Welcome To Math 34A! Differential Calculus

Instructor:

Trevor Klar, trevorklar@math.ucsb.edu
South Hall 6431X (Grad Tower, 6th floor, blue side, first door on the right)

Office Hours:

MTWR after class 2:00-3:00, and by appointment. Details on Gauchospace.

A nice thing about derivatives...

$$\frac{d}{dx}(a \cdot f(x) + b \cdot g(x)) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$$
$$= a \cdot f'(x) + b \cdot g'(x)$$

For example...

$$\begin{aligned} \tfrac{d}{dx}(3x^2 + 5x) &= 3\tfrac{d}{dx}x^2 + 5\tfrac{d}{dx}x \\ &= 3(2x) + 5(1) \\ &= 6x + 5 \end{aligned}$$

A Warning!

$$\frac{d}{dx}\left(f(x)g(x)\right) \neq f'(x) \times g'(x)$$

Example: $5x^4 = \dfrac{d}{dx}\left(x^5\right) = \dfrac{d}{dx}\left(x^2 \cdot x^3\right) \neq (2x)(3x^2) = 6x^3$

Example: Find the derivative of $(x+1)(2x+3)$

Question: $\dfrac{d}{dx}\left((x^2+1)(x^3+1)\right) = ?$

A$= 6x^3$ B$= 5x^4 + 3x^2 + 2x$ C$= x^5 + x^3 + x^2 + 1$ D$=$ Other

Answer: B

Review Examples:

(1) What is the x-coordinate of the point on the graph of $y = 4x^2 - 3x + 7$ where the graph has slope 13?

A$= 0$ B$= 1$ C$= 2$ D$= 3$ E$= 4$ C

(2) A circle is expanding so that after R seconds it has radius R cm. What is the rate of increase of area inside the circle after 2 seconds?

A$= 4\pi$ B$= 2\pi R^2$ C$= 2$ D$= 2\pi R$ E$= \pi R^2$ A

Lec 14

Review Examples:

(1) What is the x-coordinate of the point on the graph of $y = 4x^2 - 3x + 7$ where the graph has slope 13?

A$=0$ B$=1$ C$=2$ D$=3$ E$=4$

$y' = 8x - 3$

$13 = 8x - 3$

$16 = 8x$

$\boxed{x = 2}$

Differentiating $f(x) = e^{kx}$

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

versus

$$\frac{d}{dx}\left(x^{n}\right) = nx^{n-1}$$

Do not get confused and write $\frac{d}{dx}\left(e^{kx}\right) = ke^{(k-1)x}$.

Question: Find $\frac{d}{dx}\left(4e^{3x} + 5x^3\right)$

A$= 12e^{2x} + 15x^2$ B$= 12e^{3x} + 15x^3$ C$= 4e^{3x} + 15x^2$
D$= 12e^{3x} + 15x^2$ E$=$ Other D

Differentiating $f(x) = e^{kx}$

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

versus

$$\frac{d}{dx}\left(x^{n}\right) = nx^{n-1}$$

Do not get confused and write $\frac{d}{dx}\left(e^{kx}\right) = ke^{(k-1)x}$.

Question: Find $\frac{d}{dx}\left(4e^{3x} + 5x^3\right)$

A= $12e^{2x} + 15x^2$ B= $12e^{3x} + 15x^3$ C= $4e^{3x} + 15x^2$

$\boxed{\text{D} = 12e^{3x} + 15x^2}$ E= Other

$$\frac{d}{dx}\left(4e^{3x} + 5x^{3}\right) = 12e^{\overset{\text{same}}{3x}} + 15x^{\overset{-1}{2}}$$

Example

$$\boxed{\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}}$$

The temperature (in $^\circ$ C) of a cup of coffee t hours after it is made is $f(t) = 50 + 40e^{-2t}$.

(a) What is the initial temperature when the coffee is made?

A= 40 B= 50 C= 90 D= 100 C

(b) How quickly is the coffee cooling down initially? This means how many degrees per hour is the temperature going down instantaneously at $t = 0$?

A= 20 B= 40 C= 60 D= 80 E= 100 D

Example

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

The temperature (in $^\circ$ C) of a cup of coffee t hours after it is made is $f(t) = 50 + 40e^{-2t}$.

(a) What is the initial temperature when the coffee is made?

