Math 550
Homework 9
Dr. Fuller
Solutions

2. If B^3 denotes the unit ball in $\mathbf{R}^3$, then by Stokes' Theorem $\text{vol}(B^3) = \int_{B^3} dx \wedge dy \wedge dz = \int_{S^2} z\, dx \wedge dy$, where S^2 receives the orientation induced as the boundary of B^3. Parameterizing S^2 with spherical coordinates induces the opposite orientation, so we have

$$\int_{S^2} z\, dx \wedge dy = -\int_{(0,2\pi)\times(0,\pi)} g^*(z\, dx \wedge dy) = -\int_0^{\pi} \int_0^{2\pi} -\cos^2\varphi \sin\varphi \, d\theta \, d\varphi = \frac{4\pi}{3}.$$

3. The vector $c(t) = (c_1'(t), \ldots, c_n'(t))$ forms a positively oriented basis of $C_{c(t)}$, thus $\frac{1}{\|c'(t)\|}\,(c_1'(t), \ldots, c_n'(t))$ forms a positively oriented orthonormal basis there. So $1 = ds_{c(t)}(\frac{1}{\|c'(t)\|}\,(c_1'(t), \ldots, c_n'(t)))$, which implies that $ds_{c(t)}((c_1'(t), \ldots, c_n'(t))) = \|c'(t)\|$. So

$$(c^*ds)_t(1) = ds_{c(t)}(Dc_{c(t)}(1)) = ds_{c(t)}(c_1'(t), c_2'(t), \ldots, c_n'(t)) = \|c'(t)\| = \|c'(t)\| dt(1).$$

This shows that $c^*ds = \|c'(t)\|dt$, and the integral formula follows.

4. (a) Straightforward.
   (b) Combine $d^2 = 0$ with part (a):
      (i) $0 = d^2 f = d(\omega^1_{\text{grad } f}) = \omega^2_{\text{curl grad } f}$. So curl grad $f = 0$.
      (ii) $0 = d^2(\omega^1_X) = d(\omega^2_{\text{curl } X}) = \text{div curl } X\ dV$. So div curl $X = 0$.