Office Hours!

Instructor:

Peter M. Garfield, garfield@math.ucsb.edu

Office Hours:

Mondays 2–3PM

Tuesdays 10:30–11:30AM

Thursdays 1–2PM

or by appointment

Office:

South Hall 6510

§8.12: The Second Derivative

Today: We can take the derivative of a function repeatedly!

Example: If $f(x) = x^3 - 3x + 2$, then

- $\dfrac{df}{dx} = f'(x) = 3x^2 - 3$
- The second derivative of $f(x)$ is $\dfrac{d}{dx}\left(\dfrac{df}{dx}\right) = f''(x) = 6x$.

  This is written $f''(x)$ or $\dfrac{d^2f}{dx^2}$.
- The third derivative of $f(x)$ is $\dfrac{d}{dx}\left(\dfrac{d^2f}{dx^2}\right) = f'''(x) = 6$.

  This is written $f'''(x)$ or $\dfrac{d^3f}{dx^3}$.
- Keep Going! The fourth derivative is $\dfrac{d^4f}{dx^4} = f''''(x) = 0$.
- The fun ends here, for this $f(x)$ all higher derivatives are zero.

Examples

General idea: Differentiating the function n times gives us the nth derivative of f. It is written as

$$f''''\cdots''''(x) = f^{(n)}(x) = \frac{d^n f}{dx^n}.$$

(1) What is the second derivative of $3x^2 - 5x + 7$?

A$= 0$ B$= 7$ C$= 6$ D$= 3$ E$= -5$ $\boxed{\text{C}}$

(2) $\dfrac{d^2}{dx^2}\left(x^5\right) = ?$

A$= 20$ B$= 5x^4$ C$= 0$ D$= 20x^4$ E$= 20x^3$ $\boxed{\text{E}}$

(3) $\dfrac{d^2}{dx^2}\left(\sqrt{x}\right) = ?$

A$= \frac{1}{4}x^{-3/2}$ B$= \frac{-1}{4}x^{-1/2}$ C$= \frac{-1}{4}x^{-3/2}$ D$= \frac{1}{2}x^{-1/2}$ E$= 0$ $\boxed{\text{C}}$

More Examples

(4) $\dfrac{d^2}{dt^2}\left(e^{3t}\right) = ?$

A$= e^{3t}$ B$= 3e^{2t}$ C$= 9e^{3t}$ D$= 3e^{3t}$ E$= 9e^{t}$ $\boxed{\text{C}}$

(5) Find $f'''(x)$ when $f(x) = x^3$.

A$= 6x^2$ B$= 0$ C$= 3x$ D$= 3x^2$ E$= 6$ $\boxed{\text{E}}$

(6) If $f(x) = x^3 - 4x^2 + 7x - 31$, then $f''(10) = ?$

A$= 6$ B$= 3x^2 - 8x$ C$= 6x$ D$= 60$ E$= 52$ $\boxed{\text{E}}$

Example: Acceleration

The acceleration due to gravity is

$$32 \text{ feet per second per second} = 32 \text{ ft/sec}^2.$$

This means:

every second you fall,
your speed increases by 32 ft/sec $\approx$ 22 mph.

acceleration = rate of change of velocity = derivative of velocity.

velocity = rate of change of distance = derivative of distance.

Therefore

acceleration = second derivative of distance

Example: Height of ball is $h(t) = 20t - 5t^2$ meters after t seconds.

(a) Velocity of ball after t seconds is $h'(t) = 20 - 10t$ m/sec

(b) Acceleration of ball after t seconds is $h''(t) = -10$ m/sec^2

It's not the speed that kills

Suppose you hit a brick wall at 60 mph.
Question: What is your (sudden!) acceleration?

$$\begin{aligned}\left(\begin{array}{c}\text{Average rate of}\\ \text{change of velocity}\\ \text{in stopping}\end{array}\right) &= \frac{\Delta\text{ velocity}}{\Delta\text{ time}} = \frac{-60\text{ mph}}{1/10\text{ sec}}\\ &\approx \frac{-88\text{ ft/sec}}{1/10\text{ sec}} = -880\text{ ft/sec}^2.\end{aligned}$$

Since 1 gravity $= 32$ ft/sec^2, this is about

$$880\text{ ft/sec}^2 = \left(880\text{ ft/sec}^2\right) \times \frac{1\text{ gravity}}{32\text{ ft/sec}^2} \approx 28\text{ ``g''}.$$

The force which pushes you at the windshield is about 28 times your weight.
If you weigh 110 pounds, this force is about 3000 pounds $=$ 1.5 tons.

Rocket!

A rocket is fired vertically upwards. The height after t seconds is $2t^3 + 5t^2$ meters.

Question: What is the acceleration in $\mathrm{m/sec}^2$ after t seconds?

