Definition

Center of a ring R

- $\{z \in R | zr = rz \text{ for all } r \in R\}$,
- "The set of all elements which commute with R."

Proof Technique

Subring Criterion for $S \subset R$

- $S \neq \emptyset$
- $x - y \in S$ (closed under subtraction)
- $xy \in S$ (closed under multiplication)

Definition

Characteristic of a ring R

The characteristic $\mathrm{char}(R)$ is the smallest positive number n such that

$$\underbrace{1 + \cdots + 1}_{n \text{ summands}} = 0.$$

This also means that any element vanishes when added to itself this many times.

Definition

Ring

- $(R, +)$ is an abelian group (associative, identity, inverse, commutative)
- $(R, \times)$ is a monoid (associative, identity)
- $\times$ distributes over $+$ from either side. (distributive)

Definition

Unique Factorization Domain

An integral domain R in which every non-zero element $x \in R$ can be written as a product (an empty product if x is a unit) of irreducible elements p_i of R and a unit u:

$$x = u\, p_1\, p_2 \dots p_n \quad \text{with } n \geq 0$$

Definition

Principal Ideal Domain (PID)

An integral domain in which every ideal is a principal ideal.

Definition

Principal Ideal

An ideal $I \trianglelefteq R$ generated by a single element. That is if $\langle a \rangle = I$, start with $a \in R$, and make all the elements possible by multiplying something in R by a, and then make all elements possible by finite sums of those elements.

Definition

Discrete Valuation

$$v : R^{\times} \rightarrowtail \mathbb{Z} \text{ such that}$$

- v is surjective
- $v(ab) = v(a) + v(b)$
- $v(x + y) \geq \min\{v(x), v(y)\} \quad \forall x + y \neq 0$

Definition

Ideal $S \trianglelefteq R$

- $S \neq \emptyset$
- S closed under subtraction
- $rs, sr \in S \quad \forall s \in S, r \in R$. ($S$ absorbs multiplicands in R.)

Proposition

Let $\varphi : R \to S$ be a homomorphism.

What do we know about $\operatorname{Im}\varphi$ and $\ker\varphi$?

- $\operatorname{Im}\varphi \underset{\text{ring}}{\subset} S$
- $\ker\varphi \trianglelefteq R$

Definition

Augmentation ideal of RG

An element in the augmentation ideal of a group ring is of the form $\sum r_i g_i$, where $\sum r_i = 0$.

Definition

Nilradical

The nilradical of a ring is an ideal consisting of all the nilpotent elements, that is,

$$\{r \in R : r^k = 0 \text{ for some } k\}$$

Definition

Radical of ideal I

The radical of a ring ideal I is itself an ideal consisting of all the I-potent elements, that is,

$$\{r \in R : r^k \in I \text{ for some } k\}$$

Definition

Group Ring

Let R be a commutative ring with $1 \neq 0$ and G a **finite** multiplicative group. Then RG is

$$a_1g_1 + \cdots + a_ng_n \quad a_i \in R.$$

with addition defined "componentwise":
$\sum_{i=1}^{n} a_ig_i + \sum_{i=1}^{n} b_ig_i = \sum_{i=1}^{n}(a_i + b_i)g_i$
and multiplication defined by
$(ag_i)(bg_j) = (ab)(g_ig_j) = cg_k$
and extending via the distributive property (taking care if R is not commutative).

Theorem

The First Isomorphism Theorem for Rings

Let $\phi : R \to S$ be a ring homomorphism. Then

- $\ker(\phi) \trianglelefteq R$,
- $\phi(R) \underset{\text{ring}}{\subset} S$, and
- $R \big/ \ker(\phi) \cong \phi(R)$.

Theorem

The Second Isomorphism Theorem for Rings

Let $A \underset{\text{ring}}{\subset} R$, $I \trianglelefteq R$. Then

- $A + I = \{a + i : a \in A, i \in I\} \underset{\text{ring}}{\subset} R$,
- $A \cap I \trianglelefteq A$, and
- $(A+I)/I \cong A \big/ (A \cap I)$.

THEOREM

The Third Isomorphism Theorem for Rings

Let $I, J \trianglelefteq R$ with $I \subseteq J$. Then

- $J/I \trianglelefteq R/I$
- $(R/I) \big/ (J/I) \cong R/J$.