Welcome Back! Differential Calculus

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Summary of Logs

$\log(y)$ is how many tens you multiply together to get y.

$$\boxed{10^{\log(y)} = y}$$

$$\boxed{\log\left(10^{a}\right) = a}$$

$$\boxed{10^{a} \times 10^{b} = 10^{a+b}}$$

$$\boxed{\log(x \times y) = \log(x) + \log(y)}$$

$$\boxed{\left(10^{a}\right)^{p} = 10^{ap}}$$

$$\boxed{\log(a^{p}) = p\log(a)}$$

Each of these pairs of equalities says one thing!

General Compound Interest

If the interest rate is r%, then each year money multiplies by

$$m = 1 + \frac{r}{100}.$$

If you start with an initial amount A of money then after t years you have

$$A \times m^t = A \times \left(1 + \frac{r}{100}\right)^t$$

5. If you invest \$1000 at 14% interest, how much will you have 5 years later? (Guess!)

A≈ \$700 B≈ \$1400 C≈ \$1500 D≈ \$1700 E≈ \$2000 $\boxed{\text{E}}$

After 5 years, you have

$$\$1,000 \times \left(1 + \frac{14}{100}\right)^5 = \$1,000 \times (1.14)^5.$$

How much is this? Smart way: 14% in 1 year ≈ 7% per year for 2.

§7.9: Population Growth

Assume each generation of bunnies has 3 times as many bunnies as previous one. Initially have 100 bunnies. How many bunnies after n generations?

$$\text{A}= 100 \times 3n \quad \text{B}= 100 + 3n \quad \text{C}= 100(1 + 3n)$$
$$\text{D}= 100^{3n} \quad \text{E}= 100 \times 3^n$$

Answer: $\boxed{\text{E}}$

Start with 100
After 1 generation have 100×3 bunnies
After 2 generations have $100 \times 3 \times 3$ bunnies
After 3 generations have $100 \times 3 \times 3 \times 3$ bunnies
So...after n generations have

$$100 \times \underbrace{3 \times 3 \times \cdots \times 3}_{n \text{ times}} = 100 \times 3^n \text{ bunnies.}$$

More Bunnies

We saw that:

- if we start with 100 bunnies, and
- the bunny population triples every generation,

then we have 100×3^n bunnies after n generations.

1. How many generations until there are $10^7 = 10{,}000{,}000$ bunnies?

A$= \log(5/3)$ B$= 5 - \log(3)$ C$= 5/\log(3)$
D$= 5/3$ E$= 10^5/3$

A≈ 0.22 B≈ 4.52 C≈ 10.48
D≈ 1.67 E$\approx 3{,}333$ $\boxed{\text{C}}$

Flu Outbreak

2. At the start of an outbreak of H1N1 flu in a large class of students, there were 5 infected individuals. The numbers doubles every 3 days. How many days until there are 80 infected students?

A$=\log(16)/\log(2)$ B$=\log(16/2)$ C$=16/\log(2)$

D$=3\log(16)/\log(2)$ E$=\log(48/2)$ $\boxed{\text{D}}$

Doubling Time Formula

Suppose something doubles every K minutes*. If there is a mass of A at time $t = 0$, how much is there at time t minutes?

$$\text{mass after t minutes} = A \times 2^{(t/K)}$$

Idea: t/K is number of doubling periods in t minutes.

3. A disease spreads through a community. On March 1 there were 100 infected people. The number of people doubles in a 3 days. How many infected people are there t days after March 1?

A$= 2^t$ B$= 3 \times 2^{t/100}$ C$= 100 \times 2^t$ D$= 100 \times 2^{t/3}$ $\boxed{\text{D}}$

How many days until there are $1,000$ infected people?

A$= \log(10)/\log(2)$ B$= 3\log(10)/\log(2)$ C$= 3\log(5)$
D$= 3(\log(10) - \log(2))$ E$= 3\log(20)$ $\boxed{\text{B}}$

*Any time unit will work, not just minutes. Just be consistent!

A More Complicated Example

$$\text{mass after t minutes} = A \times 2^{(t/K)}$$

where

- K is the doubling time, and
- t/K is the number of doubling periods in t minutes.

