**Math 550**
**Homework 10**
Dr. Fuller
Due November 13

1. For a vector field $X = (f_x, f_y)$ on $\mathbf{R}^2$, we may define an associated 1-form, different from the one in class, by
$$\star\omega^1_X = -f_y\, dx + f_x\, dy.$$
We may also define
$$\text{div } X = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}.$$
   (a) Let $M$ be a compact 2-dimensional manifold with boundary in $\mathbf{R}^2$. Show that for all points $p \in \partial M$, the equation $\star\omega^1_X = X \cdot n\, ds$ holds.
   (b) Prove the following *Divergence form of Green's Theorem:* Let $M$ be a compact 2-dimensional manifold-with-boundary in $\mathbf{R}^2$, and let $X$ be a vector field on $M$. Then
$$\int_M \text{div } X\, dA = \int_{\partial M} X \cdot n\, ds.$$

2. Let $M$ be a compact 3-dimensional manifold-with-boundary in $\mathbf{R}^3$, with $(0,0,0) \in M - \partial M$. Consider the vector field $X(p) = \frac{p}{\|p\|^3}$ defined on $\mathbf{R}^3 - \{(0,0,0)\}$. Prove that
$$\int_{\partial M} X \cdot N\, dA = 4\pi.$$

3. (a) Show that if $X$ is a vector field on $\mathbf{R}^3$ with curl $X = 0$, then $X = \text{grad } f$ for some function $f : \mathbf{R}^3 \to \mathbf{R}$.
   (b) Show that if $X$ is a vector field on $\mathbf{R}^3$ with div $X = 0$, then $X = \text{curl } Y$ for some vector field $Y$ on $\mathbf{R}^3$ .

4. Let $\omega = \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$ be a 1-form on $\mathbf{R}^2 - \{(0,0)\}$. Prove that $\omega$ does not extend to a 1-form on $\mathbf{R}^2$.