**ALGEBRA QUALIFYING EXAM: FALL 2019.**

**Answer TWO questions from each section.**

In the first two sections, you may use standard theorems without proof, but you must state them carefully.

## 1. GROUP THEORY

G1: (a) Suppose that $G$ is a group with exactly two subgroups. Prove that $G$ is finite of prime order.
(b) Is the converse of part (a) true? Give a reason.

G2. Let $p$ be a prime number and let $1 \leq n < p^2$ be an integer. Show that every Sylow $p$-subgroup of the symmetric group $S_n$ is abelian.
Hint: Explicitly construct one Sylow $p$-subgroup by considering a subgroup generated by disjoint $p$-cycles.

G3. Suppose that $G$ is a finite group with exactly three conjugacy classes. Show that $G$ is isomorphic to the symmetric group $S_3$ or the cyclic group $\mathbb{Z}/3\mathbb{Z}$.
Hint: If the three conjugacy classes have $r$, $s$, $t$ elements, show that $r + s + t = 3$ in the abelian case and is $6$ in the nonabelian case.

## 2. RINGS AND FIELDS

RF1: An element $a$ in a ring is called nilpotent if $a^k = 0$ for some positive integer $k$. Prove that the set $N$ of nilpotent elements of a commutative ring $R$ is an ideal of $R$ and that $R/N$ has no nilpotent elements.

RF2. Let $K = \mathbb{Q}(\alpha)$, where $\alpha$ satisfies

$$\alpha^2 = 5 + 2\sqrt{5}.$$

Show that $K/\mathbb{Q}$ is a Galois extension with Galois group $\mathbb{Z}/4\mathbb{Z}$.
Hint: Suppose that $\beta$ satisfies $\beta^2 = 5 - 2\sqrt{5}$, consider $\alpha\beta$.

RF3. Let $p(x) = x^3 - x + 4$. Prove that:
a) the Galois group of $p(x)$ over $\mathbb{F}_3$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.
b) the Galois group of $p(x)$ over $\mathbb{Q}$ is isomorphic to $S_3$.

## 3. LINEAR ALGEBRA

LA1. Let $V$ be a complex vector space. Let $T$ be a linear map $T : V \to V$, such that the characteristic polynomial of $T$ is $(x-1)^5$, and the minimal polynomial is $(x-1)^3$. Assume that the rank of $T - I$ is 2, where $I$ is the identity map. Determine the rational canonical form of $T$.

LA2. Let $V$ be the set of $2 \times 2$ real matrices, considered as a 4-dimensional real vector space. For a real number $\lambda$, define a symmetric bilinear form $\langle , \rangle$ on $V$ by

$$\langle A, B\rangle = \text{Tr}(AB) + \lambda \text{Tr}(AB^t).$$

Here Tr is the trace and $B^t$ is the transpose of $B$. For which $\lambda$ is this form positive definite?

LA3. Let $V$ and $W$ be finite dimensional complex vector spaces. Assume both $V$ and $W$ have dimension $n$. Let $A$ and $B$ be linear maps $V \to W$, with $A$ an isomorphism. Show that $A + tB$ is an isormorphism for all but at most $n$ values of $t \in \mathbb{C}$.

# 1. Group Theory

G1. (a) Suppose that G is a group with exactly two subgroups. Prove that G is finite of prime order.

**Theorem (Cauchy's Theorem).** If a prime $p$ divides the order of a finite group $G$, then there exists an element $g \in G$ of order $p$.

**Theorem (Lagrange's Theorem).** If $H \leq G$ then $|G| = [G : H] \cdot |H|$. For finite groups, the order of a subgroup divides the order of the group.

**Proof** $0 \leq G$ and $G \leq G$ for any group $G$, so if $G$ has exactly two subgroups then these are the only two.[†] For any $g \in G$ note that $0 < \langle g \rangle \leq G$, so $\langle g \rangle = G$ and $G$ is cyclic.

Now observe that $G$ cannot be infinite. If it were, then $G$ is an infinite cyclic group, and every such group is isomorphic to $\mathbb{Z}$. However $0 < 2\mathbb{Z} < Z$, while $G$ cannot have a subgroup isomorphic to $2\mathbb{Z}$, since it has exactly two subgroups, 0 and $G$.

If $G$ is finite then $|G|$ is prime, since otherwise Cauchy's theorem provides an element $g \in G$ such that the order of $g$ divides $|G|$, which means $0 < \langle g \rangle < G$, a contradiction. ∎

(b) Is the converse of part (a) true? Give a reason.

**Proof** Yes, by Lagrange's theorem. If $|G| = p$ prime, any subgroup either has order $p$ or is trivial. ∎

G2. Let p be a prime number and let $1 \leq n < p^2$ be an integer. Show that every Sylow $p$-subgroup of the symmetric group $S_n$ is abelian.

**Definition.** A Sylow $p$-subgroup is a subgroup whose order is the highest power of $p$ which divides the order of the group.

**Proof** Note that $|S_n| = n!$ and since $n < p^2$, then $p$ appears exactly once in the multiplicands of $n! = \prod_{i=1}^{n}(i)$. Also $p \nmid i$ for $1 \leq i \leq n$ unless $i = p$, so the highest power of $p$ which divides $n!$ is $p$. Thus any Sylow-$p$ subgroup of $S_n$ has order $p$. Any finite group of prime order is cyclic and thus abelian, so we're done. ∎

[†] I denote proper subgroups by $<$ and subgroups by $\leq$.