Math 552
Homework 1

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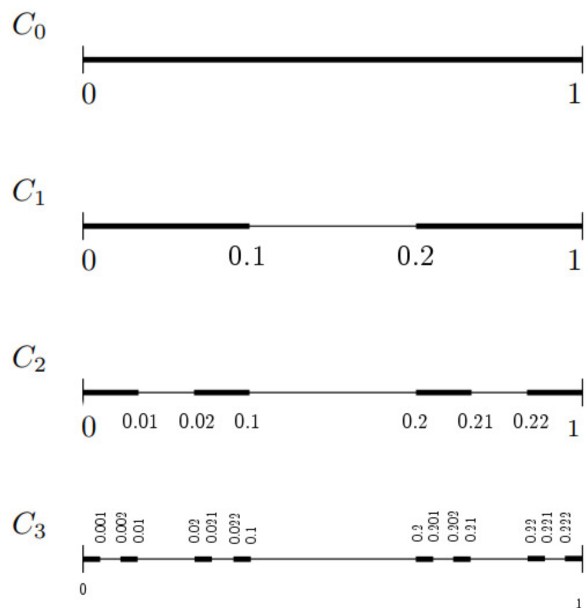
2. The Cantor set $\mathcal{C}$ can also be described in terms of ternary expansions.

   (a) Every number in $[0, 1]$ has a ternary expansion

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}, \qquad \text{where } a_k = 0, 1, \text{or } 2.$$

Note that this decomposition is not unique since, for example, $\frac{1}{3} = \sum_{k=2}^{\infty} \frac{2}{3^k}$. Prove that $x \in \mathcal{C}$ if and only if x has a representation as above where every a_k is either 0 or 2.

Proof First, there is another notation for ternary expansions which the reader may find easier to internalize, namely ternary decimals. Representing a number as a decimal in base 3 is exactly equivalent to the summation notation above; i.e. $\frac{1}{2} = \sum_{k=1}^{\infty} \frac{1}{3^k} = 0.111\ldots = 0.\overline{1}$. So the construction of the cantor set is as follows:



Now we begin the proof in earnest. For any $x \in \mathcal{C}$, one of the following must be true: Case I, there exists an expansion of x with no 1s. Case II, there exists an expansion of x with exactly one 1. Case III, there exists an expansion of x with multiple 1s.

Case III: Let $x \in \mathcal{C}$ and suppose there exists an expansion of x with multiple 1s; that is, $x = 0.\square_1 1 \square_2 1 \ldots$, where the $\square$s represent zero or more other digits of the ternary decimal form of x. Claim: x cannot be in $\mathcal{C}$, a contradiction. The construction of $\mathcal{C}$ begins by starting with the interval $[0, 1]$ and, in each step, proceeds by removing middle thirds. That is, first $(0.1, 0.2)$ is removed, then $(0.01, 0.02)$, $(0.11, 0.12)$[1], and $(0.21, 0.22)$, and so on. Thus, every such interval $(0.\square 1,\ 0.\square 2)$ is removed from the set at some point in the construction. However, when we consider our element, $x = 0.\square_1 1 \square_2 1 \ldots$, we find that $0.\square_1 1, < x < 0.\square_1 2$, and so $x \notin \mathcal{C}$. Thus Case III is impossible.

Case II: Let $x \in \mathcal{C}$ and suppose there exists an expansion of x with exactly one 1. If 1 is the terminal digit of x, then x is of the form $0.\square 1$, and we can also write x as $0.\square 0\overline{2}$. Otherwise if 1 is not the terminal digit of x then the only way $x \in \mathcal{C}$ is if $x = \square 1\overline{2}$ (using the same argument as in Case III), so we can write $x = \square 2$. Thus Case II implies Case I.

Case I: Let $x \in \mathcal{C}$ and suppose there exists an expansion of x with no 1s. This is that which was to be proven, so we are done.

$\blacksquare$

4. **Cantor-like sets.** Construct a closed set $\hat{\mathcal{C}}$ so that at the kth stage of the construction one removes 2^{k-1} centrally situated open intervals each of length ℓ_k, with

$$\ell_1 + 2\ell_2 + \ldots + 2^{k-1}\ell_k < 1.$$

   (a) If ℓ_j are chosen small enough, then $\sum_{k=1}^{\infty} 2^{k-1}\ell_k < 1$. In this case, show that $m\left(\hat{\mathcal{C}}\right) = 1 - \sum_{k=1}^{\infty} 2^{k-1}\ell_k$.

   (b) Show that if $x \in \hat{\mathcal{C}}$, then there exists a sequence of points $\{x_n\}_{n=1}^{\infty}$ such that $x \notin \hat{\mathcal{C}}$, yet $x_n \to x$ and $x_n \in I_n$, where I_n is a sub-interval in the complement of $\hat{\mathcal{C}}$ with $|I_n| \to 0$.

   (c) Prove as a consequence that $\hat{\mathcal{C}}$ is perfect, and contains no open interval.

   Proof Exercise 4(b) is poorly posed, so I don't want to rely on it. This proof will be independent. We will show that for every $x \in \hat{\mathcal{C}}$ and any $\epsilon > 0$, the open ball $B_\epsilon(x)$ contains a point in $\hat{\mathcal{C}}$ and a point not in $\hat{\mathcal{C}}$. This means that every element of $\hat{\mathcal{C}}$ is a limit point (thus $\hat{\mathcal{C}}$ is perfect), and no subset of $\hat{\mathcal{C}}$ is open (thus it contains no open intervals).

