Spanning with Power Perpetuals

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Abstract

Any continuous function can be spanned by a suitable portfolio of power perpetuals. We generate this replicating portfolio, derive the arbitrage-free funding rates, and demonstrate the replication and arbitrage-free swap fee for a Uniswap LP.

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1 Introduction

Power perpetual contracts give an exposure an asset price raised to a power. A sequence of power perpetuals can replicate any continuous payoff function by matching the exposures from its Taylor expansion. For example, constant product markets such as Uniswap can be hedged with a square-root perpetual or a sequence of integer-power perpetuals (see (2)). A European option under Black-Scholes-Merton assumptions can be hedged with a linear and a quadratic perpetual (see (3)). [1]

The arbitrage-free funding rate for any payoff is the sum of funding rates for the replicating portfolio of power perpetuals. From this perspective, the funding rate for a European option (theta) is a combination of the funding rates for the linear and quadratic perpetual. The funding rate for a constant product AMM such as Uniswap (its fair yield from swap fees) is the combination of funding rates for the sequence of power perpetuals that replicates its payoff.

We proceed first by defining the mechanism for a power perpetual. Then we construct a representation for power perpetuals in terms of power return swaps (where the floating leg of the swap is a power of the asset return), and a corresponding representation for any function of the return with a Taylor series. This gives the machinery to replicate any continuous function of an asset return as a sequence of power return swaps. The replicating portfolio of power perpetuals is a set of weights to each integer power perpetual (1-perpetual, 2-perpetual, etc) that match the Taylor series for the target return. Finally, we demonstrate the hedge for a constant product market maker using linear and quadratic perpetuals, and show a convenient representation of the arbitrage-free yield from fees as a weighed sum of funding rates of its replicating portfolio; this is approximately one eighth of the asset return variance.

[1] More generally, constant function market makers (see (1)) can be hedged with arbitrary precision with a set of power perpetuals.

2 Power perpetuals

Power perpetuals ((4), (5), (3)) are perpetual contracts that target a power of an underlying price. A first order perpetual targets the underlying price S_t, a second order perpetual targets the square of the price, S_t^2, and so forth.

Power perpetuals transfer a funding rate f_p between longs to shorts that varies with the distance between the traded price (mark) and the target price (index). A typical mechanism calculates a funding rate as:

$$f_p = \lambda \left(\frac{M_t - S_t^p}{M_t} \right)$$

Where M_t is the prevailing traded price of the power perpetual and λ is a constant.

A power perpetual can be replicated with a sequence of fixed expiry power contracts with payoffs S_T^p at expiry T. Each fixed expiry power contract can be replicated with a portfolio of futures and options (see (3)).

2.1 From power perpetuals to power swaps

The return for the long side of the contract between t_0 to t_1 is

$$\frac{S_{t_1}^p - S_{t_0}^p(1 + f_p)}{S_{t_0}^p} = (1 + r_{t_0 \to t_1})^p - f_p - 1 = \left[\sum_{k=0}^{p} \binom{p}{k} r_{t_0 \to t_1}^k \right] - f_p - 1 \tag{1}$$

Where $r_{t_0 \to t_1} = S_{t_1}/S_{t_0} - 1$, henceforth ignoring the subscript. This gives a useful representation of the funding rate as a linear function of the swap rates.

$$f_p^Q = \sum_{k=0}^{p} \binom{p}{k} s_{r^k} - 1$$

Where s_{r^p} is the fixed rate for a swap paying r^p under the risk neutral measure. So for example s_{r^1} is the funding rate for a standard total return swap.

2.2 From power swaps to general payoff functions

Any continuous function of the underlying price return $g(r)$ can be expanded as a Taylor series

$$g(r) = \sum_{n=0} \frac{g^{(n)}(0)}{n!} r^n \equiv \sum_{n=0} h_n(g) r^n \tag{2}$$

Matching terms between the Taylor expression 2 and the swap representation 1 gives the spanning portfolio for g in power perpetuals and the funding rate for $g(r)$.

3 Hedge construction

The replicating portfolio is defined in terms of a sequence of hedge ratios $h_p(g)$, exposures to exposures to each r^p. So, for example, if $h_1(g) = 0.5$ the first order (delta) hedge for g is 50%.