4. A colony of mold is growing on a cheeseburger in the back of a dorm refrigerator. When discovered it has a mass of 10 mg. One week later it was found to have a mass of 30 mg. What is the doubling time measured in days?

A= $\log(2)/\log(3)$ B= $7\log(2)/\log(3)$ C= $7\log(2/3)$ D= $7\log(3/2)$

Hint: We know A and the mass t days after discovery (for some t).

Solving $30 = 10 \times 2^{7/K}$ gives $\boxed{\text{B}}$

§7.11: Half-Life, Doubling Time

The half-life of a radioactive isotope is the time it takes for half of the isotope to decay.

Example: Isotope W has a half-life of 10 years. How much remains after 20 years? None?

$$\frac{1}{2} \times \frac{1}{2} \times (\text{amount you start with})$$

Idea: In half-life problems, convert time into half-lives.

In this problem, the half-life is 10 years. Therefore, 20 years is two half-lives.

In general: After n half-lives,

$$\text{remaining amount} = \left(\frac{1}{2}\right)^n \times (\text{amount started with})$$

5. Start with 120 grams of an isotope with a half-life of 12 years. How many grams remains after 36 years?

A= 0 B= 10 C= 15 D= 20 E= 40 $\boxed{\text{C}}$

Another Example

In general: After n half-lives,

$$\text{remaining amount} = \left(\frac{1}{2}\right)^n \times (\text{amount started with})$$

6. An isotope has a half-life of 5 years.

(a) If we start with 70 grams, how many grams will be left after t years?

$$A = 70\left(\frac{1}{2}\right)^t \quad B = 5\left(\frac{1}{2}\right)^{70t} \quad C = 70\left(\frac{1}{2}\right)^{5t}$$

$$D = 70\left(\frac{1}{2}\right)^{t/5} \quad E = 0 \quad \boxed{D}$$

(b) How many years until 10 grams remain?

$$A = 5(\log(7) - \log(2)) \quad B = \log(7)/\log(2) \quad C = 5\log(7/2)$$

$$D = 5\log(7)/\log(2) \quad E = \log(7)/(5\log(2)) \quad \boxed{D}$$

Half-Life Formula

Suppose something has a half-life of K years[†]. If there is a mass of A at time $t = 0$, how much is there at time t years?

$$\text{mass after } t \text{ years} = A \times \left(\tfrac{1}{2}\right)^{(t/K)}$$

Idea: t/K is number of half-lives in t years.

7. (Radiocarbon Dating) A bone is found with 2% of the usual amount of carbon-14 in it. The half-life of carbon-14 is 5730 years. How old (in years) is the bone?

A= $5730 \log(.01)/\log(2)$ B= $5730 \log(50)/\log(2)$

C= 5730×50 D= wicked old

Answer: $\boxed{\text{B}} \approx 32,000$ years

[†]Any time unit will work, not just years. Just be consistent!

§7.13: Logs in Other Bases

$\log(y)$ is how many tens you multiply together to get y.

$\log_2(y)$ is how many twos you multiply together to get y.

So $2^3 = 8$ means the same thing as $\log_2(8) = 3$

Examples:

$$\log_2(16) = 4 \qquad \text{because } 2^4 = 16$$

$$\log_2(32) = 5 \qquad \text{because } 2^5 = 32$$

$$\log_2(1/8) = -3 \qquad \text{because } 2^{-3} = 1/8$$

The five laws of logs work for any base b exactly the same way except...

$$\boxed{b^{\log_b(y)} = y} \qquad \boxed{\log_b\left(b^a\right) = a}$$

Summary & Examples

Important bases:

- $\log_2$ is used extensively in computer science
- $\ln = \log_e$ is used everywhere (the natural log) ($e \approx 2.718$)
  $\log_e(y) = x$ means $e^x = y$
  $\log_e(y)$ is how many e's you multiply to get y.
  Read as: " log base e of y equals x."

Examples:

$\log_3(81) =$ A$= 0$ B$= 1$ C$= 2$ D$= 3$ E$= 4$ $\boxed{\text{E}}$

$\log_5(25) =$ A$= 0$ B$= 1$ C$= 2$ D$= 3$ E$= 4$ $\boxed{\text{C}}$

Simplify $\ln\left(\left(e^{3x} \times e^{y}\right)^2\right)$

A$= 6x + y$ B$= 2x + 2y$ C$= 3x + 2y$ D$= 6x + 2y$ E$= 6xy$ $\boxed{\text{D}}$

Teaser: e is special because the derivative of e^x is e^x whatever that means.

