# Differential Equations Math 460 - Rosen, 2018

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## Contents



Note: If you find any typos in these notes, please let me know at trevor.klar.834@my.csun.edu. If you could include the page number, that would be helpful.

Note to the reader: I have highlighted topics which seem important to me, but the emphasis is mine, not Professor Fuller's. Bear that in mind when studying.

# 1 Introduction

This will be a course on ring theory and group theory, and we'll start with ring theory.

# 2 Rings

**Definition.** A ring $R$ is a set with two binary operations, $+$ and $\cdot$ which satisfy the following axioms $\forall a, b, c \in R$:

R1 $+$ commutative

R2 $+$ associative

R3 $+$ identity

R4 $+$ inverse

R5 $\cdot$ associative

R6 $\cdot$ left, right distributive

R5 $\cdot$ identity (we are assuming all rings are rings with unity)

**Definition.** If, in addition, the ring has the $\cdot$ commutative property, we say that it is a **commutative ring**.

**Theorem.** every element of a ring has a unique additive inverse:

**Proposition.** $a \cdot 0 = 0$

**Proposition.** $-a = -1 \cdot a$

**Proposition.** $(-a)b = a(-b) = -(ab)$

**Definition.** $S$ is a **subring** of $R$ if $s \subseteq R$ and $1_S = 1_R$ and $S$ with the same $+$ and $\cdot$ is ring.

To check $S$ is a subring, just check

- closure under $+, \cdot$,
- $0_R \in S$
- $1_R \in S$
- $a \in S \implies -a \in S$.

Here are some examples of rings:

1. $\mathbb{Q}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$

2. $R$ any ring, $R[x]$ = the set of all polynomials with coefficients in $R$.

$$R[x] = \{a_0 + a_1x + \cdots + a_nx^n | a_i \in R, n\text{varies}\}.$$

3. $R[x, y] = R[x][y] = R[y][x]$

**Definition.** we say that $\deg 0- = \inf$ (or that 0 has no degree), and for some polynmial $P(x)$, $\deg(P(x))$ is what you think.

**Definition.** If $R$ is a ring and there exists some $a, b \in R$ such that $a, b \neq 0$ and $ab = 0$, then we say $a$ and $b$ are **zero divisors**.

**Definition.** A commutative ring with no zero divisors is called an **integral domain**.

**Proposition (Cancellation property for integral domains).** for $R$ an I.D. and $ab = ac$ with $a \neq 0$, then $b = c$.

[jpg]

**Definition.** a **field** is a commutative ring in which every nonzero element has a multiplicative inverse (which is unique, prove it) , ex $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields.

**Proposition.** every field is an I.D, because if $ab = 0$ and $a \neq 0$, then $a^{-1}(ab) = a^{-1}0$ which means $b = 0$.

**Proposition.** If $R, S$ rings, then $R \times S$ is a ring (when equipped with componentwise addition and multiplication). By the way, unity is $(1_R, 1_S)$ and the identity is $(0_R, 0_S)$.

[jpg]
[jpg]
[jpg]

# 3 actual lecture starts here

**Definition.** A non-constant polynomial in $F[x]$ is called **irreducible** when if $f(x) = g(x)h(x)$, then either $g(x)$ or $h(x)$ is constant.

**Proposition.** Let $f(x) \in F[x]$. If $\deg f(x) = 2$ or 3, then $f(x)$ is irredeucible iff in has no roots in $F$.

**Example.** $f(x) = (x^2 + 1)^2 \in \mathbb{R}[x]$ is reducible, but has no real roots.

**Theorem (Rational Root Test).** Let $f(x) \in \mathbb{Z}[x]$ where $f(x) = a_n x^n + \cdots + a_1 x + a_0$.

$\frac{r}{s}$ (in lowest terms) is a rational root iff $r|a_0$ and $s|a_n$.

**Theorem.** Let $f(x) \in F[x]$ where $F[x]$ is a PID[a]. $f(x)$ is irreducible iff $\langle f(x) \rangle$ is maximal (and hence $F(x)/\langle f(x) \rangle$ is a field).

