Fall 2017

1. For X & Y topological spaces define what it means for a function $f: X \to Y$ to be continuous. Give the ϵ, δ definition for metric spaces. Prove that your definitions are equivalent for metric spaces.

__Definition 1__ Let X & Y be topological spaces. We say $f: X \to Y$ is __continuous__ if $\forall$ open sets $\mathcal{U}$ of Y, $f^{-1}(\mathcal{U})$ is open in X.

__Definition 2__ Let (X, d_x) & (Y, d_Y) be metric spaces. We say $f: (X, d_x) \to (Y, d_Y)$ is __continuous at x_0__ if $\forall \epsilon > 0$ $\exists$ $\delta > 0$ s.t. $\forall x \in X$, $d_x(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \epsilon$. If f is continuous $\forall x_0 \in X$ we say f is __continuous__

Now I show that if f is continuous in the 1^{st} sense for (X, d_x), (Y, d_Y), then f is continuous in the 2^{nd} sense: Let $x_0 \in X$ & let $\epsilon > 0$. Then $B_\epsilon(f(x_0))$ is open in Y. So by Definition 1, $f^{-1}(B_\epsilon(f(x_0))$ is open & contains x_0. Thus $\exists$ $\delta > 0$ s.t. $B_\delta(x_0) \subseteq B_\epsilon(f(x_0))$ (by property of the basis). Thus if $d_x(x, x_0) < \delta$ then $x \in B_\delta(x_0)$, so $f(x) \in B_\epsilon(f(x_0))$ & so $d_Y(f(x), f(x_0)) < \epsilon$. Thus f is continuous in the 2^{nd} sense at x_0. Since x_0 was arbitrary, f is continuous in the 2^{nd} sense.

Now I show that if f is continuous in the 2^{nd} sense then f is continuous in the 1^{st} sense. Let $\mathcal{U}$ be open in Y. I want to show $f^{-1}(\mathcal{U})$ is open. Let $x_0 \in f^{-1}(\mathcal{U})$. Then $f(x_0) \in \mathcal{U}$. Since $\mathcal{U}$ is open, $\exists$ $\epsilon > 0$ s.t. $B_\epsilon(f(x_0)) \subseteq \mathcal{U}$ (by property of the basis). Thus $\exists \delta > 0$ s.t. $d_x(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0))$. Thus, $\forall x \in B_\delta(x_0)$, $f(x) \in B_\epsilon(f(x_0)) \subseteq \mathcal{U}$. So $B_\delta(x_0) \subseteq f^{-1}(\mathcal{U})$ which implies $f^{-1}(\mathcal{U})$ is open. Thus these two definitions are equivalent. $\square$

2. Are the following statements true or false? Give a proof or a counterexample as appropriate.

(a) A closed bounded subset of a topological space is compact.

This is false. Consider $\mathbb{R}$ with the discrete topology. Then every subset is both open & closed. Thus $[0,1]$ is closed & bounded in this topology. However, $\{\{x\} : x \in [0,1]\}$ is an open cover with no finite subcover.

(b) The image of a closed subset under a continuous map is closed.

This is false. Let τ_1 be the discrete topology & τ_2 be the indiscrete topology. The consider the identity function: $f : (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)$. This is continuous since all subsets are open in τ_1. In part (a) I argued $[0,1]$ is closed. However $f([0,1]) = [0,1]$ is not closed in $(\mathbb{R}, \tau_2)$ since the only closed sets are $\mathbb{R}$ & $\emptyset$.

(c) If $f : X \to Y$ is a continuous surjection & Y is Hausdorff then so is X.

This is false. Let $\Omega = \{a, b, c\}$ & put the topology on $\tau_\Omega = \{\Omega, \emptyset, \{a,b\}, \{c\}\}$. Let $\Lambda = \{b, c\}$ & let it have topology $\tau_\Lambda = \{\Lambda, \emptyset, \{b\}, \{c\}\}$. Consider the map

$$g : (\Omega, \tau_\Omega) \to (\Lambda, \tau_\Lambda)$$
$$a \mapsto b$$
$$b \mapsto b$$
$$c \mapsto c$$

This is continuous since $g^{-1}(\Lambda) = \Omega$, $g^{-1}(\emptyset) = \emptyset$, $g^{-1}(\{b\}) = \{a,b\}$

& $g^{-1}(\{c\}) = \{c\}$. Also it is clearly a surjection. However, (Λ, τ_Λ) is Hausdorff & (Ω, τ_Ω) is not.

