Spring 2017

1. Prove or give a counterexample for each of the following

(a) Any quotient of a Hausdorff space is Hausdorff.

This is false. Consider the integers $\mathbb{Q}$ under the standard topology & a partition, $\sim$, where $x \sim y$ if $x = 2^k y$ for some $k \in \mathbb{Z}$. Note this is indeed a partition because:

(i) $x = 2^0 x$ so $x \sim x$

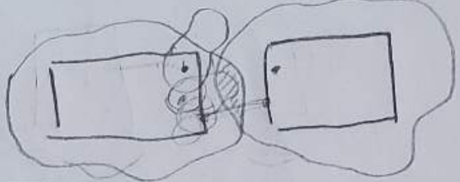
(ii) If $x \sim y$ then $x = 2^k y$, then $y = 2^{-k} x$ so $y \sim x$

(iii) If $x \sim y$ & $y \sim z$ then $x = 2^k y$ & $y = 2^\ell z$ so $x = 2^{k+\ell} z$ & $x \sim z$

Therefore we can create $\mathbb{Q}/\sim$ as a quotient space. Note that $\mathbb{Q}$ is Hausdorff since if $n \neq m \in \mathbb{Q}$, let... $d = |n-m|$. Then $B_{\frac{d}{2}}(n) \cap B_{\frac{d}{2}}(m) = \emptyset$

However, consider $\bar{0}$ & $\bar{1} \in \mathbb{Q}/\sim$. First note that $\bar{0} \neq \bar{1}$ since $0 \neq 2^k$ $\forall k \in \mathbb{Z}$. Let $\pi : \mathbb{Q} \to \mathbb{Q}/\sim$ be the quotient map. Let $\mathcal{U}$ be an open set containing 0 in $\mathbb{Q}$. Then $\exists B_\varepsilon(0) \subseteq \mathcal{U}$ for some $\varepsilon > 0$. Then $\forall N \in \mathbb{N}$ where $1/2^N < \varepsilon$, we have that $1/2^N \in B_\varepsilon(0)$. Let $\bar{\mathcal{U}}$ be an open set containing 0 in $\mathbb{Q}/\sim$, then $\pi^{-1}(\mathcal{U})$ must be open. Then $0 \in \pi^{-1}(\mathcal{U})$ so $\exists B_\varepsilon(0) \subseteq \pi^{-1}(\mathcal{U})$. Thus $\exists N$ s.t. $1/2^N < \varepsilon \Rightarrow 1/2^N \in B_\varepsilon(0) \subseteq \pi^{-1}(\mathcal{U})$ so $\pi(1/2^N) \in \mathcal{U}$. However, $\frac{1}{2^N} = 2^{-N} \cdot 1$, so $\pi(1/2^N) = \bar{1}$. Thus $\forall$ open sets of $\bar{0}$ in $\mathbb{Q}/\sim$, $\bar{1} \in \mathcal{U}$ & since $\bar{1} \neq \bar{0}$ this implies $\mathbb{Q}/\sim$ is not Hausdorff.

$\square$



(b) Any metric space is normal.

This is true. Let (X, r) be the metric space & let C & D be disjoint closed sets of X. Define for any $d \in D$

$$\text{dist}(d, C) = \inf\{d(d, c) : c \in C\}.$$

If $\text{dist}(d, C) = 0$ then $\forall \varepsilon > 0$ $B_\varepsilon(d) \cap C \neq \emptyset \Rightarrow$ $d \in \overline{C} = C$ which is a contradiction. Thus $\text{dist}(d, C) > 0$. Define $\Delta_d = \text{dist}(d, C)$. Then

Needed help on this one.

$\{B_{\frac{\Delta_d}{3}}(d)\}_{d \in D}$ is an open cover of D &

Similarly define $\Delta_c = \text{dist}(D, c) > 0$ & make $\{B_{\frac{\Delta_c}{3}}(c)\}_{c \in C}$ an open cover of C.

Suppose $\exists\, x \in B_{\frac{\Delta_d}{3}}(d) \cap B_{\frac{\Delta_c}{3}}(c)$ for some $c \in C$, $d \in D$.

Then
$$\begin{aligned} d(c, d) &\leq d(c, x) + d(x, d) \\ &< \frac{\Delta_c}{3} + \frac{\Delta_c}{3} \\ &< \frac{2\Delta_c}{3} \end{aligned}$$

which contradicts that $\text{dist}(c, D) > \Delta_c$. So $\bigcup_{d \in D} B_{\frac{\Delta_d}{3}}(d)$ & $\bigcup_{c \in C} B_{\frac{\Delta_c}{3}}(c)$ are two disjoint open sets of D & C $\Rightarrow X$ is normal. $\square$

(c) If X is a topological space & $A \subset B \subset X$ & $\bar{A}$ is the closure of A, then $\bar{A} \cap B$ is the closure of A with respect to the subspace topology on B.

