# Numerical data

- Numerical data is mainly used within scientific context in the modern computers. All the other data types are converted to numerical data to be elaborated

# How do we count?

$$252 = 2 \times 100 + 5 \times 10 + 2 \times 1$$
$$= 2 \times 10^2 + 5 \times 10^1 + 2 \times 10^0$$

The numbering system of the western world (*Hindu-Arabic numerical system*) is:

- decimal
- positional

# Numeral systems

Non positional:

- Roman numerals (ex. V, L, D)
- Arithmetic operations are difficult (ex. V + V = X)

Positional:

- Arabic (decimal)
- Mayan (Base 20)

Hybrid

- Chinese

# Positional numbering system in base B

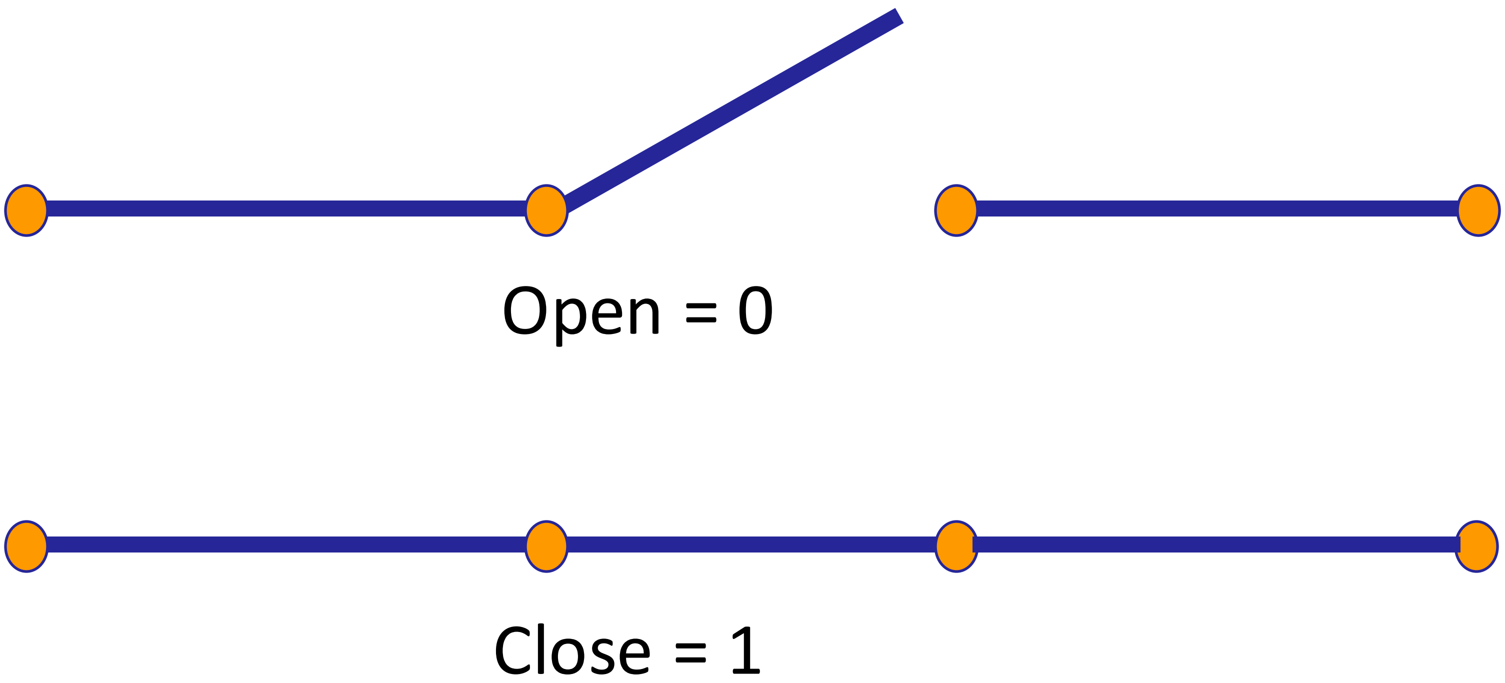
Characteristics:

- Digits: { 0, 1, 2, ..., B-1 }
- Weight of i-th digit: $B^i$
- Representation (natural numbers) on N digits

$$A = \sum_{i=0}^{N-1} a_i \cdot B^i$$

# Bit and switches

A switch has two states
(Open/Close, OFF/ON, FALSE/TRUE)



# Binary system

Base = 2

Digits= { 0, 1 }

Example:

$$101_2 = 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$
$$= 1 \times 4 + 0 + 1 \times 1$$
$$= 5_{10}$$

$$111_2 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$
$$= 1 \times 4 + 1 \times 2 + 1 \times 1$$
$$= 7_{10}$$

# Other binary numbers

| | | | | | |
|---|---|---|---|---|---|
| 0 | ... | 0 | 1000 | ... | 8 |
| 1 | ... | 1 | 1001 | ... | 9 |
| 10 | ... | 2 | 1010 | ... | 10 |
| 11 | ... | 3 | 1011 | ... | 11 |
| 100 | ... | 4 | 1100 | ... | 12 |
| 101 | ... | 5 | 1101 | ... | 13 |
| 110 | ... | 6 | 1110 | ... | 14 |
| 111 | ... | 7 | 1111 | ... | 15 |

# Terminology

**BIT (BInary digiT)** | **0** **1**

**BYTE = eight bits** | **00110110**

**WORD = n bytes** | **00001111** **10101010**

# Terminology

1 0 1 1 0 1 1 0

| MSb | LSb |
| --- | --- |
| Most Significant bit | Least Significant bit |

# Some powers of 2

| | | | | | |
|---|---|---|---|---|---|
| $2^0$ | = | 1 | $2^{11}$ | = | 2,048 |
| $2^1$ | = | 2 | $2^{12}$ | = | 4,096 |
| $2^2$ | = | 4 | $2^{13}$ | = | 8,192 |
| $2^3$ | = | 8 | $2^{14}$ | = | 16,384 |
| $2^4$ | = | 16 | $2^{15}$ | = | 32,768 |
| $2^5$ | = | 32 | $2^{16}$ | = | 65,536 |
| $2^6$ | = | 64 | $2^{17}$ | = | 131,072 |
| $2^7$ | = | 128 | $2^{18}$ | = | 262,144 |
| $2^8$ | = | 256 | $2^{19}$ | = | 524,288 |
| $2^9$ | = | 512 | $2^{20}$ | = | 1,048,576 |
| $2^{10}$ | = | 1,024 | $2^{30}$ | = | 1,073,741,824 |

# Some powers of 2

$2^0 = 1$ $\quad$ $2^{11} = 2,048$

| | | |
|---|---|---|
| $2^3=$ | 8 | |
| $2^8=$ | 256 | |
| $2^9=$ | 512 | |
| $2^{10}=$ | 1,024 | → 1Ki / 1k |
| $2^{20}=$ | 1,048,576 | → 1Mi / 1M |
| $2^{30}=$ | 1,073,741,824 | → 1Gi / 1G |

$2^{10} = 1,024$ $\quad$ $2^{30} = 1,073,741,824$

# Binary to decimal conversion

Apply the definition of the weighted sum of the binary numbers:

$$1101_2$$

$$\begin{aligned} 1101_2 &= 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\ &= 8 + 4 + 0 + 1 \\ &= 13_{10} \end{aligned}$$

# Decimal to binary conversion

Algorithm for finding the binary representation of a positive integer:

1. **Divide by 2** the original value and record the **remainder**
2. If the **quotient** is **not zero**, continue to divide the new quotient by 2 and record the remainder
3. Once the **quotient equals 0**, the binary value consists of the **remainders** listed from right to left in the order they were recorded.

| | | | | | |
|---|---|---|---|---|---|
| **13** | **6** | **3** | **1** | **0** | *quotients* |
| | **1** | **0** | **1** | **1** | *remainders* |

$13_{10} = 1101_2$

# Limits of the binary system (natural representation)

Considering natural numbers of N bits:

- 1 bit ~ 2 number ~ $\{ 0, 1 \}_2$ ~ $\{ 0, 1 \}_{10}$
- 2 bit ~ 4 number ~ $\{ 00, 01, 10, 11\}_2$ ~ $\{0,1,2,3\}_{10}$

Thus, in general for natural numbers of N bits

- distinct combinations: $2^N$
- Values intervals: $0 \leq x \leq 2^N - 1$ [ base 10 ]

  $000...0 \leq x \leq 111...1$ [ base 2 ]

# Limits of the binary system (natural representation)

| *bit* | *symbols* | *min₁₀* | *max₁₀* |
|---|---|---|---|
| 4 | 16 | 0 | 15 |
| 8 | 256 | 0 | 255 |
| 10 | 1,024 | 0 | 1,023 |
| 16 | 65,536 | 0 | 65,535 |
| 32 | 4,294,967,296 | 0 | 4,294,967,295 |

# Binary addition

Basic rules :

$$0 + 0 = 0$$

$$0 + 1 = 1$$

$$1 + 0 = 1$$

$$1 + 1 = 0 \qquad (\text{carry} = 1)$$

# Addition of binary numbers

The partial sum between the bits having the same weight is computed, including the propagation of the carry:

$$
\begin{array}{rcccccl}
 & \color{blue}{1} & \color{blue}{1} & & & & \\
 & 0 & 1 & 1 & 0 & & + \\
 & 0 & 1 & 1 & 1 & & = \\
\hline
 & 1 & 1 & 0 & 1 & &
\end{array}
$$

# Binary Subtraction

- Basic rules:

  0 - 0 = 0

  0 - 1 = 1 (borrow = 1)

  1 - 0 = 1

  1 - 1 = 0

# Binary Subtraction

The partial subtraction between the bits having the same weight is computed, including the propagation of the borrow:

$$
\begin{array}{r}
\color{red}{1\ 1\phantom{\ 0\ 1}} \\
1\ 1\ 0\ 1 \\
-\ 0\ 1\ 1\ 0 \\
\hline
0\ 1\ 1\ 1
\end{array}
$$

# Overflow

- Term **overflow** indicates the *error* that occurs, in an automatic calculation system, when the result of an operation *can not be represented* with the *same code* and *number of bits* of the operands.

# Overflow

- In pure binary addition, overflow appears when:
  - Working with a fixed number of bits
  - There is a carry from the MSb

## Overflow - example

- *Assumption*: Operation on numbers represented using only **4 bits**,

  in plain binary:

$$
\begin{array}{r l}
 & \mathbf{0101}\ + \\
 & \mathbf{1110} \\
 & \text{------------} \\
\text{overflow} \rightarrow & \mathbf{10011}
\end{array}
$$

# Octal system

- base = 8
  (Sometimes indicated with Q as octal)
- digits = { 0, 1, 2, 3, 4, 5, 6, 7 }
- Useful to write binary numbers in compact form ( 3:1 )

$$\underbrace{1\ 0}_{2}\ \underbrace{1\ 1\ 1}_{7}\ \underbrace{0\ 0\ 1}_{1}\ {}_{2} \quad \rightarrow \quad 2\ 7\ 1_{8}$$

# Hexadecimal system

- base = 16
  (Sometimes indicated with H for Hexadecimal)
- digits= { 0, 1, ..., 9, A, B, C, D, E, F }
- Useful to write binary numbers in compact form ( 4:1 )

$$\underbrace{1\ 0\ 1\ 1}_{B}\ \underbrace{1\ 0\ 0\ 1}_{9}\ {}_{2} \quad \to \quad B9_{16}$$

# Signed numbers

- A number can be:
  - positive ( + )
  - negative ( – )

- Representing the sign in binary is easy, but…
  the simpler solution is not always the best solution!
  - Sign and Magnitude (S&M)
  - One’s Complement (1C)
  - Two’s Complement (2C)