Derivatives & Differential Calculus

…are about how quickly things change.

- Need to understand PRACTICAL significance in various situations

  Spread of infectious disease, population growth, speed, acceleration, marginal rates in economics, global warming

- Calculate (or estimate) rate of change from various sources:

  graph
  table of data
  formula

- Applications:

  measure change
  predict the future
  optimization – find the best, or smallest, or biggest, or most…

This is all about *understanding* the world.

Philosophical problem

How quickly is something changing at one moment in time?

Example: Does a ball stop when I throw it straight up?

Example: How fast is the temperature rising at 7am?

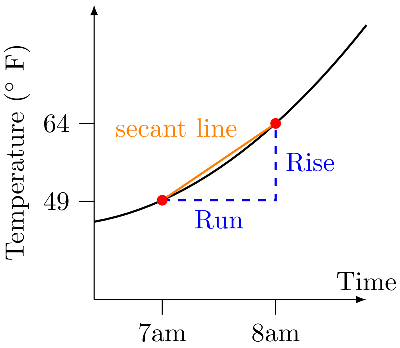
$$\begin{pmatrix} \text{change in temp} \\ \text{between 7am \& 8am} \end{pmatrix}$$

$= 64 - 49 = 15^\circ$ F

$$\begin{pmatrix} \text{average rate of} \\ \text{change in temp} \\ \text{between 7am \& 8am} \end{pmatrix}$$

$= \frac{15^\circ\ \text{F}}{1\ \text{hour}} = 15^\circ$ F/hour

= slope of secant line



Continuing Example

Similarly,

$$\left(\begin{array}{c} \text{average rate of} \\ \text{change in temp} \\ \text{between 6am \& 8am} \end{array} \right) = \frac{\text{change in temp}}{\text{time taken}}$$

Question: Suppose temperature at time t given by the formula $f(t) = t^2$. What is the average rate of change of temperature from 6am to 8am?

A $= 1$ B $= 7$ C $= 9$ D $= 14$ E $= 28$ $\boxed{\text{D}}$

Average Rate of Change

Suppose temperature at time t given by the formula $f(t) = t^2$. Using a calculator one can find the average rate of change over shorter and shorter time spans Δt, starting at 7am:

Δt	$(f(7+\Delta t) - f(7))/\Delta t$	ave rate of change ^{o}F/hr	
1	$(8^2 - 7^2)/1$	15	
0.1	$(7.1^2 - 7^2)/0.1$	14.1	
0.01	$(7.01^2 - 7^2)/0.01$	14.01	
0.001	$(7.001^2 - 7^2)/0.001$	14.001	
0.0001	$(7.0001^2 - 7^2)/0.0001$	14.0001	
0.00001	$(7.00001^2 - 7^2)/0.00001$	14.00001	
0	$(7^2 - 7^2)/0$	0/0	arghhhh

Table: Average rate of change over various time spans

What would you guess the exact instantaneous rate of change of temperature at precisely 7am is? Yes! $\boxed{14}$. But how do we get this? Answer: it is a limit!

Instantaneous Rate of Change

What does the limit

$$\lim_{\Delta t \to 0} \frac{f(7+\Delta t) - f(7)}{\Delta t}$$

mean in practice?

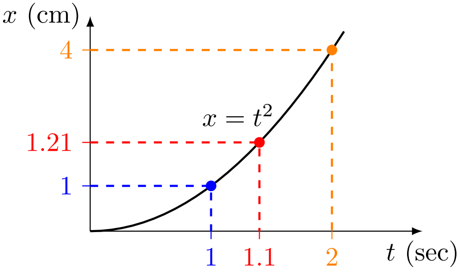
Work out the average rate of change over a very short time interval.
That is very nearly the correct answer.
The shorter the time interval you use, the more accurate you expect the answer to be.

To get the exact answer you would need to take a time interval of zero length.
This leads to the nonsense $0/0$. So you can't do this.
That is the philosophical problem.

Mathematical solution: take the limit.

