Math 550 Homework

Problems and solutions

Dr. Fuller

December 12, 2018

Math 550
Homework 1
Dr. Fuller
Due September 4, 2018

1. The set $\Lambda^n(\mathbf{R}^n)$ of all alternating, multilinear functions on $(\mathbf{R}^n)^n$ forms a vector space. (You do not have to prove this.) What is its dimension? Find a basis for this vector space.

2. Let V be an n-dimensional vector space with an inner product $\langle \ , \ \rangle$. Suppose $S \in \Lambda^n(V)$ is an alternating, multilinear function on V.

   (a) Let $(u_1, \ldots, u_n)$ be a basis for V. Suppose $(v_1, \ldots, v_n)$ is a collection of vectors in V with $v_j = \sum_i a_{ij} u_i$. Prove that $S(v_1, \ldots, v_n) = \det[a_{ij}] S(u_1, \ldots, u_n)$.

   (b) Suppose that $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are two orthonormal bases for V, with $v_j = \sum_i a_{ij} u_i$. Let $A = [a_{ij}]$. Prove that $AA^T = I$. (Hint: start by considering $\langle v_i, v_j \rangle$.)

   (c) Prove that $|S(u_1, \ldots, u_n)| = |S(v_1, \ldots, v_n)|$ for any two orthonormal bases of V.

3. Give a counterexample to show that the change of variables formula does not hold if g is not one-to-one, even if $\det Dg(x) \neq 0$ for all $x \in \Omega$. (Hint: Take $f = 1$ and $g(x,y) = (e^x \cos y, e^x \sin y)$ for a suitable region Ω.)

4. (a) Calculate $\int_{B_r} e^{-x^2-y^2} \, dx \, dy$, where $B_r = \{(x,y) : x^2 + y^2 \leq r^2\}$.

   (b) Show that $\int_{C_r} e^{-x^2-y^2} \, dx \, dy = (\int_{-r}^{r} e^{-x^2} \, dx)^2$, where $C_r = [-r, r] \times [-r, r]$.

   (c) Show that
   $$\lim_{r \to \infty} \int_{B_r} e^{-x^2-y^2} \, dx \, dy = \lim_{r \to \infty} \int_{C_r} e^{-x^2-y^2} \, dx \, dy.$$

   (d) Show that $\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$.

5. (a.) Let D be the unit ball in $\mathbf{R}^3$, and let $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$. Calculate $\int_D f$ using a change of variables.

   (b.) Let E be the ellipsoid $\{(x,y,z) \in \mathbf{R}^3 : (x^2/a^2) + (y^2/b^2) + (z^2/c^2) \leq 1\}$, where a, b, and c are positive constants. Compute the volume of E using a change of variables.

6. Let $\langle e_1, \ldots, e_n \rangle$ denote the standard basis for $\mathbf{R}^n$, and let T denote the linear operator on $\mathbf{R}^n$ defined by $T(e_1) = (1,1,1,1,\ldots,1), T(e_2) = (1,2,1,1,\ldots,1), T(e_3) = (1,2,3,1,\ldots,1), \ldots, T(e_n) = (1,2,3,4,\ldots,n)$.
   Suppose that $f : \Omega \to \mathbf{R}$ is integrable, and $\int_\Omega f = 1$. Compute $\int_{T^{-1}(\Omega)} f \circ T$.

Math 550
Homework 1
Dr. Fuller
Solutions

1. Let $S \in \Lambda^n(\mathbf{R}^n)$. Let $(\vec{v}_1, \dots, \vec{v}_n)$ be a collection of vectors with $v_j = \sum_i a_{ij} e_i$. By imitating the computation in class, we get

$$\begin{aligned} S(\vec{v}_1, \dots, \vec{v}_n) &= (\sum_\sigma (-1)^{\operatorname{sign}\sigma} a_{\sigma(1)1} \cdots a_{\sigma(n)n})\, S(\vec{e}_1, \dots, \vec{e}_n) \\ &= D(\vec{v}_1, \dots, \vec{v}_n)\, S(\vec{e}_1, \dots, \vec{e}_n). \end{aligned}$$

Thus any S is a scalar multiple of D, showing that the dimension of $\Lambda^n(\mathbf{R}^n)$ is 1, and that $\{D\}$ forms a basis.

2. (a) This also follows by imitating the calculation in class.
(b) Following the hint,

$$\delta_{i,j} = \langle \vec{v}_i, \vec{v}_j \rangle = \langle \sum_k a_{ki} \vec{u}_k, \sum_l a_{lj} \vec{u}_l \rangle = \sum_{l,k} a_{ki} a_{lj} \langle \vec{u}_k, \vec{u}_l \rangle = \sum_l a_{li} a_{lj}.$$

The last term is the (i,j)-th entry of $A^T A$, so we have $A^T A = I$.
(c) From part (b), we have $1 = \det(I) = \det(A^T A) = \det(A^T)\det(A) = \det(A)^2$. So $\det A = \pm 1$. The result then follows from part (a).

3. Following the hint, let $f = 1$ and $\Omega = \{(x,y) : 1 < |(x,y)| < e\}$. Define $g : (0,1) \times (0, 4\pi) \to \Omega$ by $f(u,v) = (e^u \cos v, e^u \sin v)$. Then

$$\int_\Omega f = \operatorname{vol}(\Omega) = \pi(e^2 - 1),$$

but

$$\int_{g^{-1}(\Omega)} (f \circ g) |\det Dg| = \int_0^{4\pi} \int_0^1 e^{2u}\, du\, dv = 2\pi(e^2 - 1).$$

4. (a) Changing variables into polar coordinates (ρ, θ) gives $\int_{B_r} e^{-x^2-y^2}\, dx\, dy = \int_0^{2\pi} \int_0^r e^{-\rho^2} \rho\, d\rho\, d\theta = \pi(1 - e^{-r^2})$.
(b) From Fubini's Theorem $\int_{C_r} e^{-x^2-y^2}\, dx\, dy = \int_{-r}^r \int_{-r}^r e^{-x^2} e^{-y^2}\, dx\, dy = (\int_{-r}^r e^{-x^2}\, dx)(\int_{-r}^r e^{-y^2}\, dy) = (\int_{-r}^r e^{-x^2}\, dx)^2$.
(c) Observe that $B_r \subset C_r \subset B_{\sqrt{2}r}$, so

$$\int_{B_r} e^{-x^2-y^2}\, dx\, dy \le \int_{C_r} e^{-x^2-y^2}\, dx\, dy \le \int_{B_{\sqrt{2}r}} e^{-x^2-y^2}\, dx\, dy.$$

The limit as $r \to \infty$ of the outer integrals in this inequality both exist and equal π (by part(a)). Thus $\lim_{r\to\infty} \int_{C_r} e^{-x^2-y^2}\, dx\, dy = \pi$ as well.
(d) Parts (b) and (c) imply that $\lim_{r\to\infty} \int_{-r}^r e^{-x^2}\, dx = \sqrt{\pi}$. Since $e^{-x^2} > 0$, this also equals $\int_{-\infty}^\infty e^{-x^2}\, dx$.

5. (a) Change variables into spherical coordinates: $g(\rho, \theta, \phi) = (a\rho \cos\theta \sin\phi, b\rho \sin\theta \sin\phi, c\rho \cos\phi)$. Direct calculation gives $|\det Dg(\rho, \theta, \phi)| = \rho^2 \sin\phi$. So $\int_D f = \int_0^\pi \int_0^{2\pi} \int_0^1 e^{\rho^3} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{4}{3}\pi(e - 1)$.

(b) Change variables using modified spherical coordinates: $g(\rho,\theta,\phi) = (a\rho\cos\theta\sin\phi, b\rho\sin\theta\sin\phi, c\rho\cos\phi)$. Direct calculation gives $|\det Dg(\rho,\theta,\phi)| = abc\rho^2\sin\phi$. So $\int_E 1 = \int_0^\pi \int_0^{2\pi} \int_0^1 abc\rho^2 \sin\phi d\rho\ d\theta\ d\phi = \frac{4}{3}\pi abc$.

6. Since T is linear, we have $\det DT(a) = \det T = (n-1)!$ for all $a \in T^{-1}(\Omega)$. Then by change of variables, $1 = \int_\Omega f = \int_{T^{-1}(\Omega)} f \circ T |\det DT| = (n-1)! \int_{T^{-1}(\Omega)} f \circ T$. Thus $\int_{T^{-1}(\Omega)} f \circ T = 1/(n-1)!$.