# Sign and magnitude representation

- One bit for the sign (usually the MSb):
  - 0 = positive sign ( + )
  - 1 = negative sign ( – )
- N-1 bits for the *absolute value* (also called *magnitude*)

| *Sign* | *Magnitude* |
|---|---|
| 1 bit | N-1 bit |

# Sign and Magnitude: Examples

Using a 3-bit representation :

$+\ 3_{10} \rightarrow \quad 011_{S\&M}$

$-\ 3_{10} \rightarrow \quad 111_{S\&M}$

$000_{S\&M} \rightarrow \quad +\ 0_{10}$

$100_{S\&M} \rightarrow \quad -\ 0_{10}$

| binary | Decimal |
|---|---|
| 000 | 0 |
| 001 | 1 |
| 010 | 2 |
| 011 | 3 |
| 100 | -0 |
| 101 | -1 |
| 110 | -2 |
| 111 | -3 |

# Sign and Magnitude issues

Issues:

- Double representation of zero (+ 0, - 0)
- Complexity of operations

  Example: A+B

| | A > 0 | A < 0 |
|---|---|---|
| B > 0 | A + B | B - \|A\| |
| B < 0 | A - \|B\| | - ( \|A\| + \|B\| ) |

# Sign and Magnitude: Limits

In a S&M representation on N bit:

$$-(2^{N-1}-1) \leq x \leq +(2^{N-1}-1)$$

Example:

- 8 bit = [ -127 ... +127 ]
- 16 bit = [ -32,767 ... +32,767 ]

# 1’s Complement

Given a binary number A of N bits, the 1's complement can be defined as :

$$\overline{A} = (2^N - 1) - A$$

Simply called *complement*.

It is usually indicated with a horizontal line upon the number or with the apostrophe (A’).

# 1's Complement

Practical Rule:

*1's complement of a binary number A can be obtained by complementing (i.e., inverting) all of its bits*

Example:

$$A = 1011 \quad \rightarrow \quad \overline{A} = 0100$$

# 1's Complement: Example

Using a 3 bit representation :

$+\ 3_{10} \rightarrow$ $011_{1C}$

$-\ 3_{10} \rightarrow$ $100_{1C}$

$000_{1C} \rightarrow$ $+\ 0_{10}$

$111_{1C} \rightarrow$ $-\ 0_{10}$

| binary | Decimal |
|---|---|
| 000 | 0 |
| 001 | 1 |
| 010 | 2 |
| 011 | 3 |
| 100 | -3 |
| 101 | -2 |
| 110 | -1 |
| 111 | -0 |

# 2's Complement

Given a binary number A of N bits, 2's complement (2C) can be defined as :

$$\overline{\overline{A}} = 2^N - A = \overline{A} + 1$$

# 2's Complement

Practical Rule:

*2's complement of a binary number A can be obtained by adding one to its one's complement*

Example:

$$A = 1011 \rightarrow \overline{A} = 0100 \rightarrow \overline{\overline{A}} = \overline{A} + 1 = 0101$$

# 2's Complement (bis)

Practical Rule (b):

*The 2's complement of a binary number is obtained starting from the LSb and copying all the bits up to the first "1" and then complementing the remaining bits.*

Example:

$$A = 10110 \rightarrow \overline{\overline{A}} = 01010$$

# 2's Complement: Example

Using a 3 bit representation :

$+\ 3_{10} \rightarrow \quad 011_{S\&M}$

$-\ 4_{10} \rightarrow \quad 100_{S\&M}$

$000_{2C} \rightarrow \quad +\ 0_{10}$

$100_{2C} \rightarrow \quad -\ 4_{10}$

| binary | Decimal |
|---|---|
| 000 | 0 |
| 001 | 1 |
| 010 | 2 |
| 011 | 3 |
| 100 | -4 |
| 101 | -3 |
| 110 | -2 |
| 111 | -1 |

# Encoding a value in 2’s complement

Algorithm for encoding a **negative** number in 2C:

1. Get the **positive pure binary value** using the **predefined bits**
2. **Complement** the number (**1C**)
3. **Add 1**.

