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# Final Exam

**1.** Let $M^n$ be an embedded submanifold of $N^m$. Show that $TM$ is an embedded submanifold of $TN$.

**Proof** Since $M$ is an embedded submanifold of $N$, then the inclusion map $\iota : M \hookrightarrow N$ is an embedding; that is, a smooth map of rank $n$ which is a homeomorphism onto its image with the subspace topology.

If we consider the global differential $d\iota : TM \to TN$, we know that $d\iota$ is smooth since $\iota$ is smooth (Proposition 3.21). Now observe that

$$\begin{aligned} d\iota(x^1, \ldots, x^n, v^1, \ldots, v^n) &= \left(\iota^1(x), \ldots \iota^n(x), \frac{\partial \iota^1}{\partial x^i}(x)v^i, \ldots, \frac{\partial \iota^n}{\partial x^i}(x)v^i\right) \\ &= (x^1, \ldots, x^n, v^1, \ldots, v^n), \end{aligned}$$

where the last line comes from the fact that $\iota^i(n)$ just gives the $i$-th coordinate of $x$ and $\frac{\partial \iota^j}{\partial x^i}(x)$ is constantly 1 if $i = j$ and constantly 0 if $i \neq j$. This means that the rank of $d\iota$ is $2n$, and $d\iota$ is the inclusion map $TM \hookrightarrow TN$. Since $\iota$ is a homeomorphism onto its image and $T_pM$ is a linear subspace of $T_pN$ for every $p \in M$, then $d\iota$ is also a homeomorphism onto its image, so $d\iota$ is an embedding, and we are done. ∎

**2.** Let $X, Y, Z$ be vector fields on $\mathbb{R}^3$ given by

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Let $A$ be the linear space spanned by $X, Y, Z$. Show that $A$ is a 3-dimensional Lie algebra with the Lie bracket of $\mathfrak{X}(\mathbb{R}^3)$.

**Proof** Since $X, Y, Z$ are clearly linearly independent, then their span over coefficients in $\mathbb{R}$ is a 3-dimensional vector space. Now we show that the bracket $[X, Y] = XY - YX$ on $A$ satisfies the desired properties. By the symmetry in their definitions, $[X, Y], [Y, Z]$, and $[Z, X]$ will all have the same properties, and while an arbitrary vector field in $A$ will be of the form $(aX + bY + cZ)$, it suffices to show that the desired properties hold for the basis vectors.

(i) Bilinearity:

$$[aX+bY,Z]f = (aX+bY)Zf - Z(aX+bY)f$$

$$\begin{aligned}
(aX+bY)Zf &= \left(ay\frac{\partial}{\partial z} - az\frac{\partial}{\partial y} + bz\frac{\partial}{\partial x} - bx\frac{\partial}{\partial z}\right)\circ\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)f \\
&= \left(bz\frac{\partial}{\partial x} - az\frac{\partial}{\partial y} + (ay-bx)\frac{\partial}{\partial z}\right)\circ\left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right) \\
&= bz\frac{\partial f}{\partial y} + bxz\frac{\partial^2 f}{\partial y\,\partial x} - bzy\frac{\partial^2 f}{\partial x^2} - axz\frac{\partial^2 f}{\partial y^2} + az\frac{\partial f}{\partial x} + ayz\frac{\partial^2 f}{\partial x\,\partial y} \\
&\quad + (ay-bx)\left(x\frac{\partial^2 f}{\partial y\,\partial z} - y\frac{\partial^2 f}{\partial x\,\partial z}\right)
\end{aligned}$$

