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Derivatives & Differential Calculus

...are about **how quickly things change**.

- Need to understand PRACTICAL significance in various situations

Spread of infectious disease, population growth, speed, acceleration, marginal rates in economics, global warming

- Calculate (or estimate) rate of change from various sources:

graph
table of data
formula

- Applications:

measure change
predict the future
optimization – find the best, or smallest, or biggest, or most...

This is all about **understanding** the world.

Philosophical problem

How quickly is something changing at **one moment** in time?

Example: Does a ball **stop** when I throw it straight up?

Example: How fast is the temperature rising at 7am?

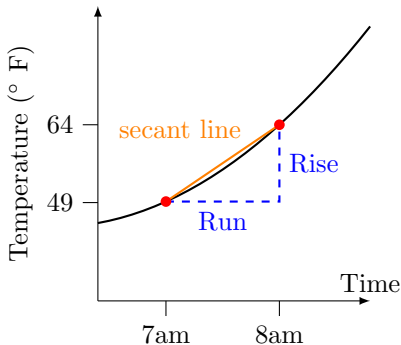
$$\left(\begin{array}{l} \text{change in temp} \\ \text{between 7am \& 8am} \end{array} \right)$$

$$= 64 - 49 = 15^\circ \text{ F}$$

$$\left(\begin{array}{l} \text{average rate of} \\ \text{change in temp} \\ \text{between 7am \& 8am} \end{array} \right)$$

$$= \frac{15^\circ \text{ F}}{1 \text{ hour}} = 15^\circ \text{ F/hour}$$

$$= \text{slope of secant line}$$



Continuing Example

Similarly,

$$\left(\begin{array}{c} \text{average rate of} \\ \text{change in temp} \\ \text{between 6am \& 8am} \end{array} \right) = \frac{\text{change in temp}}{\text{time taken}}$$

Question: Suppose temperature at time t given by the formula $f(t) = t^2$. What is the average rate of change of temperature from 6am to 8am?

(A) 1

(B) 7

(C) 9

(D) 14

(E) 28

Answer: D 14° F/hour

Average Rate of Change

Suppose temperature at time t given by the formula $f(t) = t^2$.

Using a calculator one can find the **average rate of change** over shorter and shorter time spans Δt , starting at 7am:

Δt	$(f(7 + \Delta t) - f(7))/\Delta t$	ave rate of change °F/hr
1	$(8^2 - 7^2)/1$	15
0.1	$(7.1^2 - 7^2)/0.1$	14.1
0.01	$(7.01^2 - 7^2)/0.01$	14.01
0.001	$(7.001^2 - 7^2)/0.001$	14.001
0.0001	$(7.0001^2 - 7^2)/0.0001$	14.0001
0.00001	$(7.00001^2 - 7^2)/0.00001$	14.00001
0	$(7^2 - 7^2)/0$	0/0 arghhhh

Table: Average rate of change over various time spans

What would you **guess** the **exact** instantaneous rate of change of temperature at precisely 7am is? Yes! 14. But how do we get this?

Answer: it is a **limit**!

Instantaneous Rate of Change

What does the limit

$$\lim_{\Delta t \rightarrow 0} \frac{f(7 + \Delta t) - f(7)}{\Delta t}$$

mean in practice?

Work out the average rate of change over a **very short** time interval.

That is **very nearly** the correct answer.

The shorter the time interval you use, the more accurate you expect the answer to be.

To get the **exact** answer you would need to take a time interval of zero length.

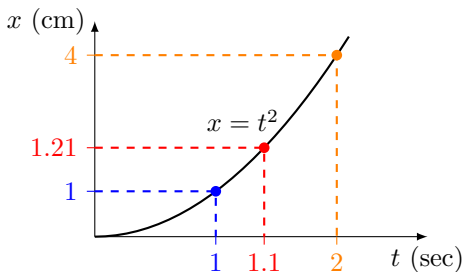
This leads to the nonsense **0/0**. So you can't do this.

That is the **philosophical** problem.

Mathematical solution: **take the limit**.

An Example

A hamster runs along the x -axis, so that after t seconds the hamster is t^2 cm from the origin. Our goal is to find the hamster's speed at time $t = 1$ sec.



$$\left(\begin{array}{c} \text{average speed from} \\ t = 1 \text{ to } t = 2 \end{array} \right) = \frac{\text{distance gone}}{\text{time taken}} = \frac{2^2 - 1^2}{2 - 1} = 3 \text{ cm/sec}$$

$$\left(\begin{array}{c} \text{average speed from} \\ t = 1 \text{ to } t = 1.1 \end{array} \right) = \frac{\text{distance gone}}{\text{time taken}} = \frac{1.1^2 - 1^2}{1.1 - 1} = 2.1 \text{ cm/sec}$$

Example Concluded

How do we work out the **exact** speed of the hamster after 1 second?

Plan:

- Find the **average speed** over a short time interval Δt , then
- Take the **limit** as $\Delta t \rightarrow 0$.

$$\begin{aligned}\left(\begin{array}{l} \text{average speed from} \\ t = 1 \text{ to } t = 1 + \Delta t \end{array} \right) &= \frac{\text{distance gone}}{\text{time taken}} \\ &= \frac{(1 + \Delta t)^2 - 1^2}{(1 + \Delta t) - 1} \\ &= \frac{(1 + 2\Delta t + (\Delta t)^2) - 1}{\Delta t} \\ &= \frac{2\Delta t + (\Delta t)^2}{\Delta t} \\ &= 2 + \Delta t\end{aligned}$$

The **limit** of this as $\Delta t \rightarrow 0$ is 2.

