Final Exam

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1. *Suppose that $f : X \to Y$ is a function.*

Prove that assuming either of the following suffice to guarantee that f is continuous:

(i) $X = \bigcup_{i=1}^{n} C_i$*, each C_i is closed in X and $f|_{C_i}$ is continuous for each i.*

Proof Let F be closed in Y. Since each $f|_{C_i}$ is continuous, then $f|_{C_i}^{-1}(F)$ is closed in X, so $\bigcup_{i=1}^{n}\left(f|_{C_i}^{-1}(F)\right)$ is also closed. Now since $X = \bigcup_{i=1}^{n} C_i$, then

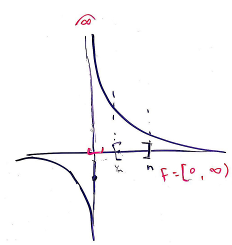
$$\begin{aligned}\bigcup_{i=1}^{n}\left(f|_{C_i}^{-1}(F)\right) &= \bigcup_{i=1}^{n}\{x \in C_i : f(x) \in F\} \\ &= \{x \in X : f(x) \in F\} \\ &= f^{-1}(F)\end{aligned}$$

then $f^{-1}(F)$ is closed for any closed F, so f is continuous. ■

(ii) $X = \bigcup_{\alpha \in A} U_\alpha$*, each U_α is open in X, A is any indexing set and $f|_{U_\beta}$ is continuous for each $\beta \in A$.*

Proof As above, for any U open in Y, $\bigcup_{\alpha \in A} f|_{U_\alpha}^{-1}(U) = f^{-1}(U)$ is open, so f is continuous. ■

Show that (i) fails if the indexing set is infinite.



Proof Let $f : \mathbb{R} \to \mathbb{R}$ (both with the usual topology) be defined by

$$f = \begin{cases} \frac{1}{x}, & x \neq 0 \\ -1, & x = 0 \end{cases},$$

and let $F = [0, \infty)$, $C_0 = (-\infty, 0]$, and for $n \in 1, 2, \ldots$, $C_n = [\frac{1}{n}, n]$. Then F, C_n are all closed, and $\bigcup_{n=0}^{\infty} C_n = \mathbb{R}$, with

$$f|_{C_n}^{-1}(F) = \begin{cases} C_n, & n > 0 \\ \emptyset, & n = 0 \end{cases}$$

which are all closed. But $f^{-1}(F) = (0, \infty)$, which is not closed, so f is not continuous. ■

2. *Prove that if $f : M \to M$ is an isometry of a compact metric space, then f is onto. (Recall that f an isometry means $d\big(f(x), f(y)\big) = d(x, y)$ for all $x, y \in M$.)*

Proof Suppose for contradiction there exists some $m \in M$ where $m \notin f(M)$. Since f is an isometry, then it is continuous[†], so since M is compact, then $f(M)$ is as well. Since $f(M)$ is a compact subset of a Hausdorff space, then it is closed. This means that $d(m, f(M)) > 0$[‡], call this distance δ. Now we make a sequence by iterating f, starting at m. Let

$$m_0 = m, \qquad\qquad m_n = f(m_{n-1}) \quad \text{for all } n \in \mathbb{N}.$$

Since $m_1 = f(m_0) \in f(M)$, then $d(m_0, m_1) \geq \delta$, and indeed $d(m_0, m_n) \geq \delta$ for all n by the same reason. Since f is an isometry, then $d(m_j, m_{n+j}) = d(m_0, m_n) \geq \delta$ for all $j, n \in \mathbb{N}$, so

$$d(m_j, m_k) \geq \delta \quad \text{whenever } j \neq k.$$

However, since M is compact then it is sequentially compact, so $\{m_n\}_{n=0}^{\infty}$ must have a subsequence $\{m_{n_i}\}_{i=0}^{\infty}$ which is convergent and thus Cauchy. So there exists some $I \in N$ such that for every $i, j > I$ with $i \neq j$,

$$d(m_{n_i}, m_{n_j}) < \varepsilon \quad \text{for any } \varepsilon > 0,$$

a contradiction.

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[†] To see this, let $\delta = \varepsilon$ and use the δ-ε definition.

[‡] See the footnote in problem 4 for a proof of this fact.

3. *Give a careful definition of* connected.

Definition. Let X be a topological space. A *separation* of X is a pair of disjoint nonempty open sets which cover X.

