Some Problems About Consecutive Products of Primes

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1 Introduction

Suppose p and q are both prime numbers with $p < q$. Consider all integers of the form $p^\alpha q^\beta$ with $\alpha, \beta \in \mathbb{N}$ and let $\{a_n\}$ be the sequence of these integers in increasing order.

Lemma 1. *If $a_k = q^n$, then $a_{k+1} \neq q^{n+1}$.*

Proof (By Contradiction)
Suppose that $a_k = q^n$ and $a_{k+1} = q^{n+1}$.
$1 < p < q \implies q^n < pq^n < q^{n+1}$ which is a contradiction as pq^n must be a term between a_k and a_{k+1} ■

Lemma 2. *There exist at most finitely many $a_k = p^n$ such that $a_{k+1} = p^{n+1}$.*

Proof Consider the smallest $n \in \mathbb{N}$ such that $p^n < q < p^{n+1}$. Note that a_n begins as $\{1, p, p^2, ..., p^n, q, p^{n+1}, ...\}$.

Claim: $\forall m \in \mathbb{N}$, $p^{n+m} < p^m q < p^{n+m+1}$. Let T(m) denote this statement. We now prove this claim by induction.
T(1) holds as $p^n < q < p^{n+1} \implies p^{n+1} < pq < p^{n+2}$.
We now assume T(m) and show T(m+1) holds.
$p^{n+m} < p^m q < p^{n+m+1} \implies p^{n+m+1} < p^{m+1} q < p^{n+m+2}$.
As such, every integer $p^{n+m} > p^n$ is followed by the term $p^m q$ before p^{n+m+1} in the sequence a_n. Thus, $a_k = p^n$ and $a_{k+1} = p^{n+1}$ can only occur at the beginning of the sequence (finitely many times) as shown above. ■