CM2208: Scientific Computing
1. Complex Numbers

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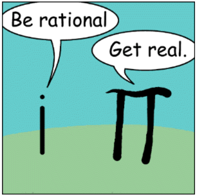


A problem when solving some equations

There are some equations, for example $x^2 + 1 = 0$, for which we cannot yet **find solutions**.

$$
\begin{aligned}
x^2 + 1 &= 0 \\
x^2 &= -1 \\
x &= \pm\sqrt{-1}?
\end{aligned}
$$

The Problem: We cannot (**yet**) find the **square root** of a **negative number** using real numbers since:

- When any real number is **squared** the result is either **positive** or **zero**, *i.e.* for all real numbers $n^2 \geq 0, n \in \mathbb{R}$[1].

[1] we use the symbol $\mathbb{R}$ to denote the **set** of all real numbers

Imaginary Numbers

We need **another category** of numbers, the **set of numbers** whose **squares** are **negative real numbers**.

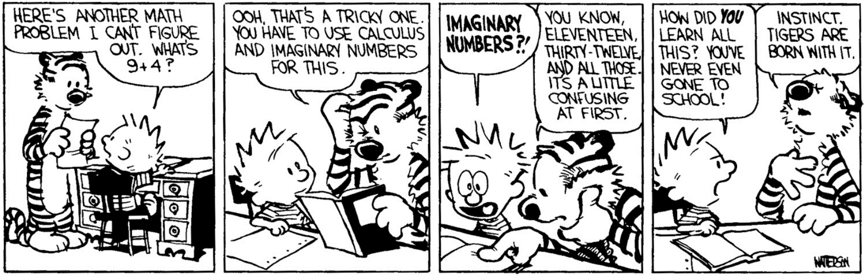
Members of this set are called **imaginary numbers**.

We define $\sqrt{-1} = i$ (or j in some texts)[2]

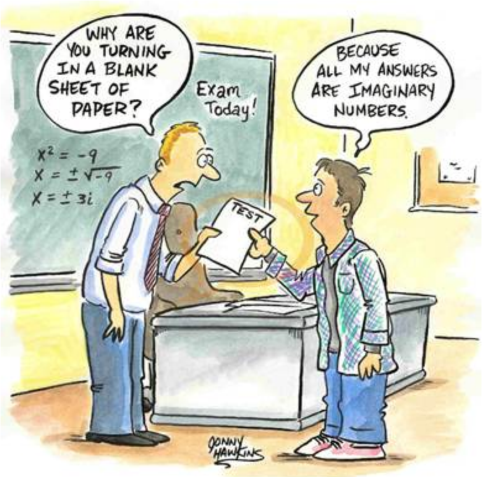
Every imaginary number can be written in the form: **ni** where *n* is **real** and $\mathbf{i} = \sqrt{-1}$

[2]If you read engineering books rather than maths books you may see j used in place of i - this is just a quirk in notation

Imaginary Numbers



Imaginary Numbers



Examples: Imaginary Numbers

Examples:

- $\sqrt{-16} = \sqrt{16 \times -1} = \sqrt{16} \times \sqrt{-1} = \pm 4\mathbf{i}$
- $\sqrt{-3} = \sqrt{3 \times -1} = \sqrt{3} \times \sqrt{-1} = \pm\mathbf{i}\sqrt{3}$
- $(-121)^{\frac{1}{2}} = \sqrt{-121} = \sqrt{123 \times -1} = \sqrt{121} \times \sqrt{-1} = \pm 11\mathbf{i}$

Imaginary Number Arithmetic: Addition

Imaginary numbers can be **added** to or **subtracted** **only** from **other imaginary numbers.**

Examples:

- $7i - 2i = 5i$
- $4i + \sqrt{3}i = (4 + \sqrt{3})i$

(Note: i behaves like a special algebraic variable)

Imaginary Number Arithmetic: Multiplication

When **imaginary numbers** are **multiplied** together the result is a **real number**.

Example:

$$2i \times 5i = 10 \times i^2$$

but we know $i = \sqrt{-1}$, and therefore $i^2 = -1$

Hence $10 \times i^2 = 10 \times -1 = \mathbf{-10}$

Imaginary Number Arithmetic: Division

Imaginary numbers when **divided** give a real number result.

- Example:

$$\frac{6i}{3i} = 2$$

Powers of i may be **simplified**

Examples:

- $i^3 = i^2 \times i = --1 \times i = -i$
- $i^{-1} = \frac{1}{i} = \frac{1}{\sqrt{-1}} = \frac{1}{\sqrt{-1}} \times \frac{\sqrt{-1}}{\sqrt{-1}} = \frac{\sqrt{-1}}{-1} = -\sqrt{-1} = -i$

Complex Numbers

Case 1: The need for Complex Numbers

Consider the quadratic equation $x^2 + 2x + 2 = 0$.

Using the quadratic formula we get:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

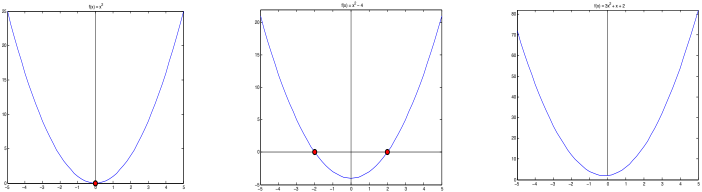
So $x = -1 + i$ or $-1 - i$

- x is now a number with a **real number** part (1) and an **imaginary number** part ($\pm i$).

x is an example of a **complex number**.

Recall: If $b^2 - 4ac < 0$ then the equation has **complex roots**.

CM2104 RECAP: The 3 possible discriminant cases visualised



(i) Two equal roots (ii) Two distinct roots (iii) No real roots

The above diagrams of $ax^2 + bx + c$ $(a \neq 0)$ show the three possible cases:

- In (i) the curve touches the x-axis, *i.e.* for $y = 0$ there are **two equal values** of x — $b^2 - 4ac = 0$.
- In (ii) the curve cuts the x-axis, *i.e.* for $y = 0$ there are **two real distinct values** of x, **i.e. real roots** — $b^2 - 4ac > 0$.
- In (iii) the does not cut the x-axis, *i.e.* for $y = 0$ there is **no real values** of x, **i.e. complex roots** — $b^2 - 4ac < 0$.