Definition. Let X be a topological space. X is *connected* if there does not exist a separation of X.

(i) *Prove that the closed interval* $[0, 1]$ *(with the usual topology) is connected.*

Proof Suppose for contradiction that A, B comprise a separation of $[0, 1]$, and let

$$\alpha = \inf(A), \quad \beta = \inf(B).$$

A and B are open sets in $[0, 1] \subset \mathbb{R}$, so each is a countable union of disjoint intervals which are open in $[0, 1]$. Thus one of A, B contains $[0, x)$[†] for some $x \in (0, 1)$, so without loss of generality say that A does, which means $\alpha = 0$ and $\beta > 0$.

Claim. $\beta \notin A \cup B$, contradicting our assumption that $\{A, B\}$ is a separation of $[0, 1]$.

Proof of claim Observe that $\beta \notin B$, since if it were then some $B_\varepsilon(\beta) \subset B$, which means there exists some $x \in B$ such that $x < \beta$, contradicting that $\beta = \inf(B)$.

However $\beta \notin A$ either, since if it were then some $B_\varepsilon(\beta) \subset A$. Since $\beta = \inf(B)$, any number greater than β is not a lower bound for B, which means there exists some $x \in (\beta, \beta + \varepsilon) \subset B_\varepsilon(\beta) \subset A$ such that $x \in B$. This contradicts that A and B are disjoint. Therefore the claim is proved and the problem follows. ■

(ii) *Show that a connected metric space with at least two points is uncountable.*

Proof Let a_0 and a_δ denote two points in the space M where $\delta = d(a, b)$. The following claim produces an injective map from the interval $(0, \delta)$ to M, which means $|M| \geq |\mathbb{R}|$.

Claim. For every real number $x \in (0, \delta)$, there exists a point a_x such that $a_x = a_y$ iff $x = y$.

Proof of claim Let $x \in (0, \delta)$ be given. Consider the sets

$$A_{<x} = \{a \in M : d(a_0, a) < x\}$$
$$A_{>x} = \{a \in M : d(a_0, a) > x\}.$$

They are open because $A_{<x}$ is an open ball and $B_{\delta - x}(a_\delta) \subset A_{>\delta}$. They are disjoint by definition, and they are nonempty because $a_0 \in A_{<x}$ and $a_\delta \in A_{>x}$. Thus they are a pair of disjoint nonempty open sets, so they cannot cover M. This means

$$\begin{aligned}(A_{<x} \cup A_{>x})^{\complement} &= \{a \in M : d(a_0, a) = x\} \\ &\neq \emptyset,\end{aligned}$$

so there exists some $a_x \in M$ with $d(a_0, a_x) = x$.

Now for any $x, y \in (0, \delta)$ with $x \neq y$, we must have that $a_x \neq a_y$ since

$$x = d(a_0, a_x), \quad y = d(a_0, a_y)$$

and d is a well-defined function. ■

[†]If this is not obvious, note that A, B are open in the subspace topology on $[0, 1]$ which means there exist U, V open in $\mathbb{R}$ such that $U \cap [0, 1] = A$ and $V \cap [0, 1] = B$. So since $0 \in U \cup V$, then $0 \in (a, b)$ for some (a, b) in either U or V. Intersecting yields the desired $[0, b)$.

4. *Let $\{C_n\}_{n\geq 1}$ be a family of closed subsets of the compact metric space X for which $\bigcap_{n\geq 1} C_n = \emptyset$. Prove that there is an $\varepsilon > 0$ so that every ball in X of radius ε misses at least one C_k.*

Proof Since $\bigcap_{n\geq 1} C_n = \emptyset$, then every $x \in X$ has some C_n which does not contain it. So let

$$\Gamma_x = \{n \in \mathbb{N} : x \notin C_n\},$$

and for each $x \in X$ and $n \in \Gamma_x$, let

$$\delta_{x,n} = \frac{1}{2} d(x, C_n) = \frac{1}{2} \inf_{y \in C_n} d(x, y)$$

Note that each $\delta_{x,n} > 0$, since $\{x\}$, and C_n are disjoint closed sets and thus have positive distance[†]. This means that the collection of open balls

$$\{B_{\delta_{x,n}}(x) : x \in X, n \in \Gamma_x\}$$

is an open cover of X, and thus has a finite subcover, call it $\{B_{\delta_i}(x_i)\}_{i=1}^N$. If we let

