Notes on James R. Munkres' Topology (2E)

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Contents

Chapter 0

Structure and reading plans

Ch 1-8 is the part I, mainly for common topology. The part II includes ch 9-14, that depends on ch 1-4, is about algebraic topology.

My plan is to read through ch 1-4 very quickly, within a weekend, and then I will start reading ch 9+ simultaneously with W.S.Massey's Algebraic topology: An induction.

Finally I wish I could finish all ch 1-8 and also some parts after ch 9.

Chapter 1

Set theory and logic

Definition 1.1. Order relation rel C on set A is called *order relation* if

1. comparability, *a.k.a.* totality for all non-eq elements, *i.e.* $\forall x, y \in A, x \neq y \Rightarrow xCy \vee yCx$
2. non-refl, *i.e.* $\forall x, \neg(xCx)$
3. trans, *i.e.* $\forall xCy \wedge yCz, xCz$

(*a.k.a.* **Linear order**, **Simple order**)

Remark. *This relation is not the same as Linear order on Wikipedia (link)*[1]*. This order is actually the strict version of the Total order on wikipedia, i.e. has non-refl property.*

Definition 1.2. Open interval if X is a set and $<$ is an order rel, and if $a < b$ we use notation (a, b) to denote $\{x \in X \mid a < x < b\}$, called *open interval.*

If $(a, b) = \varnothing$, then a is called *immediate precessor* of b and b called *immediate successor* of a.

Remark. *It makes more sense on X is a discrete set. Since if (a, b) is an open interval in $\mathbb{R}$, $(a, b) = \varnothing \Rightarrow a = b$ which makes no sense on a as an immediate precessor of b.*

Definition 1.3. Order type if A and B are two sets with $<_A$ and $<_B$. We say that A and B have same *order type* if $\exists f : A \to B$ that preserves order, *i.e.*

$$a_1 <_A b_1 \Rightarrow f(a_1) <_B f(b_1)$$

Remark. *It's just a generalization of monotone function.*

Definition 1.4. Dictionary order relation if A,B are two sets with ($<_A, <_B$), defn an order for $A \times B$ by defining

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$, or if $a_1 = a_2 \ \wedge \ b_1 <_B b_2$.

[1] `https://en.wikipedia.org/wiki/Total_order`

Definition 1.5. LUB property/GLB property For A and $<_A$, we say A has *LUB property* if

$$\forall A_0 \subset A, A \neq \varnothing \wedge \exists \text{upper bound for } A_0 \Rightarrow \exists \text{lub}\{A_0\} \in A$$

Example 1.5.1. $A = (-1, 1)$. *e.g.* $X = \left\{1 - \frac{1}{n} \mid n \in \mathbb{Z}^+\right\}$ does not have an upper bound, thus vacuously true. $\left\{-\frac{1}{n} \mid n \in \mathbb{Z}^+\right\}$ has upper bound of any number in $[0, 1) \subset A$, and $\text{lub}(X) = 0 \in (-1, 1)$.

Example 1.5.2. Counterexample. $A = (-1, 0) \cup (0, 1)$. $\left\{-\frac{1}{n} \mid n \in \mathbb{Z}^+\right\}$ has upper bound of any $(0, 1) \subset A$, while $\text{lub}(X) = 0 \notin A$.

Remark. *The completeness property of* $\mathbb{R}$ *as an axiom derives this property.*

Property 1.6. $\mathbb{R}$ field

Algebraic properties

1. assoc: $(x + y) + z = x + (y + z); (xy)z = x(yz)$
2. comm: $x + y = y + x; xy = yx$
3. id: $\exists!0, x + 0 = x$; $\exists!1, x \neq 0 \Rightarrow x1 = x$
4. inv: $\forall x, \exists!y, x + y = 0$; $\forall x \neq 0, \exists!y, xy = 1$
5. distr: $x(y + z) = xy + xz$

Mixed algebraic and order property

6. $x > y \Rightarrow x + z > y + z$; $x > y \wedge z > 0 \Rightarrow xz > yz$

Order properties

7. $<$ has LUB property
8. $\forall x < y, \exists z, x < z \wedge z < y$

1-6 make $\mathbb{R}$ a field. 1-6 + 7 make $\mathbb{R}$ an ordered field. 7-8 makes $\mathbb{R}$, called by topologists, a **Linear continuum**.

Theorem 1.7. Well ordering property $\mathbb{Z}^+$ has *Well-ordering property. i.e.* Every nonempty subset of $\mathbb{Z}^+$ has a smallest element.

Proof. We first prove that for each $n \in \mathbb{Z}^+$, the following statement holds: Every nonempty subset of $\{1, \ldots, n\}$ has a smallest element.

Let A be the set of all postive integers n for which this theorem holds. Then A contains 1, since if $n = 1$, the only possible subset is $\{1\}$ itself. Then suppose A contains n, we show that it contains $n + 1$. So let C be a nonempty subset of the set $\{1, \ldots, n + 1\}$. If C consists of the single element $n + 1$, then that element is the smallest element of C. Otherwise, consider the set $C \cap \{1, \ldots, n\}$, which is nonempty. Because $n \in A$, this set has a smallest element, which will automaticallly be the smallest element of C also. Thus A is inductive, so we conclude that $A = \mathbb{Z}^+$; hence the statement is true for all $n \in \mathbb{Z}^+$.

