Office Hours!

Instructor:

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Office Hours:

Mondays 2–3PM, Today 3–4pm

Tuesdays 10:30–11:30AM, Tomorrow 3–4pm

Thursdays 1–2PM

or by appointment

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South Hall 6510

Derivatives & Differential Calculus

…are about how quickly things change.

- Need to understand PRACTICAL significance in various situations

  Spread of infectious disease, population growth, speed, acceleration, marginal rates in economics, global warming

- Calculate (or estimate) rate of change from various sources:

  graph
  table of data
  formula

- Applications:

  measure change
  predict the future
  optimization – find the best, or smallest, or biggest, or most…

This is all about *understanding* the world.

Philosophical problem

How quickly is something changing at one moment in time?

Example: Does a ball stop when I throw it straight up?

Example: How fast is the temperature rising at 7am?

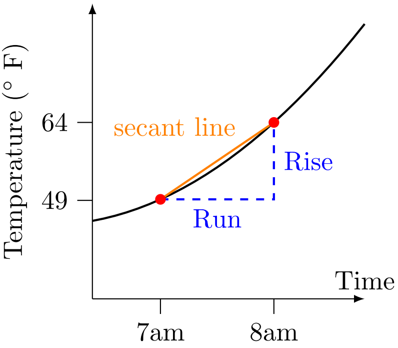
$$\begin{pmatrix} \text{change in temp} \\ \text{between 7am \& 8am} \end{pmatrix}$$

$= 64 - 49 = 15°$ F

$$\begin{pmatrix} \text{average rate of} \\ \text{change in temp} \\ \text{between 7am \& 8am} \end{pmatrix}$$

$= \frac{15° \text{ F}}{1 \text{ hour}} = 15°$ F/hour

= slope of secant line



Continuing Example

Similarly,

$$\left(\begin{array}{c}\text{average rate of}\\ \text{change in temp}\\ \text{between 6am \& 8am}\end{array}\right) = \frac{\text{change in temp}}{\text{time taken}}$$

Question: Suppose temperature at time t given by the formula $f(t) = t^2$. What is the average rate of change of temperature from 6am to 8am?

A= 1 B= 7 C= 9 D= 14 E= 28 $\boxed{\text{D}}$

Average Rate of Change

Suppose temperature at time t given by the formula $f(t) = t^2$.
Using a calculator one can find the average rate of change over shorter and shorter time spans Δt, starting at 7am:

Δt	$(f(7+\Delta t) - f(7))/\Delta t$	ave rate of change ^{o}F/hr	
1	$(8^2 - 7^2)/1$	15	
0.1	$(7.1^2 - 7^2)/0.1$	14.1	
0.01	$(7.01^2 - 7^2)/0.01$	14.01	
0.001	$(7.001^2 - 7^2)/0.001$	14.001	
0.0001	$(7.0001^2 - 7^2)/0.0001$	14.0001	
0.00001	$(7.00001^2 - 7^2)/0.00001$	14.00001	
0	$(7^2 - 7^2)/0$	0/0	arghhhh

Table: Average rate of change over various time spans

What would you guess the exact instantaneous rate of change of temperature at precisely 7am is? Yes! $\boxed{14}$. But how do we get this?
Answer: it is a limit!

Instantaneous Rate of Change

What does the limit

$$\lim_{\Delta t \to 0} \frac{f(7+\Delta t) - f(7)}{\Delta t}$$

mean in practice?

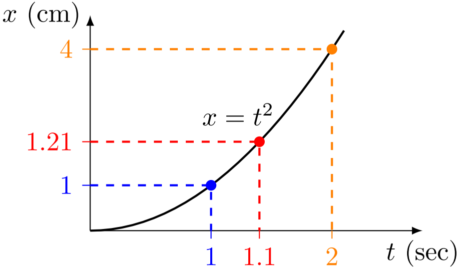
Work out the average rate of change over a very short time interval.
That is very nearly the correct answer.
The shorter the time interval you use, the more accurate you expect the answer to be.

To get the exact answer you would need to take a time interval of zero length.
This leads to the nonsense 0/0. So you can't do this.
That is the philosophical problem.

Mathematical solution: take the limit.

An Example

A hamster runs along the x-axis, so that after t seconds the hamster is t^2 cm from the origin. Our goal is to find the hamster's speed at time $t = 1$ sec.



$$\left(\begin{array}{c}\text{average speed from}\\ t = 1 \text{ to } t = 2\end{array}\right) = \frac{\text{distance gone}}{\text{time taken}} = \frac{2^2 - 1^2}{2 - 1} = 3 \ \text{ cm/sec}$$

$$\left(\begin{array}{c}\text{average speed from}\\ t = 1 \text{ to } t = 1.1\end{array}\right) = \frac{\text{distance gone}}{\text{time taken}} = \frac{1.1^2 - 1^2}{1.1 - 1} = 2.1 \text{ cm/sec}$$

Example Concluded

How do we work out the exact speed of the hamster after 1 second?

Plan:

- Find the average speed over a short time interval Δt, then
- Take the limit as $\Delta t \to 0$.

$$\begin{aligned}
\begin{pmatrix} \text{average speed from} \\ t = 1 \text{ to } t = 1 + \Delta t \end{pmatrix} &= \frac{\text{distance gone}}{\text{time taken}} \\
&= \frac{(1+\Delta t)^2 \; - \; 1^2}{(1+\Delta t) - 1} \\
&= \frac{(1 + 2\Delta t + (\Delta t)^2) \; - \; 1}{\Delta t} \\
&= \frac{2\Delta t + (\Delta t)^2}{\Delta t} \\
&= 2 + \Delta t
\end{aligned}$$

The limit of this as $\Delta t \to 0$ is 2.

