MATH 3B

Discussion Worksheet - Thursday, April 12

Fundamental Theorem of Calculus

for x specifically

- Fundamental Theorem of Calculus Part 1: If $g(x) = \int_a^x f(t)\,dt$ then $g'(x) = f(x)$
- BE CAREFUL: If $h(x) = \int_1^{\sin(x)} 4x\,dx$ then $h'(x) =$ $4\sin x \cdot \cos x$

"Think Chain Rule"

- Fundamental Theorem of Calculus Part 2: If F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- Definite vs. Indefinite Integrals: Definite integrals have limits of integration! $= \int_a^b$ vs. Indefinite integrals have no limits of integration => answers have "+ C"
- You Try!

(1) $\int_0^2 x(2+x^2)\,dx$

$= \int_0^2 2x + x^3\,dx = x^2 + \frac{x^4}{4}\Big|_0^2$

$= 2^2 + \frac{2^4}{4} = 4 + 4 = 8$

(2) Find $h'(x)$ if $h(x) = \int_0^{x^2} \sqrt{1+r^3}\,dr$

$h'(x) = \left(\sqrt{1+x^6}\right)2x$

(3) $\int \sqrt[3]{x}\,dx = \int x^{1/3}\,dx = \frac{3}{4}x^{4/3} + C$

U-Substitutions:

- Strategy: How to choose u: Set $u =$ a function such that ~~where~~ "du" appears elsewhere in the integral.

Then ① solve for dx & substitute & solve or ② substitute & solve

THEN, replace the "u's" w/ "x's"

- Example: $\int \sec^2(10x)\tan^7(10x)\,dx$

$u = \tan(10x)$

$du = 10\sec^2(10x)\,dx$, or $dx = \frac{du}{\sec^2(10x)}$

$= \int \frac{1}{10}u^7\,du = \frac{1}{80}u^8 \Rightarrow \boxed{\frac{\tan(10x)}{80}}$

- You Try!

(1) $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$

$u = x^2+1$

$du = 2x\,dx$

$= \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln|x^2+1|$

(2) $\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

Definite Integrals W/ U-Substitutions:

- Strategy: Same as above but substitute old limits into u for new limits
  ~~[illegible]~~ * no need to worry about x & u
- Example: $\int_{-\pi/40}^{\pi/40} \sec^2(10x)\tan^7(10x)\,dx$

$u = \tan(10x)$

$\frac{du}{10} = \sec^2(10x)\,dx$

$\tan^7\left(\frac{\pi}{40}\cdot 10\right) = \tan^7\left(\frac{\pi}{4}\right) = 1$

$\tan^7\left(\frac{-\pi}{40}\cdot 10\right) = \tan^7\left(\frac{\pi}{4}\right) = -1$

$\int_{-1}^{1} \frac{1}{10}u^7\,du = \frac{u^8}{80}\Big|_{-1}^{1}$

$\frac{1}{80} - \frac{1}{80} = 0$

- You Try!

(1) $\int_0^{\pi} \sec^2(t/4)\,dt = 4\int_0^{\pi/4} \sec^2(u)\,du$

$u = t/4$

$du = 1/4\,dt$

$4du = dt$

$= 4\tan(u)\Big|_0^{\pi/4}$

$= 4(1) - 4(0)$

$= \boxed{4}$

(2) $\int_0^2 (x-1)^{25}\,dx = \int_{-1}^{1} u^{25}\,dx$

$u = x - 1$

$du = dx$

$= \frac{u^{26}}{26}\Big|_{-1}^{1}$

$= \frac{1}{26} - \frac{1}{26} = 0$

Integrals of Piecewise Functions and the Absolute Value Function:

- Absolute value: $|x| = \begin{cases} -x & x < 0 \\ x & x \geqslant 0 \end{cases}$ so $\int_{-5}^{5} |x|\,dx = \int_{-5}^{0} -x\,dx + \int_0^5 x\,dx$

- Piecewise Functions (example): If $f(x) = \begin{cases} -x+3 & x \leqslant -1 \\ x^2+3 & x > -1 \end{cases}$ then

$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{-1} -x+3\,dx + \int_{-1}^{2} x^2+3\,dx$

$-\frac{x^2}{2} + 3x\Big|_{-2}^{1} + \frac{x^3}{3} + 3x\Big|_{-1}^{2}$

$-\frac{1}{2} + 3 + (+2+6) + \frac{8}{3} + 6 + \left(+\frac{1}{3} + 3\right) = -\frac{1}{2} + 11 + 12 = 23 - \frac{1}{2} = \boxed{\frac{45}{2}}$

CHALLENGE!
- You Try! $\int_{-3}^{4} |x^2 - 4|\,dx$

$\int_{-3}^{-2} x^2 - 4\,dx + \int_{-2}^{2} 4 - x^2\,dx + \int_2^4 x^2 - 4\,dx$

MATH 3B

Discussion Worksheet - Thursday, April 19

More U-Subs:

- Practice: For each of the following integrals, determine if (1) it is a u-sub problem and if so, (2) find u and (3) compute du. Be careful, some are tricky!

