Math 4A: Linear Algebra w/ Applications **Final Exam (Version 2)**
Summer 2020: Session A **Due Saturday, August 1, 12pm PDT**
Instructor: Stephen Pablo **Time Limit: 210 Minutes**

You will be assigned an exam version number in the Gauchospace gradebook which should match the version of the exam you are taking. If you have opened the incorrect version, *do not open the other version—continue taking this version!* Opening multiple versions of the exam will be considered academic dishonesty and actions may be taken at the instructors' discretion.

This exam contains 11 pages (including this cover page) and consists of 6 problems with multiple parts. Problems 1 through 5 are required and problem 6 is extra credit. Each problem starts on a new page and may continue onto the next page. The exam is worth 125 points, with possibility of up to 10 additional extra credit points for a maximum possible score of 135 points.

Before answering each question, take a moment to read (and re-read) each question to ensure you completely understand what the question is asking you to do. You are expected to justify your work completely and explain your arguments clearly. You may complete your solutions in one of the following ways:

- Save the exam .pdf to your iPad/tablet, and complete the solutions directly on the .pdf via a note-taking app.
- Print the .pdf to paper, and hand-write your solutions directly on the exam paper.
- Hand-write your solutions on your own paper. You are not required to write the problem statements, but you must organize your solutions in the order in which they appear on the exam, and clearly label where each problem (and corresponding sub-parts) begin and end.

For each problem, please write all sub-parts of the solution on the same page where the problem begins and continue onto a new blank page if you run out of room. *You must begin the solution to a new problem on a new blank page.*

Once completed, you may export your solutions to a .pdf via a note-taking app if you are using a tablet, or you may scan your handwritten solutions to a .pdf using a scanner/scanner app. I recommend the GeniusScan scanner app. Please orient your scans correctly and ensure that they are clear/legible! *Scans which are sideways/upside-down or unclear/illegible will result in a loss of 5 points from your exam score.*

Please upload your solutions as a .pdf to your Gradescope exam submission. After the upload has processed, you *must* assign the pages corresponding to each problem. *If you do not correctly assign pages you will lose 5 points from your exam score.*

You may use course notes and materials provided by your instructor, as well as any supplemental textbooks given in the syllabus to help you complete the exam. Use of a calculator is permitted, but any work verified via calculator must be justified in your solutions. Peer collaboration, the use of online resources (other than a calculator), and posting on Piazza is **strictly prohibited**.

You have 2 hours and 30 minutes to complete the exam plus an additional 1 hour to submit your work for a total of 210 minutes to complete and submit the exam from the time you begin.

Good luck!

1. (25 points) **Linear Systems**

For parts (a) through (d), consider the system of linear equations given by

$$\begin{aligned} x + y \quad &= -1 \\ 3x + y - 3z &= 5 \\ -2x - y + 2z &= -4 \end{aligned}$$

(a) (5 points) Express the system both as an augmented matrix and as a matrix equation of the form $A\vec{x} = \vec{b}$.

(b) (7 points) Apply elementary row operations to the augmented matrix until it is in *reduced* row-echelon form. (You are free to do multiple operations in one step, but please write each row-operation that you use for each step.)

(c) (8 points) Is the solution unique? Why or why not?

Express the solution to the system as a vector in $\mathbb{R}^3$.

(d) (5 points) Based on your results from parts (b) and (c), is it possible for the following system

$$\begin{aligned} x + y &= b_1 \\ 3x + y - 3z &= b_2 \\ -2x - y + 2z &= b_3 \end{aligned}$$

to be inconsistent? Why or why not?

2. (25 points) **Elementary Matrices**

Consider the following elementary row operations used to reduce A to I_2:

$$A = \begin{bmatrix} -2 & -4 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_1+2R_2 \to R_1} \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_2-3R_1 \to R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) (5 points) Briefly explain why A is invertible.

(b) (8 points) Express A^{-1} as a product of elementary matrices.

(c) (7 points) Express A as a product of elementary matrices.

(d) (5 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by $T(\vec{x}) = A\vec{x}$. Use part (c) to explain how T transforms the plane as a composition of contractions, expansions, shears, and/or reflections.

3. (17 points) **Subspaces & Linear Transformations**

We say that an $n \times n$ matrix is *skew-symmetric* if $A^T = -A$. Let W be the set of all 2×2 skew-symmetric matrices:

$$W = \left\{ A \in M_{2\times 2}(\mathbb{R}) \mid A^T = -A \right\}.$$

(a) (6 points) Show that W is a subspace of $M_{2\times 2}(\mathbb{R})$.