Now we prove the theorem. Suppose that D is a nonempty subset of $\mathbb{Z}^+$. Choose an element n of D. Then the set $A = D \cap [n]$ is nonempty, so that A has a smallest element k. The element k is automatically the smallest element of D as well. $\square$

Remark. *I don't really understand the second part of this proof. By* https://proofwiki.org*, Principle of Mathematical Induction, Well-Ordering Principle, and Principle of Complete Induction are logically equivalent. (link)*[2]

Definition 1.8. Cartesian product Let $\{A_1, \ldots, A_m\}$ be a faimily of sets indexed with the set $\{1. \ldots, m\}$. Let $X = A_1 \cup \cdots \cup A_m$. We define *cartesian product* of this indexed family, denoted by

$$\prod_{i=1}^{m} A_i \text{ or } A_1 \times \cdots \times A_m,$$

to be the set of all m-tuples $(x_1, \ldots, x_m)$ of elements of X such that $x_i \in A_i$ for each i.

Remark. *Indexing function $f : J \to \mathcal{A}$ is surjective but not necessarily injective.*

Definition 1.9. ω-tuple An *ω-tuple* of elements of set X to be a function

$$x : \mathbb{Z}^+ \to X,$$

a.k.a. sequence, or a *infinite sequence.*

Theorem 1.10. $\{0,1\}^\omega$ is uncountable. (let $X = \{0,1\}$ in the proof.)

Proof. We show that given any function $g : \mathbb{Z}^+ \to X^\omega$, g is not surjective. Four this purpose, let us denote $g(n)$ as $(x_{n1}, x_{n2}, \ldots, x_{nn}, \ldots, x_{n\omega})$, where each x_{ij} is eather 0 or 1. Then we define any element $y = (y1, \ldots, y_\omega)$ of X^ω by letting

$$y_n = \begin{cases} 0 & \text{if } x_{nn} = 1, \\ 1 & \text{if } x_{nn} = 0. \end{cases}$$

y will differ $g(n)$ for all n by a digit. Therefore $y \notin \operatorname{Im}(g)$. $\square$

Remark. *Note this proof is similar to the proof of uncountableness of $[0,1)$ using the vast digit array.*

Remark. *$\{0,1\}^\omega \simeq [0,1)$ by $f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i 2^{-i}$. (i.e. binary decimals). Then we can use the conclusion of the uncountableness of $[0,1)$ to prove this directly.*

[2] `https://proofwiki.org/wiki/Equivalence_of_Well-Ordering_Principle_and_Induction#Final_assembly`

Remark. *Think of picking a subset of $\mathbb{Z}^+$, for each $i \in \mathbb{Z}^+$ present in the subset, set $a_i = 1$, otherwise $a_i = 0$. Thus $\{0,1\}^\omega$ is just isomorphic to the power set $2^{\mathbb{Z}^+}$. By cantor's theorem, there is not surjection $f : \mathbb{Z}^+ \to 2^{\mathbb{Z}^+}$.*

Theorem 1.11. There is not surjective map $g : A \to 2^A$ for all set A. Proof: (link)[3]

Theorem 1.12. Principle of recursive definition Let A be a set; let a_0 be an element of A. Suppose ρ is a function that assigns, to each function f mapping a nonempty section of the positive integers into A, an element of A. Then there exists a unique function

$$h : \mathbb{Z}^+ \to A$$

such that

$$\begin{aligned} h(1) &= a_0, \\ h(i) &= \rho(h|\{1, \ldots, i-1\}) \text{ for } i > 1 \end{aligned}$$

The formula is called a *recursion formula* for h.

Remark. *I'm not very clear about this definition. I think the point of this definition is to indicate that there is a UNIQUE function satisified a recursive definition.*

Theorem 1.13. The following statements about set A are equivalent:

1. There exists an *injective*, not necessarily surjective (of course), function $f : \mathbb{Z}^+ \to A$.
2. There exists a bijection of A to a propert subset of A.
3. A is infinite.

Example 1.13.1.

1. $f : \mathbb{Z}^+ \hookrightarrow \mathbb{R}$.
2. $f : \mathbb{R} \to (-1, 1)$ by $f(0) = 0, f(a) = \frac{1}{a}$.
3. $\mathbb{R}$ is infinite.

Definition 1.14. Axiom of choice Given a collection $\mathcal{A}$ of disjoint nonempty sets (*i.e.* $\forall x, y \in \mathcal{A}, x \cap y = \varnothing$), there exists a set C consisting of exactly one element from each element of $\mathcal{A}$; that is, $C \subset \bigcup \mathcal{A}$ and for each $A \in \mathcal{A}$, the set $C \cap A$ contains a single element.

The set C can be thought of as having been obtained by choosing one element from each of the sets in $\mathcal{A}$.

Theorem 1.15. Well-ordering theorem Every set is well-orderable. (assuming axiom of choice)

[3] `https://proofwiki.org/wiki/Cantor's_Theorem`

Remark. *This theorem is rather against intuition. As I checked briefly on wikipedia and proofwiki, the proof (link)*[4] *is done using the axiom of choice and the Princple of Transfinite Induction. I can't comprehend the proof since I have no related knowledge about transfinite induction schema. The result is also known as* ***Zermelo's Theorem****(link)*[5]*, which is usually worded: Every set of cardinals is well-ordered with respect to $\leq$. This is weird to me since doesn't it imply that cardinals are equivalent to ordinals?*

Definition 1.16. Strict partial order A set A, a relation $\prec \subset A \times A$ is a *strict partial order* if for all elements $a, b, c \in A$,

1. Non-refl: $\neg(x \prec x)$
2. Trans: $x \prec y \wedge y \prec z \Rightarrow x \prec z$

Remark. *Compare to (non-strict) partial order, strict partial order does not have reflexivity property ($x \leq x$).*

Remark. *Compare to a simply order relation, this order relation does not require totality (or comparability).*

Theorem 1.17. The maxium principle A set A, a strict partial order $\prec$ on A. There exists a maxiaml simply ordered subset B.

