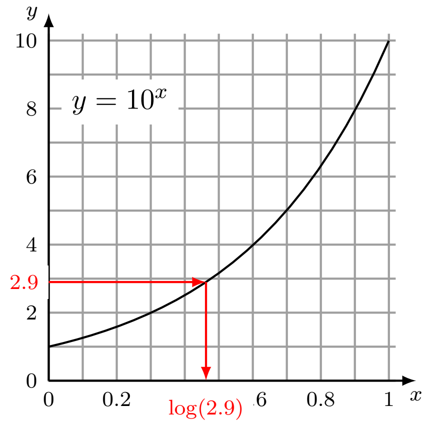
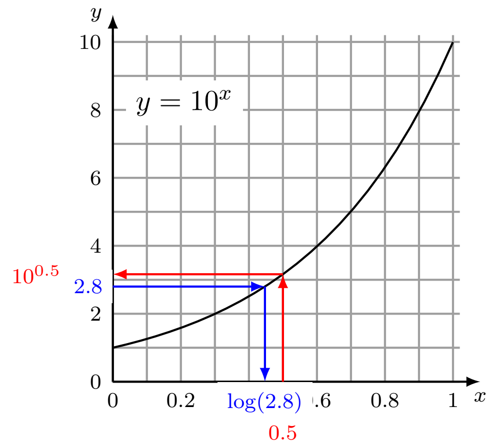
Math 34A Winter 2020
Old Final Exam #1 Solutions

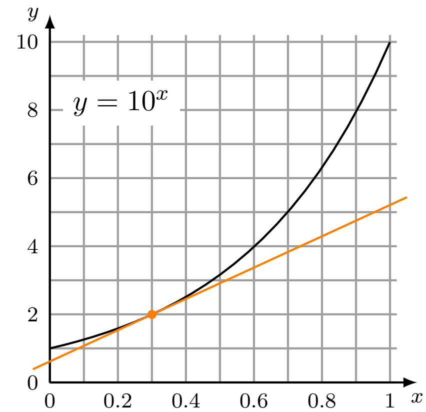
1. Here are the three graphs we'll use in solving these problems:



Part (a)



Part (b)



Part (c)

(a) To solve $10^{2x} = 2900$, we first take the log of both sides; this equation becomes $2x = \log(2900)$. We can write 2900 as 2.9×10^3, so $\log(2900) = \log(2.9 \times 10^3) = 3 + \log(2.9)$ (this is the "move the decimal point trick"). We can find this $\log(2.9)$ value from the graph; we get $\log(2.9) \approx 0.46$. Thus $2x = \log(2900) \approx 3 + 0.46 = 3.46$, so $\boxed{x \approx 1.73}$.

(b) It's difficult to compute 2.8^{10} directly, but we can compute $\log(2.8^{10})$, and then $2.8^{10} = \text{antilog}(\log(2.8^{10}))$.

We start with $\log(2.8^{10})$, which by the rules of logs is $10\log(2.8)$. From the middle graph we see that $\log(2.8) \approx 0.45$, so $\log(2.8^{10}) \approx 10(0.45) = 4.5$. Thus $2.8^{10} = \text{antilog}(\log(2.8^{10})) \approx 10^{4.5} = 10^4 \times 10^{0.5}$. Now we can read directly from the graph that $10^{0.5} \approx 3.2$, so $2.8^{10} \approx 3.2 \times 10^4 = \boxed{32,000}$.

Mathematica tells me that $2.8^{10} \approx 29,619.7$, so the percent error is about 8% (but we're logging and antilogging in a single problem).

(c) We've drawn the tangent line at $x = 0.3$ on the third graph, above. We pick two points on this line that are reasonably far apart; we'll take $(x, y) = (0.3, 2)$ and $(1, 5.2)$. Thus the slope of this line is about

$$m = \frac{5.2 - 2}{1 - 0.3} = \frac{3.2}{0.7} \approx \boxed{4.57}.$$

The actual slope of the tangent line to $y = 10^x$ at $x = 0.3$ is $m = 10^{0.3}\ln(10) \approx 4.594\,26\ldots$, so as usual we're pretty close.

2. Remember that $f(x) = 3x^5 - 7x^2$.

(a) $\dfrac{df}{dx} = f'(x) = 3 \cdot 5x^4 - 7 \cdot 2x = \boxed{15x^4 - 14x}$.

