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Final Exam

1. Definition. Let X be a topological space, with $x \in X$. A *loop* f in X is a map $f : I \to X$ with $f(0) = f(1) = x$. We call x the *basepoint* of f.

Definition. Let f and g be two loops in X, both with basepoint x_0. A *loop homotopy* between f and g is a homotopy φ_t such that $\varphi_0 = f, \varphi_1 = g$, and $\varphi_t(0) = \varphi_t(1) = x_0$ for all t.

Definition. Let (X, x_0) be a based topological space. The **fundamental group** $\pi_1(X, x_0)$[†] is the set of all loop homotopy classes $[f]$ of loops in X based at x_0, equipped with the product $[f][g] := [f \cdot g]$.[‡]

Definition. Let $p : C \to X$ be a map, and let U be open in X. We say U is *evenly covered* by p if

- $p^{-1}(U) = \coprod_\alpha \widetilde{U}_\alpha$, where each $\widetilde{U}_\alpha$ open in C, and
- each $\widetilde{U}_\alpha$ is homeomorphic to U, denoted $\widetilde{U}_\alpha \cong U$.

Definition. Let (X, x_0) be a based topological space. A **covering space** of X is a based topological space (C, c_0) together with a basepoint preserving map $p : C \to X$ such that

- for every $x \in X$, there exists a neighborhood $U \ni x$ which is evenly covered by p. [††]

Remark. We show in Hatcher that a covering map $p : C \to X$ induces an injective homomorphism $p_* : \pi_1(C) \to \pi_1(X)$, and that the image $\operatorname{im}(p_*)$ is a subgroup of $\pi_1(X)$.

Definition. Let $p : (C, c_0) \to (X, x_0)$ and $r : (D, d_0) \to (X, x_0)$ be covering spaces. We say that p **and** r **are isomorphic** if

$$\operatorname{im}(p_*) \cong \operatorname{im}(r_*)$$

2. Definition. Let $p : \widetilde{X} \to X$ be a covering space, and let $f : Y \to X$ be a map. A *lift* of f is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p \circ \widetilde{f} = f$.

Theorem. (Homotopy Lifting Property)
Given a homotopy $f_t : Y \to X$ and a map $\widetilde{f}_0 : Y \to X$ lifting f_0, there is a unique homotopy $\widetilde{f}_t$ lifting f_t which extends $\widetilde{f}_0$.

Remark. We proved this in Hatcher. Also note that we can apply this to map homotopies, path homotopies, loop homotopies, and even paths themselves which can be thought of as point homotopies.

[†] When the basepoint is irrelevant or it is clear what the basepoint is, we will suppress notation and write $\pi_1(X)$. If not specified, one can assume x_0 is the basepoint of X, a_0 is the basepoint of A, etc.

[‡] Here $\cdot$ denotes concatenation, and sometimes we will denote concatenation by juxtaposition when the meaning is clear.

[††] We often suppress notation by calling $p : C \to X$ a covering space, inferring that C is the accompanying topological space, and assuming the basepoints are named whatever makes sense.

Theorem. $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

Proof Let $p : \mathbb{R} \to \mathbb{S}^1$ be

$$p(s) = \big(\cos(2\pi s), \sin(2\pi s)\big).$$

Claim. $p : (\mathbb{R}, 0) \to \big(\mathbb{S}^1, (1,0)\big)$ is a covering space.

Proof of claim. Observe that p is a basepoint preserving map since it is continuous and $p(0) = (1,0)$. To see that every neighborhood of a point in S is evenly covered, observe that it can be factored as $p = \mathrm{proj}_{xy} \circ h$, where $h : \mathbb{R} \to \mathbb{R}^3$ and $\mathrm{proj}_{xy} : \mathrm{im}\,(h) \to \mathbb{R}^2$ are given by

$$\begin{aligned} h(s) &= \big(\cos(2\pi s), \sin(2\pi s), s\big) \\ \mathrm{proj}_{xy}(x,y,z) &= (x,y). \end{aligned}$$

Geometrically it is clear that for every $s \in \mathbb{S}^1$ with sufficiently small $U \ni s$ open in $\mathbb{S}^1$, the preimage $\mathrm{proj}_{xy}^{-1}(U)$ is a collection of disjoint stacked arcs in the helix, and $h^{-1}\big(\mathrm{proj}_{xy}^{-1}(U)\big)$ is a collection of disjoint open intervals, each of length less than 1.

