Notes for James R. Munkres' Topology (2E)

Shou

January 21, 2016

Contents

Chapter 0

Structure and reading plans

Ch 1-8 is the part I, mainly for common topology. The part II includes ch 9-14, that depends on ch 1-4, is about algebraic topology.

My plan is to read through ch 1-4 very quickly, within a weekend, and then I will start reading ch 9+ simultaneously with W.S.Massey's Agelbraic topology: An induction.

Finally I wish I could finish all ch 1-8 and also some parts after ch 9.

Chapter 1

Set theory and logic

Definition 1.1. Order relation rel C on set A is called *order relation* if

1. comparability, *i.e.* $\forall x, y \in A, x \neq y \Rightarrow xCy \vee yCx$
2. non-refl, *i.e.* $\forall x, \neg(xCx)$
3. trans, *i.e.* $\forall xCy \wedge yCz, xCz$

(*a.k.a.* linear order)

Definition 1.2. Open interval if X is a set and $<$ is an order rel, and if $a < b$ we use notation (a, b) to denote $\{x \in X \mid a < x < b\}$, called *open interval.*

If $(a, b) = \emptyset$, then a is called *immediate precessor* of b and b called *immediate successor* of a.

Remark. *It makes more sense on X is a discrete set. Since if (a, b) is an open interval in $\mathbb{R}$, $(a, b) = \emptyset \Rightarrow a = b$ which makes no sense on a as an immediate precessor of b.*

Definition 1.3. Order Type if A and B are two sets with $<_A$ and $<_B$. We say that A and B have same *order type* if $\exists f : A \to B$ that preserves order, *i.e.*

$$a_1 <_A b_1 \Rightarrow f(a_1) <_B f(b_1)$$

Remark. *It's just a generalization of monotone function.*

Definition 1.4. Dictionary order relation if A,B are two sets with ($<_A, <_B$), defn an order for $A \times B$ by defining

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$, or if $a_1 = a_2 \ \wedge \ b_1 <_B b_2$.

Definition 1.5. LUB property/GLB property For A and $<_A$, we say A has *LUB property* if

$$\forall A_0 \subset A, A \neq \emptyset \wedge \exists \text{upper bound for } A_0 \Rightarrow \exists \text{lub}\{A_0\} \in A$$

Example 1.5.1. $A = (-1, 1)$. *e.g.* $X = \{1 - \frac{1}{n} \mid n \in \mathbb{Z}^+\}$ does not have an upper bound, thus vacuously true. $\{-\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ has upper bound of any number in $[0, 1) \subset A$, and $\mathrm{lub}(X) = 0 \in (-1, 1)$.

Example 1.5.2. Counterexample. $A = (-1, 0) \cup (0, 1)$. $\{-\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ has upper bound of any $(0, 1) \subset A$, while $\mathrm{lub}(X) = 0 \notin A$.

Remark. *The completeness property of $\mathbb{R}$ as an axiom derives this property.*

Property 1.1. $\mathbb{R}$ field

Algebraic properties

1. assoc: $(x + y) + z = x + (y + z); (xy)z = x(yz)$
2. comm: $x + y = y + x; xy = yx$
3. id: $\exists!0, x + 0 = x$; $\exists!1, x \neq 0 \Rightarrow x1 = x$
4. inv: $\forall x, \exists!y, x + y = 0$; $\forall x \neq 0, \exists!y, xy = 1$
5. distr: $x(y + z) = xy + xz$

Mixed algebraic and order property

6. $x > y \Rightarrow x + z > y + z$; $x > y \wedge z > 0 \Rightarrow xz > yz$

Order properties

7. $<$ has LUB property
8. $\forall x < y, \exists z, x < z \wedge z < y$

1-6 make $\mathbb{R}$ a field. 1-6 + 7 make $\mathbb{R}$ an ordered field. 7-8 makes $\mathbb{R}$, called by topologists, a **linear continuum**.