Math 550
Homework 2
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Due September 11, 2018

1. Use a change of variables to calculate $\int_A f$, where $f(x,y,z) = (x^2+y^2)z^2$, and
$$A = \{(x,y,z) : x^2+y^2 < 1, |z| < 1\}.$$

2. Give a counterexample to show that the change of variables formula does not hold if g is not one-to-one, even if $\det Dg(x) \neq 0$ for all $x \in \Omega$. (Hint: Take $f = 1$ and $g(x,y) = (e^x \cos y, e^x \sin y)$ for a suitable region Ω.)

3. (a) Calculate $\int_{B_r} e^{-x^2-y^2}\, dx\, dy$, where $B_r = \{(x,y) : x^2+y^2 \leq r^2\}$.
   (b) Show that $\int_{C_r} e^{-x^2-y^2}\, dx\, dy = (\int_{-r}^{r} e^{-x^2}\, dx)^2$, where $C_r = [-r,r] \times [-r,r]$.
   (c) Show that
   $$\lim_{r\to\infty} \int_{B_r} e^{-x^2-y^2}\, dx\, dy = \lim_{r\to\infty} \int_{C_r} e^{-x^2-y^2}\, dx\, dy.$$
   (d) Show that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

4. (a.) Let D be the unit ball in $\mathbf{R}^3$, and let $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$. Calculate $\int_D f$ using a change of variables.
   (b.) Let E be the ellipsoid $\{(x,y,z) \in \mathbf{R}^3 : (x^2/a^2) + (y^2/b^2) + (z^2/c^2) \leq 1\}$, where a, b, and c are positive constants. Compute the volume of E using a change of variables.

5. Let $\langle e_1, \ldots, e_n\rangle$ denote the standard basis for $\mathbf{R}^n$, and let T denote the linear operator on $\mathbf{R}^n$ defined by $T(e_1) = (1,1,1,1,\ldots,1), T(e_2) = (1,2,1,1,\ldots,1), T(e_3) = (1,2,3,1,\ldots,1), \ldots, T(e_n) = (1,2,3,4,\ldots,n)$. Suppose that $f : \Omega \to \mathbf{R}$ is integrable, and $\int_\Omega f = 1$. Compute $\int_{T^{-1}(\Omega)} f \circ T$.

6. Let $p \in \mathbf{R}^4$, and let $u = (1,-1,0,2), v = (0,3,-2,1), w = (2,1,1,1)$. Compute
   (a) $(dx_1 \wedge dx_3 \wedge dx_4)_p(u,v,w)_p$
   (b) $(dx_1 \wedge dx_3 \wedge dx_1)_p(u,v,w)_p$
   (c) $((dx_1 + dx_2) \wedge dx_3 \wedge dx_4)_p(u,v,w)_p$

7. Let $p \in \mathbf{R}^3$ and $v, w \in \mathbf{R}^3_p$. Show that $(dx_p \wedge dy_p)(v,w) = dz_p(v \times w)$. (Here, "$\times$" refers to the cross product of vectors.) Express $(dx_p \wedge dz_p)(v,w)$ in terms of $dy_p(v \times w)$.

8. Suppose that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Prove that
$$(dx_{i_1} \wedge \cdots \wedge dx_{i_k})_p(e_{j_1}, \ldots, e_{j_k})_p = \begin{cases} 1 & \text{if } i_1 = j_1, i_2 = j_2, \ldots, i_k = j_k \\ 0 & \text{otherwise} \end{cases}$$

9. Let $\varphi : \mathbf{R}^n_p \times \mathbf{R}^n_p \to \mathbf{R}$ be multi-linear. Prove that $\varphi \in \Lambda^2(\mathbf{R}^n_p)$ if and only if $\varphi(v,v) = 0$ for all $v \in \mathbf{R}^n_p$.