EXAMPLE. The *unit disk* D^m, consisting of all $x \in R^m$ with

$$1 - \sum x_i^2 \geq 0,$$

is a smooth manifold, with boundary equal to S^{m-1}.

Now consider a smooth map $f : X \to N$ from an m-manifold with boundary to an n-manifold, where $m > n$.

Lemma 4. *If $y \in N$ is a regular value, both for f and for the restriction $f \mid \partial X$, then $f^{-1}(y) \subset X$ is a smooth $(m - n)$-manifold with boundary. Furthermore the boundary $\partial(f^{-1}(y))$ is precisely equal to the intersection of $f^{-1}(y)$ with ∂X.*

PROOF. Since we have to prove a local property, it suffices to consider the special case of a map $f : H^m \to R^n$, with regular value $y \in R^n$. Let $\bar{x} \in f^{-1}(y)$. If $\bar{x}$ is an interior point, then as before $f^{-1}(y)$ is a smooth manifold in the neighborhood of $\bar{x}$.

Suppose that $\bar{x}$ is a boundary point. Choose a smooth map $g : U \to R^n$ that is defined throughout a neighborhood of $\bar{x}$ in R^m and coincides with f on $U \cap H^m$. Replacing U by a smaller neighborhood if necessary, we may assume that g has no critical points. Hence $g^{-1}(y)$ is a smooth manifold of dimension $m - n$.

Let $\pi : g^{-1}(y) \to R$ denote the coordinate projection,

$$\pi(x_1, \cdots, x_m) = x_m.$$

We claim that π has 0 as a regular value. For the tangent space of $g^{-1}(y)$ at a point $x \in \pi^{-1}(0)$ is equal to the null space of

$$dg_x = df_x : R^m \to R^n;$$

but the hypothesis that $f \mid \partial H^m$ is regular at x guarantees that this null space cannot be completely contained in $R^{m-1} \times 0$.

Therefore the set $g^{-1}(y) \cap H^m = f^{-1}(y) \cap U$, consisting of all $x \in g^{-1}(y)$ with $\pi(x) \geq 0$, is a smooth manifold, by Lemma 3; with boundary equal to $\pi^{-1}(0)$. This completes the proof.

THE BROUWER FIXED POINT THEOREM

We now apply this result to prove the key lemma leading to the classical Brouwer fixed point theorem. Let X be a compact manifold with boundary.

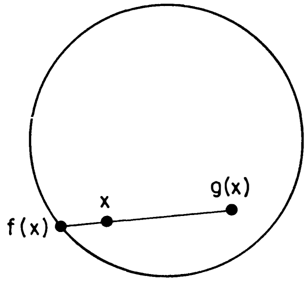

Figure 4

of D^n into points outside of D^n. To correct this we set

$$P(x) = P_1(x)/(1 + \epsilon).$$

Then clearly P maps D^n into D^n and $||P(x) - G(x)|| < 2\epsilon$ for $x \in D^n$.

Suppose that $G(x) \neq x$ for all $x \in D^n$. Then the continuous function $||G(x) - x||$ must take on a minimum $\mu > 0$ on D^n. Choosing $P : D^n \to D^n$ as above, with $||P(x) - G(x)|| < \mu$ for all x, we clearly have $P(x) \neq x$. Thus P is a smooth map from D^n to itself without a fixed point. This contradicts Lemma 6, and completes the proof.

The procedure employed here can frequently be applied in more general situations: to prove a proposition about continuous mappings, we first establish the result for smooth mappings and then try to use an approximation theorem to pass to the continuous case. (Compare §8, Problem 4.)

