# Theoretical Computerscience – Summary

WS 24/25

# Contents

# 0 Words

A word $w$ (also called String) has length $l$ and consists of symbols $\sigma \in {}^{\circ}$.

The empty word $\varepsilon$ has length 0.

The concatenation of two words $w$ and $x$ is denoted as $wx$.

$w^i$ denotes the i-fold concatenation of $w$ with itself. Therefore $w^3 = www$.

$\varepsilon w = w\varepsilon = w$.

## 0.1 Subwords

A word $w$ is a subword of a word $x$ if there exist words $u$ and $v$ such that $x = uwv$.

The empty word is a subword of every word.

The word $w$ is a subword of itself.

## 0.2 Prefixes and Suffixes

A word $w$ is a prefix of a word $x$ if there exists a word $v$ such that $x = wv$.

A word $w$ is a suffix of a word $x$ if there exists a word $u$ such that $x = uw$.

The empty word is a prefix and suffix of every word.

The word $w$ is a prefix and suffix of itself.

**TODO: Add chapter contents**

# 1 Finite Automatons

**TODO: Add chapter contents**

# 2 Regular Languages

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# 3 Regular Expressions

A regular expression always describes a regular language. If we can build a regular expression E, then L(E) ∈ REG.

## 3.1 Languages of regular expressions

If $E = \emptyset$ then $L(E) = \emptyset$
If $E = \varepsilon$ then $L(E) = \{\varepsilon\}$
If $E = \sigma$ then $L(E) = \{\sigma\}$
If $E = (E_1 + E_2)$ then $L(E) = L(E_1) \cup L(E_2)$
If $E = (E_1 E_2)$ then $L(E) = L(E_1)L(E_2)$
If $E = (E_1)^*$ then $L(E) = L(E_1)^*$

## 3.2 Identities

1. $E + F = F + E$
2. $(E + F) + G = E + (F + G)$
3. $(EF)G = E(FG)$
4. $\emptyset + E = E + \emptyset = E$
5. $\varepsilon E = E\varepsilon = E$
6. $\emptyset E = E\emptyset = \emptyset$
7. $E + E = E$
8. $(E + F)G = (EG) + (FG)$
9. $E(F + G) = (EF) + (EG)$
10. $(E^*)^* = E^*$
11. $\emptyset^* = \varepsilon$
12. $\varepsilon^* = \varepsilon$

## 3.3 Regular expressions and finite automata

If $E$ is a regular expression, then $L(E) \in$ REG.
For every deterministic finite automaton $M = (Q, \sum, \delta, q_0, Q_{acc})$ there is a regular expression $E$ with $L(M) = L(E)$.

# 4 The Pumping Lemma

**TODO: Add chapter contents**

# 5 Equivalences and Myhill Nerode

## 5.1 Equivalence Relations

**TODO: Add chapter contents**

## 5.2 Myhill Nerode

**TODO: Add chapter contents**

# 6 Limits of computability

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# 7 FOR and WHILE programs

## 7.1 WHILE programming language

Variables: $x_0, x_1, \ldots, x_n$

Constants: $0, 1, 2, \ldots$

Keywords: while, do, od

Other symbols: :=,≠, ;, +, −, [,]

$W_0$: set of all simple statements

## 7.2 FOR programming language

Variables: $x_0, x_1, \ldots, x_n$

Constants: $0, 1, 2, \ldots$

Keywords: for, do, od

Other symbols: :=,≠, ;, +, −, [,]

## 7.3 Semantics

Input is stored in $x_0, \ldots, x_{s-1}$ Output is the content of $x_0$ after execution of $P$.
The set $X = \{x_0, x_1, x_2, \ldots\}$ if akk variables is finite.

## 7.4 FOR/WHILE computable functions

1. A partial function $f : \mathbb{N}^s \to \mathbb{N}$ is WHILE computable, if there is a WHILE program P such that $f = \phi_P$.
2. f is FOR computable if $f = \phi_P$ for some FOR program P.
3. The set of all WHILE computable functions is denoted by R.
4. The set of all FOR computable functions is denoted by PR.

## 7.5 Decidable languages

The characteristic function of L is the function $\chi_L : \mathbb{N} \to \mathbb{N}$ defined by

$$\chi_L(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{if } x \notin L \end{cases}$$

A language L is decidable if $\chi_L$ is computable ($\chi_L \in R$).
The set of all decidable languages is denoted by REC.

