Math 550
Homework 3
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Due September 18

1. Suppose that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Prove that

$$(dx_{i_1} \wedge \cdots \wedge dx_{i_k})_p(e_{j_1}, \ldots, e_{j_k})_p = \begin{cases} 1 & \text{if } i_1 = j_1, i_2 = j_2, \ldots, i_k = j_k \\ 0 & \text{otherwise} \end{cases}$$

2. Let $u = (1,2,3), v = (-4,-5,-6), w = (0,0,-2)$.
   (a) Let $\omega \in \Omega^1(\mathbf{R}^3)$ be $\omega(x,y,z) = (y+z)\,dx$. Calculate $\omega(u)(v)_u$ and $\omega(v)(u)_v$.
   (b) Let $\omega \in \Omega^2(\mathbf{R}^3)$ be $\omega(x,y,z) = z\,dx \wedge dy + e^x\,dy \wedge dz$. Compute $\omega(w)(u,v)_w$.

3. Let $V(x,y,z) = 2y(e_1) - z(e_3)$ and $W(x,y,z) = z(e_1) - (e_2) + xy(e_3)$ be vector fields on $\mathbf{R}^3$. Let $\nu(x,y,z) = (y+z)\,dx$ and $\omega(x,y,z) = x^2y\,dx \wedge dy - xz\,dy \wedge dz$ be forms on $\mathbf{R}^3$.
   (a) Evaluate $\nu(1,2,3)(V(1,2,3))$
   (b) Evaluate $\omega(1,2,3)(V(1,2,3), W(1,2,3))$
   (c) The evaluations $\nu(V)$ and $\omega(V,W)$ each describe a function $\mathbf{R}^3 \to \mathbf{R}$. Find those functions.

4. Simplify the following differential forms.
   (a) $(a_1\,dx + a_2\,dy) \wedge (b_1\,dx + b_2\,dy)$ $\quad$ (a_1, a_2, b_1, b_2 are constants.)
   (b) $(x\,dx - y\,dy) \wedge (z\,dx \wedge dy + x^2\,dy \wedge dz)$
   (c) $(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$

5. (a) Let $\omega \in \Omega^k(\mathbf{R}^n)$, with k odd and $2k \leq n$. Show that $\omega \wedge \omega = 0$.
   (b) Show by example that the conclusion in part (a) is false if k is even.

6. (a) For all $p \in \mathbf{R}^n$, we can define a function $\mathbf{R}^n_p \to (\mathbf{R}^n_p)^*$ that sends any $v \in \mathbf{R}^n_p$ to the linear functional $T(w) = \langle v, w \rangle$. (Here "$\langle , \rangle$" denotes the usual inner product on $\mathbf{R}^n$.) Prove that this is an isomorphism between the vector spaces $\mathbf{R}^n_p$ and $(\mathbf{R}^n_p)^*$.
   (b) Let $X(p) = f_1(p)(e_1)_p + \ldots + f_n(p)(e_n)_p$ be a vector field on $\mathbf{R}^n$. By applying the isomorphism in part (a) at each point p, we get a 1-form ω_X on $\mathbf{R}^n$. Give a formula for ω_X.