Lemma 5. *There is no smooth map $f : X \to \partial X$ that leaves ∂X pointwise fixed.*

Proof (following M. Hirsch). Suppose there were such a map f. Let $y \in \partial X$ be a regular value for f. Since y is certainly a regular value for the identity map $f \mid \partial X$ also, it follows that $f^{-1}(y)$ is a smooth 1-manifold, with boundary consisting of the single point

$$f^{-1}(y) \cap \partial X = \{y\}.$$

But $f^{-1}(y)$ is also compact, and the only compact 1-manifolds are finite disjoint unions of circles and segments,* so that $\partial f^{-1}(y)$ must consist of an even number of points. This contradiction establishes the lemma.

In particular the unit disk

$$D^n = \{x \in R^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

is a compact manifold bounded by the unit sphere S^{n-1}. Hence as a special case we have proved that *the identity map of S^{n-1} cannot be extended to a smooth map $D^n \to S^{n-1}$.*

Lemma 6. *Any smooth map $g : D^n \to D^n$ has a fixed point* (i.e. a point $x \in D^n$ with $g(x) = x$).

Proof. Suppose g has no fixed point. For $x \in D^n$, let $f(x) \in S^{n-1}$ be the point nearer x on the line through x and $g(x)$. (See Figure 4.) Then $f : D^n \to S^{n-1}$ is a smooth map with $f(x) = x$ for $x \in S^{n-1}$, which is impossible by Lemma 5. (To see that f is smooth we make the following explicit computation: $f(x) = x + tu$, where

$$u = \frac{x - g(x)}{||x - g(x)||}, \qquad t = -x\cdot u + \sqrt{1 - x\cdot x + (x\cdot u)^2},$$

the expression under the square root sign being strictly positive. Here and subsequently $||x||$ denotes the euclidean length $\sqrt{x_1^2 + \cdots + x_n^2}$.)

Brouwer Fixed Point Theorem. *Any continuous function $G : D^n \to D^n$ has a fixed point.*

Proof. We reduce this theorem to the lemma by approximating G by a smooth mapping. Given $\epsilon > 0$, according to the Weierstrass approximation theorem,† there is a polynomial function $P_1 : R^n \to R^n$ with $||P_1(x) - G(x)|| < \epsilon$ for $x \in D^n$. However, P_1 may send points

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* A proof is given in the Appendix.

† See for example Dieudonné [7, p. 133].

§3. PROOF OF SARD'S THEOREM*

FIRST let us recall the statement:

Theorem of Sard. *Let $f: U \to R^p$ be a smooth map, with U open in R^n, and let C be the set of critical points; that is the set of all $x \in U$ with*

$$\text{rank } df_x < p.$$

Then $f(C) \subset R^p$ has measure zero.

REMARK. The cases where $n \leq p$ are comparatively easy. (Compare de Rham [29, p. 10].) We will, however, give a unified proof which makes these cases look just as bad as the others.

The proof will be by induction on n. Note that the statement makes sense for $n \geq 0$, $p \geq 1$. (By definition R^0 consists of a single point.) To start the induction, the theorem is certainly true for $n = 0$.

Let $C_1 \subset C$ denote the set of all $x \in U$ such that the first derivative df_x is zero. More generally let C_i denote the set of x such that all partial derivatives of f of order $\leq i$ vanish at x. Thus we have a descending sequence of closed sets

$$C \supset C_1 \supset C_2 \supset C_3 \supset \cdots .$$

The proof will be divided into three steps as follows:

STEP 1. The image $f(C - C_1)$ has measure zero.
STEP 2. The image $f(C_i - C_{i+1})$ has measure zero, for $i \geq 1$.
STEP 3. The image $f(C_k)$ has measure zero for k sufficiently large.

(REMARK. If f happens to be real analytic, then the intersection of

* Our proof is based on that given by Pontryagin [28]. The details are somewhat easier since we assume that f is infinitely differentiable.

the C_i is vacuous unless f is constant on an entire component of U. Hence in this case it is sufficient to carry out Steps 1 and 2.)

