Midterm Exam

0. (a) **Definition.** Let $S \subset \mathbb{R}^k$. We says that a map $\phi : S \to \mathbb{R}^q$ is *smooth* if all partial derivatives (of all orders) of ϕ exist.

(b) **Definition.** Let $X \subset \mathbb{R}^n$, and $x \in X$. A *chart of X near x* is a diffeomorphism ϕ between open sets $U \ni \phi^{-1}(x)$ and $V \ni x$ where $U \subset \mathbb{R}^k$ (or $\mathbb{H}^k$ in the case of manifolds with boundary), and $V \subset X$.

Remark. We generally assume $\phi(0) = x$, unless we have reason to do otherwise.

Definition. Let $X \subset \mathbb{R}^n$. We say that X is a *smooth k-manifold with boundary* if every $x \in X$ has a chart $\phi : U \subset \mathbb{H}^k \to V \subset X$.

Remark. Since any point x with a chart from $\mathbb{R}^k$ also has a chart from the interior of $\mathbb{H}^k$ (just shift the domain up enough), then if we just say *smooth manifold*, we mean a smooth manifold with boundary (whose boundary may or may not empty).

(c) **Definition.** From calculus, the *derivative of f at x in the direction of v* is

$$\lim_{t \to 0} \frac{f(\vec{x} + t\vec{v}) - f(\vec{x})}{t}$$

Definition. Let X be a smooth k-manifold with $x \in X$, and assume that a chart ϕ has $\phi(0) = x$. We define *the tangent space of X at x* as

$$T_x(X) = d\phi_0(\mathbb{R}^k),$$

that is, the tangent space is the image of the derivative of the chart.

(d) **Definition.** Let $f : X \xrightarrow{smooth} Y$ and let $y \in Y$. We say y is a *regular value* if, for every $x \in f^{-1}(y)$, we have that df_x is surjective. ■

1. Suppose that $M^m \subset \mathbb{R}^n$ is a smooth manifold without boundary and that $h : M \to \mathbb{R}$ is a smooth map for which 0 is a regular value. Prove that $h^{-1}([0,\infty))$ is a manifold with boundary.

Proof Let $y \geq 0$ and let $x = h^{-1}(y)$.

Case I: If y is strictly positive, then $h \in h^{-1}((0,\infty))$, which is open in M, and M is a manifold so it is locally diffeomorphic to $\mathbb{R}^m$.

This means there exists open sets U, U' such that $x \in U \subset h^{-1}((0,\infty))$ and $U' \subset \mathbb{R}^m$, and a chart $\phi : U' \to U$. As long as we choose U so that $\operatorname{diam}(U') < \infty$,[†] we can choose k large enough that $\phi'(\vec{x}) = \phi(\vec{x} - k\vec{e}_m)$ is a chart from $\widetilde{U} \subset \mathbb{H}^k \to U'$. $\square$

Case II: If $y = 0$, then y is a regular value, so dh_x has rank 1, and $\ker dh_x$ has dimension $(m-1)$. Let T be an invertible linear transformation from $\ker dh_x \to \mathbb{R}^{m-1}$, and extend it to one on all $\mathbb{R}^n$.[‡] Then define

$$\begin{aligned} H &: M \to \mathbb{R}^{m-1} \times \mathbb{R} \\ H(\xi) &= (T\xi, h(\xi)). \end{aligned}$$

