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Homework 4

Problem1. Let $\{B_n\}$ be a nested sequence of closed balls in a normed space X, where

$$B_n = \bar{B}_{r_n}(x_n), \quad \text{with} \quad r_n \geq r > 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

1. Is it true that
$$\cap_{n=1}^{\infty} B_n \neq \emptyset$$
2. Is it true that
$$B \subset \cap_{n=1}^{\infty} B_n$$
for some closed ball B with radius r?

Answer: Both are true.

Proof If $\{B_n\}$ is an increasing sequence, then $B_1 \subset \cap_{n=1}^{\infty} B_n$, so parts 1 and 2 are shown.

Otherwise, suppose $\{B_n\}$ is a decreasing sequence. Then for any fixed N, whenever $y \in B_N$ we have that $y \in B_n$ for all $n \leq N$. Let A denote the set of all limit points of the set $\{x_n\}_{n=1}^{\infty}$. Then for all $y \in A$, there exists infinitely many x_n such that $||x_n - y|| < r$, which means $y \in B_n$. Thus $A \subset \cap_{n=1}^{\infty} B_n$.

Claim. $A \neq \emptyset$.

Proof of claim. Let $m > n$. Since $B_m \subseteq B_n$, then $r_m \leq r_n$ and $x_m \in B_n$. Thus

$$||x_n - x_m|| \leq r_n - r_m.$$

Letting $n \to \infty$, $\{r_n\}$ is a decreasing sequence bounded below by r, so it converges, and thus it is Cauchy. So for any $\varepsilon > 0$, there exists $M > 0$ such that if $n, m > M$, then $||x_n - x_m|| \leq r_n - r_m < \varepsilon$, so $\{x_n\}$ is Cauchy as well. Therefore $\{x_n\}$ converges with $x_n \to x$, and $x \in A$.

Claim. $\overline{B}_r(x) \subset \cap_{n=1}^{\infty} B_n$.

Proof of claim. We know that $\{r_n\}$ converges, so without loss of generality suppose $r_n \to r$. We will show that for all $y \in \overline{B}_r(x)$, there exists $M \in \mathbb{N}$ such that for all $n > M$, we have that $y \in B_n$. This proves the claim, since we will have shown that $\overline{B}_r(x) \subset B_m$ for all $m > M$, and $\overline{B}_r(x) \subset B_m$ implies $\overline{B}_r(x) \subset B_n$ for all $n < m$. Observe:

$$y \in \overline{B}_r(x) \implies ||x - y|| \leq r, \text{ so}$$

$$\begin{aligned} ||x_n - y|| &\leq ||x_n - x|| + ||x - y|| \\ &\leq r_n - r + r \\ &= r_n \end{aligned}$$

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Problem2. Construct a Lebesgue-measurable set $A \subset [0,1]$ such that $m(A) = 1$ and A is of Baire first category in $[0,1]$.

Definition. A set A is *Baire first category* if it is a countable union of nowhere dense sets.

Answer: For $k \geq 2$, let C_k denote a fat Cantor set where at the nth stage (counting from 0) of the construction of C_k, intervals of length $2^{-k(n+1)}$ are removed. For example,

C_2 removes intervals of length	$\frac{1}{4}$,	$\frac{1}{8}$,	$\frac{1}{16}$	…
C_3 removes intervals of length	$\frac{1}{8}$,	$\frac{1}{16}$,	$\frac{1}{32}$	…
C_4 removes intervals of length	$\frac{1}{16}$,	$\frac{1}{32}$,	$\frac{1}{64}$	…
etc.				

Thus the total length of removed intervals in the construction of C_k (we will denote this S_k) is

$$\sum_{n=0}^{\infty} \frac{2^n}{2^{2n+k}} = 2^{-k+1}.$$

Since $\lim_{k\to\infty} S_k = 0$, then $\lim_{k\to\infty} \mu(C_k) = 1$. Let $A = \bigcup_{k=1}^{\infty} C_k$. Now $\mu(A) = 1$, and A is Borel by construction, so it is Lebesgue measurable. Thus it remains to be shown that each C_k is nowhere dense, so A is Baire first category.

Denote the nth stage of construction of C_k by $C_{k,n}$. If we consider $C_{k,-1}$ to be the interval $[0,1]$ with nothing removed, then for any open interval I, we know that $I \cap C_{k,-1}$ is a non-degenerate interval U whenever it is nonempty.

During the rest of the stages, a central interval in $[0,1]$ will be removed, and central intervals from the two remaining intervals will be removed, and so on. At stage n, $[0,1]$ will be divided into 2^{n+1} intervals, and as $n \to \infty$, each of the intervals in $C_{k,n}$ has length less than $2^{-(n+1)}$ which goes to 0. Thus since U has nonzero length, some part of U must be removed at some stage. Since we are removing intervals, the part removed from U must also be an interval, and so must have positive length, which means $\overline{U}$ does not contain it. Thus $U = I \cap A$ is not dense in I. Therefore each C_k is nowhere dense, and we are done. ■