Math 4B **Name:** ____________________
Summer Session A
Midterm
9 July 2020

Note: every problem has its own page, regardless of how long the solution would be.

1. (10 points each) Find the general solution of the given differential equation:

   (a) $w'' + 4w = 0$

characteristic equation: $r^2+4=0$.

$\therefore r^2 = -4 = 4i^2$

$r_1 = 2i \quad r_2 = -2i$

$\therefore \lambda = 0, \mu = 2$

$w = C_1 e^{0t}\cos(2t) + C_2 e^{0t}\sin(2t)$

$\therefore \boxed{w = C_1\cos(2t) + C_2\sin(2t)}$

(b) $y' = \dfrac{x(x^2+1)}{4y^3}$

$$\frac{dy}{dx} = \frac{x(x^2+1)}{4y^3}$$

$$4y^3\,dy = (x^3+x)\,dx$$

$$\int 4y^3\,dy = \int (x^3+x)\,dx$$

$$= \int x^3\,dx + \int x\,dx$$

$$y^4 = \frac{x^4}{4} + \frac{x^2}{2} + C$$

$$\boxed{y = \pm\left(\frac{x^4}{4} + \frac{x^2}{2} + C\right)^{\frac{1}{4}}}$$

2. (10 points each) Solve the given initial value problem:

(a) $u'' - 6u' + 9u = 0, \quad u(0) = 0, \quad u'(0) = 2$

Characteristic equation: $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r_1 = r_2 = 3$

$u = C_1 t e^{3t} + C_2 e^{3t}$

$u(0) = C_1 \cdot 0 \cdot e^0 + C_2 e^0 = C_2 = 0$

$\therefore u = C_1 t e^{3t}$

$u' = C_1 (e^{3t} + 3te^{3t})$

$u'(0) = C_1 (e^0 + 0) = C_1 = 2$

$\therefore C_1 = 2 \quad C_2 = 0$

$\boxed{u = 2te^{3t}}$

(b) $(1+2y)y' - 2x = 0, \quad y(2) = 0$

$-2x + (1+2y)y' = 0.$

Let $M(x,y) = -2x \quad N(x,y) = 1+2y$

$M_y(x,y) = 0 = N_x(x,y)$

$\therefore$ The differential equation is exact.

$\therefore f_x(x,y) = -2x$

$f_y(x,y) = 1+2y$

$f(x,y) = \int f_x(x,y) = \int -2x\, dx = -x^2 + h(y)$

$f_y(x,y) = h'(y) = 1+2y.$

$h(y) = \int h'(y) = \int (1+2y)\, dy = \int 1\, dy + \int 2y\, dy$

$= y + y^2$

$\therefore f(x,y) = -x^2 + y + y^2 = C$

$\because y(2) = 0.$

$\therefore -2^2 + 0 + 0 = C$

$\therefore C = -4$

$\therefore \boxed{-x^2 + y + y^2 = -4}$

(c) $v' + 2v = xe^{-2x}, \quad v(1) = 0$

Let $p(x) = 2 \quad g(x) = xe^{-2x}$

$\mu(x) = e^{\int 2dx} = e^{2x}$

$V = \frac{1}{e^{2x}} \int xe^{-2x} \cdot e^{2x} \, dx$

$= e^{-2x} \int x \, dx$

$= e^{-2x} \left(\frac{x^2}{2} + c \right)$

$V(1) = e^{-2} \left(\frac{1}{2} + c \right) = 0.$

$\because e^{-2} \neq 0$

$\therefore \frac{1}{2} + c = 0$

$\therefore c = -\frac{1}{2}$

$$\boxed{V = \frac{e^{-2x} \cdot (x^2 - 1)}{2}}$$

3. Given that $y_1(t) = t^2$ is a solution of

$$t^2y'' - 4ty' + 6y = 0, \quad t > 0,$$

(a) (15 points) Find a second solution $y_2(t)$ that is linearly independent from $y_1(t)$.

