Math 501
Homework 6

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1. Give an example of a closed, bounded subset of a metric space which is not compact.

   Example: Consider the real numbers $\mathbb{R}$ with the topology generated by the following metric:

   $$d(x, y) = \min(|x - y|, 1)$$

   where $|x|$ denotes the usual absolute value, not the norm with respect to d. First, we will confirm that d is in fact a metric.

   Proof

   Let $x, y, z \in \mathbb{R}$.

   - Since $|x - y| = 0$ iff $x = y$, then $d(x, y) = \min(0, 1) = 0$ iff $x = y$. This means that the definiteness property holds.
   - Since $|x - y| = |y - x|$, then $d(x, y) = \min(|x - y|, 1) = d(y, x)$. Thus, the symmetric property holds.
   - We will show that the triangle inequality holds for d, that is, $d(x, z) \leq d(x, y) + d(y, z)$.

     Case I: Either $|x - y| \geq 1$ or $|y - z| \geq 1$. Without loss of generality, suppose $|x - y| \geq 1$. Then,

     $$\begin{aligned} d(x, z) &= \min(|x - z|, 1) \\ &\leq 1 \\ &\leq 1 + d(y, z) \\ &= d(x, z) + d(y, z) \end{aligned}$$

     Case II: Both $|x - y| < 1$ and $|y - z| < 1$. Then,

     $$\begin{aligned} d(x, z) &= \min(|x - z|, 1) \\ &\leq |x - z| \\ &\leq |x - y| + |y - z| \\ &= d(x, z) + d(y, z) \end{aligned}$$

   Thus, d is a metric. $\blacksquare$

   Claim: $\mathbb{R}$ is closed and bounded under the topology generated by d, but is not compact.

   Proof

   - $\mathbb{R}$ is the entire set, so is closed.
   - Since $B_d(0, 2) = \{x \in \mathbb{R} : d(0, x) < 2\} = \mathbb{R}$, then $\mathbb{R}$ is bounded.

   Now we will show that $\mathbb{R}$ is not compact. Consider the open cover $\{U_n\}_{n \in \mathbb{Z}} = \{B_d(n, 1) : n \in \mathbb{Z}\}$[1]. We can see that $\{U_n\}_{n \in \mathbb{Z}}$ covers $\mathbb{R}$, since for any $x \in \mathbb{R}$, then $x \in B_d(\lfloor x \rfloor, 1) \in \{U_n\}_{n \in \mathbb{Z}}$.

[1] Note: Since the sets in $\{U_n\}_{n \in \mathbb{Z}}$ are open, they only contain elements whose distance from n under d is less than 1.

Now, let

$$\{U_{a_n}\}_{n=1}^{N} \subset \{U_n\}_{n\in\mathbb{Z}}$$

be a finite subcollection of the sets in $\{U_n\}_{n\in\mathbb{Z}}$, with $a_n \in \{a_1, a_2, \ldots, a_N\} \subset \mathbb{Z}$. Now,

$$\max(\{a_1, a_2, \ldots, a_N\}) + 1 \notin \bigcup_{n=1}^{N} \{U_{a_n}\},$$

so $\{U_{a_n}\}_{n=1}^{N}$ does not cover $\mathbb{R}$. Thus, $\mathbb{R}$ is not compact. ∎

Remark. Though we omit the formal proof here, this also generalizes to $\mathbb{R}^n$ with $d(x, y) = \min(d_{usual}(x, y), \sqrt{n})$. The open cover $\{U_n\}_{n\in\mathbb{Z}} = \{B_d((x_1, \ldots, x_n), \sqrt{n}) : x_1, \ldots, x_n \in \mathbb{Z}\}$ has no finite subcover, since every finite subset has an element whose center has greatest absolute value (that is, farthest from the origin under d_{usual}), and thus cannot cover $\mathbb{R}^n$.