Math 450b
Homework 9

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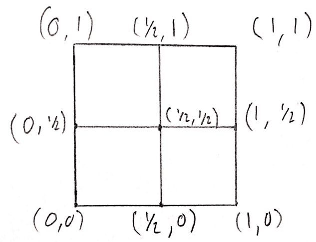
1. Let $P = (\{0, \frac{1}{2}, 1\}, \{0, \frac{1}{2}, 1\})$ be a partition of $A = [0,1] \times [0,1]$, and let $f(x,y) = x^2 + y^2$. Compute $L(f,P)$ and $U(f,P)$.

   Answer:



$$\begin{aligned} L(f,P) &= f(0,0)(\tfrac{1}{4}) + f(\tfrac{1}{2},0)(\tfrac{1}{4}) + f(0,\tfrac{1}{2})(\tfrac{1}{4}) + f(\tfrac{1}{2},\tfrac{1}{2})(\tfrac{1}{4}) \\ &= \tfrac{3}{16} \\ U(f,P) &= f(\tfrac{1}{2},\tfrac{1}{2})(\tfrac{1}{4}) + f(\tfrac{1}{2},1)(\tfrac{1}{4}) + f(0,\tfrac{1}{2})(\tfrac{1}{4}) + f(1,1)(\tfrac{1}{4}) \\ &= \tfrac{5}{4} \end{aligned}$$

2. Give an example of a rectangle $A \subset \mathbb{R}^n$ and functions f and g from A to $\mathbb{R}$ for which $M_A(f) + M_A(g) \neq M_A(f+g)$.

   Answer: Let $f, g : [0, 2\pi] \to \mathbb{R}$ be $f(x) = \sin(x)$, $g(x) = -\sin(x)$. Then

$$M_A(f) + M_A(g) = 1 + 1 \neq M_A(f+g) = 0.$$

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

   Show that f is integrable on $[0,1] \times [0,1]$ and find $\int_A f$.

   Proof Observe that for any subrectangle $S = [a,b] \times [c,d]$, either $a \neq c$ or $a \neq d$, so there exists some $(x,y) \in S$ such that $f(x,y) = 0$. Thus,

$$L(f,P) = 0$$

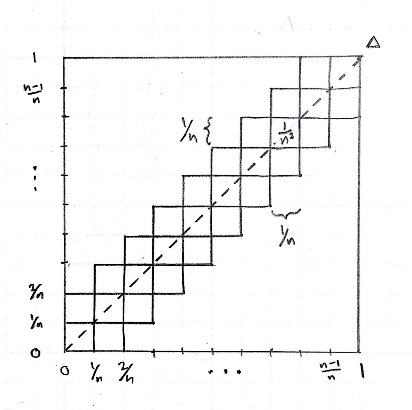
   for any partition P. To show that f is integrable, we will show that for any $\epsilon > 0$, there exists a partition P such that

$$U(f,P) - L(f,P) = U(f,P) < \epsilon.$$

Let $\epsilon > 0$ be given. Choose $n \in \mathbb{N}$ such that

$$\tfrac{3}{n} < \epsilon.$$

Let $P_0 = \{0, \frac{1}{n}, \frac{2}{n}, \dots \frac{n-1}{n}, 1\}$, and let $P = \{P_0, P_0\}$.



Denote $\Delta = \{(x,x) \in [0,1] \times [0,1]\}$, denote $S = \{S_i \in P : S_i \cap \Delta \neq \emptyset\}$ and $S' = P - S$. Note that S contains $(3n-2)$ subrectangles, and $M_{S_i}(f) = 1$ for every S_i. Also, $M_{S_i'}(f) = 0$ for every S_i' Now we calculate $U(f,P)$:

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{3n-2} M_{S_i}(f)\text{vol}(S_i) + \sum_{S'} M_{S_i'}(f)\text{vol}(S_i') \\ &= (3n-2)(1)(\tfrac{1}{n^2}) + 0 \\ &< \tfrac{3n}{n^2} \\ &= \tfrac{3}{n} \\ &< \epsilon. \end{aligned}$$

$\blacksquare$

4. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ 0 & \text{if } x \text{ rational}, y \text{ irrational} \\ {}^1/_q & \text{if } x \text{ rational}, y = {}^p/_q \text{ in lowest terms.} \end{cases}$$

Show that f is integrable, and $\int_{[0,1]\times[0,1]} f = 0$.

Proof To simplify notation, let $g : [0,1] \to \mathbb{R}$ be defined as

$$g(y) = \begin{cases} 0 & \text{if } y \text{ irrational} \\ {}^1/_q & \text{if } y = {}^p/_q \text{ in lowest terms.} \end{cases}$$

Thus,

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ g(y) & \text{otherwise.} \end{cases}$$

We will show that:

(a) $\sup\limits_P L(f,P) = 0.$

Lemma (b) For any $\epsilon > 0$, there are finitely many real numbers $y \in [0,1]$ such that $g(y) > \epsilon$.

(c) For any $\epsilon > 0$, there exists a partition P such that $U(f,P) - L(f,P) < \epsilon$.

Thus by (a) and (c), f is integrable and $\int\limits_{[0,1]^2} f = 0$.

