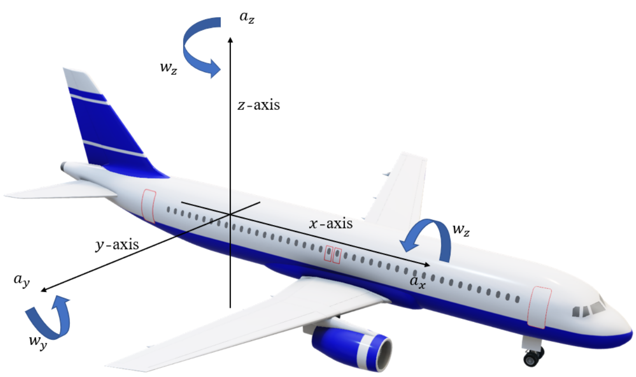
Coordinate System Transformation for Inertial Sensors

1. Inertial Measurement Unit

An inertial sensor, or inertial measurement unit (IMU), is an electronic device that measures the real-time movement status of the device. Based on the acceleration and rotation angle recorded by accelerometers and gyroscopes, the moving trace of the device can be calculated. However, the recorded data is in the device's coordinate system, and we need to transfer it into the global coordinate system, to which the moving trace belongs. This article will mainly be about how to do coordinate system transformation for IMU.

2. 6-Dof of IMU

In three-dimensional space, we all know that an object has six degrees of freedom, namely, the freedom of movement along the x, y, and z rectangular axes and the freedom of rotation around these three axes. Picture 1 shows the accelerations (a_x, a_y, a_z) and rotation anglular velocities (w_x, w_y, w_z) in the IMU coordinate system.



Picture 1

Here we use $\mathbf{a^b}(t)$ ($\mathbf{a^b} \in \mathbb{R}^3$) to represent the three-axis raw data of acceleration, and $\mathbf{w^b}(t)$ ($\mathbf{w^b} \in \mathbb{R}^3$) to represent the three-axis raw data of angle velocities, where $\mathbf{b}$ represents the body coordinate of IMU. The formulas of $\mathbf{a^b}(t)$ and $\mathbf{w^b}(t)$ are given below:

$$\mathbf{a^b}(t) = \{a_x^b(t), a_y^b(t), a_z^b(t)\}$$
$$\mathbf{w^b}(t) = \{w_x^b(t), w_y^b(t), w_z^b(t)\}$$

where x, y, z represent the three directions in the body coordinate system for the device.

To obtain the moving trajectory of IMU, not only the angular velocities are required to be transformed, but the accelerations also need the transformation, that is the orientation of the moving IMU in global coordinate.

In the following contents, we will gradually figure out the methodology of how to transform $\mathbf{a^b}, \mathbf{w^b}$ to $\mathbf{a^n}, \mathbf{w^n}$, where the uper notation $\mathbf{n}$ represents the global coordinate.

