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Homework 7

1. Show that (i) the free product $G * H$ of nontrivial groups G and H has trivial center, and (ii) the only elements of $G * H$ of finite order are conjugates of finite-order elements of G and H.

Proof (i) Let $g \in G$, $h \in H$. If z is in the center of $G * H$, then $zg = gz$ which means that z is either g, g^{-1}, or the empty word $[e]$. Similarly, $zh = hz$ which means that z is either h, h^{-1}, or $[e]$. Thus if G, H are nontrivial then for nontrivial g, h we have $g \neq g^{-1} \neq h \neq h^{-1}$, so z must be $[e]$. ■

Proof (ii) Let $x \in G * H$ with $x^n = [e]$. By simplifying, we can write x as a word of alternating letters in G and H, so $x = (g_1 h_1 g_2 \ldots g_{k-1} h_k g_k)$. Then for as many letters as possible (potentially none), we group the letters on the ends of the word which are inverses; i.e.

$$\begin{aligned}
x &= (g_1 h_1)(g_2 \ldots g_{k-1})(h_k g_k) \\
x &= (g_1 h_1)(g_2 \ldots g_{k-1})(h_1^{-1} g_1^{-1}). \\
x &= (g_1 h_1)(z)(h_1^{-1} g_1^{-1}),
\end{aligned}$$

where we let z be the alternating word $(g_2 \ldots g_{k-1})$. Now we consider x^n. We know that everything must cancel, and the ends of course cancel, which means

$$\begin{aligned}
[e] &= x^n \\
&= (g_1 h_1)(z)(h_1^{-1} g_1^{-1})\ (g_1 h_1)(z)(h_1^{-1} g_1^{-1}) \cdots (g_1 h_1)(z)(h_1^{-1} g_1^{-1}) \\
&= (g_1 h_1)(z)^n (h_1^{-1} g_1^{-1})
\end{aligned}$$

which means that z must be of order n. However, no alternating word of g's and h's with length at least 2 is of finite order, since self-multiplying would just concatenate the same word with itself. Therefore z must be a single letter of order n in either G or H, and x is a conjugate of z. ■

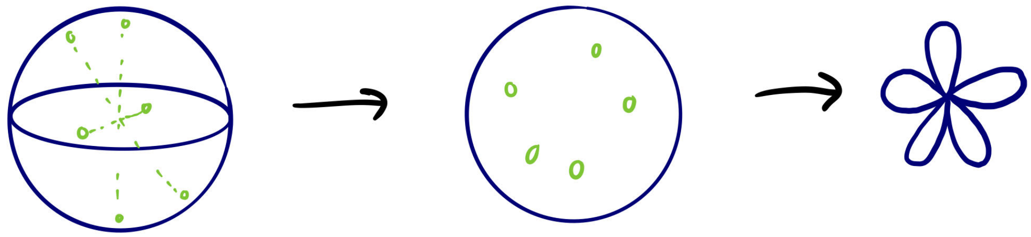
3. Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply connected if $n \geq 3$.

Proof Without loss of generality Let $X = \mathbb{R}^n - \{\vec{x}_i\}_{i=1}^k$, with basepoint the origin.

(i) To see that X is path-connected, let $x \in X$. If x is not a multiple of x_i for any i, then there is a straight-line path from x to the origin. Otherwise suppose $x \parallel x_i$ for some i. Then choose a point x' near x such that x is not a multiple of x_i for any i (we know we can do this since there are infinitely many directions orthogonal to x in which to find x', and only infinitely many x_i), and there is a straight-line path from x to x to 0. Thus for every $x \in X$ we have a path $\rho_x : I \to X$ connecting x to 0, so X is path-connected.

(ii) To see that $\pi_1(X) = 0$, observe that for any loop γ in X, we can find a homotopy from γ to the constant loop 0 by using ρ_x. Let $R : X \times I \to X$ be given by $R(x,t) = \rho_x(t)$. Then $R(\gamma(s), t)$ is the desired homotopy. ■

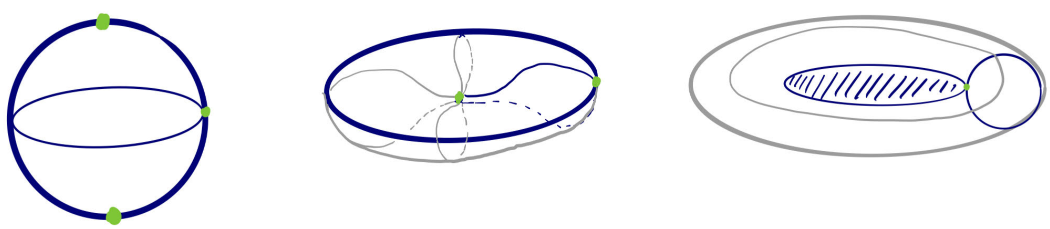
4. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$. We can deformation retract X to a ball with n lines missing, and since the origin is missing, we can deformation retract to a sphere with $2n$ antipodal points missing.



