Homework 8

1. Use theorems on the Fourier transform from the textbook and the lectures to execute the following steps:

(a) For a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $n \geq 2$, define $\frac{\partial_1}{|\nabla|}\phi$ using the Fourier transform. Prove that the operator $\frac{\partial_1}{|\nabla|} : \mathcal{S} \to X$ for, say, $X = \bar{C}_0(\mathbb{R}^n)$ the space of continuous functions limiting to 0 at infinity, is defined.

Definition. Define $\frac{\partial_1}{|\nabla|} : \mathcal{S} \to \bar{C}_0(\mathbb{R}^n)$ by

$$\frac{\partial_1}{|\nabla|}(\phi) = \left(\frac{k_1}{|k|}\hat{\phi}(k)\right)^{\vee}.$$

Proof We know that $\mathcal{S}$ is closed under Fourier so $k_1\hat{\phi}(k) \in \mathcal{S}$. So it suffices to show that we can take Fourier transforms of $\frac{1}{|k|}\phi$ for $\phi \in \mathcal{S}$, since $\left(\frac{1}{|k|}\phi\right)^{\vee}$ is a particular one. We can simply integrate to compute Fourier for any L^1 function, so it suffices to show that $\frac{1}{|k|}\phi \in L^1$. Note that

$$\int_{\mathbb{R}^n} \frac{1}{|k|}\phi = \int_{B_1} \frac{1}{|k|}\phi + \int_{B_1^{\complement}} \frac{1}{|k|}\phi.$$

Now $\int_{B_1} \frac{1}{|k|^{\alpha}}\phi$ is finite since $1 = \alpha < n \leq 2$, and $\int_{B_1^{\complement}} \frac{1}{|k|}\phi$ is finite since

$$\int_{B_1^{\complement}} \frac{1}{|k|}\phi \leq \int_{B_1^{\complement}} \phi < \infty$$

because $\phi \in \mathcal{S}$. So $\frac{1}{|k|}\phi \in L^1$ and we're done. ■

(b) Prove that if, for all $\phi \in \mathcal{S}$, we have that

$$\left|\left|\tfrac{\partial_1}{|\nabla|}\phi\right|\right|_q \leq C\,||\phi||_p$$

for some $p, q \in \mathbb{N}$, then $p = q$.

Proof Fix $\phi \in \mathcal{S}$. Let $\phi_\lambda = \phi(\frac{x}{\lambda})$ which is in $\mathcal{S}$ for all $\lambda > 0$. Then by using scaling, we find that

$$\left\|\frac{\partial_1}{|\nabla|}\phi_\lambda\right\|_p = \left\|\left(\frac{k_1}{|k|}\hat{\phi}_\lambda(k)\right)^{\vee}\right\|_p$$

$$= \left\|\left(\frac{k_1\lambda^n}{|k|}\hat{\phi}(\lambda k)\right)^{\vee}\right\|_p$$

$$= \left\|\left(\frac{k_1\lambda^n}{|k|}\hat{\phi}(k)\right)^{\vee}_{1/\lambda}\right\|$$

$$= \left\|\cancel{\lambda^{-n}}\left(\frac{k_1\cancel{\lambda^n}}{|k|}\hat{\phi}(k)\right)^{\vee}_{1/\lambda}\left(\frac{x}{\lambda}\right)\right\|_p$$

$$= \lambda^{1/p}\left\|\frac{\partial_1}{|\nabla|}\phi\right\|_p$$

LHS

$$\|\phi_\lambda\|_q = \left(\int_{\mathbb{R}^n}\left[\phi\left(\tfrac{x}{\lambda}\right)\right]^q\right)^{1/q}$$
$$= \lambda^{1/q}\left(\int_{\mathbb{R}^n}\phi^q\right)^{1/q}$$
$$= \lambda^{1/q}\|\phi\|_q$$

RHS

And thus we have, for all $\lambda > 0$,

$$\lambda^{\frac{1}{p}}\left\|\tfrac{\partial_1}{|\nabla|}\phi\right\|_p \le C\lambda^{1/q}\|\phi\|_q$$
$$\Rightarrow \lambda^{\frac{1}{p}-\frac{1}{q}}\left\|\tfrac{\partial_1}{|\nabla|}\phi\right\|_p \le C\|\phi\|_q$$
$$\Rightarrow \quad \frac{1}{p} = \frac{1}{q}$$

and so p must be equal to q. ■

(c) A fundamental and hard theorem of real variable theory states that the converse is true. However, for one particular p it is easy to prove that $\left|\left|\frac{\partial_1}{|\nabla|}\phi\right|\right|_p \le C\,||\phi||_p \quad \forall \phi \in \mathcal{S}$. Find that p and prove the estimate. What is your C?

