Math 450b
Homework 2

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1. Determine if the following examples are continuous on their domain. Justify your answers.

   (a) $f : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ given by $f(x, y) = \dfrac{xy}{x^2 + y^2}$.

   Answer: Continuous. Since xy and $x^2 + y^2$ are products and sums of continuous functions, they are continuous. Thus, f is a quotient of continuous functions, and since $\vec{0}$ is not in the domain, the denominator never vanishes. Therefore, f is continuous. ■

   (b) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

   Answer: Not continuous. We already know that f is continuous everywhere except perhaps at $\mathbf{0}$, so let's consider whether f is continuous at that point. Note that whenever $y = x \neq 0$, we have that $f(x, y) = \dfrac{x^2}{2x^2} = \dfrac{1}{2}$; however, whenever $y = -x \neq 0$, we have that $f(x, y) = \dfrac{-x^2}{2x^2} = \dfrac{-1}{2}$, thus f cannot be continuous. To see this, let $\epsilon = \frac{1}{4}$. Now, for all $\delta > 0$, we have that $\left|\left|\left(\frac{\delta}{2}, \frac{\delta}{2}\right) - \mathbf{0}\right|\right| < \delta$, but since

   $$\left|f(\mathbf{0}) - f\left(\tfrac{\delta}{2}, \tfrac{\delta}{2}\right)\right| = \left|0 - \frac{1}{2}\right| > \frac{1}{4} = \epsilon,$$

   then there is no $\delta > 0$ which lets f satisfy the definition of continuity. ■

   (c) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$

   Answer: Continuous. This function is still formed by sums, products, and a quotient of continuous functions, so by the same argument as before, it is continuous everywhere except perhaps at $\mathbf{0}$. To see that it is continuous there as well, let $\epsilon > 0$ be given, and let $\delta = \epsilon$. For $(x, y) \in B(\mathbf{0}, \delta)$ with $(x, y) \neq \mathbf{0}$, we have that $|x|, |y| < \epsilon$. So,

   $$\begin{aligned} |f(x, y)| &= \left|\tfrac{x^2 y}{x^2+y^2}\right| \\ &\leq \left|\tfrac{(x^2+y^2) y}{x^2+y^2}\right| \\ &= |y| \\ &< \epsilon \end{aligned}$$

   Thus, for any $\epsilon > 0$, $(x, y) \in B(\mathbf{0}, \delta)$ implies that $f(x, y) \in B(0, \epsilon)$, so f is continuous. ■

2. Prove that $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) = ||\mathbf{x}||$ is continuous.

   Proof Consider the functions $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $g(\mathbf{x}) = \sum_{i=1}^n x_i^2$ and $h(x) = \sqrt{x}$. Note that g is comprised of sums and products of continuous functions, and h as a well known function, so both are continuous on their domains. Now, the domain of h is the set of nonnegative reals, and the image of g is the same set, thus $h \circ g = f$ is continuous. ■

3. Suppose that $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ satisfies $||f(\mathbf{x}) - f(\mathbf{y})|| \leq K\,||\mathbf{x} - \mathbf{y}||^\alpha$, where $K > 0$ and $\alpha > 0$ are constants. Prove that f is continuous.

Proof Let $\epsilon > 0$ be given, and choose δ such that $K\delta^\alpha = \epsilon$. Thus, for any $\mathbf{x}, \mathbf{y}$ such that $||\mathbf{x} - \mathbf{y}|| < \delta$,

$$||f(\mathbf{x}) - f(\mathbf{y})|| \leq K\, ||\mathbf{x} - \mathbf{y}||^\alpha < K\delta^\alpha = \epsilon,$$

so $||f(\mathbf{x}) - f(\mathbf{y})|| < \epsilon$ and we are done. ■

4. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies:

   (i) for each fixed x_0, the function $y \mapsto f(x_0, y)$ is continuous; and

   (ii) for each fixed y_0, the function $x \mapsto f(x, y_0)$ is continuous.

   Give an example of such an f which is not continuous.

   Answer: $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

   Proof We have already shown that f is not continuous in problem 1(b). Now we will show that conditions (i) and (ii) hold. First, observe that $f(x,y) = f(y,x)$, so it suffices to prove either (i) or (ii). Now we prove (i). If $x_0 = 0$, then $f(x_0, y) \equiv 0$ is constant, so f is continuous. Now for $x_0 \neq 0$, $f(x_0, y) = \frac{x_0 y}{x_0^2 + y^2}$, which consists of sums, products, and one quotient of continuous functions, and the denominator never vanishes. Thus, $f(x_0, y)$ is continuous. ■

5. Professor Doofus mistakenly writes the following on the blackboard.

   Theorem. The following are equivalent.

   (1) $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at all $x \in \mathbb{R}^n$ (with the δ-ϵ definition)

   (2) For every open set $U \subset \mathbb{R}^n$, the image $f(U) \subset \mathbb{R}^m$ is open.

   Give an example with $m = n = 2$ which shows that Doofus is wrong.

   Example. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

   $$f(x,y) = (|x|, |y|).$$

   Claim: f is continuous at all $\mathbf{x} \in \mathbb{R}^n$.