To see that each of these disjoint open intervals is homeomorphic to its image in p, observe that h is a homeomorphism since it is continuous with continuous inverse $h^{-1}(x,y,z) = z$, and proj_{xy} is continuous because it is a projection. To see that proj_{xy} has continuous inverse on one of these small helix arcs, let $(x_1,y_1), (x_2,y_2)$ be two points in $\mathbb{S}^1$ such that the angle between them θ is less than 2π. For any (x,y) in the arc between (x_1,y_1) and (x_2,y_2) in $\mathbb{S}^1$, let $\theta(x,y)$ be the angle between (x_1,y_1) and (x,y). Denote s_0 the lower bound of the small interval whose image in h gives us the helix arc from (x_1,y_1) to (x_2,y_2). Then

$$\mathrm{proj}_{xy}^{-1}\left((x,y)\right) = \big(x, y, s_0 + \tfrac{1}{2\pi}\theta(x,y)\big)$$

is the inverse of proj_{xy} and it is continuous. Thus the claim is proved. □

Now that we have established that p is a covering map of $\mathbb{S}^1$, we classify the loops in $\pi_1(\mathbb{S}^1)$. Let f be a loop in $\mathbb{S}^1$ based at $(1,0)$. If we choose $\widetilde{f}(0) = 0$, then by the Homotopy Lifting Property, f lifts uniquely to $\widetilde{f}$ with $\widetilde{f}(1) \in p^{-1}(1,0) = \mathbb{Z}$.

Claim. $\Theta : \pi_1(\mathbb{S}^1) \to \mathbb{Z}$ is a group isomorphism given by

$$\Theta[f] = \widetilde{f}(1) \text{ where } \widetilde{f}(0) = 0.$$

Proof of claim.

- To see that Θ is well-defined, let $f \overset{\varphi}{\simeq} g$, with $\widetilde{f}$ such that $\widetilde{f}(0) = 0$. Then applying the Homotopy Lifting Property to φ we obtain $\widetilde{\varphi}$ with $\widetilde{\varphi}_0 = \widetilde{f}$, and since $\varphi_t(1)$ is fixed for all time then so is $\widetilde{\varphi}_t(1)$, so $\widetilde{\varphi}_1(1) = \widetilde{g}(1) = \widetilde{f}(1)$. Thus $\Theta[f] = \Theta[g]$.
- To see that Θ is onto, for all $n \in \mathbb{Z}$ let

$$\omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s).$$

If we choose $\widetilde{\omega}_n(0) = 0$, then by the Homotopy Lifting Property $\widetilde{\omega}_n(s) = ns$, so $\widetilde{\omega}_n(1) = n$ and $\Theta[\omega_n] = n$.

- To see that Θ is a homomorphism, note that since $\omega_1 \cdot \omega_1$ reparametrizes to ω_2, then $[\omega]^n = [\omega_n]$. Thus
$$\Theta\big([\omega_n][\omega_m]\big) = \Theta[\omega_{n+m}] = n + m.$$
Also, if f is a loop with $\widetilde{f}(1) = n$, then $\widetilde{f}$ and $\widetilde{\omega}_n$ are both paths from 0 to n, and they are homotopic by the straight-line homotopy $\Phi(t,s) = (1-t)\widetilde{f} - t\widetilde{\omega}_n$. Thus $p \circ \Phi : I \times I \to \mathbb{S}^1$ is a map with $\varphi_0 = f$, $\varphi_1 = \omega_n$. This is a homotopy exhibiting $f \simeq \omega_n$.
- To see that Θ is injective, observe that $[f] \in \ker(\Theta)$ means that $f \simeq \omega_0$ which is a constant loop, so Θ has trivial kernel.