Complex Numbers

Case 2: The need for Complex Numbers

Very Useful Mathematical Representation, to name a few:

- Widely used in many branches of Mathematics, Engineering, Physics and other scientific disciplines
  - Control theory
  - Advanced calculus: Improper integrals, Differential equations, Dynamic equations
  - Fluid dynamics — potential flow, flow fields
  - Electromagnetism and electrical engineering: Alternating current, phase induced in systems
  - Quantum mechanics
  - Relativity
  - Geometry: Fractals (*e.g.* the Mandelbrot set and Julia sets), Triangles — Steiner inellipse
  - Algebraic number theory
  - Analytic number theory
- **Signal analysis**: Essential for digital signal and image processing (**Phasors**) — **studied later**.

Definition: Complex Numbers

A **complex number** is a number of the form $z = a + bi$

- that is a **number** which has a **real** and an **imaginary** part.
- a and b can have any **real** value including 0. $(a, b \in \mathbb{R})$
- *E.g.* $3 + 2i$, $6 - 3i$, $-2 + 4i$.
- Note: the **real term** is always written **first**, even where **negative**.

Note: This means that

- when $a = 0$ we have numbers of the form bi *i.e.* only **imaginary numbers**
- when $b = 0$ we have numbers of the form a i.e. real numbers.

The **set of all complex numbers** is denoted by $\mathbb{C}$.

Real and Imaginary Parts, Notation

Mathematical Notation:

- The **set of all real numbers** is denoted by $\mathbb{R}$
- The **set of all complex numbers** is denoted by $\mathbb{C}$
- The **real part** of a complex number z is denoted by $\mathrm{Re}(z)$ or $\Re(z)$
- The **imaginary part** of a complex number z is denoted by $\mathrm{Im}(z)$ or $\Im(z)$

Example: Real and Imaginary Parts

Find the **real** and **imaginary** parts of:

- $z = 1 + 7i$ — **real part** $\Re(z) = \mathbf{1}$, **imaginary part** $\Im(z) = \mathbf{7}$
- $z = 2 - 4i$ — **real part** $\Re(z) = \mathbf{2}$, **imaginary part** $\Im(z) = \mathbf{-4}$
- $z = -3$ — **real part** $\Re(z) = \mathbf{-3}$, **imaginary part** $\Im(z) = \mathbf{0}$
- $z = i\sqrt{3}$ — **real part** $\Re(z) = \mathbf{0}$, **imaginary part** $\Im(z) = \sqrt{\mathbf{3}}$

Addition and Subtraction of Complex Numbers

Complex Numbers can be added (or subtracted) by adding (or subtracting) their real and imaginary parts separately.

Examples:

- $(2+3i)+(4-i)=6+2i$
- $(4-2i)-(3+5i)=1-7i$

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Multiplication of Complex Numbers

Complex Number **Multiplication**:

- Follows the basic laws of **polynomial multiplication** and **imaginary number multiplication** (recall $i^2 = -1$)
- Then **gather** real and imaginary terms to **simplify** the expression.

Examples:

- $2(5 - 3i) = 10 - 6i$
- $(2 + 3i)(4 - i) = 8 - 2i + 12i - 3i^2 = 8 + 10i - 3(-1) = 8 + 10i + 3 = 11 + 10i$
- $(-3-5i)(2+3i) = -6-9i-10i-15i^2 = -6-19i+15 = 9-19i$
- $(2 + 3i)(2 - 3i) = 4 - 6i + 6i - 9i^2 = 4 + 9 = 13$

Note that in the last example the product of the **two** complex numbers is a **real number**.

The Complex Conjugate

In general $(a+bi)(a-bi) = a^2 + b^2$

- A pair of complex numbers of this form are said to be **conjugate**.

Examples:

- $4+5i$ and $4-5i$ are **conjugate** **complex numbers**.
- $7-3i$ is the **conjugate** of $7+3i$

If z is a complex number ($z \in \mathbb{C}$) the **notation** for its **conjugate** is $\overline{z}$ orz^*.

Example:

- $z = 7-3i$ then $\overline{z} = 7+3i$

Division of Complex Numbers

Problem: How to evaluate/simplify:

$$z = \frac{a + bi}{\mathbf{c + di}}, a, b, c, d \in \mathbb{R}$$

Can we **express** z in the **normal** complex number form:
$\mathbf{z = e + fi}, e, f \in \mathbb{R}$?

Direct division by a complex number **cannot** be carried out:

- The **denominator** is made up of two **independent** terms
  - The **real** and **imaginary** part of the complex number $\mathbf{c + di}$
- We have to follow the basic laws of algebraic division.

The **complex conjugate** comes to the **rescue**.

Complex Number Division: Realising the Denominator

Problem: Express z (below) in the form $z = e + fi$, $a, b \in \mathbb{R}$:

$$z = \frac{a + bi}{\mathbf{c + di}}, \quad a, b, c, d \in \mathbb{R}$$

- We need to deal with the denominator, z_d. Here $z_d = c + di$.
- We can readily obtain the **complex conjugate** of z_d, $\overline{z_d} = c - di$
- We have already observed that **any complex number** $\times$ **its conjugate** is a **real number**, $z_d \times \overline{z_d} \in \mathbb{R}$: $c^2 + d^2$
- So to **remove** i from the **denominator** we can multiply **both** numerator and denominator by $\overline{z_d}$

This process is known as **realising the denominator**.

