2. Are the following statements true or false? Give a proof or counterexample as appropriate.

(a) A closed bounded subset of a topological space is compact.

This is false. Let $\mathbb{R}$ be given the discrete topology. Then all sets are closed & so the set $[0,1]$ is closed & bounded. However, the open cover $\{\{x\}\}_{x \in [0,1]}$ cannot have a finite subcover since each point is only represented once in the cover.

(b) The image of a closed subset under a continuous map is closed.

This is false. Let $f:(\mathbb{R}, \mathcal{J}_1) \to (\mathbb{R}, \mathcal{J}_2)$ be the map $f(x)=x$ where $\mathcal{J}_1$ is the discrete topology & $\mathcal{J}_2$ is the indiscrete topology. Then $f^{-1}(\mathbb{R})=\mathbb{R}$ & $f^{-1}(\emptyset)=\emptyset$ so f is continuous. However, $f(\{0\})=0$ & $\{0\}$ is closed in $(\mathbb{R}, \mathcal{J}_1)$ but is not in $(\mathbb{R}, \mathcal{J}_2)$.

(c) If $f: X \to Y$ is a continuous surjection & Y is Hausdorff then so is X.

Let $X =$  $= \{A, B, C\}$ with topology $\{X, \emptyset, \{A,B\}, \{C\}\}$

& let $Y =$  $= \{A, C\}$ with topology $\{Y, \emptyset, \{A\}, \{C\}\}$

Then $f: X \to Y$ where $f(A)=A$, $f(B)=A$ & $f(C)=C$ is continuous since $f^{-1}(Y)=X$, $f^{-1}(\emptyset)=\emptyset$, $f^{-1}(\{A\})=\{A,B\}$ & $f^{-1}(\{C\})=\{C\}$ & f is a surjection. Y is Hausdorff but X is not. False

(d) If $f: X \to Y$ is a continuous surjection & X is Hausdorff then so is Y.

This is false. Let $f: (\mathbb{R}, \mathcal{J}_1) \to (\mathbb{R}, \mathcal{J}_2)$ where $\mathcal{J}_1, \mathcal{J}_2, f$ all from part (b) which we already showed is continuous & is also surjective. $\mathcal{J}_1$ is the discrete topology so it is Hausdorff. However the only open sets in Y are $\mathbb{R}$ & $\emptyset$ so it is not.