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Homework 1

Problem1. Let X be a topological space and let μ be a measure on X such that $\mu(X) < \infty$ (in that case μ is said to be a finite measure on X). A family of μ-measurable functions $f_n : X \to \mathbb{R}$ is called **uniformly integrable in** X**,** if for any $\epsilon > 0$, there exists $M > 0$ such that

$$\int_{\{x \,:\, |f_n(x)| > M\}} |f_n(x)| d\mu < \epsilon, \quad \text{for all} \quad n = 1, 2, \ldots.$$

Similarly $\{f_n\}$ is called **uniformly absolutely continuous** if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\mu-$measurable set $A \subset X$ with $\mu(A) < \delta$ one has

$$\left| \int_A f_n(x) d\mu \right| < \epsilon, \quad \text{for all} \quad n = 1, 2, \ldots.$$

Prove that $\{f_n\}$ is uniformly integrable iff

$$\sup_n \int_X |f_n(x)| d\mu < \infty,$$

and $\{f_n\}$ is uniformly absolutely continuous.

Proof ($\Longrightarrow$) Suppose $\{f_n\}$ is uniformly integrable, and let $\varepsilon > 0$ be given. Then there exists M such that the uniform integrability property holds for f_n for all n. Let $\delta = \frac{\varepsilon}{M}$. Then for any μ-measurable set A with $\mu(A) < \delta$,

$$\begin{aligned}
\left| \int_A f_n \, d\mu \right| &= \left| \int_{A \cap \{|f_n| > M\}} f_n \quad d\mu + \int_{A \cap \{|f_n| \le M\}} f_n \quad d\mu \right| \\
&\le \left| \int_{A \cap \{|f_n| > M\}} f_n \quad d\mu \right| + \left| \int_{A \cap \{|f_n| \le M\}} f_n \quad d\mu \right| \\
&\le \int_{A \cap \{|f_n| > M\}} |f_n| \quad d\mu + \int_{A \cap \{|f_n| \le M\}} |f_n| \quad d\mu \\
&< \varepsilon + M\mu(A) \\
&< \varepsilon + M\delta \\
&= 2\varepsilon
\end{aligned}$$

and after rescaling and observing that n was arbitrary, we've shown that $\{f_n\}$ is uniformly absolutely continuous. Now to see that $\sup_n \int_X |f_n| \, d\mu < \infty$, choose any $\varepsilon > 0$ and let $M > 0$ so that the uniform integrability property hold for all f_n. Then for all n,

$$\int_X |f_n| \, d\mu = \int_{\{|f_n| > M\}} |f_n| \, d\mu + \int_{\{|f_n| \le M\}} |f_n| \, d\mu < \varepsilon + M\mu(X),$$

which is finite and constant with respect to n, so the supremum over n is finite as well. ■

Proof ($\Longleftarrow$) Let $\varepsilon > 0$. Since $\{f_n\}$ is uniformly absolutely continuous, then there exists $\delta > 0$ such that for all $A \subset X$ with $\mu(A) < \delta$, we have

$$\left|\int_A f_n \, d\mu\right| < \varepsilon \quad \forall n.$$

Let $M = \dfrac{1}{\delta} \sup_n \displaystyle\int_X |f_n| \, d\mu$. Since

$$\mu\{|f_n| > M\} \leq \frac{1}{M} \int_{\{|f_n|>M\}} |f_n| \, d\mu \leq \frac{1}{M} \sup_n \int_X |f_n| \, d\mu = \delta,$$

then $\left|\displaystyle\int_{\{|f_n|>M\}} f_n \, d\mu\right| < \varepsilon$, so

$$\begin{aligned}
\int_{\{|f_n|>M\}} |f_n| \, d\mu &= \int_{\{f_n>M\}} f_n^+ \, d\mu + \int_{\{f_n<-M\}} f_n^- \, d\mu \\
&= \left|\int_{\{f_n>M\}} f_n \, d\mu\right| + \left|\int_{\{f_n<-M\}} f_n \, d\mu\right| \\
&= \left|\int_{\{|f_n|>M\}} f_n \, d\mu\right| + \left|\int_{\{|f_n|>M\}} f_n \, d\mu\right| \\
&= 2\varepsilon
\end{aligned}$$

■

Problem2. Let X be a topological space and let μ be a finite measure on X. Let $f, f_n \colon X \to \mathbb{R}$ be $\mu-$summable on X such that the point-wise convergence $f_n(x) \to f(x)$ holds $\mu-$a.e. in X. Prove that $\{f_n\}$ is uniformly integrable iff

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)| d\mu = 0.$$

Proof ($\implies$) Let $\varepsilon > 0$. First we establish a few bounds.

(i) Since f is μ-summable, there exists $\delta_1 > 0$ such that if $\mu(A) < \delta_1$, then $\left|\int_A f\, d\mu\right| < \varepsilon_1$, where $\varepsilon_1 = \frac{\varepsilon}{6}$.