Example: Division of Complex Numbers

Express z (below) in the form $z = a + bi, a, b \in \mathbb{R}$:

$$z = \frac{2 + 9i}{5 - 2i}$$

- We need to deal with the denominator, z_d. Here $z_d = 5 - 2i$.
- Obtain **complex conjugate** of z_d, $\overline{z_d} = 5 + 2i$
- Multiply **both** numerator and denominator by $\overline{z_d}$

$$\begin{aligned} \frac{2+9i}{5-2i} \times \frac{5+2i}{5+2i} &= \frac{10 + 4i + 45i + 18i^2}{25 - 4i^2} \\ &= \frac{-8 + 49i}{29} \\ &= \mathbf{\frac{-8}{29} + \frac{49}{29}i} \end{aligned}$$

Comparing Complex Numbers: Equality

Two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$, are **equal** if and **only** if

- the **real parts** of each are **equal**

AND

- the **imaginary parts** are **equal**.

That is to say:

- $\Re(z_1) = \Re(z_2)$ or $a = c$,

AND

- $\Im(z_1) = \Im(z_2)$ or $b = d$

Example: Comparing Complex Numbers

Example:
If $x + iy = (3 - 2i)(5 + i)$ what are the **values** of x and y?

$$\begin{aligned} x + iy &= (3 - 2i)(5 + i) \\ &= 15 + 3i - 10i + 2i^2 \\ &= 13 - 7i \end{aligned}$$

So $x = 13$ and $y = -7$.

Corollary: The complex number zero

A **complex number** is **zero if and only if** the **real part** and the **imaginary part** are both **zero** *i.e.*

$$\mathbf{a} + \mathbf{bi} = \mathbf{0} \leftrightarrow \mathbf{a} = \mathbf{0} \text{ and } \mathbf{b} = \mathbf{0}.$$

Visualising Complex Numbers: The Complex Plane

A **complex number**, $z = a + ib$, is made up of two parts,

- The **real part**, a and,
- The **imaginary part**, b

One way we may visualise this is by plotting these on a 2D graph:

- The **x-axis** represents the **real** numbers, and
- The **y-axis** represents the **imaginary** numbers.

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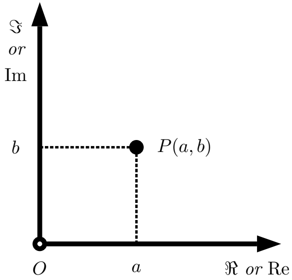
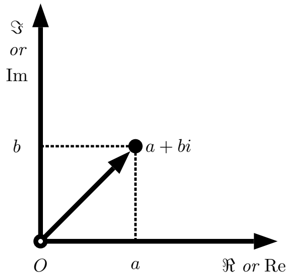
Visualising Complex Numbers: Argand Diagrams

The complex number $z = a + ib$ may then be represented in the **complex plane** by

- the point **P** whose co-ordinates are (a, b)
  or,
- the vector ***OP***, where **O** is the **point** at the **origin**, $(0, 0)$





This representation is known as the **Argand diagram**.

Exercise: Complex Numbers and Argand Diagram

Given $z_1 = 3 - 2i$ and $z_2 = 5 + 2i$ draw an Argand diagram for:

- z_1
- z_2
- $z_1 + z_2$
- $z_1 - z_2$

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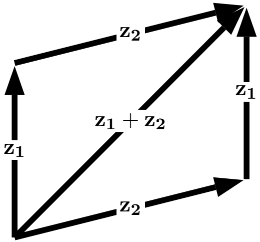
Visualising Complex Numbers: Adding Complex Numbers

Generally, given$z_1 = a_1 + b_1 i$ and $z_2 == a_2 + b_2 i$ then:
$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$



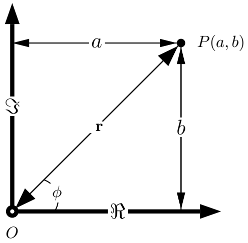
If we **plot two complex numbers** on an **Argand diagram** then we **see**

- that they form **two adjacent** sides of a **parallelogram**
- their **sum** forms the **diagonal**.
- **Basic Laws of Vector Algebra**

Visualising Complex Numbers: Polar Form

Polar Coordinates: An **alternative system** of **coordinates** in which the position of any **point** P can be **described** in terms of

- The **distance**, r, of P from the origin, O, and
- The **angle**/**direction**, ϕ, that the line ***OP*** makes with the **positive** real $\Re$-axis (or, more generally x-axis)



This is the **polar form of complex numbers**

The Polar Form of Complex Numbers

In relation to complex numbers, we call the polar coordinate terms:

- The **modulus**, r,

$$r = |z| = \sqrt{a^2 + b^2}$$

(Note this is a simple application of **Pythagoras' theorem**.)

- The **argument** or **phase**, ϕ,

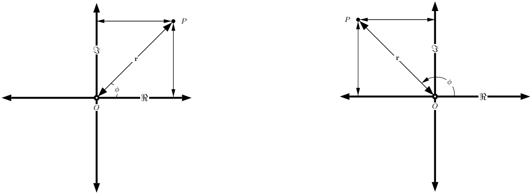
$$\text{(Simply)} \quad \phi = \arg z = \text{argument } z = \arctan\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{b}{a}\right)$$

(Note: This is a simple application of **basic trigonometry**)

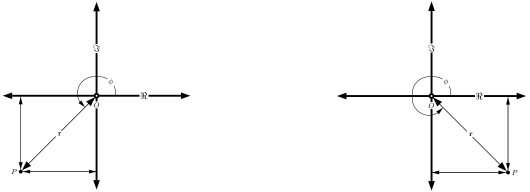
to make up what is known as the polar coordinates of a point.