A $\bar{p}$'th order replication using power perpetuals will be a set of notional amounts N_j to each j-th power perpetual (from 2 and 1):

$$\sum_{j=0}^{\bar{p}} N_j \left((1+r)^j - 1\right) = \sum_{j=0}^{\bar{p}} h_j(g) r^j$$

This works as a partial replication for functions with non-zero higher order exposures, correct up to $O(r^{p+1})$ terms.

4 Function swap rates

The funding rate for $g(r)$ is the weighted swap rate for the component hedges.

$$f_g = \sum_{j=0}^{\bar{p}} N_j \left(\left(\sum_{k=0}^{j} \binom{j}{k} s_{r^k} \right) - 1 \right)$$

The rates for the return swaps will typically know under the pricing measure. For lognormal s_r they will be functions of the non-central moments:

Order	Rate
1	$\mathrm{s}_{r^1} = c$
2	$\mathrm{s}_{r^2} = c^2 + \sigma^2$
3	$\mathrm{s}_{r^3} = c^3 + 3c\sigma^2$
4	$\mathrm{s}_{r^4} = c^4 + 6c^2\sigma^2 + 3\sigma^4$

Table 1: Power swap rates

Where σ^2 is variance of the return and c is the carry (asset yield less numeraire yield).

5 Hedging a constant product AMM with power perpetuals

The value of full range constant product AMM, excluding fees, is (see (1), (2))

$$V_t^{LP} = 2L\sqrt{S_t}$$

Where L is liquidity and S_t is the price of one unit of token0 in units of token1.

The return between two periods t_0 and t_1 is

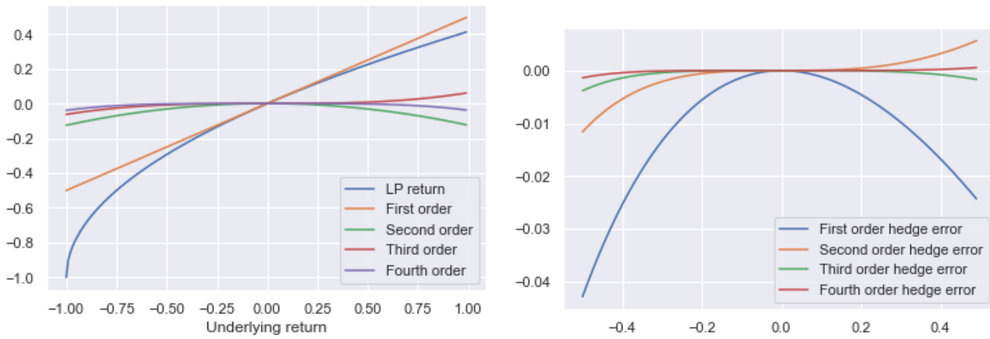
$$r_{t_0 \to t_1}^{LP} = \frac{V_{t_1}^{LP}}{V_{t_0}^{LP}} - 1 = \sqrt{\frac{S_{t_1}}{S_{t_0}}} - 1 = \sqrt{1 + r_{t_0 \to t_1}} - 1$$

where $r_{t_0 \to t_1}$ is the price return

$$r_{t_0 \to t_1} = \frac{S_{t_1}}{S_{t_0}} - 1$$

The Taylor expansion around $r = 0$ is

$$r_{t_0 \to t_1}^{LP} = \sum_{n=0} \frac{g^{(n)}(0)}{n!}|_{r=0} r_{t_0 \to t_1}^n = \frac{1}{2} r_{t_0 \to t_1} - \frac{1}{8} r_{t_0 \to t_1}^2 + \frac{3}{48} r_{t_0 \to t_1}^3 - \frac{15}{384} r_{t_0 \to t_1}^4 + O(r_{t_0 \to t_1}^5)$$



(a) (b)

Figure 1: (a) An LP and its Taylor series components (b) LP hedge error vs hedge order

This means the contract can be replicated with a 50% exposure to a return swap, a -12.5% exposure to a quadratic swap, a +3.125% exposure to a cubic swap, and so forth.

The payoff and its components are in figure 1 (a). The replicating portfolio becomes increasingly accurate with higher orders or returns in figure 1 (b).

