Math 4A: Test 2

University of California - Santa Barbara

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Instructions:

1. Now that you have opened the exam you have exactly **three (3) hours** to complete the examination and upload your solutions to Gradescope. **Every minute you are late on your exam submission will result in a reduction of 1 point from your exam score.**
2. Before answering each question, take a moment to read each question to ensure you completely understand what the question is asking you to do.
3. You are expected to justify your work completely and explain your arguments clearly. We will not just be grading on correct answers, but if steps were shown in order to get to those answers.
4. You may complete your solutions in the following ways:
   (a) Save the exam .pdf to your iPad/tablet and complete the solutions directly on the .pdf via a note-taking app
   (b) Print the .pdf to paper, and hand-write your solutions direcly on the exam paper
   (c) Hand-write your solutions on your own paper. You are not required to write the problem statements, but you must organize your solutions in the order in which they appear on the exam and clearly label where each problem (and corresponding sub-parts) begin and end.
5. For each problem, please write all sub-parts of the solution on the same page where the problem begins and continue onto a new blank page if you run out of room. *You must begin the solution to a new problem on a new blank page*
6. Once completed, you may export your solutions to a .pdf via a note-taking app or sejda.com (or a similar site or app). Be aware sejda only allows 3 tasks per hour, so you will want to make all your edits at once using the "Organize" option. Please orient your scans correctly and ensure that they are clear/legible. **Scans that are sideways/upside-down or unclear/illegible will result in a reduction of points from your exam score.**
7. Please upload your solutions to your Gradescope exam submission. After the upload has processed, you **must** assign the pages corresponding to each problem. **If you do not correctly assign pages, it will result in a reduction of points from your exam score**. Make sure to include the page with the Honor Code.
8. You may use course notes, course videos, and either textbook used for the course to help you complete the exam. Piazza will not be open for new questions, but you will be able to read what has already been written on the Piazza forum during the exam. Use of a calculator is permitted, but any work verified by a calculator must be justified in your solutions.
9. **Peer collaboration, the use of unapproved online resources (such as Chegg and Google), use of CLAS or Math Lab, and posting on the course forum is strictly forbidden. Anyone found breaking these rules will be reported to the university and receive a 0 on the exam.**
10. If at any point you have a question on the exam, do email myself and/or your TAs. We will be attentive to our emails during reasonable hours, but do understand that there is a possibility that your question may not get answered by us before your time limit is up.

Honor Code:

I have read all the above instructions. I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ____

Question 1: UCSB Test

(a) Let $A = \begin{bmatrix} 2 & -6 & 0 & 10 & -14 \\ 0 & 0 & 4 & -6 & 16 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Find rank($A$) and dim Nul($A$).

(b) If a 3×8 matrix B has rank 3, find the dimension of the nullspace of B, the dimension of the rowspace of B, and rank B^T.

Question 2: UCSB Test

Let $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be bases for $\mathbb{R}^2$. Find $\mathcal{P}_{\mathcal{C} \leftarrow \mathcal{B}}$. For $\vec{c}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\vec{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in $\mathcal{C}$ compute $[\vec{c}_1]_{\mathcal{B}}$ and $[\vec{c}_2]_{\mathcal{B}}$.

Question 3: UCSB Test

(a) Are the two vectors $\vec{u} = \begin{bmatrix} -8 \\ -2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ orthogonal? Show work.

(b) Let $\vec{u} = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$. Compute $||\vec{u}||$. Show work.

(c) Let $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$. Compute the distance between $\vec{u}$ and $\vec{v}$. Show work.

(d) Compute the orthogonal projection of $\vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ onto $\vec{v} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Show work!

(e) Write $\vec{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ as a sum of a vector in $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} \right\}$ and a vector in W^T. Show work!

Question 4: UCSB Test

Let $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

(a) Compute the eigenvalues of A. Show all work!

(b) Compute the bases for each of the eigenspaces from part (a). Show work!

(c) Find P and D such that $A = PDP^{-1}$. Show all work!

(d) Let $A = PDP^T$ where $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$. Find a formula for A^k. Show work!

Question 5: UCSB Test

Let $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_2\}$ be the standard basis for $\mathbb{R}^3$, $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be a basis for a vector space B and $T : \mathbb{R}^3 \to V$ be a linear transformation with the property that

$$T(x_1, x_2, x_3) = (x_3 - x_2)\vec{b}_1 - (x_1 + x_3)\vec{b}_2 + (x_1 - x_2)\vec{b}_3$$

(a) Compute $T(\vec{e}_1)$, $T(\vec{e}_2)$, and $T(\vec{e}_3)$

(b) Compute $[T(\vec{e}_1)]_{\mathcal{B}}$, $[T(\vec{e}_2)]_{\mathcal{B}}$, $[T(\vec{e}_3)]_{\mathcal{B}}$

(c) Find the matrix for T relative to $\mathcal{E}$ and $\mathcal{B}$

Question 6: UCSB Test

Use the Gram-Schmidt process to produce an orthonormal basis for $W = \text{span}\left\{ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} \right\}$

Question 7: UCSB Test

Let $\mathbb{P}_2$ be the inner product with inner product $< p, q >= p(0)q(0) + p(1)q(1) + p(\frac{1}{3})q(\frac{1}{3})$. Compute the following:

(a) $< p, q >$ where $p(t) = 4 + t$, $q(t) = 5 - 9t^2$

(b) $||q||$ where $q(t) = 5 - 9t^2$

Question 8: UCSB Test

On this page, I just need the answer or definition. Little to no work will need to be shown.

(a) State The Rank Theorem

(b) True or False: A matrix A is not invertible if and only if 0 is an eigenvalue of A

(c) True or False: The columns of $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ is linearly independent

(d) True or False: For an $n \times n$ matrix A, $\det(A^T) = (-1)\det(A)$

(e) True or False: An $n \times n$ matrix A is diagonalizable if and only if A has n eigenvalues, counting multiplicities

(f) True or False: The eigenspaces of a symmetric matrix are mutually orthogonal

(g) True or False: A matrix with orthonormal columns is an orthogonal matrix

(h) True or False: If $B = PDP^T$ where $P^T = P^{-1}$ and D is a diagonal matrix then B is a symmetric matrix.

(i) What does the Gram-Schmidt process do?