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Homework 3

1. Let μ be a Lebesgue measure and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable subsets of $[0,1]$. Assume the set B consists of those points $x \in [0,1]$ that belong to infinitely many of the A_n.

(i) Prove that B is Lebesgue-measurable.

Proof Let $x \in B$. Then x is in infinitely many of the A_n; so for every $k \geq 1$, $x \in A_n$ for some $n \geq k$. That is, $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$ and in fact, $B = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$. This is a Borel set, so it is Lebesgue-measurable. ∎

(ii) Prove that if $\mu(A_n) > \delta > 0$ for every $n \in N$, then $\mu(B) \geq \delta$.

Proof Let $B_k = \bigcup_{n=k}^{\infty} A_n$. Since $A_k \subseteq B_k$, then $\delta < \mu(A_k) \leq \mu(B_k)$, for all k. Now consider $\bigcap_{k=1}^{M} B_k$. Since

$$\begin{aligned} B_j \cap B_k &= \left(\bigcup_{n\geq k} A_n\right) \cap \left(\bigcup_{n\geq j} A_n\right) \\ &= \bigcup_{n\geq \max(j,k)} A_n \\ &= B_{\max(j,k)}, \end{aligned}$$

then $\bigcap_{k=1}^{M} B_k = B_M$. Then $B_k \searrow B$ and $\mu(B_1) < 1$, so $\lim_{n\to\infty} \mu(B_n) = \mu(B) \geq \delta$. ∎

(iii) Prove that if $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(B) = 0$.

Proof Let $B_k = \bigcup_{n=k}^{\infty} A_n$. Now

$$\mu(B_k) = \mu\left(\bigcup_{n=k}^{\infty} A_n\right) \leq \sum_{n=k}^{\infty} \mu(A_n).$$

Since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then $\mu(A_n) \to 0$, which means the tail of the sum also goes to 0 as $k \to \infty$. Thus $\mu(B_k) \to 0$, and $B_k \searrow B$ and $\mu(B_1) > 1$, so

$$\lim_{n\to\infty} \mu(B_n) = \mu(B) = 0.$$

∎

(iv) Give an example where $\sum_{n=1}^{\infty} \mu(A_n) = \infty$, but $\mu(B) = 0$. **Answer:** Let $A_n = [0, 1/n]$. Then $\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} 1/n = \infty$, but $B = \{0\}$ and $\mu(B) = 0$.

2. Prove that if $A \subset \mathbb{R}$ is Lebesgue-measurable with $\mu(A) > 0$, then there is a subset of A that is not Lebesgue-measurable.

Lemma. If $A \subset \mathbb{R}$ is Lebesgue-measurable with $\mu(A) > 0$, then there exists a subset $\widetilde{A} \subset A$ with $\widetilde{A}$ bounded and $\mu\left(\widetilde{A}\right) > 0$. **Proof:** Suppose not. Then for every $\widetilde{A} \subset A$, either $\widetilde{A}$ unbounded or $\mu\left(\widetilde{A}\right) = 0$. If we consider the sets

$$A_n = \big\{[n, n+1) \cap A\big\}_{n\in\mathbb{Z}},$$

then each A_n is bounded, thus it has measure zero. Since each $A - n$ is measurable and $A = \coprod\limits_{n\in Z} A_n$, then $0 = \sum\limits_{n\in\mathbb{Z}} \mu(A_n) = \mu(A) > 0$, contradiction. □

Proof By the lemma, without loss of generality we can assume that A is bounded, so let $[-a, a] \supset A$. Define an equivalence relation on A as follows. For all $x, y \in A$,

$$x \sim y \text{ if } \exists\, q \in (\mathbb{Q} \cap [-a, a]) \text{ such that } x - y = q$$

A little thought will show that $\sim$ is reflexive, symmetric, and transitive. Thus the collection of all equivalence classes $\{[x] | x \in A\}$ is a partition of A. Define V by choosing exactly one representative of each equivalence class. Then for each $x \in A$, there exists a unique $y \in V$ such that $x \sim y$, and $V \subset A$. Now all the remains is to show that V is not Lebesgue-measurable.

Suppose for contradiction that V is measurable, and consider

$$\{V + q \,|\, q \in (\mathbb{Q} \cap [-a, a])\}.$$

(From now on in this proof, we assume $q \in (\mathbb{Q} \cap [-a, a])$.) Since every $a \in A$ has a $y \in V$ such that $x \sim y$, then $A \subseteq \bigcup_q (V + q)$. And since $A \subseteq [-a, a]$ and every $q \in [-a, a]$, then $\bigcup_q (V + q) \subseteq [-2a, 2a]$. Thus by monotonicity,

$$0 < \mu(A) \le \mu\left(\bigcup_q (V + q)\right) \le 4a < \infty.$$

Since $V + q_1$ and $V + q_2$ are disjoint and measurable for all $q_1 \neq q_2$, then $\mu\left(\coprod_q (V + q)\right) = \sum_q \mu(V + q) = \sum_q \mu(V)$ since Lebesgue measure is translation-invariant. Now on one hand, if $\mu(V) > 0$ then $\sum_q \mu(V) = \infty$, but $\sum_q \mu(V) < \infty$. On the other hand, if $\mu(V) = 0$ then $\sum_q \mu(V) = 0$, but $\sum_q \mu(V) > 0$. Thus $0 < \mu(V) = 0$, contradiction. Therefore V cannot be measurable. ■

3. Let μ be the Lebesgue measure on $\mathbb{R}$. Construct a Borel set $A \subset \mathbb{R}$ such that $\mu(A) > 0$ and $\mu(A \cap I) < \mu(I)$ for every non-degenerate interval $I \subset \mathbb{R}$.

Proof Let r_k be an enumeration of the rationals, and let

$$A = (-100, -100) \setminus \bigcup_{k=1}^{\infty} B(r_k, 1/2^k).$$

Let I be any non-degenerate interval, let $a = \inf I$, and let $b = \sup I$. Then $(a, b) \subseteq I$, where $a < b$. In the case that $a = -\infty$ or $b = \infty$, then $\mu(I) = \infty$, and $\mu(A \cap I) \leq \mu(A) \leq 200$, so we're done. So consider the case where $a, b \in \mathbb{R}$. Since $\mu(I) = b - a$ and $(a, b) \subseteq I$, we will show that $\mu((a, b) \cap A) < b - a$. Choose some $r_k \in (a, b)$. Then $B(r_k, 1/2^k) \cap (a, b)$ is open, so there exists some $\epsilon > 0$ such that $B(r_k, \epsilon) \subset B(r_k, 1/2^k) \cap (a, b)$. Now, since $B(r_k, \epsilon) \subset B(r_k, 1/2^k) \subset A^{\complement}$ but $B(r_k, \epsilon) \subset (a, b)$, then

$$I \cap A \subseteq I \setminus B(r_k, \epsilon) \subset I,$$

so since all these sets are measurable,

$$\mu(I \cap A) \leq \mu(I) - \mu(B(r_k, \epsilon)) < \mu(I).$$

■

4. Let $A \subset \mathbb{R}$ be a Lebesgue-measurable set. Prove that the set

$$B = \bigcup_{x \in A} [x - 1, x + 1]$$

is Lebesgue-measurable.

Proof Observe that $B = \bigcup_{x \in A} B_1(x) \cup A - 1 \cup A + 1$. The union of balls is Borel, and translation invariance of Lebesgue measure tells us that the other two sets are measurable as well. Thus B is a union of 3 measurable sets, and thus measurable. ■