Math 221C Homework 1.

X, Y always smooth manifolds, maps are smooth etc.

1. Show that if $f : X \to X'$ and $g : Y \to Y'$ are both smooth, then so is the map $f \times g : X \times Y \to X' \times Y'$ defined by

$$(f \times g)(x, y) = (f(x), g(y))$$

2. Prove that the projection map $X \times Y \to X$ given by $(x, y) \to x$ is smooth.

3. Suppose that U is an open subset of the manifold X. Show that for all $p \in U$
$$T_p(U) = T_p(X)$$

4. If $f : X \to Y$ is a diffeomorphism, then df_x is an isomorphism for all $x \in X$. Deduce that if $\mathbb{R}^a$ is diffeomorphic to $\mathbb{R}^b$, then $a = b$.

5. A *curve* in a manifold X is smooth map $c : (-1, 1) \to X$. Define the *velocity vector* of c at $t_0 \in (-1, 1)$ to be

$$dc_{t_0}(1) \in T_{c(t_0)}(X)$$

(The point being that one should think of 1 as the unit vector in the one-dimensional vector space $\mathbb{R}$.)

Show that $T_p(X)$ is the set of velocity vectors of curves through the point p.

6. Prove that if $f : X \longrightarrow Y$ is a submersion and U is an open subset of X, then $f(U)$ is an open subset of Y.