Math 450B
Homework 3 Solutions
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1. (a) Continuous. Observe that $(x,y) \mapsto x$ and $(x,y) \mapsto y$ are linear transformations, hence continuous. The continuity of f then follows by applications of Proposition 9.

   (b) Not continuous at $(0,0)$. To see this, let $\varepsilon = \frac{1}{4}$, and observe for any $\delta > 0$, the point $(\frac{\delta}{2}, \frac{\delta}{2})$ satisfies $\|(\frac{\delta}{2}, \frac{\delta}{2}) - (0,0)\| < \delta$ but $f(\frac{\delta}{2}, \frac{\delta}{2}) = \frac{1}{2} > \frac{1}{4}$.

   (c) Continuous. At points other than $(0,0)$, this follows as in part (a). To see it is continuous at $(0,0)$, let $\varepsilon > 0$ and pick $\delta = \varepsilon$. Then for $\|(x,y)\| < \delta$, we have

$$\left|\frac{x^2 y}{x^2+y^2}\right| = \left|\frac{x^2}{x^2+y^2}\right| |y| \leq |y| \leq \|(x,y)\| < \delta = \varepsilon.$$

2. Let $\varepsilon > 0$ and pick $\delta = \varepsilon$. Then for $\|\mathbf{x} - \mathbf{a}\| < \delta$, we have

$$|\|\mathbf{x}\| - \|\mathbf{a}\|| \leq \|\mathbf{x} - \mathbf{a}\| < \delta = \varepsilon.$$

3. Let $\varepsilon > 0$ and pick $\delta = (\frac{\varepsilon}{K})^{1/\alpha}$. Then for $\|\mathbf{x} - \mathbf{a}\| < \delta$, we have

$$\|f(\mathbf{x}) - f(\mathbf{a})\| \leq K\|\mathbf{x} - \mathbf{y}\|^\alpha < K\delta^\alpha = \varepsilon.$$

4. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

5. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be $f(x,y) = (0,0)$. Then f is continuous, but for any open ball $B((a,b),r)$ we have $f(B((a,b),r)) = \{(0,0)\}$, which is not an open set.

6. Applying the definition of continuity for $\varepsilon = f(\mathbf{a})$, we get that there exists a δ such that $\mathbf{x} \in B(\mathbf{a},\delta) \cap A$ implies that $\|f(\mathbf{x}) - f(\mathbf{a})\| < f(\mathbf{a})$. Then we have

$$f(\mathbf{a}) - f(\mathbf{x}) \leq \|f(\mathbf{x}) - f(\mathbf{a})\| < f(\mathbf{a}),$$

   which implies $f(\mathbf{x}) > 0$.

7. Since A is not closed, there exists $\mathbf{x}_0 \notin A$ with $\mathbf{x}_0$ in the boundary of A. (Otherwise, every $\mathbf{x}_0 \notin A$ is in $\text{Ext}(A)$, which would imply that $\mathbf{R}^n - A$ is open, and therefore that A is closed.) We then have a well-defined function $f : A \to \mathbf{R}$ given by $f(\mathbf{x}) = \frac{1}{\|\mathbf{x}-\mathbf{x}_0\|}$. Proposition 9 and problem 2 above imply that f is continuous on A. Finally, we can see that f is unbounded by noting that for all $N > 0$, we can find $\mathbf{y} \in B(\mathbf{x}_0, \frac{1}{N}) \cap A$, and so $f(\mathbf{y}) = \frac{1}{\|\mathbf{y}-\mathbf{x}_0\|} > N$.