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# Final Exam

**1.** Let $X$ be a nonempty topological space and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of Borel regular measures on $X$. Assume for any $A \subset X$ the sequence $\mu_n(A)$ decreases and define $\mu(A) = \lim_{n\to\infty} \mu_n(A)$. Prove that if $\mu_1(X) < \infty$, then $\mu$ is a measure on $X$.

**Lemma (MCT$\searrow$)** The Monotone Convergence Theorem holds for nonnegative $\mu$-measurable functions $f_n \searrow f$[†], if $f_1$ is $\mu$-summable. **Proof** $\{(f_1 - f_n)\}_{n=1}^{\infty}$ is a nonnegative sequence of functions with $(f_1 - f_n) \nearrow (f_1 - f)$, so by the ordinary MCT

$$\lim_{n\to\infty} \int (f_1 - f_n) = \int (f_1 - f)$$

and so

$$\lim_{n\to\infty} \int f_1 - \lim_{n\to\infty} \int f_n = \int f_1 - \int f$$

Thus $\lim_{n\to\infty} \int f_n = \int f$ and MCT$\searrow$ is proved. □

**Proof** We are given that

- each $\mu_n$ is Borel regular,
- $\mu_1(X) < \infty$, and
- $\mu_n(A) \searrow \mu(A)$ for any $A \subset X$.

First, observe that $\mu_n(\emptyset) = 0$ for all $n$, so $\mu(\emptyset) = 0$. Now let $A \subset \bigcup_{i=1}^{\infty} A_i$, with $A, A_i \in X$ for all $i \in \mathbb{N}$. We need to show that

$$\mu(A) \le \sum_{i=1}^{\infty} \mu(A_i).$$

Since each $\mu_n$ is a measure,

$$\mu_n(A) \le \sum_{i=1}^{\infty} \mu_n(A_i) \tag{1}$$

for all $n$. Now since for any $A$, $\mu_n(A)$ is a decreasing real sequence bounded below by 0, then it always converges, so taking limits in both sides of (1),

$$\mu(A) \le \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_n(A_i). \tag{2}$$

Now we will view this sum as an integral. Let $f_n(x) = \begin{cases} \mu_n(A_i) \text{ where } i = \lfloor x \rfloor, & \text{if } x \ge 1 \\ 0 & \text{if } x < 1 \end{cases}$.

Then each $f_n$ is simple and nonnegative, and the Lebesgue integral of $f_n$ is

$$\int_{\mathbb{R}} f_n(x) = \sum_{i=1}^{\infty} f_n(i) = \sum_{i=1}^{\infty} \mu_n(A_i),$$

[†] $f_n \searrow f$ means that $f_n \ge f_{n+1}$ for all $n$ and $\lim_{n\to\infty} f_n = f$.

and we can substitute into (2) to find

$$\mu(A) \leq \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x). \tag{3}$$

Observe that $\mu_1(A) \leq \mu_1(X) < \infty$ and $\mu_n \searrow \mu$, so $\mu(A) < \infty$ always. Considering the $A_i$ sets, for any $n \in \mathbb{N}$ either $\sum_{i=1}^{\infty} \mu_n(A_i)$ is finite or it is infinite.

Case I: Suppose there exists some $K$ such that $\sum_{i=1}^{\infty} \mu_K(A_i)$ is finite.

Following are a few facts about the functions $f_n(x) = \mu_n(A_i)$:

(i) Since $\mu_n \searrow \mu$, then $\sum_{i=1}^{\infty} \mu_k(A_i) < \infty$ for all $k > K$.
(ii) Each $f_n$ is a nonnegative simple function, and thus measurable.
(iii) $f_k$ is $\mu$-summable for every $k > K$, since $\int_{\mathbb{R}} f_k(x) = \sum_{i=1}^{\infty} \mu_k(A_i) < \infty$.
(iv) $f_n \searrow f$, since $\mu_n \searrow \mu$.
(v) $f_1$ is bounded above by $\mu_1(X)$, since every $A_i \subset X$ and $\mu_1$ has the monotonicity property.
(vi) $f$ is measurable by (i) and (iii) above.
(vii) We can assume $f$ is $\mu$-summable, since if not then $\sum_{i=1}^{\infty} \mu(A_i) = \int_{\mathbb{R}} f = \infty > \mu(A)$ and we're done.

