Trevor Klar

Homework 4

1. Let X be a nonempty topological space and let μ be a measure on X. Prove that if the functions $f_n : X \to [-\infty, +\infty]$ are μ-measurable for $n \in \mathbb{N}$, then the set

$$A = \{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

is μ-measurable.

Proof To simplify notation, denote $f^*(x) = \limsup_{n\to\infty} f_n(x)$ and $f_*(x) = \liminf_{n\to\infty} f_n(x)$. Let $F : \mathbb{R} \to \mathbb{R}$ be

$$F(x) = f^*(x) - f_*(x)$$

Actually, F is only defined on a subset of $\mathbb{R}$, that when the limsup and the liminf are not both infinite. Observe that F is μ-measurable, since it is a sum of two μ-measurable functions, and

$$F^{-1}(\{0\}) \cup \{x \in \mathbb{R} : F(x) \text{ is undefined}\} = A,$$

since $F(x)$ is undefined exactly when $\lim_{n\to\infty} f_n(x) = \pm\infty$ and $F(x) = 0$ exactly when $\lim_{n\to\infty} f_n(x)$ exists and is finite. Thus,

$$A = F^{-1}\{0\} \cup \left(f_*^{-1}\{+\infty\} \cap f^{*-1}\{+\infty\}\right) \cup \left(f_*^{-1}\{-\infty\} \cap f^{*-1}\{-\infty\}\right).$$

Now we show that each of the above sets is μ-measurable, which means that A consists of unions and intersections of μ-measurable sets, and thus A is μ-measurable.

- F is always positive since $f^* \geq f_*$ everywhere. So $F^{-1}\{0\} = F^{-1}[-\infty, 0]$ and thus is μ-measurable.
- For any μ-measurable f (including f^* and f_*), we have that

$$f^{-1}\{\infty\} = \left(\bigcup_{n=1}^{\infty} f^{-1}[-\infty, n)\right)^{\complement}$$

$$f^{-1}\{-\infty\} = \left(\bigcup_{n=1}^{\infty} f^{-1}[n, \infty]\right)^{\complement}$$

  and thus is measurable.

$\blacksquare$

2. Prove that any Lebesgue-measurable function $f:\mathbb{R}\to\mathbb{R}$ that satisfies the relation

$$f(x+y)=f(x)+f(y) \quad \text{for all} \quad x,y\in\mathbb{R}$$

must be linear.

Proof We need to show that for all $\lambda\in\mathbb{R}$,

$$f(\lambda x)=\lambda f(x). \tag{$\dagger$}$$

First, observe that for any $n\in\mathbb{N}$,

$$f(nx)=f(\overbrace{x+\cdots+x}^{n})=\overbrace{f(x)+\cdots+f(x)}^{n}=nf(x),$$

so ($\dagger$) holds for $\lambda\in\mathbb{N}$. Next, observe that

$$f(x)=f\left(\overbrace{\frac{x}{n}+\cdots+\frac{x}{n}}^{n}\right)=\overbrace{f\left(\frac{x}{n}\right)+\cdots+f\left(\frac{x}{n}\right)}^{n}=nf\left(\frac{x}{n}\right),$$

so $\frac{1}{n}f(x)=f\left(\frac{x}{n}\right)$, which together with the previous result means that for every $\frac{p}{q}\in\mathbb{Q}$,

$$f\left(\frac{p}{q}x\right)=pf\left(\frac{x}{q}\right)=\frac{p}{q}f(x),$$

so ($\dagger$) holds for $\lambda\in\mathbb{Q}$. To prove the final result, we will need the following lemma.

Lemma. f is continuous at $x=0$.
Let $\varepsilon>0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ (thought of as the codomain of f), then $\bigcup_{i=1}^{\infty}B_{\frac{\varepsilon}{2}}(q_i)=\mathbb{R}$, where $\{q_i\}_{i=1}^{\infty}$ is an enumeration of the rationals. Since f is measurable, then every $f^{-1}\left(B_{\frac{\varepsilon}{2}}(q_i)\right)$ is measurable. Since $f^{-1}(\mathbb{R})=\mathbb{R}$ and

$$f^{-1}(\mathbb{R})=f^{-1}\left(\bigcup_{i=1}^{\infty}B_{\frac{\varepsilon}{2}}(q_i)\right)=\bigcup_{i=1}^{\infty}f^{-1}\left(B_{\frac{\varepsilon}{2}}(q_i)\right),$$

