Homework 5

Chapter 1

Theorem (Bathtub Principle). Let (Ω, Σ, μ) be a measure space and let $f : \Omega \to \mathbb{R}$ be measurable with $\mu\{f < t\}$ finite for all $t \in \mathbb{R}$. Fix $G > 0$, and define a class of measurable functions on Ω by

$$\mathcal{C} = \left\{ g \;\middle|\; 0 \le g \le 1 \quad \text{and} \quad \int_\Omega g \, d\mu = G \right\}.$$

Then the minimization problem

$$I = \inf_{g \in \mathcal{C}} \int_\Omega f g \, d\mu \tag{1}$$

is solved by

$$g = \chi_{\{f<s\}} + c\chi_{\{f=s\}} \tag{2}$$

and

$$I = \int_{\{f<s\}} f \, d\mu + cs\mu\{f = s\} \tag{3}$$

where s is the supremum of all t such that

$$\mu\{f < t\} \le G, \tag{4}$$

and c is a scalar such that

$$\mu\{f < s\} + c\mu\{f = s\} = G. \tag{5}$$

Proof We know that $\mu\{f < t\}$ is finite for all t, and since $\{f < a\} \subseteq \{f < b\}$ for $a < b$, then $\mu\{f < t\}$ increases as t increases. We would like to bound this measure, so let s be the supremum of all t such that

$$\mu\{f < t\} \le G. \tag{4}$$

Case ($s = \infty$) We assume that since g is thought to be a density, then $\mu(\Omega) \ge G$. This means that if $s = \infty$, then since $\{f < \infty\} = f^{-1}(\mathbb{R}) = \Omega$, we have that $\mu(\Omega) \le G$. Thus in (5) $c = 0$, and so equation (2) is given by

$$g = \chi_{\{f<\infty\}} = \chi_\Omega = 1.$$

Now g has integral G and it is the *only* function in $\mathcal{C}$, since any other function in $\mathcal{C}$ is equal almost everywhere or has strictly smaller integral. Thus (2) trivially solves (1), and equations (1) and (3) are both $I = \int_\Omega f$.

Case ($s < \infty$) Suppose s is finite. Then either $\mu\{f = s\} = 0$ or $\mu\{f = s\} > 0$.

Claim Either $\mu\{f < s\} = G$, or $\mu\{f < s\} + \mu\{f = s\} > G$.

Proof of Claim Suppose that $\mu\{f < s\} \ne G$. Then clearly $\mu\{f < s\} \not> G$, so $\mu\{f < s\} < G$. Since s is the least upper bound of the set, then $\mu\{f < s + \varepsilon\} > G$ for all ε. Thus

$$\mu\{f < s\} + \mu\{f = s\} > G.$$

Justified by the claim above, let $0 \le c < 1$ so that

$$\mu\{f < s\} + c\mu\{f = s\} = G. \tag{5}$$

Now define

$$g = \chi_{\{f<s\}} + c\chi_{\{f=s\}}, \tag{2}$$

and let's compute the integral in (3).

$$\begin{aligned}\int_\Omega fg\,d\mu &= \int_\Omega f\left(\chi_{\{f<s\}} + c\chi_{\{f=s\}}\right)d\mu \\ &= \int_{\{f<s\}} f\,d\mu + \int_{\{f=s\}} c\,d\mu \\ &= \int_{\{f<s\}} f\,d\mu + cs\mu\{f = s\}.\end{aligned}$$

