MATH 220B

PROBLEM SET #2

If you want to turn it in, please do so no later than Jan 31st (Friday)

The following references are to the textbook [DF04].

1. Ring of fractions and localization: Exercises 3, 4, 5, from §7.5 and 21, 22, 23 from §15.4.

Exercise A. Let R, R' be commutative rings, let $S \subset R$ be a multiplicative subset, and let $f : R \to S^{-1}R$ be the natural map $r \mapsto \frac{r}{1}$.

Show that if $g : R \to R'$ is a ring homomorphism satisfying the following three properties:

(1) for all $s \in R$, $g(s)$ is a unit in R',
(2) if $g(r) = 0$, then $rs = 0$ for some $s \in S$,
(3) every element of R' is of the form $g(r)g(s)^{-1}$ for some $r \in R$ and $s \in S$,

then there exists a unique ring isomorphism $h : S^{-1}R \xrightarrow{\sim} R'$ such that $g = h \circ f$.

Exercise B. Let R be a commutative ring, $\mathfrak{p} \subset R$ a prime ideal, and $S = R \setminus \mathfrak{p}$ (a multiplicative subset of R). Then we know that the localization $R_{\mathfrak{p}} := S^{-1}R$ is a local ring with unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}} := S^{-1}\mathfrak{p}$.

Show that there is an isomorphism

$$\mathrm{Frac}(R/\mathfrak{p}) \xrightarrow{\sim} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$$

between the field of fractions of the integral domain $R/\mathfrak{p}$ and the residue field of the local ring $R_{\mathfrak{p}}$.

2. Modules: Basic definitions and examples: Exercises 5, 8, 15, 18 from §10.1.

3. Quotient modules and homomorphisms: Exercises 4, 6, 9, 12, 13 from §10.2.

4. Generation of modules and free modules: Exercises 2, 4, 7, 12 from §10.3.

References

[DF04] David S. Dummit and Richard M. Foote, *Abstract algebra*, third ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004.