In terms of power perpetuals the replicating portfolio is 0.75 1-perpetuals and 0.125 2-perpetuals (table 2).

	Value	Return	First order	Second order	Third order
Constant product AMM	$2L\sqrt{S_t}$	$\sqrt{1+r}-1-f$	1/2	-1/8	3/48
Linear perpetual	S_t	$r - s_r$	1	0	0
Quadratic perpetual	S_t^2	$2r + r^2 - s_{r^2}$	2	1	0
Cubic perpetual	S_t^3	$r^3 + 3r^2 + 2r - s_{r^3}$	2	3	1
AMM Second-order replication	$0.75S_t - 0.125S_t^2$	$0.75(r - s^r) - 0.125(2r+r^2-s_{r^2})$	1/2	-1/8	0

Table 2: Sensitivities for the constant product AMM and power perpetuals

The first and second order sensitives are like option greeks. The AMM has a delta of 1/2 and a gamma of 1/8; a 1-perpetual has a delta of 1 and no gamma; a quadratic power perpetual has a delta of 2 and gamma of 1.

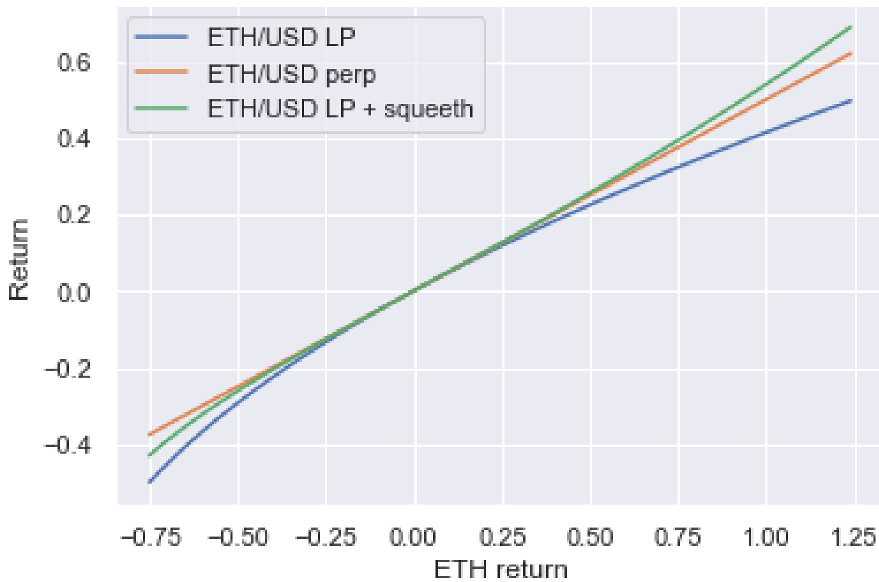


Figure 2: A hedged full range LP is close to a linear perpetual

6 A second order hedge and the fair price of fees

The swap rate for the AMM is the sum of swap rates for the Taylor expansion, so

$$f = \frac{1}{2}s_{r^1} - \frac{1}{8}s_{r^2} + \frac{3}{48}s_{r^3} - \frac{15}{384}s_{r^4} + \dots$$

where s_r and s_{r^2} are swap rates for the total return and the quadratic return, and f is the pool fee expressed as a fraction of the pool size.

For a well-behaved price process with constant variance and zero net funding cost this is

$$f^* = E^Q\left(-\frac{s_r}{2} + \frac{s_{r^2}}{8} - O(r^3)\right) \approx \frac{\sigma_r^2}{8}$$

The fair yield from fees for a full range LP on a Uniswap pool is one eighth of the variance of the price. This is exact under GBM with no drift (see (**?**)).

