Math 450B
Homework 5
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Solutions

1. Let $f : \mathbf{R} \to \mathbf{R}$ be
$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Then
$$\frac{\partial f}{\partial x}(x) = \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
Thus f is differentiable at 0, but $\frac{\partial f}{\partial x}$ is not continuous at 0; indeed, $\lim_{x \to 0} \dfrac{\partial f}{\partial x}(x)$ does not exist.

2. Assume first $m = 1$. Consider any two points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$. By applying the Mean Value Theorem, there is u_1 between x_1 and y_1 such that
$$f(y_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n) = \frac{\partial f}{\partial x_1}(u_1, x_2, \ldots, x_n)(x_1 - y_1).$$
Since $Df \equiv 0$, we have $\frac{\partial f}{\partial x_1}(u_1, x_2, \ldots, x_n) = 0$, so $f(x_1, x_2, \ldots, x_n) = f(y_1, x_2, \ldots, x_n)$. The same argument for any index j shows that $f(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) = f(y_1, \ldots, y_{j-1}, y_j, \ldots x_n)$. Thus we get
$$f(x_1, x_2, \ldots, x_n) = f(y_1, x_2, \ldots, x_n) = f(y_1, y_2, x_3, \ldots, x_n) = \cdots = f(y_1, \ldots, y_n).$$
The proof of the Lemma for a general m follows from the case $m = 1$, applied to all component functions.

3. Calculate:
$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} \frac{(x^3y^2 + 2xy^4)}{(x^2+y^2)^{3/2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$
To prove $\dfrac{\partial f}{\partial x}$ is continuous at $(0, 0)$, it is useful to observe
$$\left| \frac{\partial f}{\partial x}(x, y) \right| \leq \frac{|x|^3|y|^2}{\|(x, y)\|^3} + \frac{2|x||y|^4}{\|(x, y)\|^3} \leq |y|^2 + 2|x||y|.$$
Finally, by the symmetry of the f, the analysis of $\dfrac{\partial f}{\partial y}$ will be exactly the same.

4. It is continuous at $(0, 0)$. To show this, let $\varepsilon > 0$ and pick $\delta = \varepsilon$. Then for $\|(x, y)\| < \delta$, we have
$$\left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = \left| \frac{x}{\sqrt{x^2 + y^2}} \right| |y| \leq |y| \leq \|(x, y)\| < \delta = \varepsilon.$$
It is not differentiable at $(0, 0)$. One way to prove this is to show that $D_e f(0, 0)$ does not exist for $e = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

5. (a) $D_{-\mathbf{e}}f(\mathbf{a}) = \lim_{t\to 0} \dfrac{f(\mathbf{a}-t\mathbf{e})+f(\mathbf{a})}{t} = -\lim_{u\to 0} \dfrac{f(\mathbf{a}+u\mathbf{e})-f(\mathbf{a})}{u} = -D_{\mathbf{e}}f(\mathbf{a})$.
The middle inequality uses the substitution $u=-t$.

(b) This follows immediately from part (a), since $D_{\mathbf{e}}f(\mathbf{a}) > 0$ will imply that $D_{-\mathbf{e}}f(\mathbf{a}) < 0$.

(c) Let $f(x_1,\ldots,x_n) = x_1$, and $\mathbf{e} = (1,0,\ldots,0)$. Lots of other examples will work too.

6. $D_{\mathbf{e}}T(\mathbf{a}) = \lim_{t\to 0} \dfrac{T(\mathbf{a}+t\mathbf{e})-T(\mathbf{a})}{t} = \lim_{t\to 0} \dfrac{T(\mathbf{a})+tT(\mathbf{e})-T(\mathbf{a})}{t} = T(\mathbf{e})$.