Math 201B, Homework 4 (Metric and Normed Spaces, Baire Cathegory)

Problem1. Let $\{B_n\}$ be a nested sequence of closed balls in a normed space X, where

$$B_n = \bar{B}_{r_n}(x_n), \quad \text{with} \quad r_n \geq r > 0 \quad \text{for all} \quad n \in \mathbb{N}.$$

1. Is it true that
$$\cap_{n=1}^{\infty} B_n \neq \emptyset$$
2. Is it true that
$$B \subset \cap_{n=1}^{\infty} B_n$$
for some closed ball B with radius r?

Problem2. Construct a Lebesgue-measurable set $A \subset [0,1]$ such that $m(A) = 1$ and A is of Baire first category in $[0,1]$.

Problem3. Let $f\colon (0,1) \to \mathbb{R}$ be continuous. Prove that if $\lim_{n\to\infty} f(\frac{x}{n}) = 0$ for all $x \in (0,1)$, then $\lim_{x\to 0} f(x) = 0$.

Problem4. Let X be a real normed space and let C be a closed convex set such that $\bar{B}_{1+\epsilon}(0) \subset C + \bar{B}_1(0)$ for some $\epsilon > 0$. Does it follow that C has a nonempty interior?

Remark.
1. A set $A \subset X$ is convex if $x, y \in A$ implies $\lambda x + (1-\lambda)y \in A$ for all $\lambda \in [0,1]$.
2. The sum of two sets $A, B \subset X$ is defined as $A + B = \{a + b \ : \ a \in A, b \in B\}$.