CM2208: Scientific Computing
Fourier Transform 1:
Digital Signal and Image Processing
Fourier Theory

Prof. David Marshall

School of Computer Science & Informatics

Fourier Transform

Moving into the Frequency Domain

The **Frequency domain** can be obtained through the transformation, via **Fourier Transform (FT)**, from

- one (**Temporal (Time)** or **Spatial**) domain

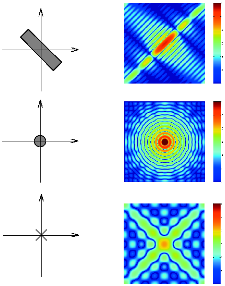
  to the other

- **Frequency** Domain
  - We do not think in terms of signal or pixel intensities but rather underlying sinusoidal waveforms of varying frequency, amplitude and phase.

Applications of Fourier Transform

Numerous Applications including:

- Essential tool for Engineers, Physicists, Mathematicians and Computer Scientists
- Fundamental tool for Digital Signal Processing and Image Processing
- Many types of Frequency Analysis:
    - **Filtering**
    - **Noise Removal**
    - Signal/Image Analysis
    - Simple implementation of **Convolution**
    - **Audio** and Image **Effects Processing**.
    - Signal/Image Restoration — *e.g.* **Deblurring**
    - Signal/Image Compression —- **MPEG** (Audio and Video), **JPEG** use related techniques.
    - Many more . . . . . .



Introducing Frequency Space

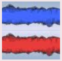
1D Audio Example

Lets consider a 1D (e.g. Audio) example to see what the different domains mean:

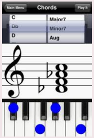
Consider a **complicated sound** such as the a **chord** played on a **piano** or a **guitar**.

We can describe this sound in two related ways:

Temporal Domain : Sample the **amplitude** of the sound many times a second, which gives an approximation to the sound as a **function** of **time**.



Frequency Domain : **Analyse** the sound in terms of the **pitches** of the notes, or **frequencies**, which make the sound up, recording the **amplitude** of **each frequency**.



Fundamental Frequencies

D♭	: 554.40Hz
F	: 698.48Hz
A♭	: 830.64Hz
C:	1046.56Hz

plus harmonics/partial frequencies ....

Back to Basics

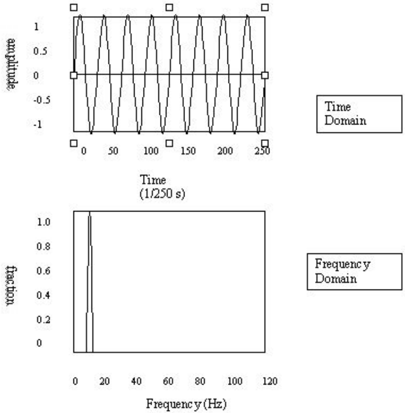
An 8 Hz Sine Wave

A signal that consists of a **sinusoidal** wave at **8 Hz**.

- 8 Hz means that wave is completing 8 cycles in 1 second
- The **frequency** of that wave is 8 Hz.

From the **frequency domain** we can see that the composition of our signal is

- **one peak** occurring with a frequency of 8 Hz — there is only one sine wave here.
  - with a **magnitude/fraction** of **1.0** i.e. it is the **whole signal**.



2D Image Example

What do Frequencies in an Image Mean?

Now images are no more complex really:

- **Brightness** along a **line** can be recorded as a set of **values** measured at **equally** spaced **distances apart**,
- **Or** equivalently, at a **set** of **spatial frequency values**.
- Each of these frequency values is a **frequency component**.
- An image is a 2D array of pixel measurements.
- We form a 2D grid of spatial frequencies.
    - A given frequency component now specifies what contribution is made by data which is changing with specified x and y direction spatial frequencies.

Frequency components of an image

What do Frequencies in an Image Mean? (Cont.)

- Large values at **high** frequency components then the data is changing rapidly on a short distance scale.
  - *e.g.* a **page of text**
  - **However**, **Noise** contributes (very) **High Frequencies** also
- Large **low** frequency components then the large scale features of the picture are more important.
  e.g. a single fairly simple object which occupies most of the image.

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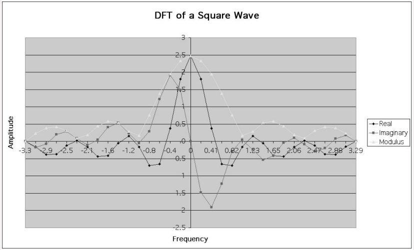
Visualising Frequency Domain Transforms

Sinusoidal Decomposition

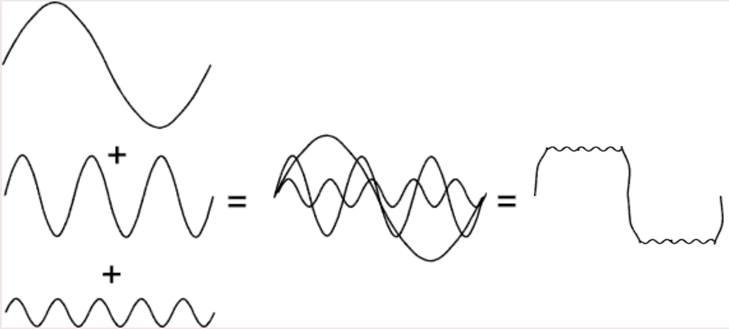
- **Any digital signal** (function) can be **decomposed** into purely **sinusoidal components**
  - Sine waves of different size/shape — varying **amplitude**, **frequency** and **phase**.
- When **added** back **together** they **reconstitute** the **original signal**.
- The **Fourier transform** is the tool that performs such an operation.



Summing Sine Waves. Example: to give a Square(ish) Wave

Digital signals are composite signals made up of many sinusoidal frequencies

- A 200Hz digital signal (**square(ish) wave**) may be a composed of 200, 600, 1000, *etc.* sinusoidal signals which sum to give:



Summary so far

So What Does All This Mean?