(ii) Since $\{f_n\}$ is a uniformly integrable sequence, then it is uniformly absolutely continuous, so there exists $\delta_2 > 0$ such that if $\mu(A) < \delta_2$, then $\left|\int_A f_n\, d\mu\right| < \epsilon_1$ for all n.

(iii) Since $\mu(X) < \infty$, then by Egoroff's Theorem there exists a set $A \subset X$ such that $f_n \to f$ uniformly on A, and $\mu(A^\complement) < \min(\delta_1, \delta_2)$.

(iv) Since $f_n \to f$ uniformly on A, then there exists $N > 0$ such that for all $n > N$, we have $|f_n - f| < \varepsilon_2$, where $\varepsilon_2 = \frac{\varepsilon}{3\mu(A)}$.

Now we apply the results above. For all $n > N$,

$$\begin{aligned}
\int_X |f_n - f|\, d\mu &= \int_A |f_n - f|\, d\mu + \int_{A^\complement} |f_n - f|\, d\mu \\
&\le \varepsilon_2 \mu(A) + \int_{A^\complement} |f_n - f|\, d\mu && \text{by (iv)} \\
&\le \varepsilon_2 \mu(A) + \int_{A^\complement} |f_n|\, d\mu + \int_{A^\complement} |f|\, d\mu && \Delta \text{ ineq.} \\
&= \varepsilon_2 \mu(A) + \left(\left| \int_{A^\complement \cap \{f_n > 0\}} f_n\, d\mu \right| + \left| \int_{A^\complement \cap \{f_n \le 0\}} f_n\, d\mu \right| \right) + \left(\left| \int_{A^\complement \cap \{f > 0\}} f\, d\mu \right| + \left| \int_{A^\complement \cap \{f \le 0\}} f\, d\mu \right| \right) \\
&\le \varepsilon_2 \mu(A) + \left(\left| \int_{A^\complement \cap \{f_n > 0\}} f_n\, d\mu \right| + \left| \int_{A^\complement \cap \{f_n \le 0\}} f_n\, d\mu \right| \right) + 2\varepsilon_1 && \text{by (i)} \\
&\le \varepsilon_2 \mu(A) + 2\varepsilon_1 + 2\varepsilon_1 && \text{by (ii)} \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

Thus $\int_X |f_n - f|\, d\mu \to 0$ as $n \to \infty$. ∎

Proof ($\Longleftarrow$) We will show that (i) $\{f_n\}$ is uniformly absolutely continuous and (ii) $\sup_n \int_X |f_n|\,d\mu < \infty$.

(i) Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} \int_X |f_n - f|\,d\mu = 0$, then there exists $N > 0$ such that if $n > N$, then $\int_X |f_n - f|\,d\mu < \varepsilon$.

Case I ($n > N$): Suppose $n > N$. Since f is μ-summable, then there exists $\delta_0 > 0$ such that for any $A \subset X$ with $\mu(A) < \delta_0$, then $\int_A |f|\,d\mu < \varepsilon$. Thus we find that

$$\varepsilon > \int_X |f_n - f|\,d\mu \geq \int_A |f_n - f|\,d\mu \geq \int_A |f_n|\,d\mu - \int_A |f|\,d\mu$$

and considering the left and right hand sides from above, we see

$$\left|\int_A f_n\,d\mu\right| \leq \int_A |f_n|\,d\mu < \int_A |f|\,d\mu + \varepsilon = 2\varepsilon.$$

Case II ($n \leq N$): For each $n = 1, \ldots, N$ we know that f_n is μ-summable, so there exists δ_n such that if $\mu(A) < \delta_n$, then

$$\left|\int_A f_n\,d\mu\right| < \varepsilon.$$

Thus we set $\delta = \min(\delta_0, \ldots, \delta_N)$ and find that for all n, if $\mu(A) < \delta$, then $\left|\int_A f_n\,d\mu\right| < \varepsilon$. $\square$

(ii) Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} \int_X |f_n - f|\,d\mu = 0$, then there exists $N > 0$ such that if $n > N$, then

$$\varepsilon > \int_X |f_n - f|\,d\mu \geq \int_X |f_n|\,d\mu - \int_X |f|\,d\mu,$$

so $\int_X |f_n|\,d\mu < \int_X |f|\,d\mu + \epsilon$. Thus

$$\sup_n \int_X |f_n|\,d\mu \leq \max\left(\int_X |f|\,d\mu + \epsilon, \int_X |f_1|\,d\mu, \ldots, \int_X |f_N|\,d\mu\right) < \infty.$$

$\blacksquare$

Problem3. Let X be a topological space and let μ be a finite measure on X. Let $f_n \colon X \to \mathbb{R}$ be $\mu-$measurable such that

$$\sup_n \int_X |f_n(x)|^{1+\delta} d\mu < \infty$$

for some $\delta > 0$. Prove that $\{f_n\}$ is uniformly integrable.