Thus we have shown that $\pi_1(\mathbb{S}^1)$ is exactly $\mathbb{Z}$, and we are done. ∎

3. Definition. Let (X, x_0) and (Y, y_0) be based topological spaces. We say that these spaces are *homotopic* and write $(X, x_0) \simeq (Y, y_0)$ if there are basepoint-preserving maps
$$f : (X, x_0) \to (Y, y_0) \text{ and } g : (Y, y_0) \to (X, x_0)$$
such that
$$f \circ g \simeq \mathbb{1} \text{ and } g \circ f \simeq 1.$$
Such f, g are called *homotopy equivalences.*

Theorem. If $f : (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group isomorphism.

Proof Let g be the homotopy inverse for f, so $f \circ g \simeq \mathbb{1}$ and $g \circ f \simeq \mathbb{1}$. We will show that the induced maps are inverse homomorphisms. Observe that $f_* g_* = (fg)_*$ since for any loop γ in (Y, y_0)
$$[\gamma] \xmapsto{(fg)_*} [fg(\gamma)]$$
$$[\gamma] \overset{g_*}{\mapsto} [g(\gamma)] \overset{f_*}{\mapsto} [fg(\gamma)]$$
and since both f, g preserve basepoints, then fg and $\mathbb{1}$ both fix their basepoint. Since $fg \simeq \mathbb{1}$, so there is a homotopy φ_t exhibiting $fg \simeq \mathbb{1}$ which fixes the basepoint for all time. Thus composing with γ yields $\varphi_t(\gamma)$ which is a loop homotopy exhibiting $fg(\gamma) \simeq \gamma$. Thus
$$[fg(\gamma)] = [\gamma],$$
and so $f_* g_* [\gamma] = [\gamma]$, so $f_* g_* = \mathbb{1}$.

The same argument shows that $g_* f_* = \mathbb{1}$, and so f_* and g_* are group isomorphisms. ∎

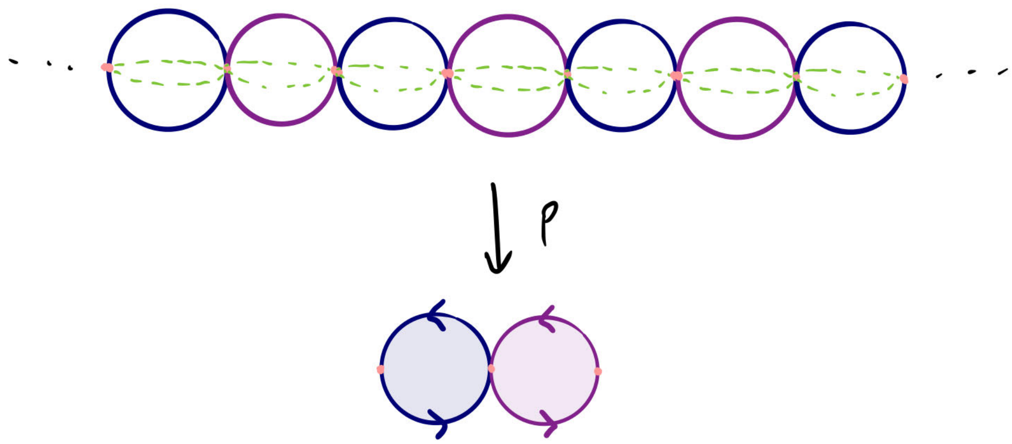
4. Let $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$.

(a) Find $\pi_1(X)$.

Answer: We know already that $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$, so applying Van Kampen's Theorem, $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2) = \mathbb{Z}_2 * \mathbb{Z}_2$.

