Properties of $p(n)$ and $\tau(n)$

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Properties of $p(n)$ and $\tau(n)$ defined by the equations

$$\sum_0^\infty p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots},$$

$$\sum_0^\infty \tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\ldots\}^{24}.$$

1 Modulus 5

Define

$$\begin{aligned} P &= 1-24\left(\tfrac{x}{1-x}+\tfrac{2x^2}{1-x^2}+\tfrac{3x^3}{1-x^3}+\ldots\right) \\ Q &= 1-240\left(\tfrac{1^3x}{1-x}+\tfrac{2^3x^2}{1-x^2}+\tfrac{3^3x^3}{1-x^3}+\ldots\right) \\ R &= 1-504\left(\tfrac{1^5x}{1-x}+\tfrac{2^5x^2}{1-x^2}+\tfrac{3^5x^3}{1-x^3}+\ldots\right) \end{aligned}$$

so that

$$Q^3-R^2 = 1728x\left\{(1-x)(1-x^2)(1-x^3)\ldots\right\}^{24}. \quad ※ \tag{1}$$

※ For an elementary proof

Further let J be any function of x with integral coefficients but not the same function throughout, and also let $\sigma_5(n)$ denote the 5th powers of the divisors of n. Then it is easy to see that

$$Q = 5J; R = P+5J, \tag{2}$$

hence

$$Q^3-R^2 = Q-P^2+5J, \tag{3}$$

but

$$Q-P^2 = 288\sum_1^\infty n\,\sigma_1(n)\,x^n; \quad ※ \tag{4}$$

※ And it is obvious that

$$\left\{(1-x)(1-x^2)(1-x^3)\ldots\right\}^{24} = \frac{(1-x^{25})(1-x^{50})(1-x^{75})\ldots}{(1-x)(1-x^2)(1-x^3)\ldots}+5J. \tag{5}$$

It follows from (1.1), (1.3)–(1.5) that

$$ask \tag{6}$$