$$\delta = \min(\delta_i),$$

Then any ball of radius δ misses at least one C_n. To see this, let $x \in X$. Since $\{B_{\delta_i}(x_i)\}_{i=1}^N$ is an open cover of X, there exists some $B_{\delta_j}(x_j)$ containing x, so

$$d(x_j, x) < \delta_j.$$

Since $\delta_j = \delta_{x_j, n_j}$ for some $n_j \in \Gamma_{x_j}$, then

$$d(x_j, C_{n_j}) = 2\delta_j,$$

so by the Triangle Inequality,

$$\begin{aligned} d(x, C_n) &\geq d(x_j, C_{n_j}) - d(x_j, x) \\ &\geq 2\delta_j - \delta_j \\ &= \delta_j \\ &\geq \delta \end{aligned}$$

and $B_\delta(x)$ misses C_{n_j}. ■

[†]This is a common theorem about closed sets in metric spaces, but I don't believe we proved it in class. To see that it holds here, suppose that $x \notin C_n$ and $d(x, C_n) = 0$. Then every ball $B_\varepsilon(x)$ contains some point in C_n because ε is not a lower bound for $\{d(x, y) : y \in C_n\}$, so $x \in \overline{C_n} = C_n$, contradiction.

5. *Define what it means for a metric space to be* complete.

Definition. Let (M, d) be a metric space. We say M is *complete* if every Cauchy sequence in M converges to a point in M.

State carefully and prove the contraction mapping theorem for metric spaces.

Theorem. (Contraction Mapping Theorem) In a complete metric space, every contraction has a fixed point. That is, let (M, d) be a complete metric space and let $f : M \to M$ be a function with the property that for all $x, y \in M$, there exists $\lambda \in (0, 1)$ such that $d\big(f(x), f(y)\big) \leq \lambda d(x, y)$. Then there exists a unique $x \in M$ with $f(x) = x$.

Proof Let x_0 be any element in M. As we did in problem 2, we iterate: let $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$, and call $\delta = d(x_0, x_1)$. Then $d(x_n, x_{n+1}) \leq \lambda^n \delta$, which means that for all $n < m \in \mathbb{N}$,

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{m-1} \lambda^i \delta \\ &\leq \sum_{i=n}^{\infty} \lambda^i \delta, \end{aligned}$$

which is a tail of a convergent geometric series, so for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\begin{aligned} \epsilon &> \sum_{i=N}^{\infty} \lambda^i \delta \\ &\geq d(x_n, x_m) \quad \text{if } n, m > N. \end{aligned}$$

Thus, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in M. Since M is complete, then $\{x_n\}_{n=0}^{\infty}$ converges to a point in M, and we can call the limit x. To see that $f(x) = x$, first note that f is continuous at every point $a \in M$, since for every $\varepsilon > 0$ if $d(a, b) < \frac{\varepsilon}{\lambda} = \delta$, then $d\big(f(a), f(b)\big) < \varepsilon$. Now suppose that $f(x) = y \neq x$, and let $\epsilon = \frac{d(x,y)}{\Omega}$ for some large number $\Omega > \frac{1}{\lambda} + 1$. Then $\delta = \frac{d(x,y)}{\Omega\lambda}$ and whenever $d(x, a) < \delta$, then $d(y, f(a)) < \frac{d(x,y)}{\Omega}$. But we can find some n such that $d(x, x_n) < \delta$, which means that both

$$\begin{aligned} d(y, x_{n+1}) &< \frac{d(x, y)}{\Omega} \quad \text{by continuity, and} \\ d(x, x_{n+1}) &< \frac{d(x, y)}{\Omega\lambda} \quad \text{since } x_n \to x, \end{aligned}$$

Which contradicts the Triangle Inequality, since $d(y, x_{n+1}) + d(x, x_{n+1}) < \frac{d(x,y)}{\Omega} + \frac{d(x,y)}{\Omega\lambda} < d(x, y)$. Finally, note that this fixed point is unique since if $x \neq y$ and x, y are fixed points, then $d(f(x), f(y)) = d(x, y) \not\leq \lambda d(x, y)$. ■

