Math 201A, Homework 2 (measure theory and measurable functions)

Problem1. Let $n \in \mathbb{R}^n$ and let $K \subset \mathbb{R}^n$ be a compact. Denote $U = \mathbb{R}^n - K$ and define for each fixed $s \in K$ the function

$$u_s(x) = \max\left(2 - \frac{|x-s|}{\mathrm{dist}(x,K)}, 0\right), \qquad x \in U.$$

Let s_i be a countable dense subset of K and define

$$\sigma(x) = \sum_{i=1}^{\infty} 2^{-i} u_{s_i}(x), \qquad x \in U.$$

It is not difficult to prove that then $0 < \sigma(x) \leq 1$ for all $x \in U$, thus we can define

$$v_i(x) = \frac{2^{-i} u_{s_i}(x)}{\sigma(x)}, \qquad x \in U.$$

Assume next $f \colon K \to \mathbb{R}$ is continuous and define

$$\bar{f}(x) = \sum_{i=1\infty} v_i(x) f(s_i), \qquad x \in U.$$

Prove that $\bar{f}(x)$ is continuous in U.

Problem2. A function $f \colon \mathbb{R}^n \to \mathbb{R}$ is called lower semi-continuous at the point $x \in \mathbb{R}^n$, if for any sequence $x_k \in \mathbb{R}^n$ with $x_k \to x$ one has

$$\liminf_{k\to\infty} f(x_k) \geq f(x).$$

Prove that any lower semi-continuous function is Borel-measurable.

Problem3. Prove the following stetments:

1. If for some $a < b$ and $a_k < b_k$, for $k = 1, 2, \ldots$ one has

$$[a,b) \subset \cup_{k=1}^{\infty} [a_k, b_k),$$

then

$$b - a \leq \sum_{k=1}^{\infty} (b_k - a_k).$$

2. If for some $[a_k, b_k)$ disjoint intervals and $c_k < d_k$, for $k = 1, 2, \ldots$ one has

$$\cup_{k=1}^{\infty}[a_k, b_k) \subset \cup_{k=1}^{\infty}[c_k, d_k),$$

then

$$\sum_{k=1}^{\infty}(b_k - a_k) \leq \sum_{k=1}^{\infty}(d_k - c_k).$$

Problem4. Prove that if a Lebesgue measurable set $A \subset \mathbb{R}$ has a positive Lebsgue measure, then the set

$$A - A = \{a - b \ : \ a, b \in A\}$$

contains a neighborhood of the origin. Is the statement true if one only assumes $\mu(A) > 0$ (i.e., A is not Lebesgue measurable)?