Math 501
Homework 9

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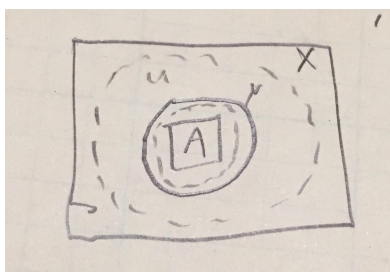
1. A space X is called *functionally normal* if for all pairs of disjoint closed sets A and B in X, there is a continuous function $f : X \to [0,1]$ with $f(a) = 0$ for all $a \in A$ and $f(b) = 0$ for all $b \in B$.

   Prove that if X is functionally normal, then X is normal.

   Proof Suppose X is functionally normal, and let A and B be two disjoint closed sets in X. Consider $f^{-1}\left([0, \frac{1}{\pi})\right)$. Since f is continuous and $[0, \frac{1}{\pi})$ is open in $[0,1]$, then $f^{-1}\left([0, \frac{1}{\pi})\right)$ is open in X. Since $f(a) = 0$ for all $a \in A$, then $A \subset f^{-1}\left([0, \frac{1}{\pi})\right)$. Similarly, $B \subset f^{-1}\left((\frac{2}{\pi}, 1]\right)$. Since f is continuous and $[0, \frac{1}{\pi}) \cap (\frac{2}{\pi}, 1] = \emptyset$, then $f^{-1}\left([0, \frac{1}{\pi})\right) \cap f^{-1}\left((\frac{2}{\pi}, 1]\right) = \emptyset$.

   Thus, for all disjoint sets A and B in X, there exist disjoint open sets $f^{-1}\left([0, \frac{1}{\pi})\right)$ and $f^{-1}\left((\frac{2}{\pi}, 1]\right)$ such that $A \subset f^{-1}\left([0, \frac{1}{\pi})\right)$ and $B \subset f^{-1}\left((\frac{2}{\pi}, 1]\right)$. ■

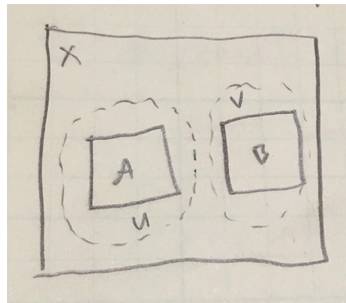
2. Let X be a space. Prove that X is normal if and only if for any closed set A and open set U with $A \subset U$, there is an open set V with $A \subset V \subset \overline{V} \subset U$.



   Proof ($\Longrightarrow$) Let X be normal, with A closed and U open such that $A \subset U$. Consider the closed set $U^{\mathsf{C}} = X - U$. Since $A \subset U$, then $A \cap U^{\mathsf{C}} = \emptyset$. Now since X is normal, there exist open sets V, V', with $A \subset V$, $U^{\mathsf{C}} \subset V'$, and $V \cap V' = \emptyset$. Currently, we have shown that $A \subset V \subset U$.

   Claim: $\overline{V} \cap V' = \emptyset$.
   To see this, suppose for contradiction that $x \in \overline{V} \cap V'$. Now, $x \notin V \cap V'$, because $V \cap V' = \emptyset$. So, $x \in V^{\ell} \cap V'$. Since x is a limit point of V and V' is an open set containing x, then $V \cap (V' - \{x\}) \neq \emptyset$, which is a contradiction. □

   Since $\overline{V} \cap V' = \emptyset$, then $\overline{V} \cap U^{\mathsf{C}} = \emptyset$, so $\overline{V} \subset U$. Thus, $A \subset V \subset \overline{V} \subset U$. ■



Proof ($\Longleftarrow$) Suppose that for any closed set A and open set U with $A \subset U$, there is an open set V with $A \subset V \subset \overline{V} \subset U$. Let A and B be closed sets in X with $A \cap B = \emptyset$.

