Math 550
Homework 1
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Solutions

1. Let $S \in \Lambda^n(\mathbf{R}^n)$. Let $(\vec{v}_1, \ldots, \vec{v}_n)$ be a collection of vectors with $v_j = \sum_i a_{ij} e_i$. By imitating the computation in class, we get

$$\begin{aligned} S(\vec{v}_1, \ldots, \vec{v}_n) &= (\sum_{\sigma} (-1)^{\operatorname{sign}\sigma} a_{\sigma(1)1} \cdots a_{\sigma(n)n})\, S(\vec{e}_1, \ldots, \vec{e}_n) \\ &= D(\vec{v}_1, \ldots, \vec{v}_n)\, S(\vec{e}_1, \ldots, \vec{e}_n). \end{aligned}$$

Thus any S is a scalar multiple of D, showing that the dimension of $\Lambda^n(\mathbf{R}^n)$ is 1, and that $\{D\}$ forms a basis.

2. (a) This also follows by imitating the calculation in class.
   (b) Following the hint,
   $$\delta_{i,j} = \langle \vec{v}_i, \vec{v}_j \rangle = \langle \sum_k a_{ki} \vec{u}_k, \sum_l a_{lj} \vec{u}_l \rangle = \sum_{l,k} a_{ki} a_{lj} \langle \vec{u}_k, \vec{u}_l \rangle = \sum_l a_{li} a_{lj}.$$
   The last term is the (i,j)-th entry of $A^T A$, so we have $A^T A = I$.
   (c) From part (b), we have $1 = \det(I) = \det(A^T A) = \det(A^T)\det(A) = \det(A)^2$. So $\det A = \pm 1$. The result then follows from part (a).

3. Following the hint, let $f = 1$ and $\Omega = \{(x,y) : 1 < |(x,y)| < e\}$. Define $g : (0,1) \times (0,4\pi) \to \Omega$ by $f(u,v) = (e^u \cos v, e^u \sin v)$. Then
$$\int_{\Omega} f = \operatorname{vol}(\Omega) = \pi(e^2 - 1),$$
but
$$\int_{g^{-1}(\Omega)} (f \circ g) |\det Dg| = \int_0^{4\pi} \int_0^1 e^{2u}\, du\, dv = 2\pi(e^2 - 1).$$

4. (a) Changing variables into polar coordinates (ρ, θ) gives $\int_{B_r} e^{-x^2-y^2}\, dx\, dy = \int_0^{2\pi} \int_0^r e^{-\rho^2} \rho\, d\rho\, d\theta = \pi(1 - e^{-r^2})$.
   (b) From Fubini's Theorem $\int_{C_r} e^{-x^2-y^2}\, dx\, dy = \int_{-r}^r \int_{-r}^r e^{-x^2} e^{-y^2}\, dx\, dy = (\int_{-r}^r e^{-x^2}\, dx)(\int_{-r}^r e^{-y^2}\, dy) = (\int_{-r}^r e^{-x^2}\, dx)^2$.
   (c) Observe that $B_r \subset C_r \subset B_{\sqrt{2}r}$, so
   $$\int_{B_r} e^{-x^2-y^2}\, dx\, dy \le \int_{C_r} e^{-x^2-y^2}\, dx\, dy \le \int_{B_{\sqrt{2}r}} e^{-x^2-y^2}\, dx\, dy.$$
   The limit as $r \to \infty$ of the outer integrals in this inequality both exist and equal π (by part(a)). Thus $\lim_{r\to\infty} \int_{C_r} e^{-x^2-y^2}\, dx\, dy = \pi$ as well.
   (d) Parts (b) and (c) imply that $\lim_{r\to\infty} \int_{-r}^r e^{-x^2}\, dx = \sqrt{\pi}$. Since $e^{-x^2} > 0$, this also equals $\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

5. (a) Change variables into spherical coordinates: $g(\rho, \theta, \phi) = (a\rho \cos\theta \sin\phi, b\rho \sin\theta \sin\phi, c\rho \cos\phi)$. Direct calculation gives $|\det Dg(\rho, \theta, \phi)| = \rho^2 \sin\phi$. So $\int_D f = \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{\rho^3} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{4}{3}\pi(e - 1)$.

(b) Change variables using modified spherical coordinates: $g(\rho,\theta,\phi) = (a\rho\cos\theta\sin\phi, b\rho\sin\theta\sin\phi, c\rho\cos\phi)$. Direct calculation gives $|\det Dg(\rho,\theta,\phi)| = abc\rho^2\sin\phi$. So $\int_E 1 = \int_0^{\pi}\int_0^{2\pi}\int_0^1 abc\rho^2\sin\phi d\rho\ d\theta\ d\phi = \frac{4}{3}\pi abc$.

6. Since T is linear, we have $\det DT(a) = \det T = (n-1)!$ for all $a \in T^{-1}(\Omega)$. Then by change of variables, $1 = \int_\Omega f = \int_{T^{-1}(\Omega)} f\circ T|\det DT| = (n-1)!\int_{T^{-1}(\Omega)} f\circ T$. Thus $\int_{T^{-1}(\Omega)} f\circ T = 1/(n-1)!$.