Now all that remains is to show that (2) solves the minimization problem

$$I = \inf_{g\in\mathcal{C}} \int_\Omega fg\,d\mu. \tag{1}$$

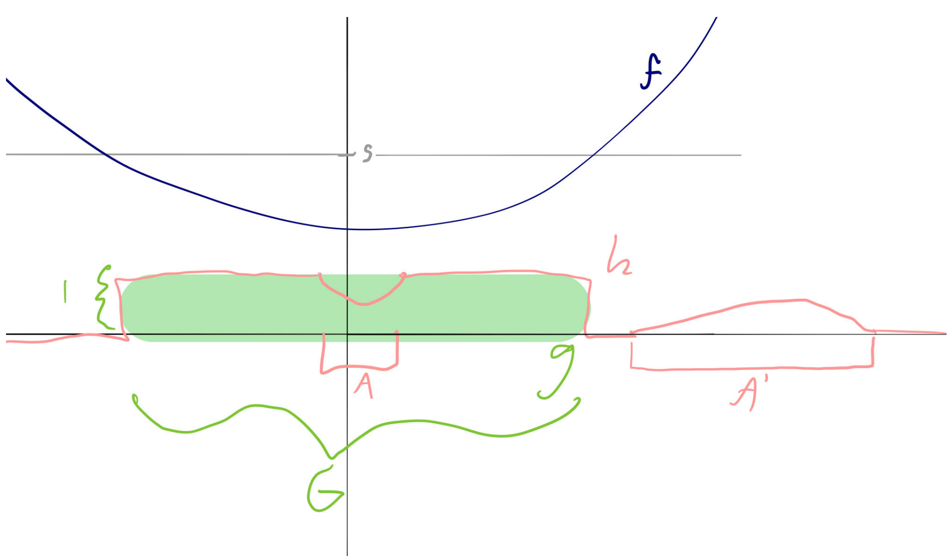
For now, suppose that $\mu\{f = s\} = 0$. Let h be any element of $\mathcal{C}$ which is distinct from g, so they differ on a set of positive measure. If $h = g$ on $\{f < s\}$ then they are the equal almost everywhere since $\int_{\{f<s\}} g\,d\mu = G$, so call

$$A = \{x \in \{f < s\} : h(x) \ne g(x)\}$$

and note that $\mu(A) > 0$ and in fact $h < g$ on A. Since this means $\int_{\{f<s\}} h\,d\mu < G$, then h and g also differ on $\{f > s\}$ on a set of positive measure. So call this set

$$A' = \{x \in \{f > s\} : h(x) \ne g(x)\}$$

and note that $\mu(A') = \mu(A)$ and $h > g$ on A'.



Now we we show that $\int_\Omega fg\,d\mu \le \int_\Omega fh\,d\mu$. First, since $\int_A (g-h) = \int_{A'} h$, then

$$\int_A f(g-h) \le \int_A s(g-h) = \int_{A'} sh \le \int_{A'} f(g-h). \tag{$\dagger$}$$

Thus

$$\begin{aligned}
\int_\Omega fg &= \int_{\{f<s\}} fg + \int_{\{f\ge s\}} fg \\
&= \left(\int_{\{f<s\}} fh + \int_A f(g-h)\right) + \left(\int_{\{f\ge s\}} fh - \int_{A'} fh\right) \\
&\le \int_{\{f<s\}} fh + \int_{A'} fh + \int_{\{f\ge s\}} fh - \int_{A'} fh && \text{by } (\dagger) \\
&= \int_{\{f<s\}} fh + \int_{\{f\ge s\}} fh + \int_{A'} fh - \int_{A'} fh \\
&= \int_\Omega fh
\end{aligned}$$

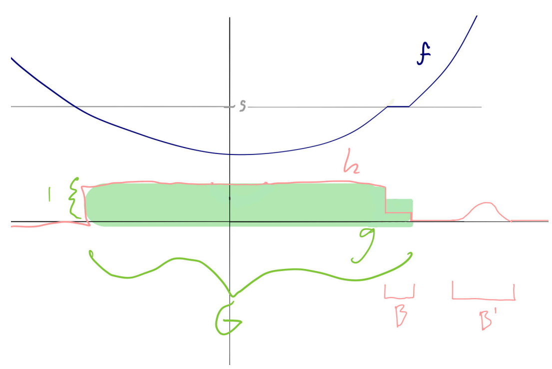
So since $\int_\Omega fg\,d\mu$ is an element of the set $\left\{\int_\Omega fh \middle| h \in \mathcal{C}\right\}$ in equation (1) and it is a lower bound of that set, then it is the infimum.

Earlier we supposed that $\mu\{f = s\} = 0$. If instead $\mu\{f = s\} > 0$, then $g = c$ (between 1 and 0) on $\{f = s\}$, so h can have three behaviors there: $h = g$ on $\{f = s\}$, there exists $B \subset \{f = s\}$ such that $h < g$ on B, or there exists $B \subset \{f = s\}$ such that $h > g$ on B.

Case I If $h = g$ on $\{f = s\}$, we can apply ($\dagger$) and use the same proof as when we assumed $\mu\{f = s\} = 0$.

