Differential Topology

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Contents

Note: If you find any typos in these notes, please let me know at trevorklar@math.ucsb.edu. If you could include the page number, that would be helpful.

Note to the reader: I have highlighted topics which seem important to me, but the emphasis is mine, not Professor Fuller's. Bear that in mind when studying.

1 Smooth Manifolds and smooth maps

The text is John W. Milnor: Topology from the Differentiable Viewpoint.

1.1 Smoothness

Definition. f **smooth** on $U \underset{\text{open}}{\subset} \mathbb{R}^n \iff$ all partials[†] exist and are continuous.

Definition. f **smooth** on arbitrary $X \subset \mathbb{R}^n \iff \exists\, F$ smooth which extends f.

[†] All partials of all orders, not just first order.

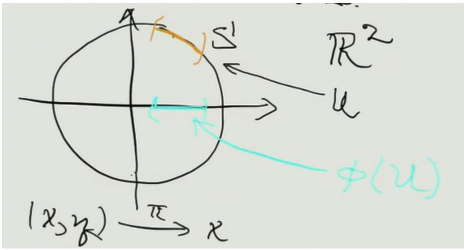
Definition. f **diffeomorphism** $\iff$ f smooth bijection w/ smooth inverse.

Definition. $X \cong Y \iff \exists$ a diffeomorphism $f : X \to Y$.

Definition. X[‡] **locally diffeomorphic** to $\mathbb{R}^k \iff$ every $x \in X$ has a neighborhood $U \cong V \underset{\text{open}}{\subset} \mathbb{R}^k$.

Definition. X is a **smooth k-manifold** $\iff$ X is locally diffeomorphic to $\mathbb{R}^k$.

Example. $\mathbb{S}^1$ is a smooth 1-manifold.



To see this, observe that $\phi(x, y) = x$ is a smooth map, and if we restrict it to a sufficiently small neighborhood of a point (x_0, y_0), then it is a diffeomorphism.[†] It has smooth inverse $x \mapsto (x, \sqrt{1 - x^2})$, so $\mathbb{S}^1$ is a smooth 1-manifold.

Example. If X is an a-manifold, and Y is a b-manifold, then $X \times Y$ is an $(a + b)$-manifold. PROOF Exercise.

1.2 Tangent Spaces and Derivatives

Definition. The **directional derivative of f at $\vec{x}$ in the direction of $\vec{v}$** is

$$df_{\vec{x}}(\vec{v}) = \lim_{t \to 0} \frac{f(\vec{x} + t\vec{v}) - f(\vec{x})}{t}$$

Remark. For a fixed $\vec{x}$, then $f_{\vec{x}} : \mathbb{R}^a \to \mathbb{R}^b$ is a linear map, so it can be written as a matrix mapping $\mathbb{R}^a \to \mathbb{R}^b$.

Definition. The **derivative of f at $\vec{x}$** is

$$df_{\vec{x}} = \left[\frac{\partial f_i}{\partial x_j}(\vec{x}) \right]$$

[‡]Assume $X \subset \mathbb{R}^n$, with the subspace topology.
[†]this gets kinda weird at $(0, 1)$, but don't worry too hard about that right now.

Definition. Recall the key fact from differential calculus is the **chain rule.**

Given appropriately composable smooth maps f, g defined on $\mathbb{R}^n$, then

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

which is sort of what you'd expect.

Goal. Find a way to formulate this for smooth manifolds. We'll need:

1. Notion of tangent spaces
2. Show that under df_x, tangent space at x to $X \mapsto$ tangent space at $f(x)$ to Y.
3. Show chain rule

Definition. Given a chart ϕ of the n-manifold X at the point p, we define the **tangent space of X at p** to be

$$T_p(X) = d\phi_0(\mathbb{R}^n),$$

that is, the tangent space is the image of the derivative of the chart.

Remark. In our context, the directional derivative and the derivative are the same thing.