PROOF OF STEP 1. This first step is perhaps the hardest. We may assume that $p \geq 2$, since $C = C_1$ when $p = 1$. We will need the well known theorem of Fubini* which asserts that *a measurable set*

$$A \subset R^p = R^1 \times R^{p-1}$$

must have measure zero if it intersects each hyperplane (constant) $\times R^{p-1}$ *in a set of* $(p - 1)$*-dimensional measure zero.*

For each $\bar{x} \varepsilon C - C_1$ we will find an open neighborhood $V \subset R^n$ so that $f(V \cap C)$ has measure zero. Since $C - C_1$ is covered by countably many of these neighborhoods, this will prove that $f(C - C_1)$ has measure zero.

Since $\bar{x} \not\varepsilon C_1$, there is some partial derivative, say $\partial f_1/\partial x_1$, which is not zero at $\bar{x}$. Consider the map $h : U \to R^n$ defined by

$$h(x) = (f_1(x), x_2, \cdots, x_n).$$

Since $dh_{\bar{x}}$ is nonsingular, h maps some neighborhood V of $\bar{x}$ diffeomorphically onto an open set V'. The composition $g = f \circ h^{-1}$ will then map V' into R^p. Note that the set C' of critical points of g is precisely $h(V \cap C)$; hence the set $g(C')$ of critical values of g is equal to $f(V \cap C)$.

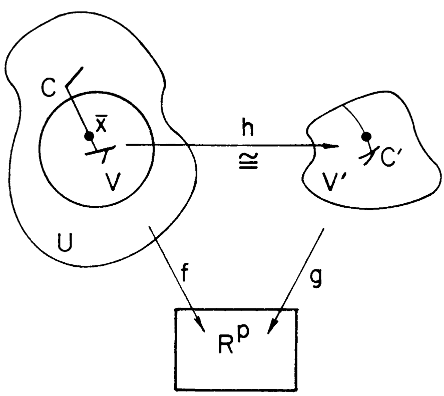

Figure 5. Construction of the map g

* For an easy proof (as well as an alternative proof of Sard's theorem) see Sternberg [35, pp. 51–52]. Sternberg assumes that A is compact, but the general case follows easily from this special case.

PROOF OF STEP 3. Let $I^n \subset U$ be a cube with edge δ. If k is sufficiently large ($k > n/p - 1$ to be precise) we will prove that $f(C_k \cap I^n)$ has measure zero. Since C_k can be covered by countably many such cubes, this will prove that $f(C_k)$ has measure zero.

From Taylor's theorem, the compactness of I^n, and the definition of C_k, we see that

$$f(x + h) = f(x) + R(x, h)$$

where

$$||R(x, h)|| \leq c\, ||h||^{k+1} \tag{1}$$

for $x \in C_k \cap I^n$, $x + h \in I^n$. Here c is a constant which depends only on f and I^n. Now subdivide I^n into r^n cubes of edge δ/r. Let I_1 be a cube of the subdivision which contains a point x of C_k. Then any point of I_1 can be written as $x + h$, with

$$||h|| \leq \sqrt{n}(\delta/r). \tag{2}$$

From 1) it follows that $f(I_1)$ lies in a cube of edge a/r^{k+1} centered about $f(x)$, where $a = 2c\,(\sqrt{n}\;\delta)^{k+1}$ is constant. Hence $f(C_k \cap I^n)$ is contained in a union of at most r^n cubes having total volume

$$V \leq r^n(a/r^{k+1})^p = a^p r^{n-(k+1)p}.$$

If $k + 1 > n/p$, then evidently V tends to 0 as $r \to \infty$; so $f(C_k \cap I^n)$ must have measure zero. This completes the proof of Sard's theorem.

