Problem Set #3
Palsson

Trevor Klar

July 5, 2018

P1) Consider the vector space $\ell^p(\mathbb{Z})$ of integer valued sequences $(a_n)_{k=1}^{\infty}$ with $\sum\limits_{k=1}^{\infty} |a_n|^p < \infty$.

(a) NTS:

i. positive definite
ii. scaling
iii. triangle inequality

Proof (Positive definite) Since

$$||\mathbf{a}||_p = \left(\sum_{k=1}^{\infty} |a_k|^p \right)^{1/p}$$

is a sum of nonnegative numbers being raised to a power, $||\mathbf{a}||_p$ is never negative. Furthermore, $||\mathbf{a}||_p = 0$ if and only if $a_k = 0$ for all k. Thus, $||\mathbf{a}||_p = 0 \iff \mathbf{a}_p = 0$. ∎

Proof (Scaling) Let $\lambda \in \mathbb{R}$. Then,

$$\begin{aligned} ||\lambda\mathbf{a}||_p &= \left(\sum_{k=1}^{\infty} |\lambda a_k|^p\right)^{1/p} \\ &= \left(\sum_{k=1}^{\infty} |\lambda|^p |a_k|^p\right)^{1/p} \\ &= \left(|\lambda|^p \sum_{k=1}^{\infty} |a_k|^p\right)^{1/p} \\ &= |\lambda| \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p} \\ &= |\lambda| \, ||\mathbf{a}||_p \end{aligned}$$

∎

The third property, the Triangle Inequality, follows immediately by Minkowski's Inequality for Sums (When $p > 1$):

Theorem (Minkowski's Inequality for Sums). *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be sequences of non-negative real numbers, and let $p > 1$ be a real number. Then,*

$$\left(\sum_{k=1}^{\infty} (a_k + b_k)^p \right)^{1/p} \leq \left(\sum_{k=1}^{\infty} (a_k)^p \right)^{1/p} + \left(\sum_{k=1}^{\infty} (b_k)^p \right)^{1/p}.$$

∎

Now we examine when $p = 1$. First observe that since $\sum\limits_{k=1}^{\infty} |a_n|^1 < \infty$ for all $\mathbf{a} \in \ell^1(\mathbb{Z})$, then all such sequences are finite. Now we can use this to find that

$$\begin{aligned} \left(\sum_{k=1}^{\infty} (a_k + b_k)^1\right)^{1/1} &= \sum_{k=1}^{\infty} (a_k + b_k) \\ &= \sum_{k=1}^{\infty} (a_k) + \sum_{k=1}^{\infty} (b_k) \\ &= \left(\sum_{k=1}^{\infty} (a_k)^1\right)^{1/1} + \left(\sum_{k=1}^{\infty} (b_k)^1\right)^{1/1} \end{aligned}$$

(b) ehh i skipped this one.

P2) Let H_n denote the linear class of functions spanned by

$$\{1, \cos(kx), \sin(kx) : k = 1, 2, \ldots, n\}.$$

(a) What is the dimension of H_n?
Answer: $2n + 1$

(b) Is H_n a subspace of H_{n+1}?
Answer: Yes, since $H_n \subset H_{n+1}$, $\vec{0} \in H_n$, and H_n is closed under addition and scalar multiplication by definition. Thus, H_n is a subspace of H_{n+1} by the subspace criterion.

(c) For what values of n is $\sin^2(x)$ an element of H_n?
Answer: $n = 2$, since $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$.