201C - FUNCTIONAL ANALYSIS — LABUTIN, SPRING 2020

Quarter of COVID-19 — Trevor Klar

Homework 2

1. Prove that both weak and weak* topologies are Hausdorff.

Proof Let $x \neq y \in X$. Define $f(y-x) = 1$ and extend linearly on the subspace spanned by $y - x$. By Hahn-Banach, we can extend f to $\varphi : X \to \mathbb{R}$ and find that $W_{\frac{1}{2}}(x;\varphi)$ and $W_{\frac{1}{2}}(y;\varphi)$ are disjoint weak neighborhoods which separate x and y. Thus the weak topology is Hausdorff. $\square$

Let $\varphi \neq \psi \in X^*$. Since they are not equal as functions, there is some $x \in X$ with $\varphi(x) \neq \psi(x)$. Denoting $r = \varphi(x) - \psi(x)$, then $W_{\frac{1}{2}}(\varphi; x)$ and $W_{\frac{1}{2}}(\psi; x)$ are disjoint weak* neighborhoods which separate ψ and φ. Thus the weak* topology is Hausdorff. $\blacksquare$

2. Let X be Banach, and let S be the unit sphere in X. Find the weak closure $\bar{S}^w$ of S.

Answer: $\bar{S}^w = \bar{B}(X)$, the closed unit ball in X.

Proof Let $x \in \bar{B}(X)$. Since any weak neighborhood W of x contains an infinite-dimensional hyperplane in X, then W also contains a point y of any magnitude greater than that of x, in particular there exists $y \in W$ with $||y|| = 1$, so $y \in S$. Thus $x \in \bar{S}^w$.

Next, let $x \notin \bar{B}(X)$. By Hahn-Banach there exist a functional φ which separates[†] the convex compact set $\{x\}$ from the convex closed set $\bar{B}(X)$, so letting $r = \varphi(x) - 1$, we have $W_r(x;\varphi)$ contains x and is disjoint with $\bar{B}(X)$, so $x \notin \bar{S}^w$. $\blacksquare$

3. (i) Show that the set of all weak* neighborhoods

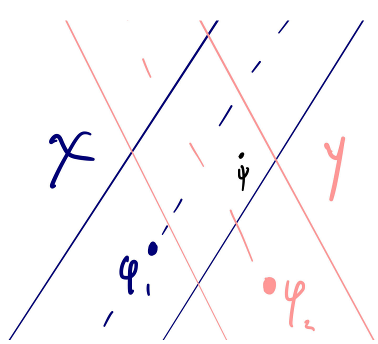
$$W(\varphi; x_1, \ldots, x_n)$$

forms a basis for a topology on X^*.

(ii) Show that convergence of a sequence $(\varphi_n)_{n=1}^{\infty}$ in this topology is equivalent to weak* convergence.

Proof (i) Denote the set of all weak* neighborhoods by $\mathcal{B}$. Note that $\mathcal{B}$ covers X^* since $W(\varphi; 0)$ is the whole space for any $\varphi \in X^*$.

Let $W(\varphi_1, x), W(\varphi_2, y) \in \mathcal{B}$.



[†]That is, maps x into $(1, \infty)$ and maps $\bar{B}(X)$ into $(0, 1)$.

For any ψ in the intersection of these two weak* neighborhoods,

$$|\langle \psi, x\rangle - \langle \varphi_1, x\rangle| < 1 \text{ and}$$
$$|\langle \psi, y\rangle - \langle \varphi_2, y\rangle| < 1,$$

so if we denote r as the smaller of the two quantities above, then

$$\psi \in W_r(\psi; x, y) \subset W(\varphi_1; x) \cap W(\varphi_2; y).$$

Since this holds for two arbitrary weak* neighborhoods, then it holds for finitely many. Therefore $\mathcal{B}$ is a basis for a topology. □

(ii) Denote convergence in the topology by $\varphi_n \xrightarrow{T} \varphi$.