   Let $x \in \hat{\mathcal{C}}$ and $\epsilon > 0$ be given. In step k of the construction, there are 2^{k-1} disjoint closed intervals in $\hat{\mathcal{C}}_k$, so their lengths can be no more than $\frac{1}{2^{k-1}}$. Thus there exists some K such that $\frac{1}{2^{K-1}} < \epsilon$. Since $x \in \hat{\mathcal{C}}$, then x is in some interval in $\hat{\mathcal{C}}_k$ for all k. Thus $B_\epsilon(x)$ contains some closed interval I in $\hat{\mathcal{C}}_K$, and in step $K+1$, an open interval $\hat{I}$ in I is removed from $\hat{\mathcal{C}}_{K+1}$. The endpoints of $\hat{I}$ are elements of $\hat{\mathcal{C}}$, and the midpoint of $\hat{I}$ is not. Since $\hat{I} \subset I \subseteq B_\epsilon(x)$, then the ball contains a point in $\hat{\mathcal{C}}$ and a point not in $\hat{\mathcal{C}}$, as desired. $\blacksquare$

   (d) Show also that $\hat{\mathcal{C}}$ is uncountable.

   Proof In each step $\hat{\mathcal{C}}_k$ of the construction of $\hat{\mathcal{C}}$, the set is composed of 2^{k-1} closed intervals. This is also true of the usual Cantor set, $\mathcal{C}$. Since any two closed intervals are isomorphic[2], then each $\hat{\mathcal{C}}_k$ is isomorphic to $\mathcal{C}_k$, and so are the limiting sets $\hat{\mathcal{C}} \cong \mathcal{C}$. We know that $\mathcal{C}$ is uncountable, so we are done. $\blacksquare$

12. Theorem 1.3 states that every open set in $\mathbb{R}$ is the disjoint union of open intervals. The analogue in $\mathbb{R}^d, d >= 2$, is generally false. Prove the following:

   (a) An open disc in $\mathbb{R}^2$ is not the disjoint union of open rectangles.

   Proof Let $C \subset \mathbb{R}^2$ be an open disc. Consider a rectangle $R \subset C$.[3] For any $x \in \partial R$, we know $x \in C$ but x cannot be an element of any open rectangle which is disjoint with R. Thus no disjoint union of open rectangles can contain x, nor can it cover C. $\blacksquare$

[1] Note, this interval was already removed in the last step, but this method of describing the removed portions is helpful, as will be seen.

[2] If this is not obvious, consider $[\alpha, \beta]$ and $[a, b]$. Map $\alpha \mapsto a$ and $\beta \mapsto b$, and extend linearly.

[3] A rectangle is not a disk, so $R \subsetneq C$.

13. The following deals with G_σ and F_δ sets.

   (b)

   (a) Give an example of an F_σ which is not a G_δ: **Answer:** $\mathbb{Q}$ is F_σ but not G_δ.

   Proof ∎

14.

14. The purpose of this exercise is to show that covering by a *finite* number of intervals will not suffice in the definition of the outer measure m_*.

The **outer Jordan content** $J_*(E)$ of a set E in $\mathbb{R}$ is defined by

$$J_*(E) = \inf \sum_{j=1}^{N} |I_j|,$$

where the inf is taken over every *finite* covering $E \subset \bigcup_{j=1}^{N} I_j$, by intervals I_j.

(a) Prove that $J_*(E) = J_*(\overline{E})$ for every set E (here $\overline{E}$ denotes the closure of E).

(b) Exhibit a countable subset $E \subset [0,1]$ such that $J_*(E) = 1$ while $m_*(E) = 0$.

(a) **Proof** ($\leq$) To see that $J_*(E) \leq J_*(\overline{E})$ for all $E \in \mathbb{R}$, observe that any cover $\phi = \{I_n\}_{n=1}^{\infty}$ which covers $\overline{E}$ also covers E. Thus $J_*(E)$ cannot be greater than $J_*(\overline{E})$. ∎

Proof ($\geq$) To show that $J_*(E) \geq J_*(\overline{E})$ for all $E \in \mathbb{R}$, we will show that for any cover $\phi = \{I_n\}_{n=1}^{\infty}$ which covers E, $\overline{\phi} = \{\overline{I_n}\}_{n=1}^{\infty}$ also covers $\overline{E}$, and $|\phi| = |\overline{\phi}|$.

Let ϕ be some cover of E as described above. To see that $\overline{\phi}$ covers $\overline{E}$, let $x \in \overline{E}$. If x is also in E, then $x \in I_n \subset \overline{I_n}$ for some n. Otherwise if $x \notin E$, then x must be a limit point of E, since $x \in \overline{E}$. Since $\overline{\phi}$ is a finite union of closed intervals, it is itself closed and thus contains all its limit points; and since $E \subset \overline{\phi}$, $\overline{\phi}$ contains all the limit points of E as well. Thus $x \in \overline{\phi}$. ∎

16. **The Borel-Cantelli lemma.** Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subset of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^d : \forall n \in \mathbb{N}, \exists k \geq n \text{ s.t. } x \in E_k\} \\ &= \limsup_{k\to\infty}(E_k). \end{aligned}$$

(a) Show that E is measurable.

(b) Prove $m(E) = 0$.

[Hint: Write $E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$.]

Proof We will show that E is a subset of a set of measure zero, and thus is measurable with $m(E) = 0$.