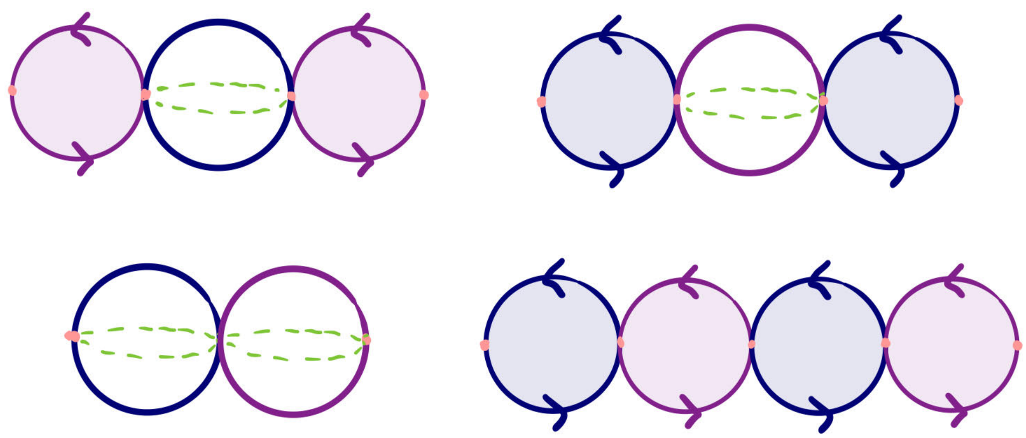
(b) Find the universal cover of X.

Answer: The following cover, where each S^2 maps to the same color $\mathbb{RP}^2$ by the real projection map.



(c) Find all of its connected 2-sheeted covers.

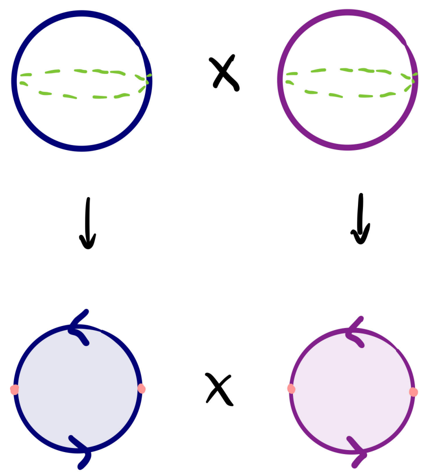
Answer: The following 4 covers with points at the end identified (thus all points in pink are identified). Each part maps to the same color part in X with either the real projection map or the identity map, as appropriate.



5. Let $X = \mathbb{RP}^2 \times \mathbb{RP}^2$.

(a) Find $\pi_1(X)$.
Answer: $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) Find the universal cover of X.
Answer: $\mathbb{S}^2 \times \mathbb{S}^2$, with the product of real projection maps.



(c) Find all of its connected 2-sheeted covers.
Answer: A multi-sheeted cover of $\mathbb{RP}^2 \times \mathbb{RP}^2$ with n sheets in the first coordinate and m sheets in the second coordinate will necessarily have nm sheets, so any two-sheeted cover with have 2 sheets in one coordinate and 1 sheet in the other. This leaves two choices for product space covers:

Theorem. (Van Kampen's Theorem [two-set version])
If

- $X = A \cup B$,
- each of A, B is open, path-connected, and contains the basepoint,
- $A \cap B$ is path-connected,

then $\pi_1(X)$ is given by
$$\pi_1(A) * \pi_1(B) \,/\, N,$$
where N is the normal subgroup generated by all elements of the form
$$i_{A*}[\gamma] i_{B*}[\gamma]^{-1}$$
for all $[\gamma] \in \pi_1(A \cap B)$.

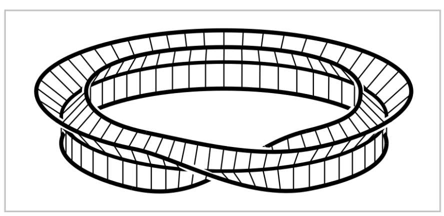
6. Calculate the fundamental group of $S(K^2 \vee T^2)$ where S denotes suspension, K^2 is the Klein bottle, and T^2 is the torus.

Answer: Let $X = K^2 \vee T^2$. We can write SX as $CX \cup CX$, where the two cones overlap a bit, say $t \in [0, 0.6)$ in the first cone and $t \in (0.4, 1]$ in the second cone. Applying Van Kampen's theorem, we find that $\pi_1(SX)$ is
$$\pi_1(CX) * \pi_1(CX) \,/\, N$$
but every cone is contractible, so $\pi_1(SX)$ is trivial. ■

7. Let X_n be the space obtained by attaching $\mathbb{D}^2$ to $\mathbb{S}^1$ along $\partial\mathbb{D}^2$ by wrapping the boundary n times around $\mathbb{S}^1$.