Let $y = v \cdot y_1 = v \cdot t^2$

$y' = v't^2 + 2vt$

$y'' = v''t^2 + 2v't + 2v't + 2v = v''t^2 + 4v't + 2v$

$t^2y'' - 4ty' + 6y = t^2(v''t^2 + 4v't + 2v) - 4t(v't^2 + 2vt) + 6vt^2$

$= v''t^4 + 4v't^3 + 2vt^2 - 4v't^3 - 8vt^2 + 6vt^2$

$= v''t^4 = 0$

$\because t > 0$

$\therefore v'' = 0.$

Let $w = v'$,

$w' = v'' = 0.$

$v' = w = \int w' = \int 0 \, dt = C$

$v = \int v' = \int C \, dt = Ct + D$

$y = vt^2 = (Ct + D)t^2 = Ct^3 + Dt^2$ is the general solution

Pick $C = 1$, $\boxed{y_2 = t^3 \text{ is a second solution.}}$

(b) (5 points) Prove that y_1 and y_2 form a fundamental set of solutions.

$y_1 = t^2$

$y_2 = t^3$

$$W[y_1, y_2] = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = t^2 \cdot 3t^2 - t^3 \cdot 2t = 3t^4 - 2t^4 = t^4$$

$\because t > 0$

$\therefore W[y_1, y_2] = t^4 \neq 0$

$\therefore$ y_1, y_2 are linearly independent.

$\therefore$ y_1 and y_2 form a fundamental set of solutions.

(c) (5 points) Write an expression for the general solution $y(t)$ to the differential equation.

According to (a),

the general solution $y(t)$ to the differential equation is $y(t) = Ct^3 + Dt^2$

(d) (BONUS - 10 points) Let's define a third solution to the differential equation to be $y_3(t) = y_2(t) + y_1(t)$ (where $y_1(t) = t^2$ and $y_2(t)$ is whatever you got in part (a)). Without calculating any specific Wronskians, do you think that the pair $\{y_3, y_1\}$ would form a fundamental set of solutions, yes or no? What about $\{y_3, y_2\}$ - would they form a fundamental set of solutions, yes or no? Explain your reasoning and thoughts. You may use linear algebra theory.

I think both $\{y_3, y_1\}$ and $\{y_3, y_2\}$ would form fundamental set of solutions.

$$W[y_3(t), y_1(t)] = \begin{vmatrix} y_2(t)+y_1(t) & y_1(t) \\ y_2'(t)+y_1'(t) & y_1'(t) \end{vmatrix}$$

$$= [y_2(t)+y_1(t)]\,y_1'(t) - [y_2'(t)+y_1'(t)]\,y_1(t)$$

$$= y_1'(t)y_2(t) + \cancel{y_1'(t)y_1(t)} - y_1(t)y_2'(t) - \cancel{y_1'(t)y_1(t)}$$

$$= y_1'(t)y_2(t) - y_1(t)y_2'(t)$$

$$= \begin{vmatrix} y_2(t) & y_1(t) \\ y_2'(t) & y_1'(t) \end{vmatrix} = W[y_2(t), y_1(t)]$$

$$W[y_3(t), y_2(t)] = \begin{vmatrix} y_2(t)+y_1(t) & y_2(t) \\ y_2'(t)+y_1'(t) & y_2'(t) \end{vmatrix} = [y_2(t)+y_1(t)]\,y_2'(t) - [y_2'(t)+y_1'(t)]\,y_2(t)$$

$$= \cancel{y_2(t)y_2'(t)} + y_1(t)y_2'(t) - \cancel{y_2'(t)y_2(t)} - y_1'(t)y_2(t)$$

$$= y_1(t)y_2'(t) - y_1'(t)y_2(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = W[y_1(t), y_2(t)]$$

According to (b), $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions, so $W[y_1(t), y_2(t)] \neq 0$, $W[y_2(t), y_1(t)] \neq 0$

$$\left.\begin{aligned} W[y_3(t), y_1(t)] &= W[y_2(t), y_1(t)] \neq 0 \\ W[y_3(t), y_2(t)] &= W[y_1(t), y_2(t)] \neq 0 \end{aligned}\right\} \Rightarrow$$

Both $\{y_3(t), y_1(t)\}$ and $\{y_3(t), y_2(t)\}$ form fundamental sets of solutions.

4. (5 points) So far, what topic or homework problem(s) from the class have you found the most challenging? Give some explanation why. (There is no right or wrong answer.)

I think reduction of order is the most challenging.
We need to compute y', y'' using the product rule,
and then substitute them into the differential equation.
I often make some mistakes in this process.
Also, sometimes the integral is hard to compute too.