Math 450B
Homework 4
Dr. Fuller
Due February 20

1. If $f, g : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ are differentiable on an open set U, and $\alpha, \beta \in \mathbf{R}$, prove that $\alpha f + \beta g : U \subset \mathbf{R}^n \to \mathbf{R}^m$ is differentiable and $D(\alpha f + \beta g)(\mathbf{a}) = \alpha Df(\mathbf{a}) + \beta Dg(\mathbf{a})$.

2. If $f : U \subseteq \mathbf{R}^n \to \mathbf{R}^m$ is a constant function, prove that $Df(\mathbf{a}) = 0$ for all $\mathbf{a} \in U$.

3. Let $f : A \subset \mathbf{R}^2 \to \mathbf{R}$ be $f(x,y) = 0$, where $A = \{(x,y) : 0 \le x \le 1,\ y = 0\}$. Prove that the derivative of f is not unique on A.

4. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be $f(x,y) = \sqrt{|xy|}$. Prove that f is not differentiable at $(0,0)$.

5. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ and suppose there is a constant M such that $\|f(\mathbf{x})\| \le M\|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbf{R}^n$. Prove that f is differentiable at $\mathbf{0}$ and $Df(\mathbf{0}) = 0$.

6. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ and suppose there is a constant M such that $\|f(\mathbf{x})\| \le M\|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbf{R}^n$. Let $g(x) = T(x) + f(x)$, where $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation. Prove that g is differentiable at $\mathbf{0}$ and $Dg(\mathbf{0}) = T$.

7. For the following functions, compute the matrix of Df with respect to the standard bases.

   (a) $f : \mathbf{R}^3 \to \mathbf{R}^2,\ f(x,y,z) = (x^4 y, xe^z)$

   (b) $f : \mathbf{R}^3 \to \mathbf{R},\ f(x,y,z) = e^{x^2+y^2+z^2}$