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date: 24.11.2024

Exercise 1+2 ODE Approximation

In this exercise we approximate numerically the differential equation

$$\frac{dV}{dt} = 1 - V - t$$

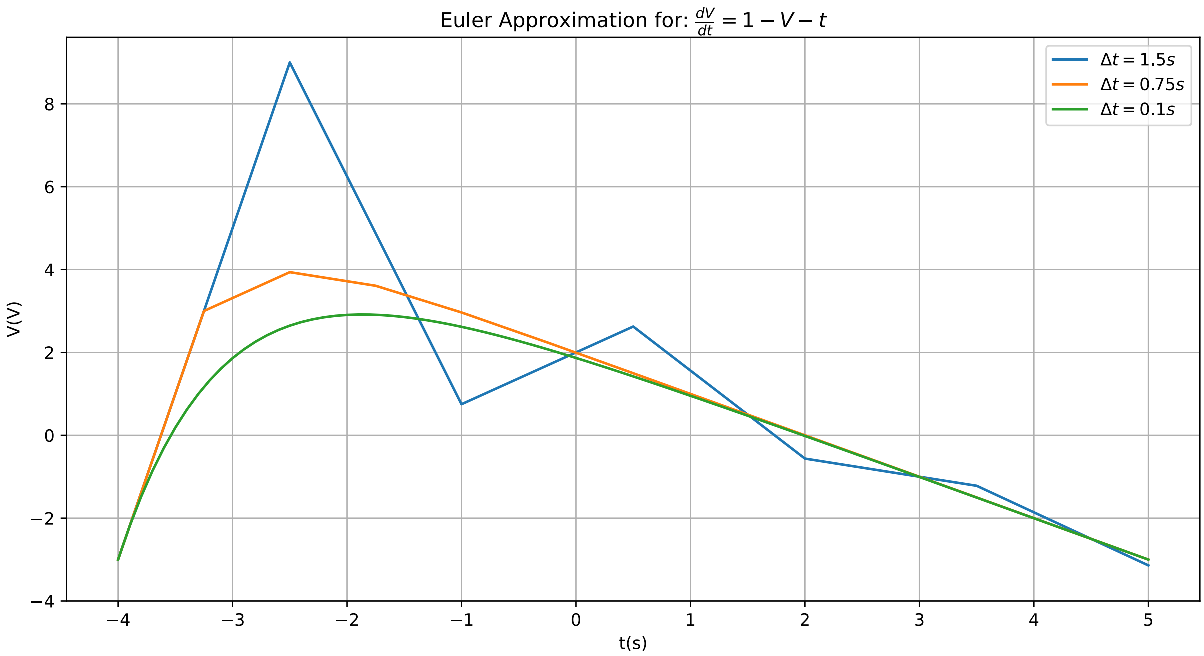
with $V_0 = V(t = 4) = -3V$
using three different methods.

Euler Method

the Euler method is a computationally simple way to approximate first order differential equations numerically.
For a first order ODE of form $\frac{dy}{dx} = F(x, y)$ we can use the following formula

$$y_n = y_{n-1} + step \cdot F(x_{n-1}, y_{n-1})$$

where step is a constant equal to $x_n - x_{n-1}$ and $F(x_0) = y_0$ are the start parameters.
Applying this to our differential equation we get this plot:



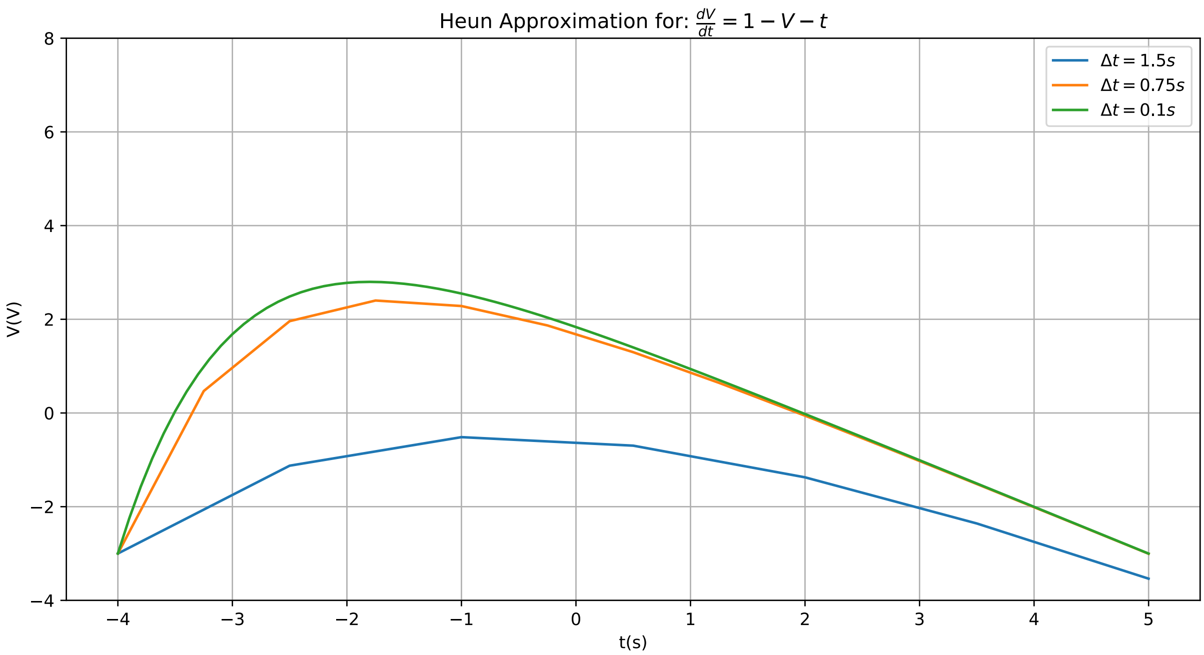
We see in the plot a much better approximation and these approximations converging the smaller the step size.
One potential problem with the Euler method is if

