09/28/21 Agenda

- review key probability definitions
- LLN / CLT
- why Monte Carlo (MC) methods are ill-suited for when we don't know if a random variable has finite expectation
- why simple MC has convergence like $\Theta(1/\sqrt{n})$
- ex 1. (oh-so tangentially related)
  MC integration estimate π (polytope ← stochastic reachability)
- online code examples of propagating uncertainty through a PDE/ODE

REVIEW

Def. a continuous random variable X is a (continuous) map $X : \Omega \longrightarrow \mathbb{R}$ where

- Ω is any nonempty set (called the event space).

The Ω represents "abstract elementary events" (something in the actual world that we observe that doesn't immediately have an associated numeric value).

Ex. $\Omega = \{$"Heads", "Tails"$\}$

stars ↑ hearts ↑

the literal events observed in the world; for this class, it's not important to consider them

What's so random?

$$X : \Omega \longrightarrow \mathbb{R}$$

- "randomness" comes from not knowing the input "event" $\underline{\omega} \in \Omega$
- we can assign a probability to certain "outcomes" occuring

$\mathbb{P}(X \leq a) =$ shorthand for: what is the chance that $X(\omega) \leq a$ for an unknown $\omega \in \Omega$?

side: measure theory

Def. A probability density function (PDF) of a random variable X is a map $f_X : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ that satisfies certain properties:

$\mathbb{R}_{\geq 0}$ ↓ non-neg. reals

★ Unless otherwise stated, all given random vars have a prob. density (or a prob. mass function)

(1) For $a, b \in \mathbb{R}$ $(a < b)$, we have

$$\mathbb{P}(\underline{a < X < b}) = \int_a^b \underline{f_X(s)}\, ds,$$

(2) $f_X \geq 0$

(3) f_X is piecewise constant

(4) $\int_{-\infty}^{\infty} f_X(s)\, ds = 1$

Ex. $X \sim N(0,1) \implies f_X : \mathbb{R} \to \mathbb{R}_{>0}$

$$f_X(s) = \frac{1}{\underline{\sqrt{2\pi}}} \exp\left(-s^2/2\right)$$

$\downarrow$

$$\mathbb{P}(\underline{X \leq a}) = \int_{-\infty}^{a} f_X(s)\, ds$$

Ex. $X \sim U[-1,1]$

$$f_X(s) = \frac{1}{2}$$

(in general, $X \sim U[c,d]$, $f_X(s) = \frac{1}{d-c}$, $c \neq d$)

Def. (First order statistics)

For a continuous RV X with PDF f_X,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} s\, \underline{f_X(s)}\, ds = \text{mean of } X$$

and

$$\mathbb{V}[X] = \int_{-\infty}^{\infty} \left(s^2 - \mathbb{E}(X)^2\right) f_X(s)\, ds = \text{variance of } X$$

Thm. (law of large numbers)

Consider some i.i.d. random variables $Y_1, \dots, Y_n$ with respect to some random variable $\underline{Y}$. Then if $\mathbb{E}[Y] < \infty$, we have that

independent and identically distributed

$$S_n := \frac{1}{n} \sum_{i=1}^{n} Y_i$$

(sample mean)

converges in some sense to the "true" mean $\mathbb{E}[Y]$. In particular,

the weak law of large numbers states that $S_n \longrightarrow \mathbb{E}(Y)$ converges in probability; that is,

$$\text{for all } \epsilon > 0, \quad \lim_{n \to \infty} \mathbb{P}(|S_n - \mathbb{E}(Y)| > \epsilon) = 0.$$

↳ some number

The strong law of large numbers says $S_n \longrightarrow \mathbb{E}(Y)$ almost surely ("with probability 1", "convergence almost everywhere"); that is,

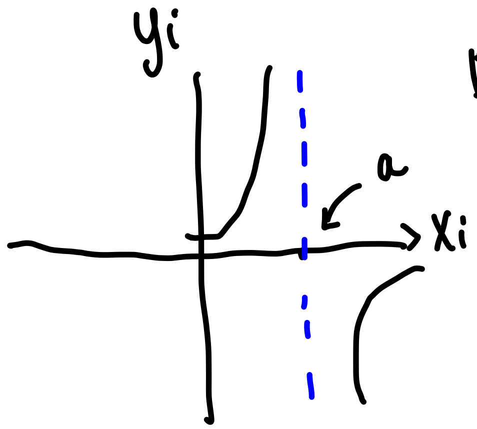
$$\mathbb{P}\left[\lim_{n \to \infty} S_n = \mathbb{E}(Y)\right] = 1.$$

Who cares if $\mathbb{E}(Y) < \infty$?

you do!

