Math 201B, Homework 1 (Theory of integration)

Problem1. Let X be a topological space and let μ be a measure on X such that $\mu(X) < \infty$ (in that case μ is said to be a finite measure on X). A family of μ-measurable functions $f_n \colon X \to \mathbb{R}$ is called **uniformly integrable in** X**,** if for any $\epsilon > 0$, there exists $M > 0$ such that

$$\int_{\{x \,:\, |f_n(x)| > M\}} |f_n(x)| d\mu < \epsilon, \quad \text{for all} \quad n = 1, 2, \ldots .$$

Similarly $\{f_n\}$ is called **uniformly absolutely continuous** if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\mu-$measurable set $A \subset X$ with $\mu(A) < \delta$ one has

$$\left| \int_A f_n(x) d\mu \right| < \epsilon, \quad \text{for all} \quad n = 1, 2, \ldots .$$

Prove that $\{f_n\}$ is uniformly integrable iff

$$\sup_n \int_X |f_n(x)| d\mu < \infty,$$

and $\{f_n\}$ is uniformly absolutely continuous.

Problem2. Let X be a topological space and let μ be a finite measure on X. Let $f, f_n \colon X \to \mathbb{R}$ be $\mu-$summable on X such that the point-wise convergence $f_n(x) \to f(x)$ holds $\mu-$a.e. in X. Prove that $\{f_n\}$ is uniformly integrable iff

$$\lim_{n \to \infty} \int_X |f_n(x) - f(x)| d\mu = 0.$$

Problem3. Let X be a topological space and let μ be a finite measure on X. Let $f_n \colon X \to \mathbb{R}$ be $\mu-$measurable such that

$$\sup_n \int_X |f_n(x)|^{1+\delta} d\mu < \infty$$

for some $\delta > 0$. Prove that $\{f_n\}$ is uniformly integrable.