Math 201A, Homework 1 (basic measure theory)

Problem1. Prove that every open set $U \subset \mathbb{R}^n$ can be expressed

1. As a countable union of open balls in $\mathbb{R}^n$.
2. As a countable union of closed balls in $\mathbb{R}^n$.

Problem2. Does there exist an infinite σ-algebra that has countably many elements?

Problem3. Is it true that if μ is a Borel measure on a nonempty set X, then for any sets $A, B \subset X$ with $\text{dist}(A, B) > 0$, one has

$$\mu(A \cup B) = \mu(A) + \mu(B)?$$

Problem4. Let X be an uncountable set and let $\mathcal{C}$ be the collection of all subsets A of X such that either A or A^c is at most countable. Prove that $\mathcal{C}$ is a $\sigma-$algebra.