Math 550
Homework 2
Dr. Fuller
Due September 11, 2018

1. Use a change of variables to calculate $\int_A f$, where $f(x,y,z) = (x^2+y^2)z^2$, and
$$A = \{(x,y,z) : x^2+y^2 < 1, |z| < 1\}.$$

2. Give a counterexample to show that the change of variables formula does not hold if g is not one-to-one, even if $\det Dg(x) \neq 0$ for all $x \in \Omega$. (Hint: Take $f = 1$ and $g(x,y) = (e^x \cos y, e^x \sin y)$ for a suitable region Ω.)

3. (a) Calculate $\int_{B_r} e^{-x^2-y^2}\, dx\, dy$, where $B_r = \{(x,y) : x^2+y^2 \leq r^2\}$.
   (b) Show that $\int_{C_r} e^{-x^2-y^2}\, dx\, dy = (\int_{-r}^{r} e^{-x^2}\, dx)^2$, where $C_r = [-r,r]\times[-r,r]$.
   (c) Show that
   $$\lim_{r\to\infty}\int_{B_r} e^{-x^2-y^2}\, dx\, dy = \lim_{r\to\infty}\int_{C_r} e^{-x^2-y^2}\, dx\, dy.$$
   (d) Show that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

4. (a.) Let D be the unit ball in $\mathbf{R}^3$, and let $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$. Calculate $\int_D f$ using a change of variables.
   (b.) Let E be the ellipsoid $\{(x,y,z) \in \mathbf{R}^3 : (x^2/a^2)+(y^2/b^2)+(z^2/c^2) \leq 1\}$, where a, b, and c are positive constants. Compute the volume of E using a change of variables.

5. Let $\langle e_1,\dots,e_n\rangle$ denote the standard basis for $\mathbf{R}^n$, and let T denote the linear operator on $\mathbf{R}^n$ defined by $T(e_1) = (1,1,1,1,\dots,1), T(e_2) = (1,2,1,1,\dots,1), T(e_3) = (1,2,3,1,\dots,1),\dots,T(e_n) = (1,2,3,4,\dots,n)$.
   Suppose that $f : \Omega \to \mathbf{R}$ is integrable, and $\int_\Omega f = 1$. Compute $\int_{T^{-1}(\Omega)} f \circ T$.

6. Let $p \in \mathbf{R}^4$, and let $u = (1,-1,0,2), v = (0,3,-2,1), w = (2,1,1,1)$. Compute
   (a) $(dx_1 \wedge dx_3 \wedge dx_4)_p(u,v,w)_p$
   (b) $(dx_1 \wedge dx_3 \wedge dx_1)_p(u,v,w)_p$
   (c) $((dx_1+dx_2) \wedge dx_3 \wedge dx_4)_p(u,v,w)_p$

7. Let $p \in \mathbf{R}^3$ and $v, w \in \mathbf{R}^3_p$. Show that $(dx_p \wedge dy_p)(v,w) = dz_p(v \times w)$. (Here, "$\times$" refers to the cross product of vectors.) Express $(dx_p \wedge dz_p)(v,w)$ in terms of $dy_p(v \times w)$.

8. Suppose that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Prove that
$$(dx_{i_1} \wedge \cdots \wedge dx_{i_k})_p(e_{j_1},\dots,e_{j_k})_p = \begin{cases} 1 & \text{if } i_1 = j_1, i_2 = j_2, \dots, i_k = j_k \\ 0 & \text{otherwise} \end{cases}$$

9. Let $\varphi : \mathbf{R}^n_p \times \mathbf{R}^n_p \to \mathbf{R}$ be multi-linear. Prove that $\varphi \in \Lambda^2(\mathbf{R}^n_p)$ if and only if $\varphi(v,v) = 0$ for all $v \in \mathbf{R}^n_p$.

Math 550
Homework 2
Dr. Fuller
Solutions

1. Change variables into cylindrical coordinates, using $g(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$. This gives $\int_A f = \int_{-1}^{1}\int_0^{2\pi}\int_0^1 (r^2z^2)(r)\,dr\,d\theta\,dz = \frac{\pi}{3}$.

6. (a) 5 (b) 0 (c) 18

7. If we write $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$, then both expressions evaluate to $v_1w_2 - v_2w_1$. Similarly, $(dx_p \wedge dz_p)(v,w) = -dy_p(v \times w)$.

9. If φ is alternating, then for all $\vec{v} \in V$ we have $\varphi(\vec{v},\vec{v}) = -\varphi(\vec{v},\vec{v})$, so $\varphi(\vec{v},\vec{v}) = 0$.

   Conversely, if $\varphi(\vec{v},\vec{v}) = 0$ for all $\vec{v} \in V$, then for any $\vec{v},\vec{w} \in V$ we have

$$0 = \varphi(\vec{v}+\vec{w},\vec{v}+\vec{w}) = \varphi(\vec{v},\vec{v}) + \varphi(\vec{v},\vec{w}) + \varphi(\vec{w},\vec{v}) + \varphi(\vec{w},\vec{w}) = \varphi(\vec{v},\vec{w}) + \varphi(\vec{w},\vec{v}),$$

   which implies $\varphi(\vec{v},\vec{w}) = -\varphi(\vec{w},\vec{v})$.

Math 550
Homework 3
Dr. Fuller
Due September 18

1. Suppose that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Prove that

$$(dx_{i_1} \wedge \cdots \wedge dx_{i_k})_p(e_{j_1}, \ldots, e_{j_k})_p = \begin{cases} 1 & \text{if } i_1 = j_1, i_2 = j_2, \ldots, i_k = j_k \\ 0 & \text{otherwise} \end{cases}$$

2. Let $u = (1,2,3), v = (-4,-5,-6), w = (0,0,-2)$.
   (a) Let $\omega \in \Omega^1(\mathbf{R}^3)$ be $\omega(x,y,z) = (y+z)\ dx$. Calculate $\omega(u)(v)_u$ and $\omega(v)(u)_v$.
   (b) Let $\omega \in \Omega^2(\mathbf{R}^3)$ be $\omega(x,y,z) = z\ dx \wedge dy + e^x\ dy \wedge dz$. Compute $\omega(w)(u,v)_w$.

3. Let $V(x,y,z) = 2y(e_1) - z(e_3)$ and $W(x,y,z) = z(e_1) - (e_2) + xy(e_3)$ be vector fields on $\mathbf{R}^3$. Let $\nu(x,y,z) = (y+z)\ dx$ and $\omega(x,y,z) = x^2y\ dx \wedge dy - xz\ dy \wedge dz$ be forms on $\mathbf{R}^3$.
   (a) Evaluate $\nu(1,2,3)(V(1,2,3))$
   (b) Evaluate $\omega(1,2,3)(V(1,2,3),W(1,2,3))$
   (c) The evaluations $\nu(V)$ and $\omega(V,W)$ each describe a function $\mathbf{R}^3 \to \mathbf{R}$. Find those functions.

4. Simplify the following differential forms.
   (a) $(a_1\ dx + a_2\ dy) \wedge (b_1\ dx + b_2\ dy)$ $(a_1, a_2, b_1, b_2$ are constants.)
   (b) $(x\ dx - y\ dy) \wedge (z\ dx \wedge dy + x^2\ dy \wedge dz)$
   (c) $(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4)$

5. (a) Let $\omega \in \Omega^k(\mathbf{R}^n)$, with k odd and $2k \leq n$. Show that $\omega \wedge \omega = 0$.
   (b) Show by example that the conclusion in part (a) is false if k is even.

6. (a) For all $p \in \mathbf{R}^n$, we can define a function $\mathbf{R}^n_p \to (\mathbf{R}^n_p)^*$ that sends any $v \in \mathbf{R}^n_p$ to the linear functional $T(w) = \langle v, w \rangle$. (Here "$\langle , \rangle$" denotes the usual inner product on $\mathbf{R}^n$.) Prove that this is an isomorphism between the vector spaces $\mathbf{R}^n_p$ and $(\mathbf{R}^n_p)^*$.
   (b) Let $X(p) = f_1(p)(e_1)_p + \ldots + f_n(p)(e_n)_p$ be a vector field on $\mathbf{R}^n$. By applying the isomorphism in part (a) at each point p, we get a 1-form ω_X on $\mathbf{R}^n$. Give a formula for ω_X.

Math 550
Homework 3
Dr. Fuller
Solutions

1. Let A be the matrix whose entry in row r, column c is $dx_{i_r}(e_{j_c})$.

   If $i_1 = j_1, i_2 = j_2, \ldots, i_k = j_k$, then $A = I$, and $\det A = 1$.

   Otherwise, suppose m is the first index with $i_m \neq j_m$ (so $i_1 = j_1, i_2 = j_2, \ldots, i_{m-1} = j_{m-1}$). If $i_m < j_m$, then $i_m \neq j_\ell$ for all $1 \leq \ell \leq k$, so $dx_{i_m}(e_{j_\ell}) = 0$ for all ℓ. Thus the i_m row of A is all zeroes, and $\det A = 0$. If $i_m > j_m$, then $j_m \neq i_\ell$ for all $1 \leq \ell \leq k$, so $dx_{i_\ell}(e_{j_m}) = 0$ for all ℓ. Thus the j_m column of A is all zeroes, and $\det A = 0$.