the changes of y between x_{n-1} to x_n are smaller than the step size, the method diverges from the solution, which we can obversve for $\Delta t = 1.5s$.

Heun Method

The Heun method is a slight improvement to the Euler method, taking not just the slope at x_{n-1} but also the slope at x_n into consideration to get the next y_n and averaging them. This makes it less susceptible to diverging as mentioned in the Euler method.

$$y_{euler} = y_{n-1} + step * F(x_{n-1}, y_{n-1})$$
$$y_n = y_{n-1} + \frac{h}{2}(F(x_{n-1}, y_{n-1}) + F(x_n, y_{euler}))$$

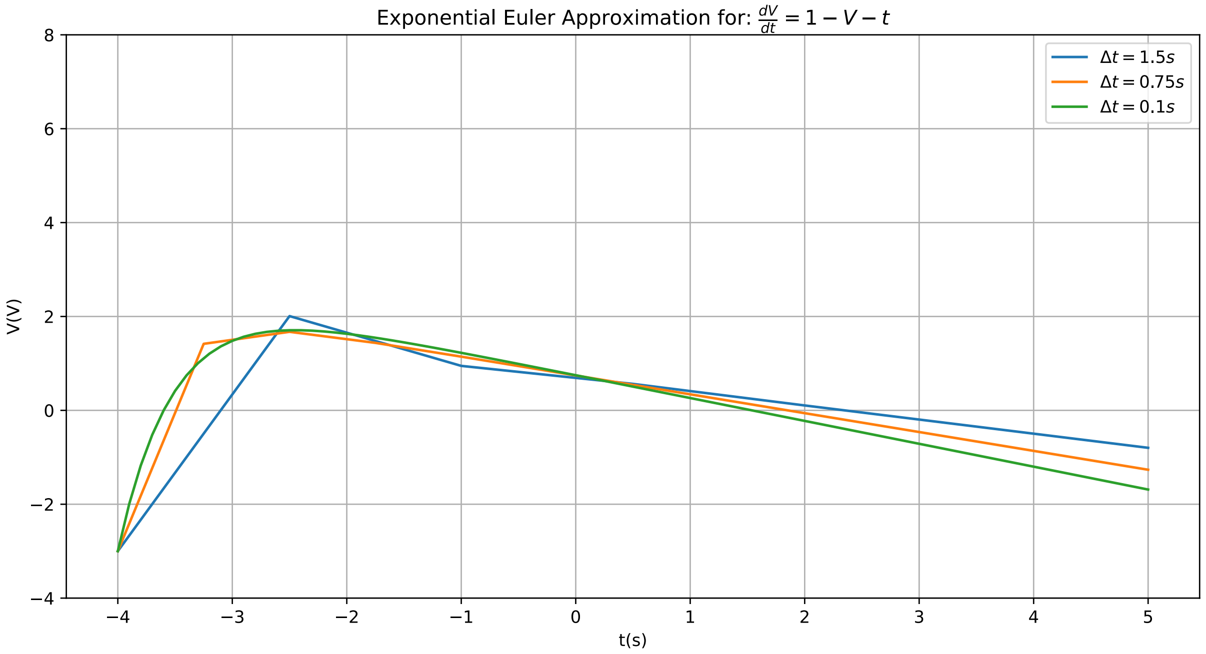


As we see in the plot, the functions converge faster and $\Delta t = 1.5s$ doesn't diverge at the start.

