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Homework 5

1. Let X be a nonempty set and let μ be a measure on X. Prove that any nonnegative μ-measurable function $f : X \to [0, \infty]$ is μ-integrable on X, i.e., the lower integral equals the upper integral:

$$\int_{*X} f \, d\mu = \int_X^* f \, d\mu.$$

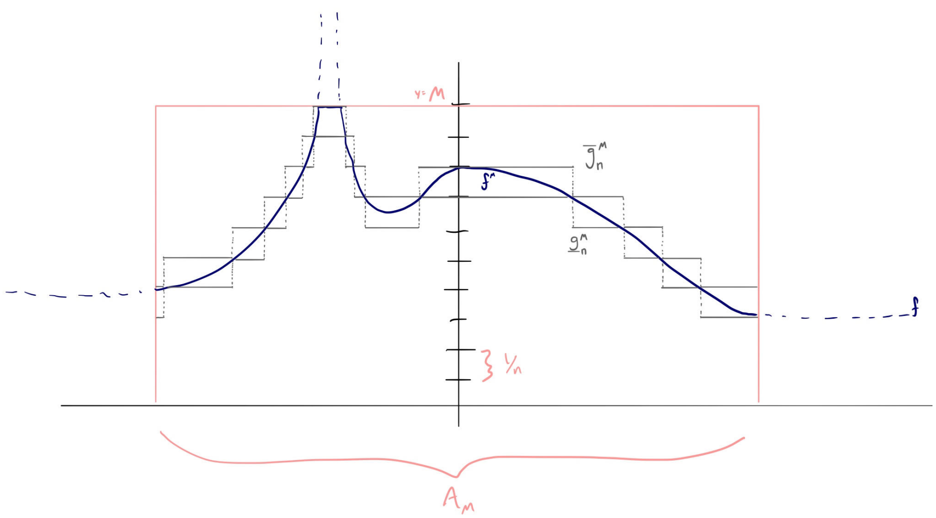
Proof Let f be nonnegative and μ-measurable, and let $A_1 \subset A_2 \subset \dots$ be any sequence of measurable sets in X such that $0 < \mu(A_i) < \infty$ for every i, and $\bigcup_{m=1}^{\infty} A_m = X$[†]. Now fix $M \in \mathbb{N}$, let f^M be

$$f^M(x) = \begin{cases} \min\big(f(x), M\big), & \text{if } x \in A_M \\ 0 & \text{otherwise.} \end{cases}$$

Thus f^M is supported on A_M and bounded above by M. Now for each $n \in \mathbb{N}$, define simple functions $\underline{g}_n^M$ and $\overline{g}_n^M$ by dividing the codomain $\mathbb{R}^+$ into intervals of length $\frac{1}{n}$. So for each $i = 1, 2, \dots$ we have

$$\underline{g}_n^M = \sum_{i=1}^{\infty} \left(\tfrac{i-1}{n}\right) \chi_{(f^M)^{-1}\left(\left[\frac{i-1}{n}, \frac{i}{n}\right]\right)}$$

$$\overline{g}_n^M = \sum_{i=1}^{\infty} \left(\tfrac{i}{n}\right) \chi_{(f^M)^{-1}\left(\left[\frac{i-1}{n}, \frac{i}{n}\right]\right)}$$



Now we can observe that for every n, $\underline{g}_n^M < f < \overline{g}_n^M$ $\mu-$a.e. and

$$\int \underline{g}_n^M \, d\mu - \int \overline{g}_n^M \, d\mu = \tfrac{1}{n}\mu(A_M),$$

[†]We can assume without loss of generality that it is possible to produce this sequence of sets since if we cannot, then for every increasing sequence of sets whose union is X, $\mu(A_M) = \infty$ for some M, which means every function which is strictly positive μ-a.e. has infinite upper and lower integral, which also gives us what we want.

so we can choose n large enough that $\frac{1}{n}\mu(A_M) < \varepsilon$ for any ε. Thus

$$\int_{*X} f^M \, d\mu = \int_X^* f^M \, d\mu.$$

To finish the proof, we let m vary over $\mathbb{N}$ and note that every f^m is μ-integrable, and $\{f^m\}_{m=1}^\infty$ is an increasing sequence of functions which converges to f, so f is integrable by MCT and

$$\int_X f \, d\mu = \lim_{m\to\infty} \int_X f^m \, d\mu.$$

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2. Let X be a nonempty set and let μ be a measure on X. Prove that if μ-measurable functions $f, g : X \to [\infty, \infty]$ are such that f is μ-summable on X, and g is bounded on X ($|g(x)| \leq M$ for all $x \in X$), then the product fg is μ-summable and

$$\int_X |fg| \, d\mu \leq M \int_X |f| \, d\mu.$$

Proof By problem 1, we know that $|f|$ and $|g|$ are integrable. So

$$\begin{aligned}\int_X |fg| \, d\mu &= \int_X |f||g| \, d\mu \\ &\leq \int_X (|f|M) \, d\mu \\ &= \int_X |Mf| \, d\mu\end{aligned}$$

and, since for any μ-summable simple function φ we know that

$$\begin{aligned}\int M\varphi \, d\mu &= \int \left(M \sum_{i=1}^\infty (a_i)\chi_{A_i} \right) d\mu \\ &= \sum_{i=1}^\infty M(a_i)\mu\left(A_i\right) \\ &= M \sum_{i=1}^\infty (a_i)\mu\left(A_i\right) \\ &= M \int \varphi \, d\mu,\end{aligned}$$