(d) If $f: X \to Y$ is a continuous surjection & X is Hausdorff then so is Y.

This is false. Let τ_1, τ_2, & f be defined as in part (b). Then $(\mathbb{R}, \tau_1)$ is Hausdorff, but $(\mathbb{R}, \tau_2)$ is not.

$(\mathbb{R}, \text{usual}) \longrightarrow (\mathbb{R}, \text{indiscrete})$

(e) If a function between Hausdorff topological spaces is continuous then the preimage of every compact set is compact.

This is false. Let τ_1 be the discrete topology & τ_2 be the standard topology on $\mathbb{R}$. Then define the identity function $h: (\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)$. Since the domain is the discrete topology, h is continuous. Both τ_1 & τ_2 are Hausdorff. However $[0,1]$ is compact in $\mathbb{R}$ (Heine-Borel) but $f^{-1}([0,1])$ is not compact in τ_1 (proved in part (a)). □

often easier to use function def'n of continuous

3. Define what it means for a topological space to be connected.

Definition Let X be a topological space. We say X is disconnected if $\exists$ nonempty open sets U & V of X such that $U \cap V = \emptyset$ & $U \cup V = X$. If X is not disconnected, we say X is connected.

(a) Show that the continuous image of a connected space is connected.

Let X & Y be topological spaces, $f: X \to Y$ be a continuous function, & let X be connected. Suppose $\exists$ nonempty open sets $\mathcal{U}, V$ of $f(X)$ such that $\mathcal{U} \cap V = \emptyset$ & $\mathcal{U} \cup V = f(X)$. Then $\mathcal{U} = \mathcal{U}' \cap f(X)$ & $V = V' \cap f(X)$ where $\mathcal{U}'$ & V' are open in Y. Since f is continuous, $f^{-1}(\mathcal{U}) = f^{-1}(\mathcal{U}') \cap X = f^{-1}(\mathcal{U}')$ & $f^{-1}(V) = f^{-1}(V') \cap X = f^{-1}(V')$ must be open. Since $\mathcal{U}$ & V are nonempty in $f(X) \Rightarrow \exists$ $f(x_1) \in \mathcal{U}$ & $f(x_2) \in V$ so $x_1 \in f^{-1}(\mathcal{U})$ & $x_2 \in f^{-1}(V)$, so both $f^{-1}(\mathcal{U})$ & $f^{-1}(V)$ are nonempty. If $x \in f^{-1}(\mathcal{U}) \cap f^{-1}(V)$ then $f(x) \in \mathcal{U} \cap V = \emptyset$ so $f^{-1}(\mathcal{U}) \cap f^{-1}(V) = \emptyset$. Further, $f^{-1}(\mathcal{U}) \cup f^{-1}(V) = f^{-1}(\mathcal{U} \cup V) = f^{-1}(f(X)) = X$. Thus this forms a separation of X contradicting that X is connected. Hence, the continuous image of a connected space is connected. □

(b) Show that if $H \subset K \subset \overline{H}$ & H is connected, then so is K.

Let X be a topological space, $H \subset K \subset \overline{H}$ & H be connected. Suppose $\exists$ a continuous function $f: K \to \{0,1\}$ where $\{0,1\}$ is given the discrete topology. Then $f|_H$ is also continuous. By Lemma 1 below, since H is connected, $f|_H$ is constant. Without loss of generality, assume $f(H) = 0$. Suppose $f^{-1}(1) \neq \emptyset$. Then $f^{-1}(1)$ is an open set in $K \subset \overline{H}$. Then $f^{-1}(1) \cap H \neq \emptyset$, but this contradicts $f(H) = 0$. So f must be the constant function & by Lemma 1, K is connected. □

Lemma 1 A space X is connected iff every continuous function $f: X \to \{0,1\}$, where $\{0,1\}$ is given the discrete topology is constant.

Proof: Suppose f is not the constant function. Then $f^{-1}(0)$ & $f^{-1}(1)$ are nonempty. Since f is continuous $f^{-1}(0)$ & $f^{-1}(1)$ are open in X. Further, $f^{-1}(0) \cap f^{-1}(1) = \emptyset$ & $f^{-1}(0) \cup f^{-1}(1) = X$, so X is not connected.

