

Theoretical Computerscience – Summary

WS 24/25

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0 Words

A word w (also called String) has length l and consists of symbols $\sigma \in \Sigma$.

The empty word ε has length 0.

The concatenation of two words w and x is denoted as wx .

w^i denotes the i -fold concatenation of w with itself. Therefore $w^3 = www$.

$\varepsilon w = w\varepsilon = w$.

0.1 Subwords

A word w is a subword of a word x if there exist words u and v such that $x = uwv$.

The empty word is a subword of every word.

The word w is a subword of itself.

0.2 Prefixes and Suffixes

A word w is a prefix of a word x if there exists a word v such that $x = wv$.

A word w is a suffix of a word x if there exists a word u such that $x = uw$.

The empty word is a prefix and suffix of every word.

The word w is a prefix and suffix of itself.

1 Finite Automatons

TODO: Add chapter contents

2 Regular Languages

TODO: Add chapter contents

3 Regular Expressions

A regular expression always describes a regular language. If we can build a regular expression E , then $L(E) \in \text{REG}$.

3.1 Languages of regular expressions

If $E = \emptyset$ then $L(E) = \emptyset$

If $E = \varepsilon$ then $L(E) = \{\varepsilon\}$

If $E = \sigma$ then $L(E) = \{\sigma\}$

If $E = (E_1 + E_2)$ then $L(E) = L(E_1) \cup L(E_2)$

If $E = (E_1 E_2)$ then $L(E) = L(E_1)L(E_2)$

If $E = (E_1)^*$ then $L(E) = L(E_1)^*$

3.2 Identities

1. $E + F = F + E$
2. $(E + F) + G = E + (F + G)$
3. $(EF)G = E(FG)$
4. $\emptyset + E = E + \emptyset = E$
5. $\varepsilon E = E\varepsilon = E$
6. $\emptyset E = E\emptyset = \emptyset$
7. $E + E = E$
8. $(E + F)G = (EG) + (FG)$
9. $E(F + G) = (EF) + (EG)$
10. $(E^*)^* = E^*$
11. $\emptyset^* = \varepsilon$
12. $\varepsilon^* = \varepsilon$

3.3 Regular expressions and finite automata

If E is a regular expression, then $L(E) \in \text{REG}$.

For every deterministic finite automaton $M = (Q, \Sigma, \delta, q_0, Q_{acc})$ there is a regular expression E with $L(M) = L(E)$.

4 The Pumping Lemma

TODO: Add chapter contents

5 Equivalences and Myhill Nerode

5.1 Equivalence Relations

TODO: Add chapter contents

5.2 Myhill Nerode

TODO: Add chapter contents

6 Limits of computability

TODO: Add chapter contents

7 FOR and WHILE programs

7.1 WHILE programming language

Variables: x_0, x_1, \dots, x_n

Constants: 0, 1, 2, ...

Keywords: while, do, od

Other symbols: $:=, \neq, :, +, -, [,]$

W_0 : set of all simple statements

7.2 FOR programming language

Variables: x_0, x_1, \dots, x_n

Constants: 0, 1, 2, ...

Keywords: for, do, od

Other symbols: $:=, \neq, :, +, -, [,]$

7.3 Semantics

Input is stored in x_0, \dots, x_{s-1} Output is the content of x_0 after execution of P .

The set $X = \{x_0, x_1, x_2, \dots\}$ if all variables is finite.

7.4 FOR/WHILE computable functions

1. A partial function $f : \mathbb{N}^s \rightarrow \mathbb{N}$ is WHILE computable, if there is a WHILE program P such that $f = \phi_P$.
2. f is FOR computable if $f = \phi_P$ for some FOR program P .
3. The set of all WHILE computable functions is denoted by R .
4. The set of all FOR computable functions is denoted by PR .

7.5 Decidable languages

The characteristic function of L is the function $\chi_L : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$\chi_L(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{if } x \notin L \end{cases}$$

A language L is decidable if χ_L is computable ($\chi_L \in R$).

The set of all decidable languages is denoted by REC .

8 Syntactic Sugar

8.1 Syntactic Sugar

Program 1: Assignments $x_i := x_j$

```
1  $x_k := 0;$ 
2  $x_i := x_j + x_k;$ 
```

8.2 Arithmetic Operations

Program 2: Multiplication $x_i := x_j \cdot x_k$

```
1  $x_i := 0;$ 
2 for  $x_j$  do
3    $x_i := x_i + x_k$ 
4 od
```

We assume that $i \neq j, k$

8.3 Pairing Functions

$$\langle x_1, x_2 \rangle = \frac{1}{2} \cdot (x_1 + x_2) \cdot (x_1 + x_2 + 1) + x_2$$

$$\pi_i(\langle x_1, x_2 \rangle) = x_i \text{ for } i = 1, 2 \text{ and } x_1, x_2 \in \mathbb{N}$$

8.4 Stacks

$S = \langle n, Y \rangle$

$n \in \mathbb{N}$ is the number of elements in the stack and $Y \in \mathbb{N}^n$ is the stack itself

