Fall 2016 Topology Qual Solution Sketches

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Note–these solutions were typed very quickly, so please let me know if you spot any egregious mistakes.

Problem 9

This question is really about recognizing the "correct" definition of a universal cover and applying it to the specific case where $\mathbb{R}$ is the universal cover of S^1.

A universal covering space of a connected space X is a connected covering space $g : \tilde{X} \to X$ such that for any other connected covering space $p : Y \to X$, there is a covering map $q : \tilde{X} \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ \tilde{X} \arrow[ur, "q"] \arrow[r, "g"'] & X \end{tikzcd}$$

In particular, a simply connected covering space $\tilde{X}$ of a connected, locally path-connected space X is a universal covering space of X. This is what we will show below.

Suppose that $g : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is a covering map, where $\tilde{X}$ is simply connected and X is connected and locally path-connected. Let $p : (Y, y_0) \to (X, x_0)$ be another covering map, where Y is connected. Then we will show there exists a covering map q such that the diagram below commutes.

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "p"] \\ (\tilde{X}, \tilde{x_0}) \arrow[ur, dashed, "q"] \arrow[r, "g"'] & (X, x_0) \end{tikzcd}$$

First note that since X is locally path-connected, so is $\tilde{X}$. Since $\tilde{X}$ is path-connected and locally path-connected, by the lifting criterion, a lift q of g exists iff $g_*(\pi_1(\tilde{X}, \tilde{x_0})) \subseteq p_*(\pi_1(Y, y_0))$, which certainly holds, since $(\tilde{X}, \tilde{x_0})$ is simply connected. We will now show that this lift q is in fact a covering map.

q **is surjective**: Let $y_1 \in Y$ and let $f : I \to Y$ be a path in Y from y_0 to y_1. Because X is locally path-connected, so is Y, and since Y is also connected this implies Y is path-connected, so such a path exists. Then $p \circ f$ is a path in X starting at $p(y_0) = x_0$. Therefore, there exists a lift h of $p \circ f$ to a path in $\tilde{X}$ starting at $\tilde{x_0}$ such that $p \circ f = g \circ h$.

Because $p \circ q = g$, we have that

$$p \circ f = g \circ h = p \circ q \circ h,$$

i.e. the diagram below commutes.

$$\begin{tikzcd} & (Y, y_0) \arrow[d, "p"] \\ I \arrow[ur, bend left, "f"] \arrow[ur, dashed, "q \circ h"'] \arrow[r, "p \circ f"'] & (X, x_0) \end{tikzcd}$$

However, this means that f and $q \circ h$ are both lifts of $p \circ f$, starting at y_0, so by uniqueness of path-lifting, we have that $f = q \circ h$, so in particular $q(h(1)) = f(1) = y_1$, and so q is surjective.

q **covers evenly**: Let $y \in Y$. We will show that y has a a neighborhood evenly covered by q. Pushforward y to $p(Y) \in X$. Since p and g are covering maps, there is a neighborhood U_1 of $p(y)$ which is evenly covered by p and a neighborhood U_2 of $p(y)$ evenly covered by g. Then $U := U_1 \cap U_2$ is evenly covered by both g and p, and by shrinking U if necessary, we may assume that U is connected. If $g^{-1}(U) = \coprod_{\alpha \in I} W_\alpha$, and $p^{-1}(U) = \coprod_{\beta \in J} V_\beta$ then let V_i denote the slice which contains y. Note that q maps each slice W_α into $\coprod V_\beta$, and since we assumed that U was connected, each slice W_α must be mapped into a single slice V_β. Hence, $q^{-1}(V_i)$ is a disjoint union of all those slices W_α such that q maps W_α into V_i.

To see that q maps each such slice W_α homeomorphically onto V_i, we note that $p|_{V_i} : V_i \to U$ and $g|_{W_\alpha} : W_\alpha \to U$ are both homeomorphisms, and that $g|_{W_\alpha} = p|_{V_i} \circ q|_{W_\alpha}$, so $q|_{W_\alpha}$ must also be a homeomorphism.