Proof Since Hausdorff-Young gives us that $\left|\left|\hat{\phi}\right|\right|_2 \le ||\phi||_2$ for all $\phi \in \mathcal{S}$, then

$$\begin{aligned}\left|\left|\left(\frac{k_1}{|k|}\hat{\phi}\right)^{\vee}\right|\right|_2 &\le \left|\left|\left(\frac{k_1}{|k|}\hat{\phi}\right)\right|\right|_2 \\ &\le \left|\left|\hat{\phi}\right|\right|_2 \quad \text{because } \frac{k_1}{|k|} \le 1 \\ &< ||\phi||_2\end{aligned}$$

So $C = 1$. ■

2. Assume $f \in C^1_{\text{loc}}(\mathbb{R} \setminus \{a\})$ and $\int_{a-1}^{a+1} |f'|\, dx < \infty$. Show that (a) one sided limits $f(a \pm 0)$ are finite and (b) use them to compute the *distributional derivative* f' on $\mathbb{R}$.

Proof To see that the one-sided limits $\lim_{t \to a^\pm} f(t)$ are finite, observe that the Fundamental Theorem of Calculus gives us, for any $t \in B_1(a)$,

$$f(t) = f(a+1) - \int_t^{a+1} f'(x)\, dx,$$

and taking limits on both sides we find that

$$\lim_{t \to a^+} f(t) = f(a+1) - \int_a^{a+1} f'(x)\, dx < \infty$$

And similarly we have

$$\lim_{t \to a^-} f(t) = f(a-1) + \int_{a-1}^{a} f'(x)\, dx < \infty$$

and (a) is proved. For use in part (b), denote $f(a^+) := \lim_{t \to a^+} f(t)$ and $f(a^-) := \lim_{t \to a^-} f(t)$. $\square$

Proof For any ϕ, the distributional derivative $Df(\phi)$ is

$$\begin{aligned}
(Df)(\phi) &= -\int_{-\infty}^{\infty} f\phi' \\
&= -\int_{-\infty}^{a^-} f\phi' - \int_{a^+}^{\infty} f\phi' \\
&\overset{\text{IBP}}{=} -\left(\big[f\phi\big]_{-\infty}^{a^-} - \int_{-\infty}^{a^-} f'\phi \right) - \left(\big[f\phi\big]_{a^+}^{\infty} - \int_{a^+}^{\infty} f'\phi \right) \\
&= -\big[f(a^-)\phi(a) - f(a^+)\phi(a)\big] + \left(\int_{-\infty}^{a^-} f'\phi + \int_{a^+}^{\infty} f'\phi \right) \\
&= \int_{-\infty}^{a^-} f'\phi + \big[f(a^+) - f(a^-)\big]\phi(a) + \int_{a^+}^{\infty} f'\phi
\end{aligned}$$

where writing a^- or a^+ in the bounds denotes an improper integral, that is, $\int_{-\infty}^{a^-} := \lim_{\substack{\alpha \searrow -\infty \\ \beta \nearrow a}} \int_\alpha^\beta$.

$\blacksquare$

Chapter 6

6. Prove that the distributional derivative of a monotone nondecreasing function on $\mathbb{R}$ is a Borel measure. [Hint: Use 6.13 and 6.22]

Proof Let f be a monotonic function, so that T_f is a distribution in $\mathcal{D}'(\mathbb{R})$, and let $j \in C_c^\infty(\mathbb{R})$ with $\int j = 1$, and for each $n \in \mathbb{N}$, let $j_n = 2^n j(2^{-n}x)$. Using Theorem 6.13, then the distributions $j_n * T_f \xrightarrow{n} T_f$ and each $j_n * T_f = T_{j_n * f}$. Note that each $j_n * f$ is itself a monotone function, since if $a < b$ then

$$j_n * f(a) = \int_{\mathbb{R}} j_n(\cdot) f(a - \cdot) < \int_{\mathbb{R}} j_n(\cdot) f(b - \cdot) = j_n * f(b)$$

because f is monotone. Now observe that since f_n has a classical derivative, then the distributional derivative $DT_{f_n} = T_{f_n'}$ and f_n' is nonnegative because f_n is monotonic. Now for all $\phi \geq 0$,

$$DT_{f_n}(\phi) = T_{f_n'}(\phi) = \int_{\mathbb{R}} f'(x)\phi(x)\, dx \geq 0$$

so every DT_{f_n} is a positive distribution, and since

$$DT_{f_n}(\phi) \xrightarrow{n} DT_f(\phi),$$

for all ϕ, then it is a positive distribution as well. Thus by Theorem 6.22, we can conclude that DT_f is a Borel measure. ■

7. Let $\mathcal{N}_T$ be the null-space of a distribution, T. Show that there is a function $\phi_0 \in \mathcal{D}$ so that every element $\phi \in \mathcal{D}$ can be written as $\phi = \lambda\phi_0 + \psi$ with $\psi \in \mathcal{N}_T$ and $\lambda \in \mathbb{R}$. One says that the null-space $\mathcal{N}_T$ has 'codimension one'.