Said differently, there exists a subset B of A such that B is simply ordered by $\prec$ and such that no subset of A that properly contains B is simply ordered by $\prec$.

Remark. *Think in this way. Draw a acyclic directed graph for A: for for an element in A draw a vertex, for each pair of vertices, draw an edge from a to b iff $a \prec b$ is minimal, i.e. $\neg\exists c : a \prec c \prec b$. Then we can say $a \prec b$ if a is connected to b by the transitivity property. This thereom actually says that there exists a maximal path in the graph.*

Definition 1.18. Upper bound and **Maximal element** Let A be set and $\prec$ be strict partial order on A. $B \subset A$. An *upper bound* on B is $c \in A$ such that $\forall b \in B : b = c \vee b \prec c$. A *maximal element* of A is an element $m \in A$ such that $\neg\exists a, m \prec a$.

Remark. *When we talk about upper bounds of A, we are under implication of a subset A of a ordered set.*

Theorem 1.19. Zorn's Lemma Let A be strictly partially ordered set. If every simply ordered subset of A has an upper bound in A, then A has a maximal element.

Remark. *This is a consequence of the maximum principle. It's easy to verify the maximum element is the upper bound of the maximal simply ordered subset.*

[4] `https://proofwiki.org/wiki/Well-Ordering_Theorem`
[5] `https://proofwiki.org/wiki/Zermelo%27s_Theorem_(Set_Theory)`

Chapter 2

Topological spaces and continuous functions

Definition 2.1. Topology A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X (*i.e.* $\mathcal{T} \subset 2^X$) having the following properties:

1. $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$,
3. The intersection of elements of any <u>finite</u> subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Remark. *After investigation, I consider there is no special meaning for a collection differing from a set. (ref. (<u>link</u>)[1], (<u>link</u>)[2])*

Definition 2.2. Topological space A *topological space* is an ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ of X. Often we omit specfic mention of $\mathcal{T}$ if no confusion will arise.

Definition 2.3. Open set If X is a topological space, we say that $U \subset X$ is an *open set* of X if $U \in \mathcal{T}$. Using this terminology, one can say a topological space is a set X together with a set of open sets. Thus the defintion of an open set is the same as topology: $\varnothing$ and X both open, arbitrary unions and finite intersections of open sets are open.

Definition 2.4. If $\mathcal{T} = 2^A$, then X is called a **Discrete topology**;
If $\mathcal{T} = \{\varnothing, X\}$, then X is called a **Trivial topology**, or **Indiscrete topology**.

Definition 2.5. Finite complement topology Let X be a set, $\mathcal{T}_f$ be the collection of all subsets U of X such that $X - U$ either is finite or is all of X. Then $\mathcal{T}_f$ is a topology on X, called the *finite complement topology.* Both X and $\varnothing$ are in $\mathcal{T}_f$ since $X - X$ is finite and $X - \varnothing$ is all of X.

[1] `http://math.stackexchange.com/a/173002`
[2] `https://proofwiki.org/wiki/Definition:Topology/Definition_1`

Now we show $\mathcal{T}_f$ is a topology. If $\{U_a\}$ is an indexed family of nonempty elements of $\mathcal{T}_f$, to show $\bigcup U_a$ is in $\mathcal{T}_f$, we compute

$$X - \bigcup U_a = \bigcap (X - U_a)$$

Which is finite since $X - U_a$ is finite for all $U_i \in \mathcal{T}_f$. To show $\bigcap U_i \in \mathcal{T}_f$, we compute

$$x - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

Note by definition of topology we only need to check finite intersection. Then the finite number of unions of finite sets is finite thus finite $\bigcup U_i \in \mathcal{T}_f$.

Definition 2.6. Finer, **Strictly finer**, **Coarser**, **Strictly coarser**, and **Comparable**

	set term	topology term
$\mathcal{T}' \supset \mathcal{T}$	superset	finer, larger, stronger
$\mathcal{T}' \supsetneq \mathcal{T}$	proper superset(?)	strictly finer
$\mathcal{T}' \subset \mathcal{T}$	subset	coarser, larger, weaker
$\mathcal{T}' \subsetneq \mathcal{T}$	proper subset	strictly coarser
$\mathcal{T}' \subset \mathcal{T}$ or $\mathcal{T} \subset T'$	-	comparable

Remark. *This is a bit not straightforward at the first sight. Remember that the essential of topology is its structure. The more open sets we have in a topology, the more fine it is.*

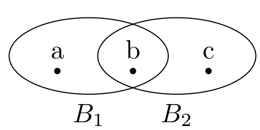
Definition 2.7. Basis A *basis* for a topology $\mathcal{B} \subset 2^X$ (called basis elements) such that

1. $\forall x \in X : \exists B \in \mathcal{B} : x \in B$
2. $\forall x \in B_1 \cap B_2 \Rightarrow \exists B_3 \in \mathcal{B} : x \in B_3 \wedge B_3 \subset B_1 \cap B_2$

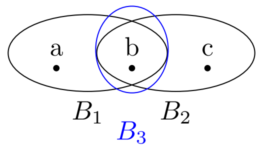
If $\mathcal{B}$ is a basis, we define *topology $\mathcal{T}$ generated by $\mathcal{B}$* to be: we say $U \subset X$ is open set (*i.e.* $U \in \mathcal{T}$), if $\forall x \in U : \exists B \in \mathcal{B} : x \in B \wedge B \subset U$.

