1. (3 points) The real 2×2 matrix A has eigenvector $\vec{v}_1 = \begin{bmatrix} 2 \\ -1+i \end{bmatrix}$ for eigenvalue $\lambda_1 = 1+3i$; and A has eigenvector $\vec{v}_2 = \begin{bmatrix} 2 \\ -1-i \end{bmatrix}$ for eigenvalue $\lambda_2 = 1-3i$. Given this information, write down the general solution to the system of ODEs

$$\vec{x}\,' = A\vec{x}$$

Solution:

From the first given complex eigenpair, the following is a *complex* solution to the system:

$$\vec{z} = e^{(1+3i)t} \begin{bmatrix} 2 \\ -1+i \end{bmatrix}$$

Simplifying,

$$\vec{z} = e^{t+3ti} \begin{bmatrix} 2 \\ -1+i \end{bmatrix}$$

Using Euler's Formula, we get

$$\vec{z} = e^{t}(\cos 3t + i\sin 3t) \begin{bmatrix} 2 \\ -1+i \end{bmatrix}$$

Next, we simplify in such a way that the real and imaginary parts of $\vec{z}$ are separated.

$$\begin{aligned}
\vec{z} =& e^t \begin{bmatrix} 2\cos 3t + 2i\sin 3t \\ (\cos 3t + i\sin 3t)(-1+i) \end{bmatrix} \\
=& e^t \begin{bmatrix} 2\cos 3t + 2i\sin t \\ -\cos 3t + i\cos 3t - i\sin 3t - \sin 3t \end{bmatrix} \\
=& e^t \begin{bmatrix} 2\cos 3t \\ -\cos 3t - \sin 3t \end{bmatrix} + ie^t \begin{bmatrix} 2\sin 3t \\ \cos 3t - \sin 3t \end{bmatrix}
\end{aligned}$$

Notice that we used the fact that $i^2 = -1$.

In class, we learned that the real and imaginary parts of a complex solution are each real solutions. Therefore

$$\vec{x}_1(t) = \mathrm{Re}[\vec{z}(t)] = e^t \begin{bmatrix} 2\cos 3t \\ -\cos 3t - \sin 3t \end{bmatrix}$$

and

$$\vec{x}_2(t) = \mathrm{Im}[\vec{z}(t)] = e^t \begin{bmatrix} 2\sin 3t \\ \cos 3t - \sin 3t \end{bmatrix}$$

are two real solutions. Notice that $\vec{x}_1$ and $\vec{x}_2$ are linearly independent and *are both real* (we want real solutions to a real system unless otherwise specified). Thus the general solution is

$$\vec{x}(t) = C_1 e^t \begin{bmatrix} 2\cos 3t \\ -\cos 3t - \sin 3t \end{bmatrix} + C_2 e^t \begin{bmatrix} 2\sin 3t \\ \cos 3t - \sin 3t \end{bmatrix}$$

Why does this work?

In the above calculations, we only worked with the first eigenpair λ_1 and $\vec{v}_1$, and got the complex solution $\vec{z}(t)$. Let's rename it $\vec{z}_1(t)$.

If we found the complex solution for the second eigenpair, λ_2 and $\vec{v}_2$, we would get another solution $\vec{z}_2(t)$. But since λ_2 is the conjugate of λ_1 and $\vec{v}_2$ is the conjugate of $\vec{v}_1$, it follows that $\vec{z}_2$ is the conjugate of $\vec{z}_1$. That is

$$\vec{z}_2(t) = e^t \begin{bmatrix} 2\cos 3t \\ -\cos 3t - \sin 3t \end{bmatrix} - ie^t \begin{bmatrix} 2\sin 3t \\ \cos 3t - \sin 3t \end{bmatrix}$$

Now we have a *complex basis* for the solution space $\{\vec{z}_1(t), \vec{z}_2(t)\}$. However, we want a *real* basis. Now, the trick is that even though $\vec{z}_1$ and $\vec{z}_2$ are complex, certain linear combinations of them are real. Since the general solution is a vector space, these linear combinations are also solutions. In particular, adding $\vec{z}_1 + \vec{z}_2$ will cancel out the imaginary parts since they are conjugates. Thus the following is a solution to the system:

$$\frac{1}{2}(\vec{z}_1 + \vec{z}_2) = e^t \begin{bmatrix} 2\cos 3t \\ -\cos 3t - \sin 3t \end{bmatrix}$$

Similarly, subtracting $\vec{z}_1 - \vec{z}_2$ will cancel out the real parts, leaving only the imaginary part. Thus the following is also a solution to the system:

$$\frac{1}{2i}(\vec{z}_1 - \vec{z}_2) = e^t \begin{bmatrix} 2\sin 3t \\ \cos 3t - \sin 3t \end{bmatrix}$$

These are of course the real and imaginary parts of $\vec{z}_1$. Now, notice that each of these two is a real solution, notice that there are two of them, and notice that they are linearly independent. Thus, they must be a basis for the general solution. We can think of this as a change of basis—changing from our original complex basis to a different complex basis that happens to be real.