An Example

A hamster runs along the x-axis, so that after t seconds the hamster is t^2 cm from the origin. Our goal is to find the hamster's speed at time $t = 1$ sec.



$$\begin{pmatrix} \text{average speed from} \\ t = 1 \text{ to } t = 2 \end{pmatrix} = \frac{\text{distance gone}}{\text{time taken}} = \frac{2^2 - 1^2}{2 - 1} = 3 \text{ cm/sec}$$

$$\begin{pmatrix} \text{average speed from} \\ t = 1 \text{ to } t = 1.1 \end{pmatrix} = \frac{\text{distance gone}}{\text{time taken}} = \frac{1.1^2 - 1^2}{1.1 - 1} = 2.1 \text{ cm/sec}$$

Example Concluded

How do we work out the exact speed of the hamster after 1 second?

Plan:

- Find the average speed over a short time interval Δt, then
- Take the limit as $\Delta t \to 0$.

$$\begin{aligned}
\left(\begin{array}{c}\text{average speed from}\\ t=1 \text{ to } t=1+\Delta t\end{array}\right) &= \frac{\text{distance gone}}{\text{time taken}} \\
&= \frac{(1+\Delta t)^2 \ - \ 1^2}{(1+\Delta t)-1} \\
&= \frac{(1+2\Delta t+(\Delta t)^2) \ - \ 1}{\Delta t} \\
&= \frac{2\Delta t+(\Delta t)^2}{\Delta t} \\
&= 2+\Delta t
\end{aligned}$$

The limit of this as $\Delta t \to 0$ is 2.

Conclusion: At $t = 1$ sec, the exact speed of the hamster is 2 cm/sec.

Hamster Summary

Soon we will calculate that...

> the exact speed of the hamster after t seconds is $2t$ cm/sec.

Summary:

$f(t) = t^2$ = distance in cm of hamster from origin after t seconds

(a function that gives the distance the hamster has traveled at time t)

$f'(t) = 2t$ = speed of hamster in cm/sec after t seconds

(called the derivative of t^2 because it can be derived or obtained from the function t^2)

Question: How many cm had the hamster run by the time its speed was 8 cm/sec?

A= 4 B= 8 C= 16 D= 32 E= 64 $\boxed{\text{C}}$

Exact Hamster Speed

Now we calculate that...

> the exact speed of the hamster after t seconds is $2t$ cm/sec.

Do this as before: working out the average speed over a short time interval Δt and taking the limit as $\Delta t \to 0$

$$\begin{aligned}
\begin{pmatrix} \text{average speed from} \\ t \text{ to } t + \Delta t \end{pmatrix} &= \frac{\text{distance gone}}{\text{time taken}} \\
&= \frac{(t + \Delta t)^2 \; - \; t^2}{(t + \Delta t) - t} \\
&= \frac{(t^2 + 2t\Delta t + (\Delta t)^2) \; - \; t^2}{\Delta t} \\
&= \frac{2t\Delta t + (\Delta t)^2}{\Delta t} \\
&= 2t + \Delta t
\end{aligned}$$

The limit of this as $\Delta t \to 0$ is $2t$.

Hamster Questions!

After t seconds, the hamster is $f(t) = t^2$ cm from origin.
(1) What is the exact speed (in cm/sec) of the hamster at $t = 2$?

A= 1 B= 2 C= 4 D= 6 E= 8 $\boxed{\text{C}}$

(2) What is the exact speed (in cm/sec) of the hamster at $t = 4$?

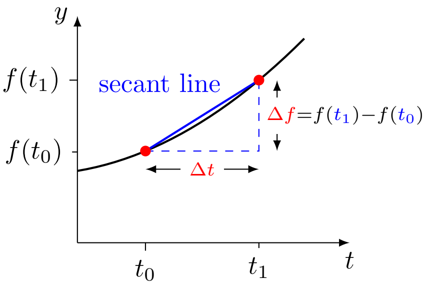
A= 1 B= 2 C= 4 D= 6 E= 8 $\boxed{\text{E}}$

(3) What is the average speed (in cm/sec) of the hamster from $t = 2$ to $t = 4$ seconds?

A= 1 B= 2 C= 4 D= 6 E= 8 $\boxed{\text{D}}$

Does this make sense?

