Modern Algebra - Castella, Winter 2020

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January 15, 2020

Contents

Note: If you find any typos in these notes, please let me know at trevorklar@math.ucsb.edu. If you could include the page number, that would be helpful.

Rings - The Boring Stuff

Definition.

1. A *ring* is a set R together with $\times$, $+$, such that
   - $(R, +)$ is an *abelian* group,
   - $\times$ is associative, and
   - $\times$ distributes over $+$.
2. R is *commutative* if multiplication is commutative.
3. R is a *ring with unity* if it has a multiplicative identity. We will assume all rings have a 1 unless otherwise specified.
4. A ring with unity R is a *division ring* if every nonzero element has a multiplicative inverse.
5. R is a *field* if it is a commutative division ring.

Definition. Let R be a ring.

1. $a, b \in R$ are *zero divisors* if $a, b \neq 0$ and either $ab = 0$ or $ba = 0$.
2. $u, v \in R$ are *units* if $uv = vu = 1$. The set of units in R is denoted $R^{\times}$.

Definition. Let R be a ring. $I \subset R$ is an *ideal* of R (denoted $I \trianglelefteq R$) if for any $r \in R$, $i \in I$,

$$ri \in I, \text{ and } ir \in I.$$

Remark. We define one-sided ideals as you would expect, though we will mostly only consider two-sided ideals.

Definition.

Theorem. For every $I \ntrianglelefteq R$, there exists a maximal ideal $J \ntrianglelefteq R$ such that $I \subset J$.

Operations on Ideals

Proposition. (Chinese Remainder Theorem) Let R be a rinf, and $a_1, \ldots a_n$ ideals in R. If a_i and a_j are coprime for all $i \neq j$, then

$$\prod_{i=1}^{n} a_i = \bigcap_{i=1}^{n} a_i$$

and the natural map $\phi : R \to R/a_1 \times \cdots \times R/a_n$ where $x \mapsto (x \mod a_1, \ldots, a$ mod $a_n)$ induces a ring isomorphism

$$R/\prod_{i=1}^{n} a_i \cong R/a_1 \times \cdots \times R/a_n.$$

(proof in photos)

Corollary 1. If $m = p_1^{r_1} \ldots p_t^{r_t}$ is the prime factorization of some $m \in \mathbb{Z}_{>0}$, we have

$$\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}\Big/p_1^{r_1} \times \cdots \times \mathbb{Z}\Big/p_t^{r_t}$$

as rings.

Definition. 1. If a and b are ideals in a ring R, the *ideal quotient* is

$$(a : b) = \{x \in R : xb \subset a\}$$

2. In particular, the *annihilator* of b is

$$\text{Ann}(b) := (0 : b) = \{x \in R : xb = 0\}$$

examples in photos

Definition. If a is a ideal of R, the *radical* of a is

$$\mathrm{rad}(a) := \{x \in R : x^n \in a \text{ for some } n > 0\}$$

note: in photos

Proposition.
1. $\mathrm{rad}(a) \supset a$
2. $\mathrm{rad}(\mathrm{rad}(a)) = \mathrm{rad}(a)$
3. $\mathrm{rad}(ab) = \mathrm{rad}(a \cap b) = \mathrm{rad}(a) \cap \mathrm{rad}(b)$
4. $\mathrm{rad}(a) = (1)$ iff $a = (1)$
5. $\mathrm{rad}(a + b) = \mathrm{rad}(\mathrm{rad}(a) + \mathrm{rad}(b))$
6. if $\wp$ is prime, then $\mathrm{rad}(\wp^n) = \wp \quad \forall n > 0$.

Proposition. $\mathrm{rad}(a)$ is the intersection of the prime ideals containing a.

Proof in photos ■