Case II Suppose there exists $B \subset \{f = s\}$ such that $h < g$ on B. Then since $g = 1$ on $\{f < s\}$, there must exist $B' \subset \{f > s\}$ such that $\int_B (g-h) = \int_{B'} h$, so

$$\int_B f(g-h) = \int_B s(g-h) = \int_{B'} sh < \int_{B'} f(g-h). \tag{$\ddagger$}$$

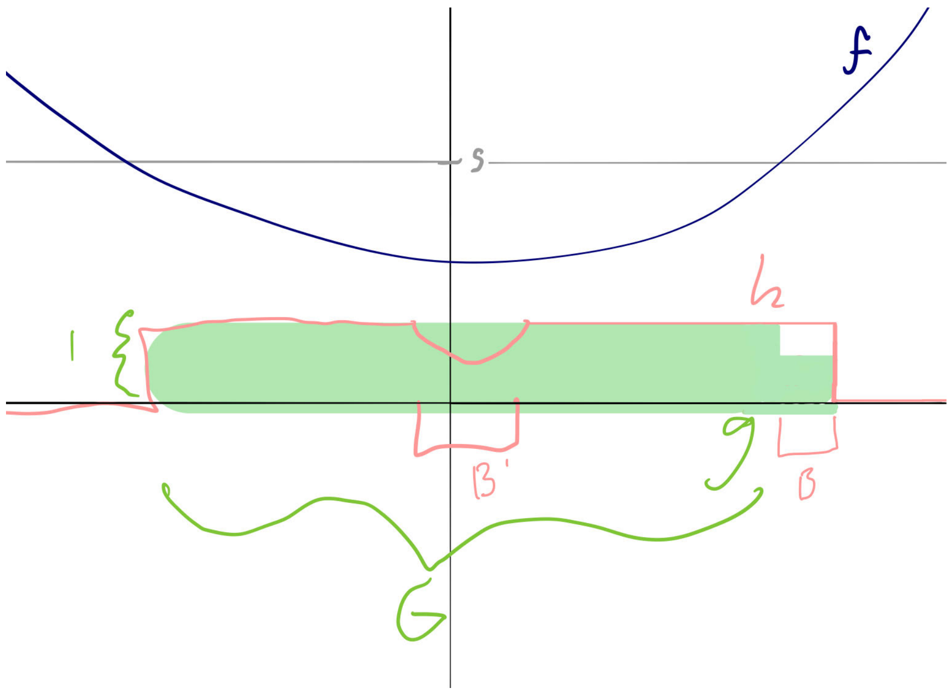


Following the same strategy of proof as when we assumed $\mu\{f = s\} = 0$, we can find that

$$\int_{\{f\ge s\}} fg < \int_{\{f\ge s\}} fh.$$

Case III Suppose there exists $B \subset \{f = s\}$ such that $h > g$ on B. Since $g = 0$ on $\{f > s\}$, there must exist $B' \subset \{f < s\}$ such that $\int_{B'}(g-h) = \int_B (h-g)$, so

$$\int_{B'} f(g-h) < \int_{B'} s(g-h) = \int_B s(h-g) = \int_B f(h-g). \tag{††}$$



Thus

$$\begin{aligned}
\int_\Omega fg &= \int_{\{f<s\}} fg + \int_{\{f=s\}} fg + \int_{\{f>s\}} fg \\
&= \left(\int_{\{f<s\}} fh + \int_{B'} f(g-h)\right) + \left(\int_{\{f=s\}} fh - \int_B f(h-g)\right) + \int_{\{f>s\}} fg \\
&< \int_{\{f<s\}} fh + \int_B f(h-g) + \int_{\{f=s\}} fh - \int_B f(h-g) + \int_{\{f>s\}} fg && \text{by (††)} \\
&= \int_{\{f<s\}} fh + \int_{\{f=s\}} fh + \int_{\{f>s\}} fg \\
&= \int_{\{f<s\}} fh + \int_{\{f=s\}} fh + \int_{\{f>s\}} fh && \text{Since } g = 0 \text{ on } \{f > s\} \\
&= \int_\Omega fh
\end{aligned}$$

