function between Hausdorff topological spaces is continuous, then the preimage of every compact set is compact.

3. Define at it means for a topological space to be *connected*.
(a) Show that the continuous image of a connected space is connected.
(b) Show that if $H \subset K \subset Closure(H)$ and H is connected, then so is K.
(c) Is $(C[0,1], sup)$ connected? .

Recall: $(C[0,1], sup)$ is the set of continuous functions on the unit interval $[0,1]$ with metric $d(f,g) = sup\{\ |f(x) - g(x)| \ \ | \ \ x \in [0,1] \ \}$.

4. Define what it means for a collection of subsets of a set X to be a *basis for a topology* on X. Give a necessary condition for a collection of subsets to be a basis for a topology.
Let X be the set of subsets of $\mathbf{N}$ (the set of positive integers). If A is a finite subset of $\mathbf{N}$, and B is a subset of $\mathbf{N}$ whose complement is finite, define a subset $[A,B]$ of X by

$$[A,B] = \{E \subset \mathbf{N} \mid \mathbf{A} \subset \mathbf{E} \subset \mathbf{B}\}$$

Show that the sets $[A,B]$ form a base for a topology on X. Prove that with this topology X is Hausdorff and disconnected. Prove that the function $f : X \times X \to X$ defined by:

$$f(E_1, E_2) = E_1 \cap E_2$$

is continuous.

5. Define *covering space*.
(a) State carefully and prove that covering spaces have the path lifting property.
(b) Suppose that $p : \tilde{X} \to X$ is a covering projection and $f : Y \to X$ is a continuous map. Show that there is a map $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$ if and only if (with appropriate basepoints) $f_*(\pi_1(Y)) \subset p_*(\pi_1(\tilde{X}))$.

6. Define *compact* and *sequentially compact*.
(a) Show that a compact subspace of a Hausdorff space is closed.
(b) Show that if M is a compact metric space, and $f : M \to M$ has the property that $d(f(x), f(y)) = d(x,y)$, for all x and y in M, then f must be surjective. (You may assume without proof that compact spaces are sequentially compact.)

7. (a) Define what it means for a metric space to be *complete*. State carefully and prove the contraction mapping theorem for metric spaces.
(b) Show that if $f(x)$ is a real differentiable function with $|f'(x)| < M$ for all real x, then for any real numbers a_{ij} and b_j, there is one and only one point $(x_1, ..., x_n)$ satisfying

$$x_i = \sum_{j=1}^{n} a_{ij} f(x_j) \quad + b_j \quad 1 \leq i \leq n$$

provided that

$$\sum_{i,j} a_{ij}^2 < 1/M^2$$