For each $(t, x_2, \cdots, x_n) \in V'$ note that $g(t, x_2, \cdots, x_n)$ belongs to the hyperplane $t \times R^{p-1} \subset R^p$: thus g carries hyperplanes into hyperplanes. Let

$$g^t : (t \times R^{n-1}) \cap V' \to t \times R^{p-1}$$

denote the restriction of g. Note that a point of $t \times R^{n-1}$ is critical for g^t if and only if it is critical for g; for the matrix of first derivatives of g has the form

$$(\partial g_i / \partial x_j) = \begin{bmatrix} 1 & 0 \\ * & (\partial g_i^t / \partial x_j) \end{bmatrix}.$$

According to the induction hypothesis, the set of critical values of g^t has measure zero in $t \times R^{p-1}$. Therefore the set of critical values of g intersects each hyperplane $t \times R^{p-1}$ in a set of measure zero. This set $g(C')$ is measurable, since it can be expressed as a countable union of compact subsets. Hence, by Fubini's theorem, the set

$$g(C') = f(V \cap C)$$

has measure zero, and Step 1 is complete.

PROOF OF STEP 2. For each $\bar{x} \in C_k - C_{k+1}$ there is some $(k+1)^{-st}$ derivative $\partial^{k+1} f_r / \partial x_{s_1} \ldots \partial x_{s_{k+1}}$ which is not zero. Thus the function

$$w(x) = \partial^k f_r / \partial x_{s_2} \cdots \partial x_{s_{k+1}}$$

vanishes at $\bar{x}$ but $\partial w / \partial x_{s_1}$ does not. Suppose for definiteness that $s_1 = 1$. Then the map $h : U \to R^n$ defined by

$$h(x) = (w(x), x_2, \cdots, x_n)$$

carries some neighborhood V of $\bar{x}$ diffeomorphically onto an open set V'. Note that h carries $C_k \cap V$ into the hyperplane $0 \times R^{n-1}$. Again we consider

$$g = f \circ h^{-1} : V' \to R^p.$$

Let

$$\bar{g} : (0 \times R^{n-1}) \cap V' \to R^p$$

denote the restriction of g. By induction, the set of critical values of $\bar{g}$ has measure zero in R^p. But each point in $h(C_k \cap V)$ is certainly a critical point of $\bar{g}$ (since all derivatives of order $\leq k$ vanish). Therefore

$$\bar{g}h(C_k \cap V) = f(C_k \cap V) \text{ has measure zero.}$$

Since $C_k - C_{k+1}$ is covered by countably many such sets V, it follows that $f(C_k - C_{k+1})$ has measure zero.

§4. THE DEGREE MODULO 2 OF A MAPPING

Consider a smooth map $f : S^n \to S^n$. If y is a regular value, recall that $\#f^{-1}(y)$ denotes the number of solutions x to the equation $f(x) = y$. We will prove that *the residue class modulo 2 of $\#f^{-1}(y)$ does not depend on the choice of the regular value y.* This residue class is called the mod 2 degree of f. More generally this same definition works for any smooth map

$$f : M \to N$$

where M is compact without boundary, N is connected, and both manifolds have the same dimension. (We may as well assume also that N is compact without boundary, since otherwise the mod 2 degree would necessarily be zero.) For the proof we introduce two new concepts.

SMOOTH HOMOTOPY AND SMOOTH ISOTOPY

Given $X \subset R^k$, let $X \times [0, 1]$ denote the subset* of R^{k+1} consisting of all (x, t) with $x \in X$ and $0 \leq t \leq 1$. Two mappings

$$f, g : X \to Y$$

are called *smoothly homotopic* (abbreviated $f \sim g$) if there exists a

* If M is a smooth manifold without boundary, then $M \times [0, 1]$ is a smooth manifold bounded by two "copies" of M. Boundary points of M will give rise to "corner" points of $M \times [0, 1]$.

smooth map $F : X \times [0, 1] \to Y$ with

$$F(x, 0) = f(x), \qquad F(x, 1) = g(x)$$

for all $x \varepsilon X$. This map F is called a *smooth homotopy* between f and g.

