Math 360
Section 1.1 Exercises

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June 21, 2018

1. (a) $b * d = e$
   (b) $c * c = b$
   (c) $[(a * c) * e] * a = [c * e] * a = a * a = a$

7. $*$ is clearly not commutative, since $1 * 2 \neq 2 * 1$. Observe the following to see that $*$ is not associative either: $(1 * 2) * 3 = -1 * 3 = -4$, whereas $1 * (2 * 3) = 1 * -1 = 2$.

9. $*$ is commutative, since $\frac{ab}{2} = \frac{ba}{2}$. Also, $*$ is associative, since $(a*b)*c = \frac{ab}{2}*c = \frac{abc}{4} = a*\frac{bc}{2} = a*(b*c)$.

11. $*$ is commutative, since $2^{ab} = 2^{ba}$. However, $*$ is not associative, since $(3 * 5) * 7 = 2^{15} * 7 = 2^{(7)(2^{15})}$, but $3 * (5 * 7) = 3 * 2^{35} = 2^{(3)(2^{35})}$.

13. For a set of 2 elements, consider the table:

$*$	a	b
a	x_1	x_2
b	x_2	x_3

There are three unique outputs, x_1, x_2, x_3, and each one can map to one of two possible values, so there are $2^3 = 8$ possible commutative binary operations on a set of 2 elements.

For 3 elements, the table gives

$*$	a	b	c
a	x_1	x_2	x_4
b	x_2	x_3	x_5
c	x_4	x_5	x_6

Thus, there are 6 unique outputs, which can each take 3 possible values. Therefore there are 3^6 possible commutative binary operations on a set of 3 elements.

As we increase the number of elements from $n - 1$ to n, we see that the number of unique outputs increases by n. Thus, the number of unique outputs for a set of n elements is given by the n-th triangle number. Thus, the number of possible commutative binary operations on a set of n elements is given by

$$n^{T_n},$$

where $T_n = \frac{(n)(n+1)}{2}$ is the n-th triangle number.

14. A binary operation $*$ on a set S is *commutative* if and only if, for all $a, b \in S$, we have $a * b = b * a$.

15. A binary operation $*$ on a set S is *associative* if and only if, for all $a, b, c \in S$, we have $(a*b)*c = a*(b*c)$.

16. A subset H of a set S is *closed* under $*$ if and only if, for all $a, b \in H$, we have $(a * b) \in H$.

For Exercises 17-22, determine if $*$ is a binary operation on S. If not, state whether Condition 1, Condition 2, or both are violated.

17. On $\mathbb{Z}^+$, define $*$ by $a * b = a - b$. **Answer:** No. Condition 2 is violated since $1, 2 \in \mathbb{Z}^+$, but $1 * 2 = -1 \notin \mathbb{Z}^+$.

19. On $\mathbb{R}$, define $*$ by $a * b = a - b$. This is of course a binary operation. For every subtraction problem, there is exactly one answer, and every difference of real numbers is a real number.

21. On $\mathbb{Z}^+$, define $a * b = c$, where c is at least 5 more than $a + b$. **Answer:** No. Condition 1 is violated, since there are infinitely many real numbers c such that $c \geq a + b + 5$.

23. Let H be the subset of $M_2(\mathbb{R})$ consisting of all matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{R}$. Is H closed under

    a. matrix addition?

    b. matrix multiplication?

    Answer to a: Yes. Adding two arbitrary elements of H, we find $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a + c & -(b + d) \\ b + d & a + c \end{bmatrix}$, which is an element of H.

    Answer to b: Yes. Multiplying two arbitrary elements of H, we find $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix}$, which is an element of H.

24. a. F
    b. T
    c. F
    d. F
    e. F
    f. T
    g. F
    h. F
    i. T
    j. F

26. Prove that if $*$ is and associative and commutative binary operation on a set S, then

$$(a * b) * (c * d) = [(d * c) * a] * b$$

for all $a, b, c, d \in S$.

Proof

$$\begin{aligned} (a * b) * (c * d) &= (c * d) * (a * b) && \text{commutative property} \\ &= (d * c) * (a * b) && \text{commutative property} \\ &= [(d * c) * a] * b && \text{associative property} \end{aligned}$$

■

In 27 and 28, prove or give a counterexample.

27. Every binary operation on a set consisting of a single element is both commutative and associative.

Proof Denote our singleton set as $S = \{e\}$, and let $*$ be a binary operation on S. Applying the definition of commutativity, we can see that $*$ is commutative: for all $a, b \in S$, we have that $a * b = b * a$. To see this, observe that since $a \in S, a = e$. Also, since $b \in S, b = e$. Thus, $a * b = e * e = b * a$.

We can apply the definition of associativity similarly if we observe that $e * e$ can only have one possible result: $e * e = e$. Thus, for all $a, b, c \in S$,

$$\begin{aligned} (a * b) * c &= (e * e) * e \\ &= e * e \\ &= e * (e * e) \\ &= a * (b * c) \end{aligned}$$

and we are done. ■

28. Every commutative binary operation on a set having just two elements is associative.

Counterexample: Consider the set $S = \{\blacksquare, \square\}$, with $*$ defined by the following table:

$*$	$\blacksquare$	$\square$
$\blacksquare$	$\square$	$\blacksquare$
$\square$	$\blacksquare$	$\blacksquare$

We will show that $(\square * \blacksquare) * \blacksquare \neq \square * (\blacksquare * \blacksquare)$, and thus the associative property does not hold.

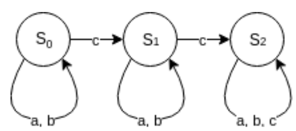
$$(\square * \blacksquare) * \blacksquare = \blacksquare * \blacksquare = \square,$$

however,

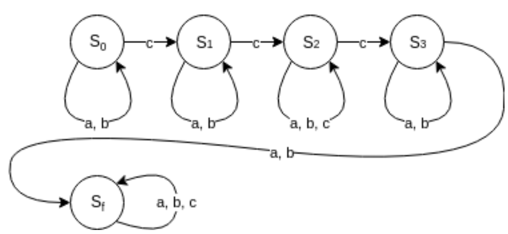
$$\square * (\blacksquare * \blacksquare) = \square * \square = \blacksquare.$$

Thus $*$ on S is not commutative (observe that the table is symmetric), but not associative. $\blacksquare$

41. This is the state diagram for a machine which determines if the input string has at least two c's. If the final state is s_2, then the input string has at least two c's.



42. This is the state diagram for a machine which determines if the input string has exactly 3 c's. If the final state is s_3, then the input string has exactly 3 c's. Note that if the final state is s_f, then the input string has greater than 3 c's, and if the final state is s_0, s_1, or s_2, then the input string has less than 3 c's.



43. This is the state diagram for a machine which determines if the number of 1's in the input string is congruent to 0, 1, or 2 modulo 3. s_0 is the initial state, and corresponds to $0 \equiv 0 \bmod 3$. The subscript of the final state gives the result.