Math 331
Theme 2 Problem

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Part 1

Part 1 is based on a problem from the text page 132, #5 (expanded):

You are asked to collect evidence for the following conjecture linking the Fibonacci sequence to the golden ratio (Φ):

Conjecture. *When Φ is raised to a positive integer power, the result can be written as $A + B\Phi$ where A and B are Fibonacci numbers.*

You are asked to find Φ^2, Φ^3, Φ^4 to gather evidence for the conjecture that each can be expressed in terms of A, B and Φ as indicated . In other words, you should express each of the powers of Φ as $A + B\Phi$. Then guess (without calculating) A and B for Φ^5 and explain the pattern emerging from your calculations.

Next, you are asked to investigate various ways to construct golden rectangles and show a geometric connection between them and Fibonacci numbers.

Step 1. (6 points) When you compute Φ^2, Φ^3, and Φ^4, use $\Phi = {}^1\!/\!_2(1+\sqrt{5})$. You must calculate the powers in terms of this value. In other words calculate the powers in terms of Φ using its square root representation. Show all your calculations.

Answer:

$$\begin{aligned}
\Phi^1 &= && && && = \tfrac{1+\sqrt{5}}{2} \\
\Phi^2 &= \left(\tfrac{1+\sqrt{5}}{2}\right)^2 &&= \tfrac{1+2\sqrt{5}+5}{4} &&= \tfrac{6+2\sqrt{5}}{4} && = \tfrac{3+\sqrt{5}}{2} \\
\Phi^3 &= \left(\tfrac{1+\sqrt{5}}{2}\right)\left(\tfrac{3+\sqrt{5}}{2}\right) &&= \tfrac{3+4\sqrt{5}+5}{4} && && = \tfrac{8+4\sqrt{5}}{4} \\
\Phi^4 &= \left(\tfrac{1+\sqrt{5}}{2}\right)\left(\tfrac{8+4\sqrt{5}}{4}\right) &&= \tfrac{8+12\sqrt{5}+20}{8} &&= \tfrac{28+12\sqrt{5}}{8} && = \tfrac{14+6\sqrt{5}}{4}
\end{aligned}$$

Step 2. (6 points) To show the evidence for the conjecture, you must write each of Φ^2, Φ^3, and Φ^4 in the form $A + B\Phi$ where A and B are Fibonacci numbers. Recall that the Fibonacci sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ etc. where each subsequent number is the sum of the previous two numbers. To help with the next step, give each of these numbers in the sequence a name that reflects its position in the sequence. For example: Let

$$F_1 = 1; F_2 = 1; F_3 = 2; F_4 = 3; F_5 = 5; F_6 = 8, \text{etc.}$$

In other words F_n= the number "sitting" in the nth position of the sequence.

Answer:

$$\begin{array}{lcll}
\Phi^1 & = & \frac{1+\sqrt{5}}{2} & = (0) + (1)\frac{1+\sqrt{5}}{2} = F_0 + F_1\Phi \\
\Phi^2 & = & \frac{3+\sqrt{5}}{2} = \frac{2}{2} + \frac{1+\sqrt{5}}{2} & = (1) + (1)\frac{1+\sqrt{5}}{2} = F_1 + F_2\Phi \\
\Phi^3 & = & \frac{8+4\sqrt{5}}{4} = \frac{4}{4} + \frac{4+4\sqrt{5}}{4} & = (1) + (2)\frac{1+\sqrt{5}}{2} = F_2 + F_3\Phi \\
\Phi^4 & = & \frac{14+6\sqrt{5}}{4} = \frac{8}{4} + \frac{6+6\sqrt{5}}{4} & = (2) + (3)\frac{1+\sqrt{5}}{2} = F_3 + F_4\Phi
\end{array}$$

Step 3. (3 points) Guess (without calculating) A and B for Φ^5. Then explain how you figured it out. In other words, explain the pattern that connects powers of the golden ratio to the Fibonacci number. You should take into consideration the power you are raising Φ to; and its relationship to the n in the F_n name of the number in the Fibonacci sequence.