Remark. *The concept of a basis of a topology is rather abstract. I think of it in this way. Since we are dealing with intersection in the definition of basis, we want to think from top to bottom, in other words, from larger sets, to their intersections.*

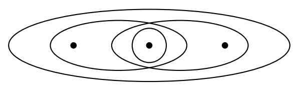
Example 2.7.1. Think of this example. In the beginning we have $X = \{a, b, c\}$ and $\mathcal{B} = \{B_1, B_2\}$ where $B_1 = \{a, b\}$ and $B_2 = \{b, c\}$.



In this way $\forall x \in X : \exists B \in \mathcal{B} : x \in B$. Now we try to satisfy the second condition and we will find that $b \in B_1 \cap B_2$ is not in any basis element who is a subset of $B_1 \cap B_2$. Now we add it as follows:



Now with $\mathcal{B} = \{B_1, B_2, B_3\}$ all above two criterion are satisified. Therefore $\mathcal{B}$ is a basis for the topology $\mathcal{T}$:



Example 2.7.2. Continuing above example. We now verify that $\mathcal{T}$ is generated by $\mathcal{B}$ by checking all open sets $U \in \mathcal{T}$ with the rule specified. Actually $\mathcal{T} = \{\varnothing, \{a, b\}, \{b, c\}, \{b\}, \{a, b, c\}\}$. I will only show how $\{a, b\} \in \mathcal{T}$ and how $\{b\} \in \mathcal{T}$ and how $\{a, c\} \notin \mathcal{T}$.

1. $U = \{a, b\} \in \mathcal{T}$: For $a \in U$, take $B = \{a, b\}$, then $B \subset U$; the same works for b. Therefore U is an open set.

2. $U = \{b\} \in \mathcal{T}$: For $b \in U$, take $B = \{b\}$, then $B \subset U$. Therefore $\{b\}$ is an open set. Note that we cannot take $B = \{a, b\}$, since $\{a, b\} \not\subset \{b\}$.

3. $U = \{a, c\} \notin \mathcal{T}$: For $a \in U$, we cannot find a $B \in \mathcal{B}$ that contains a and is a subset of $\{a, c\}$. Since the only possible subsets of $\{a, c\}$ are $\{\varnothing, \{a\}, \{c\}, \{a, c\}\}$ and neither is a basis element. Therefore $\{a, c\}$ is not an open set.

Remark. *Let $\mathcal{B}$ be set of all one-point subsets of X, then it is a basis for the discrete topology on X.*

Remark. *So my understanding of a basis of a topology is like the generator of the topology that satisfy the existence of intersection. So first of all we have to note that $\mathcal{B} \subset \mathcal{T}$.*

Using the process from the book, we can verify it. Take J to be an indexed family of $\mathcal{B}$.

It's easy to show that $\bigcup_{\alpha \in J} B_\alpha$ is in $\mathcal{T}$ if $\forall \alpha \in J : B_\alpha \in \mathcal{T}$.

It's easier to to show that $\bigcap_{\alpha \in J} B_\alpha$ is in $\mathcal{T}$ if $\forall \alpha \in J : B_\alpha \in \mathcal{T}$ since it's specified in the criteria for $\mathcal{B}$ to be a basis. (Actually basis does more than finite intersection. We can prove using induction that the intersection of any countable number of basis is in $\mathcal{T}$.

Also, $\varnothing \in \mathcal{T}$ will be vacuously true no matter what $\mathcal{B}$ we pick.

Theorem 2.8. A more visual-able theorem. $\mathcal{T}$ is the all unions of elements in $\bigcup_{B\in\mathcal{B}} B$.

Proof. Given $B = \bigcup_{\alpha\in K} B_\alpha$, since $B_\alpha \in \mathcal{T}$ and for any $x, y \in \mathcal{T}$, $x \cup y \in \mathcal{T}$. Thus $B \in \mathcal{T}$. Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$, $\exists B_x \in \mathcal{B}$ and $U = \bigcup_{x\in U} B_x$. Therefore U is some union of elements in $\mathcal{B}$. $\square$

Theorem 2.9. Let X be a topological space. $\mathcal{C} \subset \mathcal{T}$ such that for each open set $U \in \mathcal{T}$ and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for X.

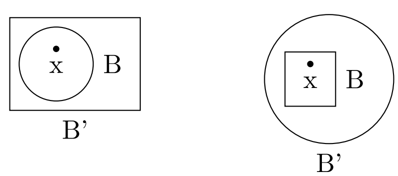
Theorem 2.10. Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for $\mathcal{T}$ and $\mathcal{T}'$, respectively. Then the following are equivalent:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$
2. $\forall x \in X, B \in \mathcal{B} : x \in B \Rightarrow \exists B' \in \mathcal{B}' : x \in B' \subset B$.

Remark. *Think of a finer topology to be a set with more elements, while inclusively. Then it works just like the defintion of "subsets". While we are now not talking about the topologies but the bases, so we only care about the elements in the bases.*

The book uses the concept of a gravel. So the pebbles forms a basis of a topology. When they get smashed into dust, they form the basis of a new topology, while finer. And the dust particles was contained inside a pebble, as says the criterion.

Example 2.10.1. One can be demonstrated that topology on $\mathbb{R}$ generated by open circles is the same topology as generated by rectangles. Below diagram shows this:



"Since for each point x and each basis element $B \in \mathcal{B}$ containing x, there is a element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$." – This applies to B to be the circluar basis and the rectangular basis. Thus we conclude that $T \subset T'$ and $T' \subset T$, thus they are equivalent.

