Math 550
Homework 7
Dr. Fuller
Due October 25

1. Let M be a k-dimensional manifold in $\mathbf{R}^n$. Prove that if there exists a nowhere zero k-form on M, then M is orientable. (Hint: recall Homework 6, problem 5.)

2. There is a general correspondence between k-forms and $(n-k)$-forms on $\mathbf{R}^n$, for all $1 \le k \le n$. Given $\omega \in \Omega^k(\mathbf{R}^n)$, we define $\star\omega \in \Omega^{n-k}(\mathbf{R}^n)$ using the rule

$$\star(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \pm\, dx_{j_1} \wedge \cdots \wedge dx_{j_{n-k}},$$

and extending linearly, where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k}$, and $\{i_1, \cdots, i_k, j_1, \cdots, j_{n-k}\} = \{1, \cdots, n\}$. The sign is chosen so that $\omega \wedge \star\omega = dx_1 \wedge \cdots \wedge dx_n$. (For example, in $\mathbf{R}^5$, $\star(dx_1 \wedge dx_4) = dx_2 \wedge dx_3 \wedge dx_5$ and $\star(dx_1 \wedge dx_3) = -dx_2 \wedge dx_4 \wedge dx_5$.)

Prove that $\star\star\,\omega = (-1)^{k(n-k)}\omega$.