Practice Problems for Final

Note: The following two formulas will be provided on the final exam:

- Euler's formula:
$$e^{it} = \cos t + i \sin t$$
- The nonhomogeneous second-order linear differential equation
$$y'' + p(t)y' + q(t)y = g(t)$$
has general solution
$$y = u_1(t)y_1(t) + u_2(t)y_2(t),$$
where $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions to the corresponding homogeneous equation
$$y'' + p(t)y' + q(t)y = 0$$
and
$$u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1, y_2](t)} dt + c_1, \qquad u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1, y_2](t)} dt + c_2.$$

1. Suppose that $y = G(x)$ is a particular solution to a differential equation of the form $y' = f(x)$. Check that $y = G(x) + C$ is as well.
2. Suppose that $y = G(x)$ is a particular solution to a differential equation of the form $y' = f(y)$. Check that $y = G(x + C)$ is as well.
3. Suppose that $y = G(x)$ is a particular solution to a differential equation of the form $y' = f(x)y$. Check that $y = CG(x)$ is as well.
4. Consider the differential equation $x\dfrac{dy}{dx} = 2y$.
   (a) Show that $y = Cx^2$ is a one-parameter family of solutions to the equation.
   (b) Determine whether there are one, more than one, or no solutions for each inital value condition below
      i. $y(0) = 1$
      ii. $y(1) = 1$
      iii. $y(-1) = 1$
   (c) Indicate what the family of curves $y = Cx^2$ looks like by drawing the curves for various values of C.

(d) Explain how the answers you got in part (b) are reflected in the visual behavior of the solution curves drawn.

5. Compute $W[y_1, y_2](t)$, where $y_1(t) = e^{\lambda t}\cos(\mu t)$, $y_2(t) = e^{\lambda t}\sin(\mu t)$.

6. For each nonhomogeneous linear equation below, determine what the guess solution y_p would be for the method of undetermined coefficients.

   (a) $y'' + 3y' = t + 1$
   (b) $y'' + 2y' + 5y = e^t\cos(2t)$
   (c) $y'' + 2y' + 5y = e^{-t}\cos(2t)$
   (d) $y'' - 4y' + 4y = e^{2t} + e^{-3t}$
   (e) $y'' - 5y' + 6y = e^{2t}\sin t$
   (f) $y'' + 4y = t^2\sin(2t) + (6t + 7)\cos(2t)$

7. Find the general solution using the method of undetermined coefficients:

   (a) $y'' - 6y' + 8y = e^{2t}$
   (b) $y'' + 2y' + y = 2e^{-t}$

8. Find the solution of the initial value problem:

$$y'' + y' - 2y = 2t, \;\; y(0) = 0, \;\; y'(0) = 1.$$

9. Find the general solution using variation of parameters:

$$4y'' - 4y' + y = 16e^{t/2}$$

10. Consider the system

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \vec{x}.$$

   (a) Find the general solution.
   (b) Solve the initial value problem with $x_1(0) = 2, x_2(0) = 4$.

11. Solve the system: $x' = 3x + 5y$, $y' = -x - y$.

12. Find the general solution to

$$\vec{x}' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \vec{x}.$$

13. Find the general solution to

$$\vec{x}' = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix} \vec{x}.$$

14. Let

$$A = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}$$

be an $n \times n$ matrix, where $a_1, a_2, \dots, a_n$ are nonzero real numbers.

(a) Find the general solution to the sysem of equations $\vec{x}' = A\vec{x}$

(b) Solve the initial value problem $x_1(0) = x_2(0) = \cdots = x_n(0) = k$, for some constant k.

(c) Solve the initial value problem $\begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$.