HW 5

Math 240

Trevor Klar

Proposition 5.21. *Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold. If any of the following holds, then S is embedded.*

(a) *S has codimension 0 in M.*
(b) *The inclusion map $S \subseteq M$ is proper.*
(c) *S is compact.*

Proof. Problem 5-3. □

Since S is an immersed submanifold, the inclusion map $\iota : S \hookrightarrow M$ is an injective immersion. By a proposition in Ch. 4, if

- Boundary(S) empty and $\dim S - \dim M = 0$, or
- ι is proper, or
- S is compact,

then ι is a smooth embedding. ▨

5-6. Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:

$$UM = \{(x, v) \in T\mathbb{R}^n : x \in M,\ v \in T_x M,\ |v| = 1\}.$$

It is called the ***unit tangent bundle of M***. Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$. (*Used on p. 147.*)

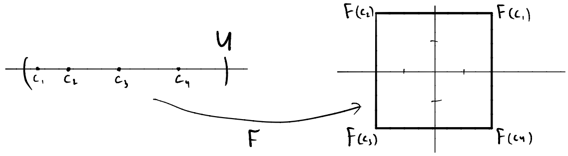
Since M is an m-dimensional submanifold, then M is the image of some embedding $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$. $\forall x \in \mathbb{R}^m$, dF_x is a smooth embedding of rank m because it is a linear transformation. the inclusion map $\iota : S^{m-1} \to \mathbb{R}^m$ is a smooth embedding of rank $m-1$ because all spheres are embedded submanifolds. Thus $\Phi : \mathbb{R}^m \times S^{m-1} \to UM$ given by

$$\Phi(x, v) = (F(x), (dF_x \circ \iota)(v))$$

is an embedding of rank $2m-1$. ▨

5-9. Let $S \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin (see Problem 3-5). Show that S does not have a topology and smooth structure in which it is an immersed submanifold of $\mathbb{R}^2$.

Suppose it does. then $\exists$ a smooth injective immersion $F : U \overset{\text{open}}{\subset} \mathbb{R}^1 \to \mathbb{R}^2$ so that $F(U) = S$. Call the preimages of the four corners of S c_1, c_2, c_3, c_4. F has rank 1 everywhere in U, including at $F^{-1}(c_1)$. However, $dF_x\left(\frac{d}{dt}\right)$ has a jump



discontinuity at $x = c_1$, so f is not smooth, contradiction. ∎

Let $O(n) = \{ A \in M_n(\mathbb{R}) \mid A^T A = I_n \}$. Show that $O(n)$ is an embedded submanifold of $M_n(\mathbb{R})$.

<u>Proof:</u> Let $F : M_n(\mathbb{R}) \longrightarrow \overset{\subset M_n(\mathbb{R})}{\mathrm{Sym}(n, \mathbb{R})}$ by $F(A) = A^T A$. We will show that F is a smooth submersion, so $O(n)$, which is the level set $F(A) = I_n$, is an embedded submanifold of $M_n(\mathbb{R})$.

Consider the differential $dF_A : T_A M_n(\mathbb{R}) \to T_{A^T A}\, \mathrm{Sym}(n, \mathbb{R})$. Let $B \in T_A M_n(\mathbb{R})$. Let $c(t) = A + tB$. This is a curve with $c(0) = A$ and $\dot{c}(0) = B$.

(leibniz rule)

Then $dF_A(B) = dF_A(\dot{c}(0)) = \left.\dfrac{d}{dt}\right|_{t=0} (F \circ c) = \left.\dfrac{d}{dt}\right|_{t=0} (A + tB)^T (A + tB) = B^T A + A^T B$.

To see that dF_A is surjective, observe that $\forall C \in \mathrm{Sym}(n, \mathbb{R})$, $A \in M_n(\mathbb{R})$, $dF_A(B) = C$ if $B = \frac{1}{2}(A^T)^{-1} C$, since

$$dF_A\left(\tfrac{1}{2}(A^T)^{-1}C\right) = A^T\left(\tfrac{1}{2}(A^T)^{-1}C\right) + \left(\tfrac{1}{2}(A^T)^{-1}C\right)^T A$$

$$= \tfrac{1}{2} A^T (A^T)^{-1} C + \tfrac{1}{2} C^T A^T (A^T)^{-1} \qquad \text{and since } C^T = C$$

$$= \tfrac{1}{2} C + \tfrac{1}{2} C$$

$$= C.$$

Thus dF_A is surjective $\forall A \in M_n(\mathbb{R})$, so F is a submersion. $\blacksquare$