Math 201A, Final Exam Problems

Problem1. Let X be a nonempty topological space and let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of Borel-regular measures on X. Assume for any $A \subset X$ the sequence $\mu_n(A)$ decreases and define $\mu(A) = \lim_{n\to\infty} \mu_n(A)$. Prove that if $\mu_1(X) < \infty$, then μ is a measure on X.

Problem2 Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue-measurable. Prove that there exists a Borel-measurable function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ a.e. in $\mathbb{R}$.

Problem3 Let X be nonempty and let μ be a measure on X. Assume $A_n \subset X$ are $\mu-$measureble for $n = 1, 2, \dots$ and assume the sequence χ_{A_n} converges in measure to some function $f\colon X \to \mathbb{R}$. Prove that there exists a μ-measurabe set $A \subset X$ such that $f = \chi_A$ $\mu-$a.e. in X.

Problem4. Let X be nonempty and let μ be a measure on X. Assume $f_n, f\colon X \to \mathbb{R}$ are $\mu-$measureble functions ($n = 1, 2, \dots$) such that for each $\epsilon > 0$ one has

$$\sum_{n=1}^{\infty} \mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \infty.$$

Prove that $f_n \to f$ $\mu-$a.e. in X.