(a) Use the Van Kampen theorem to calculate $\pi_1(X_n)$.

Answer: Write $X_n = A \cup B$, where B is the interior of $\mathbb{D}^2$, and A is constructed as follows. Take $(\bigvee_{k=1}^{n} [0, 0.1)) \times I$ to yield an n-pointed asterisk noodle, then twist $\frac{1}{n}$th of a turn and identify the ends as shown below for $n = 3$:



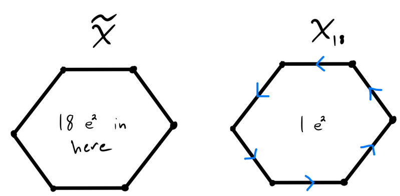
A deformation retracts to $\mathbb{S}^1$ and B is contractible, so $\pi_1(X_n)$ is
$$\mathbb{Z}(a) * 0 \,/\, N,^{\dagger}$$
so all that remains is to determine N. Since $A \cap B$ deformation retracts to the boundary circle of A, then $\pi_1(A \cap B)$ has one generator, call it $[\gamma]$. Then $i_{B*}[\gamma]$ is trivial, and $i_{A*}[\gamma] = [a^n]$, so $N = \langle a^n \rangle$ and
$$\pi_1(X_n) = \langle a \mid a^n \rangle = \mathbb{Z}_n.$$

†Here $\mathbb{Z}(a)$ denotes the free group generated by a.

(b) Find all subgroups of $\pi_1(X_{18})$ and all of the corresponding connected based covering spaces. The covers will be hard to draw but you should be able to describe them fairly precisely.

Answer: To construct the universal cover $\widetilde{X}$ of X_{18}, we build the Cayley complex: $\pi_1(X_{18}) = \langle a \mid a^{18} \rangle = \mathbb{Z}_{18}$, so we begin with 18 discrete points $0, a, a^2, \ldots, a^17$. Then we add an edge connecting each point to its image under the action of the generator a, forming an 18-sided polygon. Finally there is a loop a^{18} starting at each point, so we add 18 2-cells attached along the polygon. Thus we obtain the an 18-gon with 18 2-cells attached inside.

Motivated by this picture, we can reimagine X_{18} as the same edge a repeated 18 times in a loop, with one e^2 attached along a^{18}. That is, an 18-gon with all edges identified. (In the figures, pretend that each polygon has 18 sides.)



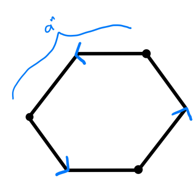
Note that $p : \widetilde{X} \to X$ is the map which is suggested by the images; a point in any of the edges of $\widetilde{X}$ is mapped to the corresponding point in the only edge of X_{18}, and a point in any of the e^2 cells of $\widetilde{X}$ maps to the corresponding point in the only e^2 of X_{18}.

The subgroups of $\langle a \mid a^{18} \rangle$ are

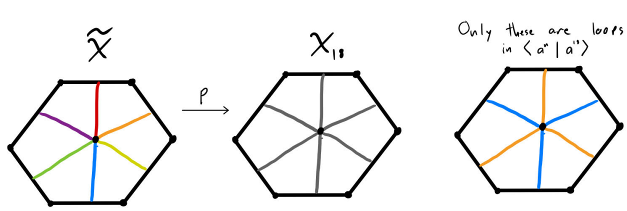
$$\langle a \mid a^{18} \rangle \qquad \langle a^2 \mid a^{18} \rangle \qquad \langle a^3 \mid a^{18} \rangle \qquad \langle a^6 \mid a^{18} \rangle \qquad \langle a^9 \mid a^{18} \rangle \qquad \langle a^{18} \mid a^{18} \rangle$$

and for each $\langle a^n \mid a^{18} \rangle$ we will construct a cover C_n of X_{18}.

