A Problem from Erdös About Products of 2 or 3 Primes

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Motivation

Take two prime numbers, say 2 and 3, and make a list of all the natural numbers which can be formed using only 2 and 3 as factors.

2	*3*	*4*	*6*	*8*	*9*	*12*	...
2^1	3^1	2^2	$2^1 3^1$	2^3	3^2	$2^2 3^1$	...

There is an interesting pattern here: there keep ocurring pairs of numbers which have only 2 or only 3 as a factor.

...	*16*	*18*	*24*	*27*	*32*	*36*	...
...	2^4	$2^1 3^2$	$2^3 3^1$	3^3	2^5	$2^2 3^2$	...

Motivation

Does this keep happening?

...	*1536*	*1728*	*1944*	*2048*	*2187*	*2304*	...
...	$2^9 3^1$	$2^6 3^3$	$2^3 3^5$	2^{11}	3^7	$2^8 3^2$	...

Does it still happen with any primes?

59	*61*	*3481*	*3599*	*3721*	*205379*	*212341*	...
59^1	61^1	59^2	$59^1 61^1$	61^2	59^3	$59^2 61^1$	...

... $59^3 61^2$ $59^2 61^3$ $59^1 61^4$ 61^5 59^6 $59^5 61^1$ ...

Motivation

What if you do it with *three* primes (i.e. 2, 3, and 5)?

$$\begin{array}{cccccccc} \ldots & \mathit{18} & \mathit{20} & \mathit{24} & \mathit{25} & \mathit{27} & \mathit{30} & \ldots \\ \ldots & 2^1 3^2 & 2^2 5^1 & 2^3 3^1 & 5^2 & 3^3 & 2^1 3^1 5^1 & \ldots \end{array}$$

Question!

Paul Erdös asked the following question about three distinct primes: If we construct a sequence of all the products of their powers, with the sequence arranged in increasing order, is it true infinitely often that consecutive terms in this sequence are both prime-powers?

Definitions

Let p, q be distinct primes, and let $m, n \in \mathbb{Z}^+$.

Definition

A pure power of p is an integer of the form p^m.

Definition

A mixed power of p and q is an integer of the form $p^m q^n$.

Definition

A critical pair of p and q is a pair of pure powers of p and q which do not have a mixed power between them.

Developing Intuition

Lemma 1

If $a_k = q^n$, then $a_{k+1} \neq q^{n+1}$.

Ex.

$$\ldots \quad 2^4 \quad 2^1 3^2 \quad 2^3 3^1 \quad 3^3 \quad \ldots$$

Here, 3^4 can't come next, because $3^3 < 3^3 2^1 < 3^4$.

Early Stages

Lemma 2

There exist at most finitely many $a_k = p^m$ such that $a_{k+1} = p^{m+1}$.

Even in a dramatic example like $p = 2$, $q = 509$, the sequence starts out all powers of 2,

$$2^1 \quad 2^2 \quad 2^3 \quad 2^4 \quad 2^5 \quad \ldots$$

but this can't continue forever.

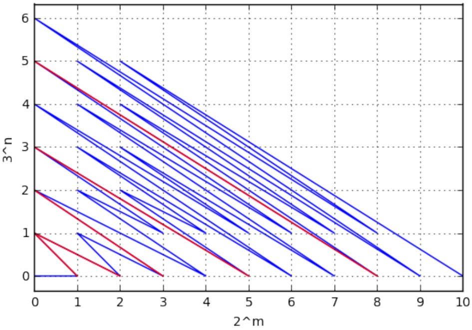
$$2^7 \quad 2^8 \quad 509^1 \quad 2^9 \quad 2^1 509^1 \quad \ldots$$

Early Stages

Lemma 3

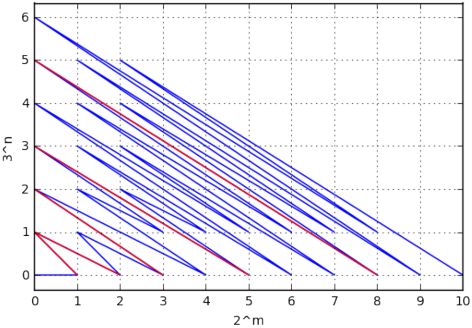
If $a_i = p^m$ and $a_{i+1} = q^n$, then m and n are relatively prime.

The "Tail" of the Sequence of Exponents



13	*47*	*169*	*611*	*2197*	*2209*	*7943*	...
13^1	47^1	13^2	$13^1 47^1$	13^3	47^2	$13^2 47^1$	...

The "Tail" of the Sequence of Exponents

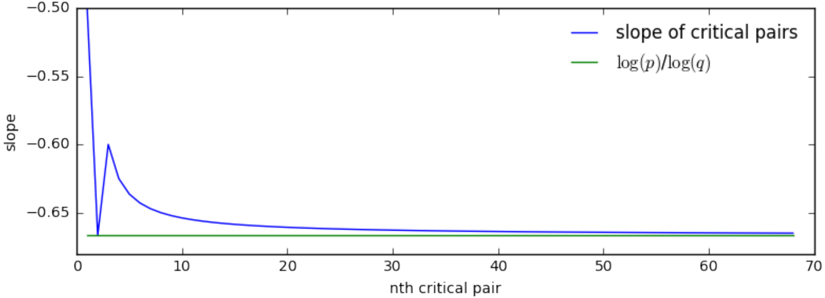


Notice how the lines seem to approach a constant slope!

As $m, n \to \infty$, we see that $\frac{m}{n} \to \frac{\log p}{\log q}$.

Converging Ratios

As $m, n \to \infty$, we see that $\frac{m}{n} \to \frac{\log p}{\log q}$.



Two Primes

After developing an understanding for the problem, we began to analyze the details that Erdös glossed over. This lead us to a proof for the following theorem:

Theorem 1

For any two distinct prime numbers p, and q, there exist infinitely many critical pairs.

Key Lemmas

Lemma 4

Consider the pure powers p^a, q^b with $p^a < q^b$ and $a, b \in \mathbb{Z}^+$. If, for all critical pairs p^s, q^t with $s < a$ and $t < b$,

$$1 < \frac{q^b}{p^a} < \frac{q^t}{p^s}, \quad s, t \in \mathbb{Z}^+$$

then p^a, q^b is a critical pair.

Lemma 5

Let α be an irrational number. Given any $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $n\alpha - \lfloor n\alpha \rfloor < \epsilon$.

Initial Proof Sketch

Maybe here we can give a few key steps to sketch our first proof? e.g., "By Contradiction", then explain the construction of our pair with a smaller ratio closer to 1, etc. Maybe state how it is reminiscent of certain proofs like the infinitely many primes proof?

Redeveloped Proof

Here we can describe the 2nd version of the proof with a graph showcasing the geometric intuition behind the proof?

Issues with 3 Primes

We have attempted the problem with various techniques: Mean Value Theorem/Taylor Polynomial bounds, Geometric Arguments, Linear Programming, Special cases (59,61,3601, ($p,q,r = pq+2$), etc. Should we mention these? Mention why it is so difficult?

Moving Forward

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