2. (a) -20 and -11.

   (b) -3

3. (a) 20

   (b) 1

   (c) $\nu(V)$ describes the function $(x,y,z) \mapsto 2y^2 + 2yz$
   $\omega(V,W)$ describes the function $(x,y,z) \mapsto 2x^2y^2 + xz^2$.

4. (a) $a_1b_2 - a_2b_1 \; dx \wedge dy$

   (b) $x^3 \; dx \wedge dy \wedge dz$

   (c) $2 \; dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$

5. (a) From Proposition 5, we have $\omega \wedge \omega = (-1)^{k^2} \omega \wedge \omega = -\omega \wedge \omega$. This implies $\omega \wedge \omega = 0$.

   (b) See 4 (c)

6. (a) It is routine to confirm that the function $\varphi : \mathbf{R}^n_p \to (\mathbf{R}^n_p)^*$ given by $\varphi(v) = T_v$, where $T_v(w) = \langle v, w \rangle$ is a linear function. If $v \in \ker \varphi$, then $\langle v, w \rangle = 0$ for all $w \in \mathbf{R}^n_p$, which implies $v = 0$. Thus φ is one-to-one. Since both $\mathbf{R}^n_p$ and $(\mathbf{R}^n_p)^*$ have dimension n, φ is an isomorphism.

   (b) $\omega_X = f_1 dx_1 \wedge \cdots \wedge f_n dx_n$

Math 550
Homework 4
Dr. Fuller
Due September 25

1. For each of the following, calculate the pullback $f^*\omega$ and simplify your answer as much as possible.

   (a) $f : \mathbf{R}^2 \to \mathbf{R}^3, f(u,v) = (\cos u, \sin u, v), \omega = z\, dx \wedge dy + y\, dz \wedge dx$

   (b) $f : \mathbf{R}^2 \to \mathbf{R}^2, f(r,\theta) = (r\cos\theta, r\sin\theta), \omega = -\dfrac{y}{x^2+y^2}\, dx + \dfrac{x}{x^2+y^2}\, dy$
   (ω is only defined on $\mathbf{R}^2 - \{(0,0)\}$.)

2. Let $g : \mathbf{R}^n \to \mathbf{R}^n$ be differentiable. Prove that $g^*(dx_1 \wedge \cdots \wedge dx_n) = \det Dg\, dx_1 \wedge \cdots \wedge dx_n$. (Hint: It enough to just check this on the standard basis $e_1, \ldots, e_n$.)

3. Let S denote the top half of the unit sphere in $\mathbf{R}^3$. Let $\omega = z^2\, dx \wedge dy$. Calculate $\int_S \omega$ using the parameterization $g(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ with $0 < \theta < 2\pi$, and $0 < \varphi < \frac{\pi}{2}$.

4. Let S be the surface in $\mathbf{R}^3$ parameterized by $g(\theta,z) = (\cos\theta, \sin\theta, z)$. where $0 < \theta < \pi$, and $0 < z < 1$. Let $\omega = xyz\, dy \wedge dz$. Calculate $\int_S \omega$.

5. Calculate the differential of each of the following.

   (a) $\omega = e^{xy}\, dx$

   (b) $\omega = x_1 x_2\, dx_3 \wedge dx_4$

   (c) $\omega = f(x,y)\, dx + g(x,y)\, dy$

   (d) $\omega = f(x,y,z)\, dy \wedge dz - g(x,y,z)\, dx \wedge dz + h(x,y,z)\, dx \wedge dy$

6. Determine if the following 2-forms are exact.

   (a) $\omega = x\, dx \wedge dy$

   (b) $\omega = z\, dx \wedge dy$

   (c) $\omega = z\, dx \wedge dy + y\, dx \wedge dz + z\, dy \wedge dz$

7. (a) Let $\alpha \in \Omega^1(\mathbf{R}^3)$ satisfy $\alpha(p) \neq 0$ for all $p \in \mathbf{R}^3$. Prove that $\ker\alpha$ is a 2-dimensional subspace (i.e. a plane) of $\mathbf{R}^3_p$ for all $p \in \mathbf{R}^3$.

   (b) Let $\alpha_1 = dz$. Sketch the planes described in part (a).

   (c) Let $\alpha_2 = x\, dy + dz$. Sketch the planes described in part (a).

   (d) Show that $\alpha_1 \wedge d\alpha_1 = 0$ and $\alpha_2 \wedge d\alpha_2 \neq 0$ (at all $p \in \mathbf{R}^3$).

Math 550
Homework 4
Dr. Fuller
Solutions

1. (a) $\sin^2 u\, du \wedge dv$

   (b) $d\theta$

2. We need to show that $g^*(dx_1 \wedge \cdots \wedge dx_n)_p(e_1, \ldots, e_n)_p = \det Dg(p)(dx_1 \wedge \cdots \wedge dx_n)_p(e_1, \ldots, e_n)_p$ for all $p \in \mathbf{R}^n$.

   Here are two ways to show that.

   Solution 1.

$$
\begin{aligned}
g^*(dx_1 \wedge \cdots \wedge dx_n)_p(e_1, \ldots, e_n)_p &= (dx_1 \wedge \cdots \wedge dx_n)_{g(p)}(Dg(p)(e_1), \ldots, Dg(p)(e_n))_{g(p)} \\
&= \det \begin{pmatrix} | & | & & | \\ Dg(p)(e_1) & Dg(p)(e_2) & \cdots & Dg(p)(e_n) \\ | & | & & | \end{pmatrix} \\
&= \det Dg(p) \\
&= \det Dg(p)(dx_1 \wedge \cdots \wedge dx_n)_p(e_1, \ldots, e_n)_p
\end{aligned}
$$

   The next-to-last equality follows from recognizing the matrix representation of $Dg(p)$ with respect to the standard basis.

   Solution 2.

$$
\begin{aligned}
g^*(dx_1 \wedge \cdots \wedge dx_n)_p(e_1, \ldots, e_n)_p &= (dx_1 \wedge \cdots \wedge dx_n)_{g(p)}(Dg(p)(e_1), \ldots, Dg(p)(e_n))_{g(p)} \\
&= (dx_1 \wedge \cdots \wedge dx_n)_{g(p)}(\textstyle\sum_i \frac{\partial g_i}{\partial x_1}(p)e_i, \ldots, \sum_i \frac{\partial g_i}{\partial x_n}(p)e_i)_{g(p)} \\
&= (\textstyle\sum_\sigma (-1)^{\text{sign } \sigma} \frac{\partial g_{\sigma(1)}}{\partial x_1}(p) \cdots \frac{\partial g_{\sigma(n)}}{\partial x_n}(p))\ (dx_1 \wedge \cdots \wedge dx_n)_{g(p)}(e_1, \ldots, e_n)_{g(p)} \\
&= \det \left[\frac{\partial g_i}{\partial x_j}(p) \right]\ (dx_1 \wedge \cdots \wedge dx_n)_{g(p)}(e_1, \ldots, e_n)_{g(p)} \\
&= \det Dg(p)(dx_1 \wedge \cdots \wedge dx_n)_p(e_1, \ldots, e_n)_p
\end{aligned}
$$