EX: transform the decimal number **-13**$_{10}$ to binary using the 2C representation on 5 bits.

1. $13_{10}$ = 01101
2. 1C = 10010
3. Add 1: 10010 + 1

- $13_{10}$ = **10011** 2C

# Getting the decimal number from a value in 2's complement

Given a negative number in **2C**, then:

1. **Complement** the number (**1C**)
2. **Add 1**
3. Convert the binary number to decimal.

EX: which is the decimal number represented on 5 bits by: **10011** (2C)

1. 1C = 01100
2. Add 1: 01100 + 1 = 01101
3. $01101_2 = 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 13$

then, **10011** (2C) = $-13_{10}$

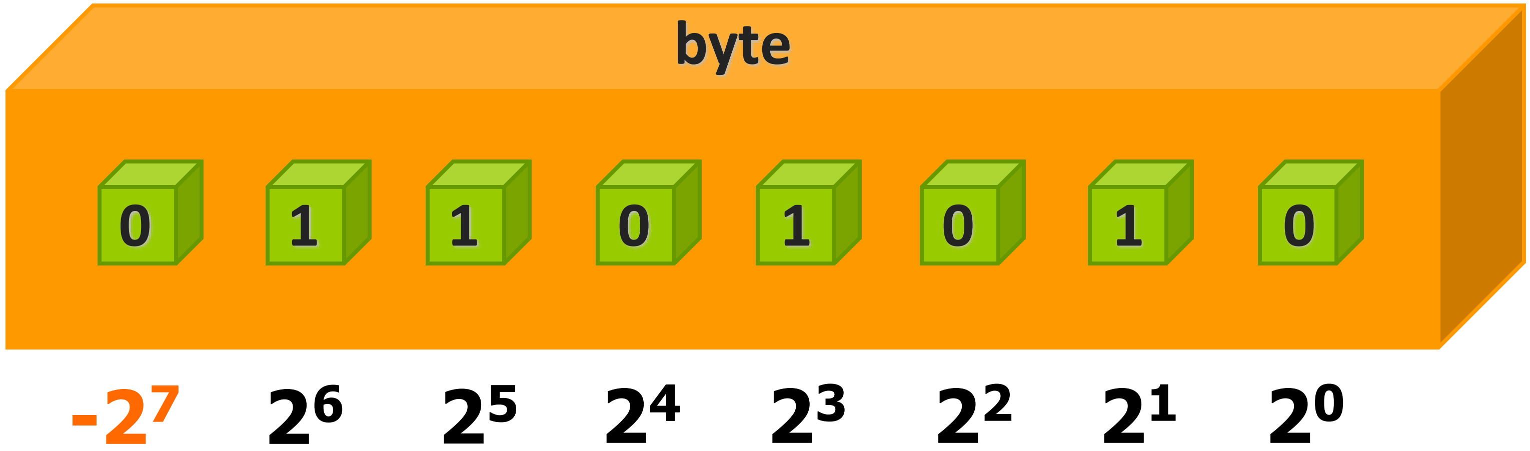
# Encoding a value in 2's complement - 2

- in a 2's complement (2C) number of N bits:
  - the MSb has a negative weight ($-2^{N-1}$)
  - the rest of the bits have a positive weight according to its position

- in this way, the MSb determines the number sign:
  - 0 = **+** 1 = –

- examples (2's complement numbers on 4 bits):
  - $1000_{2C} = -2^3 = -8_{10}$
  - $1111_{2C} = -2^3 + 2^2 + 2^1 + 2^0 = -8 +4 +2 +1 = -1_{10}$
  - $0111_{2C} = 2^2 + 2^1 + 2^0 = 7_{10}$

# Two's complement



# 2’s complement (example)

**101101**

| | | | | | |
|---|---|---|---|---|---|
| 2 | 2 | 2 | 2 | 2 | 2 |
| 5 | 4 | 3 | 2 | 1 | 0 |

$$-32+8+4+1= -19$$

# 2's Complement

- The 2's complement (2C) representation is the most adopted in today's processors, since simplifies the circuit implementation for the basic arithmetic operations
- It is possible to apply the binary rules to all of the bits composing the number