# 8 Syntactic Sugar

**TODO: Add chapter contents**

# 9 Gödel numberings

**TODO: Add chapter contents**

# 10 Diagonalisation

**TODO: Add chapter contents**

# 11 Universal WHILE Program

**TODO: Add chapter contents**

# 12 Halting Problem

**TODO: Add chapter contents**

# 13 Reductions

## 13.1 Known problems

### 13.1.1 Verification Problem $V$

The verification problem has two gödel number inputs namely $i, j$ and checks if the two programs corresponding to these gödel numbers output the same for every possible input.

### 13.1.2 Special Verification Problem $V_0$

The special verification problem has a gödel number $i$ as input and checks if the program corresponding to that gödel number will output 0 for every possible input.

### 13.1.3 Program termination $T$

The program termination problem has a gödel number $i$ as input and checks if the program corresponding to that gödel number will terminate for every possible input.

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## 13.2 Many-one Reduction

A WHILE computable total function $f : \mathbb{N} \to \mathbb{N}$ is called many-one reduction from $L$ to $L'$ if

- $L, L' \subseteq \mathbb{N}$
- $\forall x \in \mathbb{N} : x \in L \iff f(x) \in L'$

If such an $f$ exists, then $L$ is many-one reducible to $L'$, therefore we can write $L \leq L'$

## 13.3 Known Reductions

$H_0 \leq V_0$
$V_0 \leq V$
$V_0 \leq T$
$\overline{H_0} \leq V_0$

# 14 More on Reductions

**TODO: Add chapter contents**

# 15 Rice's Theorem

**TODO: Add chapter contents**

# 16 Turing Machines

**TODO: Add chapter contents**

# 17 Examples, tricks and syntactic sugar for Turing Machines

**TODO: Add chapter contents**

# 18 Church-Turing Thesis

**TODO: Add chapter contents**

# 19 Common Proof Techniques

## 19.1 Pumping Lemma

The pumping lemma can only be used to show that a language is not REG. We take a word $w \in L$ and split it into multiple sub words $u$,$v$,$w$ where $|v| \geq n$ with $n \geq 0$. Now there are words $x$,$y$,$z$ with $v = xyz$ and $|y| \geq 0$ such that $uxy^izw \in L$ for all $i \in \mathbb{N}$. Afterwards we try to find an $i$ such that the pumped word is no longer $\in L$ and thus we proved that the language is not regular.

### 19.1.1 Example

**Exercise**:
Show that the following language is not regular.
Let $A = \{1^{(3n)} \mid n \in \mathbb{N}\}$

**Solution**:
Let $n \geq 0$ be given.
We choose $u = 1^n$, $v = 1^n$, $w = 1^n$ such that $uvw = 1^{3n}$ and $|v| = n$.
Let $x$,$y$,$z$ be given as $x = 1^r$, $y = 1^s$, $z = 1^t$ with $xyz = v$ and $s \geq 0$ since $y \neq \varepsilon$ and $r + s + t = n$.
$uxy^izw = 1^n 1^r 1^{s \cdot i} 1^t 1^n = 1^n 1^{r+s+t} 1^{s \cdot (i-1)} 1^n$
We choose $i = 0$, therefore $1^n 1^{r+s+t} 1^{s \cdot (i-1)} 1^n = 1^n 1^{n-s} 1^n \notin A$ since it is not of the form $1^{3m}$ anymore for any $m \in \mathbb{N}$.
Therefore we cannot pump language $A$ and thus it is not regular.

## 19.2 Myhill Nerode

### 19.2.1 Example

**TODO: Add example for Myhill Nerode**

## 19.3 Reductions

### 19.3.1 Example

**Exercise**:
Consider the following language $L = \{i \in \mathbb{N} \mid \text{göd}^{-1}(i) \text{ outputs 42 on input 1337}\}$
Prove $L \notin \text{co}-\mathsf{RE}$

**General Ideas**:
$\overline{L}$ is the set of all programs that either diverge or output something else than 42 on input 1337.
$L$ is the set of all programs that output 42 on input 1337 which is equal to running a sub program that halts on every input and outputting 42 afterwards. Thus we could try to reduce the special halting problem $H_0$ to $L$.

