Math 4B
Summer Session B
Midterm
21 August 2020

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1. (1 point) Please write the following sentence:

   "I, [your name], understand that if I get the answer and show no work, it will be assumed that I copied off someone else and may be reported for cheating."

I, Shanna Zeng, understand that if I get the answer and show no work, it will be assumed that I copied off someone else and may be reported for cheating.

2 (10 points each) Find the general solution of the given differential equation. Write your answer in explicit form.

(a) $y' = t^2 e^y$

$e^{-y} y' = t^2$

$e^{-y} y' - t^2 = 0$

$e^{-y} dy - t^2 dt = 0$

$M(t,y) = -t^2$

$N(t,y) = e^{-y}$

$\frac{\partial N}{\partial t} = 0 \quad \frac{\partial M}{\partial y} = 0$

$\therefore$ They are exact

$F(t,y) = \int t^2 dt + g(y) = -\frac{1}{3}t^3 + g(y)$

$\frac{\partial F}{\partial y} = 0 + g'(y) = N(t,y)$

$g'(y) = e^{-y} \quad \therefore g(y) = -e^{-y}$

$F(t,y) = \frac{1}{3}t^3 - e^{-y}$

The solution of function is C

$C = -\frac{1}{3}t^3 - e^{-y}$

$e^{-y} = -\frac{1}{3}t^3 - C$

$-y = \ln(-\frac{1}{3}t^3 - C) \quad y = -\ln(-\frac{1}{3}t^3 - C)$

2. (10 points each) Find the general solution of the given differential equation. Write your answer in explicit form.

(b) $2u'' + 4u = 0$

use r to represent

$2r^2 + 4 = 0$

$r^2 + 2 = 0$

$r^2 = -2$

$r = \pm\sqrt{2}\,i$

$\begin{cases} r_1 = \sqrt{2}\,i \\ r_2 = -\sqrt{2}\,i \end{cases}$

The general solution is $u(x) = C_1 e^{\sqrt{2}i} + C_2 e^{-\sqrt{2}i}$

we can simply $e^{\sqrt{2}i}$ and $e^{-\sqrt{2}i}$ based on complex root

$u = e^{0x}(C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x)$

$u = C_1 \cos\sqrt{2}x + C_2 \sin\sqrt{2}x$

2. (10 points each) Find the general solution of the given differential equation. Write your answer in explicit form.

(c) $y' = \dfrac{x^{-6}(x-1)}{5y^4}$

$$5y^4 y' - x^{-6}(x-1) = 0$$

$$5y^4 dy - x^{-6}(x-1)dx = 0$$

$$M(x,y) = -x^{-5} + x^{-6}$$

$$N(x,y) = 5y^4$$

$$\frac{\partial N}{\partial x} = 0 \qquad \frac{\partial M}{\partial y} = 0$$

$\therefore$ They are exact.

$$F(x,y) = \int(-x^{-5} + x^{-6})\,dx + g(y) = \frac{1}{4x^4} - \frac{1}{5x^5} + g(y)$$

$$\frac{\partial F}{\partial y} = 0 + g'(y) = N(x,y)$$

$$g'(y) = 5y^4 \quad \therefore g(y) = y^5$$

$$F(x,y) = \frac{1}{4x^4} - \frac{1}{5x^5} + y^5$$

The solution is C

$$C = \frac{1}{4x^4} - \frac{1}{5x^5} + y^5$$

$$y^5 = C - \frac{1}{4x^4} + \frac{1}{5x^5} \qquad y = \left(C - \frac{1}{4x^4} + \frac{1}{5x^5}\right)^{\frac{1}{5}}$$

3. (10 points each) Solve the given initial value problem.

(a) $-w'' + 10w' - 25w = 0, \quad w(0) = 1, \quad w'(0) = -1$

use r to represent,

$-r^2 + 10r - 25 = 0$

$r^2 - 10r + 25 = 0$

$(r-5)^2 = 0$

$r = 5$

Because this is a single root,

the solution is given $w = C_1 e^{5t} + C_2 t e^{5t}$

$\because w(0) = 1$

$1 = C_1$

$w' = 5C_1 e^{5t} + 5C_2 t e^{5t} + C_2 e^{5t}$

$\because w'(0) = -1$

$-1 = 5C_1 + C_2$

$-1 = 5 + C_2$

$C_2 = -6$

$w = e^{-5t} - 6te^{-5t}$

3. (10 points each) Solve the given initial value problem.

(b) $xy' + (x+1)y = x^2e^{-x}, \quad x > 0, \quad y(3) = 0$

$$y' + \left(\frac{x+1}{x}\right)y = xe^{-x}$$

$$p(x) = \frac{x+1}{x}$$

$$Q(x) = xe^{-x}$$

$$\mu(t) = e^{\int \frac{x+1}{x}dx} = e^{x+\ln x} = xe^{x}$$

multiply it back in

$$xe^{x}y' + \left(\frac{x+1}{x}\right)xe^{x}y = x^2$$

Integrate the function.

$$(xe^{x}y)' = x^2$$

$$xe^{x}y = \frac{1}{3}x^3 + C$$

$$y = \frac{x^2}{3e^x} + \frac{C}{xe^x}$$

$\because y(3) = 0$

$$0 = \frac{9}{3e^3} + \frac{C}{3e^3}$$

$$C = -9$$

$$y = \frac{x^2}{3e^x} - \frac{9}{xe^x}$$

4. Given that $y_1(t) = t^{-3}$ is a solution of

$$t^2 y'' + 2ty' - 6y = 0, \quad t > 0,$$

(a) (15 points) Use the reduction of order method to find a second solution of the form $y_2(t) = t^k$.

