Theoretical Computerscience – Summary

WS 24/25

Contents

0 Words

A word w (also called String) has length l and consists of symbols $\sigma \in {}^{\circ}$.

The empty word ε has length 0.

The concatenation of two words w and x is denoted as wx.

w^i denotes the i-fold concatenation of w with itself. Therefore $w^3 = www$.

$\varepsilon w = w\varepsilon = w$.

0.1 Subwords

A word w is a subword of a word x if there exist words u and v such that $x = uwv$.

The empty word is a subword of every word.

The word w is a subword of itself.

0.2 Prefixes and Suffixes

A word w is a prefix of a word x if there exists a word v such that $x = wv$.

A word w is a suffix of a word x if there exists a word u such that $x = uw$.

The empty word is a prefix and suffix of every word.

The word w is a prefix and suffix of itself.

TODO: Add chapter contents

1 Finite Automatons

TODO: Add chapter contents

2 Regular Languages

TODO: Add chapter contents

3 Regular Expressions

A regular expression always describes a regular language. If we can build a regular expression E, then L(E) ∈ REG.

TODO: Add chapter contents

4 The Pumping Lemma

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5 Equivalence Relations

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6 Myhill Nerode

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7 Limits of computability

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8 FOR and WHILE programs

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10 Gödel numberings

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11 Diagonalisation

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12 Universal WHILE Program

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13 Halting Problem

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14 Reductions

14.1 Known problems

14.1.1 Verification Problem V

The verification problem has two gödel number inputs namely i, j and checks if the two programs corresponding to these gödel numbers output the same for every possible input.

14.1.2 Special Verification Problem V_0

The special verification problem has a gödel number i as input and checks if the program corresponding to that gödel number will output 0 for every possible input.

14.1.3 Program termination T

The program termination problem has a gödel number i as input and checks if the program corresponding to that gödel number will terminate for every possible input.

TODO: Add chapter contents

14.2 Many-one Reduction

A WHILE computable total function $f : \mathbb{N} \to \mathbb{N}$ is called many-one reduction from L to L' if

- $L, L' \subseteq \mathbb{N}$
- $\forall x \in \mathbb{N} : x \in L \iff f(x) \in L'$

If such an f exists, then L is many-one reducible to L', therefore we can write $L \leq L'$

14.3 Known Reductions

$H_0 \leq V_0$
$V_0 \leq V$
$V_0 \leq T$
$\overline{H_0} \leq V_0$

15 Rice’s Theorem

TODO: Add chapter contents

16 Turing Machines

TODO: Add chapter contents

17 Examples, tricks and syntactic sugar for Turing Machines

TODO: Add chapter contents

18 Church-Turing Thesis

TODO: Add chapter contents

19 Common Proof Techniques

19.1 Pumping Lemma

The pumping lemma can only be used to show that a language is not REG. We take a word $w \in L$ and split it into multiple sub words u,v,w where $|v| \geq n$ with $n \geq 0$. Now there are words x,y,z with $v = xyz$ and $|y| \geq 0$ such that $uxy^izw \in L$ for all $i \in \mathbb{N}$. Afterwards we try to find an i such that the pumped word is no longer $\in L$ and thus we proved that the language is not regular.

19.1.1 Example

Exercise:
Show that the following language is not regular.
Let $A = \{1^{(3n)} \mid n \in \mathbb{N}\}$

Solution:
Let $n \geq 0$ be given.
We choose $u = 1^n$, $v = 1^n$, $w = 1^n$ such that $uvw = 1^{3n}$ and $|v| = n$.
Let x,y,z be given as $x = 1^r$, $y = 1^s$, $z = 1^t$ with $xyz = v$ and $s \geq 0$ since $y \neq \varepsilon$ and $r + s + t = n$.
$uxy^izw = 1^n1^r1^{s \cdot i}1^t1^n = 1^n1^{r+s+t}1^{s \cdot (i-1)}1^n$
We choose $i = 0$, therefore $1^n1^{r+s+t}1^{s \cdot (i-1)}1^n = 1^n1^{n-s}1^n \notin A$ since it is not of the form 1^{3m} anymore for any $m \in \mathbb{N}$.
Therefore we cannot pump language A and thus it is not regular.