then $\int_{*X} |Mf| \, d\mu = \int_X^* |Mf| \, d\mu = M \int_{*X} |f| \, d\mu = M \int_X^* |f| \, d\mu$ so

$$\int_X |Mf| \, d\mu = M \int_X |f| \, d\mu < \infty.$$

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3. Let μ be a Radon measure and let $f : \mathbb{R}^n \to \mathbb{R}$ be μ-summable. Prove that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every μ-measurable set $A \subset \mathbb{R}^n$ with $\mu(A) < \delta$ one has

$$\int_A |f| \, d\mu < \varepsilon.$$

Proof Let $f_b = |f|\chi_{\{|f|>b\}}$. Since f is μ-summable, then $|f| < \infty$ μ-*a.e.*, so $f_b \to 0$ μ-*a.e.*. Then the sequence f_b is dominated by $|f|$, so by the Dominated Convergence Theorem,

$$\lim_{b\to\infty} \int_{\mathbb{R}^n} f_b \, d\mu = \int_{\mathbb{R}^n} \lim_{b\to\infty} f_b \, d\mu = 0.$$

So for any $\varepsilon > 0$, there exists some $b \in \mathbb{N}$ such that $\frac{\varepsilon}{2} > \int_{\mathbb{R}^n} f_b \, d\mu = \int_{\chi_{\{|f|>b\}}} |f| \, d\mu$. Now let $\delta = \frac{\varepsilon}{2b}$ and let $A \subset \mathbb{R}^n$ with $\mu(A) < \delta$. Then

$$\begin{aligned}
\int_A |f| \, d\mu &= \int_{A\cap\{|f|>b\}} |f| \, d\mu + \int_{A\cap\{|f|\leq b\}} |f| \, d\mu \\
&\leq \int_{\{|f|>b\}} |f| \, d\mu + \int_A b \, d\mu \\
&= \int_{\mathbb{R}^n} f_b \, d\mu + b\mu(A) \\
&\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2}
\end{aligned}$$

and we are done. ■

4. Let X be a nonempty set and let μ be a measure on X. Assume μ-summable functions $f, f_n : X \to [-\infty, \infty]$ are such that

$$f_n \to f \quad \mu\text{-}a.e. \text{ in } X,$$

and

$$\int_X |f_n| \, d\mu \to \int_X |f| \, d\mu.$$

Prove that

$$\int_X |f_n - f| \, d\mu \to 0.$$

Proof Since

- f, f_n are μ-measurable and $|f|, |f_n|$ are μ-summable,
- $f_n \to f$ μ-$a.e.$,
- $|f_n| \leq |f_n|$,
- $|f_n| \to |f|$ μ-$a.e.$,
- $\int_X |f_n| \, d\mu \to \int_X |f| \, d\mu$,

Then all the conditions of the Variant of Dominated Convergence Theorem from the text are satisfied, and we are done. ■

5. Compute the limit

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n \ln\left(2 + \cos\left(\frac{x}{n}\right)\right) dx.$$

Answer: $\ln 3$.

Proof Let

$$\begin{aligned} f_n &= \chi_{[0,n]} \left(1 - \frac{x}{n}\right)^n, \quad \text{and} \\ g_n &= \chi_{[0,n]} \ln\left(2 + \cos\left(\frac{x}{n}\right)\right), \quad \text{so that} \\ F_n &= f_n g_n = \chi_{[0,n]} \left(1 - \frac{x}{n}\right)^n \ln\left(2 + \cos\left(\frac{x}{n}\right)\right). \end{aligned}$$

Now the desired limit is $\lim\limits_{n\to\infty} \int_{\mathbb{R}} F_n \, d\mu$. Taking derivatives, we find that

$$\begin{aligned} \frac{d}{dn} f_n &= \left(1 - \frac{x}{n}\right)^n \left(\frac{x}{\left(1 - \frac{x}{n}\right) n} + \ln\left(1 - \frac{x}{n}\right)\right) \\ \frac{d}{dn} g_n &= \frac{x \sin\left(\frac{x}{n}\right)}{\left(\cos\left(\frac{x}{n}\right) + 2\right) n^2} \end{aligned}$$

and since we are only concerned with x, n values such that $0 < x < n$[†] then $0 < \frac{x}{n} < 1$ and so all the quantities above are positive, except the ln term. Thus we can conclude that F_n is an increasing sequence of functions if we can show that $h_n = \frac{x}{\left(1-\frac{x}{n}\right)n} + \ln\left(1 - \frac{x}{n}\right) > 0$. For any fixed n and $0 \le x < n$, $h_n(0) = 0$, and $h_n(x)$ is continuous and increasing, since $h_n' = \frac{x}{(x-n)^2}$, which is positive. Thus h_n is positive, and therefore F_n is an increasing sequence of measurable nonnegative functions.

By the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}} F_n \, dx = \int_{\mathbb{R}} \lim_{n\to\infty} F_n \, dx = \int_0^\infty e^{-x} \ln\left(2 + \cos(0)\right) dx = \ln 3$$

■

[†] For any n, the set where $x \in \{0, n\}$ has measure zero, so doesn't affect the integral; and if $x > n$, then $F_n(x) = 0$ and $F_{n+1}(x) \ge 0$.