zeige $QFT_8 |x\rangle = \frac{1}{\sqrt{8}} \left(|0\rangle + \omega_2^x |1\rangle\right)\left(|0\rangle + \omega_4^x |1\rangle\right)\left(|0\rangle + \omega_8^x |1\rangle\right)$

Nach Definition ist

$$QFT_8 |x\rangle = \frac{1}{\sqrt{8}} \sum_{k=0}^{7} \omega_8^{kx} |k\rangle$$

$$= \frac{1}{\sqrt{8}} \Big(\omega_8^0 |000\rangle + \omega_8^x |001\rangle + \omega_8^{2x} |010\rangle + \omega_8^{3x} |011\rangle + $$
$$+ \omega_8^{4x} |100\rangle + \omega_8^{5x} |101\rangle + \omega_8^{6x} |110\rangle + \omega_8^{7x} |111\rangle \Big)$$

Wir beobachten

$$\omega_8^{2x} = \exp\left(\frac{2\pi i}{8} \cdot 2x\right) = \exp\left(\frac{2\pi i}{4} x\right) = \omega_4^x$$
$$\omega_8^{3x} = \omega_8^{2x} \omega_8^x = \omega_4^x \omega_8^x$$
$$\omega_8^{4x} = \omega_8^{2x} \omega_8^{2x} = \omega_4^x \omega_4^x = \omega_4^{2x} = \omega_2^x \quad \text{(vgl Vorlesung)}$$
$$\omega_8^{5x} = \omega_8^{4x} \omega_8^x = \omega_2^x \omega_8^x$$
$$\omega_8^{6x} = \omega_8^{2x} \omega_8^{4x} = \omega_4^x \omega_2^x = \omega_2^x \omega_4^x$$
$$\omega_8^{7x} = \omega_8^{6x} \omega_8^x = \omega_2^x \omega_4^x \omega_8^x$$

Also ist

$$QFT_8 |x\rangle = \frac{1}{\sqrt{8}} \Big(|000\rangle + \omega_8^x |001\rangle + \omega_4^x |010\rangle + \omega_4^x \omega_8^x |011\rangle +$$
$$+ \omega_2^x |100\rangle + \omega_2^x \omega_8^x |101\rangle + \omega_2^x \omega_4^x |110\rangle +$$
$$+ \omega_2^x \omega_4^x \omega_8^x |111\rangle \Big)$$

$$= \frac{1}{\sqrt{8}} \left(|00\rangle + \omega_4^x |01\rangle + \omega_2^x |10\rangle + \omega_2^x \omega_4^x |11\rangle \right) \left(|0\rangle + \omega_8^x |1\rangle \right)$$

$$= \frac{1}{\sqrt{8}} \left(|0\rangle + \omega_2^x |1\rangle \right) \left(|0\rangle + \omega_4^x |1\rangle \right) \left(|0\rangle + \omega_8^x |1\rangle \right)$$

Übung 1. Berechne $R_4 \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

2. Zeige, dass $R_M = \begin{pmatrix} 1 & 0 \\ 0 & \omega_M \end{pmatrix}$ unitär ist.

zu (1) Mit $\omega_4 = i$ (beachte $\omega_4^0 = 1, \omega_4^2 = -1, \omega_4^3 = -i$) ist

$$R_4 \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

zu (2) Es gilt

$$R_M^+ R_M = \begin{pmatrix} 1 & 0 \\ 0 & \overline{\omega_M} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \omega_M \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \omega_M \overline{\omega_M} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\omega_M \overline{\omega_M} = \exp\left(\frac{2\pi i}{M}\right) \exp\left(-\frac{2\pi i}{M}\right) = \exp\left(\frac{2\pi i}{M} - \frac{2\pi i}{M}\right)$$

$$= \exp(0) = 1$$

Alternativ: Nach Serie 2 Aufgabe 3 wissen wir

$$U \in \mathbb{C}^2 \text{ unitär} \iff U = \exp(i\varphi) \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

mit $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$

Für R_M ist $\beta = 0$, $\varphi = \frac{\pi}{M}$ und $\alpha = \exp\left(-\frac{\pi i}{M}\right)$

Dann ist $\exp(i\varphi)\beta = 0$, $\exp(i\varphi)(-\overline{\beta}) = 0$, sowie

$$\exp(i\varphi)\alpha = \exp\left(\frac{i\pi}{M}\right) \exp\left(-\frac{\pi i}{M}\right) = 1,$$

$$\exp(i\varphi)\overline{\alpha} = \exp\left(\frac{\pi i}{M}\right) \overline{\exp\left(-\frac{\pi i}{M}\right)} = \exp\left(\frac{\pi i}{M}\right) \exp\left(\frac{\pi i}{M}\right)$$

$$= \exp\left(\frac{\pi i}{M} \cdot 2\right) = \omega_M$$

und mit $|z|^2 = z\overline{z} \;\forall z \in \mathbb{C}$ auch

$$|\alpha|^2 + |\beta|^2 = |\alpha|^2 = \alpha\overline{\alpha} = \exp\left(\frac{\pi i}{M}\right) \exp\left(-\frac{\pi i}{M}\right) = 1$$