Definition. Defining **the derivative of a smooth function between manifolds** $f : X \to Y$ takes a little effort. First, let ϕ, ψ be charts of X, Y that map $0 \mapsto x, 0 \mapsto y$. Then define h so that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
U \subseteq \mathbb{R}^a \arrow[u, "\phi"] \arrow[r, dashed, "h"] & V \subseteq \mathbb{R}^b \arrow[u, "\psi"']
\end{tikzcd}$$

Then we can take the derivative of h at 0 using ordinary calculus, and differentiate ϕ and ψ at 0 using the previous definition. Then we define df_x so that the following diagram commutes:

$$\begin{tikzcd}
T_xX \arrow[r, dashed, "df_x"] & T_{f(x)}Y \\
\mathbb{R}^a \arrow[u, "d\phi_0"] \arrow[r, "dh_0"] & \mathbb{R}^b \arrow[u, "d\psi_0"']
\end{tikzcd}$$

2 Immersions and Submersions

Definition. Suppose $f : X \to Y$ smooth. f is a **local diffeomorphism** at $x \in X$ if there exists an open neighborhood U of x such that $F|_U : U \to F(U)$ is a diffeomorphism.

Theorem 1. (Inverse Function Theorem) Let $f : X \to Y$ smooth.

$$\text{If } df_x : T_xX \to T_{f(x)Y} \text{ is an isomorphism (is invertible),}$$

then f is a local diffeomorphism at x.

Definition. The **canonical immersion** $\mathbb{R}^a \to \mathbb{R}^b$ is

$$(x_1, \ldots, x_a) \mapsto (x_1, \ldots, x_a, 0, \ldots, 0)$$

Definition. An **immersion at a point x** is a function f which is smooth and $df_x : T_x(X) \to T_{f(x)}(Y)$ is injective.

Theorem 2. (Local Immersion Theorem) Let $f : X \to Y$ smooth, with $\dim X < \dim Y$, so that df_x is injective. Then there exist local coordinates

$$\begin{tikzcd}
x & X \arrow[r, "f"] & Y & f(x) \\
0 \arrow[u] \arrow[r, phantom, "\in"] & U^{\text{open}} \arrow[u, "\phi"'] \arrow[r, "(*)", "u"'] & V^{\text{open}} \arrow[u, "\psi"] \arrow[r, phantom, "\ni"] & 0 \arrow[u] \\
& \mathbb{R}^a \arrow[u, phantom, "\cap"] & \mathbb{R}^b \arrow[u, phantom, "\cap"] &
\end{tikzcd}$$

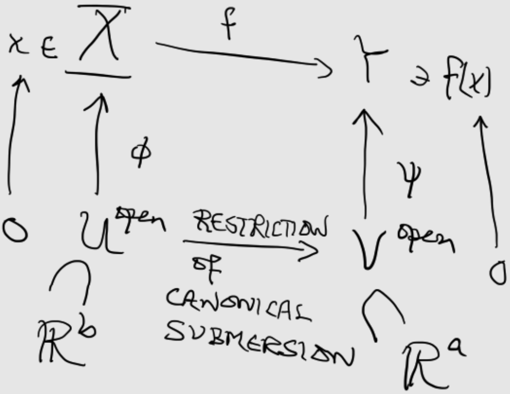
so that $U \hookrightarrow V$ by the canonical immersion.
"If f is an immersion at x, then it is an immersion near x."

Definition. An **submersion at a point x** is a function f which is smooth and $df_x : T_x(X) \to T_{f(x)}(Y)$ is surjective.