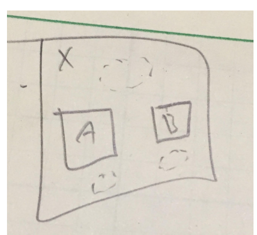
Since $A \subset B^{\mathsf{C}}$, and B^{C} is open, then there exists an open set U with $A \subset U \subset \overline{U} \subset B^{\mathsf{C}}$. Similarly, since $B \subset \overline{U}^{\mathsf{C}}$, and $\overline{U}^{\mathsf{C}}$ is open, then there exists an open set V with $B \subset V \subset \overline{V} \subset \overline{U}^{\mathsf{C}}$. Since $V \subset \overline{U}^{\mathsf{C}}$, then $V \cap \overline{U} = \emptyset$, so $V \cap U = \emptyset$.

Thus, $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. ■

3. Let X be a normal space. Prove for each pair of disjoint closed sets A and B, there are disjoint open sets U and V with $A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Proof Let A and B be closed sets in X with $A \cap B = \emptyset$. By exercise (2)($\Longrightarrow$), since X is normal, it satisfies the hypotheses for (2)($\Longleftarrow$). By (2)($\Longleftarrow$), there exist open sets U, V such that $A \subset U$, $B \subset V$, and $\overline{V} \subset \overline{U}^{\mathsf{C}}$. Thus, $\overline{U} \cap \overline{V} = \emptyset$ and we are done. ■

4. Prove that the Tietze Extension Theorem implies Urysohn's Lemma. (Remark: There exist proofs of the Tietze Extension Theorem which do not rely on Urysohn's Lemma. Thus the Tietze Extension Theorem and Urysohn's Lemma are equivalent.)



Proof Let X be a normal space, with A, B disjoint closed subsets of X. We will prove that there exists a continuous function $\varphi : X \to [0,1]$ such that $\varphi(a) = 0$ for all $a \in A$, and $\varphi(b) = 1$ for all $b \in B$.

Let $S = A \cup B$. Let $f : S \to \mathbb{R}$ be

$$\begin{aligned} f(a) &= 0 \quad \forall a \in A \\ f(b) &= 1 \quad \forall b \in B. \end{aligned}$$

Now, $f|_A$, $f|_B$ are constant functions on compact domains, so they are continuous. Thus, by the Piecing Lemma, f is continuous. Now, by Tietze's Extension Theorem, there exists a continuous function $F : X \to \mathbb{R}$ such that $f(x) = F(x)$ for all $x \in S$.

Now, define $\varphi : X \to [0,1]$ by

$$\varphi(x) = \begin{cases} 0 & F(x) \leq 0 \\ F(x) & F(x) \in [0,1] \\ 1 & F(x) \geq 1 \end{cases}.$$

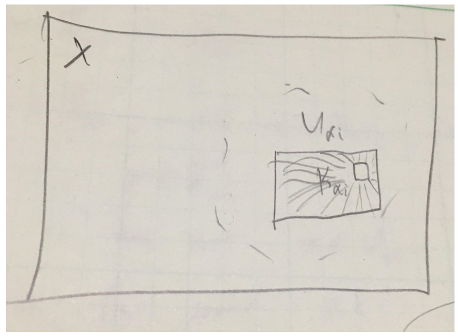
Note that since F is continuous, $F^{-1}((-\infty, 0])$ and $F^{-1}([1, \infty))$ are closed. So, since φ is constant on the above closed sets, it is continuous on them, and it is equivalent to the continuous function F on $[0,1]$. Thus, by the Piecing Lemma, φ is continuous. Therefore, we have shown that there exists continuous $\varphi : X \to [0,1]$ such that $\varphi(a) = 0$ for all $a \in A$, and $\varphi(b) = 1$ for all $b \in B$. ■