A= 40 B= 50 C= 90 D= 100

$f(0) = 50 + 40e^{(0)}$

$= 50 + 40$

$= 90$

Review · Higher Derivatives · Acceleration · Concavity

Example

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

The temperature (in $^\circ$ C) of a cup of coffee t hours after it is made is $f(t) = 50 + 40e^{-2t}$.

(a) What is the initial temperature when the coffee is made?

Same

$f'(t) = 0 - 80e^{-2t}$

A= 40 B= 50 C= 90 D= 100 C

(b) How quickly is the coffee cooling down initially? This means how many degrees per hour is the temperature going down instantaneously at $t = 0$?

A= 20 B= 40 C= 60 $\boxed{\text{D= 80}}$ E= 100

$f'(0) = -80e^{(0)}$

More Examples

$$\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$$

(1) $\dfrac{d}{dx}\left(\dfrac{3}{e^{2x}}\right) = ?$

A $= \frac{3}{2e^{2x}}$ B $= \frac{3}{2e^{x}}$ C $= \frac{6}{e^{2x}}$ D $= \frac{-6}{e^{2x}}$ D

(2) The number of grams of Einsteinium-253 after t days is $m(t) = 10e^{-t/30}$. How quickly is the mass changing (in grams per day) when $t = 0$?

A $= -1/30$ B $= -1/3$ C $= -10e^{-t/30}$ D $= -\frac{1}{3}\,e^{t/30}$ B

More Examples

$\frac{d}{dx} \frac{1}{e^{2x}} = ??$

$$\boxed{\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}}$$

(1) $\frac{d}{dx}\left(\frac{3}{e^{2x}}\right) = \frac{d}{dx}(3e^{-2x}) = -6e^{-2x}$

A $= \frac{3}{2e^{2x}}$ B $= \frac{3}{2e^{x}}$ C $= \frac{6}{e^{2x}}$ D $= \frac{-6}{e^{2x}}$

(2) The number of grams of Einsteinium-253 after t days is $m(t) = 10e^{-t/30}$. How quickly is the mass changing (in grams per day) when $t = 0$?

A $= -1/30$ $\boxed{\text{B} = -1/3}$ C $= -10e^{-t/30}$ D $= -\frac{1}{3}\,e^{t/30}$

$m'(t) = \frac{d}{dt}\left(10e^{-\frac{1}{30}t}\right) = -\frac{10}{30}e^{-t/30} = -\frac{1}{3}e^{-t/30}$

$m'(0) = -\frac{1}{3}e^{(0)} = -\frac{1}{3}$

§8.12: The Second Derivative

Today: We can take the derivative of a function repeatedly!

Example: If $f(x) = x^3 - 3x + 2$, then

- $\dfrac{df}{dx} = f'(x) = 3x^2 - 3$
- The second derivative of $f(x)$ is $\dfrac{d}{dx}\left(\dfrac{df}{dx}\right) = f''(x) = 6x$.

  This is written $f''(x)$ or $\dfrac{d^2 f}{dx^2}$.
- The third derivative of $f(x)$ is $\dfrac{d}{dx}\left(\dfrac{d^2 f}{dx^2}\right) = f'''(x) = 6$.

  This is written $f'''(x)$ or $\dfrac{d^3 f}{dx^3}$.
- Keep Going! The fourth derivative is $\dfrac{d^4 f}{dx^4} = f''''(x) = 0$.
- The fun ends here, for this $f(x)$ all higher derivatives are zero.

Examples

General idea: Differentiating the function n times gives us the nth derivative of f. It is written as

$$f^{''''\cdots''''}(x) = f^{(n)}(x) = \frac{d^n f}{dx^n}.$$

(1) What is the second derivative of $3x^2 - 5x + 7$?

A$= 0$ B$= 7$ C$= 6$ D$= 3$ E$= -5$ C

(2) $\dfrac{d^2}{dx^2}\left(x^5\right) = ?$

A$= 20$ B$= 5x^4$ C$= 0$ D$= 20x^4$ E$= 20x^3$ E

(3) $\dfrac{d^2}{dx^2}\left(\sqrt{x}\right) = ?$

A$= \frac{1}{4}x^{-3/2}$ B$= \frac{-1}{4}x^{-1/2}$ C$= \frac{-1}{4}x^{-3/2}$ D$= \frac{1}{2}x^{-1/2}$ E$= 0$ C

Examples

General idea: Differentiating the function n times gives us the nth derivative of f. It is written as

$$f''''\cdots''''(x) = f^{(n)}(x) = \frac{d^n f}{dx^n}.$$

(1) What is the second derivative of $3x^2 - 5x + 7$?