Transforming a signal into the frequency domain allows us

- **To see what sine waves make up our underlying signal**
- **E.g.**
  - One part sinusoidal wave at 50 Hz and
  - Second part sinusoidal wave at 200 Hz.
  - *Etc.*
- More **complex** signals will give more complex decompositions but the idea is exactly the same.

How is this Useful then?

Basic Idea of Filtering in Frequency Space

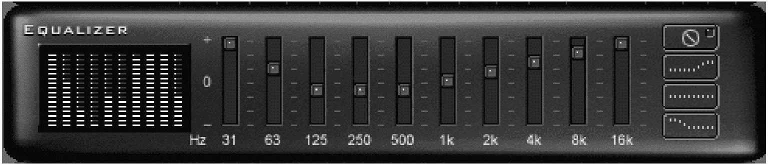
Filtering now involves **attenuating** or **removing** certain frequencies — **easily performed**:

- **Low pass filter** —
  - **Ignore high frequency** noise components — make **zero** or a **very low value**.
  - Only store lower frequency components
- **High Pass Filter** — **opposite of above**
- **Bandpass Filter** — only **allow** frequencies in a **certain range**.

Visualising the Frequency Domain

Think Graphic Equaliser

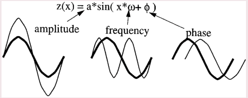
An easy way to visualise what is happening is to think of a graphic equaliser on a stereo system (or some software audio players, *e.g. iTunes*).



So are we ready for the Fourier Transform?

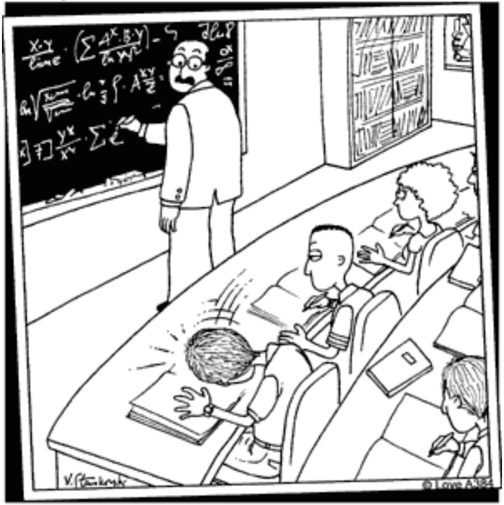
We have all the Tools....

- This lecture, so far, (hopefully) set the context for Frequency decomposition.
- Past Maths Lectures:
  - **Odd/Even Functions**: $\sin(-x) = -\sin(x)$, $\cos(-x) = \cos(x)$
  - **Complex Numbers**: **Phasor Form** $re^{i\phi} = r(\cos\phi + i\sin\phi)$
  - Calculus **Integration**: $\int e^{kxdx} = \frac{e^{kx}}{k}$
- Digital Signal Processing:
  - Basic Waveform Theory. Sine Wave $y = A.sin(2\pi.n.F_w/F_s)$ where: $A =$ amplitude, $F_w =$ wave frequency, $F_s =$ sample frequency, n is the sample index.
  - Relationship between Amplitude, Frequency and Phase:



    - Cosine is a Sine wave 90° out of phase
  - Impulse Responses
- DSP + Image Proc.: Filters and other processing, Convolution

Snapshots at jasonlove.com



Professor Herman stopped when he heard that unmistakable thud -- another brain had imploded.

Fourier Theory

Introducing The Fourier Transform

The tool which **converts** a **spatial** or **temporal** (real space) **description** of **audio**/**image** data, for example, into one in terms of its **frequency components** is called the **Fourier transform**

The new version is usually referred to as the **Fourier space description** of the data.

We then essentially process the data:

- *E.g.* for **filtering** basically this means attenuating or setting certain frequencies to zero

We then need to **convert data back** (or **invert**) to **real audio**/imagery to use in our applications.

The corresponding **inverse** transformation which turns a Fourier space description back into a real space one is called the **inverse Fourier transform**.

1D Fourier Transform

1D Case (*e.g.* Audio Signal)

Considering a **continuous** function $f(x)$ of a single variable x representing distance (or time).

The **Fourier transform** of that function is denoted $F(u)$, where u represents **spatial** (or **temporal**) **frequency** is defined by:

$$F(u) = \int_{-\infty}^{\infty} f(x) \mathbf{e}^{-2\pi \mathbf{i} \mathbf{x} \mathbf{u}} \, dx.$$

Note: In general $F(u)$ will be a complex quantity *even though* the original data is purely **real**.

- The meaning of this is that not only is the **magnitude** of each **frequency** present important, but that its **phase relationship** is **too**.
- Recall **Phasors** from **Complex Number Lectures**.
  - $e^{-2\pi ixu}$ above is a **Phasor**.

Inverse Fourier Transform

Inverse 1D Fourier Transform

The **inverse Fourier transform** for regenerating $f(x)$ from $F(u)$ is given by

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{\mathbf{2\pi ixu}} \, du,$$

which is rather similar to the (forward) Fourier transform

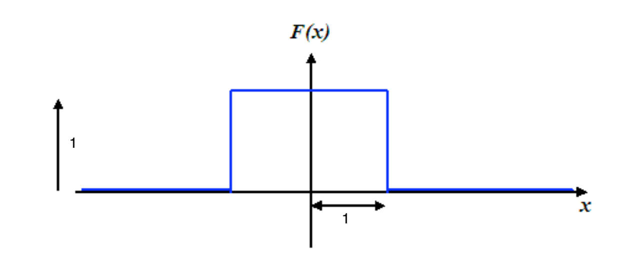
- except that the **exponential term has the opposite sign.**
- It is **not negative**

Fourier Transform Example

Fourier Transform of a Top Hat Function

Let's see how we compute a Fourier Transform: consider a particular function $f(x)$ defined as

$$f(x) = \begin{cases} 1 & \textbf{if } |x| \leq 1 \\ 0 & \textbf{otherwise,} \end{cases}$$



The Sinc Function (1)

We derive the Sinc function

So its Fourier transform is:

$$
\begin{aligned}
\mathbf{F(u)} &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i x u} \, dx \\
&= \int_{-1}^{1} 1 \times e^{-2\pi i x u} \, dx \\
&= \frac{-1}{2\pi i u} (e^{2\pi i u} - e^{-2\pi i u})
\end{aligned}
$$

Now (refer to **Complex Numbers** Lectures/**Maths Formula Sheet** Handout)

$$
\begin{aligned}
\sin\theta &= \frac{\mathbf{e^{i\theta} - e^{-i\theta}}}{\mathbf{2i}}, \text{ So:} \\
\mathbf{F(u)} &= \frac{\sin \mathbf{2\pi u}}{\mathbf{\pi u}}.
\end{aligned}
$$

In this case, $F(u)$ is purely real, which is a consequence of the original data being **symmetric** in x and $-x$.