3. Coordinate Transformation of Euler Angle

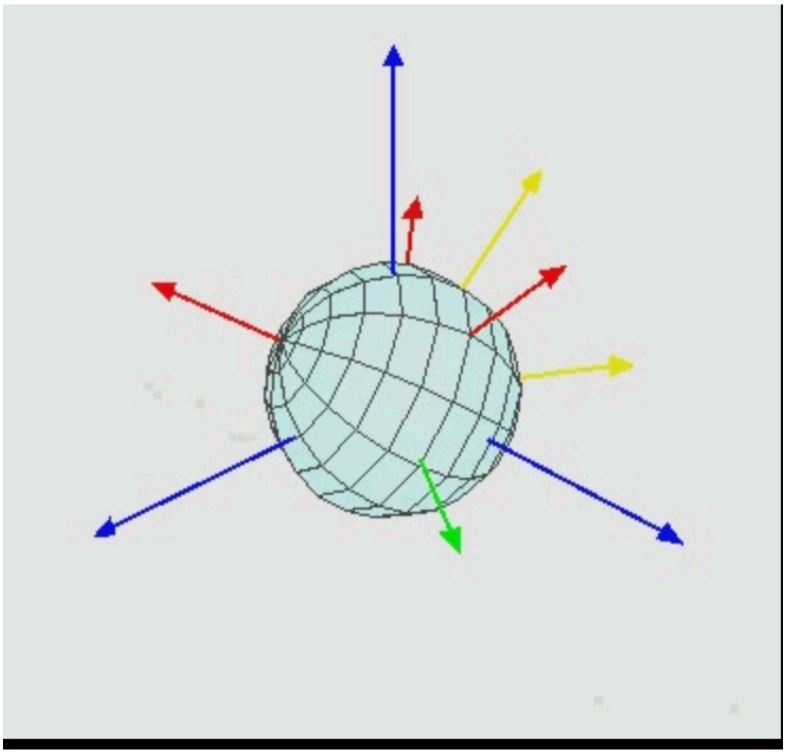


Figure 2

The **Euler angles** are three angles introduced by Leonhard Euler to describe the orientation of a rigid body with respect to a fixed coordinate system (wikipedia link). Figure 2 (you can check the link to see the gif of this image, which can help you understand better) shows the rotation process from the original coordinate (the blue one). It seems that the rotation process completes at once, but actually it can be separated into three processes, which are the rotation of x-axis, y-axis, and z-axis.

To make you understand more easily, let's take the rotation of x-axis for examle, and see what happens to the tranformation of body coordinate to the global coordinate.

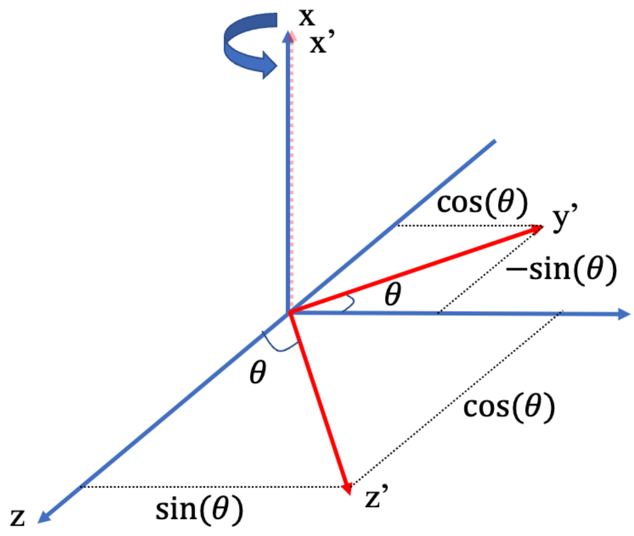


Figure 3

As Figure 3 shows, the x-axis remains the same, so you can achieve the equation $x = (1, 0, 0)(x', y', z')^T$. Similarly, you can obtain $y = (0, \cos(\theta), \sin(\theta))(x', y', z')^T$, and $z = (0, -\sin(\theta), \cos(\theta))(x', y', z')^T$. Combining these three equations together, the rotation matrix of x-axis from body coordinate to global coordinate is obtained:

$$\begin{pmatrix} x^n \\ y^n \\ z^n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x^b \\ y^b \\ z^b \end{pmatrix}$$

And $\mathbf{R}_x(\theta)$ is used to present the rotation matrix of x-axis from body frame to global frame. The rotation matrix of y-axis and z-axis can be obtained in the same way.

$$\mathbf{R}_y(\phi) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$\mathbf{R}_z(\pi_t) = \begin{pmatrix} \cos\pi & \sin\pi & 0 \\ -\sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By multiplying these three matrices, we can get the acceleration parameters for the three axes in the global coordinate system.

$$\begin{pmatrix} x^n \\ y^n \\ z^n \end{pmatrix} = \mathbf{R}_x(\theta)\mathbf{R}_y(\phi)\mathbf{R}_z(\pi) \begin{pmatrix} x^b \\ y^b \\ z^b \end{pmatrix}$$

And there raises the question, how do we get the rotation angle θ, ϕ, π? It should be noticed that when the IMU is moving, the body coordinate is only relative with the angular velocities. Assume that at time t_0, the device's orientation relative to the three coordinate axes of the global coordinate system are θ_{t_0}, ϕ_{t_0} and π_{t_0} respectively. When the time is t, we can calculate the device's orientation based on the gyroscope's data.