[a] is every field a PID? The notetaker didn't hear.

**Theorem (Binomial Theorem).**

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

**Lemma 1.** For $p$ prime, $p| \binom{p}{i}$.

**Proof** Let $m = \binom{p}{i} = \frac{p!}{i!(p-i)!}$. Then,

$$p! = mi!(p-i)!$$

and since $p$ divides $p!$ but not $i!$ or $(p-i)!$, then $p|m = \binom{p}{i}$. ■

**Theorem (Binomial Theorem mod $p$).** For $a, b \in \mathbb{Z}_p$, where $p$ prime,

$$(a+b)^p = \sum_{i=0}^{p} \binom{p}{i} a^{p-i} b^i = a^p + b^p$$

**Definition.** Let $R, R'$ be rings. A **ring homomorphism** between $R$ and $R'$ is a function $\theta : R \to R'$ such that

1. $\theta(r_1 + r_2) = \theta(r_1) + \theta(r_2)$
2. $\theta(r_1 r_2) = \theta(r_1)\theta(r_2)$
3. $\theta(1_R) = 1_{R'}$

Here are some propositions you should prove:

1. $f(0_R) =, 0_{R'}$
2. $f(-a) = -f(a)$
3. If $a$ is invertible in $R$, then $f(a)$ is invertible in $R'$. and $f(a)^{-1} = f(a^{-1})$.
4. $f$ is 1-to-1 iff $\ker f = 0_R$.

**Definition.** We say an onto homomorphism is called an **epimorphism**, and a 1-1 homomorphism is called a **monomorphism**.

**Definition.** The kernel of a homomorphism is the set of all elements in $R$ that map to $0_{R'}$ under $\theta$.

**Proposition.** The kernel of a homomorphism $\theta : R \to R'$ is an ideal of $R$.

**Proposition.** A homomorphism $\theta : R \to R'$ is 1-1 iff $\ker \theta = \{0_R\}$.

**Proposition.** Let $\theta : R \to R'$ a homomorphism. If $\ker \theta = R$, then $\theta$ is the zero map.

**Proposition.** If $f$ is onto, then $f(I) \trianglelefteq R'$.

**Definition.** If $\theta : R \to R'$ is a bijective homomorphism, then we say $\theta$ is a **ring isomorphism** and we write $R \cong R'$.

**Definition.** If $f : X \to Y$ is any function of set, then for any $B \subset Y$ let

$$f^{-1}(B) = \{x \in X | f(x) \in B\},$$

and we call this the **preimage** of $B$ under $f$. Note that the preimage of a set is a set, and you should *not* think of a preimage as a function.

**Proposition.** Suppose $f : R \to R'$ is a ring homeomorphism, and let $I' \trianglelefteq R'$. Then $f^{-1}(I') \trianglelefteq R$.

(Try the proof. Prove that $f^{-1}(I')$ is closed, is an ideal, and contains $0_R$.)

Some properties of preimages:

- $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
- $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
- If $A \subset B$, then $f^{-1}(A) \subset f^{-1}(B)$

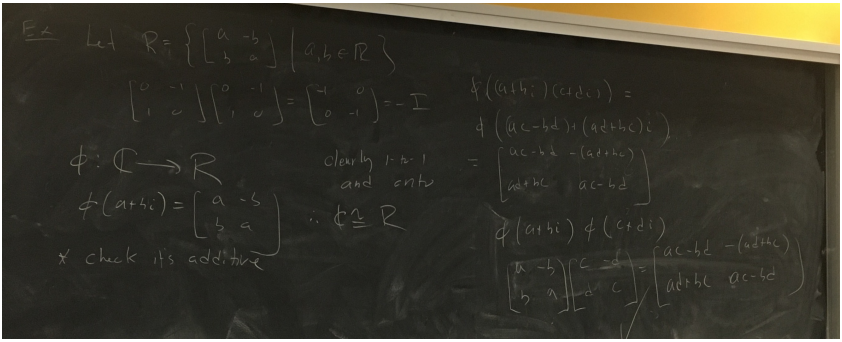
**Corollary.** $\ker f = f^{-1}(\{0_{R'}\}) \subseteq f^{-1}(I')$

**Example.**

1. Suppose $R \subset S$ (Assume $S$ is commutative). For any $s \in S$, define $\mu_s : R[x] \to S$ by
$$\mu_s\big(p(x)\big) = p(s).$$
Then we have that $\mu_s$ is a homomorphism (which we call the *substitution homomorphism*).

