Homework 2

1. Let $x \in \mathbb{R}^n$ and let $K \subset \mathbb{R}^n$ be compact. Denote $U = \mathbb{R}^n - K$ and define for each fixed $s \in K$ the function

$$u_s(x) = \max\left(2 - \frac{|x-s|}{\mathrm{dist}(x,K)}, 0\right), \quad x \in U.$$

Let s_i be a countable dense subset of K and define

$$\sigma(x) = \sum_{i=1}^{\infty} 2^{-i} u_{s_i}(x), \quad x \in U.$$

It is not difficult to prove that then $0 < \sigma(x) \leq 1$ for all $x \in U$, thus we can define

$$v_i(x) = \frac{2^{-i} u_{s_i}(x)}{\sigma(x)}, \quad x \in U.$$

Assume next $f : K \to \mathbb{R}$ is continuous and define

$$\bar{f}(x) = \sum_{i=1}^{\infty} v_i(x) f(s_i), \quad x \in U.$$

Prove that $\bar{f}(x)$ is continuous in U.

Proof We will show that u_s is continuous and

$$u_s \text{ continuous} \implies \sigma \text{ continuous} \implies v_i \text{ continuous} \implies \bar{f} \text{ continuous}.$$

- (u_s) We already know that max and euclidean distance functions are continuous, so if $\mathrm{dist}(x,K)$ is continuous, then u_s is comprised of compositions, sums, and products of continuous functions, so is continuous. So all that remains is to show that $\mathrm{dist}\,(x,K)$ is continuous. Let $x \in U = K^{\complement}$, $\epsilon > 0$ and $y \in \mathbb{R}^n$ such that $|x-y| < \frac{\epsilon}{2}$. Then for any $k \in K$,
$$|x-k| - \frac{\epsilon}{2} \leq |y-k| \leq |x-k| + \frac{\epsilon}{2}$$
by triangle inequality, so taking infs and using ϵ instead of $\frac{\epsilon}{2}$ to obtain strict inequalities, we find that
$$\mathrm{dist}\,(x,K) - \epsilon < \mathrm{dist}\,(y,K) < \mathrm{dist}\,(x,K) + \epsilon$$
so $\mathrm{dist}(x,K)$ is continuous.
- (σ) First, observe that for all $s \in K, x \in U$, $\frac{|x-s|}{\mathrm{dist}(x,K)}$ is always ≥ 1 and approaches 1 as x gets very far from K. This tells us that $0 \leq u_{s_i} \leq 1$ for every s_i. Then we can use the Weierstrauss M-test. For $x \in U$,
$$\sigma(x) = \sum_{i=1}^{\infty} 2^{-i} u_{s_i}(x) = \sum_{i=1}^{\infty} |2^{-i} u_{s_i}(x)| \leq \sum_{i=1}^{\infty} 2^{-i} = 1,$$
so since $2^{-i} u_{s_i}$ are continuous functions, then so is σ.

- (v_i) v_i is a product of continuous functions, so it is continuous whenever $\sigma(x) \neq 0$, so let's check that σ is always positive. Suppose for contradiction that there exists $x \in U$ such that $\sigma(x) = 0$[†]. Each term of σ is the product of a nonzero number with u_{s_i}, so $\sigma(x) = 0$ iff all $u_{s_i}(x) = 0$. This means that $|x - s_i| \geq 2\,\mathrm{dist}\,(x, K)$ for all s_i, which is impossible since $\{s_i\}$ is dense in K. To see the contradiction, observe that for any $k \in K$, there is a sequence $\{s_i\}_{i\in I\subset\mathbb{N}}$ which converges to k, so

$$\inf_{i\in N} |x - s_i| = \inf_{k\in K} |x - k| = \mathrm{dist}\,(x, K)\,,$$

thus there exists some s_i such that $|x - s_i| < 2\,\mathrm{dist}\,(x, K)$. Therefore σ never vanishes, and v_i is continuous.

