ALGEBRA QUALIFYING EXAM: FALL 2019.

Answer TWO questions from each section.

In the first two sections, you may use standard theorems without proof, but you must state them carefully.

1. Group Theory

G1: (a) Suppose that G is a group with exactly two subgroups. Prove that G is finite of prime order.
(b) Is the converse of part (a) true? Give a reason.

G2. Let p be a prime number and let $1 \leq n < p^2$ be an integer. Show that every Sylow p-subgroup of the symmetric group S_n is abelian.
Hint: Explicitly construct one Sylow p-subgroup by considering a subgroup generated by disjoint p-cycles.

G3. Suppose that G is a finite group with exactly three conjugacy classes. Show that G is isomorphic to the symmetric group S_3 or the cyclic group $\mathbb{Z}/3\mathbb{Z}$.
Hint: If the three conjugacy classes have r, s, t elements, show that $r + s + t = 3$ in the abelian case and is 6 in the nonabelian case.

2. Rings and Fields

RF1: An element a in a ring is called nilpotent if $a^k = 0$ for some positive integer k. Prove that the set N of nilpotent elements of a commutative ring R is an ideal of R and that R/N has no nilpotent elements.

RF2. Let $K = \mathbb{Q}(\alpha)$, where α satisfies

$$\alpha^2 = 5 + 2\sqrt{5}.$$

Show that $K/\mathbb{Q}$ is a Galois extension with Galois group $\mathbb{Z}/4\mathbb{Z}$.
Hint: Suppose that β satisfies $\beta^2 = 5 - 2\sqrt{5}$, consider $\alpha\beta$.

RF3. Let $p(x) = x^3 - x + 4$. Prove that:
a) the Galois group of $p(x)$ over $\mathbb{F}_3$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.
b) the Galois group of $p(x)$ over $\mathbb{Q}$ is isomorphic to S_3.

3. Linear Algebra

LA1. Let V be a complex vector space. Let T be a linear map $T : V \to V$, such that the characteristic polynomial of T is $(x-1)^5$, and the minimal polynomial is $(x-1)^3$. Assume that the rank of $T - I$ is 2, where I is the identity map. Determine the rational canonical form of T.

LA2. Let V be the set of 2×2 real matrices, considered as a 4-dimensional real vector space. For a real number λ, define a symmetric bilinear form $\langle , \rangle$ on V by

$$\langle A, B\rangle = \mathrm{Tr}(AB) + \lambda \mathrm{Tr}(AB^t).$$

Here Tr is the trace and B^t is the transpose of B. For which λ is this form positive definite?

LA3. Let V and W be finite dimensional complex vector spaces. Assume both V and W have dimension n. Let A and B be linear maps $V \to W$, with A an isomorphism. Show that $A + tB$ is an isormorphism for all but at most n values of $t \in \mathbb{C}$.