Graphical Approach



Δf = change in f
Δt = change in t

Many ways to say same thing:

$$\begin{pmatrix} \text{average rate of} \\ \text{change of } f \end{pmatrix} = \frac{\text{change in } f}{\text{change in } t}$$

$$= \frac{\Delta f}{\Delta t}$$

$$= \text{slope of secant line} = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

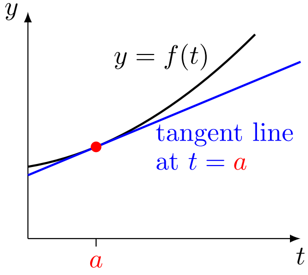
The derivative is defined to be

$$\lim_{\Delta t \to 0} \left(\frac{\Delta f}{\Delta t} \right) = \frac{df}{dt}$$

Idea: As t_1 moves closer to t_0 the secant line approaches the tangent line at t_0. This is the line with the same slope as the graph at t_0.

Understanding Derivatives

There are many ways to think about derivatives. You need to understand these to apply to problems.



slope of graph at a
= slope of tangent line
= instantaneous rate of change of f at a

$$= \left(\begin{array}{c} \text{limit of average rate of change} \\ \text{of } f \text{ over shorter and shorter} \\ \text{time intervals starting at } a \end{array} \right)$$

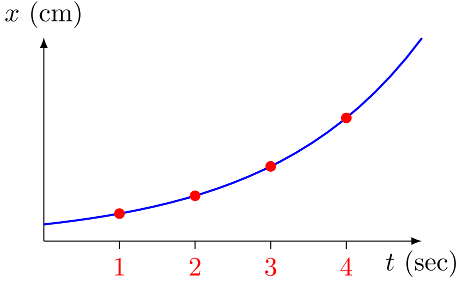
= limit of slopes of secant lines

$$= f'(a) \;=\; \left.\frac{df}{dt}\right|_{t=a}$$

Summary

- How fast something changes = rate of change
- Instantaneous rate of change is the limit of the average rate of change over shorter and shorter time spans. This gets around the $0/0$ problem.
- speed = rate of change of distance traveled.

Speed=Slope=Derivative



The graph shows the distance from the origin in cm after t seconds of a hamster. Which of the numbers below is the largest?

Hint: Speed is a slope!

A = speed of the hamster at $t = 1$

B = speed of the hamster at $t = 2$

C = speed of the hamster at $t = 3$

D = average speed of the hamster between $t = 2$ and $t = 3$

E = average speed of the hamster between $t = 3$ and $t = 4$

Answer: $\boxed{\text{E}}$

Practical Meaning

Our goal is that you understand the practical meaning of the derivative in various situations.

$$
\begin{aligned}
f(t) &= \text{temperature in } {}^\circ\text{ F at } t \text{ hours after midnight} \\
f(7) &= 48 \text{ means the temperature at } 7\text{am was } 48^\circ \text{ F} \\
f'(7) &= 3 \text{ means at } 7\text{am the temperature was rising at a rate of } 3^\circ \text{ F/hr} \\
f'(9) &= -5 \text{ means at } 9\text{am the temperature was falling at a rate of } 5^\circ \text{ F/hr} \\
&\qquad \text{or rising at a rate of } -5^\circ \text{ F/hr}
\end{aligned}
$$

$$
\begin{aligned}
g(t) &= \text{distance from origin in cm of hamster on } x\text{-axis after } t \text{ seconds} \\
g(7) &= 3 \text{ means after } 7 \text{ seconds hamster was } 3 \text{ cm from origin} \\
g'(9) &= -5 \text{ means after } 9 \text{ seconds our furry friend was running towards} \\
&\qquad \text{the origin at a speed of } 5 \text{ cm/sec}
\end{aligned}
$$

Another Context

Suppose $f(t)$ = temperature of oven in °C after t minutes.

What do $f(3) = 20$ and $f'(3) = 15$ mean?

A After 20 minutes the oven was at 3° C and heating up at a rate of 15° C/min

B After 3 minutes oven temperature was 15° C and cooling down at a rate to 20° C/min

C The oven was heating up at rate of 3° C/min after 15 minutes and also after 20 minutes

D After 3 minutes the oven was at 20° C and heating up at a rate of 15° C/min

E None of the above

Answer: $\boxed{\text{D}}$

That's it. Thanks for being here.

Lecture_9_Picture.png