Topology Exam: September 2015.

Answer exactly **seven** of the following questions.

1. For each of the following either provide a counter-example or give a proof, quoting standard results as needed.

(i) If X and Y are nonempty spaces and $X \times Y$ is compact under the product topology then X is compact.

(ii) If X and Y are compact and $Y \subseteq X$ then $X \setminus \mathrm{int}(Y)$ is compact.

(iii) If $f\colon X \to Y$ is a continuous bijection and X is Hausdorff and Y is compact, then f is a homeomorphism.

(iv) $f\colon X \to Y$ is a continuous if and only if $f^{-1}(K)$ is compact for every compact subset $K \subset Y$.

2. (i) Prove that every compact metric space complete.

(ii) Give an example of a complete metric space that is not compact.

3. A Hausdorff space X is called normal if for every pair A, B of disjoint closed subsets of X there exist disjoint open sets containing A and B respectively. Prove that every compact Hausdorff space is normal.

4. Let $f\colon X \to Y$ be a continuous onto function of compact Hausdorff spaces.

(i) Define an equivalence relation $\sim$ on X and a bijection g such that $f = g \circ p$, where p is the quotient map $p : X \to X/\sim$.

(ii) Prove that if X and Y are compact Hausdorff and $X/\sim$ has the quotient topology then g is a homeomorphism.

(iii) Define an equivalence relation on $I \times I$ and a homeomorphism from $(I \times I)/\sim$ to T^2 where $I \times I$ is the 2-dimensional square and T^2 is the 2-dimensional torus $S^1 \times S^1$.

5. (i) Give a careful definition of connected.

(ii) Prove that the closed interval $[0, 1]$ (with the usual topology) is connected.

(iii) Show that if $H \subseteq K \subseteq \overline{H}$ and H is connected then K is connected.

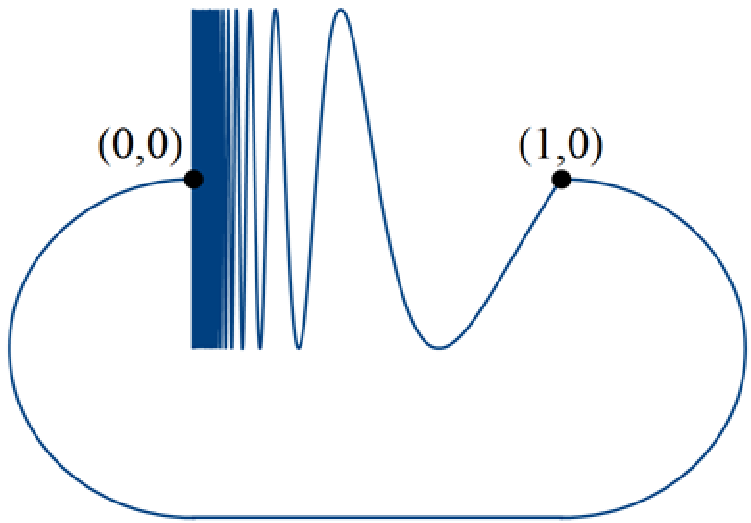
6. You may assume standard results provided you clearly state them. Let S denote the set of all closed, bounded, non-empty subsets of $\mathbb{R}$. If $A, B \in S$ then the Hausdorff distance between them, $d(A, B)$ is defined to be the infimum of all $\epsilon > 0$ such that $A \subseteq N_\epsilon(B)$ and $B \subseteq N_\epsilon(A)$. Here $N_\epsilon(C) = \{x \in \mathbb{R} \mid \exists c \in C\ |x - c| < \epsilon\}$. You may assume that this is a metric.

(a) Show that this metric is complete.

(b) Define $f\colon S \to S$ by $f(A) = \{\frac{x}{3} \mid x \in A\} \cup \{\frac{x+2}{3} \mid x \in A\}$. Show that if $A, B \in S$ then $d(f(A), f(B)) \leq (1/3)d(A, B)$.

(c) Deduce that there is a unique closed, bounded, non-empty subset of $\mathbb{R}$ that is mapped to itself.

Problems 7 and 8 use the topological space X, the closed subset of the plane shown below.



Thus, X is the union of three sets: $\{(x, \sin\frac{\pi}{x}) \mid 0 < x \leq 1\}$, $\{(0, y) \mid -1 \leq y \leq 1\}$, and a curved arc from $(0,0)$ to $(1,0)$.

7. Prove or disprove the following.

(i) X is path connected.

(ii) X is locally connected.

(iii) X is compact.

8. **(i)** Determine the fundamental group $\pi_1(X, (0,0))$.

(ii) Sketch a connected double covering of X.

(iii) State the covering space classification theorem, and explain why parts (i) and (ii) of this question are consistent with the theorem.