Conversely, let $X = \mathcal{U} \cup V$ where $\mathcal{U}$ & V are nonempty open sets with $\mathcal{U} \cap V = \emptyset$. Then define $f: X \to \{0,1\}$ by $f(x) = \begin{cases} 0 & \text{if } x \in \mathcal{U} \\ 1 & \text{if } x \in V \end{cases}$. Then $f^{-1}(\{0,1\}) = X$, $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(0) = \mathcal{U}$, & $f^{-1}(1) = V$ so f is continuous & nonconstant. □

(c) Is $(C[0,1], \sup)$ connected?

Yes! $(C[0,1], \sup)$ is connected because it is path-connected [Lemma 2]. I will first show f is path-connected to 0 for any $f \in C[0,1]$.

I'm not sure all of this is required

Define $p: [0,1] \to C[0,1]$
$t \longmapsto t f(x)$

Note $tf(x) \in C[0,1]$ $\forall t \in [0,1]$ & $p(0)=0$ & $p(1)=f(x)$.
Let $\epsilon > 0$. Then let $\delta = \epsilon / (\sup_{x \in [0,1]} f(x))$, then if $|s-t| < \delta$,

$$\sup_{x \in [0,1]} |sf(x) - tf(x)| = |s-t| \cdot \sup_{x \in [0,1]} f(x) < \epsilon$$

So p is continuous & f & 0 are path-connected.
Now let $g \in C[0,1]$, & let $q: [0,1] \to C[0,1]$ be the path between g & 0. Then $r: [0,1] \to C[0,1]$

$$r \longmapsto \begin{cases} p(1-2t) & t \in [0, \frac{1}{2}] \\ q(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

is continuous by the Pasting Lemma & $r(0) = f$ & $r(1) = g$. So $(C[0,1], \sup)$ is path-connected & thus connected.

Lemma 2 If X is a path-connected topological space, then X is connected.

Proof: Suppose $\exists$ nonempty open sets U, V s.t. $U \cap V = \emptyset$ & $U \cup V = X$. Fix $u \in U$ & $v \in V$. Then $\exists$ path $p: [0,1] \to X$. However, by (a) $p([0,1])$ will be connected (Lemma 3). However $p([0,1]) \cap U$, $p([0,1]) \cap V$ forms a separation of $p([0,1])$, which is a contradiction.

Lemma 3 $[0,1]$ is connected.

Proof: Suppose $\exists$ U, V nonempty open sets such that $U \cap V = \emptyset$ & $U \cup V = [0,1]$, with $0 \in U$. Then consider $\sup \{t \in [0,1] : t \in U\}$. Since $[0,1]$ is closed & bounded, the supremum exists. Therefore, call $s = \sup\{t \in [0,1] : t \in \mathcal{U}\}$. Then $(s,1] \subseteq V$.

* Suppose $s \in \mathcal{U}$. Then $\exists B_\varepsilon(s)$ s.t. $B_\varepsilon(s) \subseteq \mathcal{U} \Rightarrow$ s is not the supremum. However, then $s \in V$. Then $\exists \varepsilon > 0$ s.t. $B_\varepsilon(s) \subseteq V$ & so s is not the supremum. So $s \notin [0,1]$ which is a contradiction. $\square$

Basis Means every $\mathcal{U}^{open}$ is $\bigcup \mathcal{U}_\alpha$

(*) is condition needed

Not sure what is being asked here...

4. Define what it means for a collection of subsets of a set X to be a basis for a topology on X.

Let $\mathcal{B}$ be a collection of subsets of a set X. We say $\mathcal{B}$ is a basis for a topology on X if $\forall x \in X$ $\exists B_1 \in \mathcal{B}$ s.t. $x \in B_1$ & if $x \in B_1 \cap B_2$ for $B_1, B_2 \in \mathcal{B}$ then $\exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$

Give a necessary condition for a collection of subsets to be a basis for a topology (*)

Let $\mathcal{B}$ be a collection of subsets. For $\mathcal{B}$ to be a basis it must be necessary that for every open set $\mathcal{U}$ of X, $\mathcal{U} = \bigcup_{\substack{B \in \mathcal{B} \\ B \subseteq \mathcal{U}}} B$

Let X be the set of subsets of $\mathbb{N}$ (the set of positive integers.) If A is a finite subset of $\mathbb{N}$ & B is a subset of $\mathbb{N}$ whose complement is finite, define a subset [A, B] of X by

$$[A,B] = \{E \subset \mathbb{N} : A \subset E \subset B\}$$

Show that the sets [A, B] form a base for a topology on X.

