Topology 2018

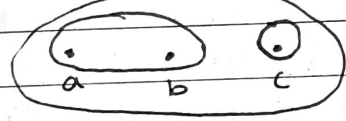
1. In each case give a proof, or give a counterexample, & prove it is a counterexample:

(a) $\text{int}(\text{int}(\mathcal{U})) = \text{int}(\mathcal{U})$

Let $x \in \text{int}(\mathcal{U})$. Then $\exists$ an open set $U_x \ni x$ s.t. $U_x \subset \mathcal{U}$. $\text{int}(\mathcal{U})$ is an open set so $\mathcal{U}_x \cap \text{int}(\mathcal{U})$ is an open set containing $x \Rightarrow x \in \mathcal{U}_x \cap \text{int}(\mathcal{U}) \subset \text{int}(\mathcal{U})$. Thus $x \in \text{int}(\text{int}(\mathcal{U}))$.

Now let $x \in \text{int}(\text{int}(\mathcal{U})) \Rightarrow \exists$ open set V_x of x s.t. $x \in V_x \subset \text{int}(\mathcal{U}) \Rightarrow x \in \text{int}(\mathcal{U})$. Thus $\text{int}(\text{int}(\mathcal{U})) = \text{int}(\mathcal{U})$. $\square$

(b) $\text{cl}(\text{cl}(\mathcal{U})) = \text{cl}(\text{int}(\mathcal{U}))$



Consider the topology on $\Omega = \{a, b, c\}$ where the open sets are $\{\emptyset, \Omega, \{a,b\}, \{c\}\}$. Let $\mathcal{U} = \{b\}$. Then $\text{int}(\mathcal{U}) = \emptyset$ & $\text{cl}(\text{int}(\mathcal{U})) = \emptyset$. On the other hand, $\text{cl}(\mathcal{U}) = \{a, b\}$ $\text{cl}(\text{cl}(\mathcal{U})) = \{a, b\}$.

(c) $\text{int}(\text{cl}(\mathcal{U})) = \text{int}(\mathcal{U})$

Consider $\mathbb{R}$ with the standard topology. Let $\mathcal{U} = \mathbb{Q}$. Then $\text{int}(\mathbb{Q}) = \emptyset$. On the other hand, $\text{int}(\text{cl}(\mathbb{Q})) = \text{int}(\mathbb{R}) = \mathbb{R}$. $\square$

(d) $\text{int}(U \times V) = \text{int}(U) \times \text{int}(V)$

Let $(x, y) \in \text{int}(\mathcal{U} \times V)$. Then $\exists$ an open set $\mathcal{W} \subseteq U \times V$ with $x \times y \in W \subseteq U \times V$. Then $\exists$ a basis element $x \times y \in \mathcal{W}_1 \times W_2 \subset W \subseteq \mathcal{U} \times V$ with $x \in W_1 \subset \mathcal{U}$ & $y \in W_2 \subset V$. So $x \times y \in \text{int}(U) \times \text{int}(V)$.
On the other hand, let $x \times y \in \text{int}(U) \times \text{int}(V)$. Then $\exists\ \mathcal{W}_1, W_2$ open with $x \in W_1 \subset \mathcal{U}$ & $y \in \mathcal{W}_2 \subset V$. Then $x \times y \subseteq W_1 \times W_2 \subseteq \mathcal{U} \times V$. Thus, $x \times y \in \text{int}(\mathcal{U} \times V)$. $\square$