The Polar Form: More on the Argument

We can measure the Argument is two ways: Both depend on which **quadrant** of complex plane the point resides in:



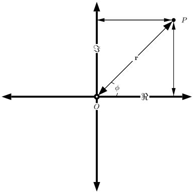
Quadrant 1 **Quadrant 2**



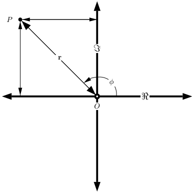
Quadrant 3 **Quadrant 4**

The Polar Form: More on the Argument

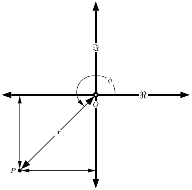
- $\phi \in [0, 2\pi)$ — All angles, ϕ, were measured anticlockwise from the +ive real axis: therefore ϕ must be in the range 0 to 2π



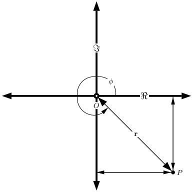
Quadrant 1



Quadrant 2



Quadrant 3



Quadrant 4

All angles given in **radians here**

The Polar Form: Argument Alternative Angle Measurement

Alternatively:

- $\phi \in (-\pi, \pi]$ — **(not illustrated)** measure smallest spanned angle from +ive real axis: ϕ measured in range $-\pi$ to π.

$$\phi = \arg z = \begin{cases} \arctan(\frac{b}{a}) & \text{if } x > 0 \\ \arctan(\frac{b}{a}) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(\frac{b}{a}) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \textbf{indeterminate} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

The polar angle for the complex number **0** is **undefined**, but usual arbitrary choice is the **angle 0**.

Examples: Modulus and Argument

Find the modulus and argument of each of the following:

- $1 + i$
  - **Modulus** $r = |1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

    Sketching the **Argand diagram** indicates that we are in the **first quadrant**, therefore positive angle, ϕ between 0 and 90.
    Argument $= \arctan(\frac{1}{1}) = 45°$or $\frac{\pi}{4}$radians

- $\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$
  - **Modulus**
    $r = |\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}| = \sqrt{(\frac{1}{\sqrt{2}})^2 + (-\frac{1}{\sqrt{2}})^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$
    Sketching the Argand diagram indicates that we are in the **fourth quadrant**, therefore angle is negative between 0 and 90 (or between 270 and 360) degrees.

    Argument $= \arctan(\frac{(-\frac{1}{\sqrt{2}})}{\frac{1}{\sqrt{2}}}) = \arctan(-1) = -45°$or $315°$

    (Radians sim.)

Exercise: Modulus and Argument

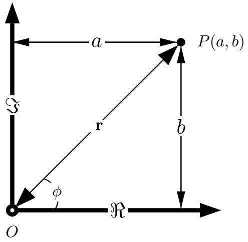
Find the modulus and argument of each of the following:

- $-1.35 + 2.56i$
- $\frac{1}{4} + \frac{\sqrt{3}}{4}i$

Converting between Cartesian and Polar forms

The **form** of a complex number in this system (polar co-ordinates) are the pairs $[r, \phi]$ or $[\text{modulus}, \text{argument}]$.
We have already seen how to **convert** from **Cartesian** (a, b) to **Polar** $[r, \phi]$ via:

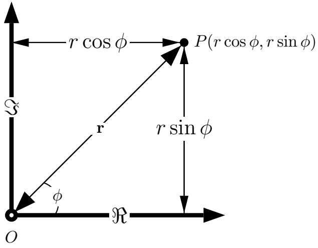
- $r = |z| = \sqrt{a^2 + b^2}$
- $\phi = \arg z = \arctan(\frac{b}{a})$



Polar to Cartesian Conversion

Can we **convert** from **Polar** $[r, \phi]$ to **Cartesian** (a, b)?

Simple trigonometry gives us the solution:



- $a = r\cos\phi$
- $b = r\sin\phi$
- Giving $\mathbf{z} = \mathbf{a} + \mathbf{bi}$

Exercise

Find the **Cartesian Co-ordinates** of the **Complex Point** $P[4, 30°]$.

Trigonometric form

From last slide

- $a = r\cos\phi$
- $b = r\sin\phi$
- Giving $\mathbf{z = a + bi}$

So if we substitute for a and b we get:

$$\begin{aligned}\mathbf{z} &= \mathbf{r}\cos\phi + \mathbf{r}\sin\phi \times \mathbf{i} \\ &= \mathbf{r}(\cos\phi + \mathbf{i}\sin\phi)\end{aligned}$$

This is known as the
trigonometric form of a complex number

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MATLAB and Complex Numbers

MATLAB knows about complex numbers

```
>> sqrt(-1)
ans = 0 + 1.0000i

% Symbolic Eqns Soln
>> syms x;
>> f = x^2 + 1;
>> solve(f)
ans =

   i
  -i

% Polynomial Roots
>> p = [1 0 1];
>> roots(p)
ans =

        0 + 1.0000i
        0 - 1.0000i
```

Declaring Complex Numbers in MATLAB

Simply use *i* in an expression or the `complex()` function

```
% Must use * operator with i even though this is not displayed
>> c1 = 3 + 4*i
c1 =

   3.0000 + 4.0000i

% MATLAB also allows the use of j
>> c2 = 2 + 4*j
c2 =

   2.0000 + 4.0000i

% What I already have a variable i (or j)  e.g. for i=1:n?
>> c3 = complex(1,2)
c3 =

   1.0000 + 2.0000i
```

MATLAB: real, imaginary, magnitude and phase

MATLAB provides functions to obtain these

```
>> c = 4+3*i
c =
   4.0000 + 3.0000i

%  Real part,  Imaginary part,  and  Absolute value
>> [real(c),  imag(c), abs(c)]
ans =
     4      3      5

%   A complex number of magnitude 11 and phase angle 0.7 radians
>> z = 11*(cos(0.7)+sin(0.7)*i)
z =
   8.4133 + 7.0864i

%   Recover the magnitude and phase of "z"
>> [abs(z), angle(z)]
ans =
   11.0000    0.7000
```

MATLAB understands Trig. form of a complex number

From the last slide example:

You can declare in trig. form but MATLAB coverts to normal representation

```
%   Trig. Form: A complex number of
%   magnitude 11 and phase angle 0.7 radians
>> z = 11*(cos(0.7)+sin(0.7)*i)
z =
   8.4133 + 7.0864i

% So Need to use abs() and angle() to
%  Recover the magnitude and phase of "z"
>> [abs(z), angle(z)]
ans =
   11.0000    0.7000
```

MATLAB Complex Arithmetic

Behaves as one would expect

```
>> c1 = 3 + 4*i;
>> c2 = 2 + 4*j;
>> c1 + c2
ans =   5.0000 + 8.0000i

>> c1 - c2
ans = 1

>> i^2
ans = -1

>> c1*c2
ans = -10.0000 +20.0000i

>> c1/c2
ans = 1.1000 - 0.2000i
```

Plotting Polar Coordinates in MATLAB

MATLAB provide useful **plotting** functions for general **Polar Coordinates**

- This is not exclusively for Complex Numbers.