Suppose $\varphi_n \xrightarrow{T} \varphi$. Then by definition, for every $\varepsilon > 0$ and $x \in X$, there exists $N > 0$ such that for all $n > N$,

$$\varphi_n \in W_\varepsilon(\varphi; x),$$

which is to say

$$|\langle \varphi_n, x\rangle - \langle \varphi, x\rangle| < \varepsilon,$$

which statement is exactly the definition of

$$\varphi_n \xrightarrow{w^*} \varphi.$$

This proof also works in reverse, so we are done. ■

4. Let (x_n) be a sequence in ℓ^1 such that $x_n \xrightarrow{w} y$ and $||x_n||_{\ell^1} \to ||y||_{\ell^1}$. Prove that $x_n \xrightarrow{\ell^1} y$.

Proof Let $\varepsilon > 0$. Since $||x_n|| \to ||y||$, there exists N_1 such that for all $n > N_1$,

$$\left| \sum_{j=1}^{\infty} |x_{nj}| - \sum_{j=1}^{\infty} |y_j| \right| < \varepsilon.$$

Since $||y|| < \infty$, then there exists J such that

$$\sum_{j=J}^{\infty} |y_j| < \varepsilon,$$

which means that for all $n > N_1$ we have $\sum_{j=J}^{\infty} |x_{nj}| < 2\varepsilon$, so

$$\begin{aligned} \sum_{j=J}^{\infty} |x_{nj} - y_j| &\le \sum_{j=J}^{\infty} |x_{nj}| + |y_j| \\ &< 3\varepsilon. \end{aligned} \tag{1}$$

Now observe that since $x_n \xrightarrow{w} y$, then in particular $\langle x_n, e_j\rangle \xrightarrow{n} \langle y, e_j\rangle$ where e_j is the functional which simply returns the j-th coordinate. This means that for all j we have $x_{nj} \xrightarrow{n} y_j$, so there exists some M_j such that if $n > M_j$, we have $|x_{nj} - y_j| < \varepsilon$. Let $N_2 = \max_{j \le J} M_j$, then for all $n > N_2$,

$$\sum_{j=1}^{J} |x_{nj} - y_j| < J\varepsilon. \tag{2}$$

Combining (1) and (2) yields $\sum_{j=1}^{\infty} |x_{nj} - y_j| < (J+3)\varepsilon$, and after rescaling, we're done. ■

5. Prove that the closed unit ball $\bar{B}(X)$ in a Banach space X is weakly closed. Prove that $\bar{B}(X^*)$ is weak* closed.

Proof We showed in problem 2 that the weak closure of $S(X)$ is $\bar{B}(X)$, so it is weakly closed. Now we show that $\bar{B}(X^*)$ is weak* closed. Observe:

- $\bar{B}(X^*)$ is weak* compact by Banach-Alaoglu.
- $\bar{B}(X^*)$ is weak* Hausdorff.
  Proof Let $\varphi \neq \psi \in \bar{B}(X^*)$. Since $\varphi \neq \psi$ as functions on X, there exists $x \in X$ with $\langle x, \varphi \rangle \neq \langle x, \psi \rangle$. Letting $a = \operatorname{avg}(\langle x, \varphi \rangle, \langle x, \psi \rangle)$ we have $\hat{x} \in X \subset X^{**}$ a linear functional on X^* such that $\langle x, \varphi \rangle < a$ and $\langle x, \psi \rangle > a$. Since $\hat{x}$ is weak* continuous by definition of the weak* topology, then $\hat{x}^{-1}(-\infty, a)$ and $\hat{x}^{-1}(a, \infty)$ are open sets which separate φ and ψ, so $\bar{B}(X^*)$ is weak* Hausdorff. □
- $\bar{B}(X^*)$ is weak* closed.
  Proof In this proof, all topological terms refer to the weak* topology. We will show that $\bar{B}(X^*)^\complement = X^* \setminus \bar{B}(X^*)$ is open. Let $\psi \notin \bar{B}(X^*)$. Since $\bar{B}(X^*)$ is Hausdorff, for every $\varphi_\alpha \in \bar{B}(X^*)$, there exist open sets U_α, V_α which separate φ_α and ψ, respectively. Since $\{U_\alpha\}$ is an open cover of $\bar{B}(X^*)$, it has a finite subcover $\{U_i\}$ with a corresponding finite collection of sets $\{V_i\}$. Since $\bigcap_i V_i \subset V_i$ for all i, then the intersection is disjoint with $\bigcup_i U_i$ which covers $\bar{B}(X^*)$. Thus $\bigcap_{i=1}^N V_i \subset V_i$ is an open subset of $\bar{B}(X^*)^\complement$ containing ψ, so we're done. ■