Let $g_n = f_{n+K}$. Now we can check that the hypotheses of MCT$\searrow$ are satisfied:

- $g_n$ are $\mu$-measurable
- $g_1$ is $\mu$-summable.
- $g_n \searrow f$

So we can apply MCT$\searrow$ and conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \lim_{n\to\infty} \int_{\mathbb{R}} g_n = \int_{\mathbb{R}} f$$

so substituting into equation (3), we find that

$$\mu(A) \leq \int_{\mathbb{R}} f(x) = \sum_{i=1}^{\infty} \mu(A_i)$$

and we are done.

Case II: Suppose $\sum_{i=1}^{\infty} \mu_n(A_i)$ is infinite for every $n$.

Since each $\mu_n$ is a Borel regular measure and $\mu_1$ is in particular, for each $A_i$ there exists a respective Borel set $B_i$ such that $B_i \subset A_i$[†] and $\mu_1(A_i) = \mu_1(B_i)$, so

$$\mu_1(A) \leq \sum_{i=1}^{\infty} \mu_1(B_i) = \sum_{i=1}^{\infty} \mu_1(A_i)$$

[†]The textbook gives the set containment the other way, but if we find Borel set $\widetilde{B}_i$ with $A_i^{\complement} \subset \widetilde{B}_i$, then $\widetilde{B}_i^{\complement}$ is our desired $B_i$.

Let $D_1 = B_1$, and $D_n = B_n \setminus \bigcup_{i=1}^{n-1} B_i$. Then $\{D_i\}_{i=1}^{\infty}$ is a disjoint collection of Borel (and thus measurable) sets with $\bigcup_{i=1}^{\infty} D_i = \bigcup_{i=1}^{\infty} B_i$. Observe that

$$\sum_{i=1}^{\infty} \mu_1(D_i) = \mu_1 \left( \bigcup_{i=1}^{\infty} D_i \right) < \infty,$$

and for any $n$, since $\mu_n$ is Borel,

$$\sum_{i=1}^{\infty} \mu_n(D_i) = \sum_{i=1}^{\infty} \left( \mu_n(B_i) - \mu_n \left( \bigcap_{j=1}^{i} B_j \right) \right) \leq \sum_{i=1}^{\infty} \mu_n(B_i),$$

so we can apply Case I to $A \subset \bigcup_{i=1}^{\infty} D_i$ to conclude that $\mu(A) \leq \sum_{i=1}^{\infty} \mu(D_i)$, and since $\mu$ has the monotonicity property[†] and $D_i \subset B_i \subset A_i$,

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(D_i) \leq \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

and we are done. ■

[†] This is because $\mu_n$ is a measure, so $\mu_n(A) \leq \mu_n(B)$ for all $n$, and taking limits, $\mu(A) \leq \mu(B)$.

**2.** Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue-measurable. Prove that there exists a Borel-measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ $\mu$-*a.e.* in $\mathbb{R}$.

**Proof** We will show that (i) every Lebesgue-measurable simple function has the desired property, and show that (ii) this implies nonnegative Lebesgue-measurable functions have the property, and thus (iii) all Lebesgue-measurable functions have the property.

(i) Let $\sigma = \sum_{i=1}^{\infty} a_i \chi_{A_i}$ be a nonnegative Lebesgue-measurable simple function with all $A_i$ sets pairwise disjoint and of finite measure†. We know that for every Lebesgue-measurable set $L$ with finite measure, there exists a compact (and thus Borel) set $K$ such that $K \subset L$ and $\mu(L \setminus K) < \varepsilon$ for every $\varepsilon$. So for each $A_i$, we find a collection of compact sets $\{K_i^n\}_{n=1}^{\infty}$ such that $K_i^n \subset A_i$ and $\mu(A_i \setminus K_i^n) < \frac{1}{k}$. Then call $K_i = \bigcup_{n=1}^{\infty} K_i^n$, and $K_i \subset A_i$, $K_i$ is Borel, and $\mu(K_i) = \mu(A_i)$.
Thus we can define $\beta = \sum_{i=1}^{\infty} a_i \chi_{K_i}$, and note that $\beta = \sigma$ $\mu$-*a.e.*, and if $\beta(x) \neq \sigma(x)$, then $\beta(x) = 0$‡.