- $f(x)$ is an **even** function.

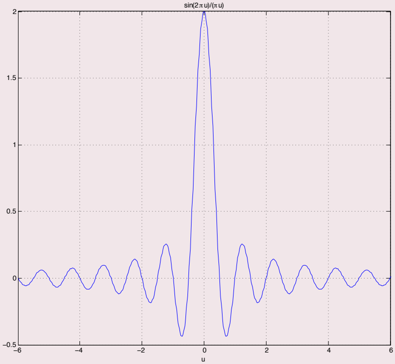
A graph of $F(u)$ is shown overleaf.

This function is often referred to as the **Sinc function**.

The Sinc Function Graph

The Sinc Function

The Fourier transform of a top hat function, the **Sinc function**:



The 2D Fourier Transform

2D Case (*e.g.* Image data)

If $f(x,y)$ is a function, for example **intensities** in an **image**, its **Fourier transform** is given by

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathbf{e}^{-2\pi \mathbf{i}(\mathbf{xu}+\mathbf{yv})}\, dx\, dy,$$

and the **inverse transform**, as might be expected, is

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{2\pi \mathbf{i}(\mathbf{xu}+\mathbf{yv})}\, du\, dv.$$

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The Discrete Fourier Transform

But All Our Audio and Image data are Digitised!!

Thus, we need a *discrete* formulation of the Fourier transform:

- **Assumes regularly spaced** data values, and
- **Returns** the **value** of the Fourier transform for a set of values in frequency space which are **equally spaced**.

This is done quite naturally by replacing the integral by a summation, to give the *discrete Fourier transform* or **DFT** for short.

1D Discrete Fourier transform (DFT)

1D Case:

In 1D it is convenient now to assume that x goes up in steps of 1, and that there are N samples, at values of x from 0 to $N-1$.

So the DFT takes the form

$$F(u) = \frac{\mathbf{1}}{\mathbf{N}} \sum_{x=0}^{N-1} f(x) e^{-2\pi ixu/\mathbf{N}},$$

while the inverse DFT is

$$f(x) = \sum_{x=0}^{N-1} F(u) e^{2\pi ixu/\mathbf{N}}.$$

NOTE: Minor changes from the continuous case are a factor of $1/N$ in the **exponential** terms, and also the factor $1/N$ in front of the forward transform which **does not appear** in the **inverse** transform.

2D Discrete Fourier transform

2D Case

The **2D DFT** works is similar.

So for an $N \times M$ grid in x and y we have

$$F(\mathbf{u}, \mathbf{v}) = \frac{1}{\mathbf{NM}} \sum_{x=0}^{N-1} \sum_{y=0}^{M-1} f(x, y) e^{-2\pi i (x\mathbf{u}/\mathbf{N} + y\mathbf{v}/\mathbf{M})},$$

and

$$f(x, y) = \sum_{u=0}^{N-1} \sum_{v=0}^{M-1} F(u, v) e^{2\pi i (xu/\mathbf{N} + yv/\mathbf{M})}.$$

Balancing the 2D DFT

Most Images are Square

Often $N = M$, and it is then it is more convenient to redefine $F(u, v)$ by multiplying it by a factor of N, so that the **forward** and **inverse** transforms are more **symmetric**:

$$F(u, v) = \frac{\mathbf{1}}{\mathbf{N}} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) e^{-2\pi i(xu+yv)/\mathbf{N}},$$

and

$$f(x, y) = \frac{\mathbf{1}}{\mathbf{N}} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u, v) e^{2\pi i(xu+yv)/\mathbf{N}}.$$

Fourier Transforms in MATLAB

fft() and fft2()

MATLAB provides functions for 1D and 2D **Discrete Fourier Transforms** (**DFT**):

fft(X)
: is the 1D discrete Fourier transform (DFT) of **vector** X. For **matrices**, the FFT operation is applied to **each column** — **NOT** a 2D DFT transform.

fft2(X)
: returns the 2D Fourier transform of matrix X. If X is a vector, the result will have the same orientation.

fftn(X)
: returns the N-D discrete Fourier transform of the **N-D array X**.

Inverse DFT
: **ifft()**, **ifft2()**, **ifftn()** perform the **inverse** DFT.

See appropriate MATLAB **help/doc** pages for **full details**.

Plenty of examples to Follow.

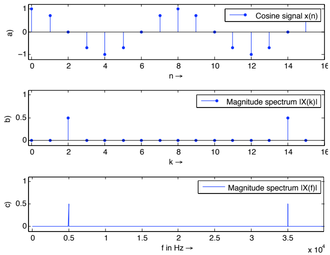
See also: **MALTAB Docs Image Processing** → **User's Guide** → **Transforms** → **Fourier Transform**

Visualising the Fourier Transform

Visualising the Fourier Transform

Having computed a DFT it might be useful to visualise its result:

- It's useful to visualise the Fourier Transform
- Standard tools
- Easily plotted in MATLAB



The Magnitude Spectrum of Fourier Transform

Recall that the Fourier Transform of our **real** audio/image data is always **complex**

- **Phasors**: This is how we encode the **phase** of the underlying signal's **Fourier Components**.

How can we visualise a complex data array?

Back to Complex Numbers:

Magnitude spectrum **Compute the absolute value of the complex data**:

$$|F(k)| = \sqrt{F_R^2(k) + F_I^2(k)} \text{ for } k = 0, 1, \ldots, N-1$$

where $F_R(k)$ is the **real** part and $F_I(k)$ is the **imaginary** part of the N sampled Fourier Transform, $F(k)$.