Definition. The **canonical submersion** $\mathbb{R}^b \to \mathbb{R}^a$ (with $a < b$) is

$$(x_1, \ldots, x_a, \ldots, x_b) \mapsto (x_1, \ldots, x_a)$$

Theorem 3. (Local Submersion Theorem) Let $f : X \to Y$ smooth with df_x surjective. Then there exist local coordinates

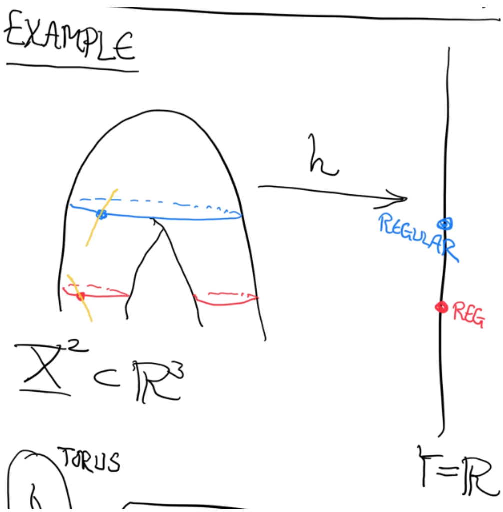


so that $U \mapsto V$ by the canonical submersion.
"If f is a submersion at x, then it is a submersion near x."

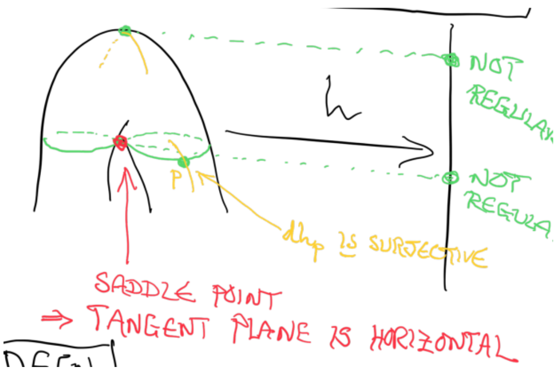
3 Regular Values

Definition. Let $f : X \to Y$ smooth. We say y is a **regular value** of f if every $x \in f^{-1}(y)$ is a submersive point (That is, df_x is surjective at those points.)

Example.



Example.

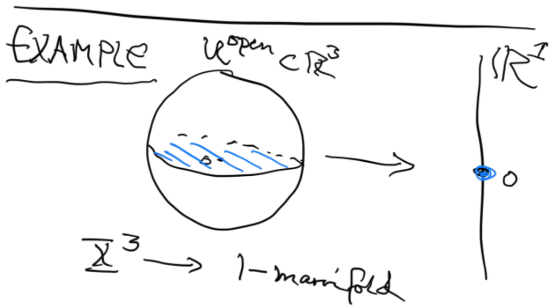


Definition. If y is not regular, then it is a **critical point** of f.

> **Theorem 4.** If $f : X \to Y$ is smooth and y is a regular value, then $f^{-1}(y)$ is a smooth manifold of dimension $\dim X - \dim Y$.
> **Reason.** For carefully chosen charts, f can be thought of as the canonical submersion from $U^b \to V^a$, so if $0 \overset{\psi}{\mapsto} y$, then $(\overbrace{0, \ldots, 0}^{a}, \overbrace{x_{a+1}, \ldots, x_b}^{b-a}) \overset{f}{\mapsto} y$.

Remark. Note that for a map into $\mathbb{R}$, it's derivative is always surjective unless it is the zero map.

Example.



Example. Let $f : \mathbb{R}^k \to \mathbb{R}$ given by

$$f(x_1, \dots x_n) = \sum_{i=1}^{k} x_i^2 = ||x|| \,.$$

Fix $p = (p_1, \dots, p_k) \in \mathbb{R}^k$. What is df_p?

$$df_p(e_1) = \lim_{t \to 0} \frac{f(te_1 + p) - f(p)}{t}$$

and

$$f(te_1 + p) = f(t + p_1, p_2, \dots, p_k) = t^2 + 2tp_1 + \overbrace{p_1^2 + p_2^2 + \dots + p_n^2}^{f(p)}$$

so

$$df_p(e_1) = \lim_{t \to 0} t + 2p_1 = 2p_1$$

which, by the way, is pretty clearly $\frac{\partial f}{\partial x_1}(p)$.