[Hint: Recall the proof that the kernel of any linear functional in any vector space has the codimension 1]

Proof Let $\widetilde{\phi} \in \mathcal{D}$ so that $T\widetilde{\phi} \neq 0$ (if this doesn't exist then $T = 0$ and $\mathcal{N}_T = \mathcal{D}$ so we're done). Denote $\phi_0 = \frac{\widetilde{\phi}}{T\widetilde{\phi}}$, so that

$$T\phi_0 = 1.$$

Then for any $\phi \in \mathcal{D}$, we can denote

$$\lambda = T\phi,$$

and observe that $T\phi = T(\lambda\phi_0)$, so $\phi - \lambda\phi_0 \in \mathcal{N}_T$. Denoting $\psi = \phi - \lambda\phi_0$, we find that

$$\phi = \lambda\phi_0 + \psi$$

and we're done. ■

8. Show that a function f is in $W^{1,\infty}(\Omega)$ if and only if $f = g$ a.e. where g is a function that is bounded and Lipschitz continuous on Ω, i.e., there exists a constant C such that

$$|g(x) - g(y)| \leq C|x - y| \quad \text{for all } x, y \in \Omega.$$

Proof ($\Longrightarrow$) Let $g \in W^{1,\infty}$. By Theorem 6.13, construct g_n to be a sequence of C^∞ functions converging to g as distributions. Since they converge as distributions to g, then they converge uniformly almost everywhere to g[†]. So there exists $N > 0$ such that for all $n > N$ and all x,

$$|g_n(x) - g(x)| < \varepsilon.$$

Then for any particular $n > N$,

$$\begin{aligned} g(x+h) - g(x) &\leq g_n(x+h) - g_n(x) + 2\varepsilon \\ &= \int_x^{x+h} g_n' \, dt + 2\varepsilon && \text{by FTC} \\ &\leq |h| \ ||g_n'||_\infty + 2\varepsilon \end{aligned}$$

□

Proof ($\Longleftarrow$)

Suppose g has Lipschitz constant C, and for each $n \in \mathbb{N}$ let

$$G_n = \frac{g(x+n^{-1}) - g(x)}{n^{-1}}$$

$\leq C$ in absolute value.

Then $\forall \phi \in L^1$, $\forall n$, $\int G_n \phi \leq \int C\phi = C\|\phi\|_1$, so every $G_n \in L^{1*} \cong L^\infty$. Note that

$$\sup_{\|\phi\|_1 = 1} \langle G_n, \phi \rangle \leq C$$

So every $\|G_n\|_\infty \leq C$. This means that the sequence $\{G_n\}_{n=1}^\infty \subset B_C(L^\infty)$, and Banach-Alaoglu gives that $B_C(L^\infty)$ is $w*$ compact,

[†]This was proved in lecture. I know how to write the proof of this fact, but I'm omitting it since we know this already.

so $\exists$ a subsequence $G_{n_k} \xrightarrow{w*} G$, that is,

$$\langle G_{n_k}, \phi \rangle \xrightarrow{k} \langle G, \phi \rangle \qquad \forall \phi \in L^1.$$

So $DG_{n_k} \xrightarrow{k} DG$, since

$$\langle DG_{n_k}, \phi \rangle = -\langle G_{n_k}, \phi' \rangle \xrightarrow{k} -\langle G, \phi' \rangle = \langle DG, \phi \rangle.$$

Note also that $G = \lim_{k\to\infty} G_{n_k} = \lim_{k\to\infty} \frac{g(x-\frac{1}{n_k}) - g(x)}{n_k^{-1}}$ which is the right derivative of g, So $G = DT_g$.
Thus $\forall \|\phi\|_1 \leq 1$ we have

$$|\langle DT_g, \phi \rangle| = |\langle G, \phi \rangle| = \lim_{n\to k} |\langle G_{n_k}, \phi \rangle| \leq \|G_{n_k}\|_\infty \|\phi\|_1 \leq C. \quad \blacksquare$$

