Math 501
Homework 7

Trevor Klar

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1. Recall for a set A in a space X, we define $\partial A = \overline{A} - \text{int}(A)$.

   (a) Prove that $x \in \partial A$ if and only if for every open set U containing x, we have $U \cap A \neq \emptyset$ and $U \cap (X - A) \neq \emptyset$.

   Proof
   Suppose $x \in \partial A$. By definition, $x \in \overline{A}$ and $x \notin \text{int}(A)$. Since $x \in \overline{A}$, then by definition either $x \in A$, or x is a limit point of A. If $x \in A$, then clearly for any open set U containing x, $U \cap A \neq \emptyset$. Otherwise if x is merely a limit point of A, then for any open U containing x,

   $$U \cap A - \{x\} \neq \emptyset \implies U \cap A \neq \emptyset.$$

   Since $x \notin \text{int}(A)$, then by definition there is no open set U containing x such that $U \subset A$. That is, for every open set U containing x,

   $$U \cap (X - A) \neq \emptyset.$$

   Since this conclusion follows from definintions, and definitions are biconditional, then the converse is also true. ■

   (b) Prove that A is open if and only if $\partial A \cap A = \emptyset$. Prove that A is closed if and only if $\partial A \subseteq A$.

   Proof Suppose A is open. Then, $A = \text{int}(A)$, so

   $$\partial A \cap A = (\overline{A} - A) \cap A = \emptyset.$$

   Suppose A is closed. Then, $A = \overline{A}$, so

   $$\partial A \cap A = (A - \text{int}(A)) \cap A = (A - \text{int}(A)) \subseteq A.$$

   Suppose $\partial A \cap A = \emptyset$. Since $\text{int}(A) \subset A \subset \overline{A}$,

   $$\partial A \cap A = (\overline{A} - \text{int}(A)) \cap A = \emptyset \implies (A - \text{int}(A)) = \emptyset \implies A \subset \text{int}(A) \implies A = \text{int}(A).$$

   Suppose $\partial A \subseteq A$. Since $\text{int}(A) \subset A \subset \overline{A}$,

   $$\partial A = (\overline{A} - \text{int}(A)) \subset A \implies \overline{A} \subset A \implies \overline{A} = A.$$

   ■

2. Let X be a metric space with metric d, and suppose that for all $x \in X$ and $r < 0$, the closed ball $\overline{B}(x, r) = \{y : d(x, y) \leq r\}$ is compact. Prove that X is d-complete.

   Proof First note that since we have assumed that all closed balls in X are compact, then they are also all complete. Let $(x_n)_{n=1}^{\infty}$ be any Cauchy sequence in X. We will show that $(x_n)_{n=1}^{\infty}$ converges. Since$(x_n)_{n=1}^{\infty}$ is Cauchy, then there exists some $N \in \mathbb{N}$ such that for all $m, n > N$, $d(x_m, x_n) < 1$. Let $n_0 = N+1$. So, for all $n > N$, $d(x_{n_0}, x_n) < 1$. Now, let $M = max\{d(x_{n_0}, x_n) : n \leq N\}$. So, for any x_n in our sequence, $d(x_{n_0}, x_n) \leq \max(M, 1)$, and thus $x_n \in \overline{B}(x_{n_0}, \max(M, 1)), \forall n \in \mathbb{N}$. Since all closed balls in X are complete, and $(x_n)_{n=1}^{\infty} \in \overline{B}(x_{n_0}, \max(M, 1))\, \forall n \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ converges. ■

3. Prove Corollary 28: Let X be a complete metric space. Show that X is not the countable union of nowhere dense closed sets.

Proof Let $\bigcup_{i=1}^{\infty} F_i$ denote any countable union of nowhere dense closed sets. Consider the complement of this union:

$$\left(\bigcup_{i=1}^{\infty} F_i\right)^{\mathsf{C}} = \bigcap_{i=1}^{\infty} \left(F_i^{\mathsf{C}}\right) = \bigcap_{i=1}^{\infty} (X - F_i)$$

Now, each $(X - F_i)$ must be a dense open set, so by the Baire Category Theorem, this countable intersection of dense open sets is dense in X. Thus, $\bigcap_{i=1}^{\infty} (X - F_i) \neq \emptyset$, so $\bigcup_{i=1}^{\infty} F_i \neq X$. ∎

3.5 Let X be a complete metric space with metric d, and suppose that X has no isolated points. Prove that X is uncountable.

Proof Suppose that X is countable. Consider the colection $\{X - \{x\}\}_{x \in X}$. Since X has no isolated points, each of these sets is dense. And since X is Hausdorff, each of these sets is open (because singletons are closed in a Hausdorff space). Now,

$$\bigcap_{x \in X} \{X - \{x\}\} = \emptyset,$$

but this countable union of dense open sets should be dense in X by the Baire Category Theorem, so we have a contradiction. ∎