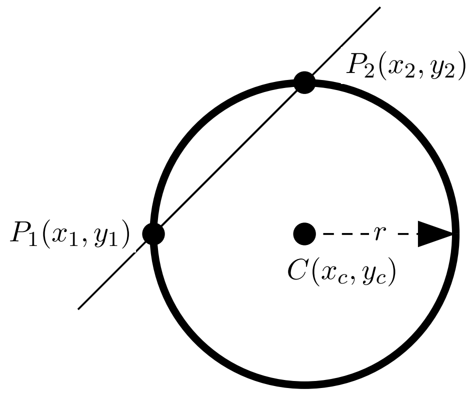
Proof of Intersection of a Line and a Circle



This problem is most easily solved if the circle is in implicit form:

$$(x - x_c)^2 + (y - y_c)^2 - r^2 = 0$$

and the line is parametric:

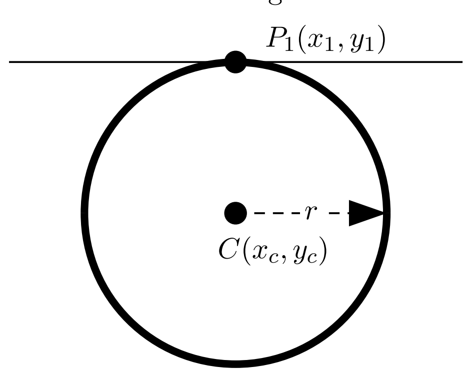
$$\begin{aligned} x &= x_0 + ft \\ y &= y_0 + gt \end{aligned}$$

Substituting for (parametric line) x and y into the circle equation gives a quadratic equation in t:

- Two roots of which give points on the line where cuts the circle.

$$t = \frac{f(x_c - x_0) + g(y_c - y_0) \pm \sqrt{r^2(f^2 + g^2) - (f(y_c - y_0) - g(x_c - x_0))^2}}{(f^2 + g^2)}$$

- The roots maybe ***coincident*** if the line is tangential to the circle.



- If roots are ***imaginary*** then there is no intersection.

Proof:

The circle is in implicit form:

$$(x - x_c)^2 + (y - y_c)^2 - r^2 = 0 \tag{1}$$

and the line is parametric:

$$\begin{aligned} x &= x_0 + ft \\ y &= y_0 + gt \end{aligned} \tag{2}$$

Substituting for (parametric line) x and y from Eqn. 2 into the circle equation, Eqn. 1, gives a quadratic equation in t gives:

$$(x_0 + ft - x_c)^2 + (y_0 + gt - y_c)^2 - r^2 = 0 \tag{3}$$

Let

$$\begin{aligned} x_d &= x_0 - x_c \\ y_d &= y_0 - y_c \end{aligned} \tag{4}$$

Then we may write Eqn. 3

$$(x_d + ft)^2 + (y_d + gt)^2 - r^2 = 0 \tag{5}$$

Expanding the quadratic parts in Eqn. 5 gives:

$$x_d^2 + 2fx_dt + f^2t^2 + y_d^2 + 2gy_dt + g^2t^2 - r^2 = 0 \tag{6}$$

Express Eqn. 6 as a quadratic in t, so gather terms in t in Eqn. 6:

$$\begin{aligned} (f^2 + g^2)t^2 + (2fx_d + 2gy_d)t + x_d^2 + y_d^2 - r^2 &= 0 \\ (f^2 + g^2)t^2 + 2(fx_d + gy_d)t + x_d^2 + y_d^2 - r^2 &= 0 \end{aligned} \tag{7}$$

Now the general solution of a quadratic of the form $ax^2 + bx + c = 0$, gives the roots, $x_i, i = 1, 2$ as:

$$x_i = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{8}$$

So in our problem (a quadratic in t) from Eqn. 7 we can see that for Eqn. 8:

$$\begin{aligned} a &= (f^2 + g^2) \\ b &= 2(fx_d + gy_d) \\ c &= x_d^2 + y_d^2 - r^2 \end{aligned}$$

Substituting for a, b and c in Eqn. 8 we get roots of t, t_i:

$$\begin{aligned} t_i &= \frac{-2(fx_d + gy_d) \pm \sqrt{(2(fx_d + gy_d))^2 - 4(x_d^2 + y_d^2 - r^2)(f^2 + g^2)}}{2(f^2 + g^2)} \\ t_i &= \frac{-2(x_d + gy_d) \pm \sqrt{4(fx_d + gy_d)^2 - 4(x_d^2 + y_d^2 - r^2)(f^2 + g^2)}}{2(f^2 + g^2)} \end{aligned} \tag{9}$$

The expression of the right hand side of Eqn. 9 can be simplified in a few ways.

Firstly we can factor 4 from the $\sqrt{}$ term and the division by 2 then simplifies the equation to:

$$\begin{aligned} t_i &= \frac{-2(x_d + gy_d) \pm 2\sqrt{(fx_d + gy_d)^2 - (x_d^2 + y_d^2 - r^2)(f^2 + g^2)}}{2(f^2 + g^2)} \\ t_i &= \frac{-(fx_d + gy_d) \pm \sqrt{(fx_d + gy_d)^2 - (x_d^2 + y_d^2 - r^2)(f^2 + g^2)}}{(f^2 + g^2)} \end{aligned} \tag{10}$$

Let us now consider the $\surd$ term and simplify this, let:

$$
\begin{aligned}
S &= (fx_d + gy_d)^2 - (x_d^2 + y_d^2 - r^2)(f^2 + g^2)) \\
S &= f^2x_d^2 + 2fgx_dy_d + g^2y_d^2 - f^2x_d^2 - f^2y_d^2 + f^2r^2 - g^2x_d^2 - g^2y_d^2 + g^2r^2
\end{aligned} \tag{11}
$$

Eqn 11 can be simplified as follows, some terms cancel out $f^2x_d^2$, $g^2y_d^2$ and others can be gathered together r^2, f, and g terms:

$$
\begin{aligned}
S &= r^2(f^2 + g^2) - (f^2y_d^2 - 2fgx_dy_d + g^2x_d^2 \\
S &= r^2(f^2 + g^2) - (fy_d - gx_d)^2 \\
S &= r^2(f^2 + g^2) - (-gx_d + fy_d)^2
\end{aligned} \tag{12}
$$

Substituting for x_d and y_d from Eqn. 4 in Eqn 12 we get:

$$
\begin{aligned}
S &= r^2(f^2 + g^2) - (-g(x_0 - x_c) + f(y_0 - y_c))^2 \\
S &= r^2(f^2 + g^2) - (g(x_c - x_0) - f(y_c - y_0))^2
\end{aligned}
$$

Substituting for S from Eqn. 13 and x_d and y_d from Eqn. 4 in the $\surd$ term in Eqn. 10 gives us the solution:

$$
t = \frac{f(x_c - x_0) + g(y_c - y_0) \pm \sqrt{r^2(f^2 + g^2) - (f(y_c - y_0) - g(x_c - x_0))^2}}{(f^2 + g^2)}
$$

Q.E.D