Real Analysis - Evans/Gariepy

Math 201, UCSB
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Contents

Note: If you find any typos in these notes, please let me know at trevorklar@math.ucsb.edu. If you could include the page number, that would be helpful.

1 General Measure Theory

1.1 Measures and Measurable Functions

1.1.1 Measures

Notation.

- Let X denote a nonempty set, the whole space.
- Let 2^X denote the collection of all subsets of X, that is, the power set of X.
- Whenever we take $\mu(A)$ for some set A, it is implied that $A \subseteq X$.
- Let $A^{\complement}$ denote $X - A$. For the complement of A in a set B (not the whole space), we will use $B - A$.
- Since $\subset$ is ambiguous, I will exclusively use $\subseteq$ and $\subsetneq$ unless I don't know (or don't care) which I aught to use.

Remark. In the statements of theorems and definitions, I have sometimes included what I call "slogans"; verbal statements of the fact which have been designed to be primarily evocative and memorable, though they are also written to be as rigorous as possible while first prioritizing intuition and ease of reading.

Definition. Let $\mu : 2^X \to [0, \infty]$ be a function. We say μ is a **measure on** X if

(i) $\mu(\emptyset) = 0$

"The empty set has measure zero"

(ii)
$$\begin{aligned} \text{If} \qquad & A \subseteq \bigcup_{n=1}^{\infty} A_n, \\ \text{then} \qquad & \mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n). \end{aligned}$$

"The measure of a set is bounded by the sum of measures for any cover of the set"

Remark. The above implies two properties of measures which are usually just listed explicitly in the definition of a measure:

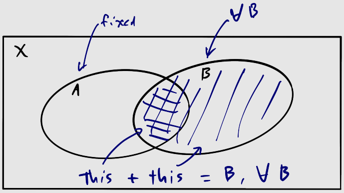
- **(Monotonicity)** If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.
- **(Subadditivity)** $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$

Definition. Let μ be a measure on X and $C \subseteq X$. Then μ *restricted to* C, written $\mu|_C$,[†] is the measure defined by

$$\mu|_C(A) := \mu(A \cap C).$$

Definition. **(Caratheodory criterion)** Let $A \subseteq X$. We say that A is **μ-measurable** if

$$\forall B \subseteq X, \quad \mu(B) = \mu(B \cap A) + \mu(B - A).$$



Theorem.

(i) A set A is μ-measurable iff $A^{\complement}$ is μ-measurable.

(ii) If $\mu(A) = 0$, then A is μ-measurable.

(iii) For any $C \subseteq X$, every μ-measurable set is also $\mu|_C$-measurable.

Proof. (i) We can prove both directions at once. Suppose A is μ-measurable. Then for any $B \subseteq X$,

$$\begin{aligned} \mu(B) &= \mu(B \cap A) + \mu(B - A) \\ &= \mu\left(B - A^{\complement}\right) + \mu\left(B \cap A^{\complement}\right) \end{aligned}$$

□

[†]The author uses a notation that looks like $\mu|\_\,C$, but (a) I don't like it and it's nonstandard, and (b) I can't figure out how to typeset it (DeTeXify didn't help). If anyone knows, let me know.

(ii) Suppose $\mu(A) = 0$. By subadditivity it is always true that

$$\mu(B) \leq \mu(B \cap A) + \mu(B - A),$$

so it suffices to prove the opposite inequality. For any $B \subseteq X$, $(B \cap A) \subseteq A$ so $\mu(B \cap A) = 0$, and $B \supseteq (B - A)$ so

$$\begin{aligned} \mu(B) &\geq \mu(B - A) \\ &= \mu(B - A) + \mu(B \cap A) \end{aligned}$$

and we're done. □

(iii) Let A be μ-measurable and let $B \subseteq X$. Then

$$\begin{aligned} \mu|_C(B) &= \mu(B \cap C) \\ &= \mu((B \cap C) \cap A) + \mu((B \cap C) - A) \\ &= \mu((B \cap A) \cap C) + \mu((B - A) \cap C) \\ &= \mu|_C(B \cap A) + \mu|_C(B - A) \end{aligned}$$

■

Corollary. The sets $\emptyset$ and X are μ-measurable.

PROOF. The empty set always has measure zero, so it is μ-measurable, and $X = \emptyset^{\complement}$, so it is also μ-measurable. ■

Exercise. Prove or disprove: For any $C \subseteq X$, if A is $\mu|_C$-measurable, then $A \cap C$ is μ-measurable. (If false, does it help if we also assume A is μ-measurable?)