A$= 0$ B$= 7$ $\boxed{\text{C}= 6}$ D$= 3$ E$= -5$

$f' = 6x - 5$

$f'' = 6$

$x^{-\frac{1}{2}-1}$

(3) $\frac{d^2}{dx^2}(\sqrt{x}) = \frac{d^2}{dx^2}\left(x^{1/2}\right) = \frac{d}{dx}\left(\frac{1}{2}x^{-1/2}\right) = -\frac{1}{4}x^{-3/2}$

A$= \frac{1}{4}x^{-3/2}$ B$= \frac{-1}{4}x^{-1/2}$ $\boxed{\text{C}= \frac{-1}{4}x^{-3/2}}$ D$= \frac{1}{2}x^{-1/2}$ E$= 0$

More Examples

(4) $\frac{d^2}{dt^2}\left(e^{3t}\right) =?$

A$= e^{3t}$ B$= 3e^{2t}$ C$= 9e^{3t}$ D$= 3e^{3t}$ E$= 9e^{t}$ C

(5) Find $f'''(x)$ when $f(x) = x^3$.

A$= 6x^2$ B$= 0$ C$= 3x$ D$= 3x^2$ E$= 6$ E

(6) If $f(x) = x^3 - 4x^2 + 7x - 31$, then $f''(10) =?$

A$= 6$ B$= 3x^2 - 8x$ C$= 6x$ D$= 60$ E$= 52$ E

Example: Acceleration

The acceleration due to gravity is

$$32 \text{ feet per second per second} = 32 \text{ ft/sec}^2.$$

This means:

every second you fall,
your speed increases by 32 ft/sec $\approx$ 22 mph.

acceleration = rate of change of velocity = derivative of velocity.

velocity = rate of change of distance = derivative of distance.

Therefore

acceleration = second derivative of distance

Example: Height of ball is $h(t) = 20t - 5t^2$ meters after t seconds.
(a) Velocity of ball after t seconds is $h'(t) = 20 - 10t$ m/sec
(b) Acceleration of ball after t seconds is $h''(t) = -10$ m/sec^2

It's not the speed that kills

Suppose you hit a brick wall at 60 mph.
Question: What is your (sudden!) acceleration?

$$\begin{pmatrix} \text{Average rate of} \\ \text{change of velocity} \\ \text{in stopping} \end{pmatrix} = \frac{\Delta \text{ velocity}}{\Delta \text{ time}} = \frac{-60 \text{ mph}}{1/10 \text{ sec}}$$
$$\approx \frac{-88 \text{ ft/sec}}{1/10 \text{ sec}} = -880 \text{ ft/sec}^2.$$

Since 1 gravity = 32 ft/sec^2, this is about

$$880 \text{ ft/sec}^2 = \left(880 \text{ ft/sec}^2\right) \times \frac{1 \text{ gravity}}{32 \text{ ft/sec}^2} \approx 28 \text{ ``g''}.$$

The force at which the brick wall pushes you is 28 times your weight.
If you weigh 110 pounds, this force is about 3000 pounds = 1.5 tons.

A Rocket

A rocket is fired vertically upwards. The height after t seconds is $2t^3 + 5t^2$ meters.

Question: What is the acceleration in m/sec^2 after t seconds?

A$= 2t^3 + 5t^2$ B$= 6t^2 + 10t$ C$= 12t + 10$ D$= 12$ E$= 0$ C

Idea:

- $h(t) =$ height in meters at time t seconds
- $h'(t) =$ velocity in m/sec at time t seconds
- $h''(t) =$ acceleration in m/sec^2 at time t seconds

More Questions:

(a) What can we say about $f(t)$ if $f'(t) = 0$ for all t?

(b) What can we say about $f(t)$ if $f''(t) = 0$ for all t?

A Rocket

A rocket is fired vertically upwards. The height after t seconds is $2t^3 + 5t^2$ meters.

f''

Question: What is the acceleration in m/sec^2 after t seconds?

A= $2t^3 + 5t^2$ B= $6t^2 + 10t$ C= $12t + 10$ D= 12 E= 0

$f' = 6t^2 + 10t$

$\boxed{f'' = 12t + 10}$

A Rocket

A rocket is fired vertically upwards. The height after t seconds is $2t^3 + 5t^2$ meters.