- ($\bar{f}$) Since f is a continuous function on a compact domain, then it is bounded. Denote the bound $B \geq f(x)$ for all $x \in K$. Then

$$\begin{aligned}
\bar{f}(x) &= \sum_{i=1}^{\infty} v_i(x) f(s_i) \\
&= \sum_{i=1}^{\infty} \frac{2^{-i} u_{s_i}(x)}{\sigma(x)} f(s_i) \\
&= \frac{1}{\sigma(x)} \sum_{i=1}^{\infty} 2^{-i} u_{s_i}(x) f(s_i) \\
&\leq \frac{1}{\sigma(x)} \sum_{i=1}^{\infty} 2^{-i} (1)(B) \\
&=\leq \frac{1}{\sigma(x)},
\end{aligned}$$

So since the functions used above are continuous, then by the Weierstrauss M-test, $\bar{f}$ is continuous. ■

[†] σ is certainly never negative because it is a sum of nonnegative numbers.

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **lower semi-continuous at the point** $x \in \mathbb{R}^n$ if, for any sequence $x_k \in \mathbb{R}^n$ with $x_k \to x$ one has

$$\liminf_{k\to\infty} f(x_k) \geq f(x).$$

2. Prove that any lower semi-continuous function is Borel measurable.

Proof Consider $f^{-1}(-\infty, a]$. If $f^{-1}(-\infty, a]$ is closed, then f is Borel measurable. Let x_n be any convergent sequence in $f^{-1}(-\infty, a]$, and say that $x_n \to \gamma$, then γ is an arbitrary limit point of $f^{-1}(-\infty, a]$. Since f is lower semi-continuous, then

$$\liminf_{n\to\infty} f(x_n) \geq f(\gamma).$$

Since $a \geq f(x_n)$ for all n, then

$$a \geq \liminf_{n\to\infty} f(x_n) \geq f(\gamma),$$

so $f^{-1}(-\infty, a]$ contains all its limit points and thus is closed. ■

3. Prove the following statements:

(i) Let $a < b$ and $a_k < b_k$ for $k \in \mathbb{N}$. If

$$[a, b) \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k),$$

then

$$b - a \leq \sum_{k=1}^{\infty} (b_k - a_k).$$

Proof Without loss of generality suppose that there are no extraneous intervals, that is, for all i, j we have $[a, b) \cap [a_i, b_i) \neq \emptyset$ and $[a_i, b_i) \not\subseteq [a_j, b_j)$. Let $\epsilon > 0$. Then $[a, b - \epsilon] \subseteq \bigcup_{k=1}^{\infty} [a_k, b_k)$, and $[a, b - \epsilon]$ is compact, so there exists a finite subcover[†]

$$[a, b - \epsilon] \subseteq \bigcup_{i=1}^{n} [a_{k_i}, b_{k_i}).$$

For any i, j such that $[a_i, b_i) \cap [a_j, b_j) \neq \emptyset$, we can write

$$[a_i, b_i) \cup [a_j, b_j) = [a_i, a_j) \cup [a_j, b_i) \cup [b_i, b_j),$$

and note that

$$\begin{aligned}(b_i - a_i) + (b_j - a_j) &= (a_j - a_i) + 2(b_i - a_j) + (b_j - b_i)\\ &> (a_j - a_i) + (b_i - a_j) + (b_j - b_i).\end{aligned}$$

[†]It's very late. I just realized that this doesn't work, because this isn't an open cover. I think that it can be fixed by using $(a_k - \frac{\epsilon}{2^k}, b_k)$, but I can't fix it tonight.

So any finite nondisjoint union of intervals $[a_i, b_i)$ can be rewritten as a finite disjoint union with smaller length. Thus we can renumber and write

$$[a, b-\epsilon] \subseteq \coprod_{i=1}^{n} [\hat{a}_i, \hat{b}_i) = \bigcup_{i=1}^{n} [a_{k_i}, b_{k_i}).$$

Since there are no extraneous intervals, then $\hat{a}_1 \leq a$, and $b - \epsilon < \hat{b}_n$, and $\hat{b}_i = \hat{a}_{i+1}$ for all i. Thus

$$(b-a) - \epsilon \leq (\hat{b}_n - \hat{a}_1) = \sum_{i=1}^{n} (\hat{b}_i - \hat{a}_i) < \sum_{i=1}^{n} (b_{k_i} - a_{k_i}) < \sum_{k=1}^{\infty} (b_k - a_k),$$