5. Let X be a compact Hausdorff space, and let $\{U_\alpha\}_{\alpha\in\Gamma}$ be an open cover of X. Prove that there exists a finite subcover $U_{\alpha_1},\dots,U_{\alpha_n}$ of X and a collection of functions $\varphi_{\alpha_1},\dots,\varphi_{\alpha_n} : X \to [0,1]$ such that

   (i) For each $i = 1, ..., n$, there exists a compact set $K_{\alpha_i} \subset U_{\alpha_i}$ such that $\varphi_{\alpha_i}(x) = 0$ for $x \in X - K_{\alpha_i}$;

   (ii) For each $x \in X$, $\sum\limits_{i=1}^{n} \varphi_{\alpha_i}(x) = 1$.

   (Remark: The φ_{α_i} and U_{α_i} are called a *partition of unity subordinate to the cover* $\{U_\alpha\}$. Hints: X is normal. Use Urysohn's Lemma to find functions f_{α_i} satisfying (i.). Let $\varphi_{\alpha_i} = \frac{f_{\alpha_i}}{\sum_j f_{\alpha_j}}$.)



Proof of (i) Since X is compact, then there exists a finite subcover $\{U_{\alpha_i}\}_{i=1}^n$. Let $\{K_{\alpha_i}\}_{i=1}^n$ be a collection of closed sets such that for each $i \in \{1,\dots,n\}$,

$$K_{\alpha_i} \subset U_{\alpha_i},$$

and $\{K\alpha_i\}_{i=1}^n$ covers X. Let $\{V_{\alpha_i}\}_{i=1}^n$ be closed sets with $V_{\alpha_i} \subset \text{int}(K_{\alpha_i}) \subset K_{\alpha_i}$ for each $i \in \{1,\dots,n\}$ (We know that these closed sets exists because X is Hausdorff, so even a singleton is closed). Now, $X - \text{int}(K_{\alpha_i})$ and V_{α_i} are both closed and $X - \text{int}(K_{\alpha_i}) \cap V_{\alpha_i} = \emptyset$ for every $i \in \{1,\dots,n\}$. This means that by Urysohn's Lemma, for each $i \in \{1,\dots,n\}$, there exists a continuous function $f_{\alpha_i} : X \to \mathbb{R}$ such that

$$f_{\alpha_i}(x) = \begin{cases} 0 & \text{if} \quad x \in X - \text{int}(K_{\alpha_i}) \\ > 0 & \text{if} \quad x \in \text{int}(K_{\alpha_i}) \\ 1 & \text{if} \quad x \in V_{\alpha_i} \end{cases}$$

Thus, we have (i), since $f_{\alpha_i}(x) = 0$ for all $x \in X - \text{int}(K_{\alpha_i})$, and $(X - K_{\alpha_i}) \subset (X - \text{int}(K_{\alpha_i}))$. ■

Proof of (ii) For each $i \in \{1,\dots,n\}$, let $\varphi : X \to [0,1]$ be

$$\varphi_{\alpha_i} = \frac{f_{\alpha_i}}{\sum_{j=1}^n f_{\alpha_j}}.$$

Remark: Since $\{K\alpha_j\}_{j=1}^n$ covers X, then for all $x \in X$, there exists some $k \in \{1,\dots,n\}$ such that $x \in \text{int}(K\alpha_k)$. Thus, $\sum\limits_{j=1}^{n} f_{\alpha_j} \neq 0$, and so φ_{α_i} is defined everywhere.

Thus, $\{\varphi_{\alpha_i}\}_{i=1}^n$ satisfies (i) and (ii), since $f_{\alpha_i}(x) = 0 \iff \varphi_{\alpha_i}(x) = 0$, and

$$\sum_{i=1}^{n} \varphi_{\alpha_i} = \sum_{i=1}^{n} \left(\frac{f_{\alpha_i}}{\sum_{j=1}^n f_{\alpha_j}} \right) = \frac{\sum_{i=1}^n f_{\alpha_i}}{\sum_{j=1}^n f_{\alpha_j}} \equiv 1.$$

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