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Homework 2

1. Let $\pi ; \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$ be the quotient map defined by

$$\pi(\vec{z}) = \{w(z^1, \dots, z^n, z^{n+1}) | w \in \mathbb{C}\} = [z^1, \dots, z^{n+1}] = [\vec{z}].$$

We will show that $\mathbb{CP}^n$ is a smooth compact topological manifold. First we will show that $\mathbb{CP}^n$ is locally Euclidean by producing a smooth atlas on it. Let $\widetilde{U}_i = \{\vec{z} \in \mathbb{C}^{n+1} | z^i \neq 0\}$, and let $U_i = \pi\left(\widetilde{U}_i\right)$. So an element of $U_i \subset \mathbb{CP}^n$ is of the form $[z^1, \dots, z^i \neq 0, \dots, z^{n+1}]$. Then we define a map $\widetilde{\varphi}_i : U_i \to \mathbb{C}^n$ by

$$\widetilde{\varphi}_i[\vec{z}] = \left(\frac{z^1}{z^i}, \dots, \hat{z}^i, \dots, \frac{z^{n+1}}{z^i}\right),$$

where hat denotes a removed quantity. We can see that $\widetilde{\varphi}_i$ is well-defined because taking $\widetilde{\varphi}(w\vec{z})$ just gives a multiple of ww^{-1} in each coordinate. Finally we push forward the map to $\mathbb{R}^{2n}$ in the obvious way: let

$$\varphi_i = \rho \circ \widetilde{\varphi}_i,$$

where $\rho(x^1 + iy^1, \dots, x^n + iy^n) = (x^1, y^1, \dots, x^n, y^n)$. Now $\{(U_i, \varphi_i)\}$ is an atlas on $\mathbb{CP}^n$. To see this, observe that $\{U_i\}$ covers C^{n+1} and each φ_i is continuous (since $z_i \neq 0$) with continuous inverse $\widetilde{\varphi}^{-1} \circ \rho^{-1}$, where $\widetilde{\varphi}_i^{-1} : \mathbb{C}^n \to U_i$ is given by

$$\widetilde{\varphi}_i^{-1}(\vec{w}) = \left[w^1, \dots, w^{i-1}, 1, w^{i+1}, \dots, w^n\right],$$

so $\mathbb{CP}^n$ is locally Euclidean. □

If π is open, then we can show that $\mathbb{CP}^n$ is second countable and Hausdorff. Let $U \subset \mathbb{CP}^n$ be open, then $\pi(U) = \{\xi U | \xi \in \mathbb{C}\}$, so $\pi^{-1}(\pi(U)) = \{\xi U | \xi \in \mathbb{C}\}$, which is open. Thus π is open. Since $\mathbb{R}^{2n+2} \cong C^{n+1} \supset \left(\mathbb{C}^{n+1} - \{\vec{0}\}\right)$ and $\mathbb{R}^{2n+2}$ is second countable, then so is $\pi\left(\mathbb{C}^{n+1} - \{\vec{0}\}\right) = \mathbb{CP}^n$.

Now we show that $\mathbb{CP}^n$ is Hausdorff. To do this, let

$$R = \{(\vec{z}, \vec{w}) | \pi(\vec{z}) = \pi(\vec{w})\},$$

where $\vec{z}, \vec{w} \in \left(\mathbb{C}^{n+1} - \{\vec{0}\}\right)$, and if R is closed, then $\pi\left(\mathbb{C}^{n+1} - \{\vec{0}\}\right) = \mathbb{CP}^n$ is Hausdorff. Consider $(\vec{z}, \vec{w}) \in R$. Since $[\vec{z}] = [\vec{w}]$, then there exists some $\xi \in \mathbb{C}$ such that $\xi\vec{z} = \vec{w}$, that is, $z^i w^j = z^j w^i$ for each $i, j = 1, \dots n+1$. Now let

$$f(\vec{z}, \vec{w}) = \sum_{1 \leq i,j \leq n+1} \left|\left|w^i z^j - z^i w^j\right|\right|^2$$

and we find that f is a continuous function which vanishes precisely on R. Thus since $\{0\}$ is closed, then so is $f^{-1}(\{0\}) = R$ and $\mathbb{CP}^n$ is Hausdorff. □

The fact that π is open also gives us that $\mathbb{CP}^n$ is compact. To see this, note that it is also continuous since $\pi^{-1}[\vec{z}] = [\vec{z}]$, where we think of $[\vec{z}]$ as an equivalence class on the left hand side and a set on the right hand side. This means that for an open subset of $\mathbb{CP}^n$, call it $U = \{[\vec{z}] | \vec{z} \in \Gamma\}$ for Γ an indexing set, $\pi^{-1}(U) = \bigcup_\Gamma [\vec{z}]$. For simplicity, we can simply write $\pi^{-1}(U) = U$, with the understanding that $U \in \mathbb{CP}^n$ is a collection of equivalence classes and $U \in C^{n+1} - \{\vec{0}\}$ consists of all the elements of those equivalence classes. Now denote the box

$$Q^{n+1}_{\mathbb{C}} = \{(w^1, \dots, w^{n+1}) \quad | \quad \mathrm{Re}(w^i), \mathrm{Im}(w^i) \in [-1, 1] \quad \forall i \in 1, \dots n+1\},$$