The MATLAB function `polar()` achieves this:

`polar()` — Polar coordinate plot

- `polar(`**Theta** , **Radius** `)` makes a plot using polar coordinates of the angle `Theta` , in radians, versus the radius `Radius`. `polar(Theta, Radius,` **S**`)` uses the **linestyle** specified in string `S`.
  - similar to `plot()` in terms of styles
- Note the order! `Theta` **first** then `Radius`!

polar() Example

Plotting Polar Representation of a Complex Number

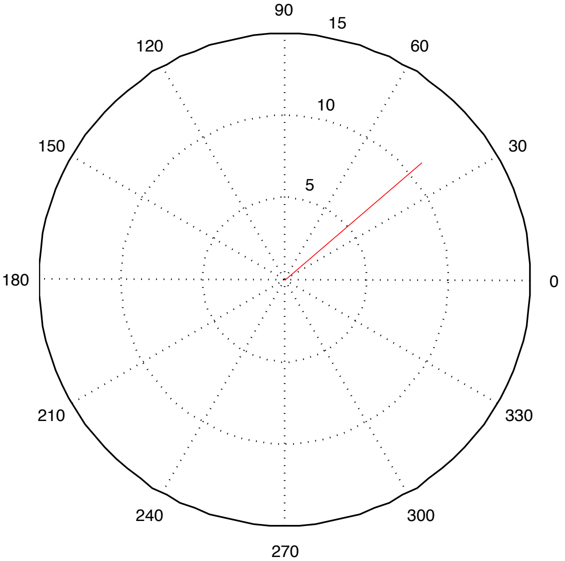
```
>> z = 11*(cos(0.7)+sin(0.7)*i)

z =

   8.4133 + 7.0864i

>> polar([0 angle(z)],[0 abs(z)],'-r');
```

`polar(angle(z),abs(z))` Plot Output



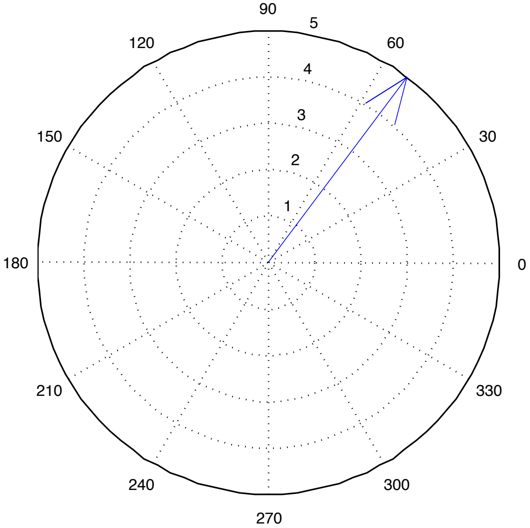
compass() Plot

The compass() knows how to plot a complex number directly:

compass() Example

```
>> c1 = 3 + 4*i;
>> compass(c1);
```

compass(c1); Plot Output



Note: c1 automatically converted to polar form

Euler's Formula: Phasor Form

Euler's Formula[3] states that we can express the trigonometric form as:

$$\mathbf{e}^{\mathbf{i}\phi} = \cos\phi + \mathbf{i}\sin\phi, \quad \phi \in \mathbb{R}$$

Exercise: Show that

$$\mathbf{e}^{-\mathbf{i}\phi} = \cos\phi - \mathbf{i}\sin\phi$$

This is also known as **phasor form** or **Phasors** , for short.

Note: Phasors and the related trigonometric form are **very important** to **Fourier Theory** which we study later.

[3] we won't prove this here. Proof here if interested

Phasor Notation

General Phasor Form: $re^{i\phi}$

More generally we use $re^{i\phi}$ where:

$$re^{i\phi} = r(\cos\phi + i\sin\phi)$$

MATLAB Speaks the Phasor Language

MATLAB Complex No. Phasor Declaration

```
>> exp( i*(pi/4) )

ans =   0.7071 + 0.7071i

>> [abs(z), angle(z)]

ans =   1.0000     0.7854
```

Phasors are stunning!

Phasers on stun!

Phasors are stunning!

Phasors are very useful mathematical tools

- Can simplify Trigonometric proofs, Trig. expression manipulation *etc*
  - Can do Trigonometry without Trigonometry (well almost!)
- Electrical Signals: Can apply simplify AC circuits to DC circuit theory (*e.g.* Ohm's Law)!
- Power engineering: Three phase AC power systems analysis
- **Signal Processing**: **Fourier Theory**, **Filters**

Trig. Example: sin and cos as functions of e

From **Euler's Formula** we can write:

$$\cos\phi = \frac{\mathbf{e}^{\mathbf{i}\phi} + \mathbf{e}^{-\mathbf{i}\phi}}{\mathbf{2}}$$

$$\sin\phi = \frac{\mathbf{e}^{\mathbf{i}\phi} - \mathbf{e}^{-\mathbf{i}\phi}}{\mathbf{2i}}$$

Prove the above

Trig. Exercise: Powers of the Trigonometric Form (de Moivre's Theorem)

If n is an **integer** then show that:

$$(\cos\theta + \mathbf{i}\sin\theta)^{\mathbf{n}} = \cos\mathbf{n}\theta + \mathbf{i}\sin\mathbf{n}\theta.$$