Let $X_i \sim N(0,1)$. Let $Y_i = \frac{1}{a - X_i}$ $a \in \mathbb{R}$ (fixed).

"distributed as"
or "is a RV whose PDF is dictated by the RHS expression"



$\mathbb{E}(Y)$ is undefined when $X_i = a$

○ We might run into trouble when our RVs involve fractions. (Importance sampling....)

How about that var(Y)?

When $var(Y) < \infty$, then the Central Limit Theorem gives that for i.i.d. Y_i,

$$\sqrt{n}\left(\frac{|S_n - \mathbb{E}(Y)|}{var(Y)}\right) \xrightarrow{dist.} N(0,1).$$

("weaker" than convergence in probability)

wikipedia

$S_n \in \left[\mathbb{E}(Y) - \frac{\sigma}{\sqrt{n}}, \mathbb{E}(Y) + \sigma/\sqrt{n}\right]$ $\sigma = \sqrt{var(Y)}$

interval length shrinks as $n \to \infty$

$$S_n \xrightarrow{\text{dist.}} N\Big(\underset{\substack{\| \\ \mathbb{E}(Y_i)}}{\mathbb{E}(Y)}, \frac{\sigma^2}{n}\Big)$$

$$\underline{|S_n - \mathbb{E}(Y)|} \leq \frac{\alpha \, \sigma}{\sqrt{n}}$$

α proportionality const. $(\alpha > 0)$

Thus,

$S_n \to \mathbb{E}(Y)$ in $\underline{\Theta(1/\sqrt{n})}$.

How does this relate to MC explicity?

input parameter X

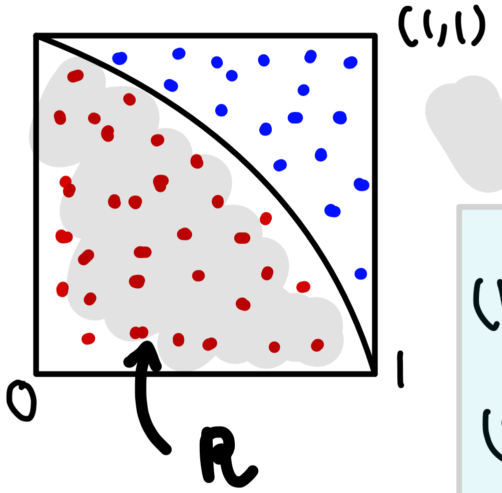
model function f

response (QoI) $y \approx f(X)$

- We care about $\mathbb{E}(Y) = \mathbb{E}(\underline{f}(X))$ ← estimating this quantity with $S_n(f) := \frac{1}{n} \sum_{i=1}^{n} f(X_i)$

Ex. MC INTEGRATION

Goal: estimate π



$= \frac{1}{4}\pi =$ area of region R

(1) Take $\vec{X}_i \sim U(0,1)$

(2) If $\vec{X} \in R$, color the dot red

else color blue

Check if $X \in R$ via checking if $(\overbrace{X_{i,1}}^{\text{components of } \vec{X} \in R})^2 + (X_{i,2}^2) \le \underline{1}$.

Define

$Y_i = \mathbb{1}_{\text{ith point in } R} = \begin{cases} 1 & \text{if } \vec{X}_i \in R \\ 0 & \text{else.} \end{cases}$

(indicator random var.)

Consider $S_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $\mathbb{E}(Y)$.

