Math 501
Homework 4

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1. Let $f : X \to Y$ be a function.

   (a) Assume $X = \bigcup_{\alpha\in\Gamma} U_\alpha$, with each U_α open, and each $f|_{U_\alpha} : U_\alpha \to Y$ continuous. Prove that f is continuous.

   Proof Let $B \in Y$ be an arbitrary open subset of Y. Since B is open, and each $f|_{U_\alpha} : U_\alpha \to Y$ is continuous, then each $f|_{U_\alpha}^{-1}(B)$ is open in U_α and, since U_α is open in X, then $f|_{U_\alpha}^{-1}(B)$ is open in X. Now,

$$\begin{aligned}\bigcup_{\alpha\in\Gamma} f|_{U_\alpha}^{-1}(B) &= \bigcup_{\alpha\in\Gamma} \left(f^{-1}(B) \cap U_\alpha\right) \\ &= \left(f^{-1}(B) \cap \bigcup_{\alpha\in\Gamma} U_\alpha\right) \\ &= f^{-1}(B) \cap X \\ &= f^{-1}(B)\end{aligned}$$

   So, $f^{-1}(B)$ is a union of open sets, which means it is open. Thus, f is continuous. ■

   (b) Assume $X = \bigcup_{\alpha\in\Gamma} A_\alpha$, with each A_α closed, and each $f|_{A_\alpha} : A_\alpha \to Y$ continuous. Is f continuous? Prove or give a counterexample.

   Counterexample. Let $f : (\mathbb{R}, usual) \to (\mathbb{R}, usual)$ be

$$f(x) = \begin{cases} 0, & x = 0 \\ \sin\left(\frac{1}{x}\right), & x \neq 0 \end{cases}$$

   and consider the collection of closed sets $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} A_n$ with $n \in Z$, and A_n defined as follows:

$$A_n = \begin{cases} \left[a, \frac{1}{a}\right], & n < 0 \\ \{0\}, & n = 0 \\ \left[\frac{1}{a}, a\right], & n > 0 \end{cases}$$

   Now, it is a common result from calculus that $\sin(\frac{1}{x})$ is continuous at every point except $x = 0$, so for all $n \neq 0$, $f|_{A_n}$ is continuous (since none of these sets contain 0). Now we will show that $f|_{A_0}$ is also continuous. For any closed set $F \in \mathbb{R}$, $f|_{A_0}^{-1}(F) = \{0\}$ if $0 \in F$, and $f|_{A_0}^{-1}(F) = \emptyset$ if $0 \notin F$. Since $\{0\}$ and $\emptyset$ are both closed, then $f|_{A_0}^{-1}(F)$ is closed, so $f|_{A_0}$ is continuous.
   The reader will recall that f can easily be shown not to be continuous by the $\delta - \epsilon$ definition, but we will make the same case using the results we have learned in topology. Consider the following preimage of a closed set:

$$f^{-1}(\{-1, 1\}) = \left\{\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \dots\right\} \cup \left\{-\frac{2}{\pi}, -\frac{2}{3\pi}, -\frac{2}{5\pi}, \dots\right\}$$

   Since $f^{-1}(\{-1, 1\})$ has a 0 as limit point, but does not contain 0, then $f^{-1}(\{-1, 1\})$ is not closed. Therefore, f is not continuous. ■

2. (a) Prove that the set of intervals of the form $[a, b)$ with $a, b \in \mathbb{R}$ are the basis for a topology on $\mathbb{R}$. We will refer to $\mathbb{R}$ with this topology as $\mathbb{R}^1_{\text{bad}}$. Show that $\mathbb{R}^1_{\text{bad}}$ is not the usual topology on $\mathbb{R}$.

Proof

- Since $[1, 0) = \{x \in \mathbb{R} : 1 \leq x < 0\} = \emptyset$, then $\emptyset \in \mathbb{R}^1_{\text{bad}}$.
- For any $x \in \mathbb{R}$, $x \in [x - 1, x + 1)$, so $\mathbb{R}^1_{\text{bad}}$ covers $\mathbb{R}$.
- For any $a, b, c, d \in \mathbb{R}$,

$$\begin{aligned} [a, b) \cap [c, d) &= \{x \in \mathbb{R} : a \leq x < b\} \cap \{x \in \mathbb{R} : c \leq x < d\} \\ &= \{x \in \mathbb{R} : \max(a, c) \leq x < \min(b, d)\} \\ &= [\max(a, c), \min(b, d)) \\ &\in \mathbb{R}^1_{\text{bad}} \end{aligned}$$

So, as desired according to Theorem 13, for any $[a, b), [c, d) \in \mathbb{R}^1_{\text{bad}}$ which both contain x, there exists $[a, b) \cap [c, d) \in \mathbb{R}^1_{\text{bad}}$ such that $x \in [a, b) \cap [c, d)$.