Clearly $C_1 = X_{18}$ and $C_{18} = \widetilde{X}$. For all other C_n, we start with $\widetilde{X}$ and identify every pair of points $\widetilde{x}, \widetilde{y} \in \widetilde{X}$ such that for any path $\widetilde{\gamma}$ between them, $\widetilde{\gamma} \overset{p_*}{\longmapsto} \gamma \in \langle a^n \mid a^{18} \rangle$. For the edges, only powers of a^n are loops, so we identify corresponding points in $\frac{18}{n}$ pieces of the perimeter.

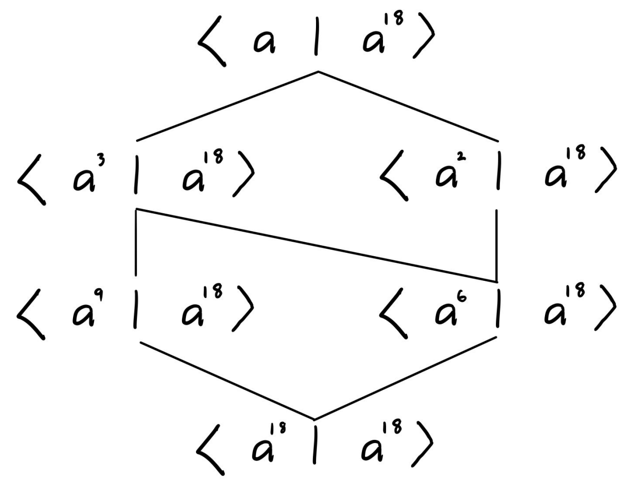


For the faces, let $\widetilde{x}_n$ be 18 corresponding points on each side of $\widetilde{X}$, and let $\widetilde{y}_n$ be 18 corresponding points on each e^2, and connect them all with paths. Concatenating any pair of them give a path which maps to a loop in X_18, but there are n distinct groups such that any pair in the group maps to a loop in $\langle a^n \mid a^{18} \rangle$.



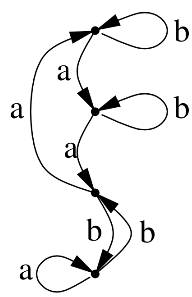
Thus we find that C_n is an $\frac{18}{n}$-gon with edges identified, and n e^2 cells attached inside. ■

(c) Draw the subgroup lattice (i.e. illustrate all of the inclusion relations between subgroups in a diagram).



■

8. Consider the cover of $\mathbb{S}^1 \vee \mathbb{S}^1$ drawn below based at its bottom vertex. Is the image of its fundamental group in $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ a normal subgroup? Why or why not?



Answer: No, because the graph does not have maximal symmetry. Observe that if we label the vertices $1, 2, 3, 4$ from top to bottom, then no deck transformation maps $1 \mapsto 4$, since 1 has a b arrow from itself to itself, and 4 does not. ■

9. (Extra credit) Use the 2-dimensional Brouwer fixed point theorem to prove that every 3×3 matrix with positive real entries has a positive real eigenvalue. [Hint: focus on how the matrix acts on rays through the origin in $\mathbb{R}^3$, particularly those that point into the first octant.]

Proof Let A be a 3×3 matrix, $\vec{x} \in \mathbb{R}^3$, and define $\varphi(\vec{x})$ to be the linear transformation $A\vec{x}$ restricted to $\mathbb{S}^2 \cap O_1$, where O_1 is the first octant. Composing with projection back to $\mathbb{S}^2$ we obtain $p \circ \varphi(\vec{x})$, a continuous function. Since $A\vec{x}$ gives a vector with all positive coordinates, $p \circ \varphi : \mathbb{S}^2 \cap O_1 \to \mathbb{S}^2 \cap O_1$. Observe that $\mathbb{S}^2 \cap O_1 \cong \mathbb{D}^2$, so by the Brouwer fixed point theorem, $p \circ \varphi$ has a fixed point $\vec{x}_0$. Since $\varphi(\vec{x}) = A\vec{x}$ maps to a point in O_1 which projects to itself, then $\varphi(\vec{x}) = \lambda\vec{x}$ with $\lambda > 0$ and $\vec{x}$ is an eigenvector of A. ■