3. $-\frac{\pi}{2}$

4. $\frac{1}{3}$

Math 550
Homework 5
Dr. Fuller
Due October 2

1. Calculate the differential of each of the following.
   (a) $\omega = e^{xy}\,dx$
   (b) $\omega = x_1x_2\,dx_3 \wedge dx_4$
   (c) $\omega = f(x,y)\,dx + g(x,y)\,dy$
   (d) $\omega = f(x,y,z)\,dy \wedge dz - g(x,y,z)\,dx \wedge dz + h(x,y,z)\,dx \wedge dy$
2. Determine if the following 2-forms are exact.
   (a) $\omega = x\,dx \wedge dy$
   (b) $\omega = z\,dx \wedge dy$
   (c) $\omega = z\,dx \wedge dy + y\,dx \wedge dz + z\,dy \wedge dz$
3. (a) Let $\alpha \in \Omega^1(\mathbf{R}^3)$ satisfy $\alpha(p) \neq 0$ for all $p \in \mathbf{R}^3$. Prove that $\ker\alpha$ is a 2-dimensional subspace (i.e. a plane) of $\mathbf{R}^3_p$ for all $p \in \mathbf{R}^3$.
   (b) Let $\alpha_1 = dz$. Sketch the planes described in part (a).
   (c) Let $\alpha_2 = x\,dy + dz$. Sketch the planes described in part (a).
   (d) Show that $\alpha_1 \wedge d\alpha_1 = 0$ and $\alpha_2 \wedge d\alpha_2 \neq 0$ (at all $p \in \mathbf{R}^3$).
4. Prove that if $\omega \in \Omega^k(\mathbf{R}^n)$ is exact and $\varphi \in \Omega^\ell(\mathbf{R}^n)$ is closed, then $\omega \wedge \varphi$ is exact.
5. Show that the image of the curve $c(t) = (\cos 2t \cos t, \cos 2t \sin t)$ for $t \in (-\pi/2, \pi/4)$ is not a 1-dimensional manifold.
6. Let $f : \mathbf{R}^3 \to \mathbf{R}$ be defined by $f(x,y,z) = x^2 + y^2 - z^2$.
   (a) For which values of a is $f^{-1}(a)$ a manifold?
   (b) Find two different values a and b so that the manifolds $f^{-1}(a)$ and $f^{-1}(b)$ are not homeomorphic, and prove that they are not homeomorphic.
7. Let S^2 denote the unit sphere in $\mathbf{R}^3$. Give a basis for the tangent space S^2_p at any $p \in S^2$.
8. Prove that the unit sphere S^{n-1} is $\mathbf{R}^n$ cannot be parameterized as a manifold by a single parameterization. Can you generalize your proof into a more general result?
9. Let V be a k-dimensional vector subspace of $\mathbf{R}^n$.
   (a) Prove that V is a k-dimensional manifold in $\mathbf{R}^n$.
   (b) Let V_p denote the tangent space to V at $p \in V$. Prove that $V_p = V$.

Math 550
Homework 5
Dr. Fuller
Solutions

1. (a) $-xe^{xy}\,dx\wedge dy$
   (b) $x_2\,dx_1\wedge dx_3\wedge dx_4 + x_1\,dx_2\wedge dx_3\wedge dx_4$
   (c) $(-\frac{\partial f}{\partial y}+\frac{\partial g}{\partial x})\,dx\wedge dy$
   (d) $(\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z})\,dx\wedge dy\wedge dz$

2. (a) Yes. For instance, $\omega = d(\frac{x^2}{2}dy)$.
   (b) No, as ω is not closed.
   (c) Yes. For instance, $\omega = d(-yz\,dx - \frac{z^2}{2}\,dy)$.

3. (a) For all p, $\alpha(p)$ is a linear transformation $\mathbf{R}^3_p \to \mathbf{R}$. Since $\alpha(p)\neq 0$, we have dim $\operatorname{im}\alpha(p)=1$. The rank-nullity theorem implies dim $\ker\alpha(p)=2$.
   (b) $\ker\alpha_1(x,y,z)$ has (e_1,e_2) as a basis. The kernel at each point is simply the (xy)-plane.
   (c) $\ker\alpha_2(x,y,z)$ has $((1,0,0),(0,1,-x))$ as a basis. How to sketch these planes will be discussed in class.
   (d) $\alpha_1\wedge d\alpha_1 = dz\wedge d(dz) = dz\wedge 0 = 0$. $\alpha_2\wedge d\alpha_2 = dx\wedge dy\wedge dz \neq 0$.

4. Since ω is exact, there is $\eta\in\Omega^{k-1}(\mathbf{R}^n)$ with $d\eta=\omega$. Then

$$d(\eta\wedge\varphi) = d\eta\wedge\varphi + (-1)^{k-1}\eta\wedge d\varphi = \omega\wedge\varphi + (-1)^{k-1}\eta\wedge 0 = \omega\wedge\varphi.$$

5. Suppose $M = c(((-\pi/2,\pi/4))$, and let $g:(a,b)\to\mathbf{R}^2$ be a local parameterization with $(0,0)\in g((a,b))$. Consider the open set $V = M\cap B((0,0),\frac{1}{4})$ in M; it is connected, and so $g^{-1}(V)$ is an interval in (a,b). But then $V-\{(0,0)\}$ has three connected components, while $g^{-1}(V-\{(0,0)\}$ will have only two. This contradicts that g^{-1} is continuous.

6. (a) When $a\neq 0$, $f^{-1}(a)$ is a hyperboloid, which is a manifold. When $a=0$, it is not a manifold. In this case, one may argue using connectedness as in the previous problem that there can be no local parameterization around $(0,0,0)\in f^{-1}(0)$.
   (b) When $a>0$, $f^{-1}(a)$ is a hyperboloid of one sheet, which is connected, but when $a<0$, $f^{-1}(a)$ is a hyperboloid of two sheets, which is not.

8. Suppose $g: U\subset\mathbf{R}^{n-1}\to\mathbf{R}^n$ is a parameterization with $g(U)=S^{n-1}$. Since S^{n-1} is compact, we have that $g^{-1}(S^{n-1})$ is a non-empty compact open subset of $\mathbf{R}^{n-1}$. Since $\mathbf{R}^{n-1}$ is connected, this is impossible.

   The argument shows that any compact k-dimensional manifold in $\mathbf{R}^n$ cannot be parameterized by a single parameterization.

Math 550
Homework 6
Dr. Fuller
Due October 9

1. Let S^2 denote the unit sphere in $\mathbf{R}^3$. Give a basis for the tangent space S^2_p at any $p \in S^2$.

2. Let V be a k-dimensional vector subspace of $\mathbf{R}^n$.

   (a) Prove that V is a k-dimensional manifold in $\mathbf{R}^n$.

   (b) Let V_p denote the tangent space to V at $p \in V$. Prove that $V_p = V$.

3. Suppose that M is a k-dimensional manifold in $\mathbf{R}^n$. Prove that the *tangent bundle*

$$TM = \{(p, v) \in M \times \mathbf{R}^n : v \in M_p\}$$

   is a $2k$-dimensional manifold in $\mathbf{R}^{2n}$.

4. Let M be a k-dimensional manifold-with-boundary. Prove that ∂M is a $(k-1)$-dimensional manifold.

5. Let $f : U \to f(U)$ and $g : V \to g(V)$ be two parameterizations of $S = f(U) = g(V)$ in $\mathbf{R}^n$, where $U, V \subset \mathbf{R}^k$. Let $\omega \in \Omega^k(S)$ be any k-form which is non-zero at $x \in S$. Prove that f and g induce the same orientation on S_x if and only if $f^*\omega(e_1, \ldots, e_k)$ and $g^*\omega(e_1, \ldots, e_k)$ have the same sign. (Hint: Recall Problem 2 from Homework 4.)

6. Let $f(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ with $0 < \theta < 2\pi$ and $0 < \varphi < \pi$. Let $g(u, v) = (u, \sqrt{1 - u^2 - v^2}, v)$ for $\{(u, v) : u^2 + v^2 < 1\}$. Do f and g induce the same orientation on $\{(x, y, z) \in S^2 : y > 0\}$? (Hint: Regard the previous problem as a criterion to compare orientations. You pick the form ω and the point of evaluation.)

7. The manifold $\partial\mathbf{H}^k$ can be oriented as the boundary of $\mathbf{H}^k$ with the usual orientation. It can also be oriented using the usual orientation of $\mathbf{R}^{k-1}$ (using the obvious identification of $\partial\mathbf{H}^k$ with $\mathbf{R}^{k-1}$). Prove that these orientations agree if and only if k is even.

Math 550
Homework 6, Addendum
Dr. Fuller

These problems will not be collected.

1. The manifold $\partial \mathbf{H}^k$ can be oriented as the boundary of $\mathbf{H}^k$ with the usual orientation. It can also be oriented using the usual orientation of $\mathbf{R}^{k-1}$ (using the obvious identification of $\partial \mathbf{H}^k$ with $\mathbf{R}^{k-1}$). Prove that these orientations agree if and only if k is even.

2. Suppose that M is an n-dimensional manifold-with-boundary in $\mathbf{R}^n$ with non-empty boundary, so that ∂M is an $(n-1)$-dimensional manifold. Assume that M is oriented with the usual orientation in $\mathbf{R}^n$. Prove that the vectors n_x and N_x (as defined in class) agree for $x \in \partial M$.

Math 550
Homework 6
Dr. Fuller
Solutions

1. Using spherical coordinates $g(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$, the tangent space at $p = g(\theta,\varphi)$ has basis
$$(\,(-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0), (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi)\,).$$

2. (a) Let $(v_1,\ldots,v_k)$ be a basis for V. Define $g : \mathbf{R}^k \to \mathbf{R}^n$ to be the linear transformation given by $g(e_i) = v_i$. Then
   (i) $g(\mathbf{R}^k) = V = V \cap \mathbf{R}^n$;
   (ii) rank $Dg(u) =$ rank $g = k$ for all $u \in \mathbf{R}^k$;
   (iii) $g^{-1} : V \to \mathbf{R}^k$ is the linear transformation given by $g^{-1}(v_i) = e_i$, which is continuous.