Definition 2.11. Standard topology, **Lower limit topology** and **K topology** The *standard topology* has the basis of all open intervals in the real line. If $\mathcal{B} = \{[a, b) \mid a, b \in \mathbb{R}\}$, the topology generated by B is then called the *lower limit topology* on $\mathbb{R}$, denoted $\mathbb{R}_l$. Let K denote the set of all numbers of the form $1/n$, $n \in \mathbb{Z}^+$, and let B' be the set of all open intervals in form of $(a, b) - K$, is called *K-topology*, denoted as $\mathbb{R}_k$.

Proposition 2.12. $\mathbb{R}_l$ and $\mathbb{R}_k$ are strictly finer than the standard $\mathbb{R}$, while are not comparable with one another.

Definition 2.13. Subbasis A subbasis S for a topology on X is a set of subsets of X whose union equals to X.

Remark. *Unlike a subset/subgroup/subspace, a subbasis is not as its name suggests to be a sbuset of some other basis. Subbasis and basis are two different way to generate a topology. A basis includes all intersection of two basis elements, while a subbasis doesn't have to. So to generate a topology using a basis, we take all union. While to generate a topology using a subbasis, we take all unions and intersection of subbasis elements. (ref. Eric Auld on Math StackExchange(link)[3], and Wikipedia(link)[4].*

Example 2.13.1. ref. (link)[5]
$\mathcal{T} = \{\varnothing, \{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$
$\mathcal{B} = \{\{0\}, \{0,1\}, \{0,2\}\}$
$\mathcal{S} = \{\{0,1\}, \{0,2\}\}$

Example 2.13.2. (ref. (link)[6]) For standard topology on $\mathbb{R}$, $\mathcal{T}$ is all open intervals (and their unions) on $\mathbb{R}$. Then,
$\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R},\, a \leq b\}$, and
$\mathcal{S} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{(b, \infty) \mid b \in \mathbb{R}\}$

Definition 2.14. Order topology Assuming we knowing about all open/closed/half-open interval concepts, *over a, possibly not continuous, set.*

Let X be a set with linear order relation with $<$. Assume $|X| > 1$. Let $\mathcal{B}$ be defined as a set of all subsets of X of the following types:

1. All valid (a,b)
2. For a_0 to be the minimal element, $[a_0, b)$ (if any)
3. For b_0 to be the maximal element, $(a, b_0]$ (if any)

Then B is a basis for a topology on X, called *order topology.*

Example 2.14.1. $\mathbb{R}$, $\mathbb{R} \times \mathbb{R}$ in dict order, $\mathbb{Z}^+$

Definition 2.15. Ray, **Open ray**, **Closed ray**

$$(a, +\infty) = \{x \mid x > a\}$$
$$(-\infty, a) = \{x \mid x < a\}$$
$$[a, +\infty) = \{x \mid x \geq a\}$$
$$(-\infty, a] = \{x \mid x \leq a\}$$

[3] `https://math.stackexchange.com/a/449577/120022`
[4] `https://en.wikipedia.org/wiki/Subbase`
[5] `http://math.stackexchange.com/a/449593/120022`
[6] `http://mathworld.wolfram.com/Subbasis.html`

Definition 2.16. Product topology Product topology on topology spaces X and Y, denoted as $X \times Y$ is the topology with the basis $\mathcal{B}$ who elements are in form of $U \times V$ where $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$. (Obviously and omitted by book the underlying set is just $X \times Y$)

Example 2.16.1. open sets on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Theorem 2.17. If $\mathcal{B}$ is a basis for topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection $\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \wedge C \in \mathcal{C}\}$ is a basis for $X \times Y$.

Definition 2.18. Projection

$$\pi_1 : X \times Y \to X, \quad \pi_2 : X \times Y \to Y$$

Projection functions are *onto.*

Definition 2.19. Subspace and **Subspace topology** For X be topological space and $\mathcal{T}$ be topology on X. If $Y \subset X$, then define

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

subspace topology of $\mathcal{T}$. $\mathcal{T}_Y$ is a topology.

Remark. *Does Y have to an open set in X (or $\mathcal{T}$)? No.*

- *remember X does not have any structure, it is $\mathcal{T}$ who gives the structure to X;*
- *it is not necessary for $Y \in \mathcal{T}$, as I will prove $\mathcal{T}_Y$ is a topology anyway below.*

Proof. $\varnothing = Y \cap \varnothing \in \mathcal{T}_Y$, $Y = Y \cap X \in \mathcal{T}_Y$, for $U_1, U_2 \in \mathcal{T}_Y$, $U_1 \cap U_2 = V_1 \cap Y \cap V_2 \cap Y = (V_1 \cap V_2) \cap Y \in \mathcal{T}_Y$ where $V_1, V_2 \in \mathcal{T}$, for $U_1, U_2 \in \mathcal{T}_Y$, $U_1 \cup U_2 = (V_1 \cap Y) \cup (V_2 \cap Y) = (V_1 \cup V_2) \cap Y \in \mathcal{T}_Y$.

This proof is not correct. A topology requires arbitrary union *i.e.* $\bigcup_{\alpha \in J}$ and finite intersection $U_1 \cap \ldots \cap U_n$. But the same rule applies. □

Theorem 2.20. Let $\mathcal{B}$ be basis for a topology of X, then $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on Y.

Theorem 2.21. If Y is subspace of X. If $U \in \mathcal{T}_Y$, and $Y \in \mathcal{T}$, then $U \in \mathcal{T}$.

This is just what I concerned about above. Pretty straightforward result.

Theorem 2.22. $A \subset X$ and $B \subset Y$, then $A \times B$ is the same topology $A \times B$ that inherits as subspace of $X \times Y$.