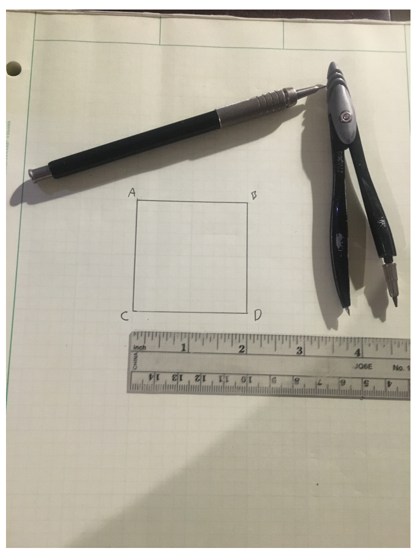
Answer: I think that

$$\Phi^5 = (3) + (5)\frac{1+\sqrt{5}}{2} = F_4 + F_5\Phi.$$

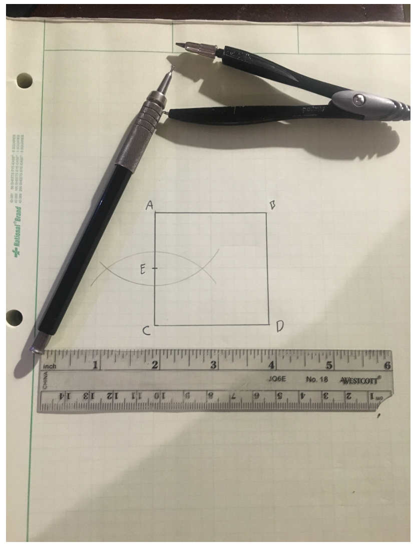
The sequence seems to be following the pattern of $\Phi^n = F_{n-1} + F_n\Phi$.

Step 4. (4 points) Use Euclid's construction (see http://mathworld.wolfram.com/GoldenRectangle.html for details) to create a golden rectangle, starting with a square of side length 2 inches. Indicate the lengths of the sides of the golden rectangle and show why it is "golden".

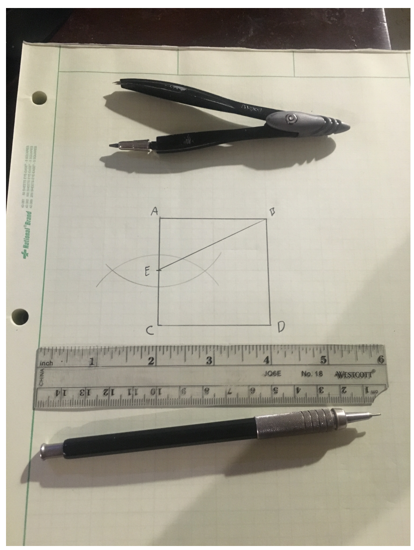
Answer: Start with square $ABDC$ with side length 2 inches.



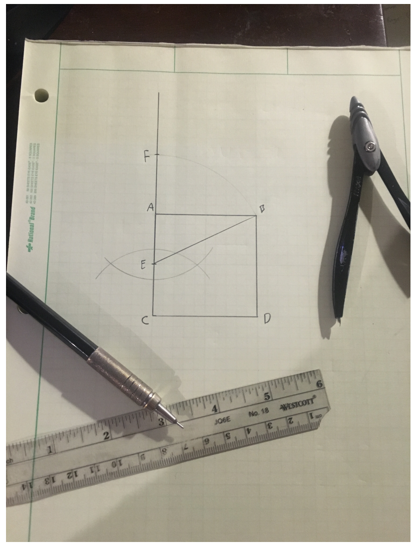
Bisect $\overline{AC}$, and call the midpoint E.



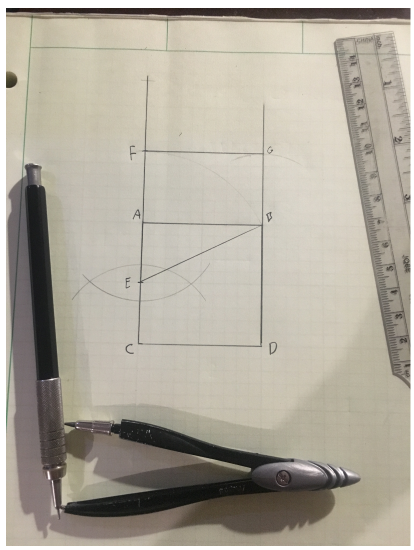
Construct $\overline{EB}$. Observe that $EF = \sqrt{5}$ by the Pythagorean Theorem, since $AE = 1$ and $AB = 2$.



Extend $\overline{EA}$ to construct $\overline{EF}$ with length $\sqrt{5}$.



Extend $\overline{DB}$ to construct $\overline{GB}$ such that $\overline{GB} = \overline{AF}$. Then construct $\overline{FG}$.

