Midterm

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1. Explain how one defines the *product topology* and the *subspace topology*. Using the definitions you have given, show that Y is homeomorphic to $\{x\} \times Y$ when equipped with the subspace topology (considered as a subset of $X \times Y$).

Definition. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. We denote the *product topology on* $X \times Y$ by $\mathcal{T}_{X\times Y}$. A set $U \subseteq X \times Y$ is open with respect to $\mathcal{T}_{X\times Y}$ if there exist open sets $U_\alpha \in \mathcal{T}_X$, $V_\alpha \in \mathcal{T}_Y$ such that

$$U = \bigcup_\alpha U_\alpha \times V_\alpha.$$

Definition. Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subseteq X$. We denote the *subspace topology on* A by $\mathcal{T}_A$. A set $U \subseteq A$ is open with respect to $\mathcal{T}_A$ if there exists a open set $\widetilde{U} \in \mathcal{T}_X$ such that

$$\widetilde{U} \cap A = U.$$

Proof Let $x \in X$. We will show that the inclusion map $\iota : Y \hookrightarrow \{x\} \times Y$ (whose inverse is the projection map $\pi : \{x\} \times Y \to Y$) is a homeomorphism.

- ι is 1-1 and onto, since

  $$\iota(y_1) = \iota(y_2) \implies (x, y_1) = (x, y_2) \implies y_1 = y_2,$$

  and for any $(x, y) \in \{x\} \times Y$, there exists $y \in Y$ with $\iota(y) = (x, y)$.
- ι and π are inverses, since for any $y \in Y$,

  $$\iota \circ \pi(x, y) = \iota(y) = (x, y), \text{ and}$$
  $$\pi \circ \iota(y) = \pi(x, y) = y.$$

- ι is an open map, since for any $V \in \mathcal{T}_Y$,

  $$\iota(V) = \{x\} \times V,$$

  and we can choose any $U \in \mathcal{T}_X$ containing x to see that

  $$U \times V \cap \{x\} \times Y = \{x\} \times V,$$

  so $\{x\} \times V$ is open in the subspace topology of the product topology.
- π is an open map, since any open set in $\{x\}\times Y$ is of the form $U \times V \cap \{x\} \times Y = \{x\} \times V$, where $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$. Then

  $$\pi\left(\{x\} \times V\right) = V,$$

  Which is open in Y.

Thus ι is a continuous bijection with continuous inverse, and therefore a homeomorphism. ∎

2. Suppose that $(X, \mathcal{T})$ is a topological space with two properties: namely X is compact and Hausdorff.

Prove that one cannot make the topology on on X either coarser (i.e. is a strict subset of $\mathcal{T}$) or finer (i.e. is a strict superset of $\mathcal{T}$) without destroying one of those properties.

Proof We will show that (i) any topology finer than $\mathcal{T}$ is not compact, and (ii) any topology courser than $\mathcal{T}$ is not Hausdorff. If we call a set "open" without reference to $\mathcal{T}$ or $\mathcal{T}'$, then it is open in both topologies. The same goes for "closed".

(i) Suppose $\mathcal{T}' \supsetneq \mathcal{T}$. Then there exists $W \in \mathcal{T}'$ which is not open in $\mathcal{T}$. To see that $(X, \mathcal{T}')$ is not compact, we will produce a covering of X consisting of sets in $\mathcal{T}'$ which has no finite subcovering. Since W is not open in $\mathcal{T}$, then $W^{\complement}$ is not closed in $\mathcal{T}$, so there exists some $x \in W$ which is also in $\overline{W^{\complement}}$. Since $(X, \mathcal{T})$ is Hausdorff, for any $y \in W^{\complement}$, we can find $U_y \ni x$, $V_y \ni y$ with U_y, V_y open and disjoint. Then

$$W \cup \bigcup_{y \in W^{\complement}} V_y$$

covers X. Suppose for contradiction that this covering has a finite subcovering, call it $W \cup \bigcup_{i=1}^{N} V_{y_i}$. Then since each V has a corresponding U, then $\bigcap_{i=1}^{N} U_{y_i}$ is an open set containing $x \in \overline{W^{\complement}}$, so

$$\bigcap_{i=1}^{N} U_{y_i} \cap W^{\complement} \neq \emptyset.$$

But $\bigcap_{i=1}^{N} U_{y_i} \subset U_{y_i}$ for all U_{y_i}, so

$$\bigcap_{i=1}^{N} U_{y_i} \cap V_{y_i} = \emptyset$$

for all V_{y_i}, so $\bigcap_{i=1}^{N} U_{y_i}$ and $\bigcup_{i=1}^{N} V_{y_i}$ are disjoint. This means that $W \cup \bigcup_{i=1}^{N} V_{y_i}$ doesn't cover $\bigcap_{i=1}^{N} U_{y_i} \cap W^{\complement}$, contradiction.

(ii) Let $\mathcal{T}' \subseteq \mathcal{T}$. We will show that if $(X, \mathcal{T}')$ is Hausdorff, then $\mathcal{T}' \supseteq \mathcal{T}$, so $\mathcal{T}' = \mathcal{T}$. Let W be open in $\mathcal{T}$, and let $x \in W$. Since $(X, \mathcal{T}')$ is Hausdorff, then for every $y \in W^{\complement}$, there exist sets $U'_y \ni x$, $V'_y \ni y$ which are open in $\mathcal{T}'$ and disjoint. Since $W^{\complement}$ is closed in $(X, \mathcal{T})$, and $(X, \mathcal{T})$ is compact, and $\{V'_y\}_{v \in W^{\complement}} \subset \mathcal{T}' \subseteq \mathcal{T}$, then we can produce a finite subcover $\{V'_{y_i}\}_{i=1}^{N}$ of $W^{\complement}$. Now for each V' we have a corresponding U', so by similar reasoning as in (i) we find that $\bigcap_{i=1}^{N} U'_{y_i}$ is open in $\mathcal{T}'$, disjoint with $\bigcup_{i=1}^{N} V'_{y_i}$ so a subset of W, and contains x by construction. Thus by the openness criterion, W is open in $\mathcal{T}'$.