In general, for $f : X^k \to Y^1$, the derivative $df_p : \mathbb{R}^k \to \mathbb{R}$ is given by the dot product

$$df_p(v) = (v_1, \dots, v_k) \cdot \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k} \right) \bigg|_p$$

I'm pretty sure that this generalizes as follows. For $f : X^a \to Y^b$, the derivative $df_p : \mathbb{R}^a \to \mathbb{R}^b$ is given by the matrix product

$$df_p(v) = vJ|_p$$

where v is a row vector and J is the Jacobian of f. In J, you want columns to have the same f-index and rows to have the same x-index.

So when is the derivative zero? When all partials are zero, i.e. $p = 0$ in this case.

Thus all nonzero numbers in $\mathbb{R}$ are regular values of f.

Example. (Lie Group)

Definition. The set of all **orthogonal real matrices**, denoted $O(n)$, is the set of all real $n \times n$ matrices such that

$$A^T A = AA^T = I_n$$

or equivalently,

$$A^{-1} = A^T$$

Remark. The Wikipedia page lists quite a few very helpful facts about orthogonal matrices.

Prove that $O(n)$ is a smooth manifold, and find its dimension.

Proof Let $f : M_n(\mathbb{R}) \to \Sigma_n(\mathbb{R})$ be the map given by

$$f(A) = A^\top A.$$

We will show that (i) $M_n(\mathbb{R})$[†] and $\Sigma_n(\mathbb{R})$[‡] are both smooth manifolds, (ii) f is a smooth map, and (iii) $O(n) = f^{-1}(I)$ with I a regular value of f, and then we're done since the preimage of a regular point is a smooth manifold.

(i) As vector spaces, $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and $\Sigma_n(\mathbb{R}) \cong \mathbb{R}^{t(n)}$ where $t(n)$ is the n-th triangle number, so they are definitely smooth manifolds.

(ii) Since the computations of $A^\top A$ just consist of multiplying and adding different elements of A, then the function f is just a polynomial in n^2 variables, so it is smooth.

(iii) We can see by inspection that $O(n) = f^{-1}(I)$, so let us show that I is a regular value of f. Fix $A \in M_n(\mathbb{R})$, and let's compute the derivative $df_A : T_A(M_n\mathbb{R}) \to T_{A^\top A}(\Sigma_n)$, and check that it is surjective whenever $A \in f^{-1}(I)$.

$$\begin{aligned} df_A(B) &= \lim_{t\to 0} \frac{f(A+tB) - f(A)}{t} \\ &= \lim_{t\to 0} \frac{(A+tB)^\top (A+tB) - A^\top A}{t} \\ &= \lim_{t\to 0} \tfrac{1}{t}\left(tB^\top A + tA^\top B + t^2 B^\top B\right) \\ &= B^\top A + A^\top B \end{aligned}$$

For any $C \in \Sigma_n(\mathbb{R})$[††] and $A \in f^{-1}(I)$, if we can find B such that $B^\top A + A^\top B = C$, then we're done. Observe that, since $A^\top = A^{-1}$, then if $B = \frac{1}{2}AC$, then

$$B^\top A + A^\top B = C.$$

Thus df_A is surjective for all $A \in f^{-1}(I)$, so we're done. ■

Remark. In particular, $O(n)$ is not only a smooth manifold, but it is a Lie group. These are important, so let's talk about them a bit.

Definition. We say a space X is a **Lie group** when:

- X is a smooth manifold
- X is a group with $xy \xmapsto{\text{smooth}} z$ and $x \xmapsto{\text{smooth}} x^{-1}$.

