2. Let $d: \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ be the function

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{x} + \frac{1}{y} & \text{if } x \neq y \end{cases}$$

Prove that $(\mathbb{Z}^+, d)$ is a metric space but not a complete metric space.

First note that by construction, if $x=y$ then $d(x,y)=0$ & if $x \neq y$, $d(x,y) = \frac{1}{x} + \frac{1}{y} \neq 0$, so we have $d(x,y)=0 \Leftrightarrow x=y$. Next, note that if $d(x,y)=0$ then $d(y,x)=0$ since $x=y$ & if $x \neq y$ $d(x,y) = \frac{1}{x} + \frac{1}{y} = \frac{1}{y} + \frac{1}{x} = d(y,x)$ so $d(y,x) = d(x,y)$ $\forall x,y \in \mathbb{Z}^+$. Finally, note that

$$d(x,y)+d(y,z) = \begin{cases} 0 & \text{if } x=y=z \\ \frac{1}{x}+\frac{1}{y} = \frac{1}{x}+\frac{1}{z} = d(x,z) & \text{if } x \neq y,\ y=z \\ \frac{1}{y}+\frac{1}{z} = \frac{1}{x}+\frac{1}{z} = d(x,z) & \text{if } x=y,\ y \neq z \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{y}+\frac{1}{z} \geq \frac{1}{x}+\frac{1}{z} = d(y,z) & \text{if } x \neq y,\ y \neq z \end{cases}$$

So $\forall x,y,z \in \mathbb{Z}^+$, $d(x,y)+d(y,z) \geq d(x,z)$ So $(\mathbb{Z}^+, d)$ is indeed a metric space.

Consider the sequence $(n)_{n \in \mathbb{Z}^+}$. This sequence is Cauchy in $(\mathbb{Z}^+, d)$ since if $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $2/N < \epsilon$ $\Rightarrow \forall n,m \geq N$,

$$d(n,m) = \begin{cases} \frac{1}{n}+\frac{1}{m} < 1/N + 1/N < \epsilon & \text{if } n \neq m \\ 0 < \epsilon & \text{if } n = m \end{cases}$$

However $(n)$ does not converge in $\mathbb{Z}^+$. $\square$