Since $\sum_{k=1}^{\infty} m(E_k)$ is a convergent sum, then for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\sum\limits_{k\geq N} m(E_k) < \epsilon$.

Then for any $x \in E$, $x \in \bigcup\limits_{k \geq N} E_k$, since the union contains some E_k with $k > N$ for any x. Now since $E \subseteq \bigcup\limits_{k \geq N} E_k$, then by sub-additivity,

$$m(E) \leq \sum_{k \geq N} m(E_k) < \epsilon$$

and we are done. ■

17. Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ with $|f_n(x)| < \infty$ for a.e. x. Show that there exists a sequence $\{c_n\}$ of positive real numbers such that

$$\frac{f_n(x)}{c_n} \to 0 \quad \text{for a.e. } x.$$

[Hint: Pick c_n such that $m\left(\left\{\left|\frac{f_n(x)}{c_n}\right| > \frac{1}{n}\right\}\right) < \frac{1}{2^n}$, and apply the Borel-Cantelli Lemma.]

Proof Consider $D_n = \{|f_n| > \frac{c_n}{n}\}$, where c_n is an as-yet undetermined positive number. We know that f_n is measurable, so $m(D_n)$ exists for all values of c_n. If we consider $m(D_n)$ as a function of c_n with n fixed, then $m(D_n)(c_n)$ is monotonically decreasing. This is because $\{|f_n| > a\} \supseteq \{|f_n| > b\}$ for all $a < b$. Now we choose c_n large enough that $m(D_n) < \frac{1}{2^n}$.[4]

Thus we have constructed a sequence of sets $\{D_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} m(D_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty,$$

so by the Borel-Cantelli Lemma, $m\left(\limsup\limits_{n\to\infty}(D_n)\right) = 0$.

Now to see that we are done, observe that for $x \notin D_n$,

$$|f_n(x)| \leq \frac{c_n}{n}, \text{ so } \left|\frac{f_n(x)}{c_n}\right| \leq \frac{1}{n}.$$

This means that for x in finitely many D_n, that is, $x \notin \limsup\limits_{n\to\infty}(D_n)$,

$$\lim_{n\to\infty} \frac{f_n(x)}{c_n} = 0.$$

We have shown that $m\left(\limsup\limits_{n\to\infty}(D_n)\right) = 0$, so we conclude that

$$\frac{f_n(x)}{c_n} \to 0 \quad \text{for a.e. } x,$$

as desired. ■

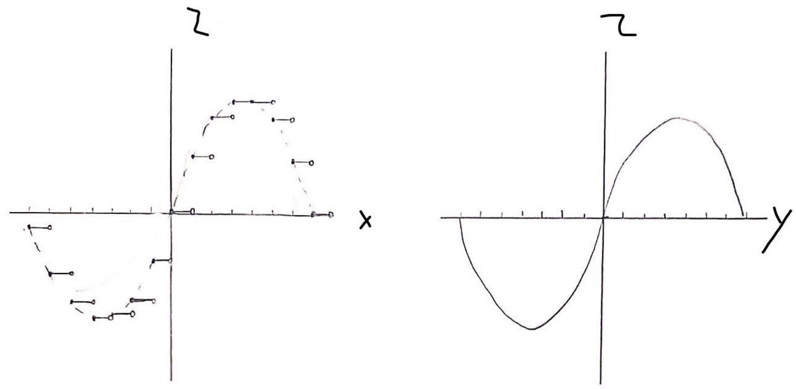
(Reading Question). Construct a function that is not measurable but finite valued everywhere.
Answer: $\chi_{\mathcal{N}}$, the characteristic function of the nonmeasurable set constructed in the text. It only takes the values $\{0,1\}$, but $\{\chi_{\mathcal{N}} > \frac{1}{2}\} = \mathcal{N}$ is not measurable.

[4] We know we can do this because if not, then $m(\{|f_n(x)| = \infty\}) > 0$.

1.3

23. Suppose $f(x, y)$ is a function on $\mathbb{R}^2$ that is separately continuous: for each fixed variable, f is continuous in the other variable. Prove that f is measurable on $\mathbb{R}^2$. [Hint: Approximate f by functions f_n which are piecewise-constant in the variable x, such that $f_n \to f$ pointwise.]

Proof First we approximate $f(x, y)$ by rounding x down to the nearest multiple of $\frac{1}{2^k}$. That is, given $n \in \mathbb{N}$ and $x \in \mathbb{R}$, there exists some $r \in \mathbb{N}$ such that $\frac{r}{2^n} \leq x < \frac{r+1}{2^n}$. So we approximate f by $f_n(x, y) = f(\frac{r}{2^n}, y)$. To visualize, following is a pair of illustrations of a possible such f_n; having some fixed y on the left, and some fixed x on the right.