and observe that $Q^{n+1}_{\mathbb{C}} \subset \mathbb{C}^{n+1} - \{0\}$ is compact. For any open cover $\{U_\alpha\}_{\alpha\in\Gamma}$ of $\mathbb{CP}^n$,

$$\pi^{-1}\left(\{U_\alpha\}_{\alpha\in\Gamma}\right) = \{U_\alpha\}_{\alpha\in\Gamma},$$

which covers $Q^{n+1}_{\mathbb{C}}$. Since $Q^{n+1}_{\mathbb{C}}$ is compact, we can produce a finite subcover $\{U_{\alpha_i}\}_{i=1}^n$, and $\pi\left(\{U_{\alpha_i}\}_{i=1}^n\right) = \{U_{\alpha_i}\}_{i=1}^n$ which is open and covers $\mathbb{CP}^n$. To see this, let $[\vec{z}] \in \mathbb{CP}^n$ and observe that $[\vec{z}] = \frac{1}{\delta}[\vec{z}] \in \mathbb{Q}^{n+1}_{\mathbb{C}}$, where $\delta = \max_i(|\mathrm{Re}(z^i)|, |\mathrm{Im}(z^i)|)$. Thus $\{U_{\alpha_i}\}_{i=1}^n$ covers $\mathbb{CP}^n$, so it is compact. □

Thus we have shown that $\mathbb{CP}^n$ is a compact topological manifold. It only remains to be shown that it is a smooth manifold, that is, that the atlas we constructed is a smooth atlas. Let's check that two arbitrary charts (U_i, φ_i) and (U_j, φ_j) are compatible. Clearly $\varphi_j \circ \varphi_i^{-1} = \rho \circ \widetilde{\varphi}_j \circ \widetilde{\varphi}_i^{-1} \circ \rho^{-1}$ is smooth if and only if $\widetilde{\varphi}_j \circ \widetilde{\varphi}_i^{-1}$ is, so we will check the latter.

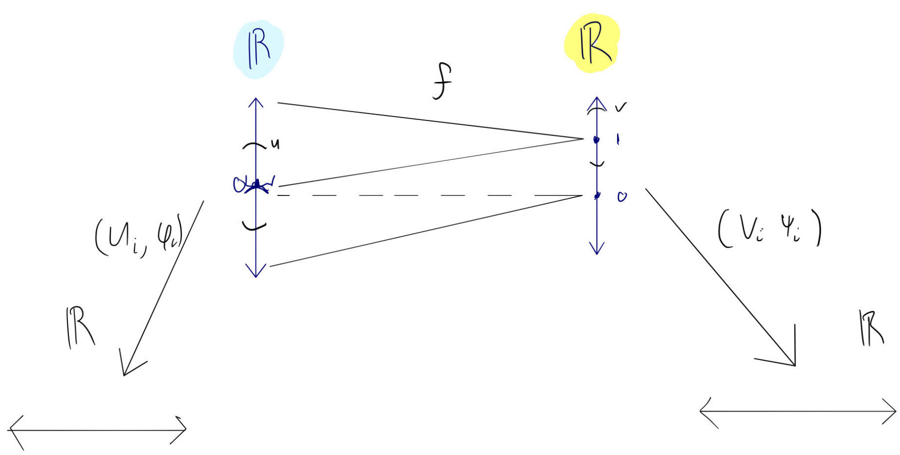
$$\widetilde{\varphi}_j \circ \widetilde{\varphi}_i^{-1}(z^1, \dots, z^n) = \left(\frac{z^1}{z^j}, \dots, \widehat{z^j}, \dots, \frac{z^{i-1}}{z^j}, \frac{1}{z^j}, \frac{z^{i+1}}{z^j} \dots, \frac{z^n}{z^j}\right),$$

which is smooth since $z^j \neq 0$ on U_j, and we are done. ■

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Consider the charts $(\mathbb{R}, \mathrm{id})$ on the domain and $(B_{\frac{1}{2}}(1), \mathrm{id}), (B_{\frac{1}{2}}(0), \mathrm{id})$ on the range.



For any $x \in \mathbb{R}$ in the domain of f, $f(x) = 0, 1$, so suppose $f(x) = 1$. Then $(\mathbb{R}, \mathrm{id})$ contains x and $(B_{\frac{1}{2}}(1), \mathrm{id})$ contains $f(x)$, and

$$\begin{aligned} \mathrm{id}\left(\mathbb{R} \cap f^{-1}\left(B_{\frac{1}{2}}(1)\right)\right) &= (\mathbb{R} \cap [0, \infty)) \\ &= [0, \infty), \end{aligned}$$

and $\mathrm{id} \circ f \circ \mathrm{id} = f$ which is constantly 1 on $[0, \infty)$, so it is smooth. Similarly, for x such that $f(x) = 0$, we can use the other chart which contains 0 and we find that $\mathrm{id} \circ f \circ \mathrm{id} = f$ is constant in that case as well. □

However, f is not a smooth function from $\mathbb{R} \to \mathbb{R}$, because for $x = 0$ in the domain, there is no chart $(U, \varphi) \ni x$ such that $f \circ \varphi^{-1}$ is smooth because f is discontinuous at 0 (the choice of chart for the range doesn't help with this). ■