Math 450b
Homework 11

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1. Show that the volume of a parallelepiped spanned by the vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$ is given by $|\det M|^{\frac{1}{2}}$, where $M = [\langle v_i, v_j \rangle]$.

   Proof Let P denote the parallelepiped in question. Observe that

$$M = [\langle v_i, v_j \rangle] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} [v_1 \ v_2 \ v_3 \ \cdots \ v_n] = T^T T,$$

   Where T is the matrix representing the linear transformation which maps the unit cube to P. So,

$$\text{vol}(P)^2 = |\det T|^2 = |\det T^T||\det T| = |\det M|,$$

   and taking square roots, we find that $\text{vol}(P) = |\det M|^{\frac{1}{2}}$, as desired. ■

2. Use a change of variables to calculate $\int_A f$, where

$$f(x, y, z) = (x^2 + y^2)z^2,$$

$$A = \{(x, y, z) : x^2 + y^2 < 1, \ |z| < 1\}.$$

   Answer: Let $g : B \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a function, and B be a set such that

$$g(r, \theta, z) = (r\cos\theta, r\sin\theta, z),$$

$$B = \{(r, \theta, z) : r < 1, \ 0 < \theta < 2\pi, \ |z| < 1\}.$$

   Observe that $g(B) = A$ with g being one-to-one and C^1 with $\det Dg \neq 0$ for all (r, θ, z) in B (we claim these facts without proof since this is a common change of variables). Then by the Change of Variables Thm,

$$\int_A f = \int_B f \circ g \, |\det Dg| = \int_{-1}^{1} \int_0^{2\pi} \int_0^1 r^2 z^2 \, |r| \, dr \, d\theta \, dz = \int_{-1}^{1} \int_0^{2\pi} \tfrac{z^2}{4} \, d\theta \, dz = 2\pi(2)\frac{1}{12} = \frac{\pi}{3}.$$

■

3. Use a change of variables to calculate $\int_A f$, where

$$f(x,y) = xy\sin(x^2-y^2),$$
$$A = \{(x,y) : 0 < y < 1, y < x, x^2-y^2 < 1\}.$$

Answer: Let $g : B \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a function, and B be a set such that

$$g(u,v) = (\sqrt{u+v^2}, v),$$
$$B = \{(u,v) : 0 < u < 1, 0 < v < 1\}.$$

Observe that $x^2-y^2 = u$, so $u < 1$, and $v = y < x = \sqrt{u+v^2}$, so $u > 0$. Thus $g(B) = A$ with g being one-to-one and C^1 for all (r,θ,z) in B. Now we compute $|\det Dg|$.

$$|\det Dg| = \left|\det\begin{bmatrix} \frac{1}{2\sqrt{u+v^2}} & \frac{v}{\sqrt{u+v^2}} \\ 0 & 1 \end{bmatrix}\right| = \left|\frac{1}{2\sqrt{u+v^2}}\right| = \frac{1}{2\sqrt{u+v^2}}$$

Then by the Change of Variables Thm,

$$\int_A f = \int_B f \circ g\,|\det Dg| = \int_0^1 \int_0^1 \frac{v\sqrt{u+v^2}\sin u}{2\sqrt{u+v^2}}du\,dv = \int_0^1 \int_0^1 \frac{v}{2}\sin u\,du\,dv = \frac{1-\cos(1)}{4}.$$

■

4. Give a counterexample to show that the change of variable formula does not hold if g is not one-to-one, even if $\det Dg \neq 0$ for all $x \in \Omega$. (Hint: Take $f = 1$ and $g(x,y) = (e^x\cos y, e^x\sin y)$ for a suitable region Ω.)

Answer: Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\begin{aligned} f &\equiv 1 \\ g(x,y) &= (e^x\cos y, e^x\sin y) \end{aligned}$$

and consider the regions

$$\begin{aligned} A &= B(\vec{0},1) - B(\vec{0},\tfrac{1}{e}) \\ \Omega &= \{(x,y) : -1 < x < 0,\ 0 < y < 4\pi\}. \end{aligned}$$

Observe that $g(\Omega) = A$, although g is not one-to-one. Now we compute $|\det Dg|$.

$$|\det Dg| = \left|\det\begin{bmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{bmatrix}\right| = \left|e^{2x}\right| = e^{2x}$$

Note that $\det Dg \neq 0$ for all $x \in \Omega$. Now we compare the two halves of the change of variables formula:

$$\int_A f \stackrel{?}{=} \int_B f \circ g\,|\det Dg|\,.$$

$$\int_A f = \mathrm{vol}(A) = \pi(1)^2 - \pi(\tfrac{1}{e})^2 = \pi - \frac{\pi}{e^2}$$

$$\int_B f \circ g\,|\det Dg| = \int_{-1}^0 \int_0^{4\pi} (1)e^{2x}\,dy\,dx = 2\left(\pi - \frac{\pi}{e^2}\right)$$

Thus, the RHS≠LHS, so the formula does not hold. ■

5. (a) Calculate $\int_{B_r} e^{-x^2-y^2}\,dx\,dy$, where $B_r = \{(x,y) : x^2+y^2 \le r\}$.
Answer: Using a change of variables to polar coordinates, we find that

$$\int_{B_r} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\int_0^{\sqrt{r}} ue^{-u^2}\,du\,d\theta = -\frac{1}{2}\int_0^{2\pi}\int_0^{\sqrt{r}} -2ue^{-u^2}\,du\,d\theta = -\frac{1}{2}\int_0^{2\pi} e^{-r}-1\,d\theta$$
$$= \pi - \pi e^{-r}$$

(b) Show that $\int_{C_r} e^{-x^2-y^2}\,dx\,dy = (\int_{-r}^{r} e^{-x^2}\,dx)^2$, where $C_r = [-r,r]\times[-r,r]$.