Show that if $f(x)$ is a real differentiable function with $|f'(x)| < M$ for all real x, then for any real numbers a_{ij} and b_j, there is one and only one point $(x_1, \dots, x_n)$ satisfying

$$x_i = \sum_{j=1}^{n} a_{ij} f(x_j) + b_j \quad \text{where } 1 \leq i \leq n,$$

provided that

$$\sum_{i,j} a_{ij}^2 < 1/M^2.$$

Proof Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the above properties, let $\mathbf{A} \in M_n(\mathbb{R})$ with entries given by the numbers a_{ij} above, and let $\vec{b} \in \mathbb{R}^n$ be have $\sum_{j=1}^n b_j$ in every coordinate. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be $\varphi(\vec{x}) = (f(x_1), \dots, f(x_n))$ Finally let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be

$$\Phi(\vec{x}) = \mathbf{A}\varphi(\vec{x}) + \vec{b}.$$

Since $\mathbb{R}^n$ equipped with the Euclidean metric is complete, we will show that Φ is a contraction, and by applying the Contraction Mapping Theorem, we will be done.

We can write $\Phi(\vec{x})$ as

$$\begin{aligned}
\Phi(\vec{x}) &= \mathbf{A}\varphi(\vec{x}) + \vec{b} \\
&= \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & & & \vdots \\ a_{i1} & & a_{ij} & & a_{in} \\ \vdots & & & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_i) \\ \vdots \\ f(x_n) \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n b_j \\ \vdots \\ \sum_{j=1}^n b_j \\ \vdots \\ \sum_{j=1}^n b_j \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=1}^n \big(a_{1j} f(x_j) + b_j\big) \\ \vdots \\ \sum_{j=1}^n \big(a_{ij} f(x_j) + b_j\big) \\ \vdots \\ \sum_{j=1}^n \big(a_{nj} f(x_j) + b_j\big) \end{bmatrix}
\end{aligned}$$

so for any $\vec{x}, \vec{y} \in \mathbb{R}^n$, the i-th coordinate of $\Phi(\vec{x}) - \Phi(\vec{y})$ is given by

$$\begin{aligned}
\pi_i\big(\Phi(\vec{x}) - \Phi(\vec{y})\big) &= \sum_{j=1}^{n} \big(a_{ij} f(x_j) + b_j\big) - \sum_{j=1}^{n} \big(a_{ij} f(y_j) + b_j\big) \\
&= \sum_{j=1}^{n} a_{ij}\big(f(x_j) - f(y_j)\big)
\end{aligned}$$

and now we check that the squared distance $||\Phi(x) - \Phi(y)||^2$ is bounded by $||\vec{x} - \vec{y}||^2$,

$$
\begin{aligned}
\sum_{i=1}^{n} \left[\pi_i\big(\Phi(\vec{x}) - \Phi(\vec{y})\big)\right]^2 &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}\big(f(x_j) - f(y_j)\big)\right)^2 && \text{and by Cauchy-Schwarz,} \\
&\le \sum_{i=1}^{n}\left(\left(\sum_{j=1}^{n} a_{ij}^2\right)\left(\sum_{j=1}^{n}\big(f(x_j) - f(y_j)\big)^2\right)\right) && \text{and since right hand factor is constant w.r.t. } i, \\
&\le \left(\sum_{i,j} a_{ij}^2\right)\left(\sum_{j=1}^{n}\big(f(x_j) - f(y_j)\big)^2\right) && \text{and since } \sum_{i,j} a_{ij}^2 < \frac{1}{M^2}, \\
&= \frac{\lambda}{M^2}\left(\sum_{j=1}^{n}(x_j - y_j)^2\left(\frac{f(x_j) - f(y_j)}{x_j - y_j}\right)^2\right) && \text{and since } |f'(x)| < M,^{\dagger} \\
&< \frac{\lambda}{M^2}\left(\sum_{j=1}^{n}(x_j - y_j)^2 M^2\right) \\
&= \lambda \sum_{j=1}^{n}(x_j - y_j)^2 \\
&= \lambda\, ||\vec{x} - \vec{y}||^2,
\end{aligned}
$$

where $\lambda = M^2 \sum_{j=1}^{n} a_{ij}^2 < 1$. Thus

$$||\Phi(x) - \Phi(y)||^2 < \lambda\, ||\vec{x} - \vec{y}||^2,$$

so Φ is a contraction, and thus has a unique fixed point.[‡] ■

[†] The difference quotient can't exceed the bound on the derivative since the Mean Value Theorem guarantees a point where the derivative is equal to the difference quotient.

[‡] I didn't expect to have to dig out my Linear Algebra knowledge from the old mental closet, and it was fun. I thoroughly enjoyed this problem.