Recall MATLAB: `Sp = abs(fft(X,N))/N;`
(**Normalised form**)

The Phase Spectrum of Fourier Transform

The Phase Spectrum

Phase Spectrum

The Fourier Transform also represent phase, the **phase spectrum** is given by:

$$\varphi = \arctan \frac{F_I(k)}{F_R(k)} \textbf{ for } k = 0, 1, \ldots, N-1$$

Recall MATLAB: `phi = angle(fft(X,N))`

Relating a Sample Point to a Frequency Point

When **plotting graphs** of *Fourier Spectra* and doing other DFT processing we may wish to **plot** the x-axis in **Hz** (**Frequency**) rather than **sample point** number $k = 0, 1, \ldots, N-1$

There is a **simple relation** between the two:

- The sample points go in steps $k = 0, 1, \ldots, N-1$
- For a given sample point k the frequency relating to this is given by:

$$f_k = k\frac{f_s}{N}$$

  where f_s is the *sampling frequency* and N the **number** of samples.
- Thus we have **equidistant frequency steps** of $\frac{f_s}{N}$ ranging from 0 Hz to $\frac{N-1}{N}f_s$ Hz

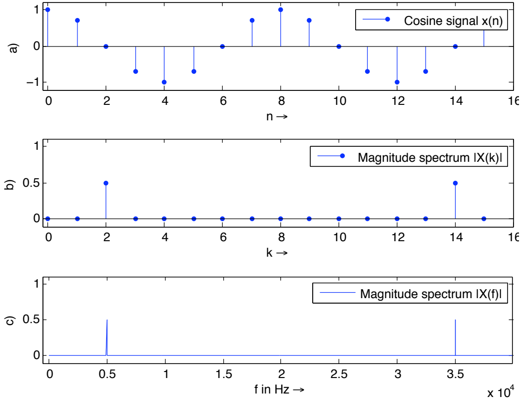
MATLAB Fourier Frequency Spectra Example

fourierspectraeg.m

```
N=16;
x=cos(2*pi*2*(0:1:N-1)/N)';

figure(1)
subplot(3,1,1);
stem(0:N-1,x,'.');
axis([-0.2 N -1.2 1.2]);
legend('Cosine signal x(n)');
ylabel('a)');
xlabel('n \rightarrow');

X=abs(fft(x,N))/N;
subplot(3,1,2);stem(0:N-1,X,'.');
axis([-0.2 N -0.1 1.1]);
legend('Magnitude spectrum |X(k)|');
ylabel('b)');
xlabel('k \rightarrow')

N=1024;
x=cos(2*pi*(2*1024/16)*(0:1:N-1)/N)';

FS=40000;
f=((0:N-1)/N)*FS;
X =abs(fft(x,N))/N;
subplot(3,1,3);plot(f,X);
axis([-0.2*44100/16 max(f) -0.1 1.1]);
legend('Magnitude spectrum |X(f)|');
ylabel('c)');
xlabel('f in Hz \rightarrow')

figure(2)
subplot(3,1,1);
plot(f,20*log10(X./(0.5)));
axis([-0.2*44100/16 max(f) ...
-45 20]);
legend('Magnitude spectrum |X(f)| ...
in dB');
ylabel('|X(f)| in dB \rightarrow');
xlabel('f in Hz \rightarrow')
```

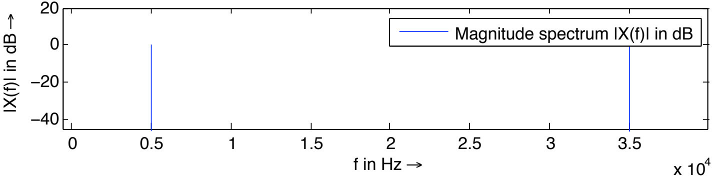
MATLAB Fourier Frequency Spectra Example Output

fourierspectraeg.m produces the following:



Magnitude Spectrum in dB

Note: It is common to plot both spectra magnitude (also frequency ranges not show here) on a dB/log scale:
(Last Plot in fourierspectraeg.m)



Time-Frequency Representation: Spectrogram

Spectrogram

It is often **useful** to look at the **frequency distribution** over a **short-time**:

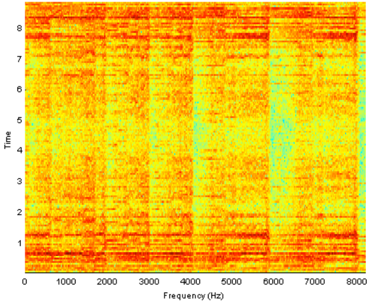
- Split signal into N segments
- Do a **windowed Fourier Transform** — **Short-Time Fourier Transform** (**STFT**)
  - Window needed to reduce *leakage* effect of doing a shorter sample SFFT.
  - Apply a **Blackman**, **Hamming** or **Hanning** Window
- MATLAB function does the job: `Spectrogram` — see `help spectrogram`
- See also MATLAB's `specgramdemo`

MATLAB spectrogram Example

spectrogrameg.m

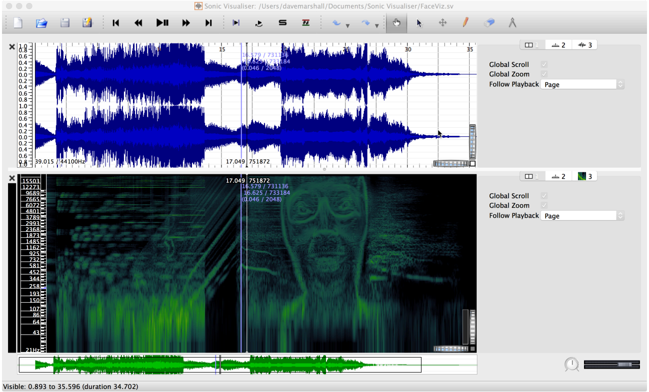
```
load('handel')
[N M] = size(y);
figure(1)
spectrogram(fft(y,N),512,20,1024,Fs);
```