(b) $f''(x) = 15 \cdot 4x^3 - 14 = \boxed{60x^3 - 14}$.

(c) From part (b), $f''(0) = 60 \cdot 0^3 - 14 = -14$, and from part (a) $f'(1) = 15 \cdot 1^4 - 14 \cdot 1 = 1$. Thus $f''(0) + f'(1) = -14 + 1 = \boxed{-13}$.

3. In this question remember that k is a constant.

(a) $\frac{d}{dx}\left(2e^{5x}+k^{-1}\right)=2\cdot 5e^{5x}+0=\boxed{10e^{5x}}$

(b) $\frac{d}{dx}\left((3x+k)^2\right)=\frac{d}{dx}\left(9x^2+6kx+k^2\right)=9\cdot 2x+6k+0=\boxed{18x+6k}$.

(c) $\frac{d}{dx}\left((3x+5)/x^2\right)=\frac{d}{dx}\left(\frac{3x+5}{x^2}\right)=\frac{d}{dx}\left(\frac{3}{x}+\frac{5}{x^2}\right)=\frac{d}{dx}\left(3x^{-1}+5x^{-2}\right)$

$$=\boxed{-3x^{-2}-10x^{-3}}=\boxed{-\frac{3}{x^2}-\frac{10}{x^3}}.$$

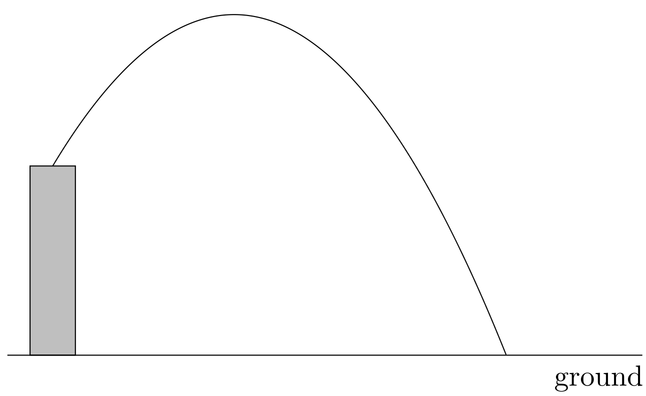
4. Remember that $y=2x^2-10x+3$.

(a) This question asks for x when $y'=2$. Since $y'=4x-10$, this equation is $4x-10=2$. Adding 10 to both sides gives us $4x=12$; dividing by 4 gives us $\boxed{x=3}$.

(b) Remember that x gives a minimum value of y if $y'=0$. Since $y'=4x-10$, this means $4x-10=0$, or $4x=10$. Thus $\boxed{x=5/2=2.5}$.

(c) The tangent line to the graph at $x=1$ goes through $(x,y)=(1,y(1))=(1,-5)$ and has slope $m=y'(10=4\cdot 1-10=-6$. The line is thus $\boxed{y=-6x+1}$.

5. (a) Here's a small picture:



Since the ball hits the ground at $t=5$, we get that the ground is at $h(5)=-25$ m. The top of the tower is $h(0)=0$ m, so the tower is $\boxed{\text{25 meters tall}}$.

(b) Since $h'(t)=20-10t$ m/s, the velocity of the ball at time 3 is $h'(3)=-10$ m/s. Thus the speed of the ball at 3 seconds is $\boxed{10 \text{ m/s}}$.

(c) How high the ball goes above the tower is the maximum value of $h(t)$. This occurs when $h'(t)=0$, or when $20-10t=0$. This is $t=2$ seconds. Thus the maximum value of $h(t)$ is $h(2)=20\cdot 2-5\cdot 2^2=20$ meters. Thus the ball goes $\boxed{\text{20 meters above the tower}}$.

(d) The ball is going down at a speed of 3 m/s when $h'(t)=-3$. Thus we need to solve $20-10t=-3$. This happens at $\boxed{t=2.3 \text{ seconds}}$.