$\blacksquare$

11. Functions in $W^{1,p}(\mathbb{R}^n)$ can be very rough for $n \geq 2$ and $p \leq n$.

(a) Construct a spherically symmetric function in $W^{1,p}(\mathbb{R}^n)$ that diverges to infinity as $x \to 0$.
Answer: Let

$$f(x) = \ln\left(|x|^{-a}\right) \chi_{B_1}$$

Where B_1 denotes the unit ball, and $a > 0$ is to be determined. Note that f is written in terms of $|x|$, so it is spherically symmetric, and as $x \to 0$, $f(x)$ clearly approaches ∞.
Claim: With the right choice of a, then $f \in W^{1,p}(\mathbb{R}^n)$.

- $f \in L^p$. On B_1, we have

$$\ln\left(|x|^{-a}\right) = \left|\ln\left(|x|^{-a}\right)\right| \leq |x|^{-a},$$

so if we choose $0 < a < \frac{n}{p}$, then

$$\left|\ln\left(|x|^{-a}\right)\right|^p \leq |x|^{-ap},$$

which is integrable over B_1.
- $\partial_i f \in L^p$. Observe that for any $i = 1 \ldots n$,

$$|\,\partial_i f|^p = \left(\frac{a x_i}{|x|^2}\right)^p \leq \frac{a^p}{|x|^p}$$

which is integrable over B_1 since $p < n$, so the claim is proved. $\blacksquare$

(b) Use this to construct a function in $W^{1,p}(\mathbb{R}^n)$ that diverges to infinity at every rational point in the unit cube.

[Hint. Write the function in (b) as a sum over the rationals. How do you prove that the sum converges to a $W^{1,p}(\mathbb{R}^n)$ function?]

Answer: Let r_j be an enumeration of the rationals. Note that the following function clearly diverges to infinity since it's value is at least that of $f(0)$ using the function f from part (b).

Let $\phi = \sum_{j=1}^{\infty} 2^{-j} f(x - r_j)$

$$\phi_n = \sum_{j=1}^{n} 2^{-j} f(x - r_j)$$

Where we work in $\mathbb{R}^d$ so that we can use n as an index.

Show: $\phi_n \nearrow \phi$ Bc $\{\phi_n\}$ cauchy in $W^{1,p}$

Choose K_0 so that $2^{-K_0+1}\|f\|_p < \varepsilon$. Then $\forall m,n > K_0$,

$$\|\phi_n - \phi_m\|_p = \left\| \sum_{j=m}^{n} 2^{-j} f(x - r_j) \right\|_p \quad \text{where } n \geq m \quad \text{Wlog}$$

$$= \left\| \sum_{j=m}^{n} 2^{-j} f(x) \right\|_p \quad \text{By } L_p \text{ shift-invariance}$$

$$= \left(\sum_{j=m}^{n} 2^{-j} \right) \|f\|_p$$

$$< 2^{-K_0+1} \|f\|_p$$

$$< \varepsilon$$

Taking ∂_i of $\phi_n = \sum_{j=1}^{n} 2^{-j} f(x - r_j)$, we find that

$$\partial_i \phi_n = \sum_{j=1}^{n} 2^{-j} \partial_i f(x - r_j).$$

For each $i = 1, \cdots, d$, Choose K_i so that $2^{-K_i+1} \|\partial_i f\|_p < \varepsilon$.

Then by the same reasoning as with f, we can see that $\forall m,n > K_i$,

$$\begin{aligned} \|\partial_i \phi_n - \partial_i \phi_m\|_p &= \left\| \sum_{j=m}^{n} 2^{-j} \partial_i f(x - r_j) \right\|_p \\ &< 2^{-K_i+1} \|\partial_i f\|_p \\ &< \varepsilon. \end{aligned}$$

thus $\forall m,n > K = \max(K_0, K_1, \cdots K_d)$,

$$\begin{aligned} \|\phi_n - \phi_m\|_{W^{1,p}} &= \|\phi_n - \phi_m\|_p + \sum_{i=1}^{d} \|\partial_i \phi_n - \partial_i \phi_m\|_p \\ &< \varepsilon + \sum_{i=1}^{d} \varepsilon \\ &= (d+1)\varepsilon \end{aligned}$$

and after rescaling, we conclude that ϕ_n is Cauchy in $W^{1,p}$. Since $W^{1,p}(\mathbb{R}^d)$ is complete, then $\phi \in W^{1,p}(\mathbb{R}^d)$ and we're done. ■