Proof

$$\int_{C_r} e^{-x^2-y^2}\,dx\,dy = \int_{-r}^{r}\int_{-r}^{r} e^{-x^2}e^{-y^2}\,dx\,dy = \int_{-r}^{r} e^{-x^2}\,dx\int_{-r}^{r} e^{-y^2}\,dy = \left(\int_{-r}^{r} e^{-x^2}\,dx\right)^2$$

■

(c) Show that

$$\lim_{r\to\infty}\int_{B_r} e^{-x^2-y^2}\,dx\,dy = \lim_{r\to\infty}\int_{C_r} e^{-x^2-y^2}\,dx\,dy$$

Proof First, observe that the LHS converges to π:

$$\lim_{r\to\infty}\int_{B_r} e^{-x^2-y^2}\,dx\,dy = \lim_{r\to\infty}\left(\pi - \pi e^{-r}\right) = \pi.$$

Now, since $B_r \subset C_r \subset B_{r\sqrt{2}}$ for any $r>0$, and $e^{-x^2-y^2} > 0$ for all $(x,y)\in\mathbb{R}^2$, then

$$\int_{B_r} e^{-x^2-y^2}\,dx\,dy \le \int_{C_r} e^{-x^2-y^2}\,dx\,dy \le \int_{B_{r\sqrt{2}}} e^{-x^2-y^2}\,dx\,dy.$$

Thus, by the squeeze theorem, since

$$\pi = \lim_{r\to\infty}\int_{B_r} e^{-x^2-y^2}\,dx\,dy = \lim_{r\to\infty}\int_{B_{r\sqrt{2}}} e^{-x^2-y^2}\,dx\,dy,$$

Then the RHS also converges to π. ■

(d) Show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

Proof Using parts (a) thorugh (c):

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2} = \sqrt{\left(\lim_{r\to\infty}\int_{-r}^{r} e^{-x^2}\,dx\right)^2} =$$
$$\sqrt{\lim_{r\to\infty}\left(\int_{-r}^{r} e^{-x^2}\,dx\right)^2} = \sqrt{\lim_{r\to\infty}\int_{C_r} e^{-x^2-y^2}\,dx\,dy} = \sqrt{\pi}$$

■

6. Let E be the ellipsoid $\{(x,y,z)\in\mathbb{R}^3 : (x^2/a^2)+(y^2/b^2)+(z^2/c^2)\le 1\}$, where a, b, and c are positive constants. Compute the volume of E using a change of variables.

Answer: Perform a change of variables using $T(u,v,w) = (au, bv, cw)$. Thus, $T(B(\vec{0},1)) = E$, so

$$\mathrm{vol}(E) = \int_E 1 = \int_{B(\vec{0},1)} |\det T| = \mathrm{vol}(B(\vec{0},1))\,|\det T| = \tfrac{4}{3}\pi abc.$$

To see that $\det T = abc$, observe that a, b, and c are the eigenvalues of T, so the determinant is equal to their product. ■

7. Let $\langle e_1, \ldots, e_n \rangle$ denote the standard basis for $\mathbb{R}^n$, and let T denote the linear operator on $\mathbb{R}^n$ defined by $T(e_1) = (1,1,1,1,\ldots,1), T(e_2) = (1,2,1,1,\ldots,1), T(e_3) = (1,2,3,1,\ldots,1), \ldots, T(e_n) = (1,2,3,4,\ldots,n)$. Suppose that $f : \Omega \to \mathbb{R}$ is integrable, and $\int_\Omega f = 1$. Compute $\int_{T^{-1}(\Omega)} f \circ T$.

Answer: First, observe that $\int_\Omega f = \int_{T^{-1}(\Omega)} f \circ T \, |\det T| = 1$. Thus, $\int_{T^{-1}(\Omega)} f \circ T = \frac{1}{|\det T|}$. So, we need to compute $|\det T|$. Let A denote the matrix representation of T with respect to the standard basis. Thus:

$$|\det A| = \left| \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 1 & 3 & \cdots & 3 \\ & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n \end{bmatrix} \right|$$

To compute the determinant, we apply row operations to reduce A to triangular form, adding $-R1+Ri$ for every row except the first. This will affect the determinant by a sign if $n-1$ is odd, but we are taking absolute value, so it doesn't matter.

$$|\det A| = \left| \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 2 & \cdots & 2 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1) \end{bmatrix} \right| = (n-1)!$$

Therefore,

$$\int_{T^{-1}(\Omega)} f \circ T = \frac{1}{(n-1)!}$$

and we are done. ■