Let $E \subseteq \mathbb{N}$. Then $[\emptyset, \mathbb{N}]$ is an open set with $\emptyset \subseteq E \subseteq \mathbb{N}$. Now let $E \in [A,B] \cap [C,D]$. Then $A \cup C$ is finite & since $A \subseteq E$ & $C \subseteq E \Rightarrow A \cup C \subseteq E$. Since $E \subseteq B$ & $E \subseteq D \Rightarrow E \subseteq B \cap D$. Since $\mathbb{N} \setminus B$ & $\mathbb{N} \setminus D$ is finite, so is $\mathbb{N} \setminus B \cup \mathbb{N} \setminus D = \mathbb{N} \setminus B \cap D$. Thus $E \in [A \cup C, B \cap D]$. For any $F \in [A \cup C, B \cap D]$, $A \cup C \subseteq F$ so $A \subseteq F$ & $C \subseteq F$. Also $F \subseteq B \cap D$ so $F \subseteq B$ & $F \subseteq D$. Thus $[A \cup C, B \cap D] \subseteq [A,B] \cap [C,D]$

So the sets [A, B] form a base for a topology.

Prove that with this topology X is Hausdorff & disconnected.

Let $E, F \subseteq \mathbb{N}$ & assume $E \neq F$. Without loss of generality $\exists\, n \in E$ where $n \notin F$. Note that $A = \{n\}$ & $B = \mathbb{N}$ satisfies that $[A, B]$ is a basis element & $\{n\} \subseteq E \subseteq \mathbb{N}$ so $E \in [A, B]$. However $\{n\}$ is not a subset of F, so $F \notin [A, B]$

Now we can create $C = \emptyset$ & $D = \mathbb{N} \setminus \{n\}$ & $[C, D]$ will be an element of the basis. Thus $\emptyset \subseteq F \subseteq \mathbb{N} \setminus \{n\}$ so $F \in [C, D]$. Further, since $n \in E$, $E \notin [C, D]$.

Now I show that $[A, B] \cap [C, D] = \emptyset$. Suppose $G \in [A, B] \cap [C, D]$. Then $G \in [A, B]$ so $n \in G$ & $G \in [C, D]$ so $n \notin G$, which is a contradiction. Thus we've found two disjoint sets $[A, B], [C, D]$ of E & F, so X is Hausdorff.

Note that $[\{1\}, \mathbb{N}] \cap [\emptyset, \mathbb{N}/\{1\}] = \emptyset$ by the previous paragraph. $\{1\} \in [\{1\}, \mathbb{N}]$ & $\{2\} \in [\emptyset, \mathbb{N}/\{1\}]$, so they are nonempty open sets. If $E \subseteq \mathbb{N}$. If $1 \in E$ then $E \in [\{1\}, \mathbb{N}]$. If $1 \notin E$ then $E \in [\emptyset, \mathbb{N}/\{1\}]$ so we've found a separation of X, i.e. $[\emptyset, \mathbb{N}/\{1\}] \amalg [\{1\}, \mathbb{N}] = X$. □

Prove that the function $f\colon X \times X \to X$ defined by $f(E_1, E_2) = E_1 \cap E_2$ is continuous.

Let $[A, B]$ be an arbitrary open set in X. Then consider $f^{-1}([A, B])$. Let $(E_1, E_2) \in f^{-1}([A, B])$. Then consider $[A, B \cup E_1] \times [A, B \cup E_2]$. Since $A \subseteq E_1 \cap E_2$

$A \subseteq E_1$ & $A \subseteq E_2$. Since $E_1 \cap E_2 \subseteq B$,
$E_1 \subseteq E_1 \cup B$ & $E_2 \subseteq E_2 \cup B$. Since
$\mathbb{N} \setminus B$ is finite, so B $\mathbb{N}\setminus(E_1 \cup B)$ & $\mathbb{N} \setminus (E_2 \cup B)$
Therefore $(E_1, E_2) \in [A, E_1 \cup B] \times [A, E_2 \cup B]$.