PROOF. False. Suppose C is not μ-measurable and let $A = X$. Then X is μ-measurable and $\mu|_C$-measurable, but $X \cap C = C$ is not μ-measurable. To see this, observe that for any B,

$$\mu((B \cap X) \cap C) + \mu((B - X) \cap C) = \mu((B \cap X) \cap C) = \mu(B \cap C),$$

so X is μ-measurable and the rest follows. ■

Definition. Let $\{A_i\}_{i=1}^{\infty}$ be a collection of sets. We say that $\{A_i\}_{i=1}^{\infty}$ is *monotonically increasing* (or simply *increasing*)if

$$A_1 \subseteq \cdots \subseteq A_i \subseteq A_{i+1} \subseteq \cdots$$

and we define *monotonically decreasing* and *decreasing* similarly. We may use *monotonic* to refer to a sequence which could be either one.

Proposition. If A_1, A_2 are μ-measurable, then

(i) $A_1 \cup A_2$ is μ-measurable

(ii) $A_1 \cap A_2$ is μ-measurable

(iii) $A_1 - A_2$ is μ-measurable

Proof. (i) Let $B \subseteq X$. Then

$$\begin{aligned} \mu(B) &= \mu(B \cap A_1) + \mu(B - A_1) \\ &= \mu(B \cap A_1) + \mu((B - A_1) \cap A_2) + \mu((B - A_1) - A_2) \\ &= \left[\mu(B \cap A_1) + \mu((B - A_1) \cap A_2)\right] + \mu(B - (A_1 \cap A_2)) \\ &= \mu(B \cup (A_1 \cap A_2)) + \mu(B - (A_1 \cap A_2)) \end{aligned}$$

so $(A_1 \cup A_2)$ is μ-measurable. □

(ii) To see that $(A_1 \cap A_2)$ is μ-measurable, observe that $(A_1 \cap A_2)^{\complement} = A_1^{\complement} \cup A_2^{\complement}$, and we know this is μ-measurable by (i). □

(iii) To see that $(A_1 - A_2)$ is μ-measurable, observe that $(A_1 - A_2) = (A_1 \cap A_2^{\complement})$, and we know this is μ-measurable by (ii). ■

Theorem (Sequences of μ-measurable sets). Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of sets which are all μ-measurable. Then

(i) The sets $\bigcap_{i=1}^{\infty} A_i$ and $\bigcup_{i=1}^{\infty} A_i$ are μ-measurable.
"Countable unions and intersections of μ-measurable sets are μ-measurable."

(ii) If the sets $\{A_i\}_{i=1}^{\infty}$ are disjoint, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty}\left(\mu(A_i)\right)$.
"Subadditivity gives equality for disjoint μ-measurable sets."

(iii) If $\{A_i\}_{i=1}^{\infty}$ is an increasing sequence, then $\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$.

(iv) If $\{A_i\}_{i=1}^{\infty}$ is decreasing sequence and $\mu(A_1)$ is finite, then $\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$.

$\left.\begin{matrix}\text{(iii)}\\ \text{(iv)}\end{matrix}\right\}$ *"Bounded monotonic sequences of sets converge in measure."*

Remark. The slogan for (iii) and (iv) above has the advantage of being easy to quote, but strictly speaking, an upper bound is not necessary for increasing sequences, since we're fine with the measure being infinity.

Proof. We will show that (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (i).

(ii) Suppose $\{A_i\}_{i=1}^{\infty}$ are μ-measurable and disjoint. Then let

$$\begin{aligned} B_1 &= A_1, \\ B_2 &= A_1 \cup A_2, \\ &\vdots \\ B_j &= \bigcup_{i-1}^{j} A_j. \end{aligned}$$

Then since each A_i is μ-measurable,

$$\begin{aligned} \mu(B_i) &= \mu(B_i \cap A_i) + \mu(B_i - A_i) \\ &= \mu(A_i) + \mu(B_{i-1}) \end{aligned}$$

and inductively, $\mu(B_i) = \mu(A_1) + \cdots + \mu(A_i)$. Thus using the definition of B_i we can write

$$\mu\left(\bigcup_{j=1}^{i} A_j\right) = \sum_{k=1}^{i} \mu(A_k)$$

for each k. Since $\bigcup\limits_{j=1}^{i} A_j \subset \bigcup\limits_{j=1}^{\infty} A_j$, then

$$\sum_{k=1}^{i} \mu(A_k) = \mu\left(\bigcup_{j=1}^{i} A_j\right) \leq \mu\left(\bigcup_{j=1}^{\infty} A_j\right)$$

by monotonicity, and since the above sum is bounded by the infinite union for all i, then so is its limit $\lim\limits_{i\to\infty} \sum\limits_{k=1}^{i} \mu(A_k)$, and

$$\sum_{k=1}^{\infty} \mu(A_k) \leq \mu\left(\bigcup_{j=1}^{\infty} A_j\right).$$

Since the reverse inequality follows from subadditivity, then (ii) is proved. $\square$