(ii) Next, let $f$ be any nonnegative Lebesgue-measurable function, and let $\sigma_n$ be a sequence of nonnegative Lebesgue-measurable simple functions with $\sigma_n \nearrow f$. By (i), produce Borel measurable functions $\beta_n$ with $\beta_n = \sigma_n$ $\mu$-*a.e.*. Since $\sigma_n \to f$ and $\beta_n = 0$ whenever $\beta_n \neq \sigma_n$, then $\beta_n$ converges to a function we can call $g = \lim_{n\to\infty} \beta_n$. To see that $g$ is Borel measurable, we show that $\liminf \beta_n$ and $\limsup \beta_n$ are Borel measurable.

$$\begin{aligned}
(\limsup_{n\to\infty} \beta_n)^{-1}(-\infty, b) &= \{x \in \mathbb{R} : \limsup_{n\to\infty} \beta_n(x) < b\} \\
&= \{x \in \mathbb{R} : \forall k > 0, \exists n > k \text{ s.t. } \beta_n(x) < b\} \\
&= \bigcap_{k=1}^{\infty} \bigcup_{n>k} \beta_n^{-1}(\infty, b)
\end{aligned}$$

which is Borel. A similar argument shows that $\liminf \beta_n$ is Borel measurable, so $g = \liminf \beta_n = \limsup \beta_n$ is as well.

(iii) Finally, we observe that if $f$ is any Lebesgue-measurable function, it can be written as $f = f^+ - f^-$ where

$$f^+(x) = \begin{cases} f(x), & f(x) \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad f^-(x) = \begin{cases} f(x), & -f(x) \leq 0 \\ 0, & \text{otherwise} \end{cases}$$

and we can use (ii) to produce Borel measurable functions $g^+$ and $g^-$ such that $g^+ = f^+$ $\mu$-*a.e.* and $g^- = f^-$ $\mu$-*a.e.*, so letting $g = g^+ - g^-$, we find that $g = f$ $\mu$-*a.e.*, and all that remains is to show that $g$ is Borel measurable:

$$g^{-1}(-\infty, b) = \{x \in \mathbb{R} : g^+(x) - g^-(x) < b\} = \begin{cases} (g^-)^{-1}(b, \infty), & \text{if } b \leq 0 \\ (g^+)^{-1}(0, b), & \text{if } b > 0 \end{cases}$$

which is a Borel set in either case. ■

†Such a disjoint collection of sets partitions $\mathbb{R}$, and if there are any with infinite measure, we can refine the partition by dividing the sets at every integer, i.e. if $A_i = (10, \infty)$, replace $A_i$ with $A_{i_1} = (10, 11], A_{i_2} = (12, 12]$, etc.

‡In case you were concerned, $\beta^{-1}(\{0\})$ is Borel even if no $a_i = 0$, since it is $\left(\bigcup_{i=1}^{\infty} K_i\right)^{\complement}$ in that case.

**3.** Let $X$ be nonempty and let $\mu$ be a measure on $X$. Assume $A_n \subset X$ are $\mu$-measurable for $n = 1, 2, \ldots$ and assume the sequence $\chi_{A_n}$ converges in measure to some function $f : X \to \mathbb{R}$. Prove that there exists a $\mu$-measurable set $A \subset X$ such that $f = \chi_A$ $\mu$-*a.e.* in $X$.