7 Example: Hedging ETH/stable pairs on Uniswap v3

A constant product AMM LP can be hedged with:

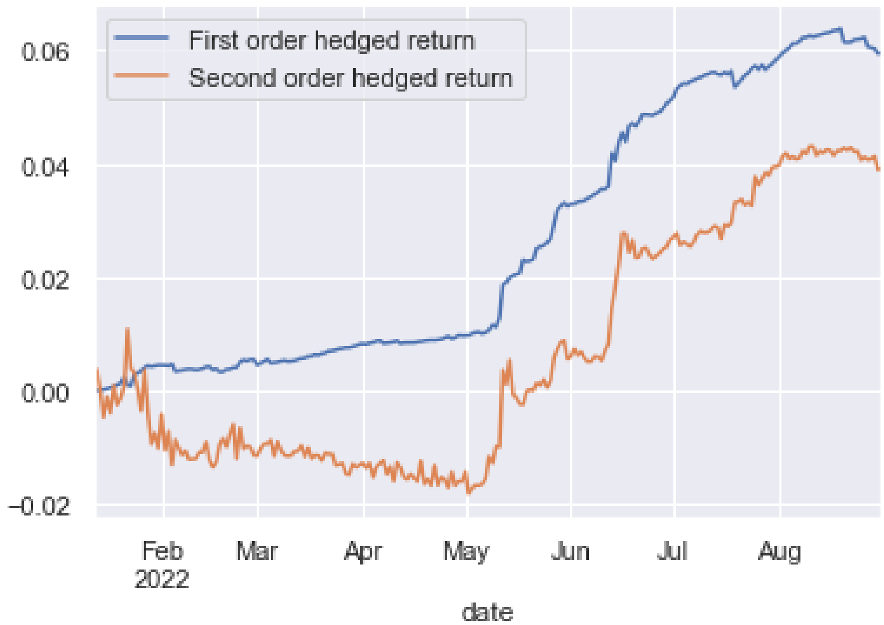


Figure 3: A Uniswap v3 ETH/USDT LP hedged to first and second order

- First order hedge: -50% notional hedge with ETH/USD perpetuals
- Second order hedge: -75% notional hedge with ETH/USD perpetuals and a 12.5% notional hedge with a quadratic power perpetual (oSQTH).

The performance of the two hedges on the ETH/USDT 5bps pool is in figure 3.

Each hedge can be viewed as receiving the pool fee and paying a hedging cost. The trade-off is typical for gamma hedging. The linear first order hedge typically is lower cost because funding is expensive, but performs worse for large jumps in the spot price.

Figure 4 below shows the 10-day moving average of the fee against the first and second order hedge costs.

References

[1] Guillermo Angeris, Akshay Agrawal, Alex Evans, Tarun Chitra, and Steven Boyd. (2022). Constant Function Market Makers: Multi-asset Trades via Convex Optimization. In:

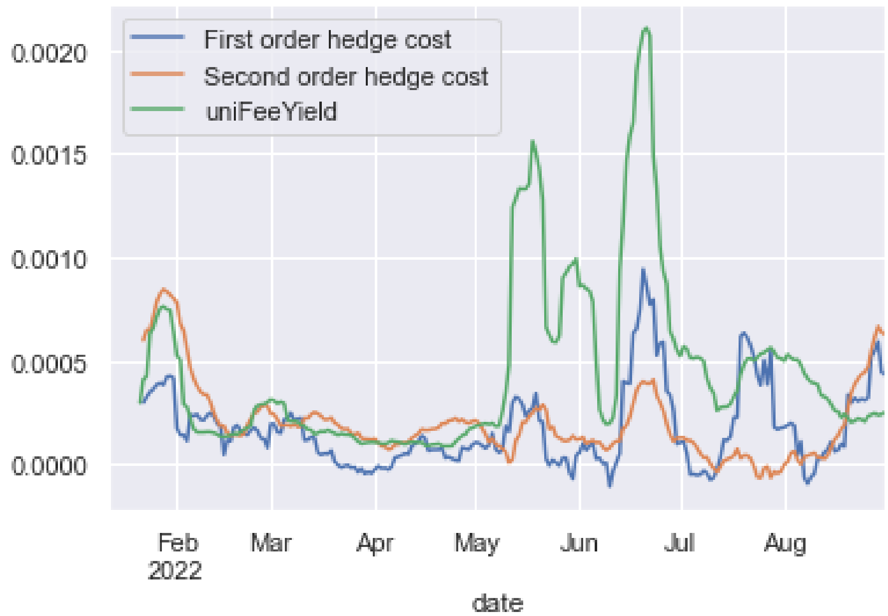


Figure 4: 10 day rolling hedge costs vs Uniswap fee yield

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