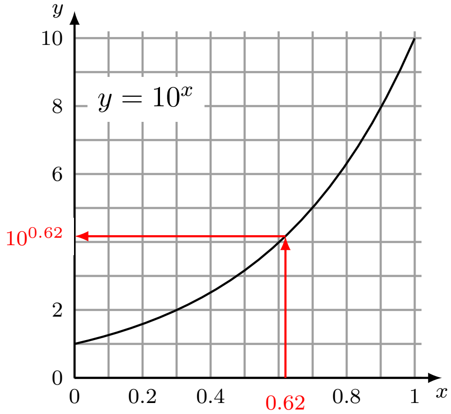
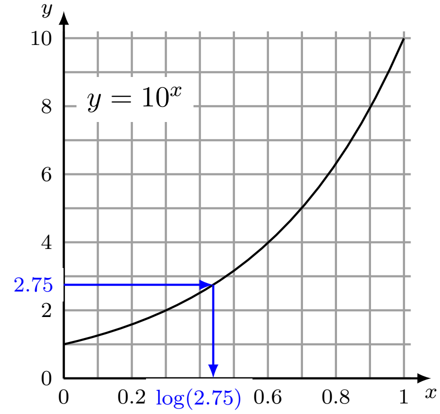
Math 34A Winter 2020
Old Final Exam #2 Solutions

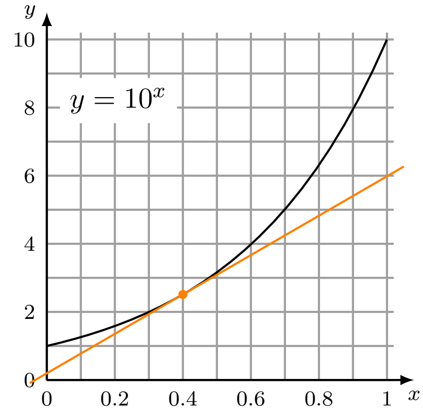
1. Here are the three graphs we'll use in solving these problems:



Part (a)



Part (b)



Part (c)

(a) To solve $\log(y) = 3.62$, we first take the antilog; this equation becomes $y = \text{antilog}(3.62) = 10^{3.62}$. We can write 3.62 as $3 + 0.62$, so $10^{3.62} = 10^{3+0.62} = 10^3 \times 10^{0.62}$. We can find this value directly from the graph; we get $10^{0.62} \approx 4.2$. Thus $y \approx 10^3 \times 4.2 = \boxed{4,200}$. (The actual value of y is about $4,168.69$.)

(b) We can compute $\log(2.75^{100})$ by using the rules of logs to simplify it to $100\log(2.75)$. From the middle graph we see that $\log(2.75) \approx 0.44$, so $\log(2.75^{100}) \approx 100(0.44) = \boxed{44}$. (The actual value of $\log(2.75^{100})$ is about 43.933 according to Mathematica.)

(c) We've drawn the tangent line at $x = 0.4$ on the third graph, above. We pick two points on this line that are reasonably far apart; we'll take $(x, y) = (0, 0.2)$ and $(1, 6)$. Thus the slope of this line is about

$$m = \frac{6 - 0.2}{1 - 0} = \frac{5.8}{1} = \boxed{5.8}.$$

The actual slope of the tangent line to $y = 10^x$ at $x = 0.4$ is $m = 10^{0.4}\ln(10) \approx 5.783\,832\ldots$, so as usual we're pretty close.

2. We write down the answers without much commentary. Remember that $f(x) = 4x^3 - 5x^2$.

(a) $\dfrac{df}{dx} = 4(3x^2) - 5(2x) = \boxed{12x^2 - 10x}$.

(b) The second derivative is the derivative of the first: $f'(x) = \dfrac{d}{dx}\big(12x^2 - 10x\big) = 12(2x) - 10(1) = \boxed{24x - 10}$.

(c) Since $f'(1) = 12(1)^2 - 10(1) = 2$ and $f''(0) = 24(0) - 10 = -10$, we get $f''(0) + f'(1) = -10 + 2 = \boxed{-8}$.

3. Again we don't say too much in computing these derivatives.

(a) $\dfrac{d}{dx}\left(2e^{kx} + k^2\right) = 2(ke^{kx}) + 0 = \boxed{2ke^{kx}}$.

