10 (a) $y' + \frac{y}{t} = \cos(t)$

Integrating factor $\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$

multiply both sides $\Rightarrow$ $ty' + y = t\cos(t)$

$[ty]' = t\cos(t)$

$\int [ty]' dt = \int t\cos(t)\, dt$

$ty = t\sin t + \cos t + C$

$y = \sin t + \frac{1}{t}\cos t + \frac{C}{t}$

As $t \to \infty$, y will not converge to anything and will oscillate like $\sin(t)$

(b) Divide by 2 to get

$$y' + \frac{y}{2} = \frac{3t}{2}.$$

The integrating factor is

$$e^{\int (1/2)\, dt} = e^{t/2},$$

so multiply by $e^{t/2}$ to get

$$\begin{aligned} e^{t/2}y' + \frac{e^{t/2}}{2}y &= \frac{3te^{t/2}}{2}, \\ (e^{t/2}y)' &= \frac{3te^{t/2}}{2}, \\ \int (e^{t/2}y)'\, dt &= \int \frac{3te^{t/2}}{2}\, dt, \\ e^{t/2}y &= 3te^{t/2} - 6e^{t/2} + C, \\ y &= 3t - 6 + Ce^{-t/2}, \end{aligned}$$

(c) $ty' - y = t^2 e^{-t}$

$y' - \frac{1}{t} y = t e^{-t}$

Integrating Factor

$\mu(t) = e^{\int -\frac{1}{t} dt} = e^{-\ln t} = e^{\ln \frac{1}{t}} = \frac{1}{t}$

Multiply by $\mu(t)$

$\frac{1}{t} y' - \frac{1}{t^2} y = e^{-t}$

$\left[\frac{1}{t} y\right]' = e^{-t}$

$\int \left[\frac{1}{t} y\right]' dt = \int e^{-t} dt$

$\frac{1}{t} y = -e^{-t} + C$

$y = -e^{-t} + Ct$

11 ~~4~~ Consider the linear DE with initial conditions:

$$\ln(t)y' + y = \cot(t); \qquad y(2) = 3.$$

Use the E/U Theorem for Linear DEs to determine the largest interval where we have a unique solution to the IVP.

Solution: First we divide by $\ln(t)$ to get the DE in the form $y' + p(t)y = g(t)$:

$$y' + \frac{1}{\ln(t)}y = \frac{\cot(t)}{\ln(t)}.$$

We are looking for the largest interval containing the point $t_0 = 2$ so that both $p(t)$ and $g(t)$ are continuous. The largest interval containing $t_0 = 2$ for which $p(t) = \frac{1}{\ln(t)}$ is continuous is $(1, \infty)$, and the largest interval containing $t_0 = 2$ for which $g(t) = \frac{\cot(t)}{\ln(t)}$ is continuous is $(1, \pi)$. Thus the largest interval where we have a unique solution to the IVP is $(1, \pi)$.

12 (a) The function g is continuous on the intervals $0 \le t < 1$ and $t > 1$. Since the initial point 0 is in the interval $0 \le t < 1$, the theorem holds on this interval. Since there is no initial point in the interval $t > 1$, only the existence part of the theorem holds on this interval. Let y_1 be the unique solution on the interval $0 \le t < 1$, and let y_2 be any solution on the interval $t > 1$. First, solve for y_1. We have

$$y_1' + 2y_1 = 1, \qquad y_1(0) = 0.$$

The integrating factor is

$$e^{\int 2\,dt} = e^{2t},$$

so multiply by e^{2t} to get

$$\begin{aligned} e^{2t}y_1' + 2e^{2t}y_1 &= e^{2t}, \\ (e^{2t}y_1)' &= e^{2t}, \\ \int (e^{2t}y_1)'\,dt &= \int e^{2t}\,dt, \\ e^{2t}y_1 &= \tfrac{1}{2}e^{2t} + A, \\ y_1 &= \tfrac{1}{2} + Ae^{-2t}, \end{aligned}$$

