Notes on James R. Munkres' Topology (2E)

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January 23, 2016

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Chapter 0

Structure and reading plans

Ch 1-8 is the part I, mainly for common topology. The part II includes ch 9-14, that depends on ch 1-4, is about algebraic topology.

My plan is to read through ch 1-4 very quickly, within a weekend, and then I will start reading ch 9+ simultaneously with W.S.Massey's Algebraic topology: An induction.

Finally I wish I could finish all ch 1-8 and also some parts after ch 9.

Chapter 1

Set theory and logic

Definition 1.1. Order relation rel C on set A is called *order relation* if

1. comparability, *a.k.a.* totality for all non-eq elements, *i.e.* $\forall x, y \in A, x \neq y \Rightarrow xCy \vee yCx$
2. non-refl, *i.e.* $\forall x, \neg(xCx)$
3. trans, *i.e.* $\forall xCy \wedge yCz, xCz$

(*a.k.a.* **Linear order**, **Simple order**)

Remark. *This relation is not the same as Linear order on Wikipedia (link)*[1]*. This order is actually the strict version of the Total order on wikipedia, i.e. has non-refl property.*

Definition 1.2. Open interval if X is a set and $<$ is an order rel, and if $a < b$ we use notation (a, b) to denote $\{x \in X \mid a < x < b\}$, called *open interval.*

If $(a, b) = \emptyset$, then a is called *immediate precessor* of b and b called *immediate successor* of a.

Remark. *It makes more sense on X is a discrete set. Since if (a, b) is an open interval in $\mathbb{R}$, $(a, b) = \emptyset \Rightarrow a = b$ which makes no sense on a as an immediate precessor of b.*

Definition 1.3. Order type if A and B are two sets with $<_A$ and $<_B$. We say that A and B have same *order type* if $\exists f : A \to B$ that preserves order, *i.e.*

$$a_1 <_A b_1 \Rightarrow f(a_1) <_B f(b_1)$$

Remark. *It's just a generalization of monotone function.*

Definition 1.4. Dictionary order relation if A,B are two sets with ($<_A, <_B$), defn an order for $A \times B$ by defining

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$, or if $a_1 = a_2 \ \wedge \ b_1 <_B b_2$.

[1] `https://en.wikipedia.org/wiki/Total_order`

Definition 1.5. LUB property/GLB property For A and $<_A$, we say A has *LUB property* if

$$\forall A_0 \subset A, A \neq \emptyset \wedge \exists \text{upper bound for } A_0 \Rightarrow \exists \text{lub}\{A_0\} \in A$$

Example 1.6. $A = (-1, 1)$. *e.g.* $X = \{1 - \frac{1}{n} \mid n \in \mathbb{Z}^+\}$ does not have an upper bound, thus vacuously true. $\{-\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ has upper bound of any number in $[0, 1) \subset A$, and $\text{lub}(X) = 0 \in (-1, 1)$.

Example 1.7. Counterexample. $A = (-1, 0) \cup (0, 1)$. $\{-\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ has upper bound of any $(0, 1) \subset A$, while $\text{lub}(X) = 0 \notin A$.

Remark. *The completeness property of* $\mathbb{R}$ *as an axiom derives this property.*

Property 1.8. $\mathbb{R}$ field

Algebraic properties

1. assoc: $(x + y) + z = x + (y + z); (xy)z = x(yz)$
2. comm: $x + y = y + x; xy = yx$
3. id: $\exists!0, x + 0 = x$; $\exists!1, x \neq 0 \Rightarrow x1 = x$
4. inv: $\forall x, \exists!y, x + y = 0$; $\forall x \neq 0, \exists!y, xy = 1$
5. distr: $x(y + z) = xy + xz$

Mixed algebraic and order property

6. $x > y \Rightarrow x + z > y + z$; $x > y \wedge z > 0 \Rightarrow xz > yz$

Order properties

7. $<$ has LUB property
8. $\forall x < y, \exists z, x < z \wedge z < y$

1-6 make $\mathbb{R}$ a field. 1-6 + 7 make $\mathbb{R}$ an ordered field. 7-8 makes $\mathbb{R}$, called by topologists, a **Linear continuum**.