(b) First we multiply this out to get $(3x + k)(3x - k) = 9x^2 - 3kx + 3kx - k^2 = 9x^2 - k^2$. Thus the derivative is

$$\frac{d}{dx}\big((3x + k)(3x - k)\big) = \frac{d}{dx}\big(9x^2 - k^2\big) = 9(2x) - 0 = \boxed{18x}.$$

(c) First we simplify $(3x^2 + 5)/x^k$ as $(3x^2 + 5)x^{-k} = 3x^2 \cdot x^{-k} + 5x^{-k} = 3x^{2-k} + 5x^{-k}$. Thus

$$\frac{d}{dx}\big((3x^2 + 5)/x^k\big) = \frac{d}{dx}\big(3x^{2-k} + 5x^{-k}\big) = 3(2-k)x^{2-k-1} + 5(-k)x^{-k-1} = \boxed{3(2-k)x^{1-k} - 5kx^{-k-1}}.$$

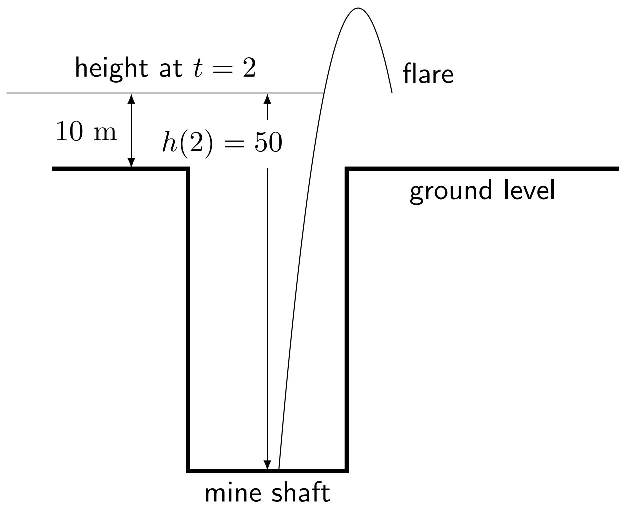
4. Since $y = x^2 - 8x + 3$, we have $y' = 2x - 8$.

(a) The slope of the graph is 1 when $y' = 1$. Since $y' = 2x - 8$, this happens when $2x - 8 = 1$. This means $2x = 9$, or $x = \boxed{9/2}$.

(b) This function is a minimum when $y' = 0$. This happens when $2x - 8 = 0$, or when $x = 8/2 = 4$. Thus the minimum value of this function is $y(4) = (4)^2 - 8(4) + 3 = \boxed{-13}$.

(c) At $x = 0$, the slope of the tangent line is $m = y'(0) = 2(0) - 8 = -8$. Since $y(0) = (0)^2 - 8(0) + 3 = 3$, the equation of the tangent line is $y - 3 = -8(x - 0)$ or, in "$y = mx + b$" form, $\boxed{y = -8x + 3}$.

5. Here's a picture of the situation:



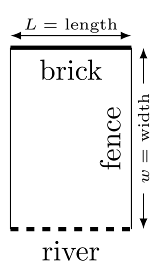
(a) After 2 seconds, the flare is at height $h(2) = 35(2) - 5(2)^2 = 50$ meters above the bottom of the shaft. Since this is 10 meters above the ground, the ground is 40 meters above the bottom of the shaft. That is, the mine is $\boxed{\text{40 meters deep}}$.

(b) The velocity after 1 second is $h'(1)$. Since $h'(t) = 35 - 10t$ m/s, the velocity after 1 second is $h'(1) = \boxed{25 \text{ m/s}}$.

(c) The acceleration after 2 seconds is $h''(2)$. Since $h''(t) = -10$ m/s^2, the acceleration after 2 seconds is $h''(2) = \boxed{-10 \text{ m/s}^2}$. (This is 10 m/s^2 *downward*.)

(d) The flare's maximum height occurs when $h'(t) = 0$. This means $35 - 10t = 0$, which happens when $t = 35/10 = \boxed{\text{3.5 seconds}}$.

6. Here is a reproduction of the picture:



(a) The total cost C of the boundary of the field is

$$\begin{aligned} C &= \$20/\text{meter} \times \Big(\text{length of fence} \Big) + \$5/\text{meter} \times \Big(\text{length of brick wall} \Big) \\ &= 20(2w) + 5(L) \\ &= 40w + 5L. \end{aligned}$$

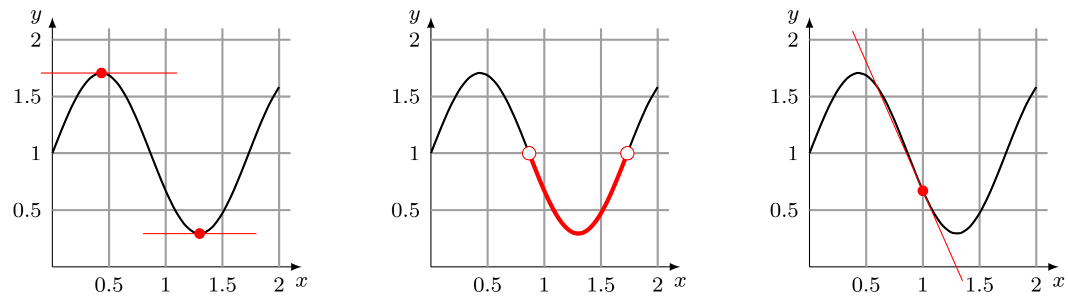
That is, the specified fence and wall will cost $\boxed{40w + 5L \text{ dollars}}$.