2. $\mu_i : \mathbb{R}[x] \to \mathbb{C}$ defined by
$$\mu_i\big(p(x)\big) = p(i)$$
Then, $(x^2 + 1) = \ker \mu_i$. (prove it using the division algorithm).

3. Consider $\mu_{2^{1/3}} : \mathbb{Q}[x] \to \mathbb{R}$. Then $\ker \mu_{2^{1/3}} = (x^3 - 2)$. (for the proof, use the division algorithm and the fact that $1, 2^{1/3}, 2^{2/3}$ are linearly independent in $\mathbb{Q}$.)

4. 


*Remark.* When a subring generated by a set, use square brackets. When you know the subring is a field, use round brackets.

i.e. $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}(\sqrt{2})$ ,but for $\mathbb{Z}[\sqrt{2}$ or $F[\sqrt{2}$ dont use () if it's not a field.

**Proposition.** Let $R$ be any ring, with $I \trianglelefteq R$. Define $\pi : R \to R/I$ by $pi(r) = r + I$. Then $\pi$ is a ring homomorphism which is onto and $\ker \pi = I$.

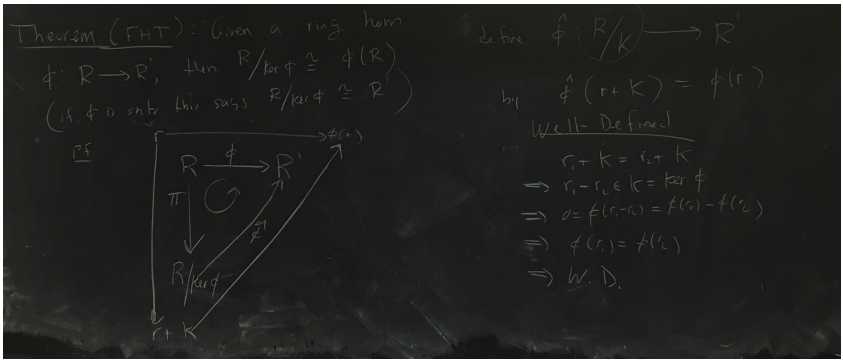
We call $\pi$ the **canonical homomorphism**.

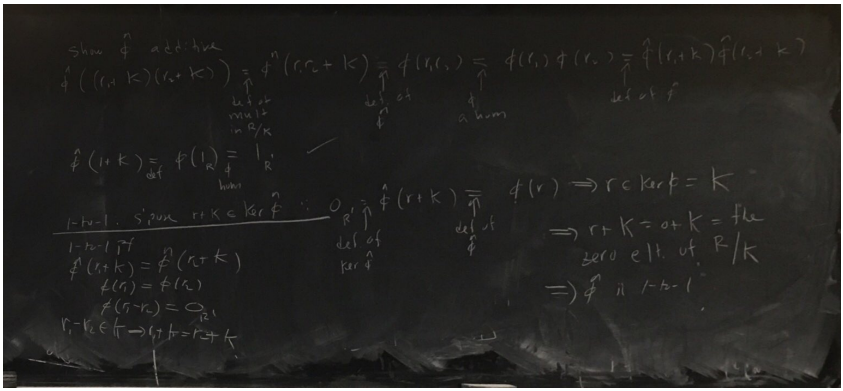
The next theorem is the converse of the above.

