Math 462 - Advanced Linear Algebra
Homework 1

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Proposition. **(1-2)** *Let there be given functions $f : S \to T$ and $g : T \to U$. Then*

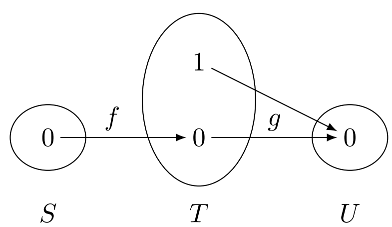
(v) if $g \circ f$ is surjective, then so is g.

Proof Since $g \circ f$ is surjective, then for any $y \in U$, there exists an $x \in S$ such that $g \circ f(x) = y$. Now x is in the domain of f, so $f(x)$ is in the domain of g, and by definition, $g(f(x)) = g \circ f(x) = y$. Thus, for any $y \in U$, there exists $f(x) \in T$ such that $g(f(x)) = y$, so g is surjective. ■

Exercises:

1. Find sets S, T, and U and functions $f : S \to T$ and $g : T \to U$ such that $g \circ f$ is injective, but g is *not* injective. (*Hint*: The choice of $S = \{0\}$, $T = \{0, 1\}$, and $U = \{0\}$ will inevitably lead to the desired result.)

   Example. Consider the following functions $f : S \to T$ and $g : T \to U$, whose definition is given by the following figure:

   In this example, $g \circ f$ is injective (vacuously) because for any two $x, y \in S$; $x \neq y \implies g \circ f(x) \neq g \circ f(y)$. This is also more intuitively clear if one recalls that injectivity is also called one-to-one, and since $g \circ f$ contains only the mapping $0 \mapsto 0$, then clearly $g \circ f$ is one-to-one.

   Secondly, g is not injective because $0 \neq 1$, however $g(0) = g(1) = 0$. ■

2. Find sets S, T, and U and functions $f : S \to T$ and $g : T \to U$ such that $g \circ f$ is surjective, but f is *not* surjective.

   Example. Actually, the same example for problem (1) will also work for this problem. The composition $g \circ f$ is surjective, because for every $y \in U$ (there is only one), there exists an $x \in S$ such that $g \circ f(x) = y$ ($x = 0$). However, f is not surjective because for $y = 1$, there does not exist any $x \in S$ such that $f(x) = y$. ■

3. Find a non-identity function $f : \mathbb{R} \to \mathbb{R}$ which is its own inverse function. That is, $f \circ f = 1_{\mathbb{R}}$ or, equivalently, $f(f(x)) = x$ for all real x.

   Example. Let $f(x) = -x$. Then, for all real x, $f(x) = -x$, and $f(f(x)) = x$. ■

5. Find a bijective map from the open interval $(-\pi/2, +\pi/2)$ to the set $\mathbb{R}$ of real numbers. This shows, by the way, that both sets have the same cardinality.

   Example. Let $f : (-\pi/2, +\pi/2) \to \mathbb{R}$ be defined as $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$.

   We will now show that f is 1-1 and onto. For any two numbers $x_1, x_2 \in (-\pi/2, +\pi/2)$,

   $$x_1 \neq x_2 \implies \tan(x_1) \neq \tan(x_2),$$

   since $\tan(x)$ is a strictly increasing function. So, f is 1-1.

   Since $\tan(x)$ for an angle x is also defined as the ratio of the opposite leg to the adjacent leg in a right triangle having one angle x, we can show that f is surjective. Let y be any arbitrary real number. Now, construct a right trangle with perpendicular sides of length y and 1. It follows that this triangle has an angle whose tangent is y. Thus, $\forall y \in \mathbb{R}, \exists x \in (-\pi/2, +\pi/2)$ such that $\tan(x) = y$, so f is surjective. ■

6. Explicitly construct a bijective function from the set of integers $\mathbb{Z}$ to the set of even integers $2\mathbb{Z}$.

   Example. Let $f : \mathbb{Z} \to 2\mathbb{Z}$ be defined as

   $$f(x) = 2x.$$

   By definition of an even integer, for every $y \in 2\mathbb{Z}$, there exists an integer x such that $2x = y$. Therefore, f is surjective.

   Secondly, if $x \neq y$, then either $x < y$ or $x > y$. Without loss of generality, call y the greater number, so that $x < y$. By the multiplication property of inequality, $2x < 2y$. Therefore, $2x \neq 2y$. We have shown that $x \neq y \implies 2x \neq 2y$, so f is injective. ■