Definition 2.23. Convex Given an ordered set X, $Y \subset X$. Say Y is *convex in* X if for any $a <_Y b$, $(a, b) \in X \Rightarrow (a, b) \in Y$.

Example 2.23.1. Here are some examples of Y convex in X.

- $X = \mathbb{R}$, $Y = [0, 1]$
- $X = \mathbb{R}$, $Y = (0, 1)$
- $X = [0, 2] \cup [3, 5]$, $Y = [1, 2] \cup [3, 4]$
- $X = [0, 2] \cup [3, 5]$, $Y = [4, 5]$

Here are some counterexamples:

- $X = \mathbb{R}$, $Y = [0, 1) \cup (1, 2]$
- $X = \mathbb{R}$, $Y = [0, 1] \cup [2, 3]$
- $X = [0, 2] \cup [3, 5]$, $Y = [1, 2] \cup [4, 5]$

Theorem 2.24. Let X be ordered set in order topology; let $Y \subset X$ be convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

Definition 2.25. Closed set A subset A of a topological space X is said to be *closed* if the set $X - A$ is open.

Remark. *A subset $A \in X$ is closed DOES NOT mean itself is not open. It means its complement is open. An example: trivial topology on $\{a, b\}$ includes $\{\varnothing, \{a, b\}\}$. Then $\{a\}$ is neither closed or open. In the discrete topology, every point is both closed and open.*

Theorem 2.26. Let X be a topological space, the following conditions hold:

1. $\varnothing$ and X are closed.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

Theorem 2.27. Let Y be subspace of X. Then a set A is closed in Y iff it squals the intersection of a closed set of X in Y.

Definition 2.28. Closure and **interior** of a Set Given $A \subset X$, the *interior* of A is definted as the union of all open sets contained in A, and the *closure* of A is defined as the intersection of all closed sets containing A.

Interior of A is denoted as $\operatorname{Int} a$, and closure of A is denoted as $\bar{A}$. I will use A° and $\overline{A}$ to denote these two in this note.

Remark. $A^\circ \subset A \subset \overline{A}$

Remark. *If A is open, $A = A^\circ$; if A is closed, $A = \overline{A}$.*

Theorem 2.29. Let Y be a subspace of X. Let A be a subset of Y. Let $\overline{A}$ be closure of A in X. Then the closure of A in Y equals $\overline{A} \cap Y$.

Remark. *When we talk about Y subspace of X and $A \subset Y$. We reserve notation $\overline{A}$ to denote the closure of A in X.*

Definition 2.30. Intersects A intersects B iff $A \cap B \neq \varnothing$.

Theorem 2.31. Let A be a subset of topology space X.

1. Then $x \in \overline{A}$ iff every open set U containing x intersects A.

2. Suppose topology of X is given by a basis $\mathcal{B}$, then $x \in A$ iff $\forall B \in \mathcal{B} : x \in B \Rightarrow B \cap A \neq \varnothing$.

Definition 2.32. Neighborhood We shorten the statement "U is an open set containing x" to the phrase "U is a neighborhood of x".

Definition 2.33. Limit point If A a subset of the topological space X and if x is a point of X, we say that x is a *limit point* (*a.k.a.* "*cluster point*") of A if every neighborhood of x intersects A in some point other than x itself.

Example 2.33.1. Consider standard topology on $\mathbb{R}$. If $A = (0, 1]$, then 0 is a limit point. So is $\frac{1}{2}$. Any $x \in [0, 1]$ is a limit point, but no other points.

Example 2.33.2. On $\mathbb{R}$, if $B = \{1/n \mid n \in \mathbb{Z}^+\}$, then 0 is the only limit point of B.

Theorem 2.34. Let A be a subset of topological space X. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$

Corollary 2.35. A subset of a topoloical space is closed iff it contains all its limit points.

Definition 2.36. Converge The convergency property of $x \in \mathbb{R}$ is generalized in topology as: One say that a sequence $x_1, x_2, \ldots$ of points of the spacec X converges to the point x of X if that corresponding to each neighborhood U of x, there is a positive integer N such that $x_n \in U$ for all $n \geq N$.

Definition 2.37. Hausdorff space A topological space X is called a *Hausdorff space* if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1, and U_2 of x_1 and x_2, respectively, that are disjoint.

Theorem 2.38. Every finite point set in a Hausdorff space X is closed.

Proof. We show any one point set $\{x_0\}$ is closed. If x is a point in X different from x_0, then x and x_0 have disjoint neighborhoods U and V, repsectively. Since U does not intersect $\{x_0\}$, the oint x cannot belong to the closure of the set $\{x_0\}$. Therefore the closure of the set $\{x_0\}$ is itself. Therefore it is closed. $\square$

Remark. *This condition that finite point sets is closed is actually weaker than the Hausdorff condition. e.g. $\mathbb{R}$ in the finite complemenet topology is not a Hausdorff space but every finite point set is closed. This condition is named as* ***T_1 axiom****.*

Theorem 2.39. Let X be a space satisfying T_1 axiom; let A be subset of X. Then the point x is a limit point of A iff every neighborhood of x contains infinitely many points of A.

Theorem 2.40. X Hausdorff space, a sequence of points in X converges to at most one point of X. **Limit** is called for that point of the sequence.

Theorem 2.41. Simply ordered set is a Hausdorff space in order topology. Product of two Hausdorff space is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.

Definition 2.42. Continuous function Let X and Y be topological spaces. A function $f : X \to Y$ is said continuous if for every open subset V of Y, the set $f^{-1}(V)$ is open subset of X.