Now we can see that

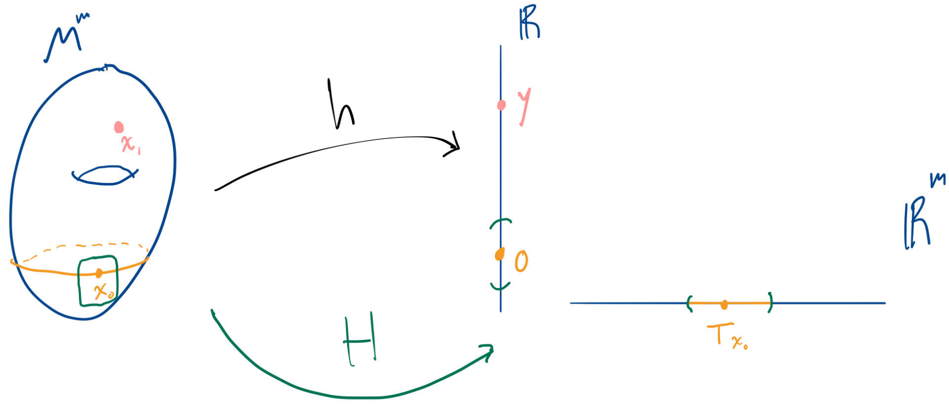
$$dH_x(v) = (Tv, dh_x(v))$$

which has rank $(m-1) + 1 = m$. Thus dh_x is an isomorphism, so by the Inverse Function Theorem there exist neighborhoods

$$U \ni x, \quad V \ni (Tx, 0)$$

where h is a diffeomorphism.

[†]We can always do this, just restrict ϕ to the unit ball centered at $\phi^{-1}(x)$.
[‡]Recall that $M \subset \mathbb{R}^n$.



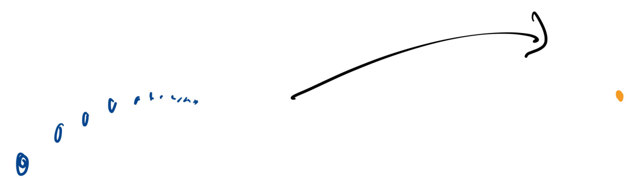
By intersecting $U \cap f^{-1}([0,\infty))$ and $V \cap \mathbb{H}^m$ and observing that the two sets correspond under H, we obtain an open neighborhood of x (with the subspace topology) which is diffeomorphic via H to an open neighborhood in $\mathbb{H}^m$. □

Thus in either case, we can produce a neighborhood of x in $f^{-1}([0,\infty))$ diffeomorphic to an open set in $\mathbb{H}^k$, so $f^{-1}([0,\infty))$ is a k-manifold with boundary. ■

2. Suppose that $f : X \to Y$ is a smooth map between compact manifolds without boundary of the same dimension. Suppose that $y \in Y$ is a regular value.

Show that $f^{-1}(y)$ is a finite set $\{x_1, \ldots x_n\}$. Show further that there is an open neighborhood V of y so that $f^{-1}(V)$ is a finite disjoint of open sets $\{U_1, \ldots, U_n\}$, so that each U_i is a neighborhood of x_i and each U_i is mapped diffeomorphically onto V by f.

Proof [Note that I have reversed the notation for U and V.] Suppose that $f^{-1}(y)$ is infinite, then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset f^{-1}(y)$. Since X is compact, then it is sequentially compact so x_n has a convergent subsequence $x_{n_k} \to \widetilde{x}$. Since f is continuous, then $f^{-1}(\{y\})$ is closed, so $\widetilde{x} \in f^{-1}(y)$.

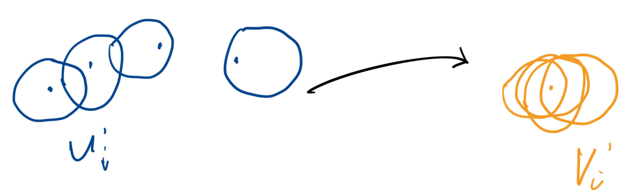


Now since y is a regular point of f, then for all $x \in f^{-1}(y)$, we have df_x is surjective, and since

$$\begin{aligned}\dim \ker df_x &= \dim Y - \dim \operatorname{Im} df_x \\ &= 0,\end{aligned}$$

then df_x is an isomorphism so by the Inverse Function Theorem, f is a local diffeomorphism at each $x \in f^{-1}(y)$. In particular f is injective on some neighborhood W of $\widetilde{x}$, but since $x_{n_k} \to \widetilde{x}$ then every neighborhood of $\widetilde{x}$ contains some x_{n_k} and $f(x_{n_k}) = f(\widetilde{x}) = y$, which contradicts that f is injective on W. Thus $f^{-1}(y)$ is finite, and from now on denote $f^{-1}(y) = \{x_i\}_{i=1}^{n}$.