$y_1(t) = t^{-3}$

let $y_2(t) = ut^{-3}$

$y_2'(t) = u't^{-3} - 3ut^{-4}$

$y_3'(t) = u''t^{-3} - 3u't^{-4} - 3u't^{-4} + 12ut^{-5}$

$= u''t^{-3} - 6u't^{-4} + 12ut^{-5}$

put into equation

$t^2(u''t^{-3} - 6u't^{-4} + 12ut^{-5}) + 2t(u't^{-3} - 3ut^{-4}) - 6ut^{-3} = 0$

$u''t^{-1} - 6u't^{-2} + 12ut^{-3} + 2u't^{-2} - 6ut^{-3} - 6ut^{-3} = 0$

$u''t^{-1} - 4u't^{-2} = 0$

$u'' - 4u't^{-1} = 0$

Let $w = u'$ and solve.

$w' - 4t^{-1}w = 0$

$w' = 4t^{-1}w$

$\frac{1}{w}\,dw = 4t^{-1}\,dt$

$\int \frac{1}{w}\,dw = \ln w$ (Because $t > 0$, w can exist without absolute sign)

$\int 4t^{-1}\,dt = 4\ln t + C$

$\ln w = 4\ln t + C$

$w = Ct^4$

Integrate w to find u

$u = \int Ct^4\,dx = \frac{1}{5}Ct^5 + D = Ct^5 + D$

Now $y_2 = uy_1$

$y_2(t) = t^{-3}(Ct^5 + D)$

$= Ct^2 + t^{-3}D$

We now only need to solution to be in the form $y_2(t) = t^k$. (t^{-3} is the first solution, no need to include it for second solution).

$y_2(t) = t^2$.

The second solution is $y_2(t) = t^2$.

(b) (5 points) Do y_1 and y_2 from part (a) form a fundamental set of solutions? Why?

I think y_1 and y_2 form a fundamental set of solutions.

In orden to form a fundamental set of solutions, they must fit requirements

a) . y_1 and y_2 are two solutions of

$$t^2y''+2ty'-6y=0$$

and

b) $W(y_1, y_2) \neq 0$.

a) already proven in the first question.

so we deal with b)

$$W(y_1,y_2) = \begin{vmatrix} t^{-3} & t^2 \\ -3t^{-4} & 2t \end{vmatrix} = 2t^{-2}+3t^{-2} = 5t^{-2} = \frac{5}{t^2}$$

Because $t>0$, so $\frac{5}{t^2}$ would never equals to zero.

Therefore, b is proven.

$\therefore$ y_1 and y_2 form a fundamental set of solutions.

(c) (5 points) Using your answers from parts (a) and (b), find the solution which satisfies the initial conditions $y(1) = 1$ and $y'(1) = 12$.

Because we proved y_1 and y_2 form a fundamental set of solutions, so the general solution is $y(t) = C_1 t^{-3} + C_2 t^2$.

$y'(t) = -3C_1 t^{-4} + 2C_2 t$

$\because y(1) = 1$

$1 = C_1 + C_2$

$\because y'(1) = 12$

$12 = -3C_1 + 2C_2$

$$\begin{cases} C_1 + C_2 = 1 \\ -3C_1 + 2C_2 = 12 \end{cases}$$

$$\begin{cases} C_1 = -2 \\ C_2 = 3 \end{cases}$$

$\therefore y(t) = -2t^{-3} + 3t^2$.

The solution that satisfies the inital condition is

$y(t) = -2t^{-3} + 3t^2$.

5. (10 points) Suppose y_1 and y_2 form a fundamental set of solutions to some differential equation. Let $y_3 = y_2 + y_1$. Do you think that the pair $\{y_3, y_1\}$ would also form a fundamental set of solutions, yes or no? Explain your reasoning and thoughts. You may use theory from linear algebra and/or differential equations.

I think the pair $\{y_3, y_1\}$ would also form a fundamental set of solutions.

In order to be the fundamental set of solution, y_3 and y_1 must fit in the requirement

a). y_1 and y_3 are two solutions of the differential equation

and

b) $W(y_1, y_3) \neq 0$.

a) y_1 is the solution that already proved.

when y_1 and y_2 form fundamental set of solution, the general solution is $y_1 + y_2$.

$y_3 = y_1 + y_2$, which is also a solution to the differential equation.

b) $W(y_1, y_3) = \begin{vmatrix} y_1, & y_3 \\ y_1', & y_3' \end{vmatrix} = y_1 y_3' - y_1' y_3$

use $y_1 + y_2$ to represent y_3

$$W(y_1, y_3) = y_1 (y_1 + y_2)' - y_1' (y_1 + y_2)$$

$$= \underline{y_1 y_1'} + y_1 y_2' - \underline{y_1' y_1} - y_1' y_2$$

$$= y_1 y_2' - y_1' y_2$$

Since y_1 and y_2 are already formed a fundamental set of the solution.

$$W(y_1, y_2) \neq 0$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \neq 0$$

we can see that $W(y_1, y_3) = W(y_1, y_2) \neq 0$

Therefore, $\{y_3, y_1\}$ fit all the requirement to be the fundamental set of solutions.