Exponential Euler

I couldn't find an Exponential Euler method online thus I made up my own:

$$y_n = e^{-step} + step \cdot e^{-step} \cdot F(x_{n-1}, y_{n-1})$$



This seems to converge fast at the start but diverge/increased error the further the bigger the x

Conclusion

The Heun method seems to be the best overall of the three as it addresses the problems from the Euler method while not being really more expensive.
At least in the solution the exponential Euler looks like it converges worse for at the end of the plot but much stronger at the start, but I couldn't replicate the formula. In my exponential euler alternative, there seems to be a strong divergence the longer the we follow the t .

it can be proven that Euler has an $O(step)$ error rate(source: 1) and Heun an $O(step^2)$ error rate(source: 2). This means that when $step$ gets halved, the error is halved for euler but only a quarter for heun. Thus heun is a much better error rate.

Exercise 3 LIF Neuron

A Leaky Integrate Fire Neuron is a simple, yet effective model to model a neuron.

$$V_{n+1} = \begin{cases} V_n + \frac{\Delta t}{C_m}(-g_{\text{leak}}(V_n - V_{\text{rest}}) + I_{\text{input}}(t_n)) & \text{if } V_n < V_{\text{thr}} \\ V_{\text{spike}} & \text{if } V_{\text{thr}} \leq V_n < V_{\text{spike}} \\ V_{\text{rest}} & \text{if } V_{\text{spike}} \leq V_n \end{cases}$$

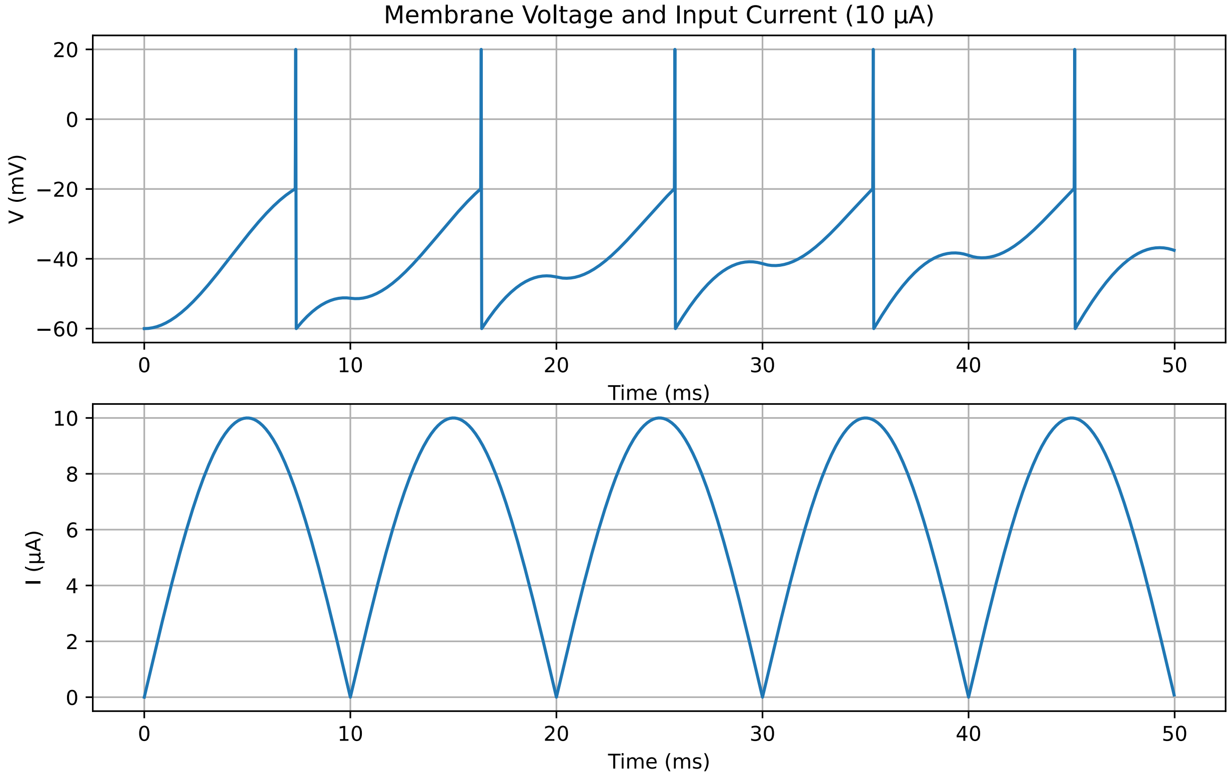
with:

- V_n : cell membrane voltage at $t = n$
- $C_m = 1\mu F$:cell membrane capacitance
- $g_{leak} = 100\mu S$: cell membrane leak conductivity
- $V_{rest} = -60mV$: cell membrane resting voltage

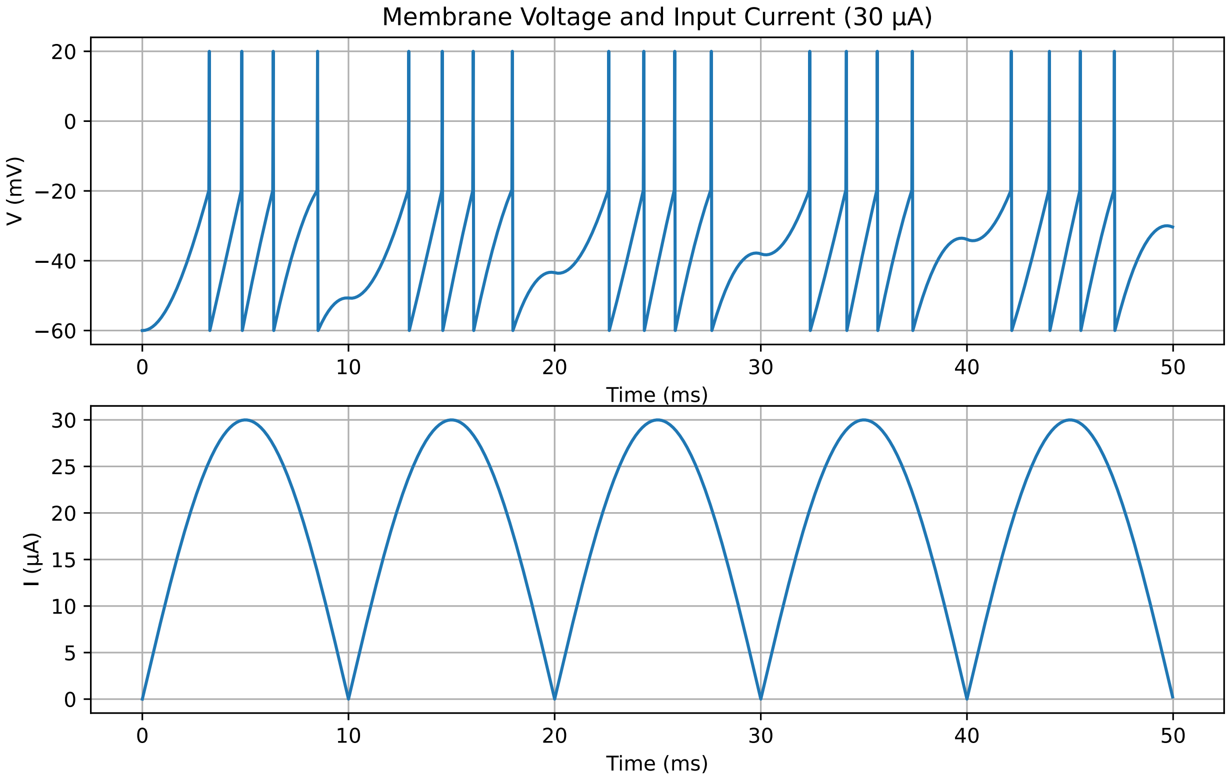
- V_{thr} = -20mV : cell membrane spiking threshold voltage
- V_{spike} = 20mV : spike voltage

in both plots we can see that the resting voltage is at $-60mV$ and the spike goes to V_{spike} in case of a spike, after an archived threshold at V_{thr}.

For our input current we use the absolute sinus with frequency $50Hz$ and one with $10\mu A$ and one with $30\mu A$ amplitude



in this graph the aspects o leaking, integrating and firing are clearly visible.
At the start and after the spike we observe a V of $0mV$. The way the V increases is also with the "speed" of the value of $I(t)$, thus it's called integrating. After archiving the spike at V_{thr} we $V = V_{rest}$ again.
If V is not 0 but I is 0 (for every $t = 10 * n, n \neq 0$) we observe the leaking, where the voltage falls off.



for $I = 30\mu A$ we observe almost the same graph as for $10\mu A$ but with the difference of rapid repeat firing. This happens as I is very high and the C_m gets "recharged" fast, allowing more spikes with within the same interval, also called a "spike train", which is also observed in the real world (source: 3)

Sources

- (1)[https://en.wikipedia.org/wiki/Euler_method#Global truncation error]
- (2) [https://ece.uwaterloo.ca/~dwharder/nm/Lecture_materials/pdfs/7.1.1.1.2%20Heun's%20method.pdf] page 20
- (3) [https://en.wikipedia.org/wiki/Action_potential]