Labutin, Spring 2020
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Homework 3

1. Think of ℓ^1 as a linear space and ℓ^∞ as its dual. Let $\bar{B}(\ell^\infty)$ be the closed unit ball with respect to the metric $||f-g||_{\ell^\infty}$. For every $f, g \in \bar{B}(\ell^\infty)$ define another metric

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} \left| f_n - g_n \right|.$$

Prove that $\sigma(\bar{B}(\ell^\infty), \ell^1)^\dagger$ coincides with the topology of the metric d.

Proof Since the two topologies are both translation invariant, it suffices to show that $W(0;p)$ is open in the d-topology and that the d-ball ${}_dB_r(0)$ is open in $\sigma(\bar{B}(\ell^\infty), \ell^1)$.

Part I: We show that $\sigma(\bar{B}(\ell^\infty), \ell^1) \subset \mathcal{T}_d(\bar{B}(\ell^\infty))$. Let $W(0,p)$ be an arbitrary subbasic weak* neighborhood in $\sigma(\bar{B}(\ell^\infty), \ell^1)$ centered at 0. Fix $f \in W(0,p)$. This means that

$$\sup_n |f_n| \leq 1$$
$$\left| \sum_n f_n p_n \right| < 1$$

Since $p \in \ell^1$, then $\sum_n |p_n|$ converges, so there exists N such that

$$\sum_{n=N}^{\infty} |p_n| < \frac{1}{4}. \tag{1}$$

Let $r > 0$ such that for all $n \leq N$,

$$|p_n| \leq \frac{2^{-n}}{4r}, \tag{2}$$

and consider ${}_dB_r(f)$.

Claim: ${}_dB_r(f) \subset W(0,p)$.

Proof Let $g \in {}_dB_r(f)$. Then

$$\sum_n 2^{-n} |f_n - g_n| < r, \text{ and} \tag{3}$$
$$\sup_n |f_n - g_n| = 2. \tag{4}$$

†the weak* topology of ℓ^∞ restricted to $\bar{B}(\ell^\infty)$

So

$$\begin{aligned}
|\langle (f-g), p\rangle| &= \left|\sum_{n=1}^{\infty}(f_n - g_n)(p_n)\right| \\
&\leq \sum_{n=1}^{\infty} |f_n - g_n||p_n| \\
&= \sum_{1}^{N} |f_n - g_n||p_n| + \sum_{N}^{\infty} |f_n - g_n||p_n| \\
&\leq \sum_{1}^{N} \frac{2^{-n}}{4r}|f_n - g_n| + \sum_{N}^{\infty} 2|p_n| && \text{applying (2) and (4)} \\
&< \frac{1}{2} + \frac{1}{2} && \text{applying (3) and (1)} \\
&= 1
\end{aligned}$$

Thus every $f \in W(0,p)$ has a d-ball containing f which is a subset of $W(0,p)$, so Part I is proved. $\square$

Part II: First note that

$$||f||_d \leq ||f||_{\ell^\infty}$$

since, if f_n is an absolutely decreasing sequence, then

$$\sum_n 2^{-n}|f_n| \leq \sup_n |f_n| \qquad \text{(with equality if } f_n \text{ is constant)},$$

and swapping any coordinates of f_n will cause $||f||_d$ to decrease while $||f||_{\ell^\infty}$ remains constant. Thus the balls ${}_{\ell^\infty}B_r(f) \subset {}_dB_r(f)$ whenever they have the same radius and center. This means that for any ball ${}_dB_r(f)$ with $g \in {}_dB_r(f)$, there is of course some

$${}_dB_{r'}(g) \subset {}_dB_r(f),$$

and

$${}_{\ell^\infty}B_{r'}(g) \subset {}_dB_{r'}(g)$$

so we're done. $\blacksquare$

2. Let $u_n \xrightarrow{w} u$ in a Banach space X, and let $\phi_n \xrightarrow{w*} \phi$ in X^*. Give an example in $X = \ell^2$ to show that $\langle \phi_n, u_n \rangle$ need not be convergent.