The empty stack is $\langle 0, 0 \rangle$

If a_1, \dots, a_n is stored in S , then $Y = \langle a_n, \langle a_{n-1}, \dots \langle a_1, 0 \rangle \dots \rangle \rangle$

Program 3: $push(S, x)$

<pre> 1 $Y := \langle x, \pi_2(S) \rangle;$ 2 $S := \langle \pi_1(S) + 1, Y \rangle;$ </pre>

Program 4: $pop(S)$

<pre> 1 $n := \pi_1(S);$ 2 if $n \neq 0$ then 3 $S := \langle n - 1, \pi_2(\pi_2(S)) \rangle;$ 4 fi </pre>

Program 5: $x = top(S)$

<pre> 1 $x := \pi_1(\pi_2(S));$ </pre>

Program 6: $b = isempty(S)$

<pre> 1 $n := \pi_1(S);$ 2 if $n \geq 0$ then 3 $b := 0;$ 4 else 5 $b := 1;$ 6 fi </pre>

8.5 Arrays

TODO: Add array implementation

9 Gödel numberings

Bijection $gd : W \rightarrow \mathbb{N}$

WHILE program U , that outputs $\phi_P(x)$ in input $i, x \in \mathbb{N}$ where $P = \text{göd}^{-1}(i)$.

9.1 Encoding

Simple Statements:

1. $x_i := x_j + x_k$ is encoded by $\langle 0, \langle i, \langle j, k \rangle \rangle \rangle_5$
2. $x_i := x_j - x_k$ is encoded by $\langle 1, \langle i, \langle j, k \rangle \rangle \rangle_5$
3. $x_i := c$ is encoded by $\langle 2, \langle i, c \rangle \rangle_5$

While loop and concatenation:

1. If $P = \text{while } x_i \neq 0 \text{ do } P_1 \text{ od}$ is encoded by $\langle 3, \langle i, \text{göd}(P_1) \rangle \rangle_5$
2. If $P = [P_1; P_2]$ then $\text{göd}(P) = \langle 4, \langle \text{göd}(P_1), \text{göd}(P_2) \rangle \rangle_5$

10 Diagonalisation

TODO: Add chapter contents

11 Universal WHILE Program

TODO: Add chapter contents

12 Halting Problem

TODO: Add chapter contents

13 Reductions

13.1 Known problems

13.1.1 Verification Problem V

The verification problem has two gödel number inputs namely i, j and checks if the two programs corresponding to these gödel numbers output the same for every possible input.

13.1.2 Special Verification Problem V_0

The special verification problem has a gödel number i as input and checks if the program corresponding to that gödel number will output 0 for every possible input.

13.1.3 Program termination T

The program termination problem has a gödel number i as input and checks if the program corresponding to that gödel number will terminate for every possible input.

TODO: Add formulas for V , V_0 and T

13.2 Many-one Reduction

A WHILE computable total function $f : \mathbb{N} \rightarrow \mathbb{N}$ is called many-one reduction from L to L' if

- $L, L' \subseteq \mathbb{N}$
- $\forall x \in \mathbb{N} : x \in L \iff f(x) \in L'$

If such an f exists, then L is many-one reducible to L' , therefore we can write $L \leq L'$

13.3 Known Reductions

$$H_0 \leq V_0$$

$$V_0 \leq V$$

$$V_0 \leq T$$

$$\overline{H_0} \leq V_0$$

14 More on Reductions

TODO: Add chapter contents

15 Rice's Theorem

TODO: Add chapter contents

16 Turing Machines

TODO: Add chapter contents

17 Examples, tricks and syntactic sugar for Turing Machines

TODO: Add chapter contents

18 Church-Turing Thesis

TODO: Add chapter contents

19 Common Proof Techniques

19.1 Pumping Lemma

The pumping lemma can only be used to show that a language is not REG. We take a word $w \in L$ and split it into multiple sub words u, v, w where $|v| \geq n$ with $n \geq 0$. Now there are words x, y, z with $v = xyz$ and $|y| \geq 0$ such that $uxy^i zw \in L$ for all $i \in \mathbb{N}$. Afterwards we try to find an i such that the pumped word is no longer $\in L$ and thus we proved that the language is not regular.

19.1.1 Example

Exercise:

Show that the following language is not regular.

Let $A = \{1^{(3n)} \mid n \in \mathbb{N}\}$

Solution:

Let $n \geq 0$ be given.

We choose $u = 1^n, v = 1^n, w = 1^n$ such that $uvw = 1^{3n}$ and $|v| = n$.

Let x, y, z be given as $x = 1^r, y = 1^s, z = 1^t$ with $xyz = v$ and $s \geq 0$ since $y \neq \epsilon$ and $r + s + t = n$.

$$uxy^i zw = 1^n 1^r 1^{s \cdot i} 1^t 1^n = 1^n 1^{r+s+t} 1^{s \cdot (i-1)} 1^n$$

We choose $i = 0$, therefore $1^n 1^{r+s+t} 1^{s \cdot (i-1)} 1^n = 1^n 1^{n-s} 1^n \notin A$ since it is not of the form 1^{3m} anymore for any $m \in \mathbb{N}$.