$$\theta_t = \theta_{t_0} + \int_{t_0}^{t} w_x^b(t)dt$$
$$\phi_t = \phi_{t_0} + \int_{t_0}^{t} w_y^b(t)dt$$
$$\pi_t = \pi_{t_0} + \int_{t_0}^{t} w_z^b(t)dt$$

The position of IMU at time t can be cumulated through double integeration.

$$\mathbf{a^n} = \mathbf{R}_x(\theta)\mathbf{R}_y(\phi)\mathbf{R}_z(\pi)\mathbf{a^b}$$

$$\mathbf{P^n} = \mathbf{P^n_{t_0}} + \int_{t_0}^{t} \mathbf{V^n}(t)\mathrm{d}t = \mathbf{P^n_{t_0}} + \int_{t_0}^{t} \left(\mathbf{V^n}(t_0) + \int_{t_0}^{t} \mathbf{a^n}(t)\mathrm{d}t\right)\mathrm{d}t$$

4. Quaternian and Coordinate Transformation

Euler's formular: e^x, is a fundamental formular. If x owns imaginary part, the formular represents the rotation on a 2-dim platform. Futher, e^x can also represents the rotation on 3-dim platform. In this case, the x is called **quaternian**.

So, what is quiaternian? Quaternian is a vector which contains the real part and the imaginary part. The math equation is given below:

$$q = [s, \mathbf{q}]$$

where s is the real part of quaternian, and $\mathbf{q}$ is the imaginary vector: $\mathbf{q} = [x, y, z] = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Now we set $\theta = |\mathbf{q}|$, hence we can obtain a sequence of equations:

$$\mathbf{q}^0 = 1$$
$$\mathbf{q}^2 = -\theta^2$$
$$\ldots\ldots$$

So, what happens to $e^{\mathbf{q}}$ when $\mathbf{q}$ is an imaginary vector. According to the Taylor expansion formula, $e^{\mathbf{q}}$ can be written as:

$$
\begin{aligned}
e^{\mathbf{q}} &= \sum_{n=0}^{\infty} \frac{\mathbf{q}^n}{n!} \\
&= 1 + \frac{\mathbf{q}}{1!} - \frac{\theta^2}{2!} - \frac{\theta^2 \mathbf{q}}{3!} + \ldots\ldots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + \frac{\mathbf{q}}{\theta}\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right) \\
&= \cos(\theta) + \frac{\mathbf{q}}{\theta}\sin(\theta) \\
&= \cos(|\mathbf{q}|) + \frac{\mathbf{q}}{|\mathbf{q}|}\sin(\mathbf{q}) \\
&= e^{\theta \mathbf{n}}
\end{aligned}
$$

And this is the famous rotation formula for unit quaternion. Usually, we will set $\mathbf{n} = \frac{\mathbf{q}}{|\mathbf{q}|}$ as the unit vector which is the axis of rotation, and the rotation angle is given as $\theta = |\mathbf{q}|$. So the formula means to do the rotation of θ degree along the axis of rotation $\mathbf{n}$ in 3D space.

For a specific vector $\mathbf{v}$, by multipying $e^{\theta \mathbf{n}}$ on the left side, we can obtain the vector $\mathbf{v}'$ after the rotation process. That is:

$$
\mathbf{v}' = e^{\theta \mathbf{n}} \mathbf{v}
$$

Euler's formula also has such a transformation form:

$$
e^{\theta \mathbf{n}} \mathbf{v} = \mathbf{v} e^{-\theta \mathbf{n}}
$$

It is easy to understand that converting from a right-handed coordinate system to a left-handed coordinate system, the rotation axis $\mathbf{n}$ or the rotation angle θ has an inversion to represent the same rotation. So we can have:

$$
\begin{aligned}
\mathbf{v}' &= e^{\frac{\theta}{2}\mathbf{n}} \mathbf{v} e^{-\frac{\theta}{2}\mathbf{n}} \\
&= \mathbf{q}\mathbf{v}\mathbf{q}^*
\end{aligned}
$$

Finally, we have the desired rotation formula:

$$
\mathbf{q} = [\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\mathbf{i}, \sin\frac{\theta}{2}\mathbf{j}, \sin\frac{\theta}{2}\mathbf{k}] = e^{\frac{\theta}{2}\mathbf{n}}
$$

You may have a question of why we don't use $e^{\theta \mathbf{n}}$ as the rotation formula, because this formula can better explain the way the rotation of the component parallel to the axis of rotation and the component perpendicular to the axis of rotation. If you are interested in this, you can collect more information to understand it. [chinese reference document].

