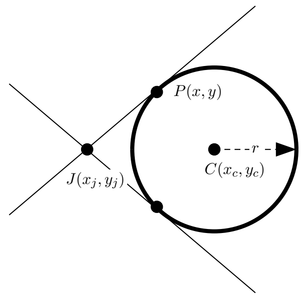
Proof of an implicit line equation through a point (not on the circle) that is tangent to a general circle

Derivation of an equation that calculates the the general equation of an implicit line through a point (not on the circle) that is tangent to a general circle:



We wish to find the implicit equation of the tangent

$$ax + by + c = 0$$

The coefficients a and b are obtained from:

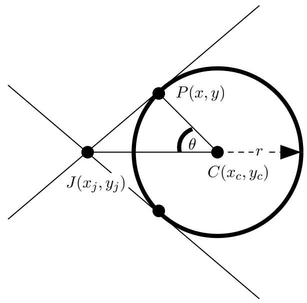
$$\begin{aligned} a &= \frac{\mp r(x_c - x_j) - (y_c - y_j)\sqrt{(x_c - x_j)^2 + (y_c - y_j)^2 - r^2}}{(x_c - x_j)^2 + (y_c - y_j)^2} \\ b &= \frac{\mp r(y_c - y_j) + (x_c - x_j)\sqrt{(x_c - x_j)^2 + (y_c - y_j)^2 - r^2}}{(x_c - x_j)^2 + (y_c - y_j)^2} \end{aligned} \tag{1}$$

c then obtained from the fact that the tangent passes through J:

$$c = -ax_j - by_j$$

See http://www.netsoc.tcd.ie/~jgilbert/maths_site/applets/circles/tangents_to_circles.html for a java applet demo.

Proof:



We know the coordinates of the circle (x_c, y_c) and its radius, r.

We also know the coordinates of the general point, $J(x_j, y_j)$.

We can therefore form a vector $\mathbf{t}$ between C and J, $\mathbf{t} = (x_t, y_t)$ where $x_t = x_c - x_j$ and $y_t = y_c - y_j$ (This is essentially using variable substitution to ease the following algebra)

We can also define a vector $\mathbf{n}$ which is from the centre C to the point of intersection $P(x, y)$ of the tangent with the circle.

By definition this vector $\mathbf{n}$ is perpendicular the the tangent as all tangents to a circle are perpendicular to a line from the center.

$\mathbf{n}$ therefore gives the parameters for the a and b coefficients of the implicit line equation of the tangent. So we may write $\mathbf{n} = (a, b)$.

Let us assume that $\mathbf{n}$ is a normal vector then $\|\mathbf{n}\| = 1$ and

$$a^2 + b^2 = 1 \tag{2}$$

Define the angle between $\mathbf{n}$ and $\mathbf{t}$ as θ.

Using the Cauchy-Schwarz inequality we may write the angle between

$$\begin{aligned} \cos\theta &= \frac{\mathbf{n.t}}{\|n\|\|t\|} \\ &= \frac{ax_t + by_t}{\|t\|} \end{aligned} \tag{3}$$

Now there also exist a right-hand triangle formed by points C,P and J. (Right angle at P).

So we may write (by simple trigonometry):

$$\cos\theta = \frac{r}{\|t\|} \tag{4}$$

So equating $\cos\theta$ in Eqns. 3 and 4 we get:

$$\begin{aligned} \frac{ax_t + by_t}{\|t\|} &= \frac{r}{\|t\|} \\ ax_t + by_t &= r \end{aligned} \tag{5}$$

From Eqn. 2 we may write a as:

$$a = \sqrt{1 - b^2} \tag{6}$$

Substituting for a from Eqn. 6 in Eqn. 5, we get:

$$\begin{aligned} \sqrt{1 - b^2}(x_t) + by_t &= r \\ \sqrt{1 - b^2}(x_t) &= r - by_t \end{aligned} \tag{7}$$

