Math 501
Homework 5

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1. Let $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ denote the unit circle in $\mathbb{R}^2$. Let $f : [0, 1) \to S^1$ be $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Show that f is not a homeomorphism.

   Proof Let $U = B((1, 0), 1) \cap S^1$.

   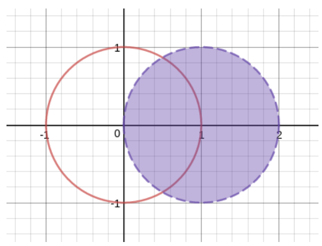

   Figure 1: S^1 in red, and $B((1, 0), 1)$ in purple.

   By definition, U is open under the subset topology in $\mathbb{R}^2$. Now, consider the preimage $f^{-1}(U)$. Since f is a parameterization of the unit circle which begins at $(0, 1)$ and proceeds counter-clockwise, approaching $(0, 1)$ again as $t \to 1$; then $f^{-1}(U) = [0, \frac{1}{6}) \cup (\frac{5}{6}, 1)$. We have already shown that intervals of the form $[a, b)$ are not open in the usual topology, thus we have an open set U whose preimage in f is not open. Therefore, f is not continuous, and not a homeomorphism. ■

2. Prove that the following are both homeomorphic to $\mathbb{R}^2$ with the usual topology: (i.) the open square $\{(x, y) : 0 < x < 1, 0 < y < 1\}$; (ii.) the open ball $B(0, 1)$.

   (i.) **Proof** Let S_q denote the open square $\{(x, y) : 0 < x < 1, 0 < y < 1\}$. Let $t : [0, 1) \to \mathbb{R}$ be defined as $t(x) = \tan(\pi x - \frac{\pi}{2})$. Let $f : S_q \to \mathbb{R}^2$ be defined as $f(x, y) = (t(x), t(y))$.

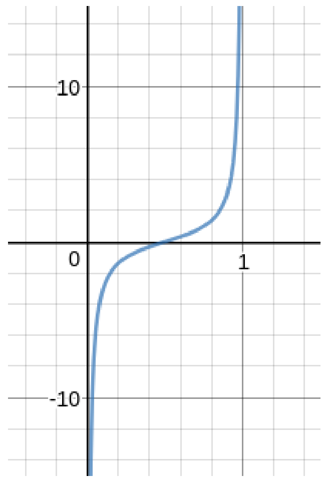

Figure 2: $\tan(\pi x - \frac{\pi}{2})$

   Now, f is a homeomorphism because

   - $t(x)$ is continuous, since it is a composition of functions which are continuous on $(0, 1)$. Also, $\arctan(x)$ is continuous everywhere, so $t^{-1}(x)$ is also continuous by the same reasoning.
   - $t(x)$ is monotonically increasing, so it is 1-1.
   - $t(x)$ is onto, since for any real number x, one can construct an angle whose tangent is $\pi x + \frac{\pi}{2}$ with two perpendicular segments of length $\pi x + \frac{\pi}{2}$ and 1, respectively.

   Thus, $t(x)$ is a continuous bijection whose inverse is also continuous. Therefore, f is a homeomorphism, since it maps each coordinate in S_q to $\mathbb{R}^2$ according to $t(x)$. ■

   (ii.) **Proof** Let S_B denote the open ball $B(0, 1)$. In this proof, it will be convenient to represent points using polar coordinates, so (r, θ) is the notation we will use. Let $t : [0, 1) \to \mathbb{R}$ be defined as $t(x) = \tan(\pi x - \frac{\pi}{2})$, the same as above. Let $f : S_B \to \mathbb{R}^2$ be defined as

$$f(r, \theta) = \left(t\left(\tfrac{1}{2}r + \tfrac{1}{2}\right), \theta\right) = \left(\tfrac{\pi}{2}r, \theta\right).$$

   Now, we have already shown that t is continuous, 1-1, and onto, and since $\left(\frac{1}{2}r + \frac{1}{2}\right)$ is a linear function it also must be a continuous bijection, and clearly so is the identity function. Therefore, since f is a composition of homeomorphisms, then it itself is a homeomorphism. ■

3. Let $\Delta = \{(x, x)\} \subset \mathbb{R}^2$. Prove that Δ as a subspace of $\mathbb{R}^2_{\text{bad}}$ is homeomorphic to $\mathbb{R}^1_{\text{bad}}$.

   Proof Let $f : \Delta \to \mathbb{R}^1$ be defined as $f(x, x) = x$.
   f is clearly 1-1 and onto, since $(a, a) \neq (b, b) \implies a \neq b$, and $\forall x \in \mathbb{R}, \exists (x, x) \in \Delta \mid f(x, x) = x$. Consider any interval $[a, b)$ which is open in $\mathbb{R}^1_{\text{bad}}$. Then, $f^{-1}([a, b))$ is the half-open line segment $\{(x, x) : a \leq x < b\} = \Delta \cap [a, b) \times [a, b)$. Therefore, f is continuous. Now we will show that $F = f^{-1}$ is continuous. $F : \mathbb{R}^1 \to \Delta$ is defined as $F(x) = (x, x)$. Let U be any set which is open in Δ. Then, by definition, $U = \Delta \cap [a, b) \times [c, d)$, where $a, b, c, d \in \mathbb{R}$. So, $U = \{(x, x) : \max(a, c) \leq x < \min(b, d)\}$. Now the preimage is $F^{-1}(U) = \{x : \max(a, c) \leq x < \min(b, d)\}$, which is open in $\mathbb{R}^1_{\text{bad}}$ by definition. Therefore, F is also continuous, and f is a homeomorphism between Δ as a subspace of $\mathbb{R}^2_{\text{bad}}$ and $\mathbb{R}^1_{\text{bad}}$. ■

4. Let $f : S^2 - \{(0,0,1)\} \to \mathbb{R}^2$ be the function pictured below, that takes a point (x,y,z) to the point $(a,b,0)$ on the xy-plane that lies along the line from $(0,0,1)$ to (x,y,z). (This function is called *stereographic projection.*) Find an explicit formula for f, and show that f is a homeomorphism between $S^2 - \{(0,0,1)\}$ and $\mathbb{R}^2$.

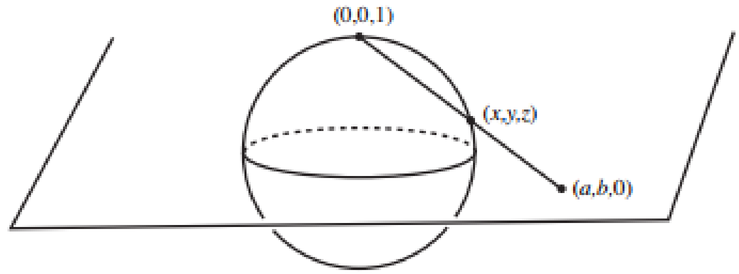

Figure 3: Stereographic projection.

Proof By parameterizing the line between $(0,0,1)$ and (x,y,z) and finding the value of t which gives a point on the xy-axis, we can obtain the function

$$f(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right),$$

and by parameterizing the line between $(0,0,1)$ and $(a,b,0)$ and finding the value of t which gives a point on the sphere, we can obtain its inverse;

$$F(a,b,0) = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right).$$

Now, it can be checked[1] that $(F \circ f)(x,y,z) = (x,y,z)$ and $(f \circ F)(a,b,0) = (a,b,0)$, therefore we can conclude that f and F are bijections. Also, since the domain of f is restricted to points in S^2, then $(1-z) \neq 0$. This means that f is a composition of functions which are continuous on its domain, and thus the function itself is continuous.
Similarly, the domain of F is restricted to the xy-plane, and F is a composition of functions which are continuous on its domain (since the denomiator of F is never zero), and thus the function itself is continuous.
Therefore, f is a homeomorphism between $S^2 - \{(0,0,1)\}$ and $\mathbb{R}^2$. ■

5. Which of the following spaces are compact? Explain your reasoning.

   (a) $\mathbb{R}^1_{\text{bad}}$
   Answer: Not compact, since the collection $\{[-n,n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}^1_{\text{bad}}$ which has no finite subcover.

   (b) $[-1,1]$ with either-or topology
   Answer: Compact, since any open set which contains 0 also contains $(-1,1)$, and now only 2 elements remain. Thus, any open cover of $([-1,1], \textit{either-or})$ has a subcover with at most 3 sets.

   (c) $[0,1]$ as subspace of $\mathbb{R}^1_{\text{bad}}$
   Answer: Not compact. First, note that $\{1\}$ is open in this topology, since $[1,2) \cap [0,1] = \{1\}$. Now, $\{[0, 1-\frac{1}{n}) : n \in \mathbb{N}\} \cup \{\{1\}\}$ is an open cover of the subspace which has no finite subcover.

[1] The algebra required for this check is *LONG*, so we have omitted it here. Also, a graphing calculator confirms that this is accurate.

(d) $\mathbb{Q} \cap [0,1]$, as a subspace of $(\mathbb{R}, usual)$.
Answer: Not compact. Consider $\left\{\mathbb{Q} \cap \left[0, \frac{1}{\Phi} - \frac{1}{n}\right) : n \in \mathbb{N}\right\} \cup \left\{\mathbb{Q} \cap \left(\frac{1}{\Phi} + \frac{1}{n}, 1\right] : n \in \mathbb{N}\right\}$, where Φ denotes the irrational number[2] $\Phi \approx 1.618$ which is the solution to the equation $\Phi^{-1} = \Phi - 1$. This collection of sets covers $\mathbb{Q} \cap [0,1]$, but any finite subcollection leaves out rational numbers in the range $\left(\frac{1}{\Phi} - \frac{1}{\max(n)}, \frac{1}{\Phi} + \frac{1}{\max(n)}\right)$.

7. Show that an infinite subset of a compact space must have a limit point.

Proof by contradiction Let S be an infinite subset of a compact space X, and suppose that S has no limit points. Then, for every $x \in S$, x is not a limit point. This means that for each such x, there exists an open set $U_x \subset X$ such that $x \in U_x$, and $U_x \cap (S - \{x\}) = \emptyset$. Thus, the collection $\{U_x\}_{x \in S}$ is an open cover of S (with respect to the subspace topology) which has no finite subcover, since each U_x is the only set in the collection which contains x. This contradicts our assumption that X is compact. ■

[2] We could have chosen any irrational number between 0 and 1, this one is just my favorite.