$$\begin{aligned}
-Z(aX+bY)f &= -\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\circ\left(bz\frac{\partial}{\partial x} - az\frac{\partial}{\partial y} + (ay-bx)\frac{\partial}{\partial z}\right)f \\
&= \left(-x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right)\circ\left(bz\frac{\partial f}{\partial x} - az\frac{\partial f}{\partial y} + ay\frac{\partial f}{\partial z} - bx\frac{\partial f}{\partial z}\right) \\
&= -bxz\frac{\partial^2 f}{\partial x\,\partial y} + axz\frac{\partial^2 f}{\partial y^2} - ax\frac{\partial f}{\partial z} - axy\frac{\partial^2 f}{\partial z\,\partial y} + bx^2\frac{\partial^2 f}{\partial z\,\partial y} \\
&\quad + byz\frac{\partial^2 f}{\partial x^2} - ayz\frac{\partial^2 f}{\partial y\,\partial x} + ay^2\frac{\partial^2 f}{\partial z\,\partial x} - by\frac{\partial f}{\partial z} - bxy\frac{\partial^2 f}{\partial z\,\partial x}
\end{aligned}$$

and all of the second-order derivatives cancel[†], the sum is given by

$$[aX+bY,Z]f = az\frac{\partial f}{\partial x} + bz\frac{\partial f}{\partial y} - (ax+by)\frac{\partial f}{\partial z}.$$

Now we check that this is equal to $a[X,Z]f + b[Y,Z]f$ :

$$\begin{aligned}
a[X,Z]f &= a(XZf - ZXf) \\
&= a\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\circ\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)f - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\circ\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)f\right] \\
&= a\left[\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\circ\left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right) - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\circ\left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)\right] \\
&= a\left[-x\frac{\partial f}{\partial z} + z\frac{\partial f}{\partial x}\right] \\
&= -ax\frac{\partial f}{\partial z} + az\frac{\partial f}{\partial x}
\end{aligned}$$

[†]Since $f \in C^\infty(\mathbb{R})$, mixed partials are equal.

$$\begin{aligned}
b[Y,Z]f &= b(YZf - ZYf)\\
&= b\left[\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\circ\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)f - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\circ\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)f\right]\\
&= b\left[z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right]\\
&= bz\frac{\partial f}{\partial y} - by\frac{\partial f}{\partial z}
\end{aligned}$$

So,

$$\begin{aligned}
[aX+bY,Z]f &= az\frac{\partial f}{\partial x} + bz\frac{\partial f}{\partial y} - (ax+by)\frac{\partial f}{\partial z}\\
&= -ax\frac{\partial f}{\partial z} + az\frac{\partial f}{\partial x} + bz\frac{\partial f}{\partial y} - by\frac{\partial f}{\partial z}\\
&= a[X,Z]f + b[Y,Z]f
\end{aligned}$$

and a similar proof will show that $[Z, aX+bY] = a[Z,X]+b[Z,Y]$. Thus, the bilinearity property is shown.

(ii) Antisymmetry:

$$\begin{aligned}
[X,Y]f &= XYf - YXf\\
&= -(-XYf + YXf)\\
&= -[Y,X]f
\end{aligned}$$

(iii) Jacobi Identity:
By examining some of the computations in the bilinearity part of this proof, we can see that

$$\begin{aligned}
[Y,X] &= Z,\\
[X,Z] &= Y, \text{ and}\\
[Z,Y] &= X.
\end{aligned}$$

Thus

$$\begin{aligned}
[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] &= [X,-X] + [Y,-Y] + [Z,-Z]\\
&= 0+0+0
\end{aligned}$$

Therefore, $A$ is a 3-dimensional vector space with a bracket operation having the bilinearity, antisymmetry, and Jacobi identity properties, so $A$ is a 3-dimensional Lie algebra. ■

**3.** a) Compute the flow of the vector field $X$ on $\mathbb{R}^2$:

$$X = y\frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

**Answer:** The vector field $X$ corresponds to the ODE system

$$\begin{aligned} \dot{y} &= -1 \\ \dot{x} &= y \end{aligned}$$

which can be solved one at a time to find the solutions $y(t) = -t + y_0$ and $x(t) = -\frac{t^2}{2} + y_0 t + x_0$. This corresponds to the flow