Squaring both sides of the equation we get:

$$\begin{aligned} (1 - b^2)x_t^2 &= r^2 - 2ry_tb + y_t^2b^2 \\ x_t^2 - x_t^2b^2 &= r^2 - 2ry_tb + y_t^2b^2 \\ r^2 - 2rby_t + y_t^2b^2 - x_t^2 + x_t^2b^2 &= 0 \\ (x_t^2 + y_t^2)b^2 - 2ry_tb + r^2 - x_t^2 &= 0 \end{aligned} \tag{8}$$

Now Eqn. 8 is a quadratic in b which may be solved as:

$$\begin{aligned} b &= \frac{2ry_t \pm \sqrt{4r^2y_t2 - 4(x_t^2 + y_t^2)(r^2 - x_t^2)}}{2(x_t^2 + y_t^2)} \\ &= \frac{ry_t \pm \sqrt{r^2y_t2 - (x_t^2 + y_t^2)(r^2 - x_t^2)}}{(x_t^2 + y_t^2)} \end{aligned} \tag{9}$$

Let us now simplify the square root part of Eqn. eqn:bsol:

$$\begin{aligned}
r^2y_t2-(x_t^2+y_t^2)(r^2-x_t^2) &= r^2y_t2-(x_t^2r^2-x_t^4+r^2y_t^2-y_t^2x_t^2) \\
&= r^2y_t2-x_t^2r^2+x_t^4-r^2y_t^2+y_t^2x_t^) \\
&= -x_t^2r^2+x_t^4+y_t^2x_t^2 \\
&= x_t^2(x_t^2+y_t^2-r^2)
\end{aligned} \tag{10}$$

So putting this back in Eqn. 9 we get

$$\begin{aligned}
b &= \frac{ry_t \pm \sqrt{x_t^2(x_t^2+y_t^2-r^2)}}{x_t^2+y_t^2} \\
&= \frac{ry_t \pm x_t\sqrt{x_t^2+y_t^2-r^2}}{x_t^2+y_t^2}
\end{aligned} \tag{11}$$

If we now substitute for $x_t = x_c - x_j$ and $y_t = y_c - y_j$ we get the required solution for b in Eqn. 1.

Well some rearranging of the $\pm$ sign (since as we are using vectors we can have only need to preserve the respective signs of the vector we can rearrange to get the formula as given in Programmer's Geometry Text Book (Page 29):

$$b = \frac{\mp r(y_c-y_j)+(x_c-x_j)\sqrt{(x_c-x_j)^2+(y_c-y_j)^2-r^2}}{(x_c-x_j)^2+(y_c-y_j)^2}$$

If we now take b from Eqn. 11 and substituting $d = \sqrt{x_t^2+y_t^2-r^2}$ and $j = x_t^2+y_t^2$ we may write b as

$$b = \frac{ry_t \pm x_t d}{j} \tag{12}$$

substitute for b in Eqn. 5

$$\begin{aligned}
ax_t + (\frac{ry_t \pm x_t d}{j})y_t = r \\
ax_t+ = r - (\frac{ry_t \mp x_t d}{j})y_t \\
ax_t+ = \frac{rj - ry_t^2 \mp x_t y_t d}{j}
\end{aligned}$$

$$a = \frac{rj - ry_t^2 \mp x_t y_t d}{j x_t} \tag{13}$$

Substitue for j in top line (and then divide out the x_t and then sub for $x_t, y_t, d$and j (bottom line):

$$\begin{aligned} a &= \frac{rxt^2 + ry_t^2 - ry_t^2 \mp x_t y_t d}{j x_t} \\ a &= \frac{rxt \mp y_t d}{j} \\ a &= \frac{rxt \mp y_t \sqrt{x_t^2 + y_t^2 - r^2}}{x_t^2 + y_t^2} \end{aligned} \tag{14}$$

We now have the required solution for a in Eqn 1.

Well some rearranging of the $\pm$ sign (since as we are using vectors we can have only need to preserve the respective signs of the vector we can rearrange to get the formula as given in Programmer's Geometry Text Book (Page 29):

$$a = \frac{\mp r(x_c - x_j) - (y_c - y_j)\sqrt{(x_c - x_j)^2 + (y_c - y_j)^2 - r^2}}{(x_c - x_j)^2 + (y_c - y_j)^2}$$

c then obtained from the fact that the tangent passes through J:

$c = -ax_j - by_j$

Q.E.D.