Therefore in any case,

$$\inf_{h\in\mathcal{C}} \int_\Omega fh\, d\mu = \int_\Omega fg\, d\mu \tag{1}$$

and we're done. ■

Chapter 4

4.3 The **weak L^p-space**, denoted $L^p_w(\mathbb{R}^n)$, is defined as the set of all measurable functions such that

$$\langle f\rangle_{p,w} = \sup_{\alpha>0} \alpha \left(\mu\{|f|>\alpha\}\right)^{1/p} < \infty \tag{3}$$

The expression given by (3) does not define a norm. For $p > 1$ there is an alternative expression, equivalent† to (3), that is indeed a norm. It is given by

$$||f||_{p,w} = \sup_A |A|^{-1/p'} \int_A |f|\,dx, \tag{5}$$

where A is any set of finite measure. Using Theorem 1.14 (bathtub principle) it is not hard to see that (3) and (5) are equivalent.

1. Prove that (5) above actually defines a norm–the weak L^p-norm.

Proof (i) $||\bullet||_{p,w}$ is absolutely homogeneous:

$$\begin{aligned} ||\lambda f||_{p,w} &= \sup_A |A|^{-1/p'} \int_A |\lambda f|\,dx \\ &= |\lambda| \sup_A |A|^{-1/p'} \int_A |f|\,dx \\ &= |\lambda|\, ||f||_{p,w} \end{aligned}$$

(ii) $||\bullet||_{p,w}$ is positive definite:
If $||f||_{p,w} = 0$, then $\int_A |f|\,dx = 0$ for all A, which means $f = 0$ almost everywhere.

(iii) $||\bullet||_{p,w}$ has the triangle inequality:

$$\begin{aligned} ||f+g||_{p,w} &= \sup_A |A|^{-1/p'} \int_A |f+g|\,dx \\ &\le \sup_A |A|^{-1/p'} \left(\int_A |f|\,dx + \int_A |g|\,dx\right) \\ &\le \sup_A \left(|A|^{-1/p'} \int_A |f|\,dx + |A|^{-1/p'} \int_A |g|\,dx\right) \\ &\le \left(\sup_A |A|^{-1/p'} \int_A |f|\,dx\right) + \left(\sup_A |A|^{-1/p'} \int_A |g|\,dx\right) \\ & ||f||_{p,w} + ||g||_{p,w} \end{aligned}$$

■

2. Prove the equivalence of the two definitions of weak L^p given in Sect. 4.3. That is, prove that

$$C_1 \langle f\rangle_{p,w} \le ||f||_{p,w} \le C_2 \langle f\rangle_{p,w},$$

where C_1 and C_2 are universal constants independent of f. Find explicit values for these constants.

†Equivalent in the sense that convergence in $\langle f\rangle$ is equivalent to convergence in $||f||$.

Proof First we will show that $||f||_{p,w} \geq C_1 \langle f \rangle_{p,w}$.

For any $\alpha > 0$, let $A_\alpha = \{|f| > \alpha\}$. Then

$$\begin{aligned} |A_\alpha|^{-1/p'} \int_{A_\alpha} |f|\, dx &\geq |A_\alpha|^{-1/p'} \int_{A_\alpha} \alpha\, dx \\ &= |A_\alpha|^{-1/p'} \alpha |A_\alpha| \\ &= \alpha |A_\alpha|^{-1/p} \\ &= \alpha \left(\mu \{|f| > \alpha\}\right)^{1/p}. \end{aligned}$$

Thus taking supremum of both sides,

$$\begin{aligned} ||f||_{p,w} &= \sup_A |A|^{-1/p'} \int_A |f|\, dx \\ &\geq \sup_{\alpha>0} |A_\alpha|^{-1/p'} \int_{A_\alpha} |f|\, dx \\ &\geq \sup_{\alpha>0} \alpha \left(\mu \{|f| > \alpha\}\right)^{1/p} \\ &= \langle f \rangle_{p,w} \end{aligned}$$

so $C_1 = 1$. $\square$

Next we show that $||f||_{p,w} \leq C_2 \langle f \rangle_{p,w}$. Before we start, observe that equation (3) gives us that

$$\langle f \rangle^p = \sup_{t>0} t^p \left|\{|f| > t\}\right|,$$

where we suppress the notation and write $\langle f \rangle^p$ to mean $(\langle f \rangle_{p,w})^p$. Thus for any particular $t > 0$, we have