This is known as **de Moivre's Theorem**

Complex Number Multiplication in Polar Form

Let $z_1 = [r_1, \phi_1]$ and $z_2 = [r_2, \phi_2]$ then
$z_1 = r_1(\cos\phi_1 + i\sin\phi_1)$ and $z_2 = r_2(\cos\phi_2 + i\sin\phi_2)$
Therefore:

$$\begin{aligned}
z_1 z_2 &= [r_1(\cos\phi_1 + i\sin\phi_1)] \times [r_2(\cos\phi_2 + i\sin\phi_2)] \\
&= r_1 r_2[(\cos\phi_1 + i\sin\phi_1) \times (\cos\phi_2 + i\sin\phi_2)] \\
&= r_1 r_2[\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 \\
&\quad +i(\cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2)]
\end{aligned}$$

From **trigonometry** we have the following relations:

- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$,
- $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$,

So **finally** we have:

$$\begin{aligned}
z_1 z_2 &= r_1 r_2[\cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2)] \\
\mathbf{z_1 z_2} &= [\mathbf{r_1 r_2}, \phi_1 + \phi_2]
\end{aligned}$$

Complex Number Multiplication via Phasors

Alternatively, we can multiply complex numbers via **Phasors**:
$z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$.
Therefore:

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\phi_1} \times r_2 e^{i\phi_2} \\ &= r_1 r_2 e^{i\phi_1} e^{i\phi_2} \end{aligned}$$

Now in general, $e^x e^y = e^{(x+y)}$

So we get: $z_1 z_2 = r_1 r_2 e^{i(\phi_1+\phi_2)}$ which (as we should expect) gives:

$$\mathbf{z_1 z_2 = [r_1 r_2, \phi_1 + \phi_2]}$$

This is a much **easier** way to prove this fact — **Agree?**[4]

[4] **This sort of algebra is important for Fourier Theory later**

Complex Number Division in Phasor Form

Sticking with the **Phasor** formulation, we can **divide** two complex numbers:
$z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$.
Therefore:

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1 e^{i\phi_1}}{r_2 e^{i\phi_2}} \\
&= \frac{r_1}{r_2}\frac{e^{i\phi_1}}{e^{i\phi_2}} \\
&= \frac{r_1}{r_2} e^{i\phi_1} e^{-i\phi_2}, \quad \text{by a same argument as in multiplication} \\
&= \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)} \\
\frac{\mathbf{z_1}}{\mathbf{z_2}} &= [\frac{\mathbf{r_1}}{\mathbf{r_2}}, \phi_\mathbf{1} - \phi_\mathbf{2}]
\end{aligned}
$$

Exercise: Prove this formula via the trigonometric polar form — **yawn**!.

Exercises: Complex Number Multiplication and Division

- If $z_1 = 3\sqrt{2} + 3\sqrt{2}i$ and $z_2 = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$, find $z_1 z_2$ and $\frac{z_1}{z_2}$, leave your answer in **polar form**.

- Evaluate $(\frac{1}{2} + \frac{\sqrt{3}}{2}i)^3$, give your answer in **Cartesian form**.

Complex Number Multiplication: Geometric Representation

Multiplying a complex number $z = x + iy$ by i **rotates** the vector representing z through 90° **anticlockwise**
Example: Let $z1 = 1$.
Then

$$z_2 = iz_1 = i.$$

- Polar form of $z_1 = [1, 0^\circ]$.
- Polar form of $z_2 = [1, 90^\circ]$, **Q.E.D**.

Back to Phase: Important Example

Concept: A **phasor** is a **complex number** used to represent a **sinusoid**.

In particular:

Sinusoid : $x(t) = M\cos(\omega t + \phi), \; -\infty < t < \infty$ — a function of **time**

Phasor : $X = Me^{i\phi} = M\cos(\phi) + iM\sin(\phi)$ — a **complex number**

Complex Numbers and Phase: Important Example

Phasors and Sinusoids are related:

$$
\begin{aligned}
\Re[Xe^{i\omega t}] &= \Re[Me^{i\phi}e^{i\omega t}] \\
&= \Re[Me^{i(\omega t+\phi)}] \\
&= \Re[M(\cos(\omega t+\phi)+i\sin(\omega t+\phi))] \\
&= M\cos(\omega t+\phi) \\
&= \mathbf{x(t)}
\end{aligned}
$$

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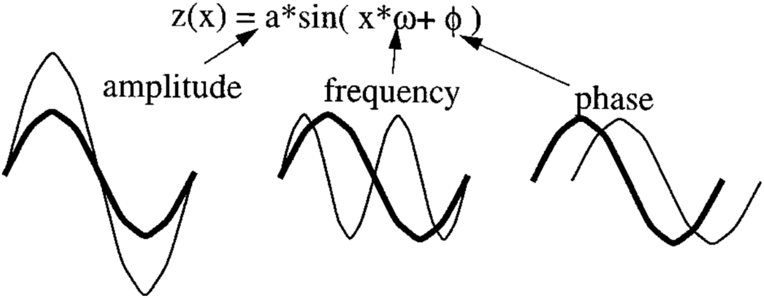
Visualising Sinusoids of differing Phase, Amplitude and Frequency



MATLAB Sine Wave Frequency and Amplitude (only)

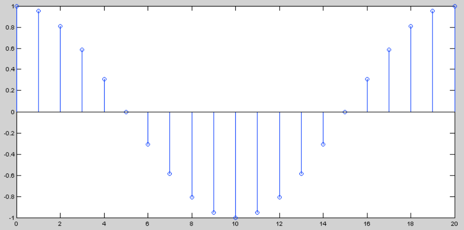
```
% Natural frequency is 2*pi radians
% If sample rate is F_s HZ then 1 HZ is 2*pi/F_s
% If wave frequency is F_w then frequency is
%        F_w* (2*pi/F_s)
% set n samples steps up to sum duration nsec*F_s where
% nsec is the duration in seconds
% So we get y = amp*sin(2*pi*n*F_w/F_s);

F_s = 11025;
F_w = 440;
nsec = 2;
dur= nsec*F_s;
n = 0:dur;

y = amp*sin(2*pi*n*F_w/F_s);
figure(1)
plot(y(1:500));
title('N second Duration Sine Wave');
```

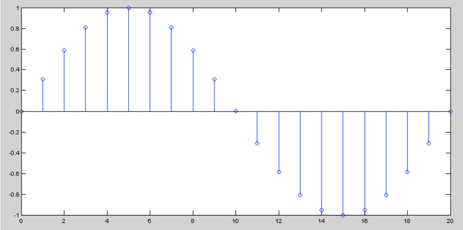
MATLAB Cos v Sin Wave

```
% Cosine is same as Sine (except 90 degrees out of phase)

yc = amp*cos(2*pi*n*F_w/F_s);

figure(2);
hold on;
plot(yc,'b');
plot(y,'r');
title('Cos (Blue)/Sin (Red) Plot (Note Phase Difference)');
hold off;
```

Sin and Cos (stem) plots

MATLAB functions `cos()` and `sin()`.