Now we ask; given some $\alpha \in \mathbb{R}$, is $\{f_n > \alpha\}$ measurable? We know that f is continuous in y for fixed x, so for some $r \in \mathbb{Z}$ we have that

$$\{y \in \mathbb{R} | f(\tfrac{r}{2^n}, y) > \alpha\}$$

is measurable. Also, for any $x \in \mathbb{R}$, we know there is some r such that $x \in \left[\frac{r}{2^n}, \frac{r+1}{2^n}\right)$, so

$$\{y \in \mathbb{R} | f_n(x, y) > \alpha\} = \{y \in \mathbb{R} | f(\tfrac{r}{2^n}, y) > \alpha\}.$$

This means that for $x \in \left[\frac{r}{2^n}, \frac{r+1}{2^n}\right)$,

$$\{(x, y) | f_n(x, y) > \alpha\} = \left(\left[\tfrac{r}{2^n}, \tfrac{r+1}{2^n}\right) \times \{y \in \mathbb{R} | f(\tfrac{r}{2^n}, y) > \alpha\}\right),$$

which is certainly measurable. Now if we consider all possible values of x, we find that

$$\{f_n > \alpha\} = \bigcup_{r=-\infty}^{\infty} \left(\left[\tfrac{r}{2^n}, \tfrac{r+1}{2^n}\right) \times \{y \in \mathbb{R} | f(\tfrac{r}{2^n}, y) > \alpha\}\right),$$

which is a countable union of measurable sets, and thus a G_σ, therefore measurable. Thus we can conclude that f_n is measurable for all $n \in \mathbb{Z}$. Since $f_n \to f$, we can conclude that f is measurable as well, and we are done. ■

25. An alternative definition of measurability is as follows: E is measurable if for every $\epsilon > 0$ there is a *closed* set F contained in E with $m_*(E - F) < \epsilon$. Show that Properties 1-4 of the Lebesgue measure still hold.

   (a)

   (b) Property 2: If $m_*(E) = 0$, then E is measurable. In particular, if $F \subseteq E$ and $m_*(E) = 0$, then F is measurable.

   Proof Suppose $m_*(E) = 0$ and let $\epsilon > 0$ be given. Then let $F = \{x\}$ for some $x \in E$, thus F is closed and $F \subset E$. Then by monotonicity, $m_*(E - F) < m_*(E) = 0 < \epsilon$. ■

   (c) Property 3: If E is a countable collection of measurable sets E_j then $\bigcap_{j=1}^{\infty} E_j$ is also measurable.

   Proof Denote $E = \bigcap_{j=1}^{\infty} E_j$ with each E_j measurable. Then there exists an F_j such that $F_j \subset E_j$ with $m_*(E_j - F_j) < \frac{\epsilon}{2^j}$. As $F_j \subset E_j$ then $\bigcap_{j=1}^{\infty} F_j \subset \bigcap_{j=1}^{\infty} E_j$. We denote $\bigcap_{j=1}^{\infty} F_j = F$ and note F is the countable intersection of closed sets thus is closed. We also observe the following:

$$E - F = \left[\bigcap_{j=1}^{\infty} E_j\right] - \left[\bigcap_{j=1}^{\infty} F_j\right] = \left[\bigcap_{j=1}^{\infty} E_j\right] \bigcap \left[\bigcap_{j=1}^{\infty} F_j\right]^c =$$

$$\left[\bigcap_{j=1}^{\infty} E_j\right] \bigcap \left[\bigcup_{j=1}^{\infty} F_j^c\right] \subseteq \left[\bigcup_{j=1}^{\infty} E_j\right] \bigcap \left[\bigcup_{j=1}^{\infty} F_j^c\right] = \bigcup_{j=1}^{\infty} \left[E_j \bigcap F_j^c\right] = \bigcup_{j=1}^{\infty} \left[E_j - F_j\right]$$

   Thus by Observation 1 (Monotonicity) and $\left[E - F\right] \subseteq \bigcup_{j=1}^{\infty} \left[E_j - F_j\right]$ then:

$$m_*\left[E - F\right] \leq m_* \bigcup_{j=1}^{\infty} \left[E_j - F_j\right] \leq \sum_{j=1}^{\infty} m_*\left[E_j - F_j\right] < \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon$$

   Thus showing E is measurable. ■

   (d) Property 4: Open sets are measurable - we were unable to prove the general case, but did prove the following special cases:

      i Open sets with finite exterior measure are measurable.

      Proof Let O be an open subset of $\mathbb{R}^d$ with $m_*(O) < \infty$ and let $\epsilon > 0$ be arbitrary. Then O is the countable union of almost disjoint cubes: $O = \cup_{j=1}^{\infty} Q_j$ and $m_*(O) = \sum_{j=1}^{\infty} |Q_j|$. Since $m_*(O) < \infty$, this is a convergent series and so the tails must go to 0. That is, there exists some $N \in \mathbb{N}$ such that $\sum_{j=N}^{\infty} |Q_j| < \epsilon$. Then, letting $F = \cup_{j=1}^{N-1} Q_j$, we have that

$$m_*(O - F) = m_*(\cup_{j=1}^{\infty} Q_j - \cup_{j=1}^{N-1} Q_j) = m_*(\cup_{j=N}^{\infty} Q_j) = \sum_{j=N}^{\infty} |Q_j| < \epsilon.$$

      Since F is the intersection of closed sets, it is closed and O is measurable.

      ■

      ii Open sets with complements of finite exterior measure are measurable.

      Proof Let O be an open subset of $\mathbb{R}^d$ with $m_*(O^c) < \infty$ and let $\epsilon > 0$ be arbitrary. Define B_R to be the open ball of radius R about the origin; clearly B_R has finite measure. Observe that we can rewrite O as

$$O = O \cap B_R \cup O \cap B_R^c.$$

      The goal is to show that this is in fact the union of two open, measurable sets and hence measurable by Property 3, but it would have to be Property 3 of the open sets definition of measurable, so I don't see why that would be available to us at all. **Can anyone confirm that this is what we were aiming for?** Proceeding nonetheless.
      The first term $O \cap B_R$ is open and has finite exterior measure: it is open because it is the intersection of two open sets and has finite exterior measure as a result of countable subadditivity. Then, the term is measurable by Property 4.i. Dr. Horn seemed to be concerned

that this was not actually open and began instead working with $O \cap Int(B_{n+1} \cap B_n$; I don't understand that construction.
Remaining questions: how does that construction help us? Why would the second term be open and of finite measure? In fact, since we're given that O^c is finite, then $O \cap B_R^c = B_R^c - O^c$ would seem to be a set of infinite exterior measure less something of finite exterior measure, and thus still having infinite exterior measure.