$$\frac{\langle f \rangle^p}{t^p} \geq \left|\{|f| > t\}\right|. \tag{†}$$

Now we begin the proof. Equation (5) gives us that

$$||f||_{p,w} = \sup_A |A|^{-1/p'} \int_A |f|\, dx, \tag{5}$$

And to bound the integral in (5) we rewrite it and split the integral at a level T (to be determined later):

$$
\begin{aligned}
\int_A |f|\,dx &= \int_0^\infty |\{|f|>t\}\cap A|\,dt \\
&= \int_0^T |\{|f|>t\}\cap A|\,dt + \int_T^\infty |\{|f|>t\}\cap A|\,dt \\
&\le T|A| + \int_T^\infty |\{|f|>t\}\cap A|\,dt \\
&\le T|A| + \int_T^\infty |\{|f|>t\}|\,dt \\
&\le T|A| + \int_T^\infty \frac{\langle f\rangle^p}{t^p}\,dt && \text{by } (\dagger) \\
&= T|A| + \frac{\langle f\rangle^p}{(p-1)\left(T^{p-1}\right)}
\end{aligned}
$$

Next, we will find a value of T to minimize the right hand side above, when everything else is held constant. Write $T|A| + \frac{\langle f\rangle^p}{(p-1)(T^{p-1})}$ as a function of T with $\beta = p-1$ and constants B_1, B_2:

$$\varphi(T) = TB_1 + \frac{B_2}{T^\beta}$$

Since $\varphi'(T) = B_1 - \beta B_2 T^{-\beta-1}$ and $-\beta B_2 T^{-\beta-1}$ is an increasing function with limit 0 as $T\to\infty$, then as long as $B_1 > 0$ (it is), then φ has exactly one minimum. Solving for T in $\varphi' = 0$ will show that we should fix

$$
\begin{aligned}
T &= \left(\frac{\beta B_2}{B_1}\right)^{\frac{1}{\beta+1}} \\
&= \left(\frac{(p-1)\langle f\rangle^p}{|A|(p-1)}\right)^{\frac{1}{p}} \\
&= \left(\frac{\langle f\rangle^p}{|A|}\right)^{\frac{1}{p}} \\
&= |A|^{-1/p}\langle f\rangle
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_A |f|\,dx &\le T|A| + \frac{\langle f\rangle^p}{(p-1)\left(T^{p-1}\right)} \\
&= |A|^{1/p'}\langle f\rangle + \frac{|A|^{1/p'}\langle f\rangle}{(p-1)} \\
&= |A|^{1/p'}\langle f\rangle\left(1 + \frac{1}{(p-1)}\right), \\
&= |A|^{1/p'}\langle f\rangle\,(p'),
\end{aligned}
$$

and finally we can conclude that

$$\begin{aligned} ||f||_{p,w} &= \sup_A |A|^{-1/p'} \int_A |f| \, dx \\ &\leq \langle f \rangle (p'), \end{aligned}$$

so $C_2 = p'$, and we're done. ■

4. Gaussian integrals appear frequently and it is important to know how to compute them.

(a) Show that

$$\int_{-\infty}^{\infty} \exp(-\lambda x^2) \, dx = \sqrt{\pi/\lambda}$$

by evaluating the square of the integral by means of polar coordinates.

Proof We will show that $\left(\int_{-\infty}^{\infty} \exp(-\lambda x^2) \, dx\right)^2 = \pi/\lambda$. First, observe that

$$\begin{aligned} \left(\int_{-\infty}^{\infty} \exp(-\lambda x^2) \, dx\right)^2 &= \left(\int_{-\infty}^{\infty} \exp(-\lambda x^2) \, dx\right)\left(\int_{-\infty}^{\infty} \exp(-\lambda y^2) \, dy\right) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-\lambda x^2) \exp(-\lambda y^2) \, dy \, dx \end{aligned}$$

and by changing to polar coordinates,

$$\begin{aligned} &= \int_0^{\infty} \int_0^{2\pi} \exp(-\lambda r^2) r \, dr \, d\theta \\ &= \left(\int_0^{\infty} \exp(-\lambda r^2) r \, dr\right)\left(\int_0^{2\pi} d\theta\right) \\ &= \left[-\frac{1}{2\lambda} e^{-\lambda r^2}\right]_{r=0}^{\infty} \Big(2\pi\Big) \\ &= \left(0 + \frac{1}{2\lambda}\right)(2\pi) \\ &= \frac{\pi}{\lambda}. \end{aligned}$$

and we're done, since taking square roots yields the desired integral. ■

(b) For A a symmetric $n \times n$ matrix whose real part is positive definite, show that

$$\int_{\mathbb{R}^n} \exp(-x^\top Ax) \, dx = \pi^{n/2}/\sqrt{\det A}.$$

In the real, symmetric case this can be done by a simple change of variables.