$$\theta_t(x, y) = (-\tfrac{t^2}{2} + ty + x,\ -t + y),$$

where integral curves are parabolas. ■

b) Let $M = M_n(\mathbb{R})$ be the space of all $n \times n$ matrices. For $A \in M$ let $V_A$ be the vector field on $M$ so that $V_A(X) = AX$, where $X \in M$ (we have used the identification $T_X M = \mathbb{R}^{n^2} = M$). Compute the flow $\theta_t$ generated by $V_A$. **Answer:** The vector field $AX$ corresponds to the linear system of ODEs $X' = AX$, and if we write $X$ as a concatenation of column vectors, we obtain

$$\begin{bmatrix} | & & | \\ (\mathbf{x}^1)' & \cdots & (\mathbf{x}^n)' \\ | & & | \end{bmatrix} = \mathbf{A} \begin{bmatrix} | & & | \\ \mathbf{x}^1 & \cdots & \mathbf{x}^n \\ | & & | \end{bmatrix},$$

and this system can be solved column by column (they will all have the same family of solutions, differing only in their initial values) as

$$(\mathbf{x}^i)' = \mathbf{A}\mathbf{x}^i \quad \text{, for } i \in 1, \ldots, n$$

according to the usual method of finding eigenvalues and eigenvectors, finding coefficients, and determining constants assuming that $\mathbf{x}^i(0) = \mathbf{x}^i_0$ for each $i$. For example,

$$\mathbf{x}^1(t) = \varphi_1(t, \mathbf{x}^1_0) + \cdots + \varphi_n(t, \mathbf{x}^1_0),$$

Where each $\varphi_j$ is $\mathbb{R}^n$-valued. Now each $\mathbf{x}^i(t)$ is given by the same set of functions with different initial points, so

$$\mathbf{x}^i(t) = \varphi_1(t, \mathbf{x}^i_0) + \cdots + \varphi_n(t, \mathbf{x}^i_0) \quad \text{for each i.}$$

To write the flow, let $\tau_t(\mathbf{x}) = \varphi_1(t, \mathbf{x}) + \cdots + \varphi_n(t, \mathbf{x})$, then

$$\theta_t(\mathbf{X}) = \begin{bmatrix} | & & | \\ \tau_t(\mathbf{x}^1) & \cdots & \tau_t(\mathbf{x}^n) \\ | & & | \end{bmatrix}$$

■

**4.** a) Give an example of complete vector field and an example of incomplete vector field. Explain why it is complete or incomplete.

**Answer:** We will discuss examples from the text, as they are readily available. Let $M = \mathbb{R}^2$, $V = \frac{\partial}{\partial x}$. Then the flow generated by $V$ is

$$\tau_t(x, y) = (x + t, y),$$

and this is a global flow, since given any $(x, y) \in M$, we can see that $\tau_t(x, y)$ determines an integral curve which is defined for all $t \in \mathbb{R}$. Thus $V$ is complete.

For an incomplete example, let $M = \mathbb{R}^2$, $V = x^2 \frac{\partial}{\partial x}$. This corresponds to the ODE system

$$\frac{dx}{dt} = x^2 \qquad\qquad \frac{dy}{dt} \equiv 0$$

which has solution[†]

$$x(t) = \frac{1}{\frac{1}{x_0} - t} \qquad\qquad y(t) = y_0$$

so the flow is given by

$$\theta_t(x, y) = \left( \frac{1}{\frac{1}{x} - t}, y \right).$$

Consider $\theta^{(1,0)}(t)$. Since

$$\theta^{(1,0)}(t) = \left( \frac{1}{1 - t}, 0 \right),$$

then the integral curve cannot be continuously extended past $t = 1$, since $x \to \infty$ as $t \nearrow 1$. ■

b) Let $X$ be a vector field on a manifold $M$, and $\gamma(t)$ an integral curve of $X$ starting at $p \in M$. If $f \in C^\infty(M)$, $f > 0$, find the integral curve of $fX$ starting at $p$.

