Math 550
Homework 4
Dr. Fuller
Due September 25

1. For each of the following, calculate the pullback $f^*\omega$ and simplify your answer as much as possible.
   (a) $f : \mathbf{R}^2 \to \mathbf{R}^3, f(u,v) = (\cos u, \sin u, v), \omega = z\, dx \wedge dy + y\, dz \wedge dx$
   (b) $f : \mathbf{R}^2 \to \mathbf{R}^2, f(r,\theta) = (r\cos\theta, r\sin\theta), \omega = -\dfrac{y}{x^2+y^2}\, dx + \dfrac{x}{x^2+y^2}\, dy$
   (ω is only defined on $\mathbf{R}^2 - \{(0,0)\}$.)
2. Let $g : \mathbf{R}^n \to \mathbf{R}^n$ be differentiable. Prove that $g^*(dx_1 \wedge \cdots \wedge dx_n) = \det Dg\, dx_1 \wedge \cdots \wedge dx_n$. (Hint: It enough to just check this on the standard basis $e_1, \ldots, e_n$.)
3. Let S denote the top half of the unit sphere in $\mathbf{R}^3$. Let $\omega = z^2\, dx \wedge dy$. Calculate $\int_S \omega$ using the parameterization $g(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ with $0 < \theta < 2\pi$, and $0 < \varphi < \frac{\pi}{2}$.
4. Let S be the surface in $\mathbf{R}^3$ parameterized by $g(\theta,z) = (\cos\theta, \sin\theta, z)$. where $0 < \theta < \pi$, and $0 < z < 1$. Let $\omega = xyz\, dy \wedge dz$. Calculate $\int_S \omega$.
5. Calculate the differential of each of the following.
   (a) $\omega = e^{xy}\, dx$
   (b) $\omega = x_1 x_2\, dx_3 \wedge dx_4$
   (c) $\omega = f(x,y)\, dx + g(x,y)\, dy$
   (d) $\omega = f(x,y,z)\, dy \wedge dz - g(x,y,z)\, dx \wedge dz + h(x,y,z)\, dx \wedge dy$
6. Determine if the following 2-forms are exact.
   (a) $\omega = x\, dx \wedge dy$
   (b) $\omega = z\, dx \wedge dy$
   (c) $\omega = z\, dx \wedge dy + y\, dx \wedge dz + z\, dy \wedge dz$
7. (a) Let $\alpha \in \Omega^1(\mathbf{R}^3)$ satisfy $\alpha(p) \neq 0$ for all $p \in \mathbf{R}^3$. Prove that $\ker\alpha$ is a 2-dimensional subspace (i.e. a plane) of $\mathbf{R}^3_p$ for all $p \in \mathbf{R}^3$.
   (b) Let $\alpha_1 = dz$. Sketch the planes described in part (a).
   (c) Let $\alpha_2 = x\, dy + dz$. Sketch the planes described in part (a).
   (d) Show that $\alpha_1 \wedge d\alpha_1 = 0$ and $\alpha_2 \wedge d\alpha_2 \neq 0$ (at all $p \in \mathbf{R}^3$).