Amplitudes of a Sine Wave

Code for sinampdemo.m

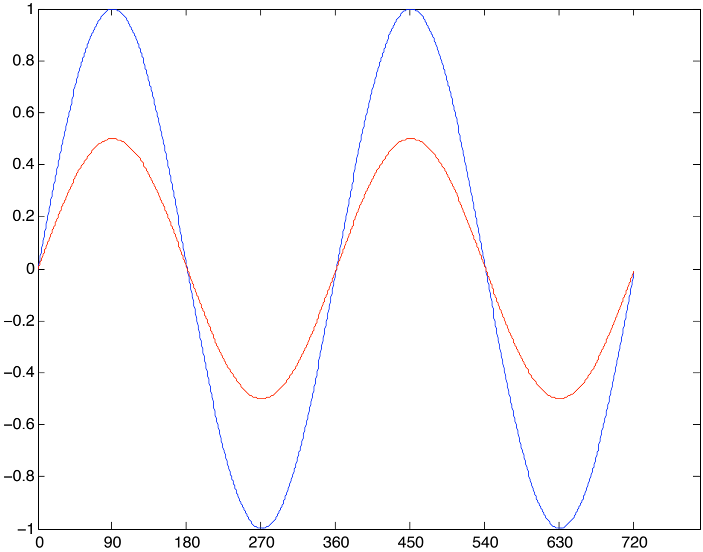
```
% Simple Sin Amplitude Demo
samp_freq = 400;
dur = 800; % 2 seconds
amp = 1; phase = 0; freq = 1;
s1 = mysin(amp,freq,phase,dur,samp_freq);

axisx = (1:dur)*360/samp_freq; % x axis in degrees
plot(axisx,s1);
set(gca,'XTick',[0:90:axisx(end)]);

fprintf('Initial Wave: \t Amplitude = ...\n', amp,
                freq, phase,...);

% change amplitude
amp = input('\nEnter Amplitude:\n\n');

s2 = mysin(amp,freq,phase,dur,samp_freq);
hold on;
plot(axisx, s2,'r');
set(gca,'XTick',[0:90:axisx(end)]);
```

mysin MATLAB code

mysin.m — a modified version of previous MATLAB sin example to account for phase

```
function s = mysin(amp,freq,phase,dur, samp_freq)
% example function to so show how amplitude,frequency
% and phase are changed in a sin function
% Inputs: amp - amplitude of the wave
%         freq - frequency of the wave
%         phase - phase of the wave in degree
%         dur - duration in number of samples
%         samp_freq - sample frequncy

x = 0:dur-1;
phase = phase*pi/180;

s = amp*sin( 2*pi*x*freq/samp_freq + phase);
```

Amplitudes of a Sine Wave: `sinampdemo` output

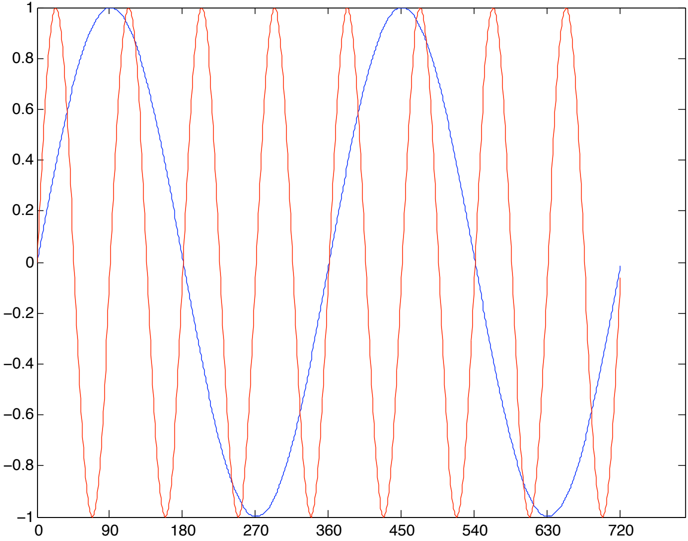


Frequencies of a Sine Wave

Code (fragment) for sinfreqdemo.m

```
% Simple Sin Frequency Demo
samp_freq = 400;
dur = 800; % 2 seconds
amp = 1; phase = 0; freq = 1;
s1 = mysin(amp,freq,phase,dur,samp_freq);

axisx = (1:dur)*360/samp_freq; % x axis in degrees
plot(axisx,s1);
set(gca,'XTick',[0:90:axisx(end)]);

fprintf('Initial Wave: \t Amplitude = %d\n', amp, freq,


% change amplitude
freq = input('\nEnter Frequency:\n\n');

s2 = mysin(amp,freq,phase,dur,samp_freq);
hold on;
plot(axisx, s2,'r');
set(gca,'XTick',[0:90:axisx(end)]);
```