19.2 Myhill Nerode

19.2.1 Example

TODO: Add example for Myhill Nerode

19.3 Reductions

19.3.1 Example

Exercise:
Consider the following language $L = \{i \in \mathbb{N} \mid \text{göd}^{-1}(i) \text{ outputs 42 on input 1337}\}$
Prove $L \notin \text{co}-\mathsf{RE}$

General Ideas:
$\overline{L}$ is the set of all programs that either diverge or output something else than 42 on input 1337.
L is the set of all programs that output 42 on input 1337 which is equal to running a sub program that halts on every input and outputting 42 afterwards. Thus we could try to reduce the special halting problem H_0 to L.

Solution:
$\forall L : L \in \mathsf{REC} \iff L \in \mathsf{RE} \land \overline{L} \in \mathsf{RE}$
$L \notin \text{co}-\mathsf{RE} \iff \overline{L} \in \mathsf{RE}$

$H_0 \notin \mathsf{REC}$ but $H_0 \in \mathsf{RE}$, therefore $\overline{H_0} \notin \mathsf{RE}$ thus reducing $\overline{H_0} \leq \overline{L}$ suffices to show that $\overline{L} \notin \mathsf{RE}$ but this is the same as $H_0 \leq L$ by equivalence.

The reduction is given by $f(i) := \text{göd}(P)$
We create the WHILE program P, that has input m. However we ignore the input m and simulate i on the input i, then return 42. Thus f is obviously WHILE computable.

We consider the following two cases:

- Let $i \in H_0$, then i halts on input i by definition. Program P halts on arbitrary inputs m and outputs 42. Therefore P also outputs 42 on our input 1337.
  Thus $\text{göd}(P) \in L$ holds.
- Let $i \notin H_0$, then i does not halt on input i by definition. Therefore Program P diverges on all inputs, which is also the case for input 1337.
  Therefore P diverges on our input 1337.
  Thus $\text{göd}(P) \notin L$ holds.

Therefore we have successfully shown that $H_0 \leq L$ is a valid reduction.

20 Useful Proofs

20.1 Regular Languages

20.1.1 Finite Set

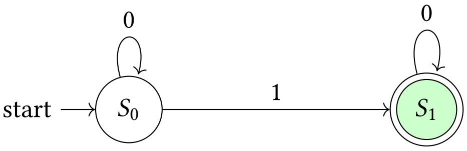
Exercise:
Show that the following language is regular over the alphabet $\{0,1\}$.
$L = \{x \mid x \text{ is prime and } x < 1'000'000'000\}$

Solution:
Since there are only finitely many prime numbers between 0 and $1'000'000'000$, the set of the words that are accepted by L is finite and thus the language is regular.

20.1.2 Finite Automaton

Exercise:
Show that the following language is regular over the alphabet $\{0,1\}$.
$L = \{0^n 1 0^m \mid n,m \in \mathbb{N}\}$

Solution:



Since we can describe the language L by the finite automaton given above, the language is regular.

20.1.3 Regular Expression

Exercise:
Show that the following language is regular over the alphabet $\{0,1\}$.
$L = \{0^n 1^m \mid n,m \in \mathbb{N}\}$

Solution:
Let $E = 0^* 1^*$ be the regular expression describing the language L.
Since we can describe the language L by the regular expression given above, the language is regular.

20.1.4 Closure Properties

20.2 Non-Regular Languages

20.2.1 Pumping Lemma

Exercise:
Show that the following language is not regular over the alphabet {0,1}.
$L = \{0^n1^n \mid n \in \mathbb{N}\}$

Solution:
Let $n \geq 0$ be given.
We choose $u = 1^n$, $v = 1^n$, $w = 1^n$ such that $uvw = 1^{3n}$ and $|v| = n$.
Let x,y,z be given as $x = 1^r$, $y = 1^s$, $z = 1^t$ with $xyz = v$ and $s \geq 0$ since $y \neq \varepsilon$ and $r + s + t = n$.
$uxy^izw = 1^n1^r1^{s \cdot i}1^t1^n = 1^n1^{r+s+t}1^{s \cdot (i-1)}1^n$
We choose $i = 0$, therefore $1^n1^{r+s+t}1^{s \cdot (i-1)}1^n = 1^n1^{n-s}1^n \notin A$ since it is not of the form 1^{3m} anymore for any $m \in \mathbb{N}$.
Therefore we cannot pump language A and thus it is not regular.

20.2.2 Myhill Nerode

Exercise:
Show that the following language is not regular over the alphabet {0,1}.
$L = \{0^n1^n \mid n \in \mathbb{N}\}$

Solution:
TODO: Add solution for Myhill Nerode