# Addition and Subtraction in 2's Complement

- Addition is done directly, without considering the signs of the operands

# Addition and Subtraction in 2's Complement

- Subtraction is done by adding the two's complement (2C) of subtrahend to minuend

$$A_{2C} - B_{2C} \rightarrow A_{2C} + \overline{\overline{B_{2C}}}$$

(note: we apply 2C transformation to B, which is already encoded in 2C)

# Addition in 2’s complement - example

`00100110 + 11001011`

```
00100110 +
11001011 =
----------------
11110001
```

verify: 38 + (– 53) = – 15

# Subtraction in 2's complement - example

```
00100110 – 11001011

    00100110 +
    00110101 =
    ---------------
    01011011
```

verify: 38 – (– 53) = 91

# Overflow 2's complement addition

Operands with different signs: overflow never occurs

Operands with the same sign: Overflow occurs if the result has different sign

In any case, carry from MSb is always neglected

# Overflow in 2's complement

| | |
|---|---|
| 0101 + | 1110 + |
| 0100 = | 1101 = |
| --------- | ---------- |
| 1001 | 11011 = |
| | 1011 |
| *overflow!* | *OK* |
| ( 5 + 4 = 9 ) | ( -2 + -3 = -5 ) |

Impossible in 2C on 4 bits

# Overflow 2's complement subtraction

| 3 – (-7) | -1 – (7) |
| --- | --- |
| 0011 + | 1111 + |
| 0111 = | 1001 = |
| ---------- | ---------- |
| 1010 | 11000 = |
| | 1000 |
| *overflow!* | *OK* ( -8 ) |

+10 is not possible in 2C on 4 bits

# Real numbers representation (Floating Point)

Scientific notation

| | |
|---|---|
| 3.14 | $0.314 \times 10^{+1}$ |
| 0.0001 | $0.1 \times 10^{-3}$ |
| 137 | $0.137 \times 10^{+3}$ |

S = sign

M = mantissa

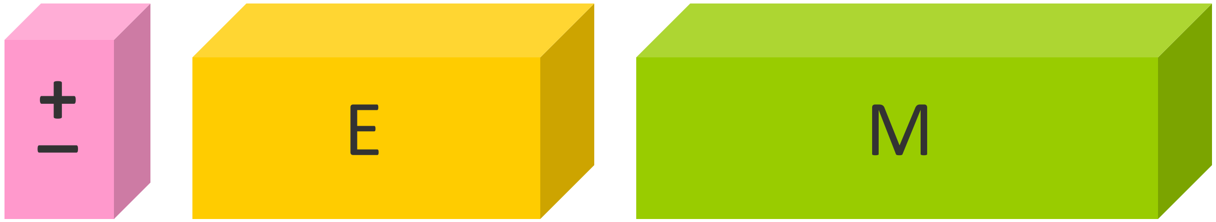
E = exponent

$$N = \pm \; M \times 2^{E}$$

# Real numbers representation (Floating Point)

The computer system stores the following elements:

- Sign
- Exponent
- Mantissa



# IEEE-754 Format

- Mantissa in the form “1. ……” (max value < 2)
- Exponent base is 2

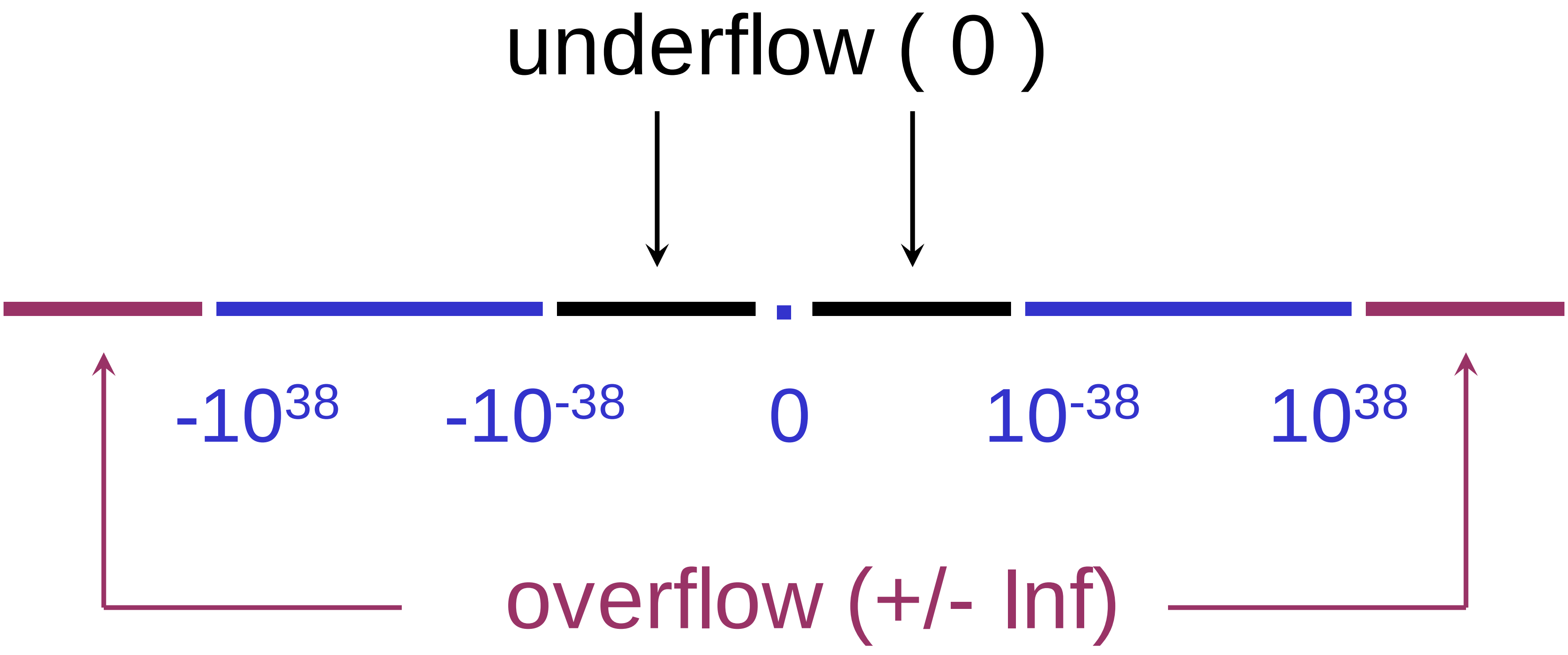
- IEEE 754 (SP) single precision, 32 bits:

| *sign* | *exponent* | *mantissa* |
|---|---|---|
| 1 bit | 8 bit | 23 bit |

- IEEE 754 (DP) double precision, 64 bit:

| *sign* | *exponent* | *mantissa* |
|---|---|---|
| 1 bit | 11 bit | 52 bit |

# IEEE-754 SP: range of values

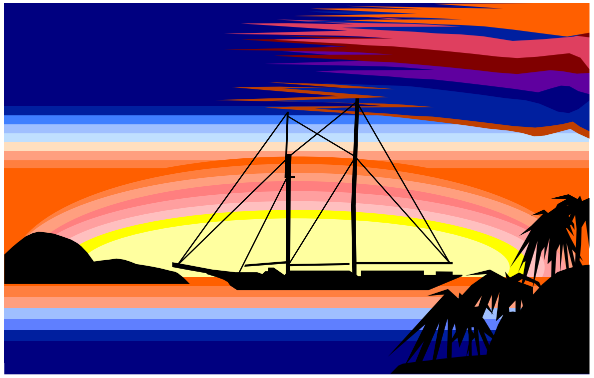


# Problems

- Fix number of encoding bits
- Numbers are represented as sequences of digits
- Problems:
  - Representation interval
    - in Python, float goes from -1.7976931348623157e308 to +1.7976931348623157e308. Outside of this range we have +/- inf
  - Precision
    - The smallest non-zero value that can be represented with Python float is +/- 2.2250738585072014e-308
    - Almost no number can be represented exactly
  - Overflow and underflow may lead to wrong results
    - (1e300 + 1) - 1e300 -> 0.0

