Math 462 - Advanced Linear Algebra
Homework 4

Trevor Klar

December 7, 2017

Exercises:

1. Let T be the $\mathbb{R}$-linear transformation given by

$$T : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (y, x, 0).$$

Compute all the eigenvalues of T and the corresponding eigenspaces.

Answer: (We use the canonical basis for $\mathbb{R}^3$.) Let $A = M(T)$.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} \\ &= -\lambda^3 + \lambda \\ &= \lambda(1 - \lambda^2) \\ &= \lambda(1 - \lambda)(1 + \lambda) \end{aligned}$$

So, the eigenvectors are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1$. A quick mental calculation will find that the 0-eigenspace is $\text{span}(0, 0, 1)$, the 1-eigenspace is $\text{span}(1, 1, 0)$, and the -1-eigenspace is $\text{span}(1, -1, 0)$.

2. Given any nonzero vector $v \in \mathbb{C}^2$, show that there exist some nonzero $\lambda \in C$ and some $\mathbb{C}$-linear transformation $T : \mathbb{C}^2 \to \mathbb{C}^2$ such that v is a λ-eigenvector of T.

Proof Let λ_0 be any complex number. Consider the dilation[1] $T(v) := \lambda_0 v$ for all $v \in \mathbb{C}^2$. By definition, every element of $\mathbb{C}^2$ is a λ_0-eigenvector of T. In case there is any doubt that T is a linear transformation, observe the matrix of T with respect to the following basis of $\mathbb{C}^2$:

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} i \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ i \end{pmatrix} \right\}$$

$$M_{\mathscr{B}}(T) = \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_0 & 0 & 0 \\ 0 & 0 & \lambda_0 & 0 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix}.$$

■

[1] We are using the term dilation loosely here to mean 'uniformly multiplying every component by a scalar'. In $\mathbb{C}^2$, this wouldn't quite appear as a dilation, as multiplication of complex numbers appears geometrically as a dilation composed with a rotation, and two complex dimensions would complicate matters even further.