Note that the relation of smooth homotopy is an equivalence relation. To see that it is transitive we use the existence of a smooth function $\varphi : [0, 1] \to [0, 1]$ with

$$\varphi(t) = 0 \quad \text{for} \quad 0 \leq t \leq \tfrac{1}{3}$$

$$\varphi(t) = 1 \quad \text{for} \quad \tfrac{2}{3} \leq t \leq 1.$$

(For example, let $\varphi(t) = \lambda(t - \frac{1}{3})/(\lambda(t - \frac{1}{3}) + \lambda(\frac{2}{3} - t))$, where $\lambda(\tau) = 0$ for $\tau \leq 0$ and $\lambda(\tau) = \exp(-\tau^{-1})$ for $\tau > 0$.) Given a smooth homotopy F between f and g, the formula $G(x, t) = F(x, \varphi(t))$ defines a smooth homotopy G with

$$G(x, t) = f(x) \quad \text{for} \quad 0 \leq t \leq \tfrac{1}{3}$$

$$G(x, t) = g(x) \quad \text{for} \quad \tfrac{2}{3} \leq t \leq 1.$$

Now if $f \sim g$ and $g \sim h$, then, with the aid of this construction, it is easy to prove that $f \sim h$.

If f and g happen to be diffeomorphisms from X to Y, we can also define the concept of a "smooth isotopy" between f and g. This also will be an equivalence relation.

Definition. The diffeomorphism f is *smoothly isotopic* to g if there exists a smooth homotopy $F : X \times [0, 1] \to Y$ from f to g so that, for each $t \varepsilon [0, 1]$, the correspondence

$$x \to F(x, t)$$

maps X diffeomorphically onto Y.

It will turn out that the mod 2 degree of a map depends only on its smooth homotopy class:

Homotopy Lemma. *Let $f, g : M \to N$ be smoothly homotopic maps between manifolds of the same dimension, where M is compact and without boundary. If $y \varepsilon N$ is a regular value for both f and g, then*

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod 2.$$

Proof. Let $F : M \times [0, 1] \to N$ be a smooth homotopy between f and g. First suppose that y is also a regular value for F. Then $F^{-1}(y)$

(For the special case $N = S^n$ the proof is easy: simply choose h to be the rotation which carries y into z and leaves fixed all vectors orthogonal to the plane through y and z.)

The proof in general proceeds as follows: We will first construct a smooth isotopy from R^n to itself which

1) leaves all points outside of the unit ball fixed, and
2) slides the origin to any desired point of the open unit ball.

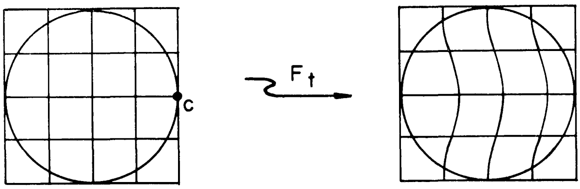

Figure 7. Deforming the unit ball

Let $\varphi : R^n \to R$ be a smooth function which satisfies

$$\varphi(x) > 0 \quad \text{for} \quad ||x|| < 1$$
$$\varphi(x) = 0 \quad \text{for} \quad ||x|| \geq 1.$$

(For example let $\varphi(x) = \lambda(1 - ||x||^2)$ where $\lambda(t) = 0$ for $t \leq 0$ and $\lambda(t) = \exp(-t^{-1})$ for $t > 0$.) Given any fixed unit vector $c \in S^{n-1}$, consider the differential equations

$$\frac{dx_i}{dt} = c_i\varphi(x_1, \cdots, x_n); \qquad i = 1, \cdots, n.$$

For any $\bar{x} \in R^n$ these equations have a unique solution $x = x(t)$, defined for all* real numbers which satisfies the initial condition

$$x(0) = \bar{x}.$$

We will use the notation $x(t) = F_t(\bar{x})$ for this solution. Then clearly

1) $F_t(\bar{x})$ is defined for all t and $\bar{x}$ and depends smoothly on t and $\bar{x}$,
2) $F_0(\bar{x}) = \bar{x}$,
3) $F_{s+t}(\bar{x}) = F_s \circ F_t(\bar{x})$.

