MATH 220B

PROBLEM SET #1

If you want to turn it in, please do so no later than Jan 22nd (Wednesday)

The following references are to the textbook [DF04].

1. Basic definitions and examples: Exercises 7, 15, 21, 26, 27 from §7.1, and Exercises 5, 6, 7[1] from §7.2.

2. Ring homomorphisms and quotient rings: Exercises 13, 14, 26 from §7.3.

3. Properties of ideals: Exercises 2, 3, 13, 14, 19, 40 from §7.4.

Exercise A. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals in a commutative ring R, and let $\mathfrak{p}$ be a prime ideal of R with $\mathfrak{p} \supset \bigcap_{i=1}^n \mathfrak{a}_i$. Show that $\mathfrak{p} \supset \mathfrak{a}_i$ for some i.

Exercise B. Prove the following properties of the radical of an ideal:

(a) $\mathrm{rad}(\mathfrak{a}) \supset \mathfrak{a}$.
(b) $\mathrm{rad}(\mathrm{rad}(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$.
(c) $\mathrm{rad}(\mathfrak{a}\mathfrak{b}) = \mathrm{rad}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{rad}(\mathfrak{a}) \cap \mathrm{rad}(\mathfrak{b})$.
(d) $\mathrm{rad}(\mathfrak{a}) = (1)$ if and only if $\mathfrak{a} = (1)$.
(e) $\mathrm{rad}(\mathfrak{a} + \mathfrak{b}) = \mathrm{rad}(\mathrm{rad}(\mathfrak{a}) + \mathrm{rad}(\mathfrak{b}))$.
(f) If $\mathfrak{p}$ is prime, then $\mathrm{rad}(\mathfrak{p}^n) = \mathfrak{p}$ for all $n > 0$.

Conclude that if $R = \mathbb{Z}$ and $\mathfrak{a} = (m)$ for $m \in \mathbb{Z}_{>1}$ with prime factorization $m = \prod_{i=1}^t p_i^{r_i}$, then

$$\mathrm{rad}(m) = (p_1 \cdots p_t).$$

5. Direct limits and inverse limits. Exercises 8, 9, 10, 11 from §7.6.

References

[DF04] David S. Dummit and Richard M. Foote, *Abstract algebra*, third ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[1] And use this exercise to show that for any $n \geq 2$, the ring $M_n(R)$ of $n \times n$ matrices with entries in a field R is a simple ring.