(b) Since the total cost is \$2,000, we have $40w + 5L = 2000$. Solving for L, we get $5L = 2000 - 40w$, or $\boxed{L = 400 - 8w}$.

(c) The area of the field is simply $A = Lw$. Plugging in the expression $L = 400 - 8w$ we just found, we get the area in terms of just the width: $A = (400 - 8w)w$ or $\boxed{A = 400w - 8w^2}$.

(d) We're trying to find w so that the area $A = 400w - 8w^2$ will be a maximum. This means we compute $A' = \frac{dA}{dw}$ and set it equal to zero. This derivative is $A' = 400 - 8(2w) = 400 - 16w$. Setting this equal to zero gives us $400 = 16w$, or $w = 400/16 = 100/4 = 25$. Thus $\boxed{w = 25 \text{ meters}}$ is the width that will maximize the area of our field.

7. Here are three views of the same graph, with various markings on them for the three parts of the problem.



(a) (b) (c)

(a) We can see that the tangent line is horizontal (that is, the slope of the graph is zero) at three points: the x values are roughly $\boxed{x \approx 0.43 \text{ and } x \approx 1.3}$.

(b) The values of x where $f''(x)$ is positive is exactly the set of x values where $f(x)$ is concave up. We've drawn on the graph where it is concave up; this is roughly $\boxed{0.9 < x < 1.7}$.

(c) To do this we ask you to draw the tangent line at $x = 1$ and measure the slope as carefully as you can. We're going to estimate that this tangent line passes through the points $(x, y) = (1.3, 0)$ and $(0.4, 2)$. Thus the slope of the tangent line – the value of the derivative at $x = 1$ – is

$$f'(1) = m \approx \frac{2 - 0}{0.4 - 1.3} = \frac{2}{-0.9} \approx -2.2.$$

That is, we've estimated that $f'(1) \approx \boxed{-2.2}$.

8. Remember that $f(x) = 20\sqrt{x}$.

(a) Since $f(x) = 20x^{1/2}$, we find that $f'(x) = 20 \cdot \frac{1}{2}x^{-1/2} = 10/x^{1/2} = 10/\sqrt{x}$. Thus $f'(4) = 10/\sqrt{4} = 10/2 = \boxed{5}$.

(b) The tangent line to $y = f(x)$ has slope $f'(4) = 5$ and passes through the point $(x, y) = (4, f(4)) = (4, 20\sqrt{4}) = (4, 40)$. Thus the equation of the tangent line is $y - 40 = 5(x - 4)$ or, equivalently, $y = 5x + 20$. Thus the tangent line approximation is $\boxed{f(x) \approx 5x + 20}$ for x near 4.

(c) Using the approximatino from part (b), $20\sqrt{5} = f(5) \approx 5(5) + 20 = 45$. (The actual value of $20\sqrt{5}$ is about 44.7214.)

9. (a) Since raising the price by a penny x times lowers the number of cookies by $5x$, she'll sell $\boxed{2,200 - 5x \text{ cookies}}$.

(b) A single cookie brings in $\$(2 + 0.01x)$, and her costs are $\frac{\$2}{10 \text{ cookies}} = \$0.20/\text{cookie}$. Thus the profit on a single cookie is $\boxed{\$1.80 + 0.01x}$.

(c) The total profit, which we'll call P, is the profit per cookie (found in part (b)) times the number of cookies (found in part (a)). That is, $P = \boxed{(1.80 + 0.01x)(2200 - 5x)}$ or $P = \boxed{3,960 + 13x - 0.05x^2}$.

(d) The value of x that maximizes our profit is the one when $P'(x) = 0$. Since $P' = 13 - 0.10x$, we get $x = \boxed{130}$.

(e) The price per cookie that maximizes her profit is $\$(2 + 0.01x) = \$2 + \$0.01 \cdot 130 = \boxed{\$3.30}$.

10. (a) In the first two hours, Jason and Marie travel

$$\left(U \, \frac{\text{km}}{\text{hr}}\right)(2 \text{ hrs}) = 2U \text{ km}.$$

Similarly, in the next three hours, Jason and Marie travel

$$\left(V \, \frac{\text{km}}{\text{hr}}\right)(3 \text{ hrs}) = 3V \text{ km}.$$

In these five hours, the two have traveled $\boxed{2U + 3V = 720}$ km. This is our first equation. The second equation is from the last sentence in our description: *They drive* 60 *km more in the last* 3 *hours than in the first* 2 *hours.* This says $\boxed{3V = 60 + 2U}$.

(b) To solve for U, we replace the "$3V$" in the first equation with the expression $3V = 60 + 2U$; thus $2U + (60 + 2U) = 720$. This simplifies to $4U + 60 = 720$, and so $4U = 660$. Thus $\boxed{U = 165 \text{ km/hr}}$.

(c) From 1pm to 2pm, the pair has traveled $(U \text{ km/hr})(1 \text{ hr}) = U \text{ km} = 165$ km. Since they started 720 km from Paris, they are $720 - 165 = \boxed{555 \text{ km}}$ from Paris at 2pm.