Remark. *Remember in the defintion, it does not say f maps all open sets in X to open sets in Y. It says f^{-1} maps all open sets in Y to open sets in X.*

Theorem 2.43. To show a function is continuous, one only need to show it's inverse image for each subbasis element is open.

Remark. *I've proven this with Prof. Erickson on his class. The final result come out to be the inversed image will become the basis. This shows one usefulness of subbasis.*

Example 2.43.1. Analysis analogy: let's study real function $f : \mathbb{R} \to \mathbb{R}$.

In analysis, we define continuity of f via the $\epsilon - \delta$ definition. Our definition will imply the $\epsilon - \delta$ definition as follows.

Definition for continuity for real valued function:
$\forall x_0 \in \mathrm{Dom}(f) : \forall \epsilon > 0 : \exists \delta : \forall x \in (x_0 - \delta, x_0 + \delta) : f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$

analysis language	topology language
$\mathbb{R}$	$\mathbb{R}$ with standard topology
$f : X \to Y$ $(X, Y : \text{Set})$	$f : X \to Y$ $(X, Y : \text{Topological space})$
open interval	open set
$\delta-$neighborhood of x_0	open set in X containing x_0
$\epsilon-$neighborhood of $f(x_0)$	open set in Y containing $f(x_0)$

Restatement of above definition for continuity in Topology language:
$\forall x_0 \in X : \forall U_Y \in \mathcal{T}_Y : \exists U_X \in \mathcal{T}_X \wedge x_0 \in U_X : \forall x \in U_X : f(x_0) \in U_Y$
Then we will discover that we can actually go more general:
$\forall U_Y \in \mathcal{T}_Y : \exists U_X \in \mathcal{T}_X : \forall x \in U_X : f(x) \in U_Y$
And even more general:
$\forall U_Y \in \mathcal{T}_Y : \exists U_X \in \mathcal{T}_X : f(U_X) = U_Y$
Then it comes to be continuity function definition in topology:
$\forall U_Y \in \mathcal{T}_Y : f^{-1}(U_Y) \in \mathcal{T}_X$

Theorem 2.44. Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:

1. f is cont.
2. For every $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$

3. For every closed set $B \subset Y$, $f^{-1}(B)$ is closed in X.
4. For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Definition 2.45. Homeomorphism Let $f : X \to Y$ be bijection. If both f and f^{-1} are continuous, then f is called a *homeomorphism*.

Remark. *One point to be noticed. This definition requires f and f^{-1} both be continuous. You may ask, if f is bijective and f is continuous, doesn't it follows automatically that f^{-1} is continuous? Of course not. Here's a counter-example.*

Take $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x$, the first is the discrete topology on $\mathbb{R}$ and the second is the standard topology $\mathbb{R}$. Then f is obviously bijective, and f is open since $f^{-1}(U)$ is always open, so f is continuous. But the converse isn't true.

Definition 2.46. An alternative definition for homeomorphism is a bijective function such that $f(U)$ is open iff U is open.

Definition 2.47. Imbedding, **Embedding** An injective continuous map is called an *imbedding* (*a.k.a. embedding.*

Definition 2.48. Unit circle, $\mathbf{S}^1$ $S^1 = \{x \times y \mid x^2 + y^2 = 1\}$

Theorem 2.49. Constructing continuous map

1. Constant map.
2. Inclusion map.
3. Composition of continuous map.
4. Restricting domain. $f : X \to Y$ is cont $\Rightarrow f|A : A \to Y$ where $A \subset X$.
5. Restricting the range.
6. Local formulation of continuity. If X can be written as union of open sets U_α, and $f|U_\alpha$ is cont. Then $f : X \to Y$ is cont.

Theorem 2.50. (**The Pasting Lemma**). Let $X = A \cup B$ where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ cont. If $f(x) = g(x)$ for every $x \in A \cap B$. Then f and g combine to give a cont func $h : X \to Y$.

Theorem 2.51. (**Maps into products**) Let $f : A \to X \times Y$ given by $f(a) = (f_1(a), f_2(a))$ then f is cont iff $f_1 : A \to X$ and $f_2 : A \to Y$ are cont. The maps f_1 and f_2 are called the **Coordinate functions** of f. (sort of like projection map).

Definition 2.52. Box topology For $\prod X_i$, either finite or infinite, if we take the cartesian product of the basis/open sets, correspondingly, we get a basis for the new topology. This is called the box topology. (Just as what we defined above for product topology)

Definition 2.53. Metric A metric on a set X is a function $d : X \times X \to \mathbb{R}$ having the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$; equal only when $x = y$
2. $d(x, y) = d(y, x)$ for all x, y
3. $d(x, y) + d(y, z) \geq d(x, z)$ for all x, y, z (**Triangle inequality**)

Definition 2.54. Epsilon ball $B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$

Definition 2.55. Metric topology If d is a metrix on X. The collection of all ϵ-balls $B_d(x, \epsilon)$ for all $x \in X$ and all $\epsilon > 0$ as basis of a topology on X. This topology is called the metric topology inducted by d.

Example 2.55.1. $\mathbb{R}$ under standard topolgoy is a metric topology induced by the metric "absolute value of difference function".

Example 2.55.2. The discrete topology for a set X is given by the metric topology induced by d defined as:

$$\begin{cases} d(x, y) = 1 & \text{if } x \neq y \\ d(x, y) = 0 & \text{if } x = y \end{cases}$$

Definition 2.56. Metrizable, **Metric space** trivial.

Definition 2.57. Bounded, **Diameter** Bounded if $d(a_1, a_2) \leq M$.
$\operatorname{diam} A = \sup \{d(a_1, a_2) \mid a_1, a_2 \in A\}$ called diameter.

