Notes on James R. Munkres' Topology (2E)

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Contents

Chapter 0

Structure and reading plans

Ch 1-8 is the part I, mainly for common topology. The part II includes ch 9-14, that depends on ch 1-4, is about algebraic topology.

My plan is to read through ch 1-4 very quickly, within a weekend, and then I will start reading ch 9+ simultaneously with W.S.Massey's Algebraic topology: An induction.

Finally I wish I could finish all ch 1-8 and also some parts after ch 9.

Chapter 1

Set theory and logic

Definition 1.1. Order relation rel C on set A is called *order relation* if

1. comparability, *a.k.a.* totality for all non-eq elements, *i.e.* $\forall x, y \in A, x \neq y \Rightarrow xCy \vee yCx$
2. non-refl, *i.e.* $\forall x, \neg(xCx)$
3. trans, *i.e.* $\forall xCy \wedge yCz, xCz$

(*a.k.a.* **Linear order**, **Simple order**)

Remark. *This relation is not the same as Linear order on Wikipedia (link)*[1]*. This order is actually the strict version of the Total order on wikipedia, i.e. has non-refl property.*

Definition 1.2. Open interval if X is a set and $<$ is an order rel, and if $a < b$ we use notation (a, b) to denote $\{x \in X \mid a < x < b\}$, called *open interval.*

If $(a, b) = \emptyset$, then a is called *immediate precessor* of b and b called *immediate successor* of a.

Remark. *It makes more sense on X is a discrete set. Since if (a, b) is an open interval in $\mathbb{R}$, $(a, b) = \emptyset \Rightarrow a = b$ which makes no sense on a as an immediate precessor of b.*

Definition 1.3. Order type if A and B are two sets with $<_A$ and $<_B$. We say that A and B have same *order type* if $\exists f : A \to B$ that preserves order, *i.e.*

$$a_1 <_A b_1 \Rightarrow f(a_1) <_B f(b_1)$$

Remark. *It's just a generalization of monotone function.*

Definition 1.4. Dictionary order relation if A,B are two sets with ($<_A, <_B$), defn an order for $A \times B$ by defining

$$a_1 \times b_1 < a_2 \times b_2$$

if $a_1 <_A a_2$, or if $a_1 = a_2 \ \wedge \ b_1 <_B b_2$.

[1] https://en.wikipedia.org/wiki/Total_order

Definition 1.5. LUB property/GLB property For A and $<_A$, we say A has *LUB property* if

$$\forall A_0 \subset A, A \neq \emptyset \wedge \exists \text{upper bound for } A_0 \Rightarrow \exists \text{lub}\{A_0\} \in A$$

Example 1.6. $A = (-1, 1)$. *e.g.* $X = \{1 - \frac{1}{n} \mid n \in \mathbb{Z}^+\}$ does not have an upper bound, thus vacuously true. $\{-\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ has upper bound of any number in $[0, 1) \subset A$, and $\text{lub}(X) = 0 \in (-1, 1)$.

Example 1.7. Counterexample. $A = (-1, 0) \cup (0, 1)$. $\{-\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ has upper bound of any $(0, 1) \subset A$, while $\text{lub}(X) = 0 \notin A$.

Remark. *The completeness property of* $\mathbb{R}$ *as an axiom derives this property.*

Property 1.8. $\mathbb{R}$ field

Algebraic properties

1. assoc: $(x + y) + z = x + (y + z); (xy)z = x(yz)$
2. comm: $x + y = y + x; xy = yx$
3. id: $\exists!0, x + 0 = x$; $\exists!1, x \neq 0 \Rightarrow x1 = x$
4. inv: $\forall x, \exists!y, x + y = 0$; $\forall x \neq 0, \exists!y, xy = 1$
5. distr: $x(y + z) = xy + xz$

Mixed algebraic and order property

6. $x > y \Rightarrow x + z > y + z$; $x > y \wedge z > 0 \Rightarrow xz > yz$

Order properties

7. $<$ has LUB property
8. $\forall x < y, \exists z, x < z \wedge z < y$

1-6 make $\mathbb{R}$ a field. 1-6 + 7 make $\mathbb{R}$ an ordered field. 7-8 makes $\mathbb{R}$, called by topologists, a **Linear continuum**.

Theorem 1.9. Well ordering property $\mathbb{Z}^+$ has *Well-ordering property. i.e.* Every nonempty subset of $\mathbb{Z}^+$ has a smallest element.