**Solution**:
$\forall L : L \in \mathsf{REC} \iff L \in \mathsf{RE} \land \overline{L} \in \mathsf{RE}$
$L \notin \text{co}-\mathsf{RE} \iff \overline{L} \in \mathsf{RE}$

$H_0 \notin \mathsf{REC}$ but $H_0 \in \mathsf{RE}$, therefore $\overline{H_0} \notin \mathsf{RE}$ thus reducing $\overline{H_0} \leq \overline{L}$ suffices to show that $\overline{L} \notin \mathsf{RE}$ but this is the same as $H_0 \leq L$ by equivalence.

The reduction is given by $f(i) := \text{göd}(P)$
We create the WHILE program $P$, that has input $m$. However we ignore the input $m$ and simulate $i$ on the input $i$, then return 42. Thus $f$ is obviously WHILE computable.

We consider the following two cases:

- Let $i \in H_0$, then $i$ halts on input $i$ by definition. Program $P$ halts on arbitrary inputs m and outputs 42. Therefore $P$ also outputs 42 on our input 1337.
  Thus $\text{göd}(P) \in L$ holds.
- Let $i \notin H_0$, then $i$ does not halt on input $i$ by definition. Therefore Program $P$ diverges on all inputs, which is also the case for input 1337.
  Therefore $P$ diverges on our input 1337.
  Thus $\text{göd}(P) \notin L$ holds.

Therefore we have successfully shown that $H_0 \leq L$ is a valid reduction.

# 20 Useful Proofs

## 20.1 Regular Languages

### 20.1.1 Finite Set

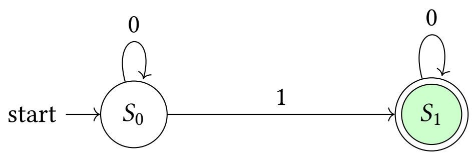
**Exercise**:
Show that the following language is regular over the alphabet $\{0,1\}$.
$L = \{x \mid x \text{ is prime and } x < 1'000'000'000\}$

**Solution**:
Since there are only finitely many prime numbers between 0 and 1′000′000′000, the set of the words that are accepted by $L$ is finite and thus the language is regular.

### 20.1.2 Finite Automaton

**Exercise**:
Show that the following language is regular over the alphabet $\{0,1\}$.
$L = \{0^n10^m \mid n,m \in \mathbb{N}\}$

**Solution**:




Since we can describe the language $L$ by the finite automaton given above, the language is regular.

### 20.1.3 Regular Expression

**Exercise**:
Show that the following language is regular over the alphabet $\{0,1\}$.
$L = \{0^n1^m \mid n,m \in \mathbb{N}\}$

**Solution**:
Let $E = 0^*1^*$ be the regular expression describing the language $L$.
Since we can describe the language $L$ by the regular expression given above, the language is regular.

### 20.1.4 Closure Properties

## 20.2 Non-Regular Languages

### 20.2.1 Pumping Lemma

**Exercise**:
Show that the following language is not regular over the alphabet $\{0,1\}$.
$L = \{0^n 1^n \mid n \in \mathbb{N}\}$

**Solution**:
Let $n \geq 0$ be given.
We choose $u = 1^n$, $v = 1^n$, $w = 1^n$ such that $uvw = 1^{3n}$ and $|v| = n$.
Let $x$,$y$,$z$ be given as $x = 1^r$, $y = 1^s$, $z = 1^t$ with $xyz = v$ and $s \geq 0$ since $y \neq \varepsilon$ and $r + s + t = n$.
$uxy^izw = 1^n 1^r 1^{s \cdot i} 1^t 1^n = 1^n 1^{r+s+t} 1^{s \cdot (i-1)} 1^n$
We choose $i = 0$, therefore $1^n 1^{r+s+t} 1^{s \cdot (i-1)} 1^n = 1^n 1^{n-s} 1^n \notin A$ since it is not of the form $1^{3m}$ anymore for any $m \in \mathbb{N}$.
Therefore we cannot pump language $A$ and thus it is not regular.

### 20.2.2 Myhill Nerode

**Exercise**:
Show that the following language is not regular over the alphabet $\{0,1\}$.
$L = \{0^n 1^n \mid n \in \mathbb{N}\}$

**Solution**:
**TODO: Add solution for Myhill Nerode**