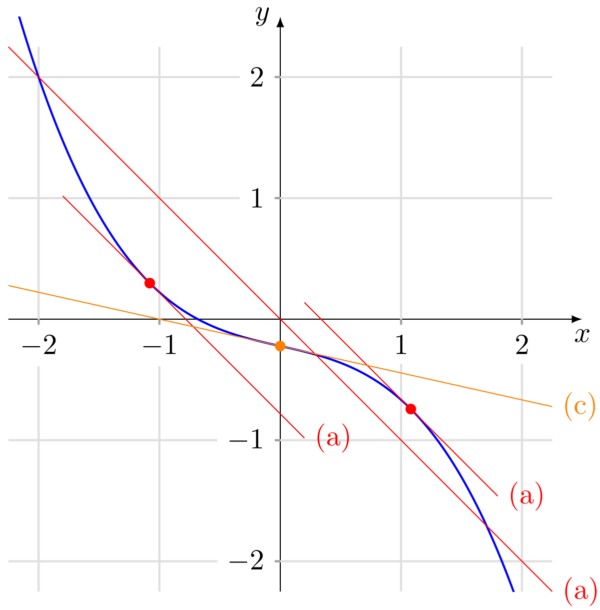
6. (a) After t hours, we have added $\left(4\,\frac{\text{liters}}{\text{hour}}\right)(t \text{ hours}) = 4t$ liters of red paint. Thus there are $\boxed{15 + 4t \text{ liters}}$ of paint after t hours.

(b) After t hours, there are $4t$ liters of red paint in $15 + 4t$ liters of paint, so the percentage of red is

$$\frac{4t\ \ell \text{ red paint}}{15 + 4t\ \ell \text{ paint}} \times 100\% = \boxed{\frac{400t}{15+4t}\%}.$$

(c) We're asked when $\dfrac{400t}{15+4t} = 25$, which is when $\boxed{t = 5/4 = 1.25 \text{ hours}}$.

7. This is again the graph of $y = f(x)$:



(a) From the graph we see that $f'(x) = -1$ when $\boxed{x \approx -1.1 \text{ and } x \approx 1.1}$. (The true values for this graph are when $x = \pm\sqrt{7/6} \approx -1.0801$.)

(b) The second derivative $f''(x)$ is positive when the graph $y = f(x)$ is concave up, which looks to be when $\boxed{x < 0}$.

(c) The slope of the graph at $x = 0$ is shown above. It looks to go through points $(x, y) = (-2, 0.2)$ and $(2, -0.6)$; thus the slope is about $\dfrac{\Delta y}{\Delta x} = \dfrac{-0.6 - 0.2}{2 - (-2)} = \boxed{-0.2}$. (This is very close to the true value, $-2/9 \approx -0.2222\ldots$.)

slope = $\boxed{}$

8. Here is the table showing the depth of water in the tank at various numbers of hours over a period of time:

hours	0	2	7	10	15	18	20
meters	20	18	15	12	8	5	4

(a) During the 20 hours, the depth of water in the tank changed at a rate of

$$\frac{4-20\text{ meters}}{20-0\text{ hours}} = -\frac{16}{20}\text{ m/hr.}$$

Since the water tank is 10 square meters in area, the total amount of water decreases at a rate of

$$\left(\frac{16}{20}\text{ m/hr}\right) \times \left(10\text{ m}^2\right) = \boxed{8\text{ m}^3/\text{hr}}.$$

(b) Here is a small chart showing the rate over the various periods of time:

time period	Δt	Δ depth	draining rate
0–2	2 hrs	-2 m	-1 m/hr
2–7	5 hrs	-3 m	$-\frac{3}{5}$ m/hr
6–10	3 hrs	-3 m	-1 m/hr
10–15	5 hrs	-4 m	$-\frac{4}{5}$ m/hr
15–18	3 hrs	-3 m	-1 m/hr
18–20	2 hrs	-1 m	$-\frac{1}{2}$ m/hr

The draining rate is least $\boxed{\text{from } t = 18 \text{ to } t = 20}$.

(c) Again, since the water tank is 10 square meters in area, the total amount of water decreases at a rate of

$$\left(\frac{1}{2}\text{ m/hr}\right) \times \left(10\text{ m}^2\right) = \boxed{5\text{ m}^3/\text{hr}}$$

during the period from $t = 18$ to $t = 20$.

9. Remember that $f(x) = 5e^x - 3x$.

(a) Since $f'(x) = 5e^x - 3$, we get $f'(0) = 5e^0 - 3 = 5 - 3 = \boxed{2}$.