   This shows that g parameterizes V as a k-dimensional manifold.

   (b) If $p = g(u)$, then $V_p = Dg(u)(\mathbf{R}^k) = g(\mathbf{R}^k) = V$.

3. Let $(p, v_p) \in TM$. Since $p \in M$, we have open sets U in $\mathbf{R}^k$ and $W \in \mathbf{R}^n$, and a local parameterization $g : U \to \mathbf{R}^n$ with
   (i) $g(U) = M \cap W$;
   (ii) rank $Dg(u) = k$ for all $u \in U$;
   (iii) $g^{-1} : g(U) \to U$ continuous.

   Define $G : U \times \mathbf{R}^k \to \mathbf{R}^n \times \mathbf{R}^n$ by $G(u,v) = (g(u), Dg(u)(v))$. We have
   (i) $g(U \times \mathbf{R}^k) = TM \cap (W \times \mathbf{R}^n)$;
   (ii) Suppose that A is the matrix which represents $Dg(u)$ with respect to the standard basis, so A has rank k. Then $DG(u,v)$ is represented in the standard basis by a $2k \times 2k$-matrix whose upper left and lower right $k \times k$ submatrices are both A. This implies that rank $DG(u,v) = 2k$.
   (iii) If $(q, v_q) \in TM$, so that $v_q \in M_q$, then $G^{-1}(q, v_q) = (g^{-1}(q), Dg^{-1}(q)(v_q))$. This shows that G^{-1} continuous.

   This confirms that G is a local parameterization around $(p, v_p) \in TM$.

4. Let $x \in \partial M$, and suppose $g : U \subset \mathbf{H}^k \to \mathbf{R}^n$ is a local parameterization, with $g(u) = x$. This means we have
   (i) $g(U) = M \cap W$ for some open set W in $\mathbf{R}^n$;
   (ii) rank $Dg(u) = k$ for all $u \in U$;
   (iii) $g^{-1} : g(U) \to U$ continuous.

   Let $i : \mathbf{R}^{k-1} \to \mathbf{R}^k$ be the inclusion map $i(u_1,\ldots,u_{k-1}) = (u_1,\ldots,u_{k-1},0)$. Let $V = i^{-1}(U)$. Then:
   (i) $(g \circ i)(V) = \partial M \cap W$;
   (ii) rank $D(g \circ i)(u) = k-1$ for all $u \in V$, since rank $D(g)(i(u)) = k$ and rank $D(i)(u) = k-1$;
   (iii) $(g \circ i)^{-1} = i^{-1} \circ g^{-1}$ continuous, as it is the composition of continuous functions.

   Thus $g \circ i : V \to \mathbf{R}^n$ defines a local parameterization around $x \in \partial M$, showing that ∂M is a $(k-1)$-dimensional manifold.

5. In class, it was shown that f and g induce the same orientation on S_x if and only if $\det(Dg^{-1} \circ Df)(u) > 0$. Now write $g^*\omega = h\, dx_1 \wedge \cdots \wedge dx_k$. Then

$$\begin{aligned} f^*\omega &= (g \circ g^{-1} \circ f)^*\omega \\ &= (g^{-1} \circ f)^* g^*\omega \\ &= (g^{-1} \circ f)^*(h\, dx_1 \wedge \cdots \wedge dx_k) \\ &= (g^{-1} \circ f)^* h\, (g^{-1} \circ f)^* dx_1 \wedge \cdots \wedge dx_k \\ &= (g^{-1} \circ f)^* h\, \det(Dg^{-1} \circ Df)\, dx_1 \wedge \cdots \wedge dx_k \\ &= \det(Dg^{-1} \circ Df)\, (h \circ g^{-1} \circ f) dx_1 \wedge \cdots \wedge dx_k \end{aligned}$$

Evaluating at u and v, where $f(u) = g(v) = x$, we get

$$\begin{aligned} f^*\omega(u)(e_1, \ldots, e_k) &= \det(Dg^{-1} \circ Df)(u)\, (h \circ g^{-1} \circ f)(u)(dx_1 \wedge \cdots \wedge dx_k)(e_1, \ldots, e_k) \\ &= \det(Dg^{-1} \circ Df)(u)\, h(v)(dx_1 \wedge \cdots \wedge dx_k)(e_1, \ldots, e_k) \\ &= \det(Dg^{-1} \circ Df)(u)\, g^*\omega(v)(e_1, \ldots, e_k). \end{aligned}$$

Thus $f^*\omega(e_1, \ldots, e_k)$ and $g^*\omega(e_1, \ldots, e_k)$ have the same sign if and only if $\det(Dg^{-1} \circ Df)(u) > 0$.

6. Yes, they induce the same orientation.

Addendum

1. Suppose $(v_1, \ldots, v_{k-1})$ is a basis for the tangent space at a point in $\partial \mathbf{H}^k$.

   Observe:

   - $(v_1, \ldots, v_{k-1})$ is a positively oriented basis for $\mathbf{R}^{k-1}$ if and only if $(v_1, \ldots, v_{k-1}, e_k)$ is a positively oriented basis for $\mathbf{R}^k$.
   - $(v_1, \ldots, v_{k-1})$ is a positively oriented basis for $\partial \mathbf{H}^k$ if and only if $(-e_k, v_1, \ldots, v_{k-1})$ is a positively oriented basis of $\mathbf{R}^k$.
   - The orientations of $(-e_k, v_1, \ldots, v_{k-1})$ and $(v_1, \ldots, v_{k-1}, e_k)$ agree if and only if k is even, since we can equate them by $k-1$ transpositions, and changing the sign of e_k.

   Combining these observations proves the statement.

Math 550
Homework 7
Dr. Fuller
Due October 25

1. Let M be a k-dimensional manifold in $\mathbf{R}^n$. Prove that if there exists a nowhere zero k-form on M, then M is orientable. (Hint: recall Homework 6, problem 5.)

2. There is a general correspondence between k-forms and $(n-k)$-forms on $\mathbf{R}^n$, for all $1 \leq k \leq n$. Given $\omega \in \Omega^k(\mathbf{R}^n)$, we define $\star\omega \in \Omega^{n-k}(\mathbf{R}^n)$ using the rule

$$\star(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \pm\, dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}},$$

and extending linearly, where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k}$, and $\{i_1, \cdots, i_k, j_1, \cdots, j_{n-k}\} = \{1, \cdots, n\}$. The sign is chosen so that $\omega \wedge \star\omega = dx_1 \wedge \cdots \wedge dx_n$. (For example, in $\mathbf{R}^5$, $\star(dx_1 \wedge dx_4) = dx_2 \wedge dx_3 \wedge dx_5$ and $\star(dx_1 \wedge dx_3) = -dx_2 \wedge dx_4 \wedge dx_5$.)

Prove that $\star\star\,\omega = (-1)^{k(n-k)}\omega$.

Math 550
Homework 7
Dr. Fuller
Solutions

1. Let ω be the given non-zero k-form on M. Suppose that $g_\alpha : U_\alpha \to M$ is a collection of local parameterizations covering M. Let $i : \mathbf{R}^k \to \mathbf{R}^k$ be $i(x_1, x_2, x_3, \ldots, x_n) = (x_2, x_1, x_3, \ldots, x_n)$. Define a collection of local parameterizations $h_\alpha : V_\alpha \to M$ by the rule

$$h_\alpha = \begin{cases} g_\alpha & \text{if } g_\alpha^* \omega(u)(e_1, \ldots, e_k) > 0 \text{ for } u \in U_\alpha \\ g_\alpha \circ i & \text{if } g_\alpha^* \omega(u)(e_1, \ldots, e_k) < 0 \text{ for } u \in U_\alpha; \end{cases}$$

   We also set $V_\alpha = U_\alpha$ in the upper case, and $V_\alpha = i(U_\alpha)$ in the lower. Then for all α, we have $h_\alpha^* \omega(v)(e_1, \ldots, e_k) > 0$ for $v \in V_\alpha$. This shows that M is orientable.

2. It suffices to verify the formula for $\omega = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. From the definition, it is clear that $\star\star\,\omega = \varepsilon\omega$, for $\varepsilon \in \{-1, 1\}$. We must verify that $\varepsilon = (-1)^{k(n-k)}$.