Thus, $\mathbb{R}^1_{\text{bad}}$ forms the basis for a topology on $\mathbb{R}$. ■

Proof Now we will show that $\mathbb{R}^1_{\text{bad}}$ is not the usual topology on $\mathbb{R}$. Consider the set $[a, b)$, for some $a, b \in \mathbb{R}$ and $a < b$. By definition, $[a, b)$ is open in $\mathbb{R}^1_{\text{bad}}$. We will show that $[a, b)$ is not open in the usual topology, and thus $\mathbb{R}_{usual}$ and $\mathbb{R}^1_{\text{bad}}$ are different. It suffices to show that no open interval (m, n) containing a is a subset of $[a, b)$.

$$a \in (m, n) \implies m < a < n \implies m < \tfrac{m+a}{2} < a < n.$$

Thus, $\frac{m+a}{2} \in (m, n)$ but $\frac{m+a}{2} \notin [a, b)$, so $(m, n) \not\subset [a, b)$. ■

(b) Prove that intervals $[a, b)$ are both open and closed in $\mathbb{R}^1_{\text{bad}}$.

Proof Any interval $[a, b)$ is open in $\mathbb{R}^1_{\text{bad}}$ by definition. If $a > b$, then $[a, b) = \emptyset$ and is closed. If $a = b$, then $[a, b) = \{x : a \leq x < a\} = \emptyset$, so $[a, b)$ is closed in this case as well.
Now, suppose $a < b$ and consider $[a, b)^{\complement} = (-\infty, a) \cup [b, \infty)$. Since $(-\infty, a) = \bigcup\limits_{n \in \mathbb{N}} [-n, a)$, and $[b, \infty) = \bigcup\limits_{n \in \mathbb{N}} [b, n)$, then $(-\infty, a) \cup [b, \infty)$ is a union of sets which are open in $\mathbb{R}^1_{\text{bad}}$. Therefore, $(-\infty, a) \cup [b, \infty)$ is also open in $\mathbb{R}^1_{\text{bad}}$, so $[a, b)$ is closed. ■

(c) Prove that every open interval (a, b) is open in $\mathbb{R}^1_{\text{bad}}$.

Proof $(a, b) = \bigcup\limits_{n \in \mathbb{N}} \left[a + \frac{1}{n}, b\right)$, so (a, b) is a union of open sets, and thus is open. ■

(d) Prove that the set of intervals of the form $[a, b)$ with $a, b \in \mathbb{Q}$ are the basis for a topology on R. Show that this topology is different from $\mathbb{R}^1_{\text{bad}}$.

Proof We will denote this topology as $\mathbb{R}^1_{\text{bad}\mathbb{Q}}$.

- Since $[1, 0) = \{x \in \mathbb{R} : 1 \leq x < 0\} = \emptyset$, then $\emptyset \in \mathbb{R}^1_{\text{bad}\mathbb{Q}}$.
- For any $x \in \mathbb{R}$, $x \in [\lfloor x \rfloor, \lfloor x \rfloor + 1)$, so $\mathbb{R}^1_{\text{bad}\mathbb{Q}}$ covers $\mathbb{R}$.
- For any $a, b, c, d \in \mathbb{Q}$,

$$\begin{aligned} [a, b) \cap [c, d) &= \{x \in \mathbb{R} : a \leq x < b\} \cap \{x \in \mathbb{R} : c \leq x < d\} \\ &= \{x \in \mathbb{R} : \max(a, c) \leq x < \min(b, d)\} \\ &= [\max(a, c), \min(b, d)) \\ &\in \mathbb{R}^1_{\text{bad}\mathbb{Q}} \end{aligned}$$

So, as desired according to Theorem 13, for any $[a, b), [c, d) \in \mathbb{R}^1_{\text{bad}\mathbb{Q}}$ which both contain x, there exists $[a, b) \cap [c, d) \in \mathbb{R}^1_{\text{bad}\mathbb{Q}}$ such that $x \in [a, b) \cap [c, d)$.