Produces the following:



Aphex Twin Spectrogram

Aphex Twin famously[1] embedded images in the spectrogram of a few tracks on his Windowlicker EP. His face on Track 2 "Formula" or "Equation" (Full title: $\Delta M_{i-1} = -\alpha \sum_{n=1}^{N} D_i[n][\sum_{\sigma \in C[i]} F_{ji}[n-1] + Fext_i[n-1]]$!!:

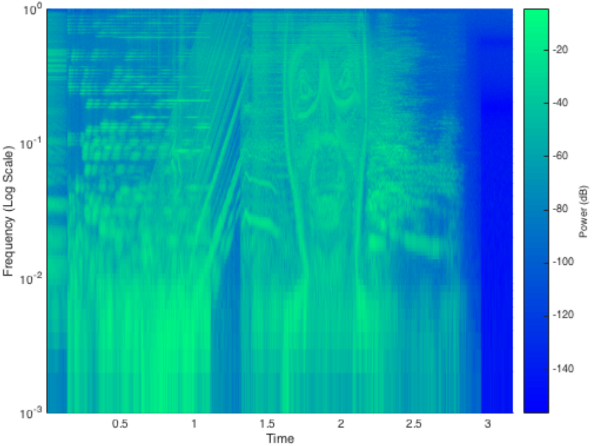


[1]See here for web link to other examples of embedded image Spectrograms

Matlab Code to show the Aphex Twin Spectrogram

Previous slide use the free and excellent **Sonic Visualiser**

We of course know how to display the image in MATLAB:



Matlab Code to show the Aphex Twin Spectrogram

Aphex_Spectrogram.m

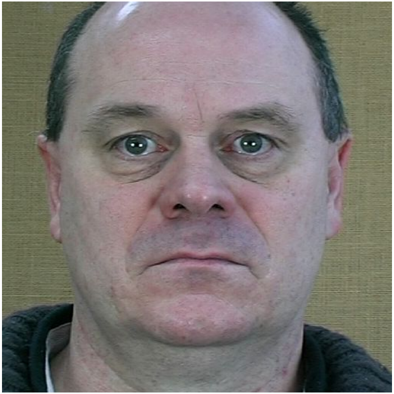
```
aphex = audioread('FormulaSnippet.wav');

mono = (aphex(:,1) + aphex(:,2))/2;

spectrogram(mono,1024,120,2048,'power','yaxis');
set(gca,'YScale','log');
colormap('winter');
xlabel('Time')
ylabel('Frequency (Log Scale)')
```

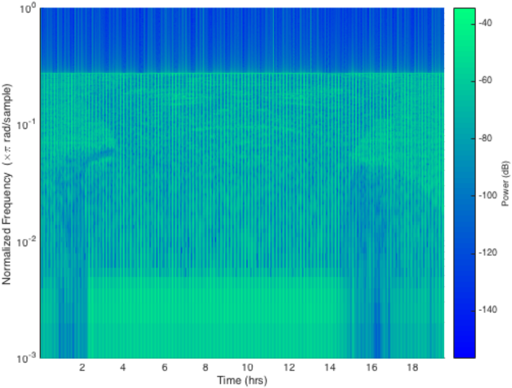
Note: we change the display of the spectrogram to a **log scale**, which looks better.

Audio clip here: FormulaSnippet.wav

So what does my face sound like?

Let's embed my face in spectrogram:





It sounds like this:

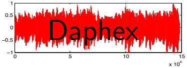


Image to Sound Conversion

Daphex.m

```
figure(1);
imshow(imread('Dave_Frame0001.jpg'));

dave_im2snd = im2sound('Dave_Frame0001', 'jpg', 44100, 40,6000,0.00002, 10);

sound(dave_im2snd,44100);

figure(2);

spectrogram(dave_im2snd,1024,120,2048,'power','yaxis');
set(gca,'YScale','log');

colormap('winter');

shg;
```

Image used here: Dave_Frame0001.jpg

Image to Sound Conversion[2]

im2sound.m (Usage)

```
function [final_sound] = im2sound(filename, ext, f_sample, f_low, ...
    f_high, amp_mod, sample_t)

%INPUTS:
%'filename' - Name of the image to be encoded (not including extension
%ext' - Extension of the image (not including "." at the beginning).
%'f_sample' - Sampling frequency (Hz)
%'f_low' - Lowest frequency (Hz) (e.g. 40)
%'f_high' - Highest frequency (Hz) (e.g. 6000)
%'amp_mod' - Multiplication factor for the amplitude.  Decrease until
%image is clear. Too high and the waveform clips.  Too low and the image
%is very dark (e.g. 0.00002)
%'sample_t' - Duration of the sample in seconds.  Longer samples have
%better quality (e.g. 10)

%OUTPUTS:
%'final_sound' - the final sound containing the image.  This is
%automatically saved to a .wav file with the original image filename
```

[2] Original Code from MATLAB Central

Image to Sound Conversion

im2sound.m (Code)

```
function [final_sound] = im2sound(filename, ext, f_sample, f_low, ...
    f_high, amp_mod, sample_t)

........

%INITIALISING VARIABLES:
%The waveform at each time point.  This is reset at the beginning of each
%time point
temp_sound = 0;
%The final waveform
final_sound = 0;


%MAIN BODY
%Loading the sample image and calculating the image size
raw_im = imread(strcat(filename,'.',ext));
size_raw_im = size(raw_im);

%Making a frequency table for the height of the image.  Each row of the
%image is assigned a particular frequency from the corresponding row of
%this table.  The frequencies are linearly distributed between the highest and
%lowest user-defined frequencies.  "f_step" is the increment between each
%adjacent frequency
f_step = (f_high - f_low)/size_raw_im(1,1);
f_table = (f_high:-f_step:f_low);
```