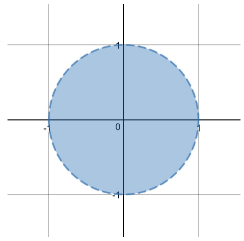
   Proof Let $\epsilon > 0$ be given. Let $\delta = \epsilon$ and let $\tilde{\mathbf{x}} \in \mathbb{R}^n$ be arbitrary. Now, for any $\mathbf{x} \in B(\tilde{\mathbf{x}}, \delta)$,

   $$\begin{aligned} \epsilon &> ||\mathbf{x} - \tilde{\mathbf{x}}|| \\ &= \sqrt{\sum_{i=1}^n (x_i - \tilde{x}_i)^2} \\ &= \sqrt{\sum_{i=1}^n |x_i - \tilde{x}_i|^2} \\ &\geq \sqrt{\sum_{i=1}^n \big||x_i| - |\tilde{x}_i|\big|^2} \\ &= \sqrt{\sum_{i=1}^n (|x_i| - |\tilde{x}_i|)^2} \\ &= ||f(\mathbf{x}) - f(\tilde{\mathbf{x}})|| \end{aligned}$$

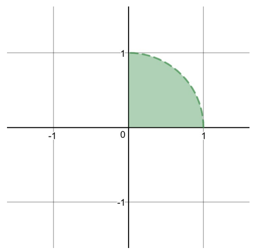
   Thus, if $\mathbf{x} \in B(\tilde{\mathbf{x}}, \delta)$, then $f(\mathbf{x}) \in B(f(\tilde{\mathbf{x}}), \epsilon)$, so f is continuous and (1) holds. ■

Claim: There exists an open set $U \subset \mathbb{R}^2$, such that the image $f(U) \subset \mathbb{R}^2$ is not open.

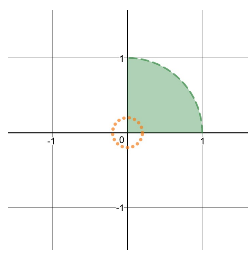
Proof Consider the open set $U = B(\mathbf{0}, 1) \subset \mathbb{R}^2$.



Now, under f, every ordered pair maps either to itself, or to a corresponding ordered pair in the first quadrant (or on its boundary); so the image of U is $f(U) = U \cap I$, where I denotes the closure of the first quadrant.



The set $f(U)$ is not open; since the origin $\mathbf{0} \in f(U)$, but every $B(\mathbf{0}, r)$ contains points in every quadrant, so no open ball $B(\mathbf{0}, r)$ is a subset of $f(U)$.



Therefore, (2) fails. Thus, (1) $\not\Rightarrow$ (2). ■

6. Suppose that $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is continuous, with $\mathbf{a} \in A$ and $f(\mathbf{a}) > 0$. Prove that there exists a $\delta > 0$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in B(\mathbf{a}, \delta) \cap A$.

   Proof Since f is continuous on A, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $||\mathbf{x} - \mathbf{a}|| < \delta$ and $\mathbf{x} \in A$, then $||f(\mathbf{x}) - f(\mathbf{a})|| < \epsilon$. Let $\epsilon = f(\mathbf{a})$. If $||f(\mathbf{x}) - f(\mathbf{a})|| < f(\mathbf{a})$, then $f(\mathbf{x}) \in B(f(\mathbf{a}), f(\mathbf{a}))$, which is the interval $(0, 2f(\mathbf{a}))$. Thus, we are done. ■

7. Suppose that $A \subset \mathbb{R}^n$ is a set which is not closed. Prove that there exists a continuous function $f : A \to \mathbb{R}$ which is unbounded. (Hint: You might find it useful to first show that the set $\mathbb{R}^n - A$ must contain a point in the boundary of A.)

   Lemma (7.1). *If a set A contains all its boundary points, then it is closed.*

   Proof A contains all of its boundary points, so A^{C} contains none of them. That is, for all $\mathbf{x} \in A^{\mathsf{C}}$, $\mathbf{x}$ is not a boundary point, so there exists some $r > 0$ such that $B(\mathbf{x}, r) \subset A^{\mathsf{C}}$. This means that A^{C} is open, by the openness criterion. Furthermore, since A^{C} is open, A is closed. ■

By the way, we have also proved the following:

Corollary (7.2). *If a set A contains none its boundary points, then it is open.*

Okay, now we are ready to prove Exercise 7.

Proof Suppose that $A \subset \mathbb{R}^n$ is not closed. By the contrapositive of Lemma (7.1), A does not contain all of its boundary points. Let $\mathbf{p}$ be a boundary point of A which is not in A. Now, let $f : A \to \mathbb{R}$ be defined as

$$f(\mathbf{x}) = \frac{1}{||\mathbf{x} - \mathbf{p}||}.$$

To see that f is unbounded, observe that for any $B > 0$, $B(\mathbf{p}, \frac{1}{B})$ contains a point in A, so there exists some $\mathbf{a} \in B(\mathbf{p}, \frac{1}{B})$ such that $||\mathbf{a} - \mathbf{p}|| < \frac{1}{B}$, so $f(\mathbf{x}) = \frac{1}{||\mathbf{a}-\mathbf{p}||} > B$. ■