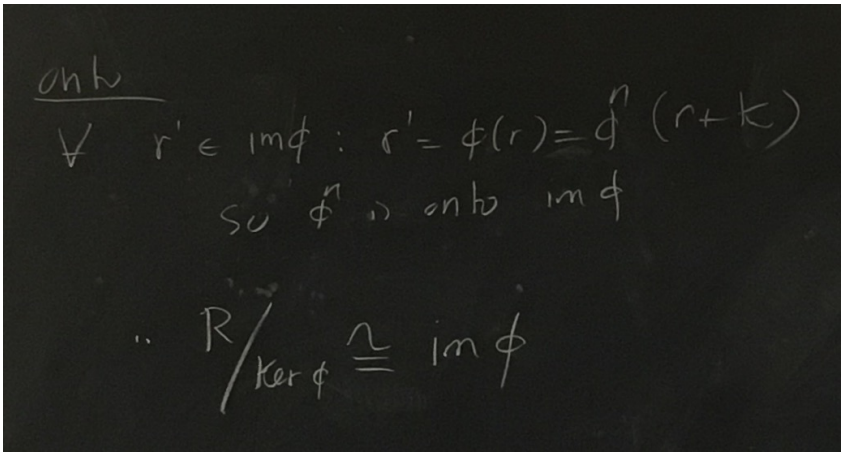
**Theorem (Fundamental Homomorphism Theorem).** Given a ring homomorphism $\phi : R \to R'$, then $R/\ker \phi \cong \phi(R)$.

This means that if $\phi : R \to R'$ is onto, then $R/\ker \phi \cong R'$.

**Proof**

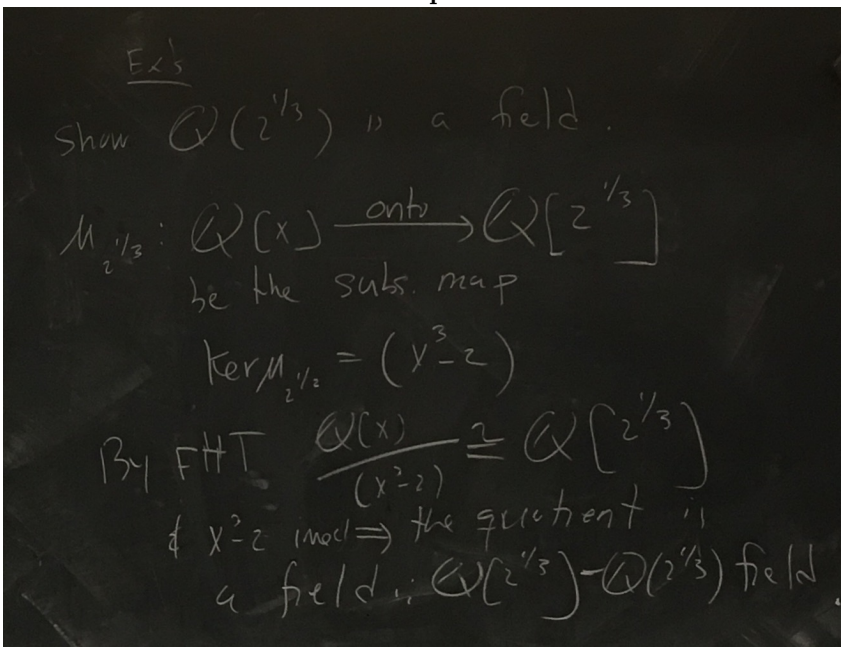


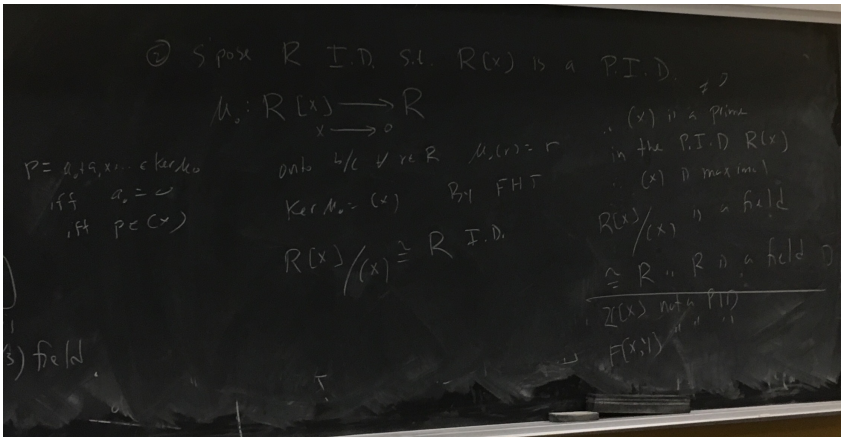




onto
$\forall\ r' \in \mathrm{im}\phi : r' = \phi(r) = \phi^n(r+k)$
so $\phi^n$ is onto $\mathrm{im}\phi$
$\therefore R/\ker\phi \cong \mathrm{im}\phi$