■

27. Suppose E_1 and E_2 are a pair of compact sets in $\mathbb{R}^d$ with $E_1 \subseteq E_2$, and let $a = m(E_1)$ and $b = m(E_2)$. Prove that for any c with $a < c < b$, there is a compact set E with $E_1 \subset E \subset E_2$ and $m(E) = c$.

 Proof For this proof, we will construct a continuous function to make a correspondence between numbers in $[a, b]$ and subsets of E_2, and use the IVT to produce the desired set.

 Let $E_1 \subset E_2 \subset \mathbb{R}^d$ with $a = m(E_1)$ and $b = m(E_2)$, and let $a < c < b$. Let $\{r_i\}_{i=0}^{\infty}$ be an enumeration of the rational numbers $\mathbb{Q} \cap [1, 2]$. Since $\mathbb{R}^d$ is a normal topological space, we can find a compact set E_{r_0} such that

$$E_1 \subsetneq E_{r_0} \subsetneq E_2,$$

 and continue this construction for all other r_i. That is, for each $i \in \mathbb{N}$ and $p, q \in \mathbb{Q} \cap [1, 2]$, we can find E_{r_i} such that

$$\text{if } p < r_i < q, \text{ then } E_p \subsetneq E_{r_i} \subsetneq E_q.$$

 Now we define a continuous function $f : [1, 2] \to [a, b]$ by

$$f(q) = m(E_q) \text{ for } q \in \mathbb{Q},$$

 and extending continuously[5]. Since f is continuous, there exists some $\gamma \in [1, 2]$ such that $f(\gamma) = c$ by the Intermediate Value Theorem. We would like to take E_γ as our desired set, but γ is probably an irrational number, so E_γ may not be defined. However, using the decimal expansion $\gamma = 1+\sum_{i=1}^{\infty} g_i(10)^{-i}$, we can obtain a countable sequence of sets $\{E_{\gamma_n}\}_{n=1}^{\infty}$ where

$$\gamma_n = \left(1 + \sum_{i=1}^{n} g_i(10)^{-i}\right) + 10^{-i}.$$

 Note that $\gamma < \gamma_n$ for all n, and $\gamma_n \to \gamma$. Finally we can define E_γ as $\bigcap_{i=1}^{\infty} E_{\gamma_i}$. Since $E_{\gamma_i} \searrow E_\gamma$ and $m(E_{\gamma_i}) < 2 < \infty$, then

$$m(E) = \lim_{i\to\infty} m(E_{\gamma_i}) = c.$$

 We know that E_γ is compact since it is the countable intersection of compact sets, and thus we are done. ■

38. Prove that $(a + b)^\gamma \geq a^\gamma + b^\gamma$ whenever $\gamma \geq 1$ and $a, b \geq 0$. Also, show that the reverse inequality holds when $0 \leq \gamma \leq 1$.
 [Hint: Integrate the inequality between $(a + t)^{\gamma-1}$ and $t^{\gamma-1}$ from 0 to b.]

 Proof Recall that (assuming $x > 0$) $f(x) = x^\alpha$ is an increasing function when $\alpha > 0$, and decreasing when $\alpha < 0$. Thus when $\gamma \geq 1$, then $(a + t)^{\gamma-1} - t^{\gamma-1} \geq 0$ for all $t, a \geq 0$; and when $\gamma \leq 1$, the inequality is reversed. Thus if $\gamma \geq 1$,

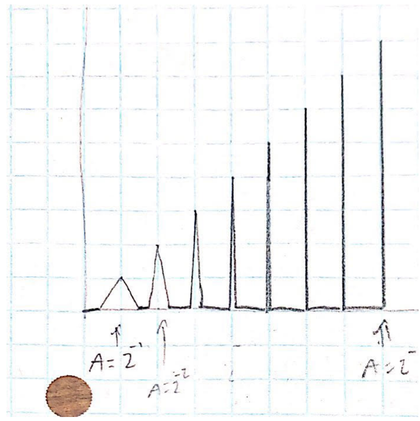
$$\begin{aligned} 0 &\leq \int_0^b (a+t)^{\gamma-1} - t^{\gamma-1}\, dt \\ &= \left[\tfrac{1}{\gamma}(a+t)^\gamma - \tfrac{1}{\gamma}t^\gamma\right]_{t=0}^{b} \\ &= \tfrac{1}{\gamma}\left[(a+b)^\gamma - b^\gamma\right] - \tfrac{1}{\gamma}a^\gamma \\ &= \tfrac{1}{\gamma}\left[(a+b)^\gamma - (a^\gamma + b^\gamma)\right]. \end{aligned}$$

 Thus, $0 \leq \frac{1}{\gamma}\left[(a + b)^\gamma - (a^\gamma + b^\gamma)\right]$, and rearranging yields the desired result. ■

[5] This function is unique since a continuous function is completely determined by its behavior on a dense subset of its domain.

