Homework 3

1. Suppose $W \subset X \times Y$ is closed, and Y is compact.

(i) Prove that $\pi_X(W)$ is closed in X.

(ii) Show this fails if Y is not compact.

2. Let X be compact, a let $V_1 \supset V_2 \supset \ldots$ be a sequence of closed nonempty sets in X. Prove that $\bigcap_{i \geq 1} V_i \neq \emptyset$.

3. Suppose $f : (X, d_X) \to (Y, d_Y)$ is continuous, and suppose X is sequentially compact. Prove that $f(X)$ is bounded.

Definition. For any set A, we define the *frontier of* A (also called the *boundary of* A) to be

$$\partial A = \overline{A} \cap \overline{A^{\complement}}.$$

That is, if $x \in \partial A$, then for every open set $U \ni x$, we have $U \cap A \neq \emptyset$ and $U \cap A^{\complement} \neq \emptyset$.

4. Prove that

(i) A is closed in $X \iff \partial A \subset A$.

(ii) $\partial A = \emptyset \iff A$ is both open and closed in X.

(iii) X is connected $\iff$ every nonempty proper subset has nonempty frontier.

5. Let $A, B \subset X$ be connected, with $A \cap \overline{B} \neq \emptyset$. Prove that $A \cup B$ is connected.