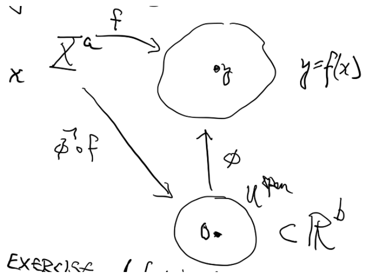
[†] Where $M_n(\mathbb{R})$ denotes the set of all real $n \times n$ matrices.
[‡] Where $\Sigma_n(\mathbb{R})$ denotes the set of symmetric $n \times n$ real matrices.
[††] The tangent space to a vector space at any point is itself.

Variation of Submersion Suppose $g_1, \dots g_\ell$ are smooth functions $X^k \to \mathbb{R}$, where $\ell \leq k$.
Question When is the set Z of common zeros of g's a reasonable object?
ANSWER. We think of $g : X^k \to \mathbb{R}^\ell$ as putting them all together, and **when every $x \in g^{-1}(0)$ is a regular value,** then Z is a smooth manifold.

$$dg_x \text{ is onto} \iff (dg_1)_x, \dots, (dg_\ell)_x \text{ are linearly independent}$$

□

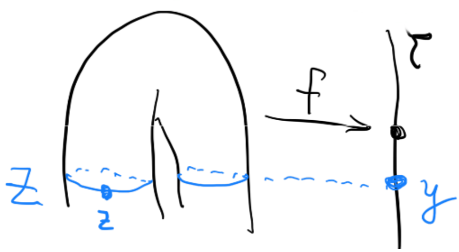
Question Given a submanifold of X, when can it be defined as the set of common zeros of some functions?
EASY CASE Suppose the submanifold of X is $f^{-1}(y)$ for some $f : X \to Y$ and y is regular. We need to show that f can be decomposed into coordinate functions $f_1, f_2, \dots f_n$. Consider a chart ϕ of Y at y:



then we can use the coordinate functions on $\mathbb{R}^b$. The reader can verify that $d(\phi^{-1} \circ f)_x$ is onto and 0 is regular for $\phi^{-1} \circ f$.

> **Lemma 5.** Let $f : X \xrightarrow{\text{smooth}} Y$ be regular at $y \in Y$, and denote $Z = f^{-1}(\{y\})$. Then
> $$T_z(Z) = \ker(df_z).$$

PROOF



It's obvious from the picture that $T_z(Z) \subset \ker(df_z)$, and since $df_z : T_zX \to T_yY$ is onto, then $\dim T_z(X \setminus Z) = \dim Y$, so

$$\dim X - \dim Y = \dim \ker(df_z)$$

and

$$\dim X - \dim Y = \dim T_z Z.$$

$\blacksquare$

4 Transversality

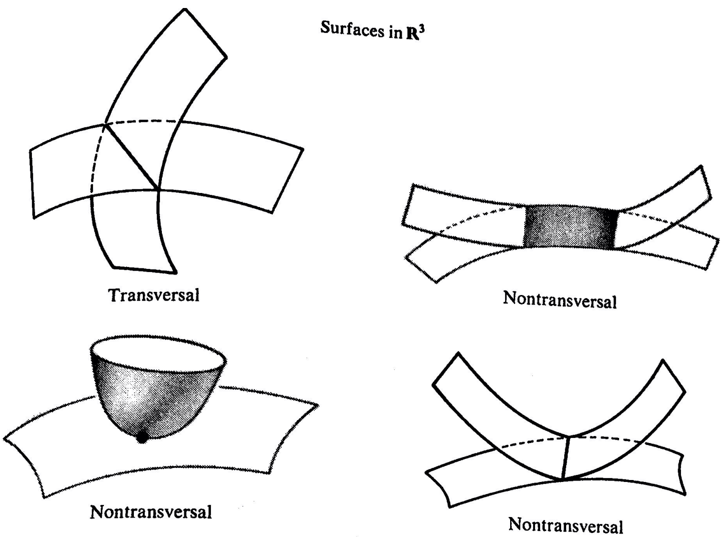
Definition. (Transverse at a point) Let $f : X \xrightarrow{\text{smooth}} Y$ with $Z \underset{\text{manif.}}{\subset} Y$. We say ***f* is transverse to *Z* at $f(x) = y$** if