(iii) Let $\{A_i\}_{i=1}^{\infty}$ be an increasing sequence of μ-measurable sets. Let

$$B_1 = A_1, \quad B_i = A_i - A_{i-1}.$$

Then $\{B_i\}_{i=1}^{\infty}$ is a disjoint μ-measurable collection. So by (ii),

$$\lim_{i\to\infty} \mu(A_i) = \sum_{i=1}^{\infty} \Big(\mu(B_i)\Big) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$$

and (iii) is proved. $\square$

(iv) Let $\{A_i\}_{i=1}^{\infty}$ be a decreasing sequence of μ-measurable sets, with $\mu(A_1)$ finite. Let

$$B_i = A_1 - A_i.$$

Then $\{B_i\}_{i=1}^{\infty}$ is an increasing collection of μ-measurable sets, So by (iii),

$$\lim_{i\to\infty} A_i = A_1 - \lim_{i\to\infty} B_i = A_1 - \bigcup_{i=1}^{\infty} B_i = \bigcap_{i=1}^{\infty} A_i$$

and (iv) is proved. □

(i) Let $\{A_i\}_{i=1}^{\infty}$ be a collection of μ-measurable sets, and let C be any set in X. Now define $B_i = \cup_{j=1}^{i} A_j$. Next we check that $\bigcup_{i=1}^{\infty} A_i$ is μ-measurable:

$$\begin{aligned}
\mu\left(C \cap \bigcup_{i=1}^{\infty} A_i\right) + \mu\left(C - \bigcup_{i=1}^{\infty} A_i\right)& \\
&= \mu|_C\left(\bigcup_{i=1}^{\infty} A_i\right) + \mu|_C\left(\bigcap_{i=1}^{\infty} A_i^{\complement}\right) \\
&= \mu|_C\left(\bigcup_{i=1}^{\infty} B_i\right) + \mu|_C\left(\bigcap_{i=1}^{\infty} B_i^{\complement}\right) \\
&= \lim_{i\to\infty} \mu|_C(B_k) + \lim_{i\to\infty} \mu|_C(B_k^{\complement}) \\
&= \lim_{i\to\infty} \left[\mu|_C(B_k) + \mu|_C(B_k^{\complement})\right] && (*) \\
&= \lim_{i\to\infty} \mu|_C(X) && (\dagger) \\
&= \lim_{i\to\infty} \mu(C) \\
&= \mu(C)
\end{aligned}$$

$(*)$ We can do this since each of the limits converge, so the sum of the limits is the limit of the sum. To see that the limits converge, observe that B_k is an increasing sequence of sets, and $B_k^{\complement}$ is a decreasing sequence restricted to C which has finite measure. $(\dagger)$ We can do this because B_k and $B_k^{\complement}$ are $\mu|_C$-measurable disjoint sets whose sum is X. Thus $\bigcup_{i=1}^{\infty} A_i$ is μ-measurable.

To see that intersections are also μ-measurable, observe that $\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^{\complement}\right)^{\complement}$. Thus (i) is proved, so we are done. ■

1.1.2 Systems of Sets

Definition. We say a collection of subsets $\mathcal{A} \subset 2^X$ is a **σ-algebra** when *"$\mathcal{A}$ contains $\emptyset$ and X and is closed under complements, countable unions, and countable intersections."* That is,

- $\emptyset, X \in \mathcal{A}$
- If $A \in \mathcal{A}$, then $A^{\complement} \in \mathcal{A}$.
- If $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.
- If $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$.

Theorem. Let μ be measure on X. The collection of all μ-measurable subsets of X is a σ-algebra.

PROOF. We have already proved the following properties:

- $\emptyset, X$ are μ-measurable by definition of a measure.
- The complement of a measurable set is μ-measurable.
- Countable unions and complements of μ-measurable sets are μ-measurable.

■

Definition. Let $\mathcal{C} \subset 2^X$ be a nonempty collection. Then the **σ-algebra generated by $\mathcal{C}$**, denoted

$$\sigma(\mathcal{C}),$$

is the closure of $\mathcal{C}$ under all the operations of a σ-algebra.

Following is the most important σ-algabra in $\mathbb{R}^n$, and probably the one you are thinking of.

Definition. Let $\mathcal{U} = \{U \subset \mathbb{R} : U \text{ is open}\}$. Then we call $\sigma(\mathcal{U})$ **the Borel σ-algebra of $\mathbb{R}^n$**.

More often, we simply refer to sets in the Borel algebra as **Borel sets**.

Another way to think of the Borel algebra is that it is the σ-algebra generated by the topology on the space.

Definition. A **Borel set** is a set formed by countably many intersections, unions, and/or complements of open and/or closed sets in X.

Definition. We say that μ is a **Borel measure** is every Borel set is μ-measurable.

Definition. A **π-system**