(7) Let $f: X \to Y$ be a quotient map. Prove that if Y is connected & each $f^{-1}(\{y\})$ is connected then X is connected.

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Suppose X is not connected, so there exists disjoint nonempty sets U & V such that $U \cup V = X$. Thus $f(U) \cup f(V) = Y$. If $\exists\, y \in f(U) \cap f(V)$ then $f^{-1}(\{y\}) \cap U$ & $f^{-1}(\{y\}) \cap V \neq \emptyset$ but this cannot happen since U & V are a separation & $f^{-1}(\{y\})$ is connected. Thus $Y = f(U) \cup f(V)$ & $f(U) \cap f(V) \neq \emptyset$. It is left to show $f(U)$ & $f(V)$ are open.

Note that $U \subseteq f^{-1}(f(U))$ & $V \subseteq f^{-1}(f(V))$. If $x \in V$ but $x \in f^{-1}(f(U))$ then $f(x) \in f(U) \cap f(V) = \emptyset$. Therefore $f^{-1}(f(U)) = U$ & $f^{-1}(f(V)) = V$, which are open.

Since f is a quotient map, a set is open in Y iff it's preimage in X is open. Thus $f(U)$ & $f(V)$ are open giving a separation of Y which contradicts that Y is connected. $\square$