Math 550
Homework 10

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1. For a vector field $X = (f_x, f_y)$ on $\mathbb{R}^2$, we may define as associated 1-form, different from the one in class, by
$$\star\omega_X^1 = -f_y\,dx + f_x\,dy.$$
We may also define
$$\mathrm{div}X = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}.$$

   (a) Let M be a compact 2-dimensional manifold with boundary in $\mathbb{R}^3$. Show that for all points $p \in \partial M$, the equation $\star\omega_X^1 = X \cdot n\,ds$ holds.

   Proof Let $p \in \partial M$, and let $v \in (\partial M)_p$. Then
$$\begin{aligned} \star\omega_X^1(p)(v) &= \big(-f_y(p)\,dx + f_x(p)\,dy\big)(v) \\ &= \det\begin{bmatrix} | & | \\ X_p & v \\ | & | \end{bmatrix} \end{aligned}$$
   Now X_p can be written as $w + \langle X_p, N_p\rangle N_p$, where $w \in (\partial M)_p$ and N_p is the unit outward normal vector of M at p. Thus we have
$$\begin{aligned} \det\begin{bmatrix} | & | \\ X_p & v \\ | & | \end{bmatrix} &= \det\begin{bmatrix} | & | \\ w + \langle X_p, N_p\rangle N_p & v \\ | & | \end{bmatrix} \\ &= \det\begin{bmatrix} | & | \\ w & v \\ | & | \end{bmatrix} + \det\begin{bmatrix} | & | \\ \langle X_p, N_p\rangle N_p & v \\ | & | \end{bmatrix} \\ &= 0 + \langle X_p, N_p\rangle \det\begin{bmatrix} | & | \\ N_p & v \\ | & | \end{bmatrix} \\ &= \langle X_p, N_p\rangle\,ds \end{aligned}$$
   and we are done. ■

   (b) Prove the following *Divergence form of Green's Theorem:* Let M be a 2-dimensional manifold with boundary in $\mathbb{R}^2$, and let X be a vector field on M. Then
$$\int_M \mathrm{div}\,X\,dA = \int_{\partial M} \langle X, n\rangle\,ds.$$

   Proof Since the differential of the RHS is
$$\begin{aligned} d\left(\langle X, n\rangle\,ds\right) &= d\big(-f_y(p)\,dx + f_x(p)\,dy\big) \\ &= -\frac{\partial f_y}{\partial y}\,dy\wedge dx + \frac{\partial f_x}{\partial x}\,dx\wedge dy \\ &= \left(\frac{\partial f_y}{\partial y} + \frac{\partial f_x}{\partial x}\right) dx\wedge dy \\ &= \mathrm{div}\,X\,dA, \end{aligned}$$

then by Stokes' Theorem,

$$\int_{\partial M} \langle X, n \rangle \, ds = \int_M d\left(\langle X, n \rangle \, ds\right) = \int_M \operatorname{div} X \, dA,$$

and we are done. ■

2. Let M be a compact 3-dimensional manifold with boundary in $\mathbb{R}^3$, with $\vec{0} \in M - \partial M$. Consider the vector field $X(p) = \frac{p}{||p||^3}$ defined on $\mathbb{R}^3 - \vec{0}$. Prove that

$$\int_{\partial M} \langle X, N \rangle \, dA = 4\pi.$$

Proof Define a manifold $M' = M - B_\epsilon(\vec{0})$. We will integrate over M' to find the integral over M. Note that $\partial M' = \partial M \cup S^2_\epsilon$, where S^2_ϵ is a sphere of radius ϵ. This means that

$$\int_{\partial M'} \langle X, N \rangle \, dA = \int_{\partial M} \langle X, N \rangle \, dA - \int_{S^2_\epsilon} \langle X, N \rangle \, dA.$$

By the Divergence form of Green's Theorem, the LHS is $\int_{M'} \operatorname{div} X \, dA$, and a straightforward calculation will show that $\operatorname{div} X = 0$. Thus we find that

$$\begin{aligned} 0 &= \int_{\partial M} \langle X, N \rangle \, dA - \int_{S^2_\epsilon} \langle X, N \rangle \, dA \\ &= \int_{\partial M} \langle X, N \rangle \, dA - 4\pi \end{aligned}$$

And we are done.

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3. (a) Show that if X is a vector field on $\mathbb{R}^3$ with $\operatorname{curl} X = 0$, then $X = \operatorname{grad} F$ for some function $F : \mathbb{R}^3 \to \mathbb{R}$.

Proof Let $X = (f_x, f_y, f_z)$. Then $\omega^1_X = f_x \, dx + f_y \, dy + f_z \, dz$. So,

$$d(\omega^1_X) = \omega^2_{\operatorname{curl} X} = 0,$$

since $\operatorname{curl} X = 0$. Thus, ω^1_X is exact by Poinacaré's Lemma. Therefore there is some function F such that $dF = \omega^1_X$, and since $dF = \omega^1_{\operatorname{grad} F}$, then $\omega^1_X = \omega^1_{\operatorname{grad} F}$. So

$$f_x \, dx + f_y \, dy + f_z \, dz = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy + \frac{\partial F}{\partial z} \, dz$$

and since dx, dy, dz are linearly independent, then we can equate the coefficients, and we are done.

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(b) Show that if X is a vector field on $\mathbb{R}^3$ with $\operatorname{div} X = 0$, then $X = \operatorname{curl} Y$ for some vector field Y on $\mathbb{R}^3$.

Proof Since $\operatorname{div} X = 0$, then $(\operatorname{div} X) \, dx \wedge dy \wedge dz = d(\omega^2_X) = 0$, so ω^2_X is exact. Then there is a one form η such that $d\eta = \omega^2_X$. Now $\eta = n_1 \, dx + n_2 \, dy + n_3 \, dz$ can be written as $\eta = \omega^1_Y$, where $Y = (n_1, n_2, n_3)$. Thus,

$$\omega^2_X = d\eta = d(\omega^1_Y) = \omega^2_{\operatorname{curl} Y},$$

so $X = \operatorname{curl} Y$, by the linear independence argument of problem 3a. ■

4. Let $\omega = \frac{-y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy$ be a 1-form on $\mathbb{R}^2 - \vec{0}$. Prove that ω does not extend to a 1-form on $\mathbb{R}^n$.

Proof Recall that we showed on a previous homework that ω is closed but not exact. If ω did extend to a 1-form on $\mathbb{R}^n$, then the extension would have to be given by the same formula, defined on all $\mathbb{R}^n$. So, it would be exact by Poincaré's Lemma, but it is not. ■