where A is a constant. Since $y_1(0) = 0$,

$$\begin{aligned} \tfrac{1}{2} + A &= 0, \\ A &= -\tfrac{1}{2}, \end{aligned}$$

and hence

$$y_1 = \tfrac{1}{2} - \tfrac{1}{2}e^{-2t}.$$

Next, solve for y_2. We have

$$\begin{aligned} y_2' + 2y_2 &= 0, \\ y_2' &= -2y_2. \end{aligned}$$

This is a separable differential equation. Separate the variables to get

$$\begin{aligned} y_2' &= -2y_2 \\ \frac{dy_2}{dt} &= -2y_2 \\ \frac{dy_2}{y_2} &= -2\,dt, \\ \int \frac{dy_2}{y_2} &= \int -2\,dt, \\ \ln y_2 &= -2t + B, \\ y_2 &= Ce^{-2t}, \end{aligned}$$

where B is a constant and $C = e^B$. Now piece together a continuous solution y to the IVP that is defined for all $t \geq 0$ by setting

$$y(t) = \begin{cases} y_1(t) & \text{if } 0 \leq t \leq 1, \\ y_2(t) & \text{if } t \geq 1 \end{cases}$$

and choosing C, if possible, so that $y_2(1) = y_1(1)$. Since $y_1(1) = \frac{1}{2} - \frac{1}{2}e^{-2}$ and $y_2(1) = Ce^{-2}$,

$$\begin{aligned} Ce^{-2} &= \tfrac{1}{2} - \tfrac{1}{2}e^{-2} \\ C &= \tfrac{1}{2}e^2 - \tfrac{1}{2}. \end{aligned}$$

Hence,

$$y(t) = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2t} & \text{if } 0 \leq t \leq 1, \\ \left(\frac{1}{2}e^2 - \frac{1}{2}\right)e^{-2t} & \text{if } t \geq 1. \end{cases}$$

(b) The function p is continuous on the intervals $0 \leq t < 1$ and $t > 1$. Since the initial point 0 is in the interval $0 \leq t < 1$, the theorem holds on this interval. Since there is no initial point in the interval $t > 1$, only the existence part of the theorem holds on this interval. Let y_1 be the unique solution on the interval $0 \leq t < 1$, and let y_2 be any solution on the interval $t > 1$. First, solve for y_1. We have

$$y_1' + 2y_1 = 0, \quad y_1(0) = 1.$$

By (a),

$$y_1 = Ae^{-2t},$$

where A is a constant. Since $y_1(0) = 1$, $A = 1$, so

$$y_1 = e^{-2t}.$$

Next, solve for y_2. We have

$$\begin{aligned} y_2' + y_2 &= 0, \\ y_2' &= -y_2. \end{aligned}$$

This is a separable differential equation. Separate the variables to get

$$\begin{aligned} y_2' &= -y_2 \\ \frac{dy_2}{dt} &= -y_2 \\ \frac{dy_2}{y_2} &= -dt, \\ \int \frac{dy_2}{y_2} &= \int -dt, \\ \ln y_2 &= -t + B, \\ y_2 &= Ce^{-t}, \end{aligned}$$

where B is a constant and $C = e^B$. Now piece together a continuous solution y to the IVP that is defined for all $t \geq 0$ by setting

$$y(t) = \begin{cases} y_1(t) & \text{if } 0 \leq t \leq 1, \\ y_2(t) & \text{if } t \geq 1 \end{cases}$$

and choosing C, if possible, so that $y_2(1) = y_1(1)$. Since $y_1(1) = e^{-2}$ and $y_2(1) = Ce^{-1}$,

$$\begin{aligned} Ce^{-1} &= e^{-2}, \\ C &= e^{-1}. \end{aligned}$$

Hence,

$$y(t) = \begin{cases} e^{-2t} & \text{if } 0 \leq t \leq 1, \\ e^{-1}e^{-t} & \text{if } t \geq 1. \end{cases}$$