(reading question) Find a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\limsup f = \infty$, but $\int\limits_{\mathbb{R}^+} f < \infty$.

Answer: Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be the function determined by countably many triangles $\{T_n\}$; each centered on an integer point $(n, 0)$, with area 2^{-n}, and height n.



For a more formal definition, consider the set of all vertices of T_n for all n, together with the origin. Define f by connecting all these points with line segments, in order of their x-coordinates. We can confirm that this function is defined as expected by observing that none of the triangles overlap: Since they are all centered on integers, their centers have distance 1, and T_1 has radius $1/2$, [6] and all future triangles have smaller width.

To see that this function has the desired properties, observe that $\limsup f = \infty$ because for all $n \in \mathbb{N}, f(n) = n$. Now observe that

$$\begin{aligned} \int_0^\infty f &= \sum_{n=1}^\infty \text{area}(T_n) \\ &= \sum_{n=1}^\infty \frac{1}{2^n} \\ &= 1 \\ &< \infty. \end{aligned}$$

■

1 Chapter 2

Reading question for 2.1 stage 1: Let f and g be simple functions with $f(x) = g(x)$ for a.e. x. Prove that $\int_{\mathbb{R}} f(x)\, dx = \int_{\mathbb{R}} g(x)\, dx$.

Proof Let

$$\sum_{k=1}^N a_k \chi_{E_k}, \; \sum_{k=1}^M b_k \chi_{F_k}$$

be the canonical forms of f and g respectively, and let $\Delta = \{x : f(x) \neq g(x)\}$. Since $f = g$ almost everywhere, then $m(\Delta) = 0$. Thus

$$\begin{aligned} f(x) &= \sum_{k=1}^N a_k \chi_{\widetilde{E_k}} + f(x)\chi_\Delta, \text{ and} \\ g(x) &= \sum_{k=1}^N b_k \chi_{\widetilde{F_k}} + g(x)\chi_\Delta, \end{aligned}$$

[6] That is, it has vertices $(n - 1/2, 0), (n, n), (n + 1/2, 0)$

where $\widetilde{E_k} = E_k - \Delta$ and $\widetilde{F_k} = F_k - \Delta$. Note that after a possible reordering, $\widetilde{E_k} = \widetilde{F_k}$ and $a_k = b_k$, since $f|_{\mathbb{R}-\Delta} = g|_{\mathbb{R}-\Delta}$ and these are the canonical forms of f and g. Thus

$$\begin{aligned}\int_{\mathbb{R}} f(x)\,dx &= \int_{\mathbb{R}} \left(\sum_{k=1}^{N} a_k \chi_{\widetilde{E_k}} + f(x)\chi_{\Delta} \right) dx \\ &= \sum_{k=1}^{N} a_k m\left(\widetilde{E_k}\right) + \int_{\Delta} f(x)\,dx\end{aligned}$$

and $\int_{\mathbb{R}} f = 0 = \int_{\mathbb{R}} g$, so

$$\begin{aligned} &= \sum_{k=1}^{N} b_k m\left(\widetilde{F_k}\right) + \int_{\Delta} g(x)\,dx \\ &= \int_{\mathbb{R}} g(x)\,dx\end{aligned}$$

and we are done. ■

10. We can break exercise 10 into two parts:

Part 1: Suppose $f > 0$, and let $E_{2^k} = \{f > 2^k\}$ and $F_k = \{2^k < f \leq 2^{k+1}\}$. If f is finite almost everywhere, note that

$$\bigcup_{k=-\infty}^{\infty} F_k = \{f > 0\}$$

and the sets F_k are disjoint. Prove that the following are equivalent:

- f is integrable,
- $\displaystyle\sum_{k=-\infty}^{\infty} 2^k m\left(F_k\right) < \infty$, and
- $\displaystyle\sum_{k=-\infty}^{\infty} 2^k m\left(E_{2^k}\right) < \infty$.

Proof YELLOW ■

Part 2: Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be:

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

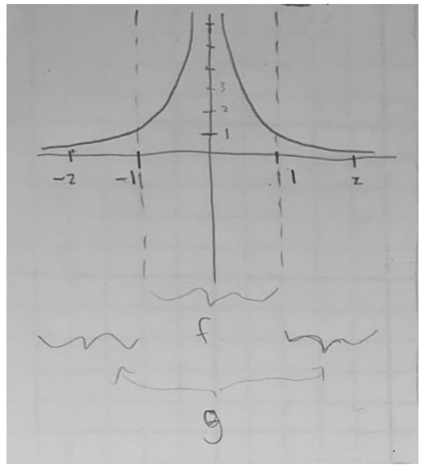
$$g(x) = \begin{cases} |x|^{-b} & \text{if } |x| > 1, \\ 0 & \text{otherwise} \end{cases}$$

for some $a, b \in \mathbb{R}^+$. Prove that

(i) f is integrable on $\mathbb{R}^d$ iff $a < d$, and

(ii) g is integrable on $\mathbb{R}^d$ iff $b > d$.

Proof Part (i): For the function f, consider E_{2^k} for $k \in \mathbb{Z}$.