Image to Sound Conversion

im2sound.m (Code)

```
%The final sound will dwell on each column of the image for a specific
%time.  This time is defined by "t_start" and "t_end".  It depends on how
%long the user determined the sound-clip should be and how wide (how many
%columns) the image is.
t_step = (sample_t/size_raw_im(1,2));

%Initial values for the start and end times.  These will be increased at
%the end of each loop iteration (when the script moves onto the next column
%of the image).
t_start = 0;
t_end = t_step;

%The loop which generates the sound file.  At each iteration it generates a
%segment of the final sound file, which is temporarily saved to
%"temp_sound".  This segment is built up of frequencies from that
%particular column of the image.
for j = 1:size_raw_im(1,2)
    %Initialising the variable (the sound for each frequency (row) is added
    %to the existing sound)
    temp_sound = 0;

    %Setting the time in matrix format
    t = t_start:1/f_sample:(t_end);
```

Image to Sound Conversion

im2sound.m (Code)

```
    %For each iteration of this loop, the script goes down the current
    %column of the image and generates a waveform of the frequency
    %specified in "f_table".  The amplitude of the waveform is determined
    %by the pixel intensity.  This generated waveform is added to all the
    %previously generated waveforms in that particular column
    for i = size_raw_im(1,1):-1:1
        temp_sound = temp_sound+ sin(2*pi*t*f_table(i))*...
            double(raw_im(i,j))*amp_mod;

    end

    %At the end of each column the segment of sound generated is added to
    %the end of the existing sound file ("final_sound").
    final_sound = cat(2,final_sound,temp_sound);

    %The temporary sound is cleared ready for the start of the next column
    clear temp_sound

    %Moving to the next time frame
    t_start = t_start + t_step;
    t_end = t_end + t_step;

end

%This saves "final_sound" to the '.wav' file of the same name as the input
%file
wavwrite(final_sound, f_sample, strcat(filename, '.wav'));
```

Properties of Fourier Transforms

Here are just a few of the other properties of Fourier transforms which are useful when reasoning, or computing, with Fourier transforms:

- Linear Operator
- Shifting
- Scaling
- Rotation
- Zeroth component
- Convolution Theorem (**see next section applications**)

Fourier Transform: Linear Operator

The Fourier transform is a *linear operator*. This means that

Theorem

If $f(x)$ and $g(x)$ are two functions with Fourier transforms $F(u)$ and $G(u)$, then the Fourier transform of $af(x) + bg(x)$ where a and b are constants is simply $aF(u) + bG(u)$.

Linear Operator Simple Example

Image 1

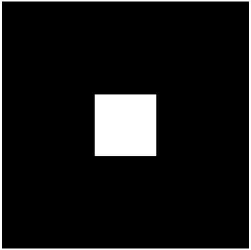


Image 2

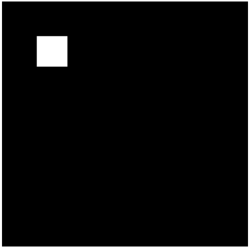
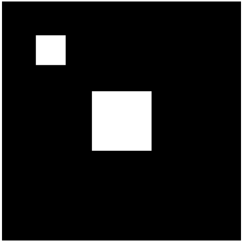
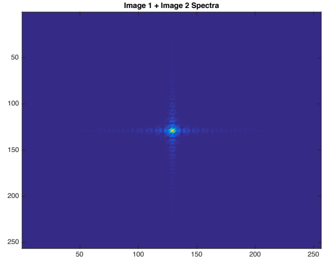
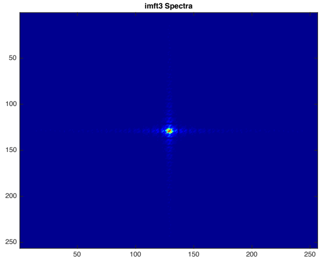


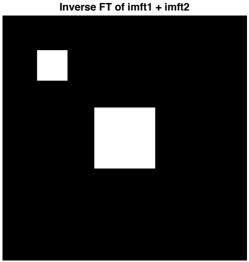
Image 3



Linear Operator Simple Example







Linear Operator Simple Example

Fourier Transform Linear Operator Demo: FFT_Linear.m

```
% Create a white box on a black background image
M= 256; N = 256;
image1 = zeros(M,N);
box = ones(64,64);
%box at centre
image1(97:160,97:160) = box;


figure(1)
imshow(image1)
title('Image 1');


% Create another  white box on a black background image
M= 256; N = 256;
image2 = zeros(M,N);
box = ones(32,32);
%box at centre
image2(37:68,37:68) = box;


figure(2)
imshow(image2)
title('Image 2');
```

Linear Operator Simple Example

Fourier Transform Linear Operator Demo: FFT_Linear.m

```
% Make composite image
image3 = image1 + image2;

figure(3)
imshow(image3)
title('Image 3');

% Compute Fourier Transforms.

imft1 = fft2(double(image1));
imft2 = fft2(double(image2));
imft3 = fft2(double(image3));

figure(4)
imagesc(abs(imft1 + imft2))
title('Image 1 + Image 2 Spectra');

figure(5)
imagesc(abs(imft3))
title('imft3 Spectra');

figure(6)
imagesc(abs(imft1 + imft2 - imft3))
title('Difference (Image 1 + Image 2) - imft3 Spectra');

figure(7)
imshow(ifft2(imft1 + imft2));
title('Inverse FT of imft1 + imft2');
```

Fourier Transform: Shifting

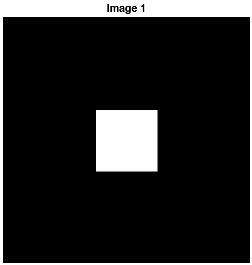
Shifting the real space data through a fixed distance has the effect that

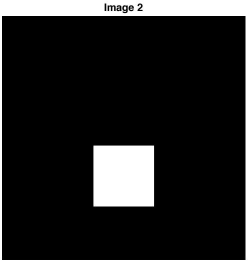
Theorem

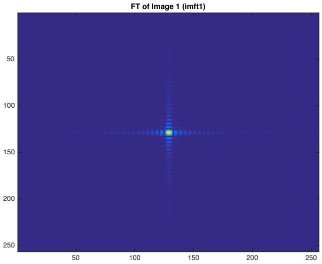
If $f(x)$ is a function with Fourier transform $F(u)$, then the Fourier transform of $f(x - x_0)$ is given by $e^{-2\pi i x_0 u} F(u)$.