Proof. We first prove that for each $n \in \mathbb{Z}^+$, the following statement holds: Every nonempty subset of $\{1, \ldots, n\}$ has a smallest element.

Let A be the set of all postive integers n for which this theorem holds. Then A contains 1, since if $n = 1$, the only possible subset is $\{1\}$ itself. Then suppose A contains n, we show that it contains $n + 1$. So let C be a nonempty subset of the set $\{1, \ldots, n + 1\}$. If C consists of the single element $n + 1$, then that element is the smallest element of C. Otherwise, consider the set $C \cap \{1, \ldots, n\}$, which is nonempty. Because $n \in A$, this set has a smallest element, which will automaticallly be the smallest element of C also. Thus A is inductive, so we conclude that $A = \mathbb{Z}^+$; hence the statement is true for all $n \in \mathbb{Z}^+$.

Now we prove the theorem. Suppose that D is a nonempty subset of $\mathbb{Z}^+$. Choose an element n of D. Then the set $A = D \cap [n]$ is nonempty, so that A has a smallest element k. The element k is automatically the smallest element of D as well. $\square$

Remark. *I don't really understand the second part of this proof. By* https://proofwiki.org*, Principle of Mathematical Induction, Well-Ordering Principle, and Principle of Complete Induction are logically equivalent. (link)*[2]

Definition 1.10. Cartesian product Let $\{A_1, \ldots, A_m\}$ be a faimily of sets indexed with the set $\{1 \ldots, m\}$. Let $X = A_1 \cup \cdots \cup A_m$. We define *cartesian product* of this indexed family, denoted by

$$\prod_{i=1}^{m} A_i \text{ or } A_1 \times \cdots \times A_m,$$

to be the set of all m-tuples $(x_1, \ldots, x_m)$ of elements of X such that $x_i \in A_i$ for each i.

Remark. *Indexing function $f : J \to \mathcal{A}$ is surjective but not necessarily injective.*

Definition 1.11. ω-tuple An *ω-tuple* of elements of set X to be a function

$$x : \mathbb{Z}^+ \to X,$$

a.k.a. sequence, or a *infinite sequence.*

Theorem 1.12. $\{0, 1\}^\omega$ is uncountable. (let $X = \{0, 1\}$ in the proof.)

Proof. We show that given any function $g : \mathbb{Z}^+ \to X^\omega$, g is not surjective. Four this purpose, let us denote $g(n)$ as $(x_{n1}, x_{n2}, \ldots, x_{nn}, \ldots, x_{n\omega})$, where each x_{ij} is eather 0 or 1. Then we define any element $y = (y1, \ldots, y_\omega)$ of X^ω by letting

$$y_n = \begin{cases} 0 & \text{if } x_{nn} = 1, \\ 1 & \text{if } x_{nn} = 0. \end{cases}$$

y will differ $g(n)$ for all n by a digit. Therefore $y \notin \operatorname{Im}(g)$. $\square$

Remark. *Note this proof is similar to the proof of uncountableness of $[0, 1)$ using the vast digit array.*

Remark. *$\{0, 1\}^\omega \simeq [0, 1)$ by $f(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} a_i 2^{-i}$. (i.e. binary decimals). Then we can use the conclusion of the uncountableness of $[0, 1)$ to prove this directly.*

[2] https://proofwiki.org/wiki/Equivalence_of_Well-Ordering_Principle_and_Induction#Final_assembly

Remark. *Think of picking a subset of* $\mathbb{Z}^+$*, for each* $i \in \mathbb{Z}^+$ *present in the subset, set* $a_i = 1$*, otherwise* $a_i = 0$*. Thus* $\{0,1\}^\omega$ *is just isomorphic to the power set* $2^{\mathbb{Z}^+}$*. By cantor's theorem, there is not surjection* $f : \mathbb{Z}^+ \to 2^{\mathbb{Z}^+}$*.*

Theorem 1.13. There is not surjective map $g : A \to 2^A$ for all set A. Proof: (link)[3]

Theorem 1.14. Principle of recursive definition Let A be a set; let a_0 be an element of A. Suppose ρ is a function that assigns, to each function f mapping a nonempty section of the positive integers into A, an element of A. Then there exists a unique function

$$h : \mathbb{Z}^+ \to A$$

such that

$$\begin{aligned} h(1) &= a_0, \\ h(i) &= \rho(h|\{1, \dots, i-1\}) \text{ for } i > 1 \end{aligned}$$

The formula is called a *recursion formula* for h.

