Topology Qualifying Exam: Spring 2017 Problem 7

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Problem. Prove that any finite sheeted covering space of a compact metric space is compact.

Solution:
Let X be a compact metric space and $p : \tilde{X} \to X$ a k-sheeted covering space for some finite k. We want to show $\tilde{X}$ is compact. Then let $\mathcal{U}$ be an open cover of $\tilde{X}$ indexed by some possibly infinite set A. Further, for each $x \in X$ let W_x be an open neighborhood of x evenly covered by p. Then the inverse image of each W_x is a collection of k open neighborhoods W_x^i each mapped homeomorphically to W_x by p. Then we have a collection of open sets $W_x^i \cap U_\alpha$ for i from 1 to k, $x \in X$ and α in A. Since each $\tilde{x} \in \tilde{X}$ is in some U_α and in some $W_{p(\tilde{x})}^i$ we have that this collection is an open cover. Then we consider the open cover of X given by $p(W_x^i \cap U_\alpha)$ and by compactness there is some finite subcover, which implies a collection $x_1, ..., x_n$ of points, and sets $U_1, ..., U_m \in \mathcal{U}$ such that the sets $p(W_{x_j}^i \cap U_s)$ for i from 1 to k, j from 1 to n and s from 1 to m cover X. The inverse image of each $p(W_{x_j}^i \cap U_s)$ is the union $\bigcup_{i=1}^k W_{x_j}^i \cap U_s$, and each of these is contained in U_s, and so the collection $U_1, ..., U_m$ is a finite subcover of $\tilde{X}$. Since $\mathcal{U}$ was arbitary, we may conclude that $\tilde{X}$ is compact.