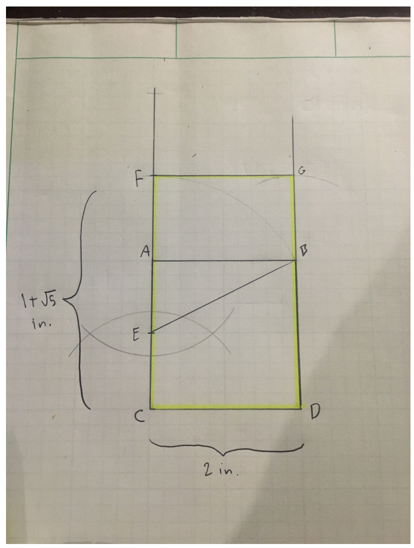


Observe that rectangle $FGDC$ is a Golden Rectangle. To see this, note that $FC = 1+\sqrt{5}$, and $CD = 2$. Thus, the ratio of side lengths is

$$\frac{1+\sqrt{5}}{2} = \Phi.$$

Also, when $FGDC$ is partitioned into square $ABDC$ and rectangle $FGBA$, we find that rectangle $FGBA$ has side lengths $FA = 1+\sqrt{5}-2 = \sqrt{5}-1$ and $AB = 2$. Thus, the ratio of its sides is

$$\frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{2+2\sqrt{5}}{4} = \frac{1+\sqrt{5}}{2} = \Phi.$$



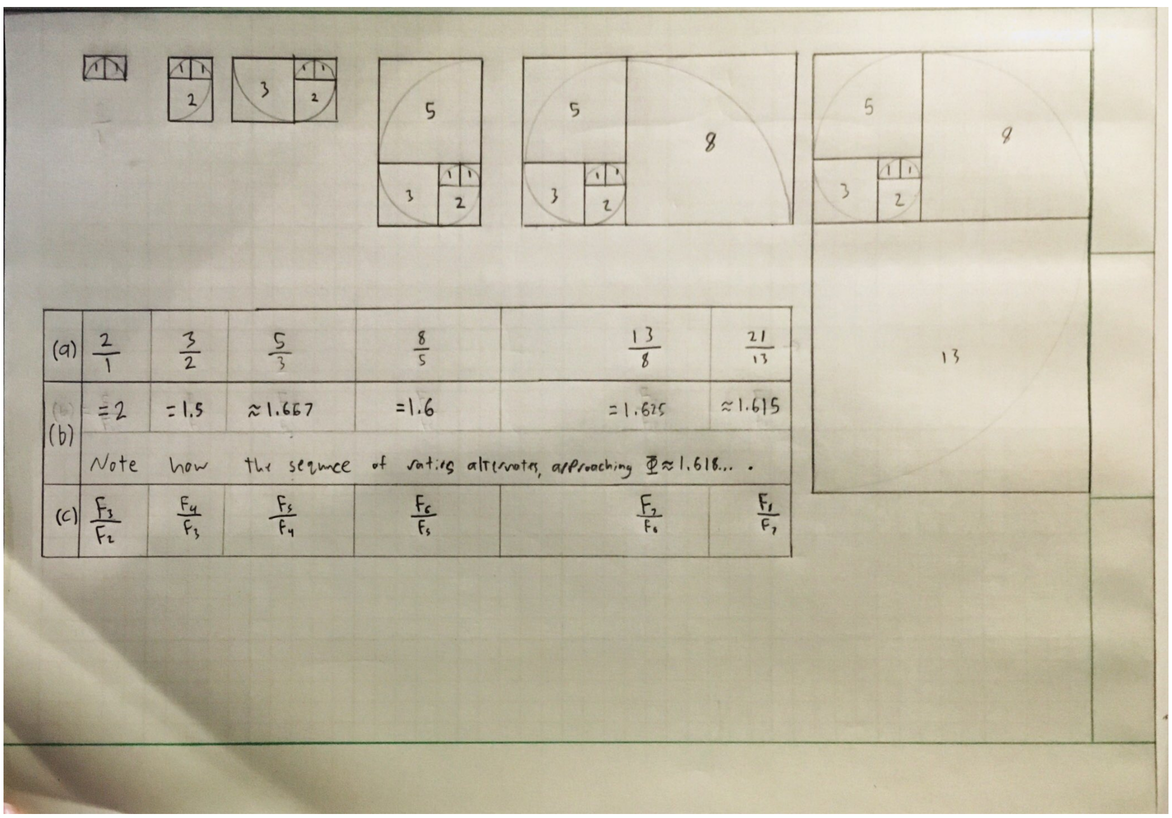
Step 5. (6 points) Illustrate a geometric connection between golden rectangles and Fibonacci numbers. Start with a square with side lengths 1 inch and add a square of the same size to form a new rectangle. Continue adding squares whose sides are the length of the longer side of the rectangle. Repeat the process at least five times. Then look at the ~~triangles~~ rectangles. Answer these questions:

(a) What is the pattern that emerges when you evaluate the ratios of the longer side to the smaller side for the rectangles you created?