$\blacksquare$

3. Let X and Y be topological spaces and let $f : X \to Y$ be a function.

(i) Show that f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$, where $\overline{A}$ denotes the closure of A.

Proof
$(\implies)$ Let $A \subset X$, let $x \in \overline{A}$, and let $V \in Y$ be any open set containing $f(x)$. Then since f is continuous, $f^{-1}(V)$ contains x and is open. Since $x \in \overline{A}$, then $f^{-1}(V) \cap A \neq \emptyset$, so there exists $a \in f^{-1}(V) \cap A$ such that $f(a) \in V \cap f(A)$. Thus $f(x) \in \overline{f(A)}$, and we are done. $\square$
$(\impliedby)$ Let $D \in Y$ be closed. Then $f\left(\overline{f^{-1}(D)}\right) \subset \overline{f(f^{-1}(D))} = \overline{D} = D$, so $\overline{f^{-1}(D)} \subseteq f^{-1}(D)$ and it is always true that $\overline{f^{-1}(D)} \supseteq f^{-1}(D)$, therefore they are equal and $f^{-1}(D)$ is closed. $\blacksquare$

(ii) Show that if f is continuous and $f(\overline{A})$ is closed, then $f(\overline{A}) = \overline{f(A)}$.

Proof We know already that $f(\overline{A}) \subseteq \overline{f(A)}$ by (i), so we need to show that $\overline{f(A)} \subseteq f(\overline{A})$. Let $y \in \overline{f(A)}$. This means that for any open $V \ni y$, we have $V \cap f(A) \neq \emptyset$.
Now suppose for contradiction that $y \notin f(\overline{A})$. Since $f(\overline{A})$ is closed, then $f(\overline{A})^{\complement}$ is open, and $y \in f(\overline{A})^{\complement}$. So there exists an open set V' such that $y \in V' \subset f(\overline{A})^{\complement}$ and furthermore that $V' \subset f(A)^{\complement}$. But since $y \in \overline{f(A)}$, then $V' \cap f(A) \neq \emptyset$, contradiction. $\blacksquare$

4. Let (X, d) be a compact metric space and let $f : X \to X$ be a map with the property that

$$d(f(x), f(y)) < d(x, y)$$

for every distinct $x, y \in X$.

Prove that (i) there is a unique point x_0 with $f(x_0) = x_0$, and (ii) show that this fails if the inequality is not always strict.

Proof (i) If such a point exists, it is unique. If not and there exist two invariant points $x_0 \neq x_1$, then $d(x_0, x_1) > d(f(x_0), f(x_1)) = d(x_0, x_1)$, contradiction.

It remains to be shown that there exists an invariant point. Observe that f is continuous (and in fact, uniformly continuous), since for any $\epsilon > 0$, if $d(x, y) < \epsilon$, then $d(f(x), f(y)) < \epsilon$.

Let $x_1 \in X$, and define $\{x_n\}_{n=1}^{\infty}$ by $x_{n+1} = f(x_n)$ for all $n > 1$. Since X is compact, then it is sequentially compact, so $\{x_n\}$ has a convergent subsequence (and without loss of generality suppose that subsequence is $\{x_n\}$ itself), and call the limit x_0. Then

$$\begin{aligned} f(x_0) &= f\left(\lim_{n\to\infty}(x_n)\right) && \text{and since } f \text{ is continuous,} \\ &= \lim_{n\to\infty} f(x_n) \\ &= \lim_{n\to\infty}(x_{n+1}) \\ &= x_0. \end{aligned}$$

(ii) If we modify the property to be $d(f(x), f(y)) \leq d(x, y)$, then we do not have uniqueness since the identity $f(x) = x$ satisfies the property. Existence is still guaranteed. $\blacksquare$

5. Let A and B be disjoint compact subspaces of a Hausdorff topological space X. Prove that there are disjoint open sets U and V such that $U \supset A$ and $V \supset B$.

Proof Let $x \in A$. Since X is Hausdorff, for every $y \in B$ there exist disjoint open sets $U_{x_y} \ni x$ and $V_{x_y} \ni y$. Then $\bigcup_{y \in B} V_{x_y}$ is a covering of the compact set B, so it has a finite subcover $\bigcup_{i=1}^{N} V_{x_{y_i}}$. Using the corresponding U sets, define

$$U_x = \bigcap_{i=1}^{N} U_{x_{y_i}} \quad \text{and} \quad V_x = \bigcup_{i=1}^{N} V_{x_{y_i}},$$

and observe that U_x and V_x are disjoint open sets such that $U_x \ni x$ and V_x covers B.[†]

Construct similarly U_x and V_x for every $x \in A$. Then $\bigcup_{x \in A} U_x$ is an open cover of the compact set A, so it has finite subcover $\bigcup_{i=1}^{N} U_{x_i}$. Using the corresponding V sets, define

$$U = \bigcup_{i=1}^{N} U_{x_i} \quad \text{and} \quad V = \bigcap_{i=1}^{N} V_{x_i},$$

and observe that since every V_{x_i} covers B and is disjoint with U_{x_i}, then $U \supset A$, $V \supset B$, and U, V are open and disjoint. ∎

[†]They are disjoint because $\forall i, U_x \subset U_{x_{y_i}} \subset V_{x_{y_i}}^{\complement}$, open because finite unions and intersections of open sets are open, and cover x and B respectively by construction.