■

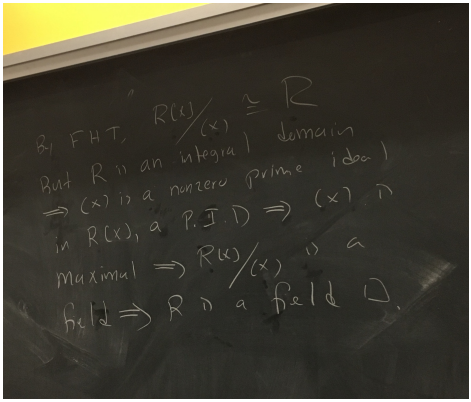
**Example.**


Ex's
Show $\mathbb{Q}(2^{1/3})$ is a field.
$\mu_{2^{1/3}} : \mathbb{Q}[x] \xrightarrow{\text{onto}} \mathbb{Q}[2^{1/3}]$
be the subs. map
$\ker\mu_{2^{1/3}} = (x^3-2)$
By FHT $\dfrac{\mathbb{Q}(x)}{(x^3-2)} \cong \mathbb{Q}[2^{1/3}]$
$\not\mid x^3-2$ irred $\Rightarrow$ the quotient is
a field $\therefore \mathbb{Q}[2^{1/3}] = \mathbb{Q}(2^{1/3})$ field



**Proposition.** If $R$ be an integral domain such that $R[x]$ is a PID, then $R$ must be a field.

**Proof** Define $\mu_0 : R[x] \to R$ by $x \mapsto 0$. So $\mu_0$ is an onto homomorphism. Also, $p(x) \in \ker \mu_0$ iff $0 = \mu_0(p_(x)) = p(0)$. Thus, $\ker \mu_0 = \langle 0 \rangle$.



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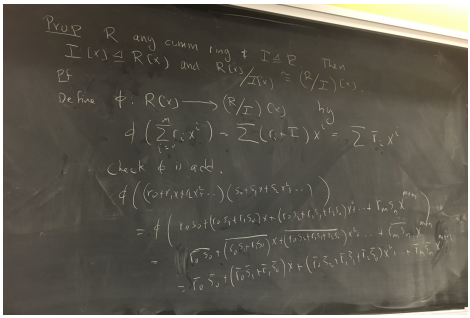
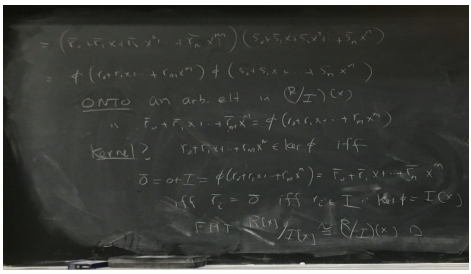
**Proposition.** Suppose $R$ is a commutative ring and $I \trianglelefteq R$. Then $I[x] \trianglelefteq R[x]$ and $R[x]/I[x] \cong (R/I)[x]$.

**Proof**





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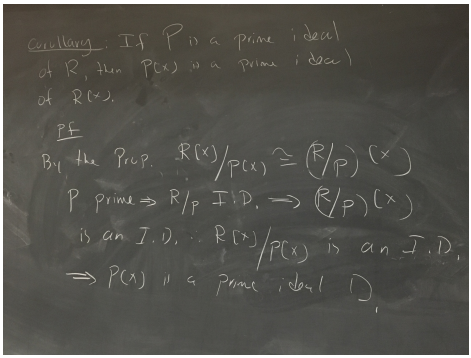
**Corollary.** If $P$ is a prime ideal of $R$, then $P[x]$ is a prime ideal of $R[x]$.