Theorem 1.9. Well ordering property $\mathbb{Z}^+$ has *Well-ordering property. i.e.* Every nonempty subset of $\mathbb{Z}^+$ has a smallest element.

Proof. We first prove that for each $n \in \mathbb{Z}^+$, the following statement holds: Every nonempty subset of $\{1, \ldots, n\}$ has a smallest element.

Let A be the set of all postive integers n for which this theorem holds. Then A contains 1, since if $n = 1$, the only possible subset is $\{1\}$ itself. Then suppose A contains n, we show that it contains $n + 1$. So let C be a nonempty subset of the set $\{1, \ldots, n + 1\}$. If C consists of the single element $n + 1$, then that element is the smallest element of C. Otherwise, consider the set $C \cap \{1, \ldots, n\}$, which is nonempty. Because $n \in A$, this set has a smallest element, which will automaticallly be the smallest element of C also. Thus A is inductive, so we conclude that $A = \mathbb{Z}^+$; hence the statement is true for all $n \in \mathbb{Z}^+$.

Now we prove the theorem. Suppose that D is a nonempty subset of $\mathbb{Z}^+$. Choose an element n of D. Then the set $A = D \cap [n]$ is nonempty, so that A has a smallest element k. The element k is automatically the smallest element of D as well. □

Remark. *I don't really understand the second part of this proof. By* https://proofwiki.org*, Principle of Mathematical Induction, Well-Ordering Principle, and Principle of Complete Induction are logically equivalent. (link)*[2]

Definition 1.10. Cartesian product Let $\{A_1, \dots, A_m\}$ be a faimily of sets indexed with the set $\{1. \dots, m\}$. Let $X = A_1 \cup \dots \cup A_m$. We define *cartesian product* of this indexed family, denoted by

$$\prod_{i=1}^{m} A_i \text{ or } A_1 \times \dots \times A_m,$$

to be the set of all m-tuples $(x_1, \dots, x_m)$ of elements of X such that $x_i \in A_i$ for each i.

Remark. *Indexing function $f : J \to \mathcal{A}$ is surjective but not necessarily injective.*

Definition 1.11. ω-tuple An *ω-tuple* of elements of set X to be a function

$$x : \mathbb{Z}^+ \to X,$$

a.k.a. sequence, or a *infinite sequence.*

Theorem 1.12. $\{0, 1\}^\omega$ is uncountable. (let $X = \{0, 1\}$ in the proof.)

Proof. We show that given any function $g : \mathbb{Z}^+ \to X^\omega$, g is not surjective. Four this purpose, let us denote $g(n)$ as $(x_{n1}, x_{n2}, \dots, x_{nn}, \dots, x_{n\omega})$, where each x_{ij} is eather 0 or 1. Then we define any element $y = (y1, \dots, y_\omega)$ of X^ω by letting

$$y_n = \begin{cases} 0 & \text{if } x_{nn} = 1, \\ 1 & \text{if } x_{nn} = 0. \end{cases}$$

y will differ $g(n)$ for all n by a digit. Therefore $y \notin \mathrm{Im}(g)$. □

Remark. *Note this proof is similar to the proof of uncountableness of $[0, 1)$ using the vast digit array.*

Remark. $\{0, 1\}^\omega \simeq [0, 1)$ *by* $f(a_1, a_2, \dots) = \sum_{i=1}^{\infty} a_i 2^{-i}$. *(i.e. binary decimals). Then we can use the conclusion of the uncountableness of $[0, 1)$ to prove this directly.*

[2] `https://proofwiki.org/wiki/Equivalence_of_Well-Ordering_Principle_and_Induction#Final_assembly`

Remark. *Think of picking a subset of* $\mathbb{Z}^+$*, for each* $i \in \mathbb{Z}^+$ *present in the subset, set* $a_i = 1$*, otherwise* $a_i = 0$*. Thus* $\{0,1\}^\omega$ *is just isomorphic to the power set* $2^{\mathbb{Z}^+}$*. By cantor's theorem, there is not surjection* $f : \mathbb{Z}^+ \to 2^{\mathbb{Z}^+}$*.*

Theorem 1.13. There is not surjective map $g : A \to 2^A$ for all set A. Proof: (link)[3]

Theorem 1.14. Principle of recursive definition Let A be a set; let a_0 be an element of A. Suppose ρ is a function that assigns, to each function f mapping a nonempty section of the positive integers into A, an element of A. Then there exists a unique function

$$h : \mathbb{Z}^+ \to A$$

such that

$$\begin{aligned} h(1) &= a_0, \\ h(i) &= \rho(h|\{1, \dots, i-1\}) \text{ for } i > 1 \end{aligned}$$

The formula is called a *recursion formula* for h.