Observe that for $k > 0$,

$$E_{2^k} = \{f > 2^k\} = \{|x|^{-a} > 2^k\} = \{|x| < 2^{-\frac{k}{a}}\} = B_{2^{-\frac{k}{a}}},$$

where $B_{2^{-\frac{k}{a}}}$ is a ball of radius $2^{-\frac{k}{a}}$ centered at the origin. The volume of such a ball in $\mathbb{R}^d$ is given by $V_d r^d$, where r is the radius and V_d is some constant which depends on the dimension d. Also observe that for $k \leq 0$, we have

$$E_{2^k} = B_1,$$

so $m\left(E_{2^k}\right) = V_d$. Now we will use these observations to calculate one of the sums from Part 1, as it relates to f:

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m\left(E_{2^k}\right) &= \sum_{k=-\infty}^{\infty} 2^k m\left(B_{2^{-\frac{k}{a}}}\right) \\ &= \sum_{k=-\infty}^{\infty} (2^k)(V_d)(2^{-\frac{kd}{a}}) \\ &= \sum_{k=-\infty}^{\infty} (V_d)(2^{1-\frac{d}{a}})^k \end{aligned}$$

which converges iff $a < d$. □

Proof Part (ii): Now we consider E_{2^k} as it applies to g. For $k \geq 0$, observe that E_{2^k} is empty, so $m\left(E_{2^k}\right) = 0$. Now we know that $g(x) = 0$ for any $x \in B_1$, so for $k < 0$,

$$\begin{aligned} E_{2^k} &= B_{2^{-\frac{k}{b}}} \cap B_1^{\complement} \\ &\subset B_{2^{-\frac{k}{b}}}, \end{aligned}$$

and we can calculate as in Part (i):

$$\begin{aligned} \sum_{k=-\infty}^{\infty} 2^k m\left(E_{2^k}\right) &= \sum_{k=-\infty}^{-1} 2^k m\left(E_{2^k}\right) \\ &< \sum_{k=-\infty}^{-1} 2^k m\left(B_{2^{-\frac{k}{b}}}\right) \\ &= \sum_{k=-\infty}^{-1} (V_d)(2^{1-\frac{d}{b}})^k \end{aligned}$$

which converges iff $b > d$. [7] ■

[7] If this is not obvious, note that k is negative, so the sum converges iff $2^{1-\frac{d}{b}} > 1$ iff $1 - \frac{d}{b} > 0$ iff $1 > \frac{d}{b}$ iff $b > d$.

11. Prove that if $f : \mathbb{R}^d \to \mathbb{R}$ is integrable on $\mathbb{R}^d$ with $\int_E f \geq 0$ for every measurable E, then $f(x) \geq 0$ for a.e. x. Also prove that if $\int_E f = 0$ for every measurable E, then $f(x) = 0$ for a.e. x.

Proof Let $E_k = \{f < -\frac{1}{k}\}$. Then $\bigcup_{k=1}^{\infty} E_k = \{f < 0\}$. For any $k > 0$,

$$\begin{array}{rcl} 0 & \leq & \int_{E_k} f(x)\,dx \\ & < & \int_{E_k} -\frac{1}{k}\,dx \\ & = & -\frac{1}{k} m\left(E_k\right). \end{array}$$

Thus $0 \leq -\frac{1}{k} m\left(E_k\right)$, and measures are nonnegative, so $m\left(E_k\right)$ must be 0. Since $m\left(E_k\right) = 0$ for all k, then by subadditivity

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m\left(E_k\right) = 0,$$

so $m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(f < 0\right) = 0$, and $f \geq 0$ for a.e. x. ■

Corollary If $\int_E f = 0$ for every measurable E, then we have that $\int_E f \geq 0$ and $\int_E -f \geq 0$, so we conclude that $f(x) \geq 0$ and $-f(x) \geq 0$ for a.e. x, thus $f(x) = 0$ almost everywhere. ■

12. Show that there are $f \in L^1(\mathbb{R}^d)$ and a sequence $\{f_n\}$ with $f_n \in L^1(\mathbb{R}^d)$ such that

$$||f - f_n||_{L^1} \to 0,$$

but $f_n(x) \to f(x)$ for no x.

[Hint: In $\mathbb{R}$, let $f_n = \chi_{I_n}$, where I_n is an appropriately chosen sequence of intervals with $m\left(I_n\right) \to 0$.]

Proof Phase I: We will first find such a sequence of functions that has the desired properties only on $[0, 1]$. By Dirichlet's Approximation Theorem, we know that for any $x \in [0, 1]$, there exist infinitely many p, q such that

$$\left|\frac{p}{q} - x\right| < \frac{1}{q^2}.$$

So we construct $\{I_n\}$ using an enumeration of the rationals in $[0, 1]$ where $r_n = \frac{p}{q}$, so that

$$I_n = \left(\frac{p}{q} - \frac{1}{q^2}\,,\, \frac{p}{q} + \frac{1}{q^2}\right).$$

Now if we consider the characterstic functions of these sets, we find that $\{\chi_{I_n}\}$ is a sequence of functions such that

$$||0 - \chi_{I_n}|| \to 0,$$

since $\int \chi_{I_n} = m\left(I_n\right) = \frac{2}{q^2} \to 0$ as $n \to \infty$. To see that this is true, recall that for any $\epsilon > 0$, there are finitely many $q \in \mathbb{N}$ such that $\frac{2}{q^2} \geq \epsilon$, and for each q there are finitely many $\frac{p}{q} \in [0, 1]$.

However, $\lim_{n\to\infty} \chi_{I_n}(x) \neq 0$, since each $x \in I_n$ for infinitely many I_n.