**Proof**



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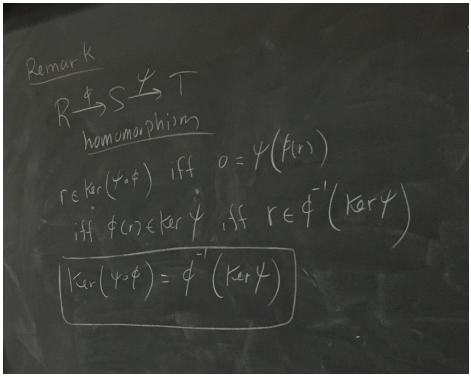
**Special Case. (reduction of coefficients mod $P$)**

$$\phi : \mathbb{Z}[x] \to \mathbb{Z}_p[x], p \text{ prime}$$

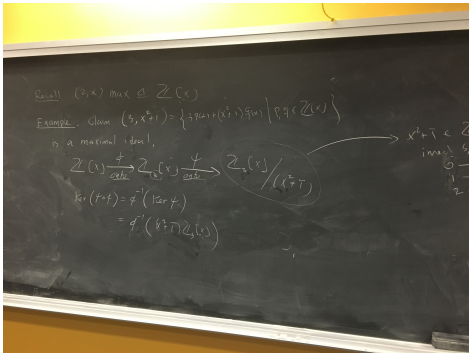
$$\sum a_i x^i \mapsto \sum \bar{a}_i x^i$$

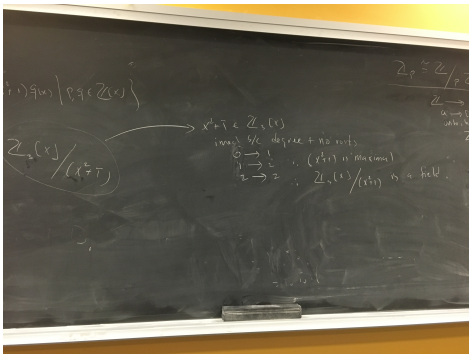
*Remark.* let $R \to^{\phi} S \to^{\psi} T$ be homomorphisms. Then

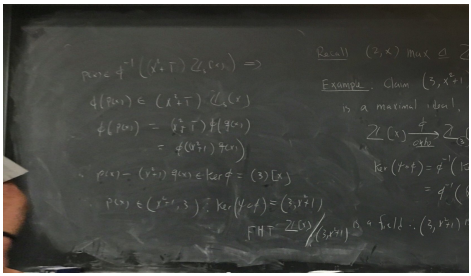
$$\ker(\psi \circ \phi) = \phi^{-1}(\ker \psi)$$

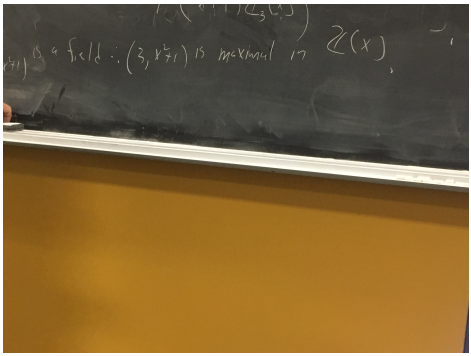



**Example.**









**Proposition.** every ideal of $\mathbb{Z}[x]$ of the form $(p, f(x))$ where $f(x)$ is irreducible mod $p$ is a maximal ideal in $\mathbb{Z}[x]$.

**Theorem (Correspondence thm).** let $\phi : R \to R'$ be an onto homomorphism. Then $I \leftrightarrow \phi(I)$ sets up a 1-to-1 correspondence between all the ideals of $R$ containing $\ker \phi$ and all the ideals of $R'$.

**Application.** $\pi : R \to^{\text{onto}} R/I$, $\ker \pi = I$.
By the correspondence thm, an arb. ideal of $R/I$ is of the form

$$\pi(J) = J/I$$

where $I \subseteq J$.

# 4 Index