Conclusion: At $t = 1$ sec, the **exact** speed of the hamster is 2 cm/sec.

Hamster Summary

Soon we will calculate that...

the **exact speed** of the hamster after t seconds is $2t$ cm/sec.

Summary:

$f(t) = t^2$ = **distance** in cm of hamster from origin after t seconds
(a function that gives the distance the hamster has traveled at time t)

$f'(t) = 2t$ = **speed** of hamster in cm/sec after t seconds
(called the **derivative** of t^2 because it can be **derived** or **obtained** from the function t^2)

Question: How many cm had the hamster run by the time its **speed** was 8 cm/sec?

(A) 4

(B) 8

(C) 16

(D) 32

(E) 64

C

Exact Hamster Speed

Now we calculate that...

the **exact speed** of the hamster after t seconds is $2t$ cm/sec.

Do this as before: working out the **average speed** over a short time interval Δt and taking the **limit** as $\Delta t \rightarrow 0$

$$\begin{aligned}\left(\begin{array}{c} \text{average speed from} \\ t \text{ to } t + \Delta t \end{array} \right) &= \frac{\text{distance gone}}{\text{time taken}} \\ &= \frac{(t + \Delta t)^2 - t^2}{(t + \Delta t) - t} \\ &= \frac{(t^2 + 2t\Delta t + (\Delta t)^2) - t^2}{\Delta t} \\ &= \frac{2t\Delta t + (\Delta t)^2}{\Delta t} \\ &= 2t + \Delta t\end{aligned}$$

The **limit** of this as $\Delta t \rightarrow 0$ is $2t$.

Hamster Questions!

After t seconds, the hamster is $f(t) = t^2$ cm from origin.

1. What is the **exact** speed (in cm/sec) of the hamster at $t = 2$?

(A) 1

(B) 2

(C) 4

(D) 6

(E) 8

C

2. What is the **exact** speed (in cm/sec) of the hamster at $t = 4$?

(A) 1

(B) 2

(C) 4

(D) 6

(E) 8

E

3. What is the **average** speed (in cm/sec) of the hamster from $t = 2$ to $t = 4$ seconds?

(A) 1

(B) 2

(C) 4

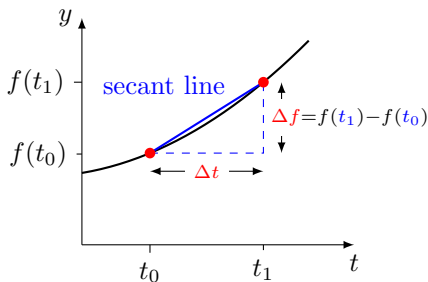
(D) 6

(E) 8

D

Does this make sense?

Graphical Approach



Δf = change in f

Δt = change in t

Many ways to say same thing:

$$\left(\begin{array}{c} \text{average rate of} \\ \text{change of } f \end{array} \right) = \frac{\text{change in } f}{\text{change in } t}$$

$$= \frac{\Delta f}{\Delta t}$$

$$= \text{slope of secant line} = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

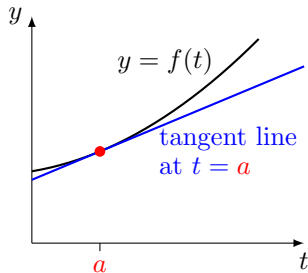
The derivative is defined to be

$$\lim_{\Delta t \rightarrow 0} \left(\frac{\Delta f}{\Delta t} \right) = \frac{df}{dt}$$

Idea: As t_1 moves closer to t_0 the secant line approaches the **tangent line** at t_0 . This is the line with the **same slope** as the graph at t_0 .

Understanding Derivatives

There are many ways to **think** about derivatives. You **need** to understand these to apply to problems.



slope of **graph** at **a**
= slope of **tangent line**
= **instantaneous rate of change** of f at a

= $\left(\begin{array}{l} \text{limit of average rate of change} \\ \text{of } f \text{ over shorter and shorter} \\ \text{time intervals starting at } a \end{array} \right)$

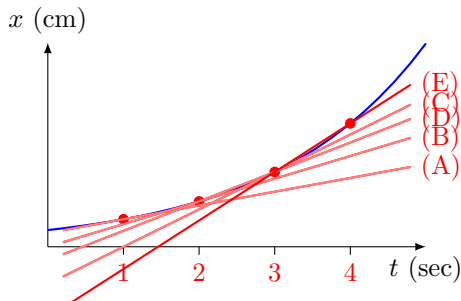
= limit of slopes of secant lines

$$= f'(a) = \left. \frac{df}{dt} \right|_{t=a}$$

Summary

- How fast something changes = **rate of change**
- **Instantaneous rate of change** is the **limit** of the average rate of change over shorter and shorter time spans. This gets around the **0/0** problem.
- **speed** = rate of change of distance traveled.

Examples



The graph shows the distance from the origin in cm after t seconds of a hamster. Which of the numbers below is the largest?

Hint: Speed is a slope!

(A) = speed of the hamster at $t = 1$

(B) = speed of the hamster at $t = 2$

(C) = speed of the hamster at $t = 3$

(D) = average speed of the hamster between $t = 2$ and $t = 3$

(E) = average speed of the hamster between $t = 3$ and $t = 4$