Then $S_n = \frac{1}{n}\sum_{i=1}^{n} y_i$ ← only nonzero if $\vec{X}_i \in R$

$$= \frac{\sum_{i \text{ s.t. } X_i \in R} (1)}{n} = \frac{\#\text{ of points in } R}{\text{total }\#\text{ of points}} = \overbrace{\left(\frac{\text{red dots}}{\text{red + blue dots}}\right)}^{\text{ratio}}$$

$$\mathbb{E}(Y_i) = \mathbb{E}[\underset{\uparrow 1}{Y_i} \mid \underline{X_i} \in R]\, \mathbb{P}(X_i \in R) + \overset{0}{\cancel{\mathbb{E}[Y_i \mid X_i \notin R]}}\, \mathbb{P}(X_i \notin R)$$

law of total expectation

$$= \mathbb{P}(X_i \in R) + 0$$

$$= \mathbb{P}(X_i \in R)$$

Y_i are i.i.d. $\Rightarrow \mathbb{E}(Y) = \mathbb{P}(\underline{X_i} \in R)$

$$S_n \longrightarrow \mathbb{E}(Y)$$

$$S_n = \frac{\#\text{ of red pts}}{\text{total }\#\text{ of points } n} \xrightarrow{\text{LLN as } n\to\infty} \mathbb{P}(X_i \in R) = \int_{s\in R} \underset{\uparrow \text{ density}}{f_{X_i}(s)}\, ds = \underline{\text{area}(R)}$$

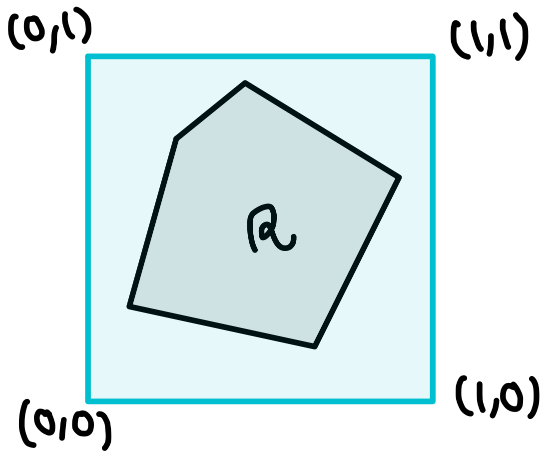
$X_i \sim U(0,1)$

$$\overbrace{\left(\frac{\#\text{ of red pts}}{\text{total }\#\text{ of points } n}\right)}^{S_n} \xrightarrow[n\to\infty]{\text{LLN}} \text{area}(R) = \pi/4$$

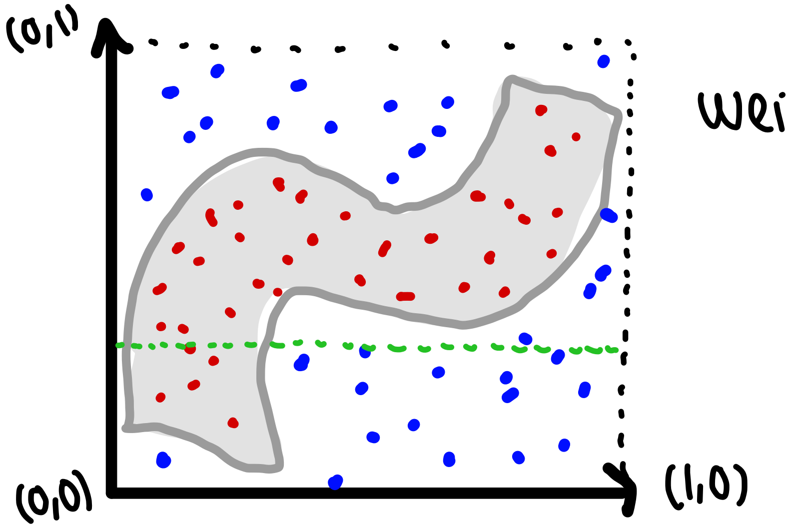
(optional lil' HW problem)

so $\pi \approx 4 S_n$ for large n.

Using MC to find the area of stranger regions



R = polytope (e.g. representing feasible initial states)



weird region

S_n = ratio of red pts to total points $\xrightarrow{n \to \infty}$ area of weird region

Take $X_i \sim U(0,1)$.

Let $Y_i = \underline{\mathbb{1} X_i \in \text{weird region}\}}$. How to efficiently evaluate if a given $X_i \in R$?

Ex:

- Inside or outside alg.

(count parity of # of times .... line intersects boundary of R to check if $\vec{X_i} \in R$)