Definition 2.58. Standard bounded metric Let X be a metric space with metric d. Define $\overline{d} : X \times X \to \mathbb{R}$ by the equation.

$$\overline{d}(x, y) = \min \{d(x, y), 1\}$$

then $\bar{d}$ is a metric that induces the same topology as d.

Definition 2.59. Norm, **Euclidean metric**, **Square metric** trivial for norm.
trivial for euclidean metric.
square metric ρ is defined by $\rho(x, y) = \max \{|x_1 - y_1|, \ldots, |x_n - y_n|\}$

Theorem 2.60. Let d and d' be two metrics on the set X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topology they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ iff for each x in X and each $\epsilon > 0$, there is a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.

Remark. *This result is obvious since finer basis determines finer topology.*

Theorem 2.61. The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Definition 2.62. Uniform metric, **Uniform topology** Given an index set J and given points $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let us define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by the equation

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup \left\{ \bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J \right\}$$

This is called the *uniform metric* on $\mathbb{R}^J$ and the topology it induces is called the *uniform topology*.

Theorem 2.63. The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology; these topologies are all different if J is infinite.

Theorem 2.64. Let $\bar{d}(a, b) = \min \{|a - b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If x and y are two points of $\mathbb{R}^\omega$, define

$$D(\mathbf{x}, \mathbf{y}) = \sup \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}$$

to be a metric on $\mathbb{R}^\omega$. Then the topology induced by D is the product topology.

Remark. *needs more explanation on these two.*

Theorem 2.65. Let $f : X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$$

Theorem 2.66. The sequence lemma Let X be topological space, let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is metrizable.

Theorem 2.67. Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.

Theorem 2.68. If X is topological space and $f, g : X \to \mathbb{R}$ are cont. Then $f + g, f - g, fg$ are cont and f/g are cont for $g \neq 0$.

Definition 2.69. Converge uniformly Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence (f_n) *converges uniformly* to the function $f : X \to Y$ if given $\epsilon > 0$, there exists an integer N such that

$$d(f_n(x), f(x)) < \epsilon$$

for all $n > N$ and all x in X.

Theorem 2.70. Uniform limit theorem Let $f_n : X \to Y$ be a sequence of functions from topological space X to metric space Y. If (f_n) converges uniformly to f then f is continuous.

Definition 2.71. Quotient map Let X and Y be topological spaces; let $p : X \to Y$ be a surjective map. The map p is said to be a quotient map if a subset U of Y is open iff $p^{-1}(U)$ is open in X.

Remark. *It sounds like definition for continuity. The difference is in continuity we specify the inverse mapping from open sets to open sets. In this one, we have to inversely map open sets to open sets, also non-open set to non-open set. This condition is sometimes called* ***Strong continuity****.*

Equivalently we can define quotient maps as $A \subset Y$ closed iff $p^{-1}(A)$ closed in X. It follows that for all $B \subset Y$:

$$f^{-1}(Y - B) = X - f^{-1}(B)$$

Definition 2.72. Saturated A set $C \subset X$ is saturated (with respect to the surjective map $p : X \to Y$) if C contains every set $p^{-1}(y)$ that C intersects. (symbolically, C is saturated if $p^{-1}(y)$

A saturated set is just a set $C \subset X$ that is an intersection of open sets of X. (link)[7]

Remark. *The definition on the book is not that clear to me. Wikipedia explains it in an intuitive way (The second line in above definition).*

Remark. *To really understand what a saturated set is, we have to do it under a quotient map $q : X \to p(X)$. A quotient map p define an equiv relation on X. Say $A \in p(X)$ consists of points $\{a, b\} \in X$, then $\{a, b\}$ is a saturated set whereas $\{a\}$ or $\{b\}$ isn't. Every saturated set A can be expressed as $q^{-1}q(A)$. (link)*[8]

Definition 2.73. Open map, **Closed map** An open map is a map that maps open sets to open sets. An closed map is a map that maps closed sets to closed sets.

Remark. *Continuous map requires $f^{-1}(U)$ open if U open. Open map requires $f(U)$ open if U open.*

Example 2.73.1. Let $X = Y = \{a, b\}, \mathcal{T}_X = \mathcal{T}_Y = \{\varnothing, \{a\}, \{a, b\}\}$

1. Open map not closed: $f\ x = a$

2. Closed map not open: $f\ x = b$

Above two maps are neither continuous

Definition 2.74. Quotient map A quotient map is a continuous surjective map that is either open or closed or both.

Definition 2.75. Quotient topology Let $p : X \to A$ be a quotient map from topological space X to set A. Then A is called a quotient topology.

[7] `https://en.wikipedia.org/wiki/Saturated_set`
[8] `http://math.stackexchange.com/a/1173764/120022`

Definition 2.76. Quotient space Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map carries each point of X to the corresponding partition in X^* that contains it. X^* is called a *quotient space* of X.

Remark. *Quotient space give rise to a quotient topology by identifying points into equivalence relations. Notice that usually only very few distinct points of X will be put into a common partition and the rest points in X will usually belong to a partition with itself only. X^* is also called **Identification space** or a **Decompostion space** of X.*

Theorem 2.77. Let $p : X \to Y$ be quotient map; Let $A \subset X$ be saturated with respect to p; then let $q : A \to p(A)$ be the map obtained by restricting p.

1. If A is either open or closed in X, then q is a quotient map.

2. If p is either an open map or a closed map, then q is a quotient map.

Theorem 2.78. Composition of quotient maps is quotient map. Can be shown by $p^{-1}(q^{-1}(U)) = (q \circ p)^{-1}(U)$.

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