Final 4b

1. ⓐ $\boxed{Ae^{-3t}}$ char. eqn. no real roots

ⓑ $(At+B)e^{3t}$ ← wait... root? $r^2-6r+9=(r-3)^2$ 3 double root

$\boxed{t^2(At+B)Ce^{3t}}$

ⓒ $A\sin(2t)+B\cos(2t)+C_3t^3+C_2t^2+C_1t+C_0$

$2i$ is a root

$\boxed{At\sin(2t)+Bt\cos(2t)+C_3t^3+C_2t^2+C_1t+C_0}$

2. ⓐ Guess: Ae^{2t} $r^2-2r-3=(r-3)(r+1)$ ✓

$y'=2Ae^{2t}$

$y''=4Ae^{2t}$

$(\cancel{4}Ae^{2t})-\cancel{2(2}Ae^{2t})-3(Ae^{2t})=3e^{2t}$ ✓

$A=-1$

$y_p=-e^{2t}$ $y_h=C_1e^{3t}+C_2e^{-t}$

$\boxed{y_g=C_1e^{3t}+C_2e^{-t}-e^{2t}}$

(2) (b) $r^2 - 5r + 6 = (r-6)(r+1)$

$$y_h = C_1 e^{6t} + C_2 e^{-t}$$

$$y_p = At^2 + Bt + C$$

$$y_p' = 2At + B$$

$$y_p'' = 2A$$

$$(2A) - 5(2At + B) + 6(At^2 + Bt + C) = 4t^2 + 3t + 1$$

$$6A = 4 \Rightarrow A = \frac{2}{3}$$

$$-10A + 6B = 3$$

$$\Rightarrow B = \left(\frac{9}{3} + \frac{20}{3}\right)\frac{1}{6} = \frac{29}{18}$$

did I mess up? (double-checked)

$$2A - 5B + 6C = 1$$

$$\frac{4}{3} - \frac{5(29)}{18} + 6C = 1$$

$$\Rightarrow \cancel{-\frac{121}{18}} \quad C = \left(\frac{18}{18} + \frac{121}{18}\right)\frac{1}{6}$$

$$C = \frac{139}{108}$$

290
− 145
− 24
~~169~~
121

121
18
139

18
× 6
108

$$y_p = \frac{2}{3}t^2 + \frac{29}{16}t + \frac{139}{108}$$

$$\boxed{y_g = C_1 e^{6t} + C_2 e^{-t} + \frac{2}{3}t^2 + \frac{29}{16}t + \frac{139}{108}}$$

3) $y'' + 9y = \cos t$

$r^2 + 9 \Rightarrow r = 3i$

$y_h = C_1 \cos(3t) + C_2 \sin(3t)$

$y_p = A\cos t + B\sin t$

$y' = -A\sin t + B\cos t$

$y'' = -A\cos t - B\sin t$

$(-A\cos t - B\sin t) + 9(A\cos t + B\sin t) = 1\cos t + 0\sin t$

$-A + 9A = 1 \Rightarrow A = \frac{1}{8}$

$-B + 9B = 0 \Rightarrow B = 0$

$y_g = C_1 \cos(3t) + C_2 \sin(3t) + \frac{1}{8}\cos t$

$y_g' = -3C_1 \sin(3t) + 3C_2 \cos(3t) - \frac{1}{8}\sin t$

$y_0 = 0 \qquad y_0' = 1$

$0 = C_1 + 0 + \frac{1}{8}$

$\Rightarrow C_1 = -\frac{1}{8}$

$1 = 0 + 3C_2 - 0$

$\Rightarrow C_2 = \frac{1}{3}$

$$\boxed{y = -\frac{1}{8}\cos(3t) + \frac{1}{3}\sin(3t) + \frac{1}{8}\cos t}$$

(4) $y'' + p\,y' + q\,y = g$

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$y'' + (-1)y' + (-2)y = 2e^{-t}$

$r^2 - r - 2 = (r-2)(r+1)$

$$W[y_1, y_2] = \begin{vmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{vmatrix} = -e^{t} - 2e^{t} = -3e^{t}$$

$y_1 = e^{2t}$ $\quad p = -1 \quad W = -3e^{t}$

$y_2 = e^{-t}$ $\quad q = -2 \quad g = 2e^{-t}$

$$u_1 = -\int \frac{e^{-t}(2e^{-t})}{-3e^{t}} + C_1 = +\frac{2}{3}\int e^{-3t}\,dt + C_1 = -\frac{2}{9}e^{-3t} + C_1$$

$$u_2 = \int \frac{e^{2t}(2e^{-t})}{-3e^{t}}\,dt + C_2 = -\frac{2}{3}\int dt + C_2 = -\frac{2}{3}t + C_2$$

$y_g = u_1 y_1 + u_2 y_2$

$= \left(-\frac{2}{9}e^{-3t} + C_1\right)\left(e^{2t}\right) + \left(-\frac{2}{3}t + C_2\right)\left(e^{-t}\right)$

$$\boxed{y_g = -\frac{2}{9}e^{-t} + C_1 e^{2t} - \frac{2}{3}te^{-t} + C_2 e^{-t}}$$