Remark. *I'm not very clear about this definition. I think the point of this definition is to indicate that there is a UNIQUE function satisified a recursive definition.*

Theorem 1.15. The following statements about set A are equivalent:

1. There exists an *injective*, not necessarily surjective (of course), function $f : \mathbb{Z}^+ \to A$.
2. There exists a bijection of A to a propert subset of A.
3. A is infinite.

Example 1.16.

1. $f : \mathbb{Z}^+ \hookrightarrow \mathbb{R}$.
2. $f : \mathbb{R} \to (-1, 1)$ by $f(0) = 0, f(a) = \frac{1}{a}$.
3. $\mathbb{R}$ is infinite.

Definition 1.17. Axiom of choice Given a collection $\mathcal{A}$ of disjoint nonempty sets (*i.e.* $\forall x, y \in \mathcal{A}, x \cap y = \emptyset$), there exists a set C consisting of exactly one element from each element of $\mathcal{A}$; that is, $C \subset \bigcup \mathcal{A}$ and for each $A \in \mathcal{A}$, the set $C \cap A$ contains a single element.

The set C can be thought of as having been obtained by choosing one element from each of the sets in $\mathcal{A}$.

Theorem 1.18. Well-ordering theorem Every set is well-orderable. (assuming axiom of choice)

[3] `https://proofwiki.org/wiki/Cantor's_Theorem`

Remark. *This theorem is rather against intuition. As I checked briefly on wikipedia and proofwiki, the proof (link)*[4] *is done using the axiom of choice and the Princple of Transfinite Induction. I can't comprehend the proof since I have no related knowledge about transfinite induction schema. The result is also known as* ***Zermelo's Theorem****(link)*[5]*, which is usually worded: Every set of cardinals is well-ordered with respect to $\leq$. This is weird to me since doesn't it imply that cardinals are equivalent to ordinals?*

Definition 1.19. Strict partial order A set A, a relation $\prec\ \subset A \times A$ is a *strict partial order* if for all elements $a, b, c \in A$,

1. Non-refl: $\neg(x \prec x)$
2. Trans: $x \prec y \wedge y \prec z \Rightarrow x \prec z$

Remark. *Compare to (non-strict) partial order, strict partial order does not have reflexivity property ($x \leq x$).*

Remark. *Compare to a simply order relation, this order relation does not require totality (or comparability).*

Theorem 1.20. The maxium principle A set A, a strict partial order $\prec$ on A. There exists a maxiaml simply ordered subset B.

Said differently, there exists a subset B of A such that B is simply ordered by $\prec$ and such that no subset of A that properly contains B is simply ordered by $\prec$.

Remark. *Think in this way. Draw a acyclic directed graph for A: for for an element in A draw a vertex, for each pair of vertices, draw an edge from a to b iff $a \prec b$ is minimal, i.e. $\neg\exists c : a \prec c \prec b$. Then we can say $a \prec b$ if a is connected to b by the transitivity property. This thereom actually says that there exists a maximal path in the graph.*

Definition 1.21. Upper bound and **Maximal element** Let A be set and $\prec$ be strict partial order on A. $B \subset A$. An *upper bound* on B is $c \in A$ such that $\forall b \in B : b = c \vee b \prec c$. A *maximal element* of A is an element $m \in A$ such that $\neg\exists a, m \prec a$.

Remark. *When we talk about upper bounds of A, we are under implication of a subset A of a ordered set.*

Theorem 1.22. Zorn's Lemma Let A be strictly partially ordered set. If every simply ordered subset of A has an upper bound in A, then A has a maximal element.

Remark. *This is a consequence of the maximum principle. It's easy to verify the maximum element is the upper bound of the maximal simply ordered subset.*

[4] `https://proofwiki.org/wiki/Well-Ordering_Theorem`
[5] `https://proofwiki.org/wiki/Zermelo%27s_Theorem_(Set_Theory)`

Chapter 2

Topological spaces and continuous functions

Definition 2.1. Topology A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X (*i.e.* $\mathcal{T} \subset 2^X$) having the following properties:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$,
3. The intersection of elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

Remark. *After investigation, I consider there is no special meaning for a collection differing from a set. (ref. (link)[1], (link)[2])*

Definition 2.2. Topological space A *topological space* is an ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ of X. Often we omit specfic mention of $\mathcal{T}$ if no confusion will arise.

