Trevor Klar

# Homework 3

**Problem1.** Let $f\colon [0,1] \to \mathbb{R}$ be continuous. Prove that for any $\epsilon > 0$ there exists a continuous function $g_\epsilon\colon [0,1] \to \mathbb{R}$ such that $g'_\epsilon(x)$ exists and equals zero a.e. (w.r.t. Lebesgue measure) in $[0,1]$ and

$$\max_{x\in[0,1]} |f(x) - g_\epsilon(x)| < \epsilon.$$

**Proof** Let $\mu$ denote the Lebesgue measure. Since $f$ is a continuous function on a compact set, then it is uniformly continuous. Given $\varepsilon > 0$, choose $\delta$ according to the uniform continuity of $f$. Let

$$g_\varepsilon(x) := f(n\delta), \text{ where } n \in \mathbb{N} \text{ is such that } n\delta \leq x < (n+1)\delta.$$

This means that for any $x \in [0,1]$ we have $|x - n\delta| < \delta$, so $|f(x) - g_\varepsilon(x)| = |f(x) - f(n\delta)| < \varepsilon$. Furthermore, $g_\varepsilon$ is a simple function, so $g'_\varepsilon = 0$ $\mu$-*a.e.* and we are done. ■

**Problem4.** Let $p \geq 1$ and let $f, g \in L^p(\mathbb{R})$. Prove that the function

$$\varphi(t) = \int_{\mathbb{R}} |f(x) + tg(x)|^p dx$$

is differentiable a.e. in $\mathbb{R}$.

Hint: Use Young's inequality: If $p, q > 0$ such that $1/p + 1/q = 1$, then for all $a, b \geq 0$ one has the estimate

$$ab \leq \frac{a^p}{p} + \frac{a^q}{q}.$$

**Lemma 1.** For $a, b \geq 0$, $p \geq 1$,

$$(a+b)^p \leq a^p + (p)(2^{p-1})(a^{p-1}b + b^p).$$

Proof of Lemma Let $a \geq 0$, $p \geq 1$ and define

$$\psi(t) = (a+t)^p - \big(a^p + (p)(2^{p-1})(a^{p-1}t + t^p)\big).$$

Observe that $\psi(0) = 0$, and

$$\psi'(t) = p\big((a+t)^{p-1} - (2^{p-1})(a^{p-1} + pt^{p-1})\big)$$

and clearly $(a+t)^{p-1} < (2^{p-1})(a^{p-1} + pt^{p-1})$, so $\psi(t) < 0$ for all $t > 1$ and the lemma is proved. □

**Proof** To suppress notation, we write $\int f\,dx$ to mean $\int_{\mathbb{R}} f(x)\,dx$. We will show that $\varphi$ is locally Lipschitz, and therefore differentiable. For all $t \in \mathbb{R}$ and $h \in (0,1)$,

$$\begin{aligned}
\frac{|\varphi(t)-\varphi(t+h)|}{|h|} &= \frac{1}{|h|}\left|\int |f+(t+h)g|^p\,dx - \int |f+tg|^p\,dx\right| \\
&= \frac{1}{|h|}\left|\int |A+hg|^p\,dx - \int |A|^p\,dx\right| && \text{where } A = f+tg \\
&\leq \frac{1}{|h|}\left|\int \big|\,|A|+|hg|\,\big|^p\,dx - \int |A|^p\,dx\right| \\
&\leq \frac{1}{|h|}\left|\int |A|^p + (2^{p-1})(p)\left(|A|^{p-1}|hg| + |hg|^p\right)dx - \int |A|^p\,dx\right| && \text{by the Lemma} \\
&= \frac{(2^{p-1})(p)}{|h|}\left|\int |A|^{p-1}|hg| + |hg|^p\,dx\right| \\
&= (2^{p-1})(p)\left|\int |A|^{p-1}|g| + |h|^{p-1}|g|^p\,dx\right|.
\end{aligned}$$

We know that $\int |h|^{p-1}|g|^p\,dx \leq |g|^p\,dx < \infty$ since $h \in (0,1)$ and $g \in L^p(\mathbb{R})$, so we need to show that $\int |A|^{p-1}|g|\,dx < \infty$. Now $p \geq 1$, so the conjugate exponent of $p$ is $q\frac{p}{p-1}$, so by Young's inequality we can write

$$\int |A|^{p-1}|g|\,dx = \int |g||A|^{p-1}\,dx \leq \int \frac{|g|^p}{p} + \frac{|A|^p}{q}\,dx.$$

We know that $\int \frac{|g|^p}{p}\,dx < \infty$ since $g \in L^p(\mathbb{R})$, so it still remains to show that $\int \frac{|A|^p}{q}\,dx < \infty$. Note that the function $|\cdot|^p : \mathbb{R} \to \mathbb{R}$ is convex, so

$$\int \frac{|A|^p}{q}\,dx = \frac{1}{q}\int |f+tg|^p\,dx \leq \frac{1}{q}\int |f|^p + t|g|^p\,dx < \infty$$

since $f, g \in L^p(\mathbb{R})$ and we're done. ■