Proof Since A is positive definite, then A is unitarily diagonalizable with positive determinant. So we can write $A = UDU^*$, and make a change of variables $x \mapsto Ux$. Then the Jacobian is $\det U = 1$, so

$$\begin{aligned}
\int_{\mathbb{R}^n} \exp\left(-x^\top Ax\right) dx &= \int_{\mathbb{R}^n} \exp\left(-(Ux)^\top (UDU^*)(Ux)\right) dx \\
&= \int_{\mathbb{R}^n} \exp\left(-x^\top (U^*U)D(U^*U)x\right) dx \\
&= \int_{\mathbb{R}^n} \exp\left(-x^\top Dx\right) dx \\
&= \int_{\mathbb{R}^n} \exp\left(\sum_{i=1}^n \lambda_i x_i^2\right) dx \qquad\qquad \text{where } \lambda_i \text{ are eigenvalues} \\
&= \prod_{i=1}^n \int_{-\infty}^{\infty} \exp\left(\lambda_i x_i^2\right) dx_i \\
&= \prod_{i=1}^n \sqrt{\pi/\lambda_i} \\
&= \sqrt{\frac{\pi^n}{\det D}} \\
&= \sqrt{\frac{\pi^n}{\det A}}
\end{aligned}$$

■

(c) For a vector v in $\mathbb{C}^n$ show, by "completing the square", that

$$\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle + 2\langle v, x\rangle)\, dx = \left(\pi^{n/2}/\sqrt{\det A}\right) \exp\left(\left\langle v, A^{-1}v\right\rangle\right).$$

Proof The expression $-\langle x, Ax\rangle + 2\langle v, x\rangle$ sort of looks like $(x+v)^2$, in an inner-producty sort of way. If we play with the numbers, we find that

$$\left\langle -x + vA^{-1},\ Ax - v\right\rangle \ = \ -\langle x, Ax\rangle + 2\langle v, x\rangle - \left\langle vA^{-1}, v\right\rangle,$$

so since $\exp(-\langle vA^{-1}, v\rangle)$ is constant with respect to x, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \exp\left(-\langle x, Ax\rangle + 2\langle v, x\rangle\right) dx \\
&\qquad = \exp\left(\left\langle vA^{-1}, v\right\rangle\right) \exp\left(-\left\langle vA^{-1}, v\right\rangle\right) \int_{\mathbb{R}^n} \exp\left(-\langle x, Ax\rangle + 2\langle v, x\rangle\right) dx \\
&\qquad = \exp\left(\left\langle vA^{-1}, v\right\rangle\right) \int_{\mathbb{R}^n} \exp\left(-\langle x, Ax\rangle + 2\langle v, x\rangle - \left\langle vA^{-1}, v\right\rangle\right) dx \\
&\qquad = \exp\left(\left\langle vA^{-1}, v\right\rangle\right) \int_{\mathbb{R}^n} \exp\left(\left\langle -x + vA^{-1},\ Ax - v\right\rangle\right) dx
\end{aligned}$$

$$= \exp\left(\left\langle vA^{-1}, v\right\rangle\right) \int_{\mathbb{R}^n} \exp\left(\left\langle -(x - vA^{-1}),\ A(x - vA^{-1})\right\rangle\right) dx \tag{i}$$

$$= \exp\left(\left\langle vA^{-1}, v\right\rangle\right) \int_{\mathbb{R}^n} \exp\left(\langle -x,\ Ax\rangle\right) dx \tag{ii}$$

$$= \exp\left(\left\langle v, A^{-1}v\right\rangle\right) \left(\pi^{n/2}/\sqrt{\det A}\right).$$

Step (i) is justified by the fact that since A is symmetric, then $vA^{-1} = A^{-1}v$ (letting any vector be a row or column vector as is convenient). In step (ii), we are making a change of variables, which comes for free since adding the constant $-vA^{-1}$ gives the same Jacobian as adding zero. ■