Definition 2.3. Open set If X is a topological space, we say that $U \subset X$ is an *open set* of X if $U \in \mathcal{T}$. Using this terminology, one can say a topological space is a set X together with a set of open sets. Thus the defintion of an open set is the same as topology: $\emptyset$ and X both open, arbitrary unions and finite intersections of open sets are open.

Definition 2.4. If $\mathcal{T} = 2^A$, then X is called a **Discrete topology**;
If $\mathcal{T} = \{\emptyset, X\}$, then X is called a **Trivial topology**, or **Indiscrete topology**.

Definition 2.5. Finite complement topology Let X be a set, $\mathcal{T}_f$ be the collection of all subsets U of X such that $X - U$ either is finite or is all of X. Then $\mathcal{T}_f$ is a topology on X, called the *finite complement topology.* Both X and $\emptyset$ are in $\mathcal{T}_f$ since $X - X$ is finite and $X - \emptyset$ is all of X.

[1] `http://math.stackexchange.com/a/173002`
[2] `https://proofwiki.org/wiki/Definition:Topology/Definition_1`

Now we show $\mathcal{T}_f$ is a topology. If $\{U_a\}$ is an indexed family of nonempty elements of $\mathcal{T}_f$, to show $\bigcup U_a$ is in $\mathcal{T}_f$, we compute

$$X - \bigcup U_a = \bigcap (X - U_a)$$

Which is finite since $X - U_a$ is finite for all $U_i \in \mathcal{T}_f$. To show $\bigcap U_i \in \mathcal{T}_f$, we compute

$$x - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i)$$

Note by definition of topology we only need to check finite intersection. Then the finite number of unions of finite sets is finite thus finite $\bigcup U_i \in \mathcal{T}_f$.

Definition 2.6. Finer, **Strictly finer**, **Coarser**, **Strictly coarser**, and **Comparable**

	set term	topology term
$\mathcal{T}' \supset \mathcal{T}$	superset	finer, larger, stronger
$\mathcal{T}' \supsetneq \mathcal{T}$	proper superset(?)	strictly finer
$\mathcal{T}' \subset \mathcal{T}$	subset	coarser, larger, weaker
$\mathcal{T}' \subsetneq \mathcal{T}$	proper subset	strictly coarser
$\mathcal{T}' \subset \mathcal{T}$ or $\mathcal{T} \subset T'$	-	comparable

Remark. *This is a bit not straightforward at the first sight. Remember that the essential of topology is its structure. The more open sets we have in a topology, the more fine it is.*

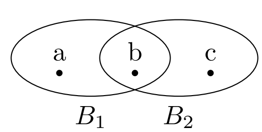
Definition 2.7. Basis A *basis* for a topology $\mathcal{B} \subset 2^X$ (called basis elements) such that

1. $\forall x \in X : \exists B \in \mathcal{B} : x \in B$
2. $\forall x \in B_1 \cap B_2 \Rightarrow \exists B_3 \in \mathcal{B} : x \in B_3 \wedge B_3 \subset B_1 \cap B_2$

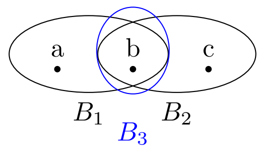
If $\mathcal{B}$ is a basis, we define *topology $\mathcal{T}$ generated by $\mathcal{B}$* to be: we say $U \subset X$ is open set (*i.e.* $U \in \mathcal{T}$), if $\forall x \in U : \exists B \in \mathcal{B} : x \in B \wedge B \subset U$.

Remark. *The concept of a basis of a topology is rather abstract. I think of it in this way. Since we are dealing with intersection in the definition of basis, we want to think from top to bottom, in other words, from larger sets, to their intersections.*

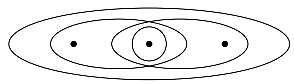
Example 2.8. Think of this example. In the beginning we have $X = \{a, b, c\}$ and $\mathcal{B} = \{B_1, B_2\}$ where $B_1 = \{a, b\}$ and $B_2 = \{b, c\}$.