Thus, $\mathbb{R}^1_{\text{bad}\mathbb{Q}}$ forms the basis for a topology on $\mathbb{R}$. □

Now we will show that $\mathbb{R}^1_{\text{bad}\mathbb{Q}} \neq \mathbb{R}^1_{\text{bad}}$. Consider the set $[\pi, 5)$. By definition, $[\pi, 5)$ is open in $\mathbb{R}^1_{\text{bad}}$. Now, $[\pi, 5)$ is not itself a basic open set in $\mathbb{R}^1_{\text{bad}\mathbb{Q}}$, nor is it a union of basic sets in $\mathbb{R}^1_{\text{bad}\mathbb{Q}}$, since any union of rational intervals $[a, b)$ must either disclude π, or include reals which are less than π. ■

3. (a) Show that the set of half-open rectangles of the form $\{(x, y) \in \mathbb{R}^2 : a \leq x < b, c \leq y < d\}$ form the basis for a topology on $\mathbb{R}^2$. We will refer to $\mathbb{R}^2$ endowed with this topology as $\mathbb{R}^2_{\text{bad}}$.

   Notation. Let $[a, b) \times [c, d)$ denote a set $\{(x, y) \in \mathbb{R}^2 : a \leq x < b, c \leq y < d\} \in \mathbb{R}^2_{\text{bad}}$.

   Proof

   - $[0, 0) \times [0, 0) = \emptyset$, so $\emptyset \in \mathbb{R}^2_{\text{bad}}$.
   - Let $(x, y) \in \mathbb{R}^2$. Then, $(x, y) \in [x, x + 1) \times [y, y + 1)$, so $\mathbb{R}^2_{\text{bad}}$ covers $\mathbb{R}^2$.
   - For any $x_1, \dots x_4, y_1, \dots y_4 \in \mathbb{R}$,

   $$\begin{aligned} [x_1, x_2) \times [y_1, y_2) \cap [x_3, x_4) \times [y_3, y_4) &= [\max(x_1, x_3), min(x_2, x_4)) \times [\max(y_1, y_3), min(y_2, y_4)) \\ &\in \mathbb{R}^2_{\text{bad}} \end{aligned}$$

   So, as desired according to Theorem 13, this set of half-open rectangles is the basis for a topology on $\mathbb{R}^2$. ■

   (b) Let L_1 denote the line $y = -x$ in $\mathbb{R}^2$. Show that the subspace topology on L_1, as a subspace of $\mathbb{R}^2_{\text{bad}}$, is the discrete topology.

   Proof Let $(x, -x)$ be any point on the line $y = -x$. Now, since the singleton $\{(x, -x)\} = [x, x + 1) \times [-x, -x + 1) \cap L_1$, then $\{(x, -x)\}$ is open in L_1. Thus, for any set $S \subset L_1$, the union $\bigcup_{(x,-x) \in S} \{(x, -x)\} = S$ is open. ■

   (c) Let L_2 denote the line $y = x$ in $\mathbb{R}^2$. Show that the subspace topology on L_2, as a subspace of $\mathbb{R}^2_{\text{bad}}$, is not the discrete topology.

   Proof To show that the subspace topology on L_2 is not the discrete topology, it suffices to produce a set which is not open. Consider the singleton $\{(0, 0)\}$. If $\{(0, 0)\}$ is open in L_2, then for any $(x, y) \in \{(0, 0)\}$, there exists a basic open set U containing (x, y) such that $U \cap L_2 = \{(0, 0)\}$. Let $[a, b) \times [c, d)$ be an any set containing the origin which is a basic open set in $\mathbb{R}^2_{\text{bad}}$. Since $(0, 0) \in [a, b) \times [c, d)$, then $b > 0$ and $d > 0$. Let $p = \min(b, d)$ Thus, $(\frac{p}{2}, \frac{p}{2}) \in [a, b) \times [c, d) \cap L_2$, but $(\frac{b}{2}, \frac{d}{2}) \notin \{(0, 0)\}$. Thus, there is no basic open set whose intersection with L_2 is $\{(0, 0)\}$, so $\{(0, 0)\}$ is not open. ■

5. Let X be a set, and let $\{0, 1\}^X$ denote the set of all functions $X \to \{0, 1\}$.

   (a) Prove that the collection of sets of the form $U(x, \epsilon) = \{f \in \{0, 1\}^X : f(x) = \epsilon\}$, for all $x \in X$ and $\epsilon \in \{0, 1\}$ forms a subbasis for a topology on $\{0, 1\}^X$.