A$= 2t^3 + 5t^2$ B$= 6t^2 + 10t$ C$= 12t + 10$ D$= 12$ E$= 0$ $\boxed{\text{C}}$

Idea:

- $h(t) =$ height in meters at time t seconds
- $h'(t) =$ velocity in m/sec at time t seconds
- $h''(t) =$ acceleration in $\mathrm{m/sec}^2$ at time t seconds

More Questions:

(a) What can we say about $f(t)$ if $f'(t) = 0$ for all t?

(b) What can we say about $f(t)$ if $f''(t) = 0$ for all t?

Application 2: Concavity

$$\frac{df}{dx} = \text{rate of change of } f(x)$$

$$\text{and so } \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \text{rate of change of } \frac{df}{dx}$$

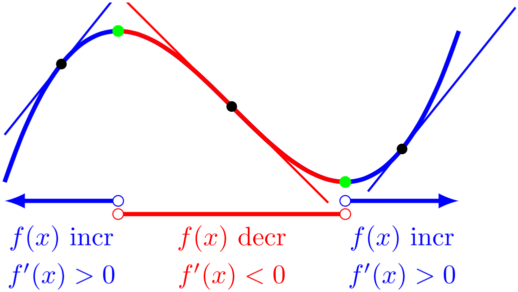
Conclusion:

> The second derivative tells you how quickly the rate of change is changing.

Uses of second derivative:

- We've seen: acceleration is the rate of change of velocity
  So: acceleration is the second derivative of distance traveled.
- Is the graph concave up or concave down?
- Are things changing for better or worse?

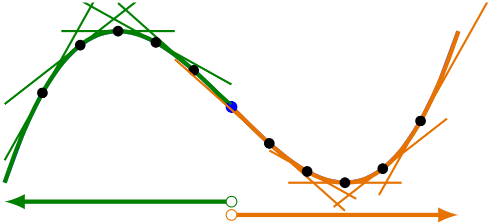
Meanings: The First Derivative



Point:

$$f'(x) > 0 \iff f(x) \text{ is increasing}$$
$$f'(x) < 0 \iff f(x) \text{ is decreasing}$$

Meanings: The Second Derivative



Point:

$$f''(x) > 0 \iff f'(x) \text{ is increasing}$$
$$\iff f(x) \text{ is concave up}$$

$$f''(x) < 0 \iff f'(x) \text{ is decreasing}$$
$$\iff f(x) \text{ is concave down}$$

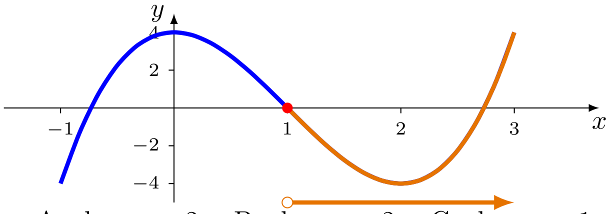
Concavity

$$f''(x) > 0 \iff f(x) \text{ is concave up}$$
$$f''(x) < 0 \iff f(x) \text{ is concave down}$$

(1) For which values of x is $f(x) = x^3 - 6x^2 + 3x + 2$ concave up?

A when $x = 0$ B when $x < 6$ C when $x > 6$
D when $x < 2$ E when $x > 2$ $\boxed{\text{E}}$

(2) Where is $f''(x) > 0$?



A when $x < 2$ B when $x > 2$ C when $x < 1$
D when $x > 1$ E when $-0.7 < x < 1$ $\boxed{\text{D}}$

Review Problems

(1) An oil slick in the shape of a rectangle is expanding. After t hours the length is $30t$ meters and the width is $50t$ meters. How quickly is the area increasing in m^2/hour after 2 hours?

A= 800 B= 1500 C= 3200 D= 6000 E= Other $\boxed{\text{D}}$

(2) Suppose $f'(1) = 4$ and $g'(1) = 3$. What is the rate of change of $f(x) + 2g(x)$ when $x = 1$?

A $= 3$ B $= 4$ C $= 7$ D $= 10$ E $= 14$ $\boxed{\text{D}}$

More Review Problems

(a) What is the x-coordinate of the point on the graph $y = 2x^2 + 5x - 7$ where the slope is 11?

$$\mathrm{A} = 1 \quad \mathrm{B} = 3/2 \quad \mathrm{C} = 2 \quad \mathrm{D} = 5/3 \quad \mathrm{E} = 0 \quad \boxed{\mathrm{B}}$$

(b) What is the value of x at the point on the graph $y = 4x^2 + 16x$ where the tangent line is horizontal?

$$\mathrm{A}= 2 \quad \mathrm{B}= 0 \quad \mathrm{C}= -2 \quad \mathrm{D}= -4 \quad \boxed{\mathrm{C}}$$

(c) $\dfrac{d}{dx}\left(\dfrac{3}{x^4}\right) =?$

$$\mathrm{A}= \frac{3}{4x^3} \quad \mathrm{B}= \frac{12}{x^5} \quad \mathrm{C}= -\frac{3}{4x^3} \quad \mathrm{D}= -\frac{12}{x^5} \quad \boxed{\mathrm{D}}$$