(1) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$ (2) $\displaystyle\int \frac{3+\sqrt{x}}{x^3}\,dx$

Contrast

compare

(3) $\displaystyle\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta$ (4) $\displaystyle\int_{-1}^{2} (t - 2|t|)\,dt$

(5) $\displaystyle\int \frac{x}{1+x^4}\,dx$ (6) $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx$

CHANGE TO DEFINITE INTEGRAL

Area Between Curves:

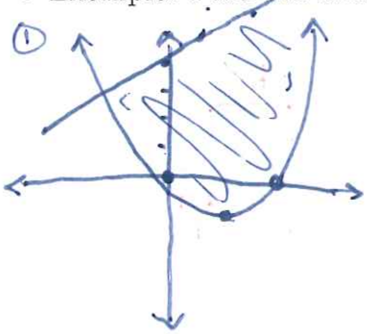
- The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is given by $A = \int_a^b |f(x) - g(x)|\,dx$

  * we can drop the absolute value if we take $f(x)$ to be the curve that lies above $g(x)$

- Strategy: (1) Sketch the graph of each curve
  (2) if you aren't given a region to integrate over, you need to find the points of intersection of your 2 curves
  (3) Integrate using the definition above (lose the abs value if you integrate top - bottom)

- Example: Find the area between the curves given by $y = x^2 - 2x$ and $y = x + 4$.

(1)



(2) $x^2 - 2x = x + 4 \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x-4)(x+1) = 0$

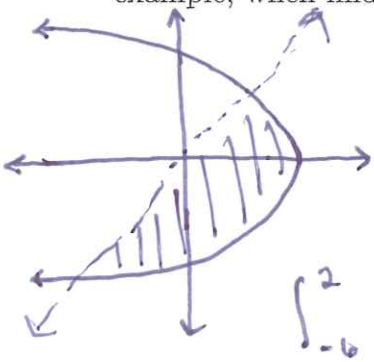
$x = 4,\ x = -1$

(3) $\displaystyle\int_{-1}^{4} x + 4 - x^2 + 2x\,dx = \int_{-1}^{4} x^2 + 3x + 4\,dx$

$\displaystyle = \frac{x^3}{3} + \frac{3x^2}{2} + 4x \Big|_{-1}^{4} = \frac{4^3}{3} + \frac{3\cdot 16}{2} + 16 + \frac{1}{3} - \frac{3}{2} + 4$

$\displaystyle = \frac{128}{6} + 24 + 20 - \frac{7}{6} = \boxed{\frac{121}{6} + 44}$

- Sometimes, curves are more easily described as functions of x in terms of y. For example, when finding the area between the curves $4x + y^2 = 12$ and $x = y$.



$4x+y^2=12$

$\boxed{x=-\frac{1}{4}y^2+3}$

$-\frac{1}{4}y^2+3=y \Rightarrow y^2-12=-4y$

$\Rightarrow y^2+4y-12=0 \Rightarrow (y+6)(y-2)=0$

$y=-6,\ 2$

$$\int_{-6}^{2} -\frac{1}{4}y^2+3-y\,dy = \frac{-y^3}{12}-\frac{y^2}{2}+3y\Big|_{-6}^{2} =$$

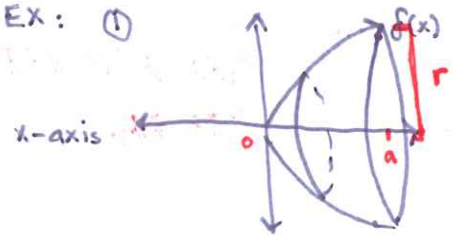
$$-\frac{2^3}{12}-2+6-\left(\frac{6^3}{12}-18-18\right) = -\frac{3}{4}+4+18 = 22-\frac{3}{4}$$

Volumes of Solids:

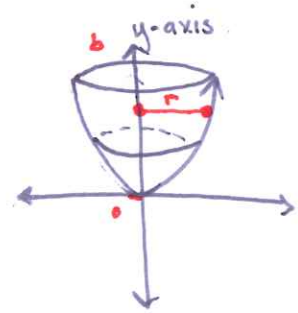
- Given a solid, we like to first think about a *cross section* of the surface which is the intersection of a plane w/ your solid (this plane is usually orthogonal to the axis of symmetry).

- Disk Method:

Ex: ①



②



Cross section:  disk $a=\pi r^2$

Cross section:  disk $a=\pi r^2$

$$A=\int_0^a \pi\left(f(x)\right)^2 dx$$

* function w.r.t. x

* limit w.r.t. x

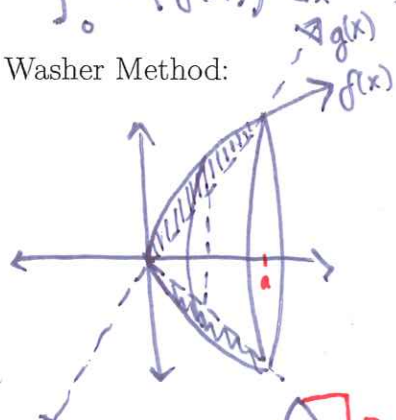
$$A=\int_0^b \pi\left(h(y)\right)^2 dy$$

* "y" function

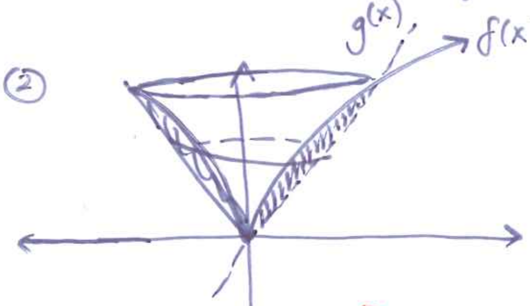
* "y" limits

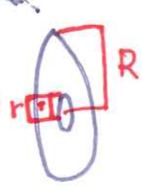
- Washer Method:

①

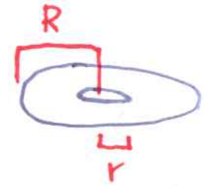


②



Cross section 

$\pi\left(R^2-r^2\right)=A=$



$$A=\int_0^a \pi\left(f(x)^2-g(x)^2\right) dx$$

$$A=\int_0^b \pi\left(h(y)^2-p(y)^2\right) dy$$

$h = g$, $p = f$