Math 450B
Homework 5
Dr. Fuller

This assignment will not be collected.

1. Give an example which shows that the condition for differentiability given in Theorem 20 is not necessary.

2. Suppose that $f : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable, and $Df(\mathbf{a}) = \mathbf{0}$ for all $\mathbf{a}$. Prove that f is a constant function.

3. Use Theorem 20 to show that $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x,y) = \begin{cases} \frac{(xy)^2}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$.

4. Determine the continuity and differentiability at $(0,0)$ of $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(Hint: for differentiability, consider $D_{\mathbf{e}}f(0,0)$ for $\mathbf{e} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.)

5. (a) Suppose that the directional derivative $D_{\mathbf{e}}f(\mathbf{a})$ exists. Prove that $D_{-\mathbf{e}}f(\mathbf{a})$ exists and calculate it in terms of the former.
   (b) Show that there is no function $f : \mathbf{R}^n \to \mathbf{R}$ so that for some $\mathbf{a} \in \mathbf{R}^n$ we have $D_{\mathbf{e}}f(\mathbf{a}) > 0$ for all unit vectors $\mathbf{e} \in \mathbf{R}^{\mathbf{n}}$.
   (c) Show that there can, however, be a function $f : \mathbf{R}^n \to \mathbf{R}$ so that for some unit vector $\mathbf{e} \in \mathbf{R}^n$ we have $D_{\mathbf{e}}f(\mathbf{a}) > 0$ for all $\mathbf{a} \in \mathbf{R}^n$.

6. Let $T : \mathbf{R}^n \to \mathbf{R}$ be a linear transformation. Show that $D_{\mathbf{e}}T(\mathbf{a})$ exists for all $\mathbf{a} \in \mathbf{R}^n$ and all unit $\mathbf{e} \in \mathbf{R}^n$, and calculate it.