Phase II: Now we extend the sequence of functions to work on $\mathbb{R}^+$. For any $x \in \mathbb{R}^+$, we can write x as $m + \alpha$, where m is given by truncating x, that is, $m = \lfloor x \rfloor$ and $\alpha = x - m$. Then we define

$$f_n(x) = \frac{1}{2^m} \chi_{I_n}(\alpha).$$

To see that $\int f_n \to 0$, observe that

$$\int_{\mathbb{R}} f_n(x)\,dx = \int_{\mathbb{R}} \frac{1}{2^m} \chi_{I_n}(\alpha)\,dx \leq \int_{\mathbb{R}} \frac{1}{2^m} \chi_{[0,1]}(\alpha)\,dx = \sum_{m=0}^{\infty} \frac{1}{2^m} = 2 < \infty,$$

so then since $\int \chi_{I_n} \to 0$, then so does $\int f_n \to 0$ as $n \to \infty$. Also, $f_n(x) \not\to 0$ for the same reasons as in phase 1, since we are using the same map everywhere, and the only difference is in the magnitude.

Phase III: Finally, we extend the sequence of functions to all of $\mathbb{R}$ merely by considering $f_n(|x|)$, and we are done. ■

2.7

Proof of (a) and (b) (I could only think of a proof that accomplishes both parts simultaneously.) Let

$$E_n = \bigcup_{k\in\mathbb{Z}} \left(f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) \times \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right).$$

We would like to say that $E_n \to \Gamma$ and $m(E_n) \to 0$ so we are done, but we need to be a bit more careful than that. First, we bound E_n and Γ, so that

$$E_n^b = E_n \cap \left([-\tfrac{b}{2}, \tfrac{b}{2}]^d \times \mathbb{R}\right), \text{ and}$$

$$\Gamma^b = \Gamma \cap \left([-\tfrac{b}{2}, \tfrac{b}{2}]^d \times \mathbb{R}\right).$$

Then we have that $m\left(E_n^b\right) = \frac{b^d}{2^n}$, so as $n \to \infty$ we have $E_n^b \searrow \Gamma^b$ and $m\left(\Gamma^b\right) = 0$. [8] Now this holds for any $b \in \mathbb{N}$, so then $\Gamma^b \nearrow \Gamma$ and $m\left(\Gamma^b\right) = 0$ for all b. Therefore

$$\lim_{b\to\infty} m\left(\Gamma^b\right) = m(\Gamma) = 0$$

and we are done. ■

3.14 (b) Let $F : [a, b] \to \mathbb{R}$ be increasing and bounded, and let $J(x) = \sum_{n=1}^{\infty} \alpha_n j_n(x)$ be the jump function associated with F. Show that

$$\limsup_{h\to 0} \frac{J(x+h) - J(x)}{h}$$

is measurable.

Proof Let $J_N = \sum_{n=1}^{N} \alpha_n j_n(x)$. Now define

$$F_{k,m}^N(x) = \sup_{\frac{1}{k}<|h|<\frac{1}{m}} \left| \frac{J_N(x+h) - J_N(x)}{h} \right|.$$

As $k \to \infty$,

$$F_{k,m}^N \nearrow F_m^N = \sup_{0<|h|<\frac{1}{m}} \left| \frac{J_N(x+h) - J_N(x)}{h} \right|, \text{and}$$

■

Proof Since F is bounded and increasing, we know it can have at most countably many discontinuities; so let $\{x_n\}_{n=1}^{\infty}$ denote the set of discontinuities of F.
Claim: If $x \in \{x_n\}_{n=1}^{\infty}$, then the desired lim sup is ∞, and it vanishes for all other x.
To see this, let $x \in \{x_n\}_{n=1}^{\infty}$. This means that since x is a point of jump discontinuity, then $|J(x+h) - J(x)| > 0$ for any $h > 0$ (or if not, then it works for any $h < 0$), so

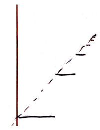
$$\limsup_{h\to 0} \frac{J(x+h) - J(x)}{h} = \infty.$$

Now suppose that $x \notin \{x_n\}_{n=1}^{\infty}$. Either x is a limit point of $\{x_n\}_{n=1}^{\infty}$, or it is not. If it is not, then $J(x)$ is constant near x, so $J'(x)$ exists and is 0, which means the desired lim sup vanishes as well.
Otherwise suppose $x \notin \{x_n\}_{n=1}^{\infty}$ but x is a limit point of $\{x_n\}_{n=1}^{\infty}$. Since F is bounded and $J \leq F$, then $\sum_{n=1}^{\infty} \alpha_n j_n(x) < \infty$ for any x, which means $\alpha_n \to 0$. Thus for any subsequence $x_{n_i} \to x$, $J(x_{n_i}) - J(x_{n_{i+1}}) \to 0$, so the lim sup vanishes in this case as well.

THIS TOTALLY DOESN'T WORK!

This is a freshman calculus mistake, I should have known better. Just because the numerator goes to 0 doesn't mean the quotient goes to zero, since the denominator goes to 0 for sure, and thus we get $0/0$ which is an indeterminate form. In fact, you can construct a counterexample.

[8] The bound was needed here because $m\left(\Gamma^b\right) = \lim_{n\to\infty} m\left(E_n^b\right)$ only when some E_n^b has finite measure.



Just let $\{x_n\}_{n=1}^{\infty} = 1/2^n$ and $\alpha_n = x_n$. Then as $x \to 1$, $J(x) \to x$ and the desired lim sup at $x = 1$ is 1. ■