Proof of (a) Observe that for any subrectangle $S = [a,b] \times [c,d]$, there exists an irrational number γ such that $a < \gamma < b$, thus $f(\gamma, c) = m_S(f)\text{vol}(S) = 0$. Since this holds for arbitrary subrectangles, $L(f,P) = 0$ for any partition P. □

Proof of Lemma (b). Let $\epsilon > 0$ be given. For any $g(y) = \frac{1}{n} > \epsilon$, we have that $n < \frac{1}{\epsilon}$, and there are finitely many such $n \in \mathbb{N}$. Also, given any natural number n, there are finitely many fractions $\frac{k}{n} \in [0,1]$; that is, $k = 1, 2, \ldots, n$. Furthermore, $g\left(\frac{k}{n}\right) \geq \frac{1}{n}$ for all such fractions. Thus,

$$\left\{\tfrac{k}{n} : g\left(\tfrac{k}{n}\right) \geq \epsilon, \quad k, n \in \mathbb{N}, \quad k \leq n\right\}$$

is finite. □

Proof of (c) By Lemma (b), let $\{y_1, y_2, \ldots, y_n\}$ denote the rational numbers in $[0,1]$ such that

$$g(y_i) > \tfrac{\epsilon}{2}$$

where $i = 1, \ldots, n$. Note that for ϵ smaller than 2, $y_n = 1$. Let

$$\delta = \min\left\{\frac{\epsilon}{4n}, \frac{|0 - y_1|}{2}, \frac{|y_i - y_{i+1}|}{2}\right\}.$$

This ensures that

$$2\delta \leq \frac{\epsilon}{2n},$$

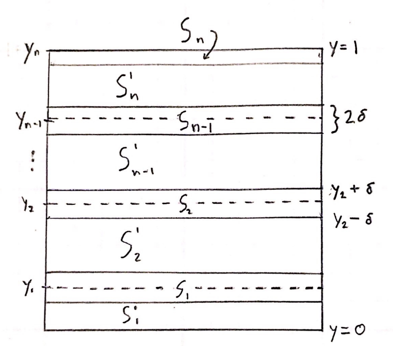
and that P_y (defined below) is truly a partition, with terms ordered as expected. Let

$$\begin{aligned} P_x &= \{0, 1\} \\ P_y &= \{0, (y_1 - \delta), (y_1 + \delta), (y_2 - \delta), (y_2 + \delta), \ldots, (y_{n-1} - \delta), (y_{n-1} + \delta), (1 - \delta), 1\}, \end{aligned}$$

and let $P = \{P_x, P_y\}$. We denote the constituent subrectangles S_i and S_i' as follows:

$$S_i = \begin{cases} [0,1] \times [(y_i - \delta), (y_i + \delta)] & 1 \leq i < n \\ [0,1] \times [(1 - \delta), 1] & i = n \end{cases}$$

$$S_i' = \begin{cases} [0,1] \times [0, (y_1 - \delta)] & i = 1 \\ [0,1] \times [(y_{i-1} + \delta), (y_i - \delta)] & 1 < i \leq n \end{cases}$$



Now, observe that

$$\begin{aligned}
U(f,P) &= \sum_{i=1}^{n} M_{S_i}(f)\mathrm{vol}(S_i) + \sum_{i=0}^{n} M_{S_i'}(f)\mathrm{vol}(S_i') \\
&\leq \sum_{i=1}^{n}(1)2\delta + \sum_{i=0}^{n} \tfrac{\epsilon}{2}\mathrm{vol}(S_i') \\
&\leq \sum_{i=1}^{n} \tfrac{\epsilon}{2n} + \tfrac{\epsilon}{2}\sum_{i=0}^{n} \mathrm{vol}(S_i') \\
&< \sum_{i=1}^{n} \tfrac{\epsilon}{2n} + \tfrac{\epsilon}{2}(1) \\
&= \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

In evaluating $M_{S_i}(f)$ and $M_{S_i'}(f)$ above, we used the fact that f is bounded above by 1, and the fact that every y_i such that $g(y_i) > \frac{\epsilon}{2}$ is contained in exactly one S_i. Thus, $M_{S_i}(f) \leq 1$ and $M_{S'}(f) \leq \frac{\epsilon}{2}$ for all appropriate i.

Therefore, given any $\epsilon > 0$, we have produced a partition P such that $U(f,P) - L(f,P) < \epsilon$, so we are done. ■

5. (a) Let $A \subset \mathbb{R}^n$ be a rectangle, and assume that $f : A \to \mathbb{R}$ is integrable and satisfies $F \geq 0$ on A. Prove that $\int_A f \geq 0$.

   Proof For any subrectangle S of any partition P of the domain A,

   $f(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in S$, so
   $m_S(f) \geq 0$ for any $S \in P$, so
   $L(f,P) \geq 0$ for any partition P of A, so

   since $0 \leq L(f,P) \leq \int_A f$, then we are done. ■

   (b) Assume in addition that f is continuous and f is positive at some point in A. Prove that $\int_A f > 0$.

   Proof Let a denote the assumed element of A such that $f(a) > 0$. Let $\epsilon = f(a)$. Since f is continuous, there exists some $\delta > 0$ such that if $x \in A$ and $||x - a|| < \delta$, then $|f(x) - f(a)| < \epsilon$, so $f(x) > 0$. Now, let P be a partition of A containing the subrectangle

   $$S_0 = \prod_{i=1}^{n} \left[\left(a_i - \tfrac{\delta}{\sqrt{n}}\right), \left(a_i + \tfrac{\delta}{\sqrt{n}}\right)\right].$$

   Since $S_0 \subset B(a,\delta)$,

   $$m_{S_0}(f) > 0,$$

   and by (a),

   $$m_S(f) \geq 0$$

   for all other S, so

   $$L(f,P) < \int_A f$$

   and we are done. ■