⑤ Let $\vec{x}_1 = \begin{bmatrix} a e^{k_1 t} \\ b e^{k_2 t} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} c e^{k_1 t} \\ d e^{k_2 t} \end{bmatrix}$

ⓐ $$W = \begin{vmatrix} a e^{k_1 t} & c e^{k_1 t} \\ b e^{k_2 t} & d e^{k_2 t} \end{vmatrix} = \boxed{(ad - bc)\, e^{(k_1 + k_2)t}}$$

ⓑ $\boxed{\text{the value of } t \text{ has no effect.}}$

ⓒ $\boxed{\text{If } ad - bc = 0 \text{ then } \vec{x}_1, \vec{x}_2 \text{ are dependent.}}$

⑥ $\xi = \begin{bmatrix} -1 \\ 2 + 3i \end{bmatrix} \quad \lambda = 1 + 3i$

$$\cancel{\boxed{y_g = C_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} e^{3t}}}$$

$$\begin{bmatrix} -1 \\ 2 + 3i \end{bmatrix} e^{t} \left[\cos 3t + i \sin 3t\right]$$

Real: $$\begin{bmatrix} -\cos(3t) \\ 2\cos(3t) - 3\sin(3t) \end{bmatrix} e^{t}$$

Im: $$\begin{bmatrix} -\sin(3t) \\ 2\sin(3t) + 3\cos(3t) \end{bmatrix} e^{t}$$

$$\boxed{C_1 [\text{Real}] + C_2 [\text{Im}]}$$

7 (a) $x' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} x$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)+1$$
$$= 3 - 4\lambda + \lambda^2 + 1$$
$$= \lambda^2 - 4\lambda + 4$$
$$= (\lambda-2)^2$$
$$\Rightarrow \lambda = 2 \quad (\text{mult. } 2)$$

Find e-vectors:

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0 \qquad \vec{\xi} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \vec{\eta} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\boxed{y_g = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + C_2\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)e^{2t}}$$

7 (b) $x' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 3\lambda + 2$$

$-(-3\lambda)$

$(-\lambda^3 + 2)$

$$\begin{array}{r} -\lambda^2 + \lambda + 2 \\ (\lambda+1)\overline{\big)\, -\lambda^3 + 0\lambda^2 + 3\lambda + 2} \\ + \; +\lambda^3 + \lambda^2 \qquad\qquad \\ \hline \lambda^2 + 3\lambda \qquad \\ + \; -\lambda^2 + -\lambda \qquad \\ \hline 2\lambda + 2 \end{array}$$

$$\lambda^2-\lambda-2=(\lambda-2)(\lambda+1)$$

$$\text{so } \lambda=-1^{(2)},\ 2.$$

Let $\lambda=2$.

$$\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}\vec{\xi}_1=\vec{0} \quad\Rightarrow\quad \vec{\xi}_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Let $\lambda=-1$

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\vec{\xi}_2=\vec{0} \quad\Rightarrow\quad \vec{\xi}_2=\begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad \vec{\xi}_3=\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$\boxed{y_g=C_1\begin{bmatrix}1\\1\\1\end{bmatrix}e^{2t}+C_2\begin{bmatrix}1\\-1\\0\end{bmatrix}e^{-t}+C_3\begin{bmatrix}1\\0\\-1\end{bmatrix}e^{-t}}$$

8

a

$x_1=y$
$x_2=y'$
$x_3=y''$

$$2t^3y'-2y^{(3)}=-3t^4$$

$$2tx_2-2x_3'=-3t^4$$

$$+2x_3'=+3t^4+2tx_2$$

$$x_3'=\tfrac{3}{2}t^4+tx_2$$

$$\boxed{\begin{aligned}x_1'&=x_2\\x_2'&=x_3\\x_3'&=\frac{3t^4}{2}+tx_2\end{aligned}}$$

b

$$\boxed{\vec{x}'=\begin{bmatrix}0&1&0\\0&0&1\\0&t&0\end{bmatrix}\vec{x}+\begin{bmatrix}0\\0\\\frac{3t^4}{2}\end{bmatrix}}$$

9) a)

$$C_1\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}e^{t}+C_2\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}e^{2t}+C_3\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}e^{3t}+C_4\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}e^{4t}+C_5\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}e^{5t}$$

b)

$$\begin{bmatrix}5\\0\\0\\0\\0\end{bmatrix}e^{t}+\begin{bmatrix}0\\5\\0\\0\\0\end{bmatrix}e^{2t}+\begin{bmatrix}0\\0\\5\\0\\0\end{bmatrix}e^{3t}+\begin{bmatrix}0\\0\\0\\5\\0\end{bmatrix}e^{4t}+\begin{bmatrix}0\\0\\0\\0\\5\end{bmatrix}e^{5t}$$

hr min

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