This is true. Let C be the closure of A with respect to the subspace topology on B. Then note that $\bar{A} \cap B$ is a closed set containing A by definition of the subspace topology so $C \subseteq \bar{A} \cap B$. Further since C is closed wrt the subspace topology, $C = \tilde{C} \cap B$ where $\tilde{C}$ is closed in X. Since $A \subseteq C \Rightarrow A \subseteq \tilde{C}$, so $\bar{A} \subseteq \tilde{C}$ which gives that $\bar{A} \cap B \subseteq \tilde{C} \cap B = C$. Thus $C = \bar{A} \cap B$. $\square$

2. Suppose X is a compact topological space, Y is a topological space, & $\mathcal{C}$ is an open cover of $X \times Y$. Prove that for all $y \in Y$ $\exists$ an open neighborhood $\mathcal{U}$ of y s.t. $X \times \mathcal{U}$ is contained in the union of finitely many sets from $\mathcal{C}$.

Let $\mathcal{C} = \{C_\alpha\}_{\alpha \in \Lambda}$ be the open cover of $X \times Y$. Let $\bar{y} \in Y$. Since $X \times \{\bar{y}\}$ is homeomorphic to X & X is compact, so is $X \times \{\bar{y}\}$. Note that by definition of the subspace topology, $\forall (x,y) \in C_\alpha$ $\exists$ $\mathcal{U}_x$ open in X, V_y open in Y s.t. $(x,y) \in \mathcal{U}_x^y \times V_y^x \subseteq C_\alpha$. Therefore, create

$\mathcal{U} = \{\mathcal{U}_x^{\bar{y}} \times V_{\bar{y}}^x\}_{x \in X}$ where $(x,\bar{y}) \in \mathcal{U}_x^{\bar{y}} \times V_{\bar{y}}^x \subseteq C_\alpha$ for any $\alpha \in \Lambda$. This is an open cover of $X \times \{\bar{y}\}$ so $\exists$ a finite subcover, say $\{\mathcal{U}_{x_i}^{\bar{y}} \times V_{\bar{y}}^{x_i}\}_{i=1}^n$. Thus the corresponding $\{C_i\}_{i=1}^n$ form an open cover of $X \times \{\bar{y}\}$

Therefore $\bigcup \mathcal{U}_{x_i}^{\bar{y}} \supseteq X$ & $V = \bigcap_{i=1}^n V_{\bar{y}}^{x_i} \ni \bar{y}$ & $X \times V \subseteq \bigcup_{i=1}^n C_i$,

as desired. $\square$

4. Prove that, in any topological space, the intersection of two open, dense sets is open & dense. Prove that, in any complete metric space, the intersection of countably many open dense sets is nonempty.

Let X be a topological space, & let A & B be two open dense sets. Since A & B are open, $A \cap B$ is open. Let $x \in X$. Then any open set $\mathcal{U}$ of x satisfies that $A \cap \mathcal{U} \neq \emptyset$. Let $y \in A \cap \mathcal{U}$, which is an open set. Then since B is dense $B \cap (A \cap \mathcal{U}) \neq \emptyset$. Thus $(A \cap B) \cap \mathcal{U} \neq \emptyset \Rightarrow A \cap B$ is dense.

Now let (A_n) be a collection of open & dense sets. Then let $x \in X$, & $\mathcal{U}$ be an open set containing x. Since A_1 is dense $\Rightarrow$ $A_1 \cap \mathcal{U} \neq \emptyset$. This is an open set, call it $\mathcal{U}_1$ & A_2 is nonempty, thus $A_2 \cap \mathcal{U}_1 \neq \emptyset$. Now define $\mathcal{U}_2 = A_2 \cap \mathcal{U}_1$ which is open. Continuing this process inductively, assume $A_n \cap \mathcal{U}_{n-1} \neq \emptyset$. Thus since A_{n+1} is dense, $A_{n+1} \cap \mathcal{U}_n \neq \emptyset$. Thus $\forall n$, $(A_n \cap \cdots \cap A_1) \cap \mathcal{U} \neq \emptyset$.

Not sure how to make countable argument

Now consider $\bigcap_{n \in \mathbb{N}} A_n \cap \mathcal{U}$. If

5. Prove that $\mathbb{R}$ is connected. Prove that if a topological space X has a connected dense subset then X is connected.

