Topology Exam: Spring 2018.

Answer SIX of the NINE questions

1. In each case give a proof, or give a counterexample and prove it is a counterexample.
(a) $\mathrm{int}(\mathrm{int}\, U) = \mathrm{int}(U)$
(b) $\mathrm{cl}(\mathrm{cl}(U)) = \mathrm{cl}(\mathrm{int}(U))$
(c) $\mathrm{int}(\mathrm{cl}(\mathrm{int}(U))) = \mathrm{int}(U)$
(d) $\mathrm{int}(U \times V) = \mathrm{int}(U) \times \mathrm{int}(V)$

2. An *ultrafilter on a set* X is a collection $\mathcal{U}$ of subsets of X with the properties that
(i) If $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$
(ii) If $U \subset X$ then $U \in \mathcal{U}$ or $X \setminus U \in \mathcal{U}$.
(iii) $\emptyset \notin \mathcal{U}$

   (a) Show that if $V \supset U$ and $U \in \mathcal{U}$ then $V \in \mathcal{U}$.

   (b) Show that $\mathcal{T} := \mathcal{U} \cup \{\emptyset\}$ is a topology on X.

   (c) If X is infinite show that the topology is connected, but not compact and not Hausdorff.

3. Suppose (X, d_X) is *separable* (there is a countable dense subset) metric space and (Y, d_Y) is a compact metric space. Let Z be the metric space consisting of the set of all continuous maps $f : X \to Y$ with metric

$$d_Z(f, g) = \sup_{x \in X} d_Y(fx, gx)$$

   Let $W \subset Z$ be the subspace of all 2-Lipschitz maps ($f : X \to Y$ is 2-Lipschitz if $d_Y(fx_1, fx_2) \leq 2d_X(x_1, x_2)$ for all $x_1, x_2 \in X$). For each of Z and W either prove it is sequentially compact, or give a counterexample and prove it is one.

4. Let $X \subset \mathbb{R}^2$ be the subspace $\{(x, (1/x)\sin(1/x)) : x > 0\} \cup \{(x, y) : x \leq 0\}$. (a) Is X connected? (b) Is X path connected? In each case prove your answer is correct.

5. Prove that the product of two compact Hausdorff spaces is compact and Hausdorff.

6. Let $A = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y \leq 1\}$ and $B = \{(x, y, z) \in A \times J : yz = x\}$ where $J \subset \mathbb{R}$ is an interval defined below, each topologized as a subspace of Euclidean space. Define $\pi : B \to A$ by $\pi(x, y, z) = (x, y)$. Let $X = B/\pi$ be the the quotient space.
(a) If $J = [0, 1]$ prove that X is homeomorphic to A. [hint: is X compact?]
(b) If $J = [0, 1)$ prove X is not homeomorphic to A. [hint: is X compact?]

Turn Over (the exam)

7. Let (X, d) be a metric space. Recall that a metric completion of X is a complete metric space (X', d') and a map $f\colon X \to X'$ such that f is an isometry onto $f(X)$ and $f(X)$ is dense in X'.

   (a) Carefully define a metric space (X', d') and map $f\colon X \to X'$ that forms a completion of X. State, without proof, *everything* that must be checked to confirm that your definition is a metric completion of X.

   (b) Now fill in the following details. Assuming d' forms a metric, prove that f is an isometry onto $f(X)$ and that $f(X)$ is dense in X'.

   (c) Let $X = \{(\cos t, \sin t) : 0 < t < 2\pi\}$ with the metric $d((\cos a, \sin a), (\cos b, \sin b)) = |b-a|$. State, without proof, what well-known metric space the metric completion of (X, d) is isometric to.

8. Let $X = \mathbb{R}^n \setminus \{0\}$ with the subspace topology from $\mathbb{R}^n$. Let $\sim$ be the equivalence relation on X given by $\vec{x} \sim \vec{y}$ if and only if $\vec{x} = t\vec{y}$ for some $t > 0$.
   Prove that $X/\sim$ is homeomorphic to $S^{n-1} := \{\vec{x} \in \mathbb{R}^n \; : \; \|\vec{x}\| = 1 \}$ with the subspace topology from $\mathbb{R}^n$.

9. Suppose $p : Y \to X$ is a covering space.
   (a) Prove, using only the definition of covering space, that if $f : [0,1] \to X$ is continuous, and $t_0 = 0$, and $y \in Y$ with $p(y) = f(t_0)$, then there is a continuous map $F : [0,1] \to Y$ such that $p \circ F = f$ and $F(t_0) = y$.
   (b) Modify (a) so that $[0,1]$ is replaced by $S^1 := \{(x,y) \in \mathbb{R}^2 \; : \; x^2 + y^2 = 1\}$ with the topology as a subspace of $\mathbb{R}^2$ and $t_0 = (1,0)$. Does such F always exist? Prove your answer is correct.