Therefore we cannot pump language A and thus it is not regular.

19.2 Myhill Nerode

19.2.1 Example

TODO: Add example for Myhill Nerode

19.3 Reductions

19.3.1 Example

Exercise:

Consider the following language $L = \{i \in \mathbb{N} \mid \text{göd}^{-1}(i) \text{ outputs 42 on input 1337}\}$

Prove $L \notin \text{co-RE}$

General Ideas:

\bar{L} is the set of all programs that either diverge or output something else than 42 on input 1337.

L is the set of all programs that output 42 on input 1337 which is equal to running a sub program that halts on every input and outputting 42 afterwards. Thus we could try to reduce the special halting problem H_0 to L .

Solution:

$$\forall L : L \in \text{REC} \iff L \in \text{RE} \wedge \bar{L} \in \text{RE}$$

$$L \notin \text{co-RE} \iff \bar{L} \in \text{RE}$$

$H_0 \notin \text{REC}$ but $H_0 \in \text{RE}$, therefore $\overline{H_0} \notin \text{RE}$ thus reducing $\overline{H_0} \leq \bar{L}$ suffices to show that $\bar{L} \notin \text{RE}$ but this is the same as $H_0 \leq L$ by equivalence.

The reduction is given by $f(i) := \text{göd}(P)$

We create the WHILE program P , that has input m . However we ignore the input m and simulate i on the input i , then return 42. Thus f is obviously WHILE computable.

We consider the following two cases:

- Let $i \in H_0$, then i halts on input i by definition. Program P halts on arbitrary inputs m and outputs 42. Therefore P also outputs 42 on our input 1337.
Thus $\text{göd}(P) \in L$ holds.
- Let $i \notin H_0$, then i does not halt on input i by definition. Therefore Program P diverges on all inputs, which is also the case for input 1337.
Therefore P diverges on our input 1337.
Thus $\text{göd}(P) \notin L$ holds.

Therefore we have successfully shown that $H_0 \leq L$ is a valid reduction.

20 Useful Proofs

20.1 Regular Languages

20.1.1 Finite Set

Exercise:

Show that the following language is regular over the alphabet $\{0,1\}$.

$$L = \{x \mid x \text{ is prime and } x < 1'000'000'000\}$$

Solution:

Since there are only finitely many prime numbers between 0 and $1'000'000'000$, the set of the words that are accepted by L is finite and thus the language is regular.

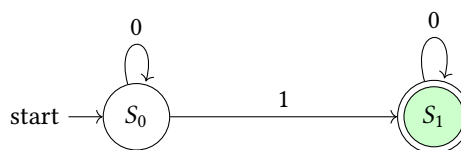
20.1.2 Finite Automaton

Exercise:

Show that the following language is regular over the alphabet $\{0,1\}$.

$$L = \{0^n 10^m \mid n, m \in \mathbb{N}\}$$

Solution:



Since we can describe the language L by the finite automaton given above, the language is regular.

20.1.3 Regular Expression

Exercise:

Show that the following language is regular over the alphabet $\{0,1\}$.

$$L = \{0^n 1^m \mid n, m \in \mathbb{N}\}$$

Solution:

Let $E = 0^*1^*$ be the regular expression describing the language L .

Since we can describe the language L by the regular expression given above, the language is regular.

20.1.4 Closure Properties

20.2 Non-Regular Languages

20.2.1 Pumping Lemma

Exercise:

Show that the following language is not regular over the alphabet $\{0,1\}$.

$$L = \{0^n 1^n \mid n \in \mathbb{N}\}$$

Solution:

Let $n \geq 0$ be given.

We choose $u = 1^n$, $v = 1^n$, $w = 1^n$ such that $uvw = 1^{3n}$ and $|v| = n$.

Let x,y,z be given as $x = 1^r$, $y = 1^s$, $z = 1^t$ with $xyz = v$ and $s \geq 0$ since $y \neq \varepsilon$ and $r + s + t = n$.

$$uxy^i z w = 1^n 1^r 1^{s \cdot i} 1^t 1^n = 1^n 1^{r+s+t} 1^{s \cdot (i-1)} 1^n$$

We choose $i = 0$, therefore $1^n 1^{r+s+t} 1^{s \cdot (i-1)} 1^n = 1^n 1^{n-s} 1^n \notin A$ since it is not of the form 1^{3m} anymore for any $m \in \mathbb{N}$.

Therefore we cannot pump language A and thus it is not regular.

20.2.2 Myhill Nerode

Exercise:

Show that the following language is not regular over the alphabet $\{0,1\}$.

$$L = \{0^n 1^n \mid n \in \mathbb{N}\}$$

Solution:

TODO: Add solution for Myhill Nerode

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