First I prove the following lemma:

<u>Lemma</u> $[0,1] \subseteq \mathbb{R}$ is connected

Suppose $\exists$ a disconnection, i.e. $S \amalg T = [0,1]$ where S & T are open & nonempty. Without loss of generality, let $0 \in S$. Then consider
$\sup \{t \in [0,1] : [0,t) \subseteq S\}$
Since T is nonempty & $[0,1]$ is bounded, this supremum exists. If $t \in S$ then since S is open $\exists$ $(t-\varepsilon, t+\varepsilon) \subseteq S$ for some $\varepsilon > 0$, but then $t+\varepsilon > t$ & $[0, t+\varepsilon) \subseteq S$ which is a contradiction. Therefore $t \in T$. However, then $\exists$ $\delta > 0$ s.t. $(t-\delta, t+\delta) \subseteq T$, so $t - \frac{\delta}{2} \in [0,t) \subseteq S$ which contradicts T & S be disconnected.
Finally, note that this set $\{t \in [0,1] \cdot [0,t) \subseteq S\}$ is nonempty since $0 \in S$ & S is open, so $\exists$ $\rho > 0$ s.t. $[0, \rho) \subset S$.
Thus we've reached a contradiction, so $[0,1]$ is connected.

Now let's prove $\mathbb{R}$ is connected. Note that $\mathbb{R}$ is homeomorphic to $(0,1)$. Let $S \amalg T = (0,1)$ be a separation. Let $s \in S$ & $t \in T$ & WLOG let $s < t$. Then $S \cap [s,t] \amalg T \cap [s,t]$ is a separation of $[s,t]$. Note that with minor modification our lemma can show all closed intervals are connected. So this is a contradiction. So $(0,1)$ is not connected $\Rightarrow$ $\mathbb{R}$ is not either. $\square$

6. State & prove the contraction mapping theorem. Give an example of a complete metric space X & a function $f: X \to X$ s.t. $d(f(x), f(y)) < d(x,y)$ $\forall x, y \in X$ but f has no fixed point.

Definition Let (X,d) be a metric space. We call $f: X \to X$ if $\exists\, \delta \in [0,1)$ s.t. $\forall x, y \in X$,

$$\delta d(x,y) \geq d(f(x), f(y))$$

Let X be a complete metric space. Then any contraction has a fixed point & this point is unique.

Proof: Let $f: X \to X$ be the contraction. Pick $x_0 \in X$ & define a sequence $x_n = f^n(x_0)$ (where the exponent denotes composition). Note that x_n is a Cauchy sequence since if $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\sum_N^\infty \delta^i < \frac{\varepsilon}{d(f(x_0), x_0)}$ & so $\forall n, m \geq N$ (& assume $n > m$)

$$\begin{aligned} d(x_n, x_m) = d(f^n(x_0), f^m(x_0)) &\leq d(f^n(x_0), f^{n-1}(x_0)) + \dots + d(f^{m+1}(x_0), f^m(x_0)) \\ &< \delta^{n-1} d(f(x_0), x_0) + \dots + \delta^m d(f(x_0), x_0) \\ &= d(f(x_0), x_0) \sum_{i=m}^{n-1} \delta^i \\ &\leq d(f(x_0), x_0) \sum_{i=N}^{\infty} \delta^i \\ &< \varepsilon \end{aligned}$$

Since X is complete $\Rightarrow$ (x_n) converges, say to x^*. So $\forall \varepsilon > 0$ $\exists K$ s.t. $\forall k \geq K$ $d(x^*, f^k(x_0)) < \varepsilon/2$ & $\exists M \in \mathbb{N}$ s.t. $\forall m \geq M$ $d(f(x^*), f^m(x_0)) \leq \delta d(x^*, f^{m-1}(x_0)) < \frac{\varepsilon}{2}$

Thus $\forall k \geq \max\{K, M\}$,

$$\begin{aligned} d(x^*, f(x^*)) &\leq d(x^*, f^k(x_0)) + d(f^k(x_0) + f(x^*)) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

So $x^* = f(x^*)$ & f has a fixed point.