Remark. *I'm not very clear about this definition. I think the point of this definition is to indicate that there is a UNIQUE function satisified a recursive definition.*

Theorem 1.15. The following statements about set A are equivalent:

1. There exists an *injective*, not necessarily surjective (of course), function $f : \mathbb{Z}^+ \to A$.
2. There exists a bijection of A to a propert subset of A.
3. A is infinite.

Example 1.16.

1. $f : \mathbb{Z}^+ \hookrightarrow \mathbb{R}$.
2. $f : \mathbb{R} \to (-1, 1)$ by $f(0) = 0, f(a) = \frac{1}{a}$.
3. $\mathbb{R}$ is infinite.

Definition 1.17. Axiom of choice Given a collection $\mathcal{A}$ of disjoint nonempty sets (*i.e.* $\forall x, y \in \mathcal{A}, x \cap y = \emptyset$), there exists a set C consisting of exactly one element from each element of $\mathcal{A}$; that is, $C \subset \bigcup \mathcal{A}$ and for each $A \in \mathcal{A}$, the set $C \cap A$ contains a single element.

The set C can be thought of as having been obtained by choosing one element from each of the sets in $\mathcal{A}$.

Theorem 1.18. Well-ordering theorem Every set is well-orderable. (assuming axiom of choice)

[3] `https://proofwiki.org/wiki/Cantor's_Theorem`

Remark. *This theorem is rather against intuition. As I checked briefly on wikipedia and proofwiki, the proof (link)*[4] *is done using the axiom of choice and the Princple of Transfinite Induction. I can't comprehend the proof since I have no related knowledge about transfinite induction schema. The result is also known as* ***Zermelo's Theorem****(link)*[5]*, which is usually worded: Every set of cardinals is well-ordered with respect to* $\leq$*. This is weird to me since doesn't it imply that cardinals are equivalent to ordinals?*

Definition 1.19. Strict partial order A set A, a relation $\prec \subset A \times A$ is a *strict partial order* if for all elements $a, b, c \in A$,

1. Non-refl: $\neg(x \prec x)$
2. Trans: $x \prec y \wedge y \prec z \Rightarrow x \prec z$

Remark. *Compare to (non-strict) partial order, strict partial order does not have reflexivity property* ($x \leq x$)*.*

Remark. *Compare to a simply order relation, this order relation does not require totality (or comparability).*

Theorem 1.20. The maxium principle A set A, a strict partial order $\prec$ on A. There exists a maxiaml simply ordered subset B.

Said differently, there exists a subset B of A such that B is simply ordered by $\prec$ and such that no subset of A that properly contains B is simply ordered by $\prec$.

Remark. *Think in this way. Draw a acyclic directed graph for A: for for an element in A draw a vertex, for each pair of vertices, draw an edge from a to b iff* $a \prec b$ *is minimal, i.e.* $\neg\exists c : a \prec c \prec b$*. Then we can say* $a \prec b$ *if a is connected to b by the transitivity property. This thereom actually says that there exists a maximal path in the graph.*

Definition 1.21. Upper bound and **Maximal element** Let A be set and $\prec$ be strict partial order on A. $B \subset A$. An *upper bound* on B is $c \in A$ such that $\forall b \in B : b = c \vee b \prec c$. A *maximal element* of A is an element $m \in A$ such that $\neg\exists a, m \prec a$.

Remark. *When we talk about upper bounds of A, we are under implication of a subset A of a ordered set.*

Theorem 1.22. Zorn's Lemma Let A be strictly partially ordered set. If every simply ordered subset of A has an upper bound in A, then A has a maximal element.

Remark. *This is a consequence of the maximum principle. It's easy to verify the maximum element is the upper bound of the maximal simply ordered subset.*

[4] `https://proofwiki.org/wiki/Well-Ordering_Theorem`
[5] `https://proofwiki.org/wiki/Zermelo%27s_Theorem_(Set_Theory)`

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