Homework 6

Chapter 5

2. Prove the Reimann-Lebesgue lemma mentioned in Section 5.1., i.e., for $f \in L^1(\mathbb{R}^n)$,

$$\hat{f}(k) \to 0 \text{ as } k \to \infty.$$

[*Hint.* 5.3(1) is useful.

5.3 THEOREM (Plancherel's theorem)

If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\widehat{f}$ is in $L^2(\mathbb{R}^n)$ and the following formula of Plancherel holds:

$$\|\widehat{f}\|_2 = \|f\|_2. \tag{1}$$

Proof I didn't have time to do all of these problems, since this week we have a midterm in this class and another class, in addition to the regular work which I can barely keep up with. ■

5. Complete the proof of Theorem 5.8, i.e., work out the approximation argument mentioned at the end of Sect. 5.8:

5.8 THEOREM (Convolutions)

Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, and let $1 + 1/r = 1/p + 1/q$. Suppose $1 \le p, q, r \le 2$. Then

$$\widehat{f * g}(k) = \widehat{f}(k)\,\widehat{g}(k). \tag{1}$$

PROOF. By Young's inequality, Theorem 4.2, $f * g \in L^r(\mathbb{R}^n)$. By Theorem 5.7, $\widehat{f} \in L^{p'}(\mathbb{R}^n)$ and $\widehat{g} \in L^{q'}(\mathbb{R}^n)$, so $\widehat{f}\,\widehat{g} \in L^{r'}(\mathbb{R}^n)$ by Hölder's inequality. Since $h := f * g$ is in $L^r(\mathbb{R}^n)$, $\widehat{h} \in L^{r'}(\mathbb{R}^n)$ by Theorem 5.7. If both f and g are also in $L^1(\mathbb{R}^n)$, then (1) is true by 5.1(8). The theorem follows by an approximation argument that is left to the reader. ■

Proof Let $h, u \in C_0^\infty(\mathbb{R}^n)$. In particular, $h, u \in L^1(\mathbb{R}^n)$, so $\widehat{h * u} = \hat{h}\hat{u}$ by (1). Now consider the operators

$$T : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \text{ given by}$$
$$(f, g) \mapsto f * g$$

$$S : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^{r'}(\mathbb{R}^n) \text{ given by}$$
$$(f, g) \mapsto \hat{f}\hat{g}$$

$$\hat{\bullet} : L^{r'} \to C_0, \text{ the Fourier Transform,}$$

all of which are clearly continuous operators. Observe that $\hat{\bullet} \circ T = S$ on $C_0^\infty \times C_0^\infty$, and thus we have two continuous maps which agree on a dense subset of their domains, so they agree everywhere. ■

6. For $f \in C_c^\infty(\mathbb{R}^n)$ show that its Fourier transform $\widehat{f}$ is also in C^∞ (in fact $\widehat{f}$ is analytic). Show also that $g_a(k) := |\ |k|^a \widehat{f}(k)|$ is a bounded function for each $a > 0$.

Proof I didn't have time to do all of these problems, since this week we have a midterm in this class and another class, in addition to the regular work which I can barely keep up with. ■

9. Verify that 5.6(1) cannot hold when $\rho > 2$ by considering Gaussian functions, as in 5.2(1), with $\lambda = a + ib$ and with $a > 0$.

5.6 THE FOURIER TRANSFORM IN $L^p(\mathbb{R}^n)$

One way to extend the Fourier transform for $p < \infty$ would be to imitate the $L^2(\mathbb{R}^n)$ construction. The goal would then be to find a constant $C_{p,q}$ such that for every $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ the Fourier transform is in $L^q(\mathbb{R}^n)$ and satisfies

$$\|\widehat{f}\|_q \leq C_{p,q}\|f\|_p. \tag{1}$$

Using the continuity argument of Theorem 5.3 (and the density of $L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$) one can then extend the Fourier transform to all of $L^p(\mathbb{R}^n)$ and (1) will continue to hold.

5.2 THEOREM (Fourier transform of a Gaussian)

For $\lambda > 0$, denote by g_λ **the Gaussian function** *on $\mathbb{R}^n$ given by*

$$g_\lambda(x) = \exp[-\pi\lambda|x|^2] \tag{1}$$

for $x \in \mathbb{R}^n$. Then

$$\widehat{g}_\lambda(k) = \lambda^{-n/2}\exp[-\pi|k|^2/\lambda].$$

Proof I didn't have time to do all of these problems, since this week we have a midterm in this class and another class, in addition to the regular work which I can barely keep up with. ■

Problems from the PDF

Assignment 6

2. This problem is based on the scaling.

(a) Let for some $1 \le p \le \infty$ the inequality (Sobolev inequality)

$$||u||_p \le C(n,p)\,||\nabla u||_1$$

hold for all $u \in C_0^\infty(\mathbb{R}^n), n \ge 2$, with the constant C independent of u. What are the possible values (the *necessary* conditions) of p?