In this way $\forall x \in X : \exists B \in \mathcal{B} : x \in B$. Now we try to satisfy the second condition and we will find that $b \in B_1 \cap B_2$ is not in any basis element who is a subset of $B_1 \cap B_2$. Now we add it as follows:



Now with $\mathcal{B} = \{B_1, B_2, B_3\}$ all above two criterion are satisified. Therefore $\mathcal{B}$ is a basis for the topology $\mathcal{T}$:



Example 2.9. Continuing above example. We now verify that $\mathcal{T}$ is generated by $\mathcal{B}$ by checking all open sets $U \in \mathcal{T}$ with the rule specified. Actually $\mathcal{T} = \{\emptyset, \{a, b\}, \{b, c\}, \{b\}, \{a, b, c\}\}$. I will only show how $\{a, b\} \in \mathcal{T}$ and how $\{b\} \in \mathcal{T}$ and how $\{a, c\} \notin \mathcal{T}$.

1. $U = \{a, b\} \in \mathcal{T}$: For $a \in U$, take $B = \{a, b\}$, then $B \subset U$; the same works for b. Therefore U is an open set.

2. $U = \{b\} \in \mathcal{T}$: For $b \in U$, take $B = \{b\}$, then $B \subset U$. Therefore $\{b\}$ is an open set. Note that we cannot take $B = \{a, b\}$, since $\{a, b\} \not\subset \{b\}$.

3. $U = \{a, c\} \notin \mathcal{T}$: For $a \in U$, we cannot find a $B \in \mathcal{B}$ that contains a and is a subset of $\{a, c\}$. Since the only possible subsets of $\{a, c\}$ are $\{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and neither is a basis element. Therefore $\{a, c\}$ is not an open set.

Remark. *Let $\mathcal{B}$ be set of all one-point subsets of X, then it is a basis for the discrete topology on X.*

Remark. *So my understanding of a basis of a topology is like the generator of the topology that satisfy the existence of intersection. So first of all we have to note that $\mathcal{B} \subset \mathcal{T}$.*

Using the process from the book, we can verify it. Take J to be an indexed family of $\mathcal{B}$.

It's easy to show that $\bigcup_{\alpha \in J} B_\alpha$ is in $\mathcal{T}$ if $\forall \alpha \in J : B_\alpha \in \mathcal{T}$.

It's easier to to show that $\bigcap_{\alpha \in J} B_\alpha$ is in $\mathcal{T}$ if $\forall \alpha \in J : B_\alpha \in \mathcal{T}$ since it's specified in the criteria for $\mathcal{B}$ to be a basis. (Actually basis does more than finite intersection. We can prove using induction that the intersection of any countable number of basis is in $\mathcal{T}$.

Also, $\emptyset \in \mathcal{T}$ will be vacuously true no matter what $\mathcal{B}$ we pick.

Theorem 2.10. A more visual-able theorem. $\mathcal{T}$ is the all unions of elements in $\bigcup_{B\in\mathcal{B}} B$.

Proof. Given $B = \bigcup_{\alpha\in K} B_\alpha$, since $B_\alpha \in \mathcal{T}$ and for any $x, y \in \mathcal{T}$, $x \cup y \in \mathcal{T}$. Thus $B \in \mathcal{T}$. Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$, $\exists B_x \in \mathcal{B}$ and $U = \bigcup_{x\in U} B_x$. Therefore U is some union of elements in $\mathcal{B}$. □

Definition 2.11. Order topology Assuming we knowing about all open/closed/half-open interval concepts, *over a, possibly not continuous, set.*

Let X be a set with linear order relation with $<$. Assume $|X| > 1$. Let $\mathcal{B}$ be defined as a set of all subsets of X of the following types:

1. All valid (a, b)
2. For a_0 to be the minimal element, $[a_0, b)$ (if any)
3. For b_0 to be the maximal element, $(a, b_0]$ (if any)

Then B is a basis for a topology on X, called *order topology.*

Example 2.12. $\mathbb{R}$, $\mathbb{R}\times\mathbb{R}$ in dict order, $\mathbb{Z}^+$

Definition 2.13. Ray, **Open ray**, **Closed ray**

$$(a, +\infty) = \{x \mid x > a\}$$
$$(-\infty, a) = \{x \mid x < a\}$$
$$[a, +\infty) = \{x \mid x \geq a\}$$
$$(-\infty, a] = \{x \mid x \leq a\}$$

Definition 2.14. Product topology Product topology on topology spaces X and Y, denoted as $X \times Y$ is the topology with the basis $\mathcal{B}$ who elements are in form of $U \times V$ where $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$. (Obviously and omitted by book the underlying set is just $X \times Y$)

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