(b) (5 points) Find a basis for W and determine $\dim(W)$.

(c) (6 points) Suppose $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is a linear transformation given by

$$T(A) = A^T + A.$$

Is T injective? Is T surjective? Why or why not?

You do not need to verify that T is linear.

4. (30 points) **Eigenvalues, Eigenvectors, & Diagonalization**

   For parts (a) through (d), consider the matrix

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 3 & 2 \\ -2 & -4 & -3 \end{bmatrix}.$$

   (a) (7 points) Compute the characteristic polynomial $p(\lambda)$ of A and then give the corresponding characteristic equation of A.

   (b) (5 points) Use the characteristic equation of A to find the eigenvalues of A. State the algebraic multiplicity of each eigenvalue.

   Hint: You should have *at most* two distinct eigenvalues. If you have three distinct eigenvalues, then you have made an error computing $p(\lambda)$.

(c) (10 points) For each eigenvalue λ of A, find a basis for the corresponding eigenspace $E^{(\lambda)}$. Use the bases you obtain to find the geometric multiplicity of each eigenvalue.

(d) (8 points) Is A diagonalizable? If so, construct an invertible matrix P which diagonalizes A, then give the corresponding diagonalization of A. If not, explain why.

5. (28 points) **Quickies!**

The following questions can be answered with minimal computation and/or a short argument. For True or False questions, briefly explain why the statement is true or give a specific counter example/explanation which shows that the statement is false. *All solutions must provide a brief argument for full credit.*

(a) (4 points) **True or False?** If A is any $m \times n$ matrix, then the matrices $A^T A$ and AA^T are both symmetric.

(b) (4 points) **True or False?** The matrix $A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ -1 & 0 & 4 & 0 & 0 \\ 6 & -9 & 4 & 3 & 0 \\ 7 & 3 & -2 & 8 & -5 \end{bmatrix}$ is diagonalizable.

(c) (4 points) Suppose A, B and C are 3×3 matrices and $\det(A) = -1$, $\det(B) = 3$, $\det(C) = 6$. Find $\det(2A^3CB^{-1})$.

(d) (4 points) **True or False?** If A is a 3×3 matrix with characteristic polynomial $p(\lambda) = \lambda^3 + \lambda^2 - 3\lambda$, then A is invertible.

(e) (4 points) The set of ordered pairs of real numbers

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

is a vector space with vector addition $\oplus$ and scalar multiplication $\odot$ defined by

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \oplus \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} := \begin{bmatrix} x_1 + x_2 - 1 \\ y_1 + y_2 - 1 \end{bmatrix}, \qquad k \odot \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} k(x-1)+1 \\ k(y-1)+1 \end{bmatrix}.$$

Verify that there exists a vector $\vec{0} \in V$ such that $\vec{v} \oplus \vec{0} = \vec{0} \oplus \vec{v} = \vec{v}$, for any $\vec{v} \in V$.

(f) (4 points) Suppose A is a 2×2 matrix which is diagonalized by $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, with $P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ and diagonalization given by $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$.

Compute A^4.

(g) (4 points) Suppose $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ are bases for $\mathbb{R}^2$ and suppose the following augmented matrices are row-equivalent

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -1 & -4 \\ 0 & 1 & 2 & 7 \end{array}\right].$$

If $[\vec{v}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, then find $[\vec{v}]_C$.

No credit will be awarded if you use row-reduction to solve the problem.

6. (10 points) **(Extra Credit) Critical Thinking**

Choose **one** of the following problems to solve:

○ Suppose that A is an $n \times n$ matrix such that *every* nonzero vector in $\mathbb{R}^n$ is an eigenvector of A. Show that A is a scalar multiple of the identity matrix I_n.

(*Hint:* Think of any vector $\vec{v} \in \mathbb{R}^n$ as a linear combination of the standard basis vectors $\vec{e_1}, \vec{e_2}, \ldots, \vec{e_n}$.)

○ Suppose that A is an $n \times n$ matrix such that $A^2 = A$. Show that any eigenvalue of A is either 0 or 1.

Mark the problem you are solving in the checkbox above. If you write more than one solution, clearly highlight the solution that you want graded and omit the solution(s) that you don't want graded, otherwise you will only receive credit for the lowest scoring solution.

For full credit, you must write a complete solution with clear and rigorous exposition. Partial credit may be awarded for reasonable progress toward a complete solution.