Question: What is the acceleration in m/sec^2 after t seconds?

A= $2t^3 + 5t^2$ B= $6t^2 + 10t$ C= $12t + 10$ D= 12 E= 0 C

Idea:

- $h(t)$ = height in meters at time t seconds
- $h'(t)$ = velocity in m/sec at time t seconds
- $h''(t)$ = acceleration in m/sec^2 at time t seconds

More Questions:

(a) What can we say about $f(t)$ if $f'(t) = 0$ for all t? constant

(b) What can we say about $f(t)$ if $f''(t) = 0$ for all t? straight line

Application 2: Concavity

$$\frac{df}{dx} = \text{rate of change of } f(x)$$

$$\text{and so } \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \text{rate of change of } \frac{df}{dx}$$

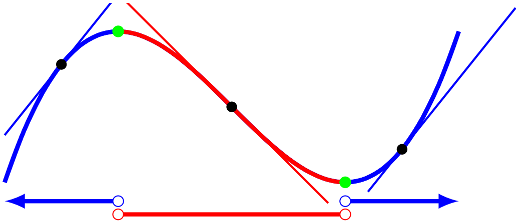
Conclusion:

> The second derivative tells you how quickly the rate of change is changing.

Uses of second derivative:

- We've seen: acceleration is the rate of change of velocity
  So: acceleration is the second derivative of distance traveled.
- Is the graph concave up or concave down?
- Are things changing for better or worse?

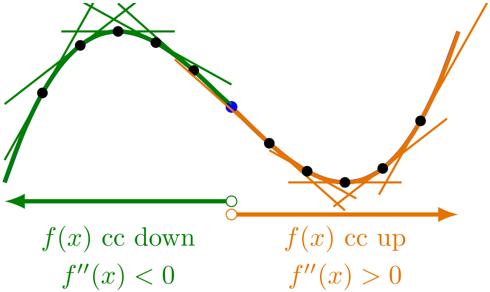
Meanings: The First Derivative



Point:

$$f'(x) > 0 \iff f(x) \text{ is increasing}$$
$$f'(x) < 0 \iff f(x) \text{ is decreasing}$$

Meanings: The Second Derivative



Point:

$$\begin{aligned} f''(x) > 0 &\iff f'(x) \text{ is increasing} \\ &\iff f(x) \text{ is concave up} \\[4pt] f''(x) < 0 &\iff f'(x) \text{ is decreasing} \\ &\iff f(x) \text{ is concave down} \end{aligned}$$

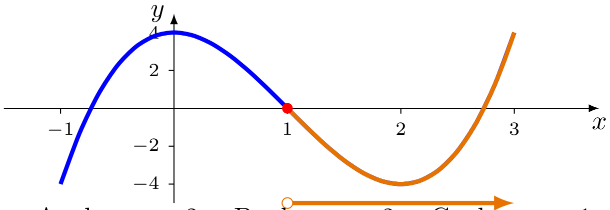
Concavity

$$f''(x) > 0 \iff f(x) \text{ is concave up}$$
$$f''(x) < 0 \iff f(x) \text{ is concave down}$$

(1) For which values of x is $f(x) = x^3 - 6x^2 + 3x + 2$ concave up?

A when $x = 0$ B when $x < 6$ C when $x > 6$
D when $x < 2$ E when $x > 2$ E

(2) Where is $f''(x) > 0$?



A when $x < 2$ B when $x > 2$ C when $x < 1$
D when $x > 1$ E when $-0.7 < x < 1$ D

Concavity

$$f''(x) > 0 \iff f(x) \text{ is concave up}$$
$$f''(x) < 0 \iff f(x) \text{ is concave down}$$

(1) For which values of x is $f(x) = x^3 - 6x^2 + 3x + 2$ concave up?

A when $x = 0$ B when $x < 6$ C when $x > 6$
D when $x < 2$ E when $x > 2$

where is $f'' > 0$?

$f'(x) = 3x^2 - 12x + 3$

$f''(x) = 6x - 12$

$6x - 12 > 0$
$+12 \quad +12$
$6x > 12$
$x > 2$

If mult. by negative or take reciprocals,

$< \longrightarrow >$

$> \longrightarrow <$