Math 462 - Advanced Linear Algebra
Assignment 2

Trevor Klar

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Exercises:

1. Let $T : V \to V'$ be a surjective linear transformation of finite-dimensional vector spaces. Show that there exists a subspace W of V such that

$$V = \operatorname{Ker}(T) \oplus W \text{ with } W \cong V'.$$

Proof Let $\{v_1, \ldots, v_n\}$ be a basis for $\operatorname{Ker}(T)$. Now extend this basis so that it spans V, as follows: $\{v_1, \ldots, v_n, w_1, \ldots, w_k\}$. Now, $\{v_1, \ldots, v_n\} \subset \operatorname{Ker}(T)$, so $T(\{v_1, \ldots, v_n\}) = 0$. This means that, since $\{v_1, \ldots, v_n, w_1, \ldots, w_k\}$ spans V and T is surjective, then $T(\{v_1, \ldots, v_n, w_1, \ldots, w_k\}) = T(\{w_1, \ldots, w_k\}) \cup \{0\}$ spans V'. Thus, $\{w_1, \ldots, w_k\}$ spans some $W \subset V$ and $T(\{w_1, \ldots, w_k\}) \cup \{0\}$ spans V', so $w \cong V'$. ■

2. Let k be a field. Let $\{0\}$ denote the zero vector space over k. A sequence of vector spaces of the form

$$\{0\} \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{n-1}} V_n \xrightarrow{\alpha_n} \{0\}$$

where each α_i is a k-linear transformation with $\operatorname{im}(a_i) = \ker(a_{i+1})$ is called an *exact sequence*.

Show that $\sum\limits_{i=0}^{n} (-1)^i \dim V_i = 0$.

Proof

$$\begin{aligned}
\sum_{i=0}^{n} (-1)^i \dim V_i &= \sum_{i=0}^{n} (-1)^i \dim(\operatorname{im}\alpha_i) + \sum_{i=0}^{n} (-1)^i \dim(\ker\alpha_i) \quad \text{(By Rank-Nullity Thm)} \\
&= \sum_{i=0}^{n} (-1)^i \dim(\ker\alpha_{i+1}) + \sum_{i=0}^{n} (-1)^i \dim(\ker\alpha_i) \quad \text{(Substitution)} \\
&= \sum_{i=1}^{n+1} (-1)^{i-1} \dim(\ker\alpha_i) + \sum_{i=0}^{n} (-1)^i \dim(\ker\alpha_i) \quad \text{(Change of index)} \\
&= (-1)^n \dim(\ker\alpha_{n+1}) + \sum_{i=1}^{n} (-1)^{i-1} \dim(\ker\alpha_i) + \sum_{i=0}^{n} (-1)^i \dim(\ker\alpha_i) + \dim(\ker\alpha_0) \\
&= (-1)^n \dim(\operatorname{im}\alpha_n) + \sum_{i=1}^{n} (-1)^{i-1} \dim(\ker\alpha_i) + \sum_{i=0}^{n} (-1)^i \dim(\ker\alpha_i) + \dim(\ker\alpha_0) \\
&= 0 + \left(\sum_{i=1}^{n} (-1)^{i-1} \dim(\ker\alpha_i) + \sum_{i=0}^{n} (-1)^i \dim(\ker\alpha_i) \right) + 0 \\
&= 0
\end{aligned}$$

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