13 (a) Let $v = y/x$. Then $y = xv$, so $y' = xv' + v$, and

$$\begin{aligned} xv' + v &= \frac{3(xv)^2 - x^2}{2x(xv)}, \\ xv' + v &= \frac{3v^2 - 1}{2v}, \\ 2xvv' + 2v^2 &= 3v^2 - 1, \\ 2xvv' - v^2 &= -1. \end{aligned}$$

Now let $u = v^2$. Then $u' = 2vv'$, and

$$xu' - u = -1.$$

This is a linear differential equation. Divide by x to get

$$u' - \frac{u}{x} = -\frac{1}{x}.$$

The integrating factor is

$$e^{\int(-1/x)\,dx} = e^{-\ln x} = \frac{1}{x},$$

so multiply by $1/x$ to get

$$\begin{aligned}
\frac{u'}{x}-\frac{u}{x^2}&=-\frac{1}{x^2},\\
\left(\frac{u}{x}\right)'&=-\frac{1}{x^2},\\
\int\left(\frac{u}{x}\right)'dx&=\int-\frac{1}{x^2}\,dt,\\
\frac{u}{x}&=\frac{1}{x}+C,\\
u&=1+Cx,
\end{aligned}$$

where C is a constant. Since $u=v^2$ and $v=y/x$, $u=(y/x)^2=y^2/x^2$. Hence,

$$\begin{aligned}
\frac{y^2}{x^2}&=1+Cx,\\
y^2&=x^2+Cx^3.
\end{aligned}$$

(b) Let $v=x+y$. Then $v'=1+y'$, so

$$v'=v^2.$$

This is a separable differential equation. Separate the variables to get

$$\begin{aligned}
v'&=v^2,\\
\frac{dv}{dx}&=v^2\\
\frac{dv}{v^2}&=dx,\\
\int\frac{dv}{v^2}&=\int dx,\\
-\frac{1}{v}&=x+C,\\
v&=-\frac{1}{x+C},
\end{aligned}$$

where C is a constant. Since $v=x+y$,

$$\begin{aligned}
x+y&=-\frac{1}{x+C},\\
y&=-x-\frac{1}{x+C}.
\end{aligned}$$

(c) Let $v = y^2$. Then $v' = 2yy'$, so

$$v' = \cos v.$$

This is a separable differential equation. Separate the variables to get

$$\begin{aligned}
\frac{dv}{dx} &= \cos v, \\
\frac{dv}{\cos v} &= dx, \\
\sec v \, dv &= dx, \\
\int \sec v \, dv &= \int dx, \\
\ln(\tan v + \sec v) &= x + B, \\
\tan v + \sec v &= Ce^x,
\end{aligned}$$

where B is a constant and $C = e^B$. Since $v = y^2$,

$$\tan y^2 + \sec y^2 = Ce^x.$$

(14) Consider the following autonomous DE:

$$y' = y^2(4 - y^2).$$

Use qualitative information to sketch solution curves to this equation.

(a) Find the equilibrium solutions and classify them as stable, semistable or unstable.

(b) Find a formula for y'' and use this to determine the concavity of solutions for certain values of y.

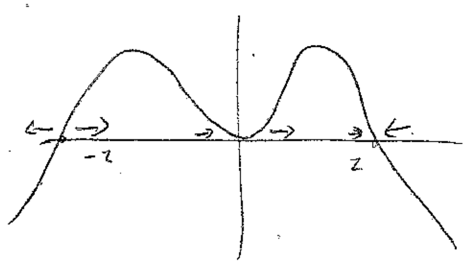
(c) Sketch several graphs of solutions in the ty-plane.

Solution: (a) Setting

$$y' = y^2(4 - y^2) = y^2(2 - y)(2 + y) = 0,$$

we see that the equilibrium solutions are $y_1(t) = -2$, $y_2(t) = 0$, and $y_3(t) = 2$.

Since y' is negative on $-\infty < y < -2$, positive on $-2 < y < 0$, positive on $0 < y < 2$, and negative on $2 < y < \infty$, we have that $y_1(t) = -2$ is unstable, $y_2(t) = 0$ is semistable, and $y_3(t) = 2$ is stable (see graph).

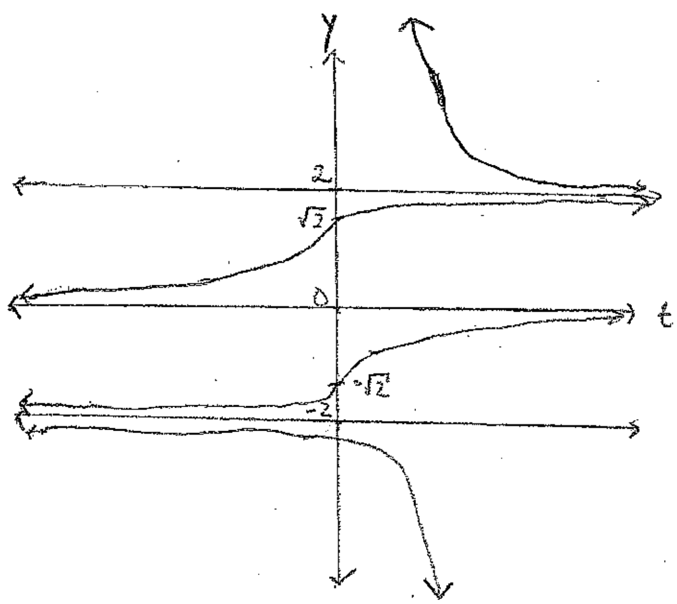


(b) We use the chain and product rules to compute that

$$\begin{aligned} y'' &= 2yy'(4-y^2)+y^2(-2yy') \\ &= 8yy'-4y^3y' \\ &= 4yy'(2-y^2) \\ &= 4y(y^2(4-y^2))(2-y^2) \\ &= 4y^3(2-y)(2+y)(\sqrt{2}-y)(\sqrt{2}+y) \end{aligned}$$

Thus solutions will be concave down for $-\infty < y < -2$, $-\sqrt{2} < y < 0$ and $\sqrt{2} < y < 2$, and concave up for $-2 < y < -\sqrt{2}$, $0 < y < \sqrt{2}$, and $2 < y < \infty$.

(c)



(15) In both cases the mass and spring constant (k) will be the same. The motion is unforced and undamped so the DE describing the position of the block is:

$$m\ddot{x} + kx = 0.$$

Solutions will be of the form

$$x(t) = c_1 \cos\left(\sqrt{\tfrac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\tfrac{k}{m}}\, t\right).$$

The difference between the two cases is the initial conditions

$$x_1(0) = L \quad \dot{x}_1(0) = 0 \qquad \text{vs.} \qquad x_2(0) = 2L \quad \dot{x}_2(0) = 0.$$

So the two solutions are

$$x_1(t) = L \cos\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

$$x_2(t) = 2L \cos\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

The two motions have the same period, but the amplitude of the second is twice that of the first.

(16) The DE describing the position of the block is

$$\ddot{x} + b\dot{x} + kx = 0$$

We need to find k:

$$mg = kx$$

$$9.8 = k(4.9m) \Rightarrow k = 2.$$

The DE is

$$\ddot{x} + b\dot{x} + 2x = 0$$

the characteristic polynomial is $r^2 + br + 2$

roots $r_{1,2} = \dfrac{-b \pm \sqrt{b^2-8}}{2}$

(a) If $b^2 > 8 \Rightarrow$ real roots and overdamped

If $b^2 < 8 \Rightarrow$ complex roots and underdamped

If $b^2 = 8$, $b = \sqrt{8} \Rightarrow$ repeated roots and critically damped.

(b) If $b = 3$ $r_{1,2} = -1, -2$ so the general solution is $x(t) = c_1 e^{-t} + c_2 e^{-2t}$

$$\left.\begin{array}{l} x(0) = 0 \\ \dot{x}(0) = 2 \end{array}\right\} \Rightarrow x(t) = 2e^{-t} - 2e^{-2t}$$

(c) *NOTE: This problem should say "... cross the $\underline{t}$-axis only once"

To cross the t-axis, $x(t) = 0$.

$$2e^{-t} - 2e^{-2t} = 0 \Rightarrow e^{-t} = e^{-2t} \Rightarrow e^{t} = 1,$$

this only happens for $t = 0$, so the block won't cross equilibrium again.

17

(a) Verify that the functions $y_1(x) = -\dfrac{1}{x}$ and $y_2(x) = -\dfrac{1}{x+2}$ are solutions to the differential equation

$$y'' - 2y^3 = 0. \tag{1}$$

(b) Verify that an arbitrary linear combination of y_1 and y_2 is **not** a solution to the differential equation (1).
(c) It seems that this equation doesn't satisfy the principle of superposition from section 3.2 of your textbook. Explain why not.

Solution:
(a) We have $y_1'' - 2y_1^3 = -\frac{2}{x^3} + -2(\frac{-1}{x})^3 = 0$ and $y_2'' - 2y_2^3 = -\frac{2}{(x+2)^3} - 2(\frac{-1}{x+2})^3 = 0$.

(b) We'll show that $y_1 + y_2$ doesn't satisfy the DE:
$(y_1 + y_2)'' - 2(y_1 + y_2)^3 = \frac{-2}{x^3} + \frac{-2}{(x+2)^2} - 2(\frac{-1}{x} + \frac{-1}{x+2})^3 = \frac{12(x+1)}{x^2(x+2)^2} \neq 0$ (omitting much simplification).

(c) The principle of superposition from section 3.2 states that if y_1 and y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$ then so is any linear combination of y_1 and y_2. Note that $y'' - 2y^3 = 0$ can't be written in the form $y'' + p(t)y' + q(t)y = 0$ so the principle of superposition doesn't apply.

18 Consider the IVP $t^2y'' - t(t+2)y' + (t+2)y = 0$, $y(1) = a$, $y'(1) = b$

(a) Verify that $y_1(t) = t$ is a solution to the DE. For which values of a and b is the solution to the IVP a scalar multiple of y_1?

(b) Explain how you know that the fundamental set for this DE will have at least one other solution. Use Theorem 3.2.1 from your book in your argument.

(c) Use reduction of order to find the second solution for the fundamental set for this DE.

(d) Solve the IVP for $a = 1$ and $b = 0$.

Solution:

(a) First verify that $y_1(t) = t$ is a solution to the DE by plugging the function $y_1(t) = t$ in the DE as following. Because $y_1'(t) = 1$ and $y_1''(t) = 0$, when we plug $y_1(t) = t$ in the DE we have:

$$LHS = t^2y_1'' - t(t+2)y_1' + (t+2)y_1 = t^2 \times 0 - t(t+2) + (t+2)t = 0 = RHS.$$

Thus the function $y_1(t) = t$ is a solution to the DE.

Now we consider the IVP:

$$\begin{cases} y'' - 2y' + 2y = 0 \\ y(1) = a, \quad y'(1) = b \end{cases} \tag{2}$$

First if a multiple $y(t) = \lambda t$ of y_1 is a solution to the IVP (2) for some scalar $\lambda \in \mathbb{R}$, then $a = y(1) = \lambda$ and $b = y'(1) = \lambda$, i.e. $a = b$.

On the other hand, if in the IVP (2) $a = b = \lambda \in \mathbb{R}$, then the scalar multiple $y(t) = \lambda t$ of y_1 is a solution to the IVP (2). Now let's verify this. First when we plug the function $y(t) = \lambda t$ in the DE, we have:

$$LHS = t^2y'' - t(t+2)y' + (t+2)y = t^2 \times 0 - t(t+2)\lambda + (t+2)\lambda t = 0 = RHS,$$

since $y'(t) = \lambda$ and $y''(t) = 0$. Thus the function $y(t) = \lambda t$ is a solution to the DE. Then since $y(1) = \lambda \times 1 = \lambda = a$ and $y'(t) = \lambda = b$, in particular $y'(1) = b$, the function $y(t) = \lambda t$ satisfies the initial conditions in the IVP (2). Hence the scalar multiple $y(t) = \lambda t$ of y_1 is a solution to the IVP (2). And by the Uniqueness part of the Existence and Uniqueness Theorem 3.2.1 from the textbook, the function $y(t) = \lambda t$ is THE solution to the IVP (2). Thus when $a = b$, the solution to the IVP (2) is a scalar multiple of y_1. Actually we have proved that the solution to the IVP (2) is a scalar multiple of y_1 if and only if $a = b$.

(b) Now let's explain why the fundamental set for this DE will have at least one other solution by considering the following IVP.

$$\begin{cases} t^2y'' - t(t+2)y' + (t+2)y = 0 \\ y(1) = 1, \quad y'(1) = 2 \end{cases} \tag{3}$$

Then in order to use the Theorem 3.2.1 on the textbook, we rewrite the IVP (3) as the following form:

$$\begin{cases} y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0 \\ y(1) = 1, \quad y'(1) = 2. \end{cases} \tag{4}$$

Now since functions $\frac{t+2}{t}$ and $\frac{t+2}{t^2}$ are both continuous on the interval $(0.5, 1.5)$ containing $t = 1$, by the Theorem 3.2.1, IVP (4) has a solution $y_2(t)$. And since $t \neq 0$ for $t \in (0.5, 1.5)$, $y_2(t)$ is also a solution to the IVP (3). In particular, y_2 is a solution to the DE. Moreover since in the IVP (3) $a = 1 \neq 2 = b$, i.e. $a \neq b$, by the conclusion in the solution of part (a), we have that y_2 can not be a scalar multiple of y_1. Then y_1 and y_2 are linearly independent solutions to the DE. Thus the fundamental set for the DE will have at least one other solution.

(c) Now let's use reduction of order to find the second solution for the fundamental set for this DE. We set $y(t) = v(t)y_1 = v(t)t$, then

$$y' = v't + v, \quad y'' = v''t + 2v'.$$

Substituting for y, y' and y'' in the DE, we have

$$\begin{aligned} & t^2(v''t + 2v') - t(t+2)(v't + v) + (t+2)vt \\ = & t^3v'' + 2t^2v' - t^3v' - t^2v - 2t^2v' - 2tv + t^2v + 2tv \\ = & t^3v'' - t^3v' \\ = & 0. \end{aligned}$$

Thus we have

$$v'' - v' = 0.$$

So $v' = Ce^t$ and hence $v(t) = Ce^t$. It follows that $y(t) = Ce^tt$ is a solution to DE, where C is an arbitrary constant. Then we can take the function $y = te^t$ as the second solution for the fundamental set for the DE.

(d) We need to solve the IVP:

$$\begin{cases} t^2y'' - t(t+2)y' + (t+2)y = 0 \\ y(1) = 1, \quad y'(1) = 0. \end{cases} \tag{5}$$

By the answer in part (c), we assume $y(t) = C_1t + C_2te^t$ is a solution to the IVP (5). Then since $y' = C_1 + C_2e^t + C_2te^t$, by the initial conditions, we have

$$\begin{cases} C_1 + C_2e = 1, \\ C_1 + 2C_2e = 0 \end{cases}$$

So $C_1 = 2$ and $C_2 = -\frac{1}{e}$. Hence $y(t) = 2t - \frac{1}{e}te^t$ is the solution to the IVP (5).

19 Consider the IVP $y'' - 2y' + 2y = 0$, $y(0) = 2$ and $y'(0) = 0$.

(a) Find a fundamental set of *complex* solutions to the DE. Then find a solution to the IVP from among those complex solutions.

(b) Use Euler's Formula to find a fundamental set of *real* solutions to the DE.

(c) Find a solution to the IVP from among your real solutions. Then use Euler's formula to show that this solution is the same as the solution you found in part (a).

Solution:

(a) The characteristic equation to the DE $y'' - 2y' + 2y = 0$ is

$$r^2 - 2r + 2 = 0.$$

By solving the characteristic equation, we obtain $r = 1+i$ or $r = 1-i$. Hence $\{e^{t+it},\ e^{t-it}\}$ is a fundamental set of complex solutions to the DE. Thus the general solution to the DE is given by $y = c_1e^{t+it} + c_2e^{t-it}$, where c_1 and c_2 are arbitrary real constants. Then by using the initial conditions, we obtain:

$$\begin{cases} c_1 + c_2 = 2, \\ (1+i)c_1 + (1-i)c_2 = 0. \end{cases}$$

By solving for c_1 and c_2 in the family of equations, we obtain:

$$\begin{cases} c_1 = 1 + i, \\ c_2 = 1 - i. \end{cases}$$

Hence the solution to the IVP is $y(t) = (1+i)e^{t+it} + (1-i)e^{t-it}$.

(b) By using Euler's formula we have:

$$\begin{cases} e^{t+it} = e^t cos(t) + ie^t sin(t), \\ e^{t-it} = e^t cos(t) - ie^t sin(t). \end{cases}$$

Thus $\{e^t cos(t),\ e^t sin(t)\}$ is a fundamental set of real solutions to the DE.

(c) By using the fundamental set of solutions to the DE obtained in part (b), the general solution to the DE is given by $y(t) = c_1e^t cos(t) + c_2e^t sin(t)$, where c_1 and c_2 are arbitrary real constants. By using the initial conditions in IVP, we obtain

$$\begin{cases} c_1 = 2, \\ c_1 + c_2 = 0. \end{cases}$$

So we have

$$\begin{cases} c_1 = 2, \\ c_2 = -2. \end{cases}$$

Thus the solution to the IVP is $y(t) = 2e^t cos(t) - 2e^t sin(t)$. Now let's verify that this solution is actually the same as the solution we got in part (a). The solution obtained in part (a) is $y(t) = (1+i)e^{t+it} + (1-i)e^{t-it}$. By using Euler's formula, we have

$$\begin{aligned} y(t) &= (1+i)e^{t+it} + (1-i)e^{t-it} \\ &= (1+i)(e^t cos(t) + ie^t sin(t)) + (1-i)(e^t cos(t) - ie^t sin(t)) \\ &= e^t cos(t) + ie^t sin(t) + ie^t cos(t) - e^t sin(t) + e^t cos(t) - ie^t sin(t) - ie^t cos(t) - e^t sin(t) \\ &= 2e^t cost(t) - 2e^t sin(t). \end{aligned}$$

Hence two solutions are the same.

9. Consider the forced spring-mass system governed by the equation

$$y'' + 2y' + 2y = 2\cos(t)$$

(a) What is the mass of the object, the damping coefficient, the spring constant and the external force for this system?

(b) Find the general solution to the differential equation. Identify the fundamental set for solutions to the corresponding homogeneous equation and the particular solution to the nonhomogeneous equation that appears in your general solution.

(c) Find *a different* fundamental set for solutions to the homogeneous equation and *a different* particular solution and express the general solution in terms of these functions.

(d) Solve the DE with initial conditions $y(0) = 2$ and $y'(0) = 0$.

Solution:

(a) For this spring-mass system, the mass of the object is 1, the damping coefficient is 2, the spring constant is 2 and the external force is 2cos(t).

(b) The corresponding homogeneous equation is $y'' + 2y' + 2y = 0$. Its characteristic equation is

$$r^2 + 2r + 2 = 0.$$

By solving for r in this characteristic equation we obtain that $r = -1 + i$ or $r = -1 - i$. Thus $\{e^{-t}cos(t), \ e^{-t}sin(t)\}$ is a fundamental set for solutions to the corresponding homogeneous equation. So the general solution to the corresponding homogeneous equation is given by $y_h(t) = c_1 e^{-t}cos(t) + c_2 e^{-t}sin(t)$. Then let's use the Method of Undetermined Coefficients to find a particular solution to the nonhomogeneous DE. We set