   We have that
$$\omega \wedge \star\omega = \star\omega \wedge \star\star\,\omega = (-1)^{k(n-k)} \star\star\omega \wedge \star\omega.$$

   The first equality follows because both are equal to $dx_1 \wedge \cdots \wedge dx_n$. Then

$$0 = (\omega - (-1)^{k(n-k)} \star\star\omega) \wedge \star\omega = (1 - (-1)^{k(n-k)}\varepsilon)\ \omega \wedge \star\omega = (1 - (-1)^{k(n-k)}\varepsilon)\ dx_1 \wedge \cdots \wedge dx_n.$$

   Thus $1 - (-1)^{k(n-k)}\varepsilon = 0$, which implies $\varepsilon = (-1)^{k(n-k)}$.

Math 550
Homework 8
Dr. Fuller
Due October 30

1. Let $M \subset \mathbf{R}^3$ be the manifold bounded above by the sphere $x^2+y^2+z^2=a^2$ and below by $z=0$. Let

$$\omega = xz\, dy \wedge dz + yz\, dz \wedge dx + (x^2+y^2+z^2)\, dx \wedge dy.$$

   Compute $\int_{\partial M} \omega$ both directly and using Stokes' Theorem. (Answer: πa^4.)

2. Let C be the intersection of the sphere $x^2+y^2+z^2=1$ and the plane $x+y+z=0$, oriented counterclockwise as viewed from above the xy-plane. Use Stokes' Theorem to evaluate

$$\int_C z^3\, dx.$$

3. Show that

$$\omega = \frac{x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

   on $\mathbf{R}^3 - \{(0,0,0)\}$ is closed but not exact.

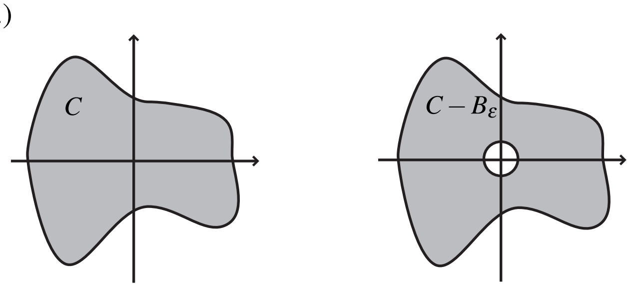
4. Show that Stokes' Theorem is false if M is not compact.

5. Let M be a compact k-manifold without boundary. Show that $\int_M d\omega = 0$ for all $\omega \in \Omega^{k-1}(M)$. Give a counterexample if M is not compact.

6. Suppose that C is a compact 2-dimensional manifold-with-boundary in $\mathbf{R}^2$, and assume $(0,0) \notin \partial C$. Let $\omega = \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$. Prove that

$$\int_{\partial C} \omega = \begin{cases} 0 & \text{if } (0,0) \notin C, \\ 2\pi & \text{if } (0,0) \in C. \end{cases}$$

   (Hint: If $(0,0) \in C$, then ω is not defined on C, so consider $C - B_\varepsilon$, where B_ε is an open ball centered at the origin with $B_\varepsilon \subset C$.)



Math 550
Homework 8
Dr. Fuller
Solutions

2. To apply Stokes', note that C is the boundary of a disk D in the plane $x+y+z=0$, and we compute $\int_D d(z^3dx) = -\int_D 3z^2\, dx\wedge dz = \frac{\pi}{2\sqrt{3}}$.

   (To compute the integral, the challenge is to parameterize D. This can be done by finding an explicit orthonormal basis $\{\vec{u},\vec{v}\}$ for the vector space $x+y+z=0$, and defining $g(r,\theta) = (r\cos\theta)\vec{u} + (r\sin\theta)\vec{v}$ for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.)

3. Direct calculation gives $d\omega = 0$. But $\int_{S^2} \omega|_{S^2} = \pm 4\pi$ (depending on a choice of orientation of S^2), so by the Corollary to Stokes' Theorem, ω is not closed.

4. Examples abound. For instance, if M is the open upper hemisphere of S^2 parameterized and oriented by $g(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$ with $0 < \theta < 2\pi$, and $0 < \varphi < \frac{\pi}{2}$, then $\int_M d(x\,dy) = \int_M dx\wedge dy = \frac{1}{2}$.

   However, $\partial M = \emptyset$, so $\int_{\partial M} x\,dy = 0$.

5. This follows immediately from Stokes' Theorem: $\int_M d\omega = \int_{\partial M}\omega = 0$.

   The counterexample given in the previous problem also works here, with $\omega = x\,dy$.

6. Direct calculation gives $d\omega = 0$.

   If $(0,0) \notin C$, then ω is defined on C, and we may use Stokes' Theorem to get $\int_{\partial C}\omega = \int_C d\omega = 0$.

   If $(0,0) \in C$, then $(0,0) \notin C - B_\varepsilon$, so we may use Stokes' Theorem on $C - B_\varepsilon$ to get

   $$0 = \int_{C-B_\varepsilon} d\omega = \int_{\partial(C-B_\varepsilon)} \omega = \int_{\partial C}\omega + \int_{\partial B_\varepsilon}\omega = \int_{\partial C}\omega - 2\pi.$$

   In the above, $\int_{\partial B_\varepsilon}\omega = -2\pi$ comes from direct calculation, keeping in mind that we must use the boundary orientation on ∂B_ε inherited from the standard orientation on C. This requires a clockwise orientation on ∂B_ε, and the use of a parameterization such as $g(\theta) = (\varepsilon\sin\theta, \varepsilon\cos\theta)$.

Math 550
Homework 9
Dr. Fuller
Due November 6

1. Professor Doofus gives the following "rule" for canceling differential forms $\omega \in \Omega^k(\mathbf{R}^n)$ and $\alpha, \beta \in \Omega^\ell(\mathbf{R}^n)$:

$$\text{If } \omega \wedge \alpha = \omega \wedge \beta, \text{ then } \alpha = \beta.$$

Give an example which shows that Doofus is mistaken.

2. Compute the volume of the unit ball in $\mathbf{R}^3$ by integrating an appropriate 2-form over the unit sphere in $\mathbf{R}^3$.

3. Suppose C is a 1-dimensional manifold in $\mathbf{R}^n$, oriented by a parameterization $c : [a,b] \to M$. Prove that $\int_{[a,b]} c^* ds = \int_a^b [\sum_{i=1}^n (c_i'(t))^2]^{\frac{1}{2}}\, dt$.

4. Let X be a vector field on $\mathbf{R}^3$, and let ω_X^1 and ω_X^2 denote the associated 1- and 2-forms, respectively. Let $f : \mathbf{R}^3 \to \mathbf{R}$ be a 0-form.
   (a) Show that: $df = \omega^1_{\text{grad } f}$, $d(\omega_X^1) = \omega^2_{\text{curl } X}$, and $d(\omega_X^2) = \text{div } X\, dx \wedge dy \wedge dz$.
   (b) Prove that curl grad $f = 0$ and div curl $X = 0$.

5. For a vector field $X = (f_x, f_y)$ on $\mathbf{R}^2$, we may define an associated 1-form, different from the one in class, by

$$\star\omega_X^1 = -f_y\, dx + f_x\, dy.$$

We may also define

$$\text{div } X = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}.$$

   (a) Let M be a compact 2-dimensional manifold with boundary in $\mathbf{R}^2$. Show that for all points $p \in \partial M$, the equation $\star\omega_X^1 = X \cdot n\, ds$ holds.
   (b) Prove the following *Divergence form of Green's Theorem:* Let M be a 2-dimensional manifold-with-boundary in $\mathbf{R}^2$, and let X be a vector field on M. Then

$$\int_M \text{div } X\, dA = \int_{\partial M} X \cdot n\, ds.$$

6. Let M be a compact 3-dimensional manifold-with-boundary in $\mathbf{R}^3$, with $(0,0,0) \in M - \partial M$. Consider the vector field $X(p) = \frac{p}{\|p\|^3}$ defined on $\mathbf{R}^3 - \{(0,0,0)\}$. Prove that

$$\int_{\partial M} X \cdot N\, dA = 4\pi.$$

Math 550
Homework 9
Dr. Fuller
Solutions

2. If B^3 denotes the unit ball in $\mathbf{R}^3$, then by Stokes' Theorem $\text{vol}(B^3) = \int_{B^3} dx \wedge dy \wedge dz = \int_{S^2} z\, dx \wedge dy$, where S^2 receives the orientation induced as the boundary of B^3. Parameterizing S^2 with spherical coordinates induces the opposite orientation, so we have

$$\int_{S^2} z\, dx \wedge dy = -\int_{(0,2\pi)\times(0,\pi)} g^*(z\, dx \wedge dy) = -\int_0^{\pi}\int_0^{2\pi} -\cos^2\varphi \sin\varphi\, d\theta\, d\varphi = \frac{4\pi}{3}.$$

3. The vector $c(t) = (c_1'(t), \ldots, c_n'(t))$ forms a positively oriented basis of $C_{c(t)}$, thus $\frac{1}{\|c'(t)\|}\,(c_1'(t), \ldots, c_n'(t))$ forms a positively oriented orthonormal basis there. So $1 = ds_{c(t)}(\frac{1}{\|c'(t)\|}\,(c_1'(t), \ldots, c_n'(t)))$, which implies that $ds_{c(t)}((c_1'(t), \ldots, c_n'(t))) = \|c'(t)\|$. So

$$(c^*ds)_t(1) = ds_{c(t)}(Dc_{c(t)}(1)) = ds_{c(t)}(c_1'(t), c_2'(t), \ldots, c_n'(t)) = \|c'(t)\| = \|c'(t)\|dt(1).$$

This shows that $c^*ds = \|c'(t)\|dt$, and the integral formula follows.