Next, since f is a local diffeomorphism at each $x \in f^{-1}(y)$,



there exist $U_i' \ni x_i$ such that f is a diffeomorphism on U_i'. Since every manifold is Hausdorff[†], then we can separate the finite set of points $\{x_i\}$ by disjoint open sets U_i'', and $\{U_i' \cap U_i''\}_{i=1}^{n}$ are disjoint open sets where f is a diffeomorphism onto its image, but they may not all have the same image. So let

$$V = \bigcap_{i=1}^{n} f\left(U_i' \cap U_i''\right)$$

and then $f^{-1}(V) = \coprod_{i=1}^{n} U_i$ is a disjoint collection of open neighborhoods, one for each x_i, where each U_i is diffeomorphic to V, as desired. ∎

[†] Since it is locally diffeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$

3. Prove (a) that $O(n) = \left\{A \in M(n,\mathbb{R}) \mid A^\top A = I\right\}$ is a manifold. (b) Compute its dimension and identify $T_I(O(n))$.

Proof (a) Let $f : M_n(\mathbb{R}) \to \Sigma_n(\mathbb{R})$ be the map given by

$$f(A) = A^\top A.$$

We will show that (i) $M_n(\mathbb{R})$[†] and $\Sigma_n(\mathbb{R})$[‡] are both smooth manifolds, (ii) f is a smooth map, and (iii) $O(n) = f^{-1}(I)$ with I a regular value of f, and then we're done since the preimage of a regular point is a smooth manifold.

(i) As vector spaces, $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and $\Sigma_n(\mathbb{R}) \cong \mathbb{R}^{t(n)}$ where $t(n)$ is the nth triangle number, so they are definitely smooth manifolds.

(ii) Since the computations of $A^\top A$ just consist of multiplying and adding different elements of A, then the function f is just a polynomial in n^2 variables, so it is smooth.

(iii) We can see by inspection that $O(n) = f^{-1}(I)$, so let us show that I is a regular value of f. Fix $A \in M_n(\mathbb{R})$, and let's compute the derivative $df_A : T_A(M_n) \to T_{A^\top A}(\Sigma_n)$, and check that it is surjective whenever $A \in f^{-1}(I)$.

$$\begin{aligned} df_A(B) &= \lim_{t\to 0} \frac{f(A+tB) - f(A)}{t} \\ &= \lim_{t\to 0} \frac{(A+tB)^\top (A+tB) - A^\top A}{t} \\ &= \lim_{t\to 0} \tfrac{1}{t}\left(tB^\top A + tA^\top B + t^2 B^\top B\right) \\ &= B^\top A + A^\top B \end{aligned}$$

For any $C \in \Sigma_n(\mathbb{R})$[††] and $A \in f^{-1}(I)$, if we can find B such that $B^\top A + A^\top B = C$, then we're done. Observe that, since $A^\top = A^{-1}$, then if $B = \frac{1}{2}AC$, then

$$B^\top A + A^\top B = C.$$

Thus df_A is surjective for all $A \in f^{-1}(I)$, so (a) is proved. □

Proof (b) The kernel of df_I gives us the desired information here, so let's compute it. For any $B \in M_n(\mathbb{R})$,

$$df_I(B) = B^\top + B,$$

so the kernel is the set of all antisymmetric matrices, the matrices such that

$$B = -B^\top$$

which I denote $\mathfrak{Z}_n$. Observe that this is a vector space since $\mathfrak{Z}_n \subset M_n$ and for all $\lambda \in \mathbb{R}$, and $B, C \in \mathfrak{Z}_n$,

$$\begin{gathered} (B+C)^\top + (B+C) = B^\top + B + C^\top + C = 0, \text{ and} \\ (\lambda B)^\top + (\lambda B) = \lambda(B^\top + B) = \lambda(0) = 0. \end{gathered}$$

Since the diagonal entries are all zero, and each entry b_{ij} for $i > j$ is determined by b_{ji}, then $\mathfrak{Z}_n$ is isomorphic to $\mathbb{R}^{t(n-1)}$, so $\dim(\mathfrak{Z}_n) = t(n-1)$ and $T_I(O(n)) = \ker df_I = \mathfrak{Z}_n$. ■

[†] Where $M_n(\mathbb{R})$ denotes the set of all real $n \times n$ matrices.
[‡] Where $\Sigma_n(\mathbb{R})$ denotes the set of symmetric $n \times n$ real matrices.
[††] The tangent space to a vector space at any point is itself.