The multiplication of quaternions is slightly different from that of vectors. We will give the equivalent formula for your better understanding.

$$\mathbf{q} = [q_0, q_1\mathbf{i}, q_2\mathbf{j}, q_3\mathbf{k}]$$

$$\begin{aligned}
\mathbf{v}' &= \mathbf{q}\mathbf{v}\mathbf{q}^* \\
\mathbf{v}' &= L(\mathbf{q})R(\mathbf{q}^*)\mathbf{v} \\
\mathbf{v}' &= \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \mathbf{v}
\end{aligned}$$

You don't need to understand this formula in depth, that it will cost you a lot of time.

Now that we have a solution to transform a vector in 3D space with a quaternion, but our task has not been done yet. It is obvious that the quaternion $\mathbf{q}$ is always changing as time flys. How can we use the received angular velocities to update $\mathbf{q}$ at every timeslot? Back to the equation we have introduced: $\mathbf{q} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{n}$, if you take the derivative of both sides of the equation, you can obtain the following results:

$$\dot{\mathbf{q}} = -\frac{1}{2}\sin\frac{\theta}{2}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{1}{2}\cos\frac{\theta}{2}\frac{\mathrm{d}\theta}{\mathrm{d}t}\mathbf{n} + \sin\frac{\theta}{2}\cdot\frac{\mathbf{n}}{\mathrm{d}t}$$

It is for sure that the rotation axis $\mathbf{n}$ will remain the same, so the last part is 0, hence we can update the above equation:

$$\begin{aligned}
\dot{\mathbf{q}} &= -\frac{1}{2}\sin\frac{\theta}{2}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{1}{2}\cos\frac{\theta}{2}\frac{\mathrm{d}\theta}{\mathrm{d}t}\mathbf{n} \quad (\mathbf{n}\cdot\mathbf{n} = -1) \\
&= \frac{\dot{\theta}}{2}\mathbf{n}\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{n}\right) \\
&= \frac{\dot{\theta}}{2}\mathbf{n}\cdot\mathbf{q} \\
&= \frac{1}{2}\omega_{nb}^{n}\mathbf{q}
\end{aligned}$$

where ω_{nb}^{n} is the angular velocity matrix in global coordinate.

As we have discussed before, to rotate a vector from one coordinate to another one, we can just implement the formula: $\mathbf{v}' = \mathbf{q}\mathbf{v}\mathbf{q}^*$. So, it is obvious to get:

$$\omega_{nb}^{n} = \mathbf{q}\omega_{nb}^{b}\mathbf{q}^*$$

And because the multiply result of a quaternion and its conjugate version is 1, we can further obtain:

$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{q}\omega_{nb}^{b}$$

According to the quaternion multiplication rule (you can search for more information about the standards), we can rewrite the above formula in a matrix form:

$$
\dot{\mathbf{q}} = \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{1}{2}\Omega(\omega_t)\mathbf{q}
$$

With the formulas introduced above, the update formula for rotation quaternion is thus obtained:

$$
\begin{aligned}
\mathbf{q}_t &= \mathbf{q}_{t-1} + \dot{\mathbf{q}}\delta t \\
&= \mathbf{q}_{t-1} + \frac{1}{2}\mathbf{q}_{t-1}\omega_{nb}^{n}\delta t \\
&= \mathbf{q}_{t-1}\mathbf{I}_4 + \frac{1}{2}\delta t \cdot \Omega(\omega_t)\mathbf{q}_{t-1} \\
&= \mathbf{q}_{t-1}(\mathbf{I}_4 + \frac{1}{2}\delta t \cdot \Omega(\omega_t))
\end{aligned}
$$

These formulas enable us to tranfer the accelerations from body frame to global frame, along with the orientation of the IMU. The rest of the work is very simple. You only need a double integral to get the position of the IMU to the ground, which is the same as the last step in the third part.