Now let $(F_1, F_2) \in [A, E_1 \cup B] \times [A, E_2 \cup B]$
Then $A \subseteq F_1$ & $A \subseteq F_2$, so $A \subseteq F_1 \cap F_2$. Also
$F_1 \subseteq E_1 \cup B$ & $F_2 \subseteq E_2 \cup B$, so $F_1 \cap F_2 \subseteq B \cup (E_2 \cap E_1)$

Since $E_2 \cap E_1 \subseteq B$, this gives $F_1 \cap F_2 \subseteq B$.
Therefore $[A, E_1 \cup B] \times [A, E_2 \cup B] \subseteq f^{-1}([A,B])$
containing (E_1, E_2), so $f^{-1}([A,B])$ is open
& f is continuous. $\square$

5. Define covering space.

Let X be a topological space. A covering space is a topological space $\tilde{X}$ & a continuous map $p: \tilde{X} \to X$ s.t. $\exists$ an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Delta}$ for X such that $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$ each of which are mapped homeomorphically onto U_α by p $\forall \alpha \in \Delta$.

Path components?

(a) State carefully & prove that covering spaces have the path-lifting property.

Let $p: \tilde{X} \to X$ be covering & let $f: [0,1] \to X$ be a path. Then there exists a unique $F: [0,1] \to \tilde{X}$ s.t. $f = p \circ F$ & $f(0) = F(0)$.

Proof: Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be the open cover of X s.t. $p^{-1}(U_\alpha)$ is a disjoint union of open sets of $\tilde{X}$ each of which are homeomorphic to U_α by p.

Let $t \in [0,1]$. Then $f(t) \in U_t \in \mathcal{U}$. Then consider $\{f^{-1}(U_t)\}_{t \in [0,1]}$ which is an open cover of $[0,1]$ which is compact, so $\exists$ finite subcover $\{f^{-1}(U_{t_i})\}_{i=1}^{n}$. Therefore, we can partition $[0,1]$ by $0 = s_0 < s_1 < \cdots < s_m$ s.t. $[s_j, s_{j+1}] \subseteq f^{-1}(U_{t_i})$ for some i $\forall\, 0 \le j \le m-1$

Note that $F(0)$ is defined. Suppose up to j, $F([0, s_j])$ is defined. Now $f([s_j, s_{j+1}]) \subseteq U_{t_i}$ & $p^{-1}(U_{t_i})$ is a disjoint union of open sets in $\tilde{X}$, so s_j is in one of them, say V_j. Since $[s_j, s_{j+1}]$ is connected, $[s_j, s_{j+1}] \subseteq V_j$. Therefore,

$p^{-1}|_{U_{\alpha_i}}$ is a homeomorphism onto V_{j_i}. Thus
$F = p^{-1}|_{U_{\alpha_i}} \circ f|_{[s_j, s_{j+1}]} : [s_j, s_{j+1}] \to U_{t_i}$ is a continuous
function. Thus we can use induction & define F on $[0,1]$ & $\Rightarrow p \circ F = f$. $\square$

(b) Suppose that $p: \tilde{X} \to X$ is a covering projection & $f: Y \to X$ is a continuous map. Show that there is a map $\tilde{f}: Y \to \tilde{X}$ s.t. $p \circ \tilde{f} = f$ if & only if (with appropriate base points) $f_*(\pi_1(Y)) \subset p_*(\pi_1(\tilde{X}))$

(a) Define compact & sequentially compact.

Definition 1 Let X be a topological space. Let $A \subseteq X$, we say A is compact if for every open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Delta}$ there exists a finite subcover of A

Definition 2 Let X be a topological space. Let $A \subseteq X$. We say A is sequentially compact if $\forall$ (x_n) sequences in A there exists a convergent subsequence.

(b) Show that a compact subspace of a Hausdorff space is closed.

Let X be Hausdorff & let $A \subseteq X$ be compact. Let $x \in X \setminus A$. Since X is Hausdorff $\forall\, a \in A$ $\exists$ disjoint open sets U_x^a & V_x^a of x & a respectively. Then $\{V_x^a\}_{a \in A}$ is an open cover of A so there exists a finite subcover, say $\{V_x^{a_i}\}_{i=1}^n$. Therefore, the corresponding $U_x^{a_i}$ we have $\bigcap_{i=1}^n U_x^{a_i}$ is an open set of x & $\bigcap_{i=1}^n U_x^{a_i} \cap A \subseteq \bigcap_{i=1}^n U_x^{a_i} \cap \bigcup_{i=1}^n V_x^{a_i} = \emptyset$.

Therefore $X \setminus A$ is open, so A is closed. $\square$

(b) Show that if M is a compact metric space & $f: M \to M$ has the property that $d(f(x), f(y)) = d(x,y)$ $\forall\, x, y \in M$, then f must be surjective (you may assume both are sequentially compact)

Good with this now?