(b) The tangent line at $x = 0$ has slope $f'(0) = 2$ and goes through the point $(x, y) = (0, f(0)) = (0, 5)$. Thus the tangent line is $y - 5 = 2(x - 0)$ or $y = 2x + 5$. Thus the tangent line approximation to $y = f(x)$ at $x = 0$ is $\boxed{f(x) \approx 2x + 5}$.

(c) From part (b), $f(0.1) \approx 2(0.1) + 5 = \boxed{5.2}$.

10. Initially Jason is in Paris and Marie is in Rome. The road from Rome to Paris is 1100 km long. They both start driving at the same time. Jason drives at speed J for the first hour, then speeds up to speed $2J$. Marie drives at constant speed M. They meet after 3 hours. After 1 hour of driving, they are 820 km apart.

(a) In the first 3 hours, they travel 1100 km. Jason drives

$$\left(J\ \frac{\text{km}}{\text{hr}}\right)(1\ \text{hr}) + \left(2J\ \frac{\text{km}}{\text{hr}}\right)(2\ \text{hrs}) = 5J\ \text{miles}$$

and, similarly, Marie drives

$$\left(M\ \frac{\text{km}}{\text{hr}}\right)(3\ \text{hrs}) = 3M\ \text{miles}.$$

Thus our first equation is $\boxed{5J + 3M = 1100}$. In the first hour, they travel $1100 - 820 = 280$ km, and they drive

$$\left(J\ \frac{\text{km}}{\text{hr}}\right)(1\ \text{hr}) + \left(M\ \frac{\text{km}}{\text{hr}}\right)(1\ \text{hr}) = J + M\ \text{miles}.$$

Thus our second equation is $\boxed{J + M = 280}$. (Another good second equation is $\boxed{4J + 2M = 820}$.)

(b) We'd like to solve for M in the system of two equations from part (a). One way to do this is to plug $J = 180 - M$ into the first equation and solve for M. Another is to subtract the first from 5 times the second equation to eliminate the J terms, which is what we do:

$$\begin{array}{r} 5J + 5M = 1400 \\ -(5J + 3M = 1100) \\ \hline 2M = 300. \end{array}$$

Thus $\boxed{M = 150\ \text{km/hr}}$.

11. (a) The total cost of the fence is \$5 per meter times the length of fencing, namely $2w$ meters; thus the total cost of fence is $\$10w$. The total cost of the brick is \$40 per meter times the length of fencing, namely $2L$ meters; thus the total cost of brick is $\$80L$. Hence the total cost is $\boxed{\$(10w + 80L)}$.

(b) Since $wL = 230$, we have $L = 230/w$. Thus the total length (in meters) of fencing (fence and brick) needed is $2w + 2L = 2w + 2 \cdot 230/w = \boxed{2w + 460/w\ \text{meters}}$.

(c) From part (a), the total cost of brick and fence is $\$10w + 80L = \$10w + 80 \cdot 230/w = \boxed{\$(10w + 18,400/w)}$.

(d) This asks us to minimize the function $C = 10w + 18,400/w$. This is a minimum when $C' = 0$. Since $C' = 10 - 18,400/w^2$, we find this is zero when $w^2 = 1840$, or when $\boxed{w = \sqrt{1840}}$. (Since $1840 = 16 \cdot 115$, we could write this as $w = 4\sqrt{115}$ or $w \approx 42.9$, but $\sqrt{1840}$ is fine.)

12. (a) If tickets are sold for $\$(200 + x)$ each, then we've raised the price x dollars, and thus we've lowered the number of tickets by $4x$. Thus there are $\boxed{2400 - 4x\ \text{tickets sold}}$.

(b) The total amount of money Fred gets by selling tickets for $\$(200 + x)$ each is the number of tickets, $2400 - 4x$, times this price. Thus the total amount of money generated is

$$\begin{aligned} \text{money} &= (200 + x)(2400 - 4x) = 480,000 - 800x + 2400x - 4x^2 \\ &= \boxed{480,000 + 1600x - 4x^2}. \end{aligned}$$

(c) We'd like to maximize $y = 480,000 + 1600x - 4x^2$, so we solve for x with $y' = 0$. Since $y' = 1600 - 8x$, we get $x = 1600/8 = 200$. Thus we should raise the price by \$200, so the price should be $\boxed{\$400\ \text{per ticket}}$.