then $\bigcup_{i=1}^{\infty}f^{-1}\left(B_{\frac{\varepsilon}{2}}(q_i)\right)$ covers $\mathbb{R}$ so at least one of them has positive measure, by subadditivity. Say $f^{-1}\left(B_{\frac{\varepsilon}{2}}(q_k)\right)$ does and call it A. Since $\mu(A)>0$ then $A-A$ contains a neighborhood of zero, call it $B_\delta(0)$. Any element x of $B_\delta(0)\subset A-A$ can be written in the form $x=a_1-a_2$, so $f(x)=f(a_1-a_2)=f(a_1)-f(a_2)$. That is,

$$f(B_\delta(0))\subseteq f(A)-f(A)\subset B_{\frac{\varepsilon}{2}}(q_k)-B_{\frac{\varepsilon}{2}}(q_k)\subset B_\varepsilon(0),$$

so f is continuous at 0. $\square$

We can use this to show that f is continuous everywhere. For any $x \in \mathbb{R}$,

$$\begin{aligned}\lim_{h\to 0}[f(x+h)] &= \lim_{h\to 0}[f(x)+f(h)] \\ &= \lim_{h\to 0} f(x) + \lim_{h\to 0} f(h) \\ &= f(x)+f(0) \\ &= f(x)+0\end{aligned}$$

Let $\lambda \in \mathbb{R}$, and let $\{q_n\}_{n=1}^{\infty}$ be a sequence of rational numbers that converges to λ. Then,

$$\begin{aligned}f(\lambda x) &= f\left(\lim_{n\to\infty} q_n x\right) \qquad \text{and since } f \text{ is continuous,} \\ &= \lim_{n\to\infty} f\left(q_n x\right) \\ &= \lim_{n\to\infty} q_n f\left(x\right) \\ &= \lambda f(x)\end{aligned}$$

so (†) holds for $\lambda \in \mathbb{R}$, and we are done. ∎

3. Let $f : (0,1) \to \mathbb{R}$ be such that for every $x \in (0,1)$ there exists $\delta_x > 0$ and a Borel-measurable function $g_x : \mathbb{R} \to \mathbb{R}$ (both dependent on x), such that $f(y) = g_x(y)$ for all $y \in D_x$, where $D_x = B_{\delta_x}(x) \cap (0,1)$. Prove that f is Borel-measurable.

Proof Observe that $\{D_x\}_{x\in(0,1)}$ is an open cover of $(0,1)$. Since we know that any open set is a countable union of open balls with rational radii and centers, we can produce a countable subcover. Let $\{B_i\}_{i=1}^{\infty}$ be an enumeration of the rational balls in $\mathbb{R}$, and let

$$\Gamma = \{i \in \mathbb{N} \mid B_i \subset D_x \text{ for some } x \in (0,1)\}.$$

Then $\{B_i\}_{i\in\Gamma}$ covers $(0,1)$ and for each $i \in \Gamma$, we can choose a corresponding x_i such that $B_i \subset D_{x_i}$. Then $\{D_{x_i}\}_{i\in\Gamma}$ is a countable subcover of $\{D_x\}_{x\in(0,1)}$.

Let $I \in \mathbb{R}$ be any open interval. Then

$$\begin{aligned} f^{-1}(I) &= \bigcup_{i\in\Gamma} \left(f\big|_{D_{x_i}} \right)^{-1} (I) \\ &= \bigcup_{i\in\Gamma} \left(g_{x_i}\big|_{D_{x_i}} \right)^{-1} (I) \\ &= \bigcup_{i\in\Gamma} \left\{ y \in \left(B_{\delta_{x_i}}(x_i) \cap (0,1) \right) \mid g_{x_i}(y) \in I \right\} \\ &= \bigcup_{i\in\Gamma} B_{\delta_{x_i}}(x_i) \cap (0,1) \cap g_{x_i}^{-1}(I) \end{aligned}$$

which is a Borel set since g_x is Borel-measurable for all x, so it is a countable union of Borel sets. Thus f is Borel-measurable. ■

4. Give an example of a collection of Lebesgue-measurable functions $\{f_\alpha\}_{\alpha\in A}$ where each $f_\alpha : \mathbb{R} \to \mathbb{R}$ and the function

$$g(x) = \sup_{\alpha\in A} f_\alpha(x), \qquad x \in \mathbb{R}$$

is finite for all $x \in \mathbb{R}$ but g is not Lebesgue-measurable. Here A is a nonempty index set.
Answer: Let $V \subset [0,1]$ be a Vitali set, and for each $\alpha \in V$, let $f_\alpha = \chi_{\{\alpha\}}$. Then

$$g(x) = \sup_{\alpha\in V} f_\alpha(x) = \chi_V$$

which is not Lebesgue-measurable, as we know.