   Proof Let $\mathscr{S}$ be the collection of all sets of the form $U(x, \epsilon) = \{f \in \{0, 1\}^X : f(x) = \epsilon\}$, with $x \in X$ and $\epsilon \in \{0, 1\}$. Let $\mathscr{B}$ be the collection of all finite intersections of sets in $\mathscr{S}$.

   - For some $x_0 \in X$, consider the sets $U(x_0, 1)$ and $U(x_0, 0)$. $U(x_0, 1) \cap U(x_0, 0) = \emptyset$, so $\emptyset \in \mathscr{B}$.
   - Let x_0 be an arbitrary element of X, and let f be an arbitrary function $f : X \to \{0, 1\}$ where $f(x_0) = \epsilon_0$. Since $f \in U(x_0, \epsilon_0)$ by definition, then $\mathscr{B}$ covers $\{0, 1\}^X$.

   Thus, by Theorem 14, $\mathscr{S}$ forms a subbasis for a topology on $\{0, 1\}^X$. ■

   (b) Under what conditions are two basic open sets in this topology disjoint?

   Answer: Since every basic open set is a finite intersection of sets of the form $U(x, \epsilon)$, every basic open set $U \in \mathscr{B}$ has the following property: U has a nonempty "characteristic set" $C \subset X$ such that for any fixed $x \in C$, $f(x) = g(x)$ for all $f, g \in U$. That is, all functions in U are equal at

every point in C.
Thus, two basic open sets U, V in $\mathscr{B}$ are disjoint if and only if their characteristic sets, $C(U), C(V)$ are equal; and for any $f \in U$ and $g \in V$, $f(x) \neq g(x)$ for all $x \in C(U) = C(V)$.

(c) Is this topology Hausdorff?
Answer: Yes.

Proof Let f and g be any two distinct functions in $\{0,1\}^X$. Since they are distinct, there exists at least one $x \in X$ such that $f(x) \neq g(x)$. Without loss of generality, suppose $f(x) = 1$ and $g(x) = 0$. Therefore, the basic open sets $U(x,1)$ and $U(x,0)$ contain f and g, respectively. Since their characteristic sets are equal but they contain functions which are not equal at $x \in C$, we can conclude that $U(x,1)$ and $U(x,0)$ are disjoint. Therefore, this topology is Hausdorff. ■

6. (a) Show that the collection consisting of $\emptyset$ and the set of all intervals $[a,b]$ with $a < b$ does not form the basis for a topology on $\mathbb{R}$.

Proof In order for this collection of sets to be a basis for some topology on $\mathbb{R}$, it must be true that for any two basic sets U, V with $x \in U \cap V$, there exists another basic set W such that $x \in W \subset U \cap V$. However, consider the basic sets

$$[j,k] \text{ and } [k,m].$$

The element k is in the intersection $[j,k] \cap [k,m] = \{k\}$, but the set $\{k\}$ cannot contain any interval $[a,b]$; since $a < b$ implies that $[a,b]$ contains more than just one element. ■

(b) Show that the collection consisting of $\emptyset$ and the set of all intervals $[a,b]$ with $a < b$ does form a subbasis for a topology on $\mathbb{R}$. That topology is one we have seen before. Identify it.

Claim: Let $\mathscr{S}$ be the collection consisting of $\emptyset$ and the set of all intervals $[a,b]$ with $a < b$, and let $\mathscr{B}$ be the collection of all finite intersections of sets in $\mathscr{S}$. Then, $\mathscr{B}$ is a basis for the discrete topology on $\mathbb{R}$.

Proof First, since $[1,2] \cap [3,4] = \emptyset$, then $\emptyset \in \mathscr{B}$. Now, Let S be an arbitrary subset of $\mathbb{R}$, and let x be any real number such that $x \in S$. We can see that $\{x\} \in \mathscr{B}$ by observing that $[x-,x] \cap [x, x+1] = \{x\}$, so we can take the union $\bigcup_{x \in S} \{x\} = S$. Thus, S is a union of open sets, so S is open. Therefore, since any arbitrary $S \subset \mathbb{R}$ is open in this topology, then $\mathscr{B}$ is a basis for the discrete topology on $\mathbb{R}$. ■