Math 550
Homework 6
Dr. Fuller
Solutions

1. Using spherical coordinates $g(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$, the tangent space at $p = g(\theta,\varphi)$ has basis
$$(\,(-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0), (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi)\,).$$

2. (a) Let $(v_1,\ldots,v_k)$ be a basis for V. Define $g : \mathbf{R}^k \to \mathbf{R}^n$ to be the linear transformation given by $g(e_i) = v_i$. Then
   (i) $g(\mathbf{R}^k) = V = V \cap \mathbf{R}^n$;
   (ii) rank $Dg(u) =$ rank $g = k$ for all $u \in \mathbf{R}^k$;
   (iii) $g^{-1} : V \to \mathbf{R}^k$ is the linear transformation given by $g^{-1}(v_i) = e_i$, which is continuous.

   This shows that g parameterizes V as a k-dimensional manifold.

   (b) If $p = g(u)$, then $V_p = Dg(u)(\mathbf{R}^k) = g(\mathbf{R}^k) = V$.

3. Let $(p, v_p) \in TM$. Since $p \in M$, we have open sets U in $\mathbf{R}^k$ and $W \in \mathbf{R}^n$, and a local parameterization $g : U \to \mathbf{R}^n$ with

   (i) $g(U) = M \cap W$;
   (ii) rank $Dg(u) = k$ for all $u \in U$;
   (iii) $g^{-1} : g(U) \to U$ continuous.

   Define $G : U \times \mathbf{R}^k \to \mathbf{R}^n \times \mathbf{R}^n$ by $G(u,v) = (g(u), Dg(u)(v))$. We have

   (i) $g(U \times \mathbf{R}^k) = TM \cap (W \times \mathbf{R}^n)$;
   (ii) Suppose that A is the matrix which represents $Dg(u)$ with respect to the standard basis, so A has rank k. Then $DG(u,v)$ is represented in the standard basis by a $2k \times 2k$-matrix whose upper left and lower right $k \times k$ submatrices are both A. This implies that rank $DG(u,v) = 2k$.
   (iii) If $(q, v_q) \in TM$, so that $v_q \in M_q$, then $G^{-1}(q, v_q) = (g^{-1}(q), Dg^{-1}(q)(v_q))$. This shows that G^{-1} continuous.

   This confirms that G is a local parameterization around $(p, v_p) \in TM$.

4. Let $x \in \partial M$, and suppose $g : U \subset \mathbf{H}^k \to \mathbf{R}^n$ is a local parameterization, with $g(u) = x$. This means we have

   (i) $g(U) = M \cap W$ for some open set W in $\mathbf{R}^n$;
   (ii) rank $Dg(u) = k$ for all $u \in U$;
   (iii) $g^{-1} : g(U) \to U$ continuous.

   Let $i : \mathbf{R}^{k-1} \to \mathbf{R}^k$ be the inclusion map $i(u_1,\ldots,u_{k-1}) = (u_1,\ldots,u_{k-1},0)$. Let $V = i^{-1}(U)$. Then:

   (i) $(g \circ i)(V) = \partial M \cap W$;
   (ii) rank $D(g \circ i)(u) = k-1$ for all $u \in V$, since rank $D(g)(i(u)) = k$ and rank $D(i)(u) = k-1$;
   (iii) $(g \circ i)^{-1} = i^{-1} \circ g^{-1}$ continuous, as it is the composition of continuous functions.

   Thus $g \circ i : V \to \mathbf{R}^n$ defines a local parameterization around $x \in \partial M$, showing that ∂M is a $(k-1)$-dimensional manifold.

5. In class, it was shown that f and g induce the same orientation on S_x if and only if $\det(Dg^{-1} \circ Df)(u) > 0$.

   Now write $g^*\omega = h\, dx_1 \wedge \cdots \wedge dx_k$. Then

$$\begin{aligned}
f^*\omega &= (g \circ g^{-1} \circ f)^*\omega \\
&= (g^{-1} \circ f)^* g^*\omega \\
&= (g^{-1} \circ f)^*(h\, dx_1 \wedge \cdots \wedge dx_k) \\
&= (g^{-1} \circ f)^* h\, (g^{-1} \circ f)^* dx_1 \wedge \cdots \wedge dx_k \\
&= (g^{-1} \circ f)^* h\, \det(Dg^{-1} \circ Df)\, dx_1 \wedge \cdots \wedge dx_k \\
&= \det(Dg^{-1} \circ Df)\, (h \circ g^{-1} \circ f) dx_1 \wedge \cdots \wedge dx_k
\end{aligned}$$

   Evaluating at u and v, where $f(u) = g(v) = x$, we get

$$\begin{aligned}
f^*\omega(u)(e_1, \ldots, e_k) &= \det(Dg^{-1} \circ Df)(u)\, (h \circ g^{-1} \circ f)(u)(dx_1 \wedge \cdots \wedge dx_k)(e_1, \ldots, e_k) \\
&= \det(Dg^{-1} \circ Df)(u)\, h(v)(dx_1 \wedge \cdots \wedge dx_k)(e_1, \ldots, e_k) \\
&= \det(Dg^{-1} \circ Df)(u)\, g^*\omega(v)(e_1, \ldots, e_k).
\end{aligned}$$

   Thus $f^*\omega(e_1, \ldots, e_k)$ and $g^*\omega(e_1, \ldots, e_k)$ have the same sign if and only if $\det(Dg^{-1} \circ Df)(u) > 0$.

6. Yes, they induce the same orientation.

Addendum

1. Suppose $(v_1, \ldots, v_{k-1})$ is a basis for the tangent space at a point in $\partial \mathbf{H}^k$.

   Observe:

   - $(v_1, \ldots, v_{k-1})$ is a positively oriented basis for $\mathbf{R}^{k-1}$ if and only if $(v_1, \ldots, v_{k-1}, e_k)$ is a positively oriented basis for $\mathbf{R}^k$.
   - $(v_1, \ldots, v_{k-1})$ is a positively oriented basis for $\partial \mathbf{H}^k$ if and only if $(-e_k, v_1, \ldots, v_{k-1})$ is a positively oriented basis of $\mathbf{R}^k$.
   - The orientations of $(-e_k, v_1, \ldots, v_{k-1})$ and $(v_1, \ldots, v_{k-1}, e_k)$ agree if and only if k is even, since we can equate them by $k-1$ transpositions, and changing the sign of e_k.

   Combining these observations proves the statement.