Since this holds for all $\epsilon > 0$, we can let $\epsilon \to 0$ and find that

$$b - a \leq \sum_{k=1}^{\infty} (b_k - a_k),$$

as desired. ■

(ii) Let $[a_k, b_k)$ be disjoint intervals and $c_k < d_k$ for all k. If

$$\bigcup_{k=1}^{\infty} [a_k, b_k) \subseteq \bigcup_{k=1}^{\infty} [c_k, d_k),$$

then

$$\sum_{k=1}^{\infty} (b_k - a_k) \leq \sum_{k=1}^{\infty} (d_k - c_k).$$

Proof For every $k, i \in \mathbb{N}$, if $[a_k, b_k) \cap [c_i, d_i) \neq \emptyset$ and $[a_{k+1}, b_{k+1}) \cap [c_i, d_i) \neq \emptyset$, then split $[c_i, d_i)$ at $\frac{b_k + a_{k+1}}{2}$, that is, remove $[c_i, d_i)$ from the collection and replace it with $[c_i, \frac{b_k + a_{k+1}}{2})$ and $[\frac{b_k + a_{k+1}}{2}, d_i)$. Then after renumbering, we have that

$$\bigcup_{k=1}^{\infty} [a_k, b_k) \subseteq \bigcup_{1 \leq i, k < \infty} [\hat{c}_{k_i}, \hat{d}_{k_i}) = \bigcup_{k=1}^{\infty} [c_k, d_k),$$

where $[a_k, b_k) \subseteq \bigcup_{i=1}^{\infty} [\hat{c}_{k_i}, \hat{d}_{k_i})$ for all k. We know from the previous problem that

$$(b_k - a_k) \leq \sum_{i=1}^{\infty} (\hat{d}_{k_i} - \hat{c}_{k_i})$$

for all k, so

$$\begin{aligned}
\sum_{k=1}^{\infty} (b_k - a_k) &\leq \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} (\hat{d}_{k_i} - \hat{c}_{k_i}) \right) \\
&= \sum_{1 \leq i, k < \infty} (\hat{d}_{k_i} - \hat{c}_{k_i}) \\
&= \sum_{k=1}^{\infty} (d_k - c_k),
\end{aligned}$$

and we're done. ■

4. Prove that if a Lebesgue measurable set $A \subset \mathbb{R}$ has positive Lebesgue measure, then the set

$$A - A = \{a - b : a, b \in A\}$$

contains a neighborhood of the origin. Is the statement true if one only assumes $m(A) > 0$ (i.e., A is not Lebesgue measurable)?

Proof Since A is Lebesgue measurable, then we can approximate A with a compact set $K \subseteq A$ and an open set $U \supseteq A$ such that $m(U) - m(K) < \epsilon$, for any $\epsilon > 0$. Since K compact and $U^{\complement}$ closed with $K, U^{\complement}$ disjoint, then $\operatorname{dist}\left(K, U^{\complement}\right) > 0$. If we let $0 < \delta < \operatorname{dist}\left(K, U^{\complement}\right)$, then

$$K + (-\delta, \delta) \subset U$$

because $\operatorname{dist}\left(k, U^{\complement}\right) > \delta$ for all $k \in K$. Now we will show that for any r with $|r| < \delta$, that $K \cap K + r \neq \emptyset$ and $B_\delta(0) \subset K - K \subset A - A$. Suppose for contradiction that $|r| < \delta$ and $K \cap K + r = \emptyset$. Since $K, K + r$ are measurable and disjoint, and Lebesgue measure is translation invariant, then

$$m(K \cup (K + r)) = 2m(K).$$

Since $K \cup (K + r) \subseteq U$, then

$$m(K \cup (K + r)) \leq m(K) + \epsilon,$$

But for $\epsilon < m(K)$, this is a contradiction. ■

Answer: If one does not assume that A is measurable, the result does not hold. For example, let $A = \mathcal{V}$, a Vitali set in $[0, 1]$ constructed in the usual way. Then $m(\mathcal{V}) = 1 > 0$, but $\mathcal{V} - \mathcal{V}$ contains no rational numbers except 0 by the construction of $\mathcal{V}$.