6. Prove the statement from the lectures: Let X be a separable Banach space with a dense set $U = \{u_n\}_{n \in \mathbb{N}}$. Then the weak* topology restricted to $\bar{B}(X^*)$ denoted $\sigma(\bar{B}(X^*), X)$, coincides with the topology of the metric

$$d(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \frac{|(\phi - \psi)(u_n)|}{1 + |(\phi - \psi)(u_n)|}.$$

Proof Since we are working in $\bar{B}(X^*)$, then every functional has norm at most 1. Since the two topologies are both translation invariant, it suffices to show that $W(0; p)$ is open in the d-topology and that the d-ball ${}_dB_r(0)$ is open in the weak* topology.

Let $W(0,p)$ be an arbitrary subbasic weak* neighborhood in $\sigma(\bar{B}(X^*), X)$, centered at 0. Let

$$\phi \in W(0,p).$$

We will produce a d-ball ${}_dB_r(0) \subset W(0,p)$. Since U is dense in X, then there exists some u_N such that

$$||u_N - p|| < \tfrac{1}{3}.$$

Let

$$r = \frac{1}{(3)(2^{N+1})}.$$

Then if $\psi \in {}_dB_r(0)$, then

$$\sum_{n=1}^{\infty} 2^{-n} \frac{|(\psi)(u_n)|}{1 + |(\psi)(u_n)|} < r,$$

and since the whole sum is bounded by r, then in particular so is each term since they are all positive. Thus

$$\begin{aligned} 2^{-N} \frac{|\psi(u_N)|}{1 + |\psi(u_N)|} &< \frac{1}{(3)(2^{N+1})} \\ \implies \qquad 2|\psi(u_N)| &< \frac{1 + |\psi(u_N)|}{3} \\ \implies \qquad |\psi(u_N)| &< \frac{1}{5}. \end{aligned}$$

Now we observe that $\psi \in W(0,p)$:

$$\begin{aligned} |\psi(p)| &\leq |\psi(p - u_N)| + |\psi(u_N)| \\ &< ||\psi|| \, ||u_N - p|| + \frac{1}{5} \\ &< \frac{1}{3} + \frac{1}{5} \\ &< 1 \end{aligned}$$

Thus the weak* topology is a subset of the d-topology. □

Let ${}_dB_r(0)$ be an arbitrary d-ball centered at 0. Then fix $\phi \in {}_dB_r(0)$, and observe that

$$||\phi||_d = \sum_{n=1}^{\infty} 2^{-n} \frac{|\phi(u_n)|}{1 + |\phi(u_n)|} < r$$

and since this sum converges, there exists $N > 0$ such that $\sum_{n=1}^{\infty} 2^{-n} \frac{|\phi(u_n)|}{1+|\phi(u_n)|} < \varepsilon$ for all $\varepsilon > 0$. Let

$$\delta = \min_{n \leq N} \left(\frac{r}{(N)2^{-n+1} - r} \right),$$

so that for all $n < N$,

$$2^{-n} \frac{\delta}{1 + \delta} < \frac{r}{2N}.$$

Consider $W_\delta(0; u_1, u_2, \dots, u_N)$. For any ϕ in this weak neighborhood, $|\phi(u_n)| < \delta$ for all $n < N$, so

$$\begin{aligned}
\sum_{n=1}^{\infty} 2^{-n} \frac{|\phi(u_n)|}{1 + |\phi(u_n)|} &= \sum_{n=1}^{N} 2^{-n} \frac{|\phi(u_n)|}{1 + |\phi(u_n)|} + \sum_{n=N}^{\infty} 2^{-n} \frac{|\phi(u_n)|}{1 + |\phi(u_n)|} \\
&\leq \sum_{n=1}^{N} \frac{r}{2N} + \varepsilon \\
&= \frac{r}{2} + \varepsilon \\
&< r.
\end{aligned}$$

Thus the d-topology is a subset of the weak* topology. ■