Math 450B
Homework 3
Dr. Fuller
Due February 13

1. Determine if the following examples are continuous on the indicated domain. Justify your answers.

   (a) $f : \mathbf{R}^2 - \{\mathbf{0}\} \to \mathbf{R}$ given by $f(x,y) = \dfrac{xy}{x^2+y^2}$

   (b) $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

   (c) $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

2. Prove that $f : \mathbf{R}^n \to \mathbf{R}$ given by $f(x) = \|x\|$ is continuous.

3. Suppose that $f : A \subseteq \mathbf{R}^n \to \mathbf{R}^m$ satisfies $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq K\|\mathbf{x}-\mathbf{y}\|^{\alpha}$, where $K > 0$ and $\alpha > 0$ are constants. Prove that f is continuous.

4. Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ satisfies:

   (i.) for each fixed x_0, the function $y \mapsto f(x_0, y)$ is continuous; and

   (ii.) for each fixed y_0, the function $x \mapsto f(x, y_0)$ is continuous.

   Give an example of such an f which is not continuous.

5. Professor Doofus mistakenly writes the following on the blackboard.

   Theorem 11. The following are equivalent.

   (1) $f : \mathbf{R}^n \to \mathbf{R}^m$ is continuous (with the ε-δ definition)

   (2) For every open set $U \subseteq \mathbf{R}^n$, the image $f(U) \subseteq \mathbf{R}^m$ is open.

   Give an example with $m = n = 2$ which shows that Doofus is wrong.

6. Suppose that $f : A \subseteq \mathbf{R}^n \to \mathbf{R}$ is continuous, with $\mathbf{a} \in A$ and $f(\mathbf{a}) > 0$. Prove that there exists $\delta > 0$ such that $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in B(\mathbf{a}, \delta) \cap A$.

7. Suppose that $A \subset \mathbf{R}^n$ is a set which is not closed. Prove that there exists a continuous function $f : A \to \mathbf{R}$ which is unbounded. (Hint: You might find it useful to first show that the set $\mathbf{R}^n - A$ must contain a point in the boundary of A.)