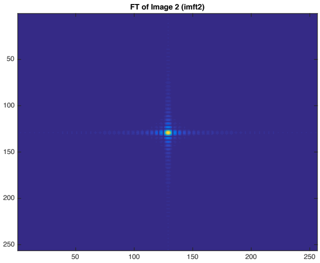
- **Note**: that the exponential term here has **unit** magnitude for all values of u.
  - Thus, the **magnitude** of the **resulting** Fourier transform is left **unchanged**, although the **relative** sizes of the real and imaginary parts are **altered**.

Shifting Operator Simple Example









Shifting Operator Simple Example

Inverse FT of imft3

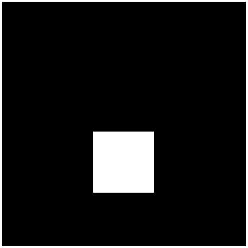
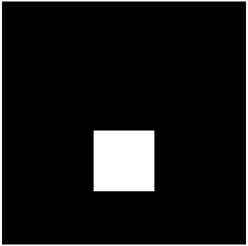
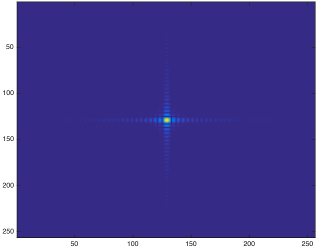


Image 2

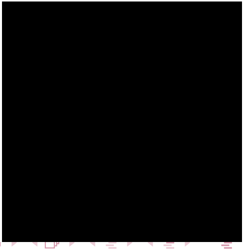


Both Images are the same.

FT of imft3 (imft1 shifted in Fourier Space



Residual of IFT of imft3 and image2



Shifting Operator Simple Example

Fourier Transform Shifting Operator Demo: FFT_Shift.m

```
% Define Initial box
x0 = 97;
y0 = 97;
xwidth = 64;
ywidth = 64 ;

% Create a white box on a black background image
M= 256; N = 256;
image1 = zeros(M,N);
box = ones(xwidth,ywidth);
%box at centre
image1(x0:x0+xwidth-1,y0:y0+ywidth-1) = box;

figure(1)
imshow(image1)
title('Image 1');

% Define shift
dx= 40;
dy= 0;

% Shift Image 1
image2 = zeros(M,N);
image2(x0+dx:x0+dx+xwidth-1,y0+dy:y0+dy+ywidth-1) = box;

figure(2)
imshow(image2)
title('Image 2');
```

Shifting Operator Simple Example

Fourier Transform Shifting Operator Demo: FFT_Shift.m

```
% Compute  Fourier Transfoms.
imft1 = fft2(double(image1));
imft2 = fft2(double(image2));

figure(3)
imagesc(abs(fftshift(imft1)));
title('FT of Image 1 (imft1)');

figure(4)
imagesc(abs(fftshift(imft3)));
title('FT of Image 2 (imft2)');

% Define shift in frequency domain
[yF,xF] = meshgrid(-M/2:M/2-1,-N/2:N/2-1);

% Perform the shift
imft3=imft1.*exp(-1i*2*pi.*(xF*dx+yF*dy)/256);

figure(5)
imagesc(abs(fftshift(imft3)));
title('FT of imft3 (imft1 shifted in Fourier Space');

figure(6)
imshow(ifft2(imft3));
title('Inverse FT of imft3');

figure(7)
imshow(ifft2(imft3)-image2);
title('Residual of IFT of imft3 and image2');
```

Fourier Transform: Scaling

If we scale the spacing of the real space data in distance, we have that

Theorem

If $f(x)$ is a function with Fourier transform $F(u)$, then the Fourier transform of $f(ax)$ where a is a real constant is given by $\frac{1}{|a|}F(\frac{u}{a})$.

In other words:

- if we **spread out** the data in **real space**, the data is **compressed** into more **compact** region of **Fourier space**.
- This is **intuitive** — if we **double** the spacing of a **grid pattern**, its **spatial frequency** is **halved**.
- **Note** that the **magnitude** of the Fourier representation is also **affected**.

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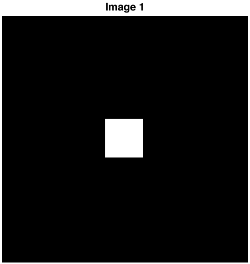
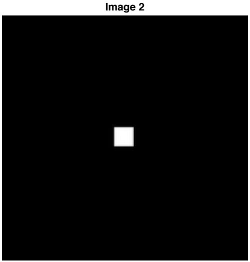
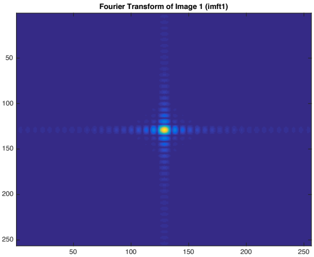
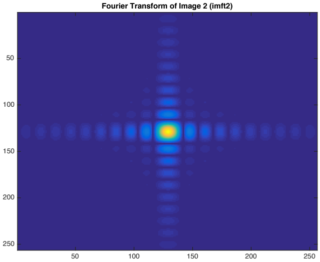
Scaling Operator Simple Example









Scaling Operator Simple Example

Fourier Transform Scaling Operator Demo: FFT_Scale.m

```
% Define Initial box
xwidth = 40;
ywidth = 40;

% Create a white box on a black background image
M= 256; N = 256;

x0 = M/2 - xwidth/2;
y0 = N/2 - ywidth/2;

image1 = zeros(M,N);
box = ones(xwidth,ywidth);
%box at centre
image1(x0:x0+xwidth-1,y0:y0+ywidth-1) = box;

% Define scale
scalex = 0.5;
scaley = 0.5;

% Image 2 = Scaled Image 1
image2 = zeros(M,N);
[px,qx] = rat(scalex);
[py,qy] = rat(scaley);
```