Further, this point is unique. Suppose x^* & x^{**} are both fixed points. Then

$$d(x^*, x^{**}) = d(f(x^*), f(x^{**})) \leq \delta\, d(x^*, x^{**})$$

& since $\delta \neq 1$ we have a contradiction. $\square$

Let $f: [1,\infty) \to [1,\infty)$ with $[0,\infty)$ given the subspace topology wrt $\mathbb{R}$ in the standard topology. Then define

$$f(x) = x + \frac{1}{x}$$

Note that for any $x, y \in [1,\infty)$

$$|f(x) - f(y)| = \left| x + \frac{1}{x} - \left(y + \frac{1}{y}\right)\right|$$

$$= \left| \frac{x^2y + y - xy^2 - x}{xy} \right| = \left| \frac{xy(x-y) - (x-y)}{xy} \right|$$

$$= \left| \frac{(xy-1)(x-y)}{xy} \right|$$

$$= \left| \frac{(xy-1)}{xy} \right| \cdot |x-y|$$

$$< |x-y|$$

However, there is no fixed point since if

$$x = x + \frac{1}{x} \Rightarrow 0 = \frac{1}{x}. \quad \square$$

7. Prove that any finite sheeted covering space of a compact metric space is compact.

Let $p: \tilde{X} \to X$ be a finite-sheeted covering space & let X be compact. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Delta}$ be an open cover of $\tilde{X}$. Since p is a covering map $\exists$ an open cover $V = \{V_\beta\}_{\beta \in \Delta}$ s.t. $p^{-1}(V_\beta)$ is a disjoint union of n open sets of $\tilde{X}$ each of which map homeomorphically onto V_β by p. Since X is compact, there exists a finite subcover $\{V_i\}_{i=1}^{n}$ of V. Thus $\{p^{-1}(V_i)\}_{i=1}^{n}$ is a finite cover of $\tilde{X}$.

Each $p^{-1}(V_i) = \coprod_{j=1}^{m} W_j$ & W_j is homeomorphic to V_i

Stuck

Not sure if X [illegible]

Attempt 2:

~~Let $p: \tilde{X} \to X$ be a finite-sheeted covering space & let X be compact. Let $V = \{V_\alpha\}_{\alpha \in \Delta}$ be the open cover s.t. $p^{-1}(V_\alpha)$ is a disjoint union of n open sets each homeomorphic to V_α by p. Find $W_\alpha \subseteq V_\alpha$ s.t. $\overline{W_\alpha} \subseteq V_\alpha$, $\overline{W_\alpha}$ is closed in X & thus compact. Now $\{W_\alpha$~~

8. Let M^6 be the 6-manifold $\mathbb{RP}^2 \times \mathbb{RP}^2 \times S^2$. Calculate $\pi_1(M^6)$. How many covers does M^6 have? Roughly describe each cover & the subgroup with which it corresponds.

$\mathbb{RP}^2$ & S^2 are path-connected, so

$$\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2 \times S^2) = \pi_1(\mathbb{RP}^2) \times \pi_1(\mathbb{RP}^2) \times \pi_1(S^2)$$

$$\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times 0$$

$$\cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Do I need to prove this?

Do I need to prove $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$?

Since $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2 \Rightarrow$ there are two connected coverings of $\mathbb{RP}^2$ (up to isomorphism) $i : \mathbb{RP}^2 \to \mathbb{RP}^2$ where i is the identity & $a : S^2 \to \mathbb{RP}^2$ where a is the antipodal map, which is a 2-sheeted cover.

Thus $\mathbb{RP}^2 \times \mathbb{RP}^2 \times S^2$ has 4 covers:

$i \times i \times i : \mathbb{RP}^2 \times \mathbb{RP}^2 \times S^2 \to \mathbb{RP}^2 \times \mathbb{RP}^2 \times S^2$

$i \times a \times i : \mathbb{RP}^2 \times S^2 \times S^2 \to \mathbb{RP}^2 \times \mathbb{RP}^2 \times S^2$

$a \times i \times i : S^2 \times \mathbb{RP}^2 \times S^2 \to \mathbb{RP}^2 \times \mathbb{RP}^2 \times S^2$

$a \times a \times i : S^2 \times S^2 \times S^2 \to \mathbb{RP}^2 \times \mathbb{RP}^2 \times S^2$

However $\mathbb{RP}^2 \times S^2 \times S^2 \cong S^2 \times \mathbb{RP}^2 \times S^2$ so there are 3 coverings up to isomorphism.

$a \times a \times i$ will correspond to $\{(0,0)\} \leq \mathbb{Z}_2 \times \mathbb{Z}_2$

$a \times i \times i$ will correspond to $\langle (1,0) \rangle \leq \mathbb{Z}_2 \times \mathbb{Z}_2$

$i \times a \times i$ will correspond to $\langle (0,1) \rangle \leq \mathbb{Z}_2 \times \mathbb{Z}_2$

$i \times i \times i$ will correspond to $\mathbb{Z}_2 \times \mathbb{Z}_2$

Not sure if this is enough/correct.