Math 460
Homework 4

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1. Show $f(x) = x^4 + x^3 + 1$ is irreducible over $\mathbb{Q}$, then answer the following:

   Proof $f(x)$ is irreducible by the Rational Root Theorem, since the only possible roots in $\mathbb{Q}$ are ± 1, and neither is a root of $f(x)$. ■

   (i) If u is a root of $f(x)$ in an extension field of $\mathbb{Q}$, determine $[\mathbb{Q}(u) : \mathbb{Q}]$ and give a $\mathbb{Q}$-basis for $\mathbb{Q}(u)$.
   Answer Since $\deg f(x) = 4$, then $[\mathbb{Q}(u) : \mathbb{Q}] = 4$ and $1, u, u^2, u^3$ is a $\mathbb{Q}$-basis for $\mathbb{Q}(u)$.

   (ii) Express each of the following elements in terms of a basis (You should not have to solve for the scalars for these): $u^{-1}, (u^2)^{-1}, (u^3)^{-1}$.
   Answer $u^{-1} = -u^3 - u^2, \quad (u^2)^{-1} = -u^2 - u, \quad (u^3)^{-1} = -u - 1.$

   (iii) Express $(1-u)^{-1}$ as a linear combination of the basis elements (You will have to solve for the scalars for this).
   Answer Let's solve.

$$\begin{aligned} 1 &= (1-u)(a + bu + cu^2 + du^3) \\ &= a + (b-a)u + (c-b)u^2 + (d-c)u^3 - du^4 \\ &= a + (b-a)u + (c-b)u^2 + (d-c)u^3 + d(u^3+1) \\ &= (a+d) + (b-a)u + (c-b)u^2 + (2d-c)u^3, \end{aligned}$$

   so $a = b = c$, thus $a + d = 1$ and $2d = a$, which gives $a = b = c = 2/3, d = 1/3$. Thus, $(1-u)^{-1} = \frac{2}{3} + \frac{2}{3}u + \frac{2}{3}u^2 + \frac{1}{3}u^3$.

2. Determine whether the following polynomials are irreducible over the indicated fields. If irreducible, give a reason. If reducible, factor it into irreducible factors.

   (i) $x^{10} + 2x + 6, \mathbb{Q}$
   Answer Irreducible by Eisenstein's Criterion, since 2 divides 2 and 6 but not 1, and 2^2 does not divide 6.

   (ii) $x^4 + 2, \mathbb{Z}_3$
   Answer Reducible. Since 1 is a root and $\mathbb{Z}_3$ is a field, then by the Division Algorithm $x^4 + 2 = (x-1)q(x) + r(x)$, so $r(1) = 0$ and $x - 1 = x + 2$ is a factor. Synthetic division $\mod 3$ will find the following factorization: $x^4 + 2 = (x+2)(x+1)(x^2+1)$

   (iii) $x^4 + 3x^2 + 1, \mathbb{Q}$
   Answer Irreducible since it is positive for all real x, so it has no roots. Further, the Rational Root test gives ± 1 as the only possible rational roots, and computation eliminates $(x \pm 1)$ as factors.

   (iv) $x^5 + 5x^3 + 4$
   Answer Using the linear substitution $\phi\big(p(x)\big) = p(x+1)$, we find that

$$\phi\big(x^5 + 5x^3 + 4\big) = x^5 + 5x^4 + 15x^3 + 25x^2 + 20x + 10,$$

   and since 5 divides all but the leading coefficient and 25 does not divide the constant term, we have that $x^5 + 5x^3 + 4$ is irreducible by Eisenstein's Criterion.

(v) $x^4 + x^2 + 1$

Answer Irreducible. It has no roots, so it has no linear or 3rd degree factors. It is quadratic in form, and the quadratic formula shows that it factors as

$$\left(x^2 + \frac{1}{3} + \frac{\sqrt{3}}{2}i\right)\left(x^2 + \frac{1}{3} - \frac{\sqrt{3}}{2}i\right)$$

which can be further factored in the complex numbers, but the non-real complex numbers are closed under square roots. Thus, $x^4 + x^2 + 1$ can be factored as $(x - z_1)(x - z_2)(x - z_3)(x - z_4)$, where all z_i are non-real complex numbers.

(vi) $x^4 + 2x^2 + 3, \mathbb{Z}_5$

Answer Computation checks that 1 and 2 are not roots, and since the function is even, neither are 3 and 4. Though the polynomial is quadratic in form, it does not factor as $(ax^2 + b)(cx^2 + d)$, since no two elements of $\mathbb{Z}_5$ add to 2 and multiply to 3. If this polynomial does reduce, its factors as the product of two general quadratics, but I don't know how to calculate that.

One thing worth pointing out is that if we apply the linear substitution $x = x + 1$, we find that the polynomial becomes $x^4 + 4x^3 + 8x^2 + 8x + 6$, which is irreducible over $\mathbb{Q}$ by Eisenstein's Criterion. I think that this implies that it is irreducible over $\mathbb{Z}_5$ as well, but I'm not sure.