This sphere is diffeomorphic via spherical projection to a disk with $2n - 1$ points missing, which deformation retracts to a wedge of $2n - 1$ circles. Thus $\pi_1(\mathbb{R}^3 - X)$ is the free group on $2n - 1$ generators. ■

7. Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Put a cell complex structure on X and use it to compute $\pi_1(X)$.

Answer: First observe that X the following shapes have the same homotopy type:



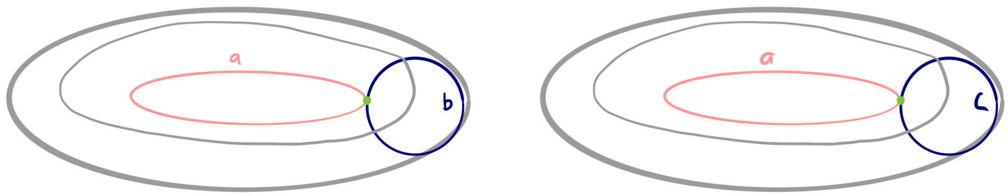
The last figure is the same because the center disc is contractible, and modding it out yields the middle figure. It is on the last figure that we put a cell structure. X has one 0-cell, two 1-cells and a 2-cell to form a torus, with one additional 2-cell attached spanning the center. We know that a torus has fundamental group $\mathbb{Z} \times \mathbb{Z}$, and if we attach the additional 2-cell along the loop $(1, 0)$, then

$$\pi_1(X) = \mathbb{Z} \times \mathbb{Z} \big/ \langle (1,0) \rangle = \mathbb{Z} \times \mathbb{Z} / \mathbb{Z} = \mathbb{Z}.$$

■

8. Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ with the corresponding circle in the other torus.

Answer: We can put a cell structure on this space as a wedge of 3 circles a, b, c, with 2-cells attached along $ab\overline{a}\overline{b}$ and $ac\overline{a}\overline{c}$, respectively.

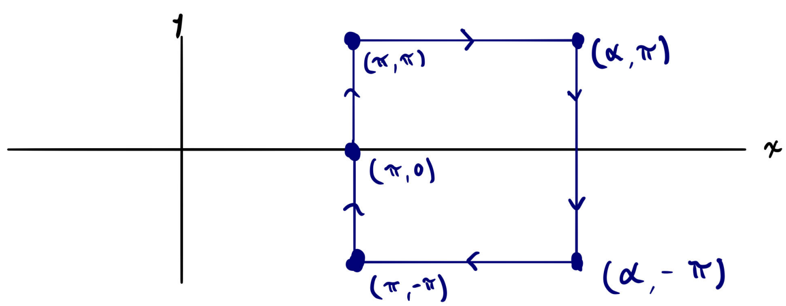


Thus $\pi_1(X) = \mathop{\ast}\limits_{i=1}^{3} \mathbb{Z}_i \Big/ \begin{matrix} aba^{-1}b^{-1} \\ aca^{-1}c^{-1} \end{matrix}$. ■

Remark: Is the above notation equivalent to $\pi_1(X) = \langle a, b, c \mid ab = ba, ac = ca\rangle$? I'm not totally clear on how that notation works.

17. Show that $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$ is uncountable.

Proof Let $(\pi, 0)$ be the basepoint. For every irrational number α, let γ_α be the straight line path from $(\pi, 0)$ to (π, π) to (α, π) to $(\alpha, -\pi)$ to $(\pi, -\pi)$ to $(\pi, 0)$.



Clearly $\gamma_\alpha \not\simeq \gamma_\beta$ for $\alpha \neq \beta$, since any homotopy between the corresponding third segments of the paths would have to pass through points in $\mathbb{Q}^2$ as the x-coordinates moved from α to β. Thus we have an injective function from irrational numbers to loops in $\pi_1(X)$, so we're done. ■

Collaborators:

None for this homework.