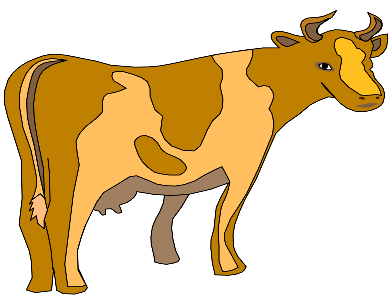
      (1e300 – 1e300) + 1 -> 1.0

# Elaborating non-numerical information

# Non-numerical information

If in a finite quantity, you can put in correspondence with integer numbers.



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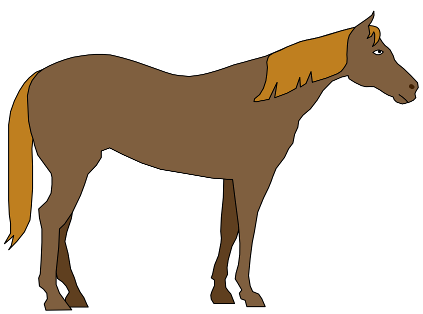
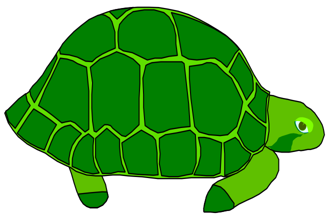


01



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# Numerical data representation

- Assuming a data composed of N bits…
  - It is possible to encode $2^N$ different elements
  - Useful for different numerical representations

- Example (using 3 bits):

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Binariy elements | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
| Natural numbers | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| Relative num. (S&M) | +0 | +1 | +2 | +3 | -0 | -1 | -2 | -3 |
| Relative num.(1C) | +0 | +1 | +2 | +3 | -3 | -2 | -1 | -0 |
| Relative num.(2C) | +0 | +1 | +2 | +3 | -4 | -3 | -2 | -1 |

# Characters

- There exist standard encodings that describe how to map characters to numbers:
  - ASCII (American Standard Code for Information Interchange)
  - EBCDIC (Extended BCD Interchange Code)
  - UNICODE

# ASCII Code

- Used also in telecommunications
- Uses 7 bit (8 bit in the extended version) for the representation:
  - 52 alphabetic characters (`a`...`z`, `A`...`Z`)
  - 10 digits (`0`...`9`)
  - Punctuation symbols (`, ; ! ?`...)
  - Control characters

```
CR       ( 13 )         Carriage Return
LF,NL    ( 10 )         New Line, Line Feed
FF,NP    ( 12 )         New Page, Form Feed
HT       ( 9 )          Horizontal Tab
VT       ( 11 )         Vertical Tab
NUL      ( 0 )          Null
BEL      ( 7 )          Bell
EOT      ( 4 )          End-Of-Transmission
. . .    . . .    . . .
```

# ASCII Code (Cont.)

| | | | |
|---|---|---|---|
| 01000001 | A | 00100000 | |
| 01110101 | u | 01110100 | t |
| 01100111 | g | 01110101 | u |
| 01110101 | u | 01110100 | t |
| 01110010 | r | 01110100 | t |
| 01101001 | i | 01101001 | i |
| 00100000 | | 00100001 | ! |
| 01100001 | a | | |

# Extended ASCII Code

- The characters with ASCII code higher than 127 are not standard.
- In practice, the characters with MSb=1 are used for:
  - Control characters
  - Local alphabets (example: ñ à á ä ã æ )
  - Graphic characters (example: § ¨ ♥ ♠ ∝ ⇔ √ )

# UNICODE and UTF-8

Unicode expresses all the chars of all the world languages (21-bit per character, more than one million possible characters).

UTF-8 is the most used Unicode format *enconding* (a way to *represent* the UNICODE *representations*):

- 1 byte for US-ASCII chars(MSb=0)
- 2 byte for Latin chars with diacritical symbols, