Problem Set #1
Palsson

Trevor Klar

July 3, 2018

P1) Show that given N points in the plane there exists a point that determines at least $\sim\sqrt{N}$ distinct distances as $N \to \infty$.

[Hint: Out of the N points pick two, call them p and q. Draw s number of circles around p and t number of circles around q. Note that the number of distinct circles around p corresponds to the distinct distances from p. Now note that all the other points are on the intersection of circles. How many intersections could you possibly have?]

P2) Show from the definition of an inner product that

$$\langle \alpha_1 f_1 + \beta_1 g_1,\ \alpha_2 f_2 + \beta_2 g_2 \rangle = \alpha_1 \overline{\alpha_2} \langle f_1, f_2 \rangle + \alpha_1 \overline{\beta_2} \langle f_1, g_2 \rangle + \beta_1 \overline{\alpha_2} \langle g_1, f_2 \rangle + \beta_1 \overline{\beta_2} \langle g_1, g_2 \rangle .$$

Proof The properties we have in our definition are (1) positive definiteness, (2) Commutativity with conjugate, and (3) linearity in the first argument.

$$\begin{aligned}
\langle (\alpha_1 f_1 + \beta_1 g_1),\ (\alpha_2 f_2 + \beta_2 g_2) \rangle &= \langle \alpha_1 f_1, (\alpha_2 f_2 + \beta_2 g_2) \rangle + \langle \beta_1 g_1, (\alpha_2 f_2 + \beta_2 g_2) \rangle \\
&= \alpha_1 \langle f_1, (\alpha_2 f_2 + \beta_2 g_2) \rangle + \beta_1 \langle g_1, (\alpha_2 f_2 + \beta_2 g_2) \rangle \\
&= \alpha_1 \overline{\langle (\alpha_2 f_2 + \beta_2 g_2), f_1 \rangle} + \beta_1 \overline{\langle (\alpha_2 f_2 + \beta_2 g_2), g_1 \rangle} \\
&= \alpha_1 \overline{\langle \alpha_2 f_2, f_1 \rangle + \langle \beta_2 g_2, f_1 \rangle} + \beta_1 \overline{\langle \alpha_2 f_2, g_1 \rangle + \langle \beta_2 g_2, g_1 \rangle} \\
&= \alpha_1 \overline{\langle \alpha_2 f_2, f_1 \rangle} + \alpha_1 \overline{\langle \beta_2 g_2, f_1 \rangle} + \beta_1 \overline{\langle \alpha_2 f_2, g_1 \rangle} + \beta_1 \overline{\langle \beta_2 g_2, g_1 \rangle} \\
&= \alpha_1 \overline{\alpha_2 \langle f_2, f_1 \rangle} + \alpha_1 \overline{\beta_2 \langle g_2, f_1 \rangle} + \beta_1 \overline{\alpha_2 \langle f_2, g_1 \rangle} + \beta_1 \overline{\beta_2 \langle g_2, g_1 \rangle} \\
&= \alpha_1 \overline{\alpha_2} \langle f_1, f_2 \rangle + \alpha_1 \overline{\beta_2} \langle f_1, g_2 \rangle + \beta_1 \overline{\alpha_2} \langle g_1, f_2 \rangle + \beta_1 \overline{\beta_2} \langle g_1, g_2 \rangle
\end{aligned}$$

$\blacksquare$

P3) Prove the parallelogram equality

$$||f+g||^2 + ||f-g||^2 = 2\,||f||^2 + 2\,||g||^2$$

Proof

$$\begin{aligned}
||f+g||^2 + ||f-g||^2 &= \langle f+g, f+g \rangle + \langle f-g, f-g \rangle \\
&= \langle f, f+g \rangle + \langle g, f+g \rangle + \langle f, f-g \rangle + \langle -g, f-g \rangle \\
&= \overline{\langle f+g, f \rangle} + \overline{\langle f+g, g \rangle} + \overline{\langle f-g, f \rangle} + \overline{\langle f-g, -g \rangle} \\
&= \overline{\langle f, f \rangle + \langle g, f \rangle} + \overline{\langle f, g \rangle + \langle g, g \rangle} + \overline{\langle f, f \rangle + \langle -g, f \rangle} + \overline{\langle f, -g \rangle + \langle -g, -g \rangle} \\
&= \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle + \langle f, f \rangle + \overline{-\langle g, f \rangle} + \langle -g, f \rangle + \overline{-\langle g, -g \rangle} \\
&= \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle + \langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + -\langle -g, g \rangle \\
&= \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle + \langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + \langle g, g \rangle \\
&= \langle f, f \rangle + \langle f, f \rangle + \langle g, g \rangle + \langle g, g \rangle + \langle f, g \rangle - \langle f, g \rangle + \langle g, f \rangle - \langle g, f \rangle \\
&= ||f||^2 + ||f||^2 + ||g||^2 + ||g||^2 + 0 + 0 \\
&= 2\,||f||^2 + 2\,||g||^2
\end{aligned}$$

$\blacksquare$

P5) Let f and g be elements of $C[0,1]$ defined by $f(x) = 1$ and $g(x) = x$. Find the projection of f in the direction of g.

Answer:

$$
\begin{aligned}
\mathrm{proj}_g(f) &= \langle f, g \rangle \, g \\
&= \left(\frac{1}{1-0} \int_0^1 f(x)\overline{g(x)} dx \right) g(x) \\
&= \left(\int_0^1 x \right) x \\
&= x \left[\frac{1}{2} x^2 \right]_0^1 \\
&= \frac{1}{2} x
\end{aligned}
$$