**Proof** Since $\chi_{A_n} \xrightarrow{\mu} f$, then there exists a subsequence $\chi_{A_{n_k}} \to f$ $\mu$-*a.e.*. Thus we can let

$$A = \{x \in X : \lim_{k\to\infty} \chi_{A_{n_k}}(x) = 1\}$$

That is, $A^{\complement}$ contains all $x \in X$ where $\lim\limits_{k\to\infty} \chi_{A_{n_k}}(x) = 0$ or DNE. Now observe that

$$\chi_A = f \ \mu\text{-}a.e.,$$

Since $\chi_A = \lim\limits_{k\to\infty} \chi_{A_{n_k}}$ except when the limit DNE, and the limit certainly does not agree with $f$ when it DNE, so

$$\mu(\{x \in X : \lim_{k\to\infty} \chi_{A_{n_k}}(x) \text{ DNE}\}) = 0.$$

Thus $\chi_A = \lim\limits_{k\to\infty} \chi_{A_{n_k}}$ $\mu$-*a.e.* and $\lim\limits_{k\to\infty} \chi_{A_{n_k}} = f$ $\mu$-*a.e.*, so $\chi_A = f$ $\mu$-*a.e.*

To see that $A$ is measurable, observe that $\chi_A = \lim\limits_{k\to\infty} \chi_{A_{n_k}}$ $\mu$-*a.e.* and each $\chi_{A_{n_k}}$ is a measurable function, so their limit is measurable. Thus

$$\{x \in X : \tfrac{1}{2} < \chi_A(x) < \tfrac{3}{2}\} = A$$

is measurable, and we're done. ∎

**4.** Let $X$ be nonempty and let $\mu$ be a measure on $X$. Assume $f_n, f : X \to \mathbb{R}$ are $\mu$-measurable functions such that for each $\varepsilon > 0$ one has

$$\sum_{n=1}^{\infty} \mu\big(\{x : |f_n(x) - f(x)| > \varepsilon\}\big) < \infty.$$

Prove that $f_n \to f$ $\mu$-*a.e.* in $X$.

**Proof** Let $\varepsilon > 0$. Since the sum is finite, then the tail of the sum goes to zero, so the terms go to zero. That is, since

$$\sum_{n=1}^{\infty} \mu\big(\{x : |f_n(x) - f(x)| > \varepsilon\}\big) < \infty, \quad \text{then}$$
$$\lim_{k\to\infty} \sum_{n=k}^{\infty} \mu\big(\{x : |f_n(x) - f(x)| > \varepsilon\}\big) = 0, \quad \text{so}$$
$$\lim_{n\to\infty} \mu\big(\{x : |f_n(x) - f(x)| > \varepsilon\}\big) = 0, \quad \text{so}$$
$$f_n \xrightarrow{\mu} f.$$

Since $f_n \xrightarrow{\mu} f$, then there exists a subsequence $f_{n_k} \to f$ $\mu$-*a.e.*

To see that the more general case of $f_n \to f$ $\mu$-*a.e.* holds, suppose not. Denote

$$A_\delta = \{x \in X : \lim_{k\to\infty} f_{n_k}(x) \neq f(x)\}.$$

We know that

$$\sum_{n=1}^{\infty} \mu(B_n^\varepsilon) < \infty, \text{ where}$$
$$B_n^\varepsilon = \{x : |f_n(x) - f(x)| > \varepsilon\}$$

For any $x \in X$, if $\lim\limits_{n\to\infty} |f_n(x) - f(x)|$ exists and nonzero, then $x \in A_\delta$. So we can observe the following about "the bad set" of $f_n$:

$$\mu\,(A) > 0, \text{ where}$$
$$A = \Big\{x \in X : \lim_{n\to\infty} |f_n(x) - f(x)| \text{ DNE}\Big\}.$$

Let $x \in A \setminus A_\delta$, so $f_{n_k}(x) \to f(x)$, but $f_n(x) \not\to f(x)$. Then there exists a subsequence $f_{n_j}(x)$ such that $f_{n_j}(x) \to L \neq f(x)$, where $L \in [-\infty, \infty]$. Then

$$\lim_{j\to\infty} |f_{n_j}(x) - f(x)| = |L - f(x)|$$

so for small $\varepsilon$, there exists $J \in \mathbb{N}$ such that $|f_{n_j}(x) - f(x)| > \varepsilon$ for every $j > J$. This means $x$ is in infinitely many $B_n^\varepsilon$, so $(A \setminus A_\delta) \subset \limsup_{n\to\infty} B_n^\varepsilon$ and by the Borel-Cantelli Lemma, they both have measure zero. This contradicts that $A$ has positive measure, since $A \subset (A \setminus A_\delta) \cup A_\delta$. ∎