* Compare [22, §2.4].

is a compact 1-manifold, with boundary equal to

$$F^{-1}(y) \cap (M \times 0 \cup M \times 1) = f^{-1}(y) \times 0 \cup g^{-1}(y) \times 1.$$

Thus the total number of boundary points of $F^{-1}(y)$ is equal to

$$\#f^{-1}(y) + \#g^{-1}(y).$$

But we recall from §2 that a compact 1-manifold always has an even number of boundary points. Thus $\#f^{-1}(y) + \#g^{-1}(y)$ is even, and therefore

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod 2.$$

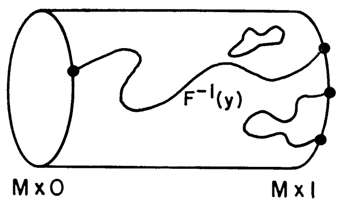

Figure 6. The number of boundary points on the left is congruent to the number on the right modulo 2

Now suppose that y is not a regular value of F. Recall (from §1) that $\#f^{-1}(y')$ and $\#g^{-1}(y')$ are locally constant functions of y' (as long as we stay away from critical values). Thus there is a neighborhood $V_1 \subset N$ of y, consisting of regular values of f, so that

$$\#f^{-1}(y') = \#f^{-1}(y)$$

for all $y' \in V_1$; and there is an analogous neighborhood $V_2 \subset N$ so that

$$\#g^{-1}(y') = \#g^{-1}(y)$$

for all $y' \in V_2$. Choose a regular value z of F within $V_1 \cap V_2$. Then

$$\#f^{-1}(y) = \#f^{-1}(z) \equiv \#g^{-1}(z) = \#g^{-1}(y)),$$

which completes the proof.

We will also need the following:

Homogeneity Lemma. *Let y and z be arbitrary interior points of the smooth, connected manifold N. Then there exists a diffeomorphism $h: N \to N$ that is smoothly isotopic to the identity and carries y into z.*

Therefore each F_t is a diffeomorphism from R^n onto itself. Letting t vary, we see that each F_t is smoothly isotopic to the identity under an isotopy which leaves all points outside of the unit ball fixed. But clearly, with suitable choice of c and t, the diffeomorphism F_t will carry the origin to any desired point in the open unit ball.

Now consider a connected manifold N. Call two points of N "isotopic" if there exists a smooth isotopy carrying one to the other. This is clearly an equivalence relation. If y is an interior point, then it has a neighborhood diffeomorphic to R^n; hence the above argument shows that every point sufficiently close to y is "isotopic" to y. In other words, each "isotopy class" of points in the interior of N is an open set, and the interior of N is partitioned into disjoint open isotopy classes. But the interior of N is connected; hence there can be only one such isotopy class. This completes the proof.

We can now prove the main result of this section. Assume that M is compact and boundaryless, that N is connected, and that $f : M \to N$ is smooth.

Theorem. *If y and z are regular values of f then*

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \quad \text{(modulo 2)}.$$

This common residue class, which is called the mod 2 degree *of f, depends only on the smooth homotopy class of f.*

PROOF. Given regular values y and z, let h be a diffeomorphism from N to N which is isotopic to the identity and which carries y to z. Then z is a regular value of the composition $h \circ f$. Since $h \circ f$ is homotopic to f, the Homotopy Lemma asserts that

$$\#(h \circ f)^{-1}(z) \equiv \#f^{-1}(z) \quad \text{(mod 2)}.$$

But

$$(h \circ f)^{-1}(z) = f^{-1}h^{-1}(z) = f^{-1}(y),$$

so that

$$\#(h \circ f)^{-1}(z) = \#f^{-1}(y).$$

Therefore

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \quad \text{(mod 2)},$$

as required.

Call this common residue class $\deg_2(f)$. Now suppose that f is smoothly homotopic to g. By Sard's theorem, there exists an element $y \in N$