**Answer:** Assume that $fX$ denotes multiplication, so $(f \cdot X)|_p = f|_p \cdot X|_p$[‡] for any point $p \in M$. Since $\gamma' = X|_\gamma$, then

$$(f \cdot X)|_\gamma = f|_\gamma \cdot X|_\gamma = f(\gamma) \cdot \gamma'.$$

We seek a function $\Gamma : \mathbb{R} \to M$ such that $\Gamma' = (f \cdot X)|_\Gamma$, and $f(\gamma) \cdot \gamma'$ looks like a chain rule derivative:

$$\begin{aligned}(f \circ \gamma)(t) \cdot \gamma'(t) &= f\big(\gamma^1(t), \dots, \gamma^n(t)\big) \cdot \left(\tfrac{\partial \gamma^1}{\partial t} + \cdots + \tfrac{\partial \gamma^n}{\partial t}\right) \\ &= \sum_{i=1}^{n} \frac{\partial \gamma^i}{\partial t} \cdot f\big(\gamma^1(t), \dots, \gamma^n(t)\big)\end{aligned}$$

[†] Assuming $x_0 \neq 0$. Otherwise if $x_0 = 0$ then $x = 0$ for all time.

[‡] This is one of those times when using juxtaposition for multiplication, composition, and evaluation makes me a go a liiiiiiitle bit crazy.

Thus if there exists a function $F$ such that[†]

$$\frac{\partial F}{\partial x^i} = \frac{\partial F}{\partial x^j} = f \quad \text{for all } 1 \le i, j \le n,$$

then we can let $\Gamma = F \circ \gamma$ and we're done. Fortunately, since $f$ is smooth, we can just take indefinite integrals with respect to each $x^i$ and combine the antiderivatives to obtain $F$:

$$\begin{aligned} F(x^1, \dots, x^n) &= \int f(x^1, \dots, x^n)\, dx^1 + g(x^2, \dots, x^n) + C_1 \\ &\vdots \\ &= \int f(x^1, \dots, x^n)\, dx^i + g(x^1, \dots, \hat{x}^i, \dots, x^n) + C_i \\ &\vdots \\ &= \int f(x^1, \dots, x^n)\, dx^n + g(x^1, \dots, x^{n-1}) + C_{n-1} \end{aligned}$$

Combining these together yields a function $F \in C^\infty(M)$ such that

$$(F(\gamma)' = f(\gamma) \cdot \gamma' = fX|_\gamma,$$

So $\Gamma = F \circ \gamma$ is the desired integral curve. ∎

c) Give an explanation why the following statement could be true: for any manifold $M$ and any vector field $X$ on $M$, there is a $f \in C^\infty(M)$, $f > 0$, such that $fX$ is complete.

**Answer:** Anywhere that an integral curve shoots off to infinity in finite time, you can multiply the vector field by a function which goes to zero faster than the integral curve would go to infinity. Anywhere an integral curve goes to a point not in the manifold, you can similarly scale down the vector field so that it takes infinite time to get there.

**5.** Let $M^n$ be a compact manifold which carries $n$ vector fields $X_1, \dots, X_n$ such that $[X_i, X_j] = 0$ for all $i, j = 1, \dots, n$ and $X_1, \dots, X_n$ are pointwise linearly independent. Let $\theta_t^1, \dots, \theta_t^n$ be the flow generated by $X_1, \dots, X_n$ respectively.

(i) Show that $F : \mathbb{R}^n \to M$ defined by $F(x_1, \dots, x_n) = \theta_{x_1}^1 \circ \cdots \circ \theta_{x_n}^n(p)$ for some fixed $p \in M$ is well-defined and a submersion. Conclude that $F$ is a local diffeomorphism.

**Proof** Since $[X_i, X_j] = 0$ for all $i, j = 1, \dots, n$, then the vector fields all commute, so they are invariant under each other's flows. This means that regardless of which vector field we call $X_1, X_2$, etc, we are always referring to the same function when we say $F$ is the composition of all $n$ flows starting at $p$.

$F$ is a submersion because $[X_i, X_j] = 0$ for all $i, j = 1, \dots, n$ and $X_1, \dots, X_n$ are pointwise linearly independent, so each flow is locally moving in a linearly independent direction, which means $F$ has rank $n$. Since $F$ is a map of constant rank between two $n$-dimensional manifolds, then it is a local diffeomorphism. ∎

[†]Here we are using $x^i \dots x^n$ to denote the coordinate functions on $M$.