$$\mathrm{Im}\,(df_x) \oplus T_y Z = T_y Y.$$

Definition. (Transverse to a submanifold) Let $f : X \xrightarrow{\text{smooth}} Y$ with $Z \underset{\text{manif.}}{\subset} Y$. We write $f \pitchfork Z$ and say ***f* is transverse to *Z*** if for all $x \in f^{-1}(Z)$,

$$\mathrm{Im}\,(df_x) \oplus T_y Z = T_y Y.$$

where $y = f(x)$.



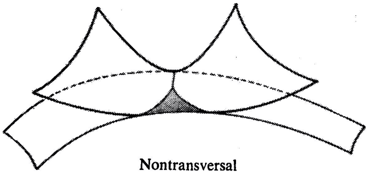
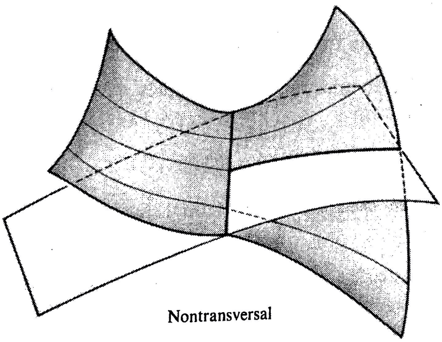


Figure 1-18



Why do we care? Because transversals give us nice, *stable* intersections.

Definition. Let P be a property of some function f. We say that $P(f)$ is **stable** if for any homotopy f_t of f, there exists $\epsilon > 0$ such that

$$P(f_t) \text{ is true for all } 0 \leq t < \epsilon.$$

Example. Let $f : X \to Y$, and let $P(f)$ be the property that f is a local diffeomorphism at x. Then P is a stable property.

Proof (sketch) WLOG suppose $f : \mathbb{R}^a \to \mathbb{R}^a$ (since we can always take charts). Since f is a diffeomorphism, then the Jacobian

$$\left.\left(\frac{\partial f_\alpha}{\partial x_\beta}\right)\right|_{\mathbf{x}=0}$$

has det $\neq 0$. So since a small homotopy of f changes $\det f$ by a small amount (IFT), then f_ϵ is a local diffeomorphism as well. ■

Theorem 6. Let X be compact, and consider functions $X \to Y$. The following function classes are stable:

(a) Local diffeomorphism

(b) Immersion

(c) Submersion

(d) Maps transverse to $\underset{\text{subspace}}{Z \subset Y}$

(e) Embeddings (Injective Immersions)

(f) Diffeomorphisms.

Remark. It would really be a nightmare if $f : X \xrightarrow{\text{smooth}} Y$, but f had no regular values. Fortunately, Sard's Theorem will imply that regular values are dense.

Definition.

SEMI-RELEVANT DIGRESSION

$C \subset \mathbb{R}^k$ C HAS $\boxed{\text{MEASURE ZERO}}$ IF $\forall\ \epsilon > 0\ \exists$ COVERING

$C \subset \bigcup_{i \geq 0} (k\text{-cubes})_i$

$\sum_{i \geq 0} \mathrm{vol}((k\text{-cubes})_i) < \epsilon$

Definition.

Y IS A k-manifold, $C \subset Y$ HAS MEASURE ZERO IF C COVERED BY $\phi_i : U_i \to C$ SO THAT $\phi_i^{-1}(C) \subset U_i$ THE SUM OF THESE HAS $\mathbb{R}^k$ MEASURE ZERO.