4. (a) Straightforward.
   (b) Combine $d^2 = 0$ with part (a):
      (i) $0 = d^2 f = d(\omega^1_{\text{grad } f}) = \omega^2_{\text{curl grad } f}$. So curl grad $f = 0$.
      (ii) $0 = d^2(\omega^1_X) = d(\omega^2_{\text{curl } X}) = \text{div curl } X\; dV$. So div curl $X = 0$.

Math 550
Homework 10
Dr. Fuller
Due November 13

1. For a vector field $X = (f_x, f_y)$ on $\mathbf{R}^2$, we may define an associated 1-form, different from the one in class, by
$$\star\omega_X^1 = -f_y\, dx + f_x\, dy.$$
We may also define
$$\text{div } X = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}.$$
   (a) Let M be a compact 2-dimensional manifold with boundary in $\mathbf{R}^2$. Show that for all points $p \in \partial M$, the equation $\star\omega_X^1 = X \cdot n\, ds$ holds.
   (b) Prove the following *Divergence form of Green's Theorem:* Let M be a compact 2-dimensional manifold-with-boundary in $\mathbf{R}^2$, and let X be a vector field on M. Then
$$\int_M \text{div } X\, dA = \int_{\partial M} X \cdot n\, ds.$$

2. Let M be a compact 3-dimensional manifold-with-boundary in $\mathbf{R}^3$, with $(0,0,0) \in M - \partial M$. Consider the vector field $X(p) = \frac{p}{\|p\|^3}$ defined on $\mathbf{R}^3 - \{(0,0,0)\}$. Prove that
$$\int_{\partial M} X \cdot N\, dA = 4\pi.$$

3. (a) Show that if X is a vector field on $\mathbf{R}^3$ with curl $X = 0$, then $X = \text{grad } f$ for some function $f : \mathbf{R}^3 \to \mathbf{R}$.
   (b) Show that if X is a vector field on $\mathbf{R}^3$ with div $X = 0$, then $X = \text{curl } Y$ for some vector field Y on $\mathbf{R}^3$.

4. Let $\omega = \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$ be a 1-form on $\mathbf{R}^2 - \{(0,0)\}$. Prove that ω does not extend to a 1-form on $\mathbf{R}^2$.

Math 550
Homework 10, Addendum
Dr. Fuller

These problems will not be collected.

1. Let M be a compact oriented k-dimensional manifold without boundary, with volume form ν. Prove that $\int_M \nu > 0$.
2. Let M be a compact oriented k-dimensional manifold without boundary. Prove that M is not contractible.

Math 550
Homework 10
Dr. Fuller
Solutions

1. (a) We may write $X(p) = w + (X_p \cdot n_p)n_p$, where $w \in \partial M_p$ and n_p is the unit outward normal at p. (Recall that in this case, the two different outward normal vectors n_p and N_p coincide.) Let $u = (u_1, u_2) \in \partial M_p$. Then

$$\begin{aligned}
(\star\omega_X^1)_p(u) &= (-f_y(p)\,dx + f_x(p)\,dy)(u_1,u_2) \\
&= \det\begin{pmatrix} f_x(p) & u_1 \\ f_y(p) & u_2 \end{pmatrix} \\
&= \det\begin{pmatrix} | & | \\ X_p & u \\ | & | \end{pmatrix} \\
&= \det\begin{pmatrix} | & | \\ w + (X_p \cdot n_p)n_p & u \\ | & | \end{pmatrix} \\
&= X_p \cdot n_p \det\begin{pmatrix} | & | \\ n_p & u \\ | & | \end{pmatrix} = X_p \cdot n_p\, ds(u).
\end{aligned}$$

(b) Direct calculation gives $d(\star\omega_X^1) = \text{div } X\, dA$. So

$$\int_M \text{div } X\, dA = \int_M d(\star\omega_X^1) = \int_{\partial M} \star\omega_X^1 = \int_{\partial M} X \cdot n\, ds.$$

2. Direct calculation gives $\text{div } X = 0$.

Since X is defined on M^3, we apply the Divergence Theorem to $M - B_\varepsilon^3$, where B_ε^3 is a small open ball of radius ε centered at $(0,0,0)$. Then

$$0 = \int_{M-B_\varepsilon^3} \text{div } X\, dV = \int_{\partial(M-B_\varepsilon^3)} X \cdot N\, dA = \int_{\partial M} X \cdot N\, dA + \int_{S_\varepsilon^2} X \cdot N\, dA = \int_{\partial M} X \cdot N\, dA - 4\pi.$$

To see that $\int_{S_\varepsilon^2} X \cdot N\, dA = -4\pi$, note that the boundary orientation induced on S_ε^2 from $M - B_\varepsilon^3$ is opposite the orientation induced from the standard orientation on B_ε^3. Also, if $p = (x,y,z) \in S_\varepsilon^2$, then $X(p) = \frac{1}{\varepsilon^3}(x,y,z)$ and $N(p) = \frac{1}{\varepsilon}(x,y,z)$, so $X(p) \cdot N(p) = \frac{1}{\varepsilon^4}(x,y,z)\cdot(x,y,z) = \frac{1}{\varepsilon^4}\varepsilon^2 = \frac{1}{\varepsilon^2}$. Thus

$$\int_{S_\varepsilon^2} X \cdot N\, dA = -\frac{1}{\varepsilon^2}\int_{S_\varepsilon^2} dA = -\frac{1}{\varepsilon^2} 4\pi\varepsilon^2 = -4\pi.$$

3. (a) $d(\omega_X^1) = \omega_{\text{curl } X}^2 = 0$. Then ω_X^1 is exact by the Poincare Lemma, so there exists f with $\omega_X^1 = df = \omega_{\text{grad } f}^1$. Thus $X = \text{grad } f$.

(b) $d(\omega_X^2) = \text{div } X = 0$. Then ω_X^2 is exact by the Poincare Lemma, so there exists a 1-form η with $\omega_X^2 = d\eta$. Note that if we write $\eta = f_x\, dx + f_y\, dy + f_z\, dz$, then $\eta = \omega_Y^1$ for the vector field $Y = (f_x, f_y, f_z)$. Then we have $\omega_X^2 = d\eta = d(\omega_Y^1) = \omega_{\text{curl } Y}^2$. Thus $X = \text{curl } Y$.

4. Suppose that ω extends to a 1-form $\widetilde{\omega}$ on $\mathbf{R}^2$. Then at $p \neq (0,0,0)$, we have $d\widetilde{\omega}(p) = d\omega(p) = 0$. Thus the coefficient functions of $d\widetilde{\omega}$ are identically zero on $\mathbf{R}^2 - \{(0,0)\}$, and by continuity on $\mathbf{R}^2$ as well. So $d\widetilde{\omega}((0,0,0)) = 0$. Thus $\widetilde{\omega}$ is closed. By the Poincare Lemma, $\widetilde{\omega}$ is exact. But this would mean that its restriction ω to $\mathbf{R}^2 - \{(0,0)\}$ is exact, a contradiction to the fact that ω has been shown to be otherwise.

Addendum

1. Let $g : U \to \mathbf{R}^n$ be a local parameterization of M which induces the given orientation. Then we can write $g^*\nu = f\ dx_1 \wedge \cdots \wedge dx_k$, and since $g^*\nu(u)(e_1,\ldots,e_k) > 0$ for all $u \in U$, we have

$$f(u) = f(u)\ dx_1 \wedge \cdots \wedge dx_k(e_1,\ldots,e_k) = g^*\nu(u)(e_1,\ldots,e_k) > 0$$

for all $u \in U$. Therefore,

$$\int_{g(U)} \nu = \int_U g^*\nu = \int_U f dx_1 \cdots dx_k > 0.$$

Finally, taking $\{\varphi\}$ to be a partition of unity subordinate to the cover of M by $\{g_\alpha(U_\alpha)\}$, this implies $\int_M \nu = \sum_\varphi \int_M \varphi\nu > 0$.

2. If M were contractible, then the identity $i : M \to M$ would be homotopic to a constant function $c : M \to M$. Then if ν is the volume form on M, we have $\int_M \nu = \int_M i^*\nu = \int_M c^*\nu = 0$. This contradicts the previous problem.