Proof For all $\lambda > 0$, let $u_\lambda := u\left(\frac{x}{\lambda}\right)$. Now $u_\lambda \in C_0^\infty$, so the Sobolev inequality holds for u_λ. Consider the right hand side,

$$\begin{aligned}
||\nabla u||_1 &= \int \left|\nabla\left[u\left(\tfrac{x}{\lambda}\right)\right]\right| dx \\
&= \int \frac{1}{\lambda}\left|(\nabla u)\left(\tfrac{x}{\lambda}\right)\right| dx && \text{by the chain rule} \\
&= \lambda^{n-1}\int |(\nabla u)(x)|\,dx && \text{by change of variables} \\
&= \lambda^{n-1}\,||\nabla u||_1\,.
\end{aligned}$$

Now consider the left hand side,

$$\begin{aligned}
||u_\lambda||_p &= \left(\int_{R^n}\left|u\left(\tfrac{x}{\lambda}\right)\right|^p dx\right)^{1/p} \\
&= \left(\lambda^n\int_{R^n}|u(x)|^p\,dx\right)^{1/p} && \text{by change of variables} \\
&= \lambda^{n/p}\left(\int_{R^n}|u(x)|^p\,dx\right)^{1/p} \\
&= \lambda^{n/p}\,||u||_p\,.
\end{aligned}$$

Substituting and collecting λ on the left side gives

$$\lambda^{\frac{n}{p}-n+1} \|u\|_p \leq C \|\nabla u\|_1 ,$$

and this is true for all $\lambda > 0$, even while the right hand side is finite and constant with respect to λ. Thus

$$\tfrac{n}{p} - n + 1 \leq 0,$$

otherwise $\lambda^{\frac{n}{p}-n+1} \|u\|_p \to \infty$ as $\lambda \to \infty$. However we also have that

$$\tfrac{n}{p} - n + 1 \geq 0,$$

otherwise $\lambda^{\frac{n}{p}-n+1} \|u\|_p \to \infty$ as $\lambda \to 0$. Thus $\frac{n}{p} - n + 1 = 0$, so $p = \frac{n}{n-1} = n'$. □

(b) Let B_r be a ball centered at the origin in $\mathbf{R}^n$, $n < p < \infty$. It is known (Morrey's inequality) that for all $u \in C^1(B_1)$ the inequality

$$\sup_{x,y\in B_{\frac{1}{2}}} |u(x) - u(y)| \leq C(n,p) \left(\int_{B_1} |\nabla u|^p \, dx \right)^{1/p}$$

holds. Prove that for any $r > 0$, any $u \in C^1(B_r)$ the inequality

$$\sup_{x,y\in B_{\frac{r}{2}}} |u(x) - u(y)| \leq C(n,p,r) \left(\int_{B_r} |\nabla u|^p \, dx \right)^{1/p}$$

holds and find the exact dependence of $C(n,p,r)$ on r.

Proof Suppose that $u \in C^1(B_r)$ for $r > 0$, and let $\rho : \mathbb{R}^n \to \mathbb{R}^n$ be the function which scales by r, that is $r(x) = rx$. Notice that $u \circ \rho \in C^1(B_1)$. Thus

$$\begin{aligned}
\sup_{x,y\in B_{r/2}} |u(x) - u(y)| &= \sup_{x,y\in B_{1/2}} |u(rx) - u(ry)| \\
&= \sup_{x,y\in B_{1/2}} |(u\circ\rho)(x) - (u\circ\rho)(y)| \\
&\leq C(n,p) \left(\int_{B_1} \left|\nabla(u\circ\rho)(x)\right|^p dx \right)^{1/p} && \text{by Morrey's inequality} \\
&\leq C(n,p) \left(\int_{B_1} r^p \left|(\nabla u)(rx)\right|^p dx \right)^{1/p} && \text{by chain rule} \\
&= C(n,p) r \left(\int_{B_1} \left|(\nabla u)(rx)\right|^p dx \right)^{1/p} \\
&= C(n,p) r^{1-n} \left(\int_{B_1} |\nabla u(x)|^p \, dx \right)^{1/p} && \text{by change of variables}
\end{aligned}$$

Thus $C(p,n,r) = C(n,p)r^{1-n}$. ■