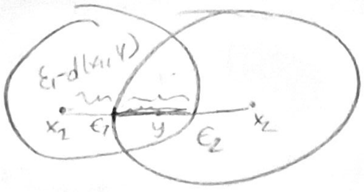
Fall 2018



1. Let (X,d) be a metric space. Define the metric topology on X & prove it is a topology. Prove that if $A \subset X$ & $x \in X$ then x is in the closure of A iff x is the limit of a sequence of points in A.

Define $B_\epsilon(x) = \{y \in X : d(x,y) < \epsilon\}$. Note that $\{B_\epsilon(x)\}_{x \in X}^{\epsilon > 0}$ this is a basis on X since $x \in B_\epsilon(x)$ & if $y \in B_{\epsilon_1}(x_1) \cap B_{\epsilon_2}(x_2)$ let $\bar{d} = \min\{\epsilon_1 - d(y,x_1), \epsilon_2 - d(y,x_2)\}$. Then consider $B_{\bar{d}}(y)$. If $z \in B_{\bar{d}}(y)$ then

$$\begin{aligned} d(z,x_1) &\leq d(z,y) + d(y,x_1) \\ &< \bar{d} + d(y,x_1) \\ &\leq \epsilon_1 - d(y,x_1) + d(y,x_1) \\ &= \epsilon_1 \end{aligned}$$

$$\begin{aligned} \& \quad d(z,x_2) &\leq d(z,y) + d(y,x_2) \\ &< \bar{d} + d(y,x_2) \\ &\leq \epsilon_2 - d(y,x_2) + d(y,x_2) \\ &= \epsilon_2 \end{aligned}$$

Then we define the metric topology on X as the topology generated by the basis $\{B_\epsilon(x)\}_{x \in X, \epsilon > 0}$.

Let $x \in \bar{A}$. Then $\forall B_{\frac{1}{n}}(x)$, $B_{\frac{1}{n}}(x) \cap A$ contains a point other than x. Define $(x_n) \subseteq A$ where $x_n \in B_{1/n}(x) \cap A$ & $x_n \neq x$. Let $\epsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $1/N < \epsilon$, & $\forall n \geq N$, $1/n < \epsilon$. Thus, $\forall n \geq N$ $d(x_n, x) < 1/n < \epsilon$, so $(x_n) \to x$.

On the other hand, let $(x_n) \to x$. Let $\mathcal{U}$ be an open set of x. Since $(x_n) \to x$ $\exists$ infinitely many points of $x_n \in A$ in $\mathcal{U}$. Since $x_n \neq x \Rightarrow \mathcal{U} \cap A \setminus \{x\} \neq \emptyset$. So $x \in \bar{A}$. □