Math 550
Homework 11
Dr. Fuller
Due December 4, 2018

1. Suppose $g : U \subset \mathbf{R}^2 \to \mathbf{R}^3$ is a parameterization of an oriented surface $g(U)$ in $\mathbf{R}^3$. Prove
$$\int_U g^* dA = \int_U \|Dg(u,v)(e_1) \times Dg(u,v)(e_2)\| \, du \, dv.$$
(Remark: This problem is a companion to Homework 9, problem 3. Together, they show that the definitions of line integrals and surface integrals from elementary vector calculus can be viewed as integrals of volume forms on 1- and 2-dimensional manifolds, repectively.)

2. Let $f : S^{2k} \to S^{2k}$ be a C^∞ function. Prove that there exists $\vec{x} \in S^{2k}$ with either $f(\vec{x}) = \vec{x}$ or $f(\vec{x}) = -\vec{x}$.

3. Let $n \geq 2$, and suppose $f : D^n \to \mathbf{R}^n$ is C^∞, with $\|f(\vec{x}) - \vec{x}\| < 1$ for all $\vec{x} \in S^{n-1}$. Prove that there exists $\vec{x} \in D^n$ such that $f(\vec{x}) = 0$.

4. Prove that if M contractible, then M is simply connected.

5. Show that the converse of Exercise 4 is false.

6. (a) Suppose that ω_1 and ω_2 are cohomologous k-forms on a compact oriented k-dimensional manifold M. Prove that
$$\int_M \omega_1 = \int_M \omega_2.$$
   (b) Show that integration over M defines a linear functional
$$\int_M : H^k(M) \to \mathbf{R}.$$
   (c) Suppose that M bounds; that is, suppose M is the boundary of some compact oriented $(k+1)$-dimensional manifold. Show that $\int_M$ is zero.

7. (a) Prove that a closed n-form ω on S^n is exact if and only if $\int_{S^n} \omega = 0$.
   (b) Prove that the linear function $\int_{S^n} : H^n(S^n) \to \mathbf{R}$ is an isomorphism.

8. For $\ell > 0$, prove
$$H^\ell(\mathbf{R}^k - \{(-1,0,\ldots,0),(1,0,\ldots,0)\} \cong \begin{cases} \mathbf{R}^2 & \text{if } \ell = k-1, \\ 0 & \text{if } \ell \neq k-1. \end{cases}$$

Math 550
Homework 11
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Solutions

1. Let $\vec{x} = g(u,v)$. We know that $(Dg(u,v)(e_1), Dg(u,v)(e_2))$ is a positively oriented basis of the tangent space $g(U)_{\vec{x}}$. Thus the unit outward normal to $g(U)$ at $\vec{x}$ is $N_{\vec{x}} = \frac{Dg(u,v)(e_1)\times Dg(u,v)(e_2)}{\|Dg(u,v)(e_1)\times Dg(u,v)(e_2)\|}$. Then for all $(u,v)\in U$

$$\begin{aligned}
g^*dA(u,v)(e_1,e_2) &= dA(\vec{x})(Dg(u,v)(e_1), Dg(u,v)(e_2)) \\
&= \det\begin{pmatrix} | & | & | \\ N_{\vec{x}} & Dg(u,v)(e_1) & Dg(u,v)(e_2) \\ | & | & | \end{pmatrix} \\
&= \det\begin{pmatrix} | & | & | \\ \frac{Dg(u,v)(e_1)\times Dg(u,v)(e_2)}{\|Dg(u,v)(e_1)\times Dg(u,v)(e_2)\|} & Dg(u,v)(e_1) & Dg(u,v)(e_2) \\ | & | & | \end{pmatrix} \\
&= \|Dg(u,v)(e_1)\times Dg(u,v)(e_2)\| \\
&= \|Dg(u,v)(e_1)\times Dg(u,v)(e_2)\|\, du\wedge dv(e_1,e_2).
\end{aligned}$$

   This shows that $g^*dA = \|Dg(u,v)(e_1)\times Dg(u,v)(e_2)\|\, du\wedge dv$, and the integral formula follows.

2. For every $\vec{x}\in S^{2k}$, define $X(\vec{x}) = P(f(\vec{x}))$, where $P : \mathbf{R}^{2k+1}_{\vec{x}} \to S^{2k}_{\vec{x}}$ is the projection of $\mathbf{R}^{2k+1}_{\vec{x}}$ onto the subspace $S^{2k}_{\vec{x}}$. (In more detail, if we write $f(\vec{x}) = \vec{w} + \lambda N_{\vec{x}}$, for unique $\vec{w}\in S^{2k}_{\vec{x}}$ and $\lambda\in\mathbf{R}$, then $P(f(\vec{x})) = \vec{w}$.)
   X defines a vector field on S^{2k}. By Theorem 28, $0 = X(\vec{x}) = P(f(\vec{x}))$ for some $\vec{x}$, and at that point we have $f(\vec{x}) = \pm N_{\vec{x}} = \pm\vec{x}$.

3. Suppose $f(\vec{x}) \neq 0$ for all $\vec{x}\in D^n$. Then we can define $\frac{f}{\|f\|} : D^n \to S^{n-1}$, and also consider its restriction $\frac{f}{\|f\|} : S^{n-1}\to S^{n-1}$. Since $\|f(\vec{x}) - \vec{x}\| < 1$ for all $\vec{x}\in S^{n-1}$, we can define a homotopy

$$H(x,t) = \frac{t\vec{x} + (1-t)\frac{f}{\|f\|}(\vec{x})}{\|t\vec{x} + (1-t)\frac{f}{\|f\|}(\vec{x})\|}$$

   between $\frac{f}{\|f\|} : S^{n-1}\to S^{n-1}$ and the identity $S^{n-1}\to S^{n-1}$.

   Then if ν is the volume form on S^{n-1}, we have

$$\int_{S^{n-1}} \nu = \int_{S^{n-1}} (f/\|f\|)^*\nu = \int_{D^n} d(f/\|f\|)^*\nu = \int_{D^n} (f/\|f\|)^* d\nu = 0.$$

   This contradicts that $\int_{S^{n-1}}\nu > 0$.

4. Let $x_0\in M$. Since M is contractible, there exists a homotopy $H : M\times[0,1]\to M$ with $H(x,0) = x$ and $H(x,1) = x_0$, for all $x\in M$.

   To verify M is path connected, let $x_1, x_2\in M$. Define $\gamma_1 : [0,1]\to M$ by $\gamma_1(t) = H(x_1,t)$; this is a path from x_1 to x_0. Similarly, $\overline{\gamma_2}(t) = H(x_2, 1-t)$ gives a path from x_0 to x_2. We may concatenate the two paths to get a path from x_1 to x_2.

   Let $\gamma : S^1\to M$ be any closed curve. Then $G : S^1\times[0,1]\to M$ given by $G(s,t) = H(\gamma(s),t)$ gives a homotopy between γ and the constant curve at x_0.

5. It was shown in class that S^k is simply connected for $k \geq 2$. But each such S^k is a compact oriented manifold without boundary, and therefore not contractible by Homework 10, addendum problem 2.

6. (a) Since ω_1 and ω_2 are cohomologous, we have $\omega_1 = \omega_2 + d\eta$. So

$$\int_M \omega_1 = \int_M \omega_2 + \int_M d\eta = \int_M \omega_2 + \int_{\partial M = \emptyset} \eta = \int_M \omega_2.$$

   (b) Part (a) shows that $\int_M : H^k(M) \to \mathbf{R}$ given by $\int_M([\omega]) = \int_M \omega$ is well-defined. The linearity of $\int_M$ follows immediately from the linearity of the integral.

   (c) If $M = \partial W$, then by Stokes' Theorem we have $\int_M([\omega]) = \int_M \omega = \int_W d\omega = 0$.

7. (a) Since $H^n(S^n)$ has dimension 1 and is generated by the class $[\nu]$ of the volume form, we can write $\omega = r\nu + d\eta$, for some $r \in \mathbf{R}$ and some $(n-1)$-form η. Then

$$\int_{S^n} \omega = r \int_{S^n} \nu + \int_{S^n} d\eta = r \int_{S^n} \nu + \int_{D^{n+1}} d^2\eta = r \int_{S^n} \nu.$$

   This implies $\int_{S^n} \omega = 0$ if and only if $r = 0$ if and only if $\omega = d\eta$.

   (b) Part (a) shows that $\ker \int_{S^n} = \{[0]\}$, so $\int_{S^n}$ is one-to-one. Then since $H^n(S^n)$ and $\mathbf{R}$ are both 1 dimensional, $\int_{S^n}$ is an isomorphism.