Scaling Operator Simple Example

Fourier Transform Scaling Operator Demo: FFT_Scale.m

```
in_image2 = resample(resample(double(image1),px,qx)',py,qy)';

xrange = int16(M/2-M*scalex/2:M/2+M*scalex/2-1);
yrange = int16(N/2-N*scaley/2:N/2+N*scaley/2-1);
image2(xrange,yrange)= in_image2;

figure(1)
imshow(image1);
title('Image 1');

figure(2)
imshow(image2);
title('Image 2');

% Compute  Fourier Transfoms.
imft1 = fft2(double(image1));
imft2 = fft2(double(image2));

figure(3)
imagesc(abs(fftshift(imft1)))
title('Fourier Transform of Image 1 (imft1)');

figure(4)
imagesc(abs(fftshift(imft2)))
title('Fourier Transform of Image 2 (imft2)');
```

Fourier Transform: Rotation

One useful property in two dimensions is that:

- if we **rotate** the **real space** data, its **Fourier transform** is **rotated** by the **same angle**, or more exactly

Theorem

If $f(x, y)$ is a function with Fourier transform $F(u, v)$, on expressing these functions in terms of polar coordinates r, θ, ρ, ϕ where $x = r\cos\theta, y = r\sin\theta, u = \rho\cos\phi, v = \rho\sin\phi$ so that $f(x, y)$ and $F(u, v)$ become $f(r, \theta)$ and $F(\rho, \phi)$ respectively, the Fourier transform of $f(r, \theta + \omega)$ where ω is a constant is given by $F(\rho, \phi + \omega)$.

- Basically this means the **two-dimensional** Fourier transform is an intrinsic property of the data which is **independent** of our choice of axis directions.

Rotation Operator Simple Example

Image 1

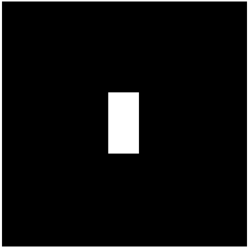
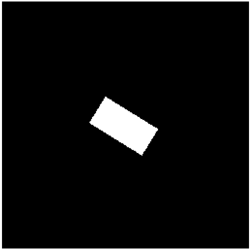
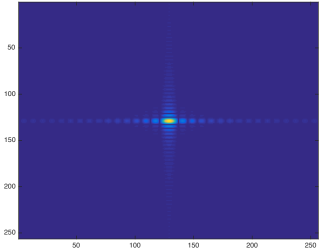


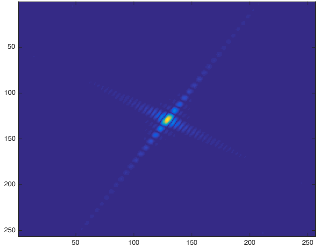
Image 2



Fourier Transform of Image 1 (imft1)



Fourier Transform of Image 2 (imft2)



Rotation Operator Simple Example

Fourier Transform Rotate Operator Demo: FFT_Rotate.m

```
% Define Initial box
xwidth = 64;
ywidth = 32;

% Create a white box on a black background image
M= 256; N = 256;
image1 = zeros(M,N);

x0 = M/2 - xwidth/2;
y0 = N/2 - ywidth/2;

box = ones(xwidth,ywidth);
%box at centre
image1(x0:x0+xwidth-1,y0:y0+ywidth-1) = box;

% Define Rotation in degrees
rot = 45;

image2 = imrotate(image1, rot,'bilinear','crop');

figure(1); imshow(image1);
title('Image 1');

figure(2); imshow(image2);
title('Image 2');
```

```
% Compute  Fourier Transfoms.
imft1 = fftshift(fft2(double(image1)));
imft2 = fftshift(fft2(double(image2)));


figure(3); imagesc(abs(imft1))
title('Fourier Transform of
       Image 1 (imft1)');

figure(4); imagesc(abs(imft2))
title('Fourier Transform of I
       mage 2 (imft2)');
```

Fourier Transform: Zeroth Component

A final property is that the **zeroth component** of the **Fourier space representation** is just the **average data value** (**apart** from a factor of $1/N$, in two dimensions – assuming balanced FT)). This is illustrated below for the two-dimensional DFT case:

$$F(u,v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) e^{-2\pi i (xu+yv)/N}, \tag{1}$$

so

$$F(0,0) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x,y) = N\bar{f}(x,y). \tag{2}$$

Simple Zeroth Component Example

Illustration of zeroth element, F(1) (1D) or F(1,1) (2D), being the mean of the data plus a factor N: FFT_zeroth.m

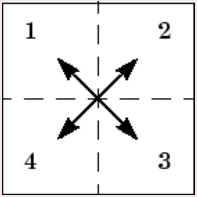
```
% 1D Example
y = 1:100
yft = fft(y);
yft(1)/100  % Zeroth Element of y
mean(y) % Mean of Data

% 2D Example
im = imread('cameraman.tif');

[N M] = size(im);

imft = fft2(im);

imft(1,1)/(N*M) % Zeroth Element of imft
mean(mean(im)) % Mean of Data
```

FT is not balanced in MATLAB

Corollary: Shifting the Fourier Transform, `fftshift()`

Centring the Frequency of a Fourier Transform

- Most computations of FFT represent the frequency from 0 — $N-1$ samples (similarly in 2D, 3D *etc.*) with corresponding frequencies ordered accordingly — the 0 frequency is not really the **centre**.
- We frequently like to visualise the FFT as the **centre of the spectrum**.
- In 1D (Audio/Vector): **swaps the left and right halves of the vector**
- Similarly in 2D (Image/Matrix) we swap the first quadrant with the third and the second quadrant with the fourth:



The fftshift() MATLAB Command

help fftshift()

Y = fftshift(X) rearranges the outputs of **fft**, **fft2**, and **fftn** by **moving** the zero-frequency component to the **center of the array**.

It is useful for **visualising** a Fourier transform with the zero-frequency component in the **middle** of the spectrum.

For **vectors**, **fftshift(X)** **swaps** the **left** and **right** halves of X.

For **matrices**, **fftshift(X)** swaps the **first** quadrant with the **third** and the **second** quadrant with the **fourth**.