Theorem 7. Let $f : X \xrightarrow{\text{smooth}} Y$ with $Z \underset{\text{manif.}}{\subset} Y$. If $f \pitchfork Z$, then $f^{-1}(Z)$ is a smooth manifold.

Proof Dr. Long referred to this theorem as if we had proved it in lecture, but I don't see it in my notes here. Perhaps another time I'll write out the proof. ■

Theorem 8 (Sard's Theorem). Let $f : X \xrightarrow{\text{smooth}} Y$. Then the critical values of f have measure zero in Y.

Proof The proof of this theorem is a bunch of analysis, which (a) sucks, and (b) is kinda beside the point of this course. So we'll just regard Sard's Theorem to be an "axiom". ■

Corollary. Let $f : X \xrightarrow{\text{smooth}} Y$. Then the regular values of f are dense in Y.

Proof Since the set of critical value C has measure zero, then it can't contain any k-cubes. Thus any k-cube contains a regular value, so we're done. ■

Remark. Our goal is a version of the Whitney Embedding Theorem, but first we need to make a relevant digression:

Definition (Tangent bundle). Let $X^k \subset \mathbb{R}^n$ be a smooth k-manifold. We form the **tangent bundle of X**, by

$$\begin{aligned} TX &= \{(x, v) \in X \times \mathbb{R}^n \mid v \in T_x X\} \\ &= \{(x, v) \mid x \in X,\ v \in T_x X\} \end{aligned}$$

Remark. You can prove that this is a smooth $2k$-manifold.

Definition. Let $f : X \xrightarrow{\text{smooth}} Y$. Then we can define an obvious smooth map $df : TX \to TY$ by

$$df(x, v) = \big(f(x), df_x(v)\big)$$

Remark. This satisfies the chain rule:

$$d(g \circ f) = dg \circ df$$

and also dI_X is the identity on TX:

$$dI_X = I_{TX}$$

and

$$\text{if } f : X \xrightarrow{\text{diffeo}} Y, \text{ then } df : TX \xrightarrow{\text{diffeo}} TY.$$

Theorem 9 (Whitney Embedding Theorem). (Slightly Weak Version) Let X be a compact[a] smooth k-manifold. Then X embeds in $\mathbb{R}^{2k+1}$.

[a]In the full theorem, we don't require that X is compact, and X embeds in $\mathbb{R}^{2k}$.

Proof see lec 9. it's a good proof, but for the sake of time I didn't type it here. ■

5 Manifolds with Boundary

Remark. In the big picture, we're planning to talk about homotopies of smooth maps between smooth manifolds, which means we're going to have to care about the manifold I and products of I with other manifolds, but I is a manifold *with boundary*, so we'll have to develop some theory about how those manifolds work.

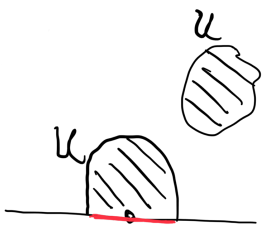
Definition. We define the ***k*-dimensional half space** as

$$\mathbb{H}^k = \{x_1, x_2, \ldots, x_k \mid h_k \geq 0\}.$$

And its boundary is

$$\partial\mathbb{H}^k = \{x_1, x_2, \ldots, x_k \mid h_k = 0\}.$$

Definition. $X \subset \mathbb{R}^N$ is a ***k*-manifold with boundary** if every point $x \in X$ has a chart ϕ from $U \to X$ where $U \underset{\text{open}}{\subset} \mathbb{H}^k$.



Remark. If we have an arbitrary chart ϕ of X near x, we can almost assume without loss of generality that $\phi(\vec{0}) = x$, but instead we need that $\phi(\vec{0}, \lambda) = x$ where the last coordinate λ could be anything nonnegative.

Definition. The **boundary of X**

$$\partial X = \left\{ x \in X \;\middle|\; x = \phi(t) \text{ for some chart } \phi \text{ and some } t \in \partial \mathbb{H}^k \right\}.$$

Remark. From here on out, we assume that every map is smooth and every manifold is with boundary (unless specified otherwise).

