Math 450
Homework 1
Dr. Fuller
Due January 30

1. Let $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$. Prove that $|\langle \mathbf{x}, \mathbf{y} \rangle| = \|\mathbf{x}\| \|\mathbf{y}\|$ if and only if $\mathbf{y} = r\mathbf{x}$ for some $r \in \mathbf{R}$.

2. Let $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ be nonzero. Prove that $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

3. Let $\mathbf{x} = (1, 1, \ldots, 1)$ and $\mathbf{y} = (1, 2, \ldots, n)$ in $\mathbf{R}^n$. Let θ_n be the angle between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n$. Find $\lim_{n \to \infty} \theta_n$.

4. ($\square$) Decide if the following subsets of $\mathbf{R}^n$ are open and/or closed. (Draw pictures, and give answers. No proofs necessary.)
   (a) $\{(x, y) : xy = 0\} \subset \mathbf{R}^2$
   (b) $\{(x, y) : xy \neq 0\} \subset \mathbf{R}^2$
   (c) $\{(x, y, z) : x^2 + y^2 < 1 \text{ and } z = 0\} \subset \mathbf{R}^3$
   (d) $\{(x, y, z) : x^2 + y^2 < 1\} \subset \mathbf{R}^3$
   (e) $\{(x_1, \ldots, x_n) : \text{each } x_i \in \mathbf{Q}\} \subset \mathbf{R}^n$

5. ($\square$) Let S be an $(n-1)$-dimensional vector subspace of $\mathbf{R}^n$. Prove that S is not an open set.

6. ($\square$) Let $x \in \mathbf{R}^n$, $r \geq 0$, and define $\overline{B}(x, r) = \{y \in \mathbf{R}^n : \|x - y\| \leq r\}$. Prove that $\overline{B}(x, r)$ is closed.

7. (a) Prove that $\mathbf{R}^n$ is an open set.
   (b) Let $\{U_\alpha\}_{\alpha \in \Gamma}$ be a collection of an arbitrary number of open sets in $\mathbf{R}^n$. Prove that $\bigcup_{\alpha \in \Gamma} U_\alpha$ is an open set.
   (c) Let U_1 and U_2 be open sets in $\mathbf{R}^n$. Prove that $U_1 \cap U_2$ is open.

8. Let $\{C_\alpha\}_{\alpha \in \Gamma}$ be a collection of an arbitrary number of closed sets in $\mathbf{R}^n$.
   (a) Prove that $\bigcap_{\alpha \in \Gamma} C_\alpha$ is an closed set.
   (b) Professor Doofus writes that in addition $\bigcup_{\alpha \in \Gamma} C_\alpha$ is a closed set. Give an example which shows that Doofus is wrong.