Conclusion: At $t = 1$ sec, the exact speed of the hamster is 2 cm/sec.

Hamster Summary

Soon we will calculate that…

> the exact speed of the hamster after t seconds is $2t$ cm/sec.

Summary:

$f(t) = t^2$ = distance in cm of hamster from origin after t seconds

(a function that gives the distance the hamster has traveled at time t)

$f'(t) = 2t$ = speed of hamster in cm/sec after t seconds

(called the derivative of t^2 because it can be derived or obtained from the function t^2)

Question: How many cm had the hamster run by the time its speed was 8 cm/sec?

A= 4 B= 8 C= 16 D= 32 E= 64 $\boxed{\text{C}}$

Exact Hamster Speed

Now we calculate that…

> the exact speed of the hamster after t seconds is $2t$ cm/sec.

Do this as before: working out the average speed over a short time interval Δt and taking the limit as $\Delta t \to 0$

$$\begin{aligned}
\left(\begin{array}{c}\text{average speed from}\\ t \text{ to } t+\Delta t\end{array}\right) &= \frac{\text{distance gone}}{\text{time taken}}\\
&= \frac{(t+\Delta t)^2 \ - \ t^2}{(t+\Delta t)-t}\\
&= \frac{(t^2+2t\Delta t+(\Delta t)^2) \ - \ t^2}{\Delta t}\\
&= \frac{2t\Delta t+(\Delta t)^2}{\Delta t}\\
&= 2t+\Delta t
\end{aligned}$$

The limit of this as $\Delta t \to 0$ is $2t$.

Hamster Questions!

After t seconds, the hamster is $f(t) = t^2$ cm from origin.
(1) What is the exact speed (in cm/sec) of the hamster at $t = 2$?

A= 1 B= 2 C= 4 D= 6 E= 8 $\boxed{\text{C}}$

(2) What is the exact speed (in cm/sec) of the hamster at $t = 4$?

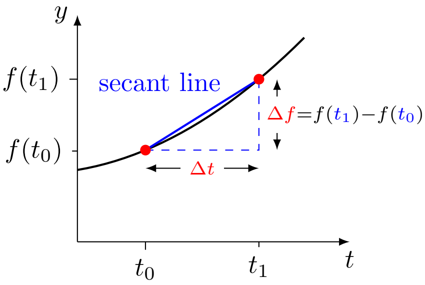
A= 1 B= 2 C= 4 D= 6 E= 8 $\boxed{\text{E}}$

(3) What is the average speed (in cm/sec) of the hamster from $t = 2$ to $t = 4$ seconds?

A= 1 B= 2 C= 4 D= 6 E= 8 $\boxed{\text{D}}$

Does this make sense?

Graphical Approach



Δf = change in f
Δt = change in t

Many ways to say same thing:

$$\left(\begin{array}{c}\text{average rate of}\\ \text{change of } f\end{array}\right) = \frac{\text{change in } f}{\text{change in } t}$$

$$= \frac{\Delta f}{\Delta t}$$

$$= \text{slope of secant line} = \frac{f(t_1) - f(t_0)}{t_1 - t_0}$$

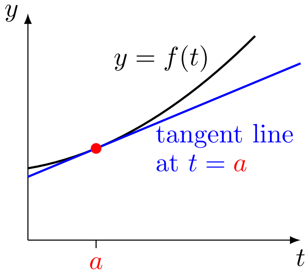
The derivative is defined to be

$$\lim_{\Delta t \to 0} \left(\frac{\Delta f}{\Delta t} \right) = \frac{df}{dt}$$

Idea: As t_1 moves closer to t_0 the secant line approaches the tangent line at t_0. This is the line with the same slope as the graph at t_0.

Understanding Derivatives

There are many ways to think about derivatives. You need to understand these to apply to problems.



slope of graph at a

= slope of tangent line

= instantaneous rate of change of f at a

$$= \left(\begin{array}{c} \text{limit of average rate of change} \\ \text{of } f \text{ over shorter and shorter} \\ \text{time intervals starting at } a \end{array} \right)$$

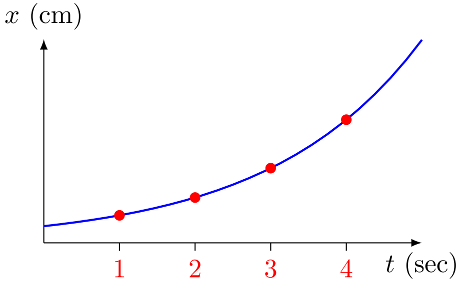
= limit of slopes of secant lines

$$= f'(a) \;=\; \left.\frac{df}{dt}\right|_{t=a}$$

Summary

- How fast something changes = rate of change
- Instantaneous rate of change is the limit of the average rate of change over shorter and shorter time spans. This gets around the $0/0$ problem.
- speed = rate of change of distance traveled.

Examples



The graph shows the distance from the origin in cm after t seconds of a hamster. Which of the numbers below is the largest?

Hint: Speed is a slope!

A = speed of the hamster at $t = 1$

B = speed of the hamster at $t = 2$

C = speed of the hamster at $t = 3$

D = average speed of the hamster between $t = 2$ and $t = 3$

E = average speed of the hamster between $t = 3$ and $t = 4$