Frequencies of a Sine Wave: `sinfreqdemo` output

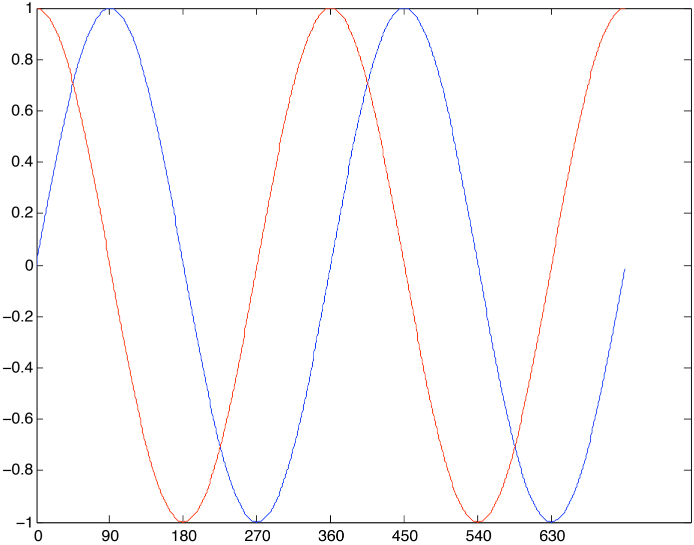


Phase of a Sine Wave

sinphasedemo.m (Fragment)

```
% Simple Sin Phase Demo
samp_freq = 400;
dur = 800; % 2 seconds
amp = 1; phase = 0; freq = 1;
s1 = mysin(amp,freq,phase,dur,samp_freq);

axisx = (1:dur)*360/samp_freq; % x axis in degrees
plot(axisx,s1);
set(gca,'XTick',[0:90:axisx(end)]);

ffprintf('Initial Wave: \t Amplitude = %d\n', amp, freq,


% change amplitude
phase = input('\nEnter Phase:\n\n');

s2 = mysin(amp,freq,phase,dur,samp_freq);
hold on;
plot(axisx, s2,'r');
set(gca,'XTick',[0:90:axisx(end)]);
```

Phase of a Sine Wave: `sinphasedemo` output



Sum of Two Sinusoids of Same Frequency (1)

Hopefully we now have a good understanding and can visualise Sinusoids of different phase, amplitude and frequency.

Back to Phasors: $X = Me^{i\phi} = M\cos(\phi) + iM\sin(\phi)$

Consider two sinusoids: Same frequency, ω but different phase, θ and ϕ and amplitude, A and B

$\mathbf{A}\cos(\omega\mathbf{t} + \theta)$, and

$\mathbf{B}\cos(\omega\mathbf{t} + \phi)$

Let's add them together

Sum of Two Sinusoids of Same Frequency (2)

$$\begin{aligned} A\cos(\omega t+\theta)+B\cos(\omega t+\phi) &= \Re[Ae^{i(\omega t+\theta)}+Be^{i(\omega t+\phi)}] \\ &= \Re[e^{i\omega t}(Ae^{i\theta}+Be^{i\phi})] \end{aligned}$$

Now let $Ae^{i\theta}+Be^{i\phi}=Ce^{i\gamma}$ for some C and γ, then

$$\begin{aligned} \Re[e^{i\omega t}(Ae^{i\theta}+Be^{i\phi})] &= \Re[e^{i\omega t}(Ce^{i\gamma})] \\ &= C\cos(\omega t+\gamma) \end{aligned}$$

Sum of Two Sinusoids of Same Frequency (3)

Trigonometry Equation

$$A\cos(\omega t + \theta) + B\cos(\omega t + \phi) = C\cos(\omega t + \gamma)$$

Equivalent Complex Number Equation

$$Ae^{i\theta} + Be^{i\phi} = Ce^{i\gamma}$$

Which is neater? **Let's see**

Example: Sum of Two Sinusoids of Same Frequency (1)

Simplify

$$5\cos(\omega t + 53^\circ) + \sqrt{2}\cos(\omega t + 45^\circ)$$

Hard way via trigonometry

- Use the cosine addition formula three times
  - see **maths formula sheet handout** for formula
- Third time to simplify the result.
- Not difficult but **tedious**!

Example: Sum of Two Sinusoids of Same Frequency (2)

Easy Way Phasors

$$\begin{aligned}\Re[5e^{i53^\circ} + \sqrt{2}e^{i45^\circ}] &= (3+4i)+(1+i)\\ &= (4+5i)\\ &= 6.4e^{i51^\circ}\end{aligned}$$

So:

$$\begin{aligned}5\cos(\omega t + 53^\circ) + \sqrt{2}\cos(\omega t + 45^\circ) &= \Re[6.4e^{i(wt+51^\circ)}]\\ &= 6.4\cos(\omega t + 51^\circ)\end{aligned}$$

This is a **very important example** - make sure you **understand it.**

Another Example (1)

Simplify

$$\cos(\omega \mathbf{t} + \mathbf{30}^\circ) + \cos(\omega \mathbf{t} + \mathbf{150}^\circ) + \sin(\omega \mathbf{t})$$

First trick to note:

$$\sin(\omega t) = \cos(\omega t - 90^\circ)$$

So now simplify:

$$\cos(\omega \mathbf{t} + \mathbf{30}^\circ) + \cos(\omega \mathbf{t} + \mathbf{150}^\circ) + \cos(\omega \mathbf{t} - \mathbf{90}^\circ)$$

Hard way via trigonometry

- Use the cosine addition formula three times
  - see **maths formula sheet handout** for formula
- Not difficult but **tedious**!

Another Example (2)

Easy Way Phasors

$$
\begin{aligned}
e^{i30^\circ} + e^{i150^\circ} + e^{-i90^\circ} &= e^{i90}(= i) \\
\text{So we get:} & \\
\Re[e^{i90}] &= \cos(90^\circ) \\
&= 0
\end{aligned}
$$

So:

$$\cos(\omega \mathbf{t} + \mathbf{30}^\circ) + \cos(\omega \mathbf{t} + \mathbf{150}^\circ) + \cos(\omega \mathbf{t} - \mathbf{90}^\circ) = 0$$

or

$$\cos(\omega \mathbf{t} + \mathbf{30}^\circ) + \cos(\omega \mathbf{t} + \mathbf{150}^\circ) + \sin(\omega \mathbf{t}) = 0$$

This fact is used in **three-phase AC** to conserve current flow