Proposition 10. Suppose $g : X \to \mathbb{R}$ is a map, $\partial X = \emptyset$ and g has 0 as a regular value. Then $g^{-1}([0, \infty))$ is a manifold with boundary and $\partial = g^{-1}(0)$.

Proof Exercise for the reader. ■

Corollary. The unit ball B^n is a manifold, and $\partial B^n = S^{n-1}$.

Proposition 11. If X is a smooth manifold without boundary, and Y is a smooth manifold with boundary, then $X \times Y$ is a smooth manifold with boundary, and $\partial(X \times Y) = X \times \partial Y$.

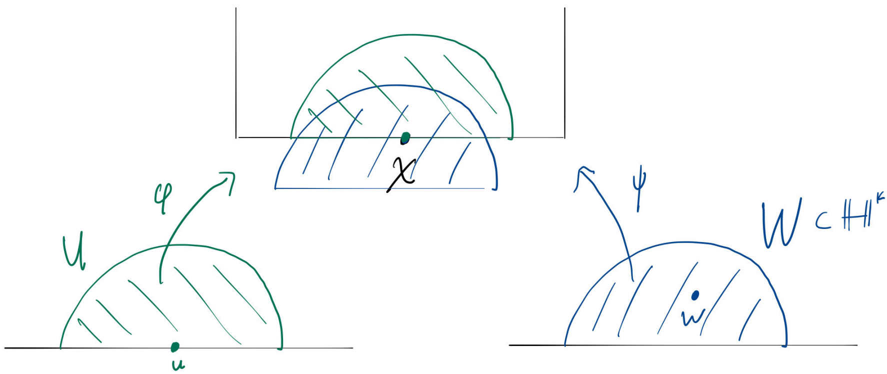
Just considering the charts will hand this right to you. For ϕ chart of X on $U \in \mathbb{R}^k$, ψ chart of Y on $V \in \mathbb{H}^\ell$, then $\phi \times \psi$ is a chart of the product on $H^{k+\ell}$.

Proposition 12. Let X^k be a manifold. Then ∂X is a $(k-1)$-manifold without boundary.

Proof

Lemma If $x \in \phi(\partial U)$ for some chart $\phi : U \to X$, then it is in the boundary for *any* chart.

Proof of Lemma Here's the nightmare: that some chart gives x as a boundary point, but some other chart does not:



but fortunately $g = \phi^{-1} \circ \psi$ is a diffeomorphism and in particular a homeomorphism, so it preserves the boundary. $\square$

To see that $\dim(\partial X) = (k-1)$, just note that any chart ϕ for X near $x \in \partial X$ is a chart for ∂X where the domain is $\mathbb{R}^{k-1} \times \{0\}$, so we're done. $\blacksquare$

Remark. We want to deal with homotopies $M \times I \to Y$ and we want to know, what is a regular value? Now we have that $M \times I$ is a manifold with boundary, and so some subtle weirdness happens there. We would like that if $Z \subset Y$ submanifold with $\partial Z = \emptyset$, then $f^{-1}(Z)$ is a good object. So we want:

- $f^{-1}(Z)$ a manifold
- $\partial f^{-1}(Z) \subset \partial(M \times I)$ (Called properly embedded)

Theorem 13. Let $f : X \xrightarrow{\text{smooth}} Y$, with $\partial Y = \emptyset$ (∂X may not be empty). Let also $Z \underset{\text{manif.}}{\subset} Y$ with $\partial Y = \emptyset$. Suppose

- $f : X \to Y$ has $f \pitchfork Z$.
- $\partial f : \partial X \to Y$ has $\partial f \pitchfork Z$.

Then

- $f^{-1}(Z)$ is a manifold with boundary and
- $\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X$.

Proof See the textbook. $\blacksquare$