Prove that if either $u_n \to u$ or $\phi_n \to \phi$ strongly, then $\langle \phi_n, u_n \rangle \to \langle \phi, u \rangle$.

Example. Recall that $\ell^{2*} = \ell^2$. Let $u_j = (-1)^j e_j$ and let $\phi_n = e_n$. Then $u_j \xrightarrow{w} 0$ and $\phi_n \xrightarrow{w*} 0$, but $\langle \phi_k, u_k \rangle$ alternates between 1 and -1, and doesn't converge. $\square$

Remark. Wait, why doesn't it work that $\langle \phi_n, u_j \rangle \xrightarrow{j} \langle \phi_n, u \rangle \xrightarrow{n} \langle \phi, u \rangle$; using weak convergence followed by weak* convergence?

It does. However $\langle \phi_k, u_k \rangle \xrightarrow{k} \langle \phi, u \rangle$ requires that we can get $\langle \phi_k, u_k \rangle$ arbitrarily close to $\langle \phi, u \rangle$ without taking *either one* of the limits.

Proof Case I: Suppose $||x_n|| \to ||x||$. Then there exists $N > 0$ such that $n > N \implies ||x_n - x|| < \varepsilon$ for all $\varepsilon > 0$. Since $\phi_j \xrightarrow{w*} \phi$, then by the Uniform Boundedness Principle $||\phi_j|| \leq C$. Thus

$$\begin{aligned} |(\phi_n - \phi)(x_n - x)| &\leq \phi_n(x_n - x) + \phi(x_n - x) \\ &\leq C\varepsilon + C\varepsilon = 2C\varepsilon, \end{aligned}$$

and after rescaling, we're done. $\square$

Case II: If on the other hand $||\phi_n \to \phi||$, then $\hat{x}_n \xrightarrow{w**} \hat{x}$, and we can use the same proof as above. ■

3. Let $\Omega \subset \mathbb{R}^n$ such that $|\Omega| < \infty$, and let (f_n) be a sequence in $L^p(\Omega)$. Suppose $f_n \to 0$ μ-*a.e.* in Ω, and $f_n \xrightarrow{w} 0$.[†]

Prove that for $p = 2$ and $q = 1$[‡] one has $||f_n||_{L^q(\Omega)} \to 0$.

Proof Let $\varepsilon > 0$. Since $f_n \xrightarrow{w} 0$, then by the Uniform Boundedness Principle $||f_n||_2 < B$. By Egoroff's Theorem, $\exists\, N > 0$ such that $\forall\, n > N$ $\int_U |f_n| < \varepsilon$ where $\mu(U^\complement) < \varepsilon$. So $\forall\, n > N$,

$$\begin{aligned} \int_\Omega |f_n| &= \int_U |f_n| + \int_{U^\complement} |f_n| \\ &\leq \varepsilon + \int_{U^\complement} |f_n(1)| \\ &\leq \varepsilon + ||f_n||_2 \left(\mu\left(U^\complement\right)\right)^{1/2} \\ &\leq \varepsilon + B\sqrt{\varepsilon} \end{aligned}$$

and after rescaling, we're done. ■

[†] $|\omega|$ denotes the Lebesgue measure of Ω, and $p < 1$.
[‡] This actually holds for any $1 \leq q < p$.

5. Let X be reflexive. Prove that any closed subspace W of X is also reflexive.

Proof We know that X is reflexive iff $\bar{B}(X)$ is weakly compact, and in the subspace topology $\bar{B}(W)$ is exactly $W \cap \bar{B}(X)$, so it suffices to show that $W \cap \bar{B}(X)$ is weakly compact. Since W is closed and convex, then by Hahn-Banach there exists a functional φ separating $p \in W^{\complement}$ from W, so $W(p, \varphi) \subset W^{\complement}$ and W is weakly closed. Thus $\bar{B}(W)$ is a weak closed subset of the weak compact set $\bar{B}(X)$, so $\bar{B}(W)$ is weak compact. Therefore W is reflexive. ■