(b) Show that the longer you continue the process, the larger and larger rectangles that are formed will successively be approximating a golden rectangle.

(c) What pattern emerges in the lengths of the longer sides of successive rectangles ?

Answer:



(a)	$\frac{2}{1}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{13}{8}$	$\frac{21}{13}$
(b)	$=2$	$=1.5$	≈ 1.667	$=1.6$	$=1.625$	≈ 1.615
	Note how the sequence of ratios alternates, approaching $\Phi \approx 1.618\ldots$.					
(c)	$\frac{F_3}{F_2}$	$\frac{F_4}{F_3}$	$\frac{F_5}{F_4}$	$\frac{F_6}{F_5}$	$\frac{F_7}{F_6}$	$\frac{F_8}{F_7}$

Part 2

The essay begins on the following page.

The Golden Ratio and its Application in the Classroom

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The Golden Ratio and its Application in the Classroom

Stephen Brown's article "From the Golden Rectangle and Fibonacci to Pedagogy and Problem Posing", is about how Fibonacci Numbers and the Golden Ratio can be used in the classroom to encourage exploration, perseverance in problem solving, and mathematical thinking. The article is organized as a list of problems, questions which can be posed to students, and I found a lot of material which I can apply in my high school classroom.

The Fibonacci Sequence is a sequence of numbers which starts with 1, 1, … and each successive term is found by adding the two previous terms. So the sequence begins

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89… ,

and from this sequence, we find a very interesting number, the Golden Ratio, whose name is Φ (phi, pronounced /faɪ/). The value of Φ is approximately 1.6180399, and if we examine the ratios of consecutive terms in the Fibonacci sequence,

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \ldots$$

we find that those ratios get closer and closer to the Golden Ratio.

I love the Golden Ratio, and I especially love sharing it with my students. Actually, before beginning this assignment, I had forgotten that the square root form of the number existed. I usually think of it as "the number whose reciprocal is equal to 1 minus itself." This is a cool definition to show school children, because they almost invariably think of nonintegers as decimals. If they have some experience with reciprocals on a calculator, they know that the decimal representation of a number is usually very different from its reciprocal; and they are often shocked to type 1÷1.61803399= and find the answer to be 0.61803399. Before I read this article, I was thinking of Φ as a transcendental number. A transcendental number is a number (like π) which is not the solution to an algebra problem, that is, it can only be written down as its name or a decimal approximation. Brown takes the reader through an algebraic solution to the equation 1-Φ=1/Φ, which is actually a quadratic in form. A quadratic equation is an equation of the form $ax^2+bx+c=0$. With a little rearranging, 1-Φ=1/Φ can be put into that form, and there is a formula (the quadratic formula) which gives the solution(s) to any quadratic equation. This is where the radical form of Φ comes from; Φ=½(1+√5). (Brown, 181)

Brown also suggests giving students an introduction to continued fractions, by using some clever (and very unorthodox) substitutions (Brown, p. 182). He starts by setting up an equation to find 1/Φ,

$$\frac{1}{X} = \frac{X}{1-X} \;,$$

but instead of using the method described in the previous paragraph, he cross-multiplies and does some rearranging to find

$$X = \frac{1}{1+X} \;.$$

He then shows how we can substitute this equation into *itself* to yield

$$X = \frac{1}{1+X} = \frac{1}{1+\boxed{\dfrac{1}{1+X}}} \;,$$

and by doing this over and over, we find

$$X = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}} .$$

Now at this point, all the students in the room will be looking at the teacher sideways, thinking "Yep, he's definitely lost his marbles." But, there is a fairly intuitive way to handle this. We just try to evaluate this fraction by going partway, and lopping off the rest.

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}$$

Doing this repeatedly for more and more of the fraction, to get an approximation of the number we're looking for. Here's what we get:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}.$$

This is that wonderful Fibonacci Sequence, popping up again!

Brown goes on to discuss ideas in Geometry, Number Theory, and some of the Philosophy of Math (Brown, p. 185-189). This is a great article with some really cool classroom ideas, and indeed, just cool ideas for any mathematically-inclined person. Brown shows how the Fibonacci Sequence and the Golden Mean can be an excellent exploration tool to build mathematical maturity in young people.

Works Cited

Brown, Stephen I. (1976). FROM THE GOLDEN RECTANGLE AND FIBONACCI TO PEDAGOGY AND PROBLEM POSING. *The Mathematics Teacher.,69*(3), 180-188.

Davis, P. J., Hersh, R., & Marchisotto, E. (1995). The Mathematical Experience.

Weisstein, Eric W. "Golden Rectangle." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/GoldenRectangle.html