$$\frac{du}{dt} - \frac{d^2u}{dx^2} = f(x,t)$$

$$f(x,t) = (\Omega^2 - 1)e^{-t}\sin(\Omega x)$$

Analytic Solution

$$\boxed{u(x,t) = e^{-t}\sin(\Omega x)}$$

$(x,t) \in [0,1] \times [0,1)$

$u(x,0) = \sin \Omega(x)$

$u(0,t) = u(1,t) = 0$

$N = 11$ Nodes

where then can be spaced

by any $0:x:N$

$f(x,t) =$ user input

1). Weak Form

Test Function

$v(x)$

$$\frac{du}{dt} - \frac{d^2u}{dx^2} = f(x,t)$$

$$\int_0^1 \frac{du}{dt} v(x) dx - \int_0^1 \frac{d^2u}{dx^2} v(x) dx = \int_0^1 f(x,t) v(x) dx$$

I.B.P

$u = v(x) \quad du = v'(x) dx$

$v = \frac{du}{dx} \quad dv = \frac{d^2u}{dx^2} dx$

$$\int_0^1 \frac{\partial u}{\partial t} v(x) dx - v(x) \frac{\partial u}{\partial x}\Big|_0^1 + \int_0^1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx = \int_0^1 f(x,t) v(x) dx$$

## apply Boundary conditions

$u(x,0) = \sin(\Omega x)$

$u(0,t) = u(1,t) = 0$

$$\int_0^1 \frac{\partial u}{\partial t} v(x)\, dx - \cancel{v(x) \frac{\partial u}{\partial x}\Big|_0^1}^{\,0} + \int_0^1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}\, dx = \int_0^1 f(x,t)\, v(x)\, dx$$

$$\boxed{v(x) = \phi_i(x)}$$

$$\int_0^1 \frac{\partial u}{\partial t} \phi_i(x)\, dx + \int_0^1 \frac{\partial u}{\partial x} \frac{\partial \phi_i}{\partial x}\, dx - \int_0^1 f(x,t)\, \phi_i(x)\, dx = 0$$

$\phi_i(x) = \phi_j(x) = \sin(\Omega x)$

# Forward Euler

$$R_i = \int_0^1 \left( \frac{u(t+\Delta t) - u(t)}{\Delta t} \right) \phi_i(x)\, dx + \int_0^1 \frac{\partial u}{\partial x} \frac{\partial \phi_i}{\partial x}\, dx - \int_0^1 f(x,t)\, \phi_i(x)\, dx$$

$$\Downarrow$$

$$\frac{1}{\Delta t} \underbrace{\sum_{j=1}^{N} u_j^{n+1}}_{\vec{u}^{\,n+1}} \underbrace{\int_0^1 \phi_j \phi_i \, dx}_{m_{ij}} - \frac{1}{\Delta t} \underbrace{\sum_{j=1}^{N} u_j^{n}}_{\vec{u}^{\,n}} \underbrace{\int_0^1 \phi_i \phi_j \, dx}_{m_{ij}} + \sum_{j=1}^{N} u_j^{n} \underbrace{\int_0^1 \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x}\, dx}_{k_{ij}} - \underbrace{\int_0^1 f^n(x,t)\, \phi_i(x)\, dx}_{\vec{F}_i^{\,n}} = 0$$

$$\Downarrow$$

$$\frac{1}{\Delta t} M \vec{u}^{\,n+1} - \frac{1}{\Delta t} M \vec{u}^{\,n} + K \vec{u}^{\,n} - \vec{F}^{\,n} = 0$$

$$\Downarrow$$

$$\boxed{\vec{u}^{\,n} \left[ I - \Delta t\, M^{-1} K \right] + \Delta t\, M^{-1} \vec{F}^{\,n}}$$

**Backward Euler (same steps as forward)**

$$\underbrace{\left[\frac{1}{\Delta t} M + K\right]}_{B}$$

$$\boxed{\vec{u}^{(n+1)} = \frac{1}{\Delta t} B^{-1} M \vec{u}^{(n)} + B^{-1} \vec{F}^{(n+1)}}$$

# Mapping

$e: (0,1)$ $\quad$ $\hat{e}: (-1,1)$

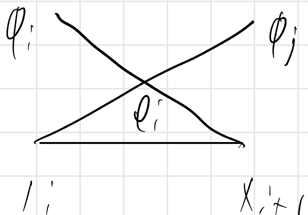
$$\int_e \phi_i \phi_j \, dx \Rightarrow \int_{\hat{e}} \hat{\phi}_i \hat{\phi}_j \, |\det J| \, d\xi$$

$$\int_e \phi_i' \phi_j' \, dx \Rightarrow \int_{\hat{e}} \hat{\phi}_i' \hat{\phi}_j' \, |\det J| \, d\xi$$

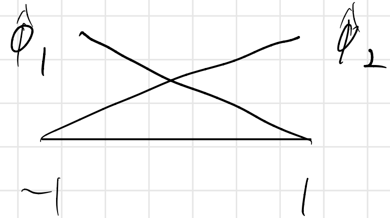
$$\hat{\phi}_i = \frac{1-\xi}{2} \qquad \hat{\phi}_i' = -\frac{1}{2}$$

$$\hat{\phi}_j = \frac{1+\xi}{2} \qquad \hat{\phi}_j' = \frac{1}{2}$$

$|\det| =$




$\Rightarrow$




$$\frac{dx}{d\xi} = \frac{h}{2}$$

$$\frac{d\xi}{dx} = \frac{2}{h}$$

$$K_{ij} = \int_{-1}^{1} \left(\frac{\partial \xi}{\partial x} \hat{\phi}_i'\right)\left(\frac{d\xi}{dx} \hat{\phi}_j'\right) \frac{dx}{d\xi}\, d\xi$$

$$K_{ij} = \int_{-1}^{1} \left(\frac{2}{h} \cdot -\frac{1}{2}\right)\left(\frac{2}{h} \cdot \frac{1}{2}\right) \frac{h}{2}\, d\xi$$

$$= \boxed{-\frac{1}{2h} \int_{-1}^{1} d\xi}$$

**Using quadrature for local elements**

or $\frac{dx}{d\xi} \sum_{i}^{n=2} w_i \hat{f}(\xi) \hat{\phi}_i(\xi)$

$$M_{ij} = \int_{-1}^{1} \hat{\phi}_i \hat{\phi}_j \, |\det J| \, d\xi = \int_{-1}^{1} \left(\frac{1-\xi}{2}\right)\left(\frac{1+\xi}{2}\right) \frac{h}{2}\, d\xi$$

$$= \boxed{\frac{h}{8} \int_{-1}^{1} (1-\xi)^2 \, d\xi}$$