# §1. SMOOTH MANIFOLDS AND SMOOTH MAPS

FIRST let us explain some of our terms. $R^k$ denotes the $k$-dimensional euclidean space; thus a point $x \in R^k$ is an $k$-tuple $x = (x_1, \cdots, x_k)$ of real numbers.

Let $U \subset R^k$ and $V \subset R^l$ be open sets. A mapping $f$ from $U$ to $V$ (written $f : U \to V$) is called *smooth* if all of the partial derivatives $\partial^n f/\partial x_{i_1} \cdots \partial x_{i_n}$ exist and are continuous.

More generally let $X \subset R^k$ and $Y \subset R^l$ be arbitrary subsets of euclidean spaces. A map $f : X \to Y$ is called *smooth* if for each $x \in X$ there exist an open set $U \subset R^k$ containing $x$ and a smooth mapping $F : U \to R^l$ that coincides with $f$ throughout $U \cap X$.

If $f : X \to Y$ and $g : Y \to Z$ are smooth, note that the composition $g \circ f : X \to Z$ is also smooth. The identity map of any set $X$ is automatically smooth.

DEFINITION. A map $f : X \to Y$ is called a *diffeomorphism* if $f$ carries $X$ homeomorphically onto $Y$ and if both $f$ and $f^{-1}$ are smooth.

We can now indicate roughly what *differential topology* is about by saying that it studies those properties of a set $X \subset R^k$ which are invariant under diffeomorphism.

We do not, however, want to look at completely arbitrary sets $X$. The following definition singles out a particularly attractive and useful class.

DEFINITION. A subset $M \subset R^k$ is called a *smooth manifold* of *dimension* $m$ if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open subset $U$ of the euclidean space $R^m$.

Any particular diffeomorphism $g : U \to W \cap M$ is called a *parametrization* of the region $W \cap M$. (The inverse diffeomorphism $W \cap M \to U$ is called a system of *coordinates* on $W \cap M$.)

origin that is parallel to this. (Compare Figures 1 and 2.) Similarly one thinks of the nonhomogeneous linear mapping from the tangent hyperplane at $x$ to the tangent hyperplane at $y$ which best approximates $f$. Translating both hyperplanes to the origin, one obtains $df_x$.

Before giving the actual definition, we must study the special case of mappings between open sets. For any open set $U \subset R^k$ the *tangent space* $TU_x$ is defined to be the entire vector space $R^k$. For any smooth map $f\colon U \to V$ the *derivative*

$$df_x : R^k \to R^l$$

is defined by the formula

$$df_x(h) = \lim_{t \to 0} (f(x + th) - f(x))/t$$

for $x \in U$, $h \in R^k$. Clearly $df_x(h)$ is a linear function of $h$. (In fact $df_x$ is just that linear mapping which corresponds to the $l \times k$ matrix $(\partial f_i/\partial x_j)_x$ of first partial derivatives, evaluated at $x$.)

Here are two fundamental properties of the derivative operation:

1 (Chain rule). *If $f : U \to V$ and $g : V \to W$ are smooth maps, with $f(x) = y$, then*

$$d(g \circ f)_x = dg_y \circ df_x.$$

In other words, to every commutative triangle

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & W \end{tikzcd}$$

of smooth maps between open subsets of $R^k$, $R^l$, $R^m$ there corresponds a commutative triangle of linear maps

$$\begin{tikzcd} & R^l \arrow[dr, "dg_y"] & \\ R^k \arrow[ur, "df_x"] \arrow[rr, "d(g \circ f)_x"'] & & R^m. \end{tikzcd}$$

2. *If $I$ is the identity map of $U$, then $dI_x$ is the identity map of $R^k$. More generally, if $U \subset U'$ are open sets and*

$$i : U \to U'$$

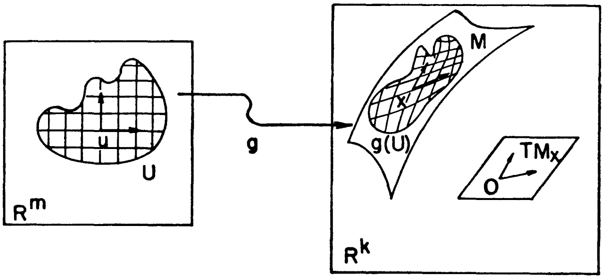


*Figure 1. Parametrization of a region in M*

Sometimes we will need to look at manifolds of dimension zero. By definition, $M$ is a manifold of dimension zero if each $x \in M$ has a neighborhood $W \cap M$ consisting of $x$ alone.

EXAMPLES. The unit sphere $S^2$, consisting of all $(x, y, z) \in R^3$ with $x^2 + y^2 + z^2 = 1$ is a smooth manifold of dimension 2. In fact the diffeomorphism

$$(x, y) \to (x, y, \sqrt{1 - x^2 - y^2}),$$

for $x^2 + y^2 < 1$, parametrizes the region $z > 0$ of $S^2$. By interchanging the roles of $x$, $y$, $z$, and changing the signs of the variables, we obtain similar parametrizations of the regions $x > 0$, $y > 0$, $x < 0$, $y < 0$, and $z < 0$. Since these cover $S^2$, it follows that $S^2$ is a smooth manifold.

More generally the sphere $S^{n-1} \subset R^n$ consisting of all $(x_1, \cdots, x_n)$ with $\sum x_i^2 = 1$ is a smooth manifold of dimension $n - 1$. For example $S^0 \subset R^1$ is a manifold consisting of just two points.

A somewhat wilder example of a smooth manifold is given by the set of all $(x, y) \in R^2$ with $x \neq 0$ and $y = \sin(1/x)$.

## TANGENT SPACES AND DERIVATIVES

To define the notion of *derivative* $df_x$ for a smooth map $f : M \to N$ of smooth manifolds, we first associate with each $x \in M \subset R^k$ a linear subspace $TM_x \subset R^k$ of dimension $m$ called the *tangent space* of $M$ at $x$. Then $df_x$ will be a linear mapping from $TM_x$ to $TN_y$, where $y = f(x)$. Elements of the vector space $TM_x$ are called *tangent vectors* to $M$ at $x$.

Intuitively one thinks of the $m$-dimensional hyperplane in $R^k$ which best approximates $M$ near $x$; then $TM_x$ is the hyperplane through the *is the inclusion map, then again $di_x$ is the identity map of $R^k$.*

Note also:

3. *If $L : R^k \to R^l$ is a linear mapping, then $dL_x = L$.*

As a simple application of the two properties one has the following:

ASSERTION. *If $f$ is a diffeomorphism between open sets $U \subset R^k$ and $V \subset R^l$, then $k$ must equal $l$, and the linear mapping*

$$df_x : R^k \to R^l$$

*must be nonsingular.*

PROOF. The composition $f^{-1} \circ f$ is the identity map of $U$; hence $d(f^{-1})_y \circ df_x$ is the identity map of $R^k$. Similarly $df_x \circ d(f^{-1})_y$ is the identity map of $R^l$. Thus $df_x$ has a two-sided inverse, and it follows that $k = l$.

A partial converse to this assertion is valid. Let $f : U \to R^k$ be a smooth map, with $U$ open in $R^k$.

**Inverse Function Theorem.** *If the derivative $df_x : R^k \to R^k$ is nonsingular, then $f$ maps any sufficiently small open set $U'$ about $x$ diffeomorphically onto an open set $f(U')$.*

(See Apostol [2, p. 144] or Dieudonné [7, p. 268].)

Note that $f$ may not be one-one in the large, even if every $df_x$ is nonsingular. (An instructive example is provided by the exponential mapping of the complex plane into itself.)

Now let us define the *tangent space* $TM_x$ for an arbitrary smooth manifold $M \subset R^k$. Choose a parametrization

$$g : U \to M \subset R^k$$

of a neighborhood $g(U)$ of $x$ in $M$, with $g(u) = x$. Here $U$ is an open subset of $R^m$. Think of $g$ as a mapping from $U$ to $R^k$, so that the derivative

$$dg_u : R^m \to R^k$$

is defined. *Set $TM_x$ equal to the image $dg_u(R^m)$ of $dg_u$.* (Compare Figure 1.)

We must prove that this construction does not depend on the particular choice of parametrization $g$. Let $h : V \to M \subset R^k$ be another parametrization of a neighborhood $h(V)$ of $x$ in $M$, and let $v = h^{-1}(x)$. Then $h^{-1} \circ g$ maps some neighborhood $U_1$ of $u$ diffeomorphically onto a neighborhood $V_1$ of $v$. The commutative diagram of smooth maps ween open sets

$$\begin{tikzcd} & R^k & \\ U_1 \arrow[ur, "g"] \arrow[rr, "h^{-1} \circ g"'] & & V_1 \arrow[ul, "h"'] \end{tikzcd}$$

es rise to a commutative diagram of linear maps

$$\begin{tikzcd} & R^k & \\ R^m \arrow[ur, "dg_u"] \arrow[rr, "d(h^{-1} \circ g)_u"', "\cong"] & & R^m, \arrow[ul, "dh_v"'] \end{tikzcd}$$

l it follows immediately that

$$\text{Image } (dg_u) = \text{Image } (dh_v).$$

ıs $TM_x$ is well defined.

'ROOF THAT $TM_x$ IS AN $m$-DIMENSIONAL VECTOR SPACE. Since

$$g^{-1} : g(U) \to U$$

, smooth mapping, we can choose an open set $W$ containing $x$ and mooth map $F : W \to R^m$ that coincides with $g^{-1}$ on $W \cap g(U)$. ting $U_0 = g^{-1}(W \cap g(U))$, we have the commutative diagram

$$\begin{tikzcd} & W \arrow[dr, "F"] & \\ U_0 \arrow[ur, "g"] \arrow[rr, "\text{inclusion}"] & & R^m \end{tikzcd}$$

therefore

$$\begin{tikzcd} & R^k \arrow[dr, "dF_x"] & \\ R^m \arrow[ur, "dg_u"] \arrow[rr, "\text{identity}"] & & R^m . \end{tikzcd}$$

s diagram clearly implies that $dg_u$ has rank $m$, and hence that its ge $TM_x$ has dimension $m$.

Iow consider two smooth manifolds, $M \subset R^k$ and $N \subset R^l$, and a transformation by going around the bottom of the diagram. That is:

$$df_x = dh_v \circ d(h^{-1} \circ f \circ g)_u \circ (dg_u)^{-1}.$$

This completes the proof that

$$df_x : TM_x \to TN_y$$

is a well-defined linear mapping.

As before, the derivative operation has two fundamental properties:

1. (Chain rule). *If $f : M \to N$ and $g : N \to P$ are smooth, with $f(x) = y$, then*

$$d(g \circ f)_x = dg_y \circ df_x.$$

2. *If $I$ is the identity map of $M$, then $dI_x$ is the identity map of $TM_x$. More generally, if $M \subset N$ with inclusion map $i$, then $TM_x \subset TN_x$ with inclusion map $di_x$.* (Compare Figure 2.)

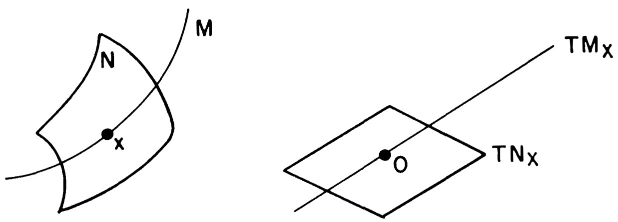


*Figure 2. The tangent space of a submanifold*

The proofs are straightforward.

As before, these two properties lead to the following:

ASSERTION. *If $f : M \to N$ is a diffeomorphism, then $df_x : TM_x \to TN_y$ is an isomorphism of vector spaces. In particular the dimension of $M$ must be equal to the dimension of $N$.*

## REGULAR VALUES

Let $f : M \to N$ be a smooth map between manifolds of the same dimension.* We say that $x \in M$ is a *regular point* of $f$ if the derivative

* This restriction will be removed in §2.

smooth map

$$f : M \to N$$

with $f(x) = y$. The *derivative*

$$df_x : TM_x \to TN_y$$

is defined as follows. Since $f$ is smooth there exist an open set $W$ containing $x$ and a smooth map

$$F : W \to R^l$$

that coincides with $f$ on $W \cap M$. *Define $df_x(v)$ to be equal to $dF_x(v)$ for all $v \in TM_x$.*

To justify this definition we must prove that $dF_x(v)$ belongs to $TN_y$ and that it does not depend on the particular choice of $F$.

Choose parametrizations

$$g : U \to M \subset R^k \quad \text{and} \quad h : V \to N \subset R^l$$

for neighborhoods $g(U)$ of $x$ and $h(V)$ of $y$. Replacing $U$ by a smaller set if necessary, we may assume that $g(U) \subset W$ and that $f$ maps $g(U)$ into $h(V)$. It follows that

$$h^{-1} \circ f \circ g : U \to V$$

is a well-defined smooth mapping.

Consider the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{F} & R^l \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle h} \\ U & \xrightarrow{h^{-1} \circ f \circ g} & V \end{array}$$

of smooth mappings between open sets. Taking derivatives, we obtain a commutative diagram of linear mappings

$$\begin{array}{ccc} R^k & \xrightarrow{dF_x} & R^l \\ {\scriptstyle dg_u}\uparrow & & \uparrow{\scriptstyle dh_v}, \\ R^m & \xrightarrow{d(h^{-1} \circ f \circ g)_u} & R^n \end{array}$$

where $u = g^{-1}(x)$, $v = h^{-1}(y)$.

It follows immediately that $dF_x$ carries $TM_x = \text{Image } (dg_u)$ into $TN_y = \text{Image } (dh_v)$. Furthermore the resulting map $df_x$ does not depend on the particular choice of $F$, for we can obtain the same linear

$df_x$ is nonsingular. In this case it follows from the inverse function theorem that $f$ maps a neighborhood of $x$ in $M$ diffeomorphically onto an open set in $N$. The point $y \varepsilon N$ is called a *regular value* if $f^{-1}(y)$ contains only regular points.

If $df_x$ is singular, then $x$ is called a *critical point* of $f$, and the image $f(x)$ is called a *critical value.* Thus each $y \varepsilon N$ is either a critical value or a regular value according as $f^{-1}(y)$ does or does not contain a critical point.

*Observe that if $M$ is compact and $y \varepsilon N$ is a regular value, then $f^{-1}(y)$ is a finite set (possibly empty).* For $f^{-1}(y)$ is in any case compact, being a closed subset of the compact space $M$; and $f^{-1}(y)$ is discrete, since $f$ is one-one in a neighborhood of each $x \varepsilon f^{-1}(y)$.

For a smooth $f : M \to N$, with $M$ compact, and a regular value $y \varepsilon N$, *we define $\#f^{-1}(y)$ to be the number of points in $f^{-1}(y)$.* The first observation to be made about $\#f^{-1}(y)$ is that it is locally constant as a function of $y$ (where $y$ ranges only through regular values!). I.e., *there is a neighborhood $V \subset N$ of $y$ such that $\#f^{-1}(y') = \#f^{-1}(y)$ for any $y' \varepsilon V$.* [Let $x_1, \cdots, x_k$ be the points of $f^{-1}(y)$, and choose pairwise disjoint neighborhoods $U_1, \cdots, U_k$ of these which are mapped diffeomorphically onto neighborhoods $V_1, \cdots, V_k$ in $N$. We may then take

$$V = V_1 \cap V_2 \cap \cdots \cap V_k - f(M - U_1 - \cdots - U_k).]$$

## THE FUNDAMENTAL THEOREM OF ALGEBRA

As an application of these notions, we prove the fundamental theorem of algebra: *every nonconstant complex polynomial $P(z)$ must have a zero.*

For the proof it is first necessary to pass from the plane of complex numbers to a compact manifold. Consider the unit sphere $S^2 \subset R^3$ and the stereographic projection

$$h_+ : S^2 - \{(0, 0, 1)\} \to R^2 \times 0 \subset R^3$$

from the "north pole" $(0, 0, 1)$ of $S^2$. (See Figure 3.) We will identify $R^2 \times 0$ with the plane of complex numbers. The polynomial map $P$ from $R^2 \times 0$ itself corresponds to a map $f$ from $S^2$ to itself; where

$$f(x) = h_+^{-1} P h_+(x) \quad \text{for} \quad x \neq (0, 0, 1)$$

$$f(0, 0, 1) = (0, 0, 1).$$

It is well known that this resulting map $f$ is smooth, even in a neighbor-

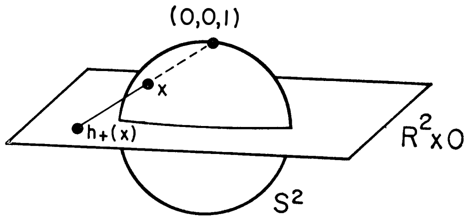


*Figure 3. Stereographic projection*

hood of the north pole. To see this we introduce the stereographic projection $h_-$ from the south pole $(0, 0, -1)$ and set

$$Q(z) = h_- f h_-^{-1}(z).$$

Note, by elementary geometry, that

$$h_+ h_-^{-1}(z) = z/|z|^2 = 1/\bar{z}.$$

Now if $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, with $a_0 \neq 0$, then a short computation shows that

$$Q(z) = z^n/(\bar{a}_0 + \bar{a}_1 z + \cdots + \bar{a}_n z^n).$$

Thus $Q$ is smooth in a neighborhood of 0, and it follows that $f = h_-^{-1} Q h_-$ is smooth in a neighborhood of $(0, 0, 1)$.

Next observe that $f$ has only a finite number of critical points; for $P$ fails to be a local diffeomorphism only at the zeros of the derivative polynomial $P'(z) = \sum a_{n-j}\, j z^{j-1}$, and there are only finitely many zeros since $P'$ is not identically zero. The set of regular values of $f$, being a sphere with finitely many points removed, is therefore connected. Hence the locally constant function $\# f^{-1}(y)$ must actually be constant on this set. Since $\# f^{-1}(y)$ can't be zero everywhere, we conclude that it is zero nowhere. Thus $f$ is an onto mapping, and the polynomial $P$ must have a zero.

has rank less than $n$ (i.e. is not onto). Then $C$ will be called the set of *critical points,* $f(C)$ the set of *critical values,* and the complement $N - f(C)$ the set of *regular values* of $f$. (This agrees with our previous definitions in the case $m = n$.) Since $M$ can be covered by a countable collection of neighborhoods each diffeomorphic to an open subset of $R^m$, we have:

**Corollary** (A. B. Brown). *The set of regular values of a smooth map $f : M \to N$ is everywhere dense in $N$.*

In order to exploit this corollary we will need the following:

**Lemma 1.** *If $f : M \to N$ is a smooth map between manifolds of dimension $m \geq n$, and if $y \in N$ is a regular value, then the set $f^{-1}(y) \subset M$ is a smooth manifold of dimension $m - n$.*

PROOF. Let $x \in f^{-1}(y)$. Since $y$ is a regular value, the derivative $df_x$ must map $TM_x$ onto $TN_y$. The null space $\mathfrak{N} \subset TM_x$ of $df_x$ will therefore be an $(m - n)$-dimensional vector space.

If $M \subset R^k$, choose a linear map $L : R^k \to R^{m-n}$ that is nonsingular on this subspace $\mathfrak{N} \subset TM_x \subset R^k$. Now define

$$F : M \to N \times R^{m-n}$$

by $F(\xi) = (f(\xi), L(\xi))$. The derivative $dF_x$ is clearly given by the formula

$$dF_x(v) = (df_x(v), L(v)).$$

Thus $dF_x$ is nonsingular. Hence $F$ maps some neighborhood $U$ of $x$ diffeomorphically onto a neighborhood $V$ of $(y, L(x))$. Note that $f^{-1}(y)$ corresponds, under $F$, to the hyperplane $y \times R^{m-n}$. In fact $F$ maps $f^{-1}(y) \cap U$ diffeomorphically onto $(y \times R^{m-n}) \cap V$. This proves that $f^{-1}(y)$ is a smooth manifold of dimension $m - n$.

As an example we can give an easy proof that the unit sphere $S^{m-1}$ is a smooth manifold. Consider the function $f : R^m \to R$ defined by

$$f(x) = x_1^2 + x_2^2 + \cdots + x_m^2.$$

Any $y \neq 0$ is a regular value, and the smooth manifold $f^{-1}(1)$ is the unit sphere.

If $M'$ is a manifold which is contained in $M$, it has already been noted that $TM'_x$ is a subspace of $TM_x$ for $x \in M'$. The orthogonal complement of $TM'_x$ in $TM_x$ is then a vector space of dimension $m - m'$ called *the space of normal vectors to $M'$ in $M$ at $x$.*

In particular let $M' = f^{-1}(y)$ for a regular value $y$ of $f : M \to N$.

# §2. THE THEOREM OF SARD AND BROWN

IN GENERAL, it is too much to hope that the set of critical values of a smooth map be finite. But this set will be "small," in the sense indicated by the next theorem, which was proved by A. Sard in 1942 following earlier work by A. P. Morse. (References [30], [24].)

**Theorem.** *Let $f : U \to R^n$ be a smooth map, defined on an open set $U \subset R^m$, and let*

$$C = \{x \in U \mid \text{rank } df_x < n\}.$$

*Then the image $f(C) \subset R^n$ has Lebesgue measure zero.**

Since a set of measure zero cannot contain any nonvacuous open set, it follows that the complement $R^n - f(C)$ must be everywhere dense† in $R^n$.

The proof will be given in §3. It is essential for the proof that $f$ should have many derivatives. (Compare Whitney [38].)

We will be mainly interested in the case $m \geq n$. If $m < n$, then clearly $C = U$; hence the theorem says simply that $f(U)$ has measure zero.

More generally consider a smooth map $f : M \to N$, from a manifold of dimension $m$ to a manifold of dimension $n$. Let $C$ be the set of all $x \in M$ such that

$$df_x : TM_x \to TN_{f(x)}$$

* In other words, given any $\epsilon > 0$, it is possible to cover $f(C)$ by a sequence of cubes in $R^n$ having total $n$-dimensional volume less than $\epsilon$.

† Proved by Arthur B. Brown in 1935. This result was rediscovered by Dubovickiĭ in 1953 and by Thom in 1954. (References [5], [8], [36].)

**Lemma 2.** *The null space of $df_x : TM_x \to TN_y$ is precisely equal to the tangent space $TM'_x \subset TM_x$ of the submanifold $M' = f^{-1}(y)$. Hence $df_x$ maps the orthogonal complement of $TM'_x$ isomorphically onto $TN_y$.*

Proof. From the diagram

$$\begin{tikzcd} M' \arrow[r, "i"] \arrow[d] & M \arrow[d, "f"] \\ y \arrow[r] & N \end{tikzcd}$$

we see that $df_x$ maps the subspace $TM'_x \subset TM_x$ to zero. Counting dimensions we see that *$df_x$ maps the space of normal vectors to $M'$ isomorphically onto $TN_y$.*

## MANIFOLDS WITH BOUNDARY

The lemmas above can be sharpened so as to apply to a map defined on a smooth "manifold with boundary." Consider first the closed half-space

$$H^m = \{(x_1, \cdots, x_m) \in R^m \mid x_m \geq 0\}.$$

The *boundary* $\partial H^m$ is defined to be the hyperplane $R^{m-1} \times 0 \subset R^m$.

Definition. A subset $X \subset R^k$ is called a *smooth m-manifold with boundary* if each $x \in X$ has a neighborhood $U \cap X$ diffeomorphic to an open subset $V \cap H^m$ of $H^m$. The *boundary* $\partial X$ is the set of all points in $X$ which correspond to points of $\partial H^m$ under such a diffeomorphism.

It is not hard to show that $\partial X$ is a well-defined smooth manifold of dimension $m - 1$. The *interior* $X - \partial X$ is a smooth manifold of dimension $m$.

The tangent space $TX_x$ is defined just as in §1, so that $TX_x$ is a full $m$-dimensional vector space, even if $x$ is a boundary point.

Here is one method for generating examples. Let $M$ be a manifold without boundary and let $g : M \to R$ have 0 as regular value.

**Lemma 3.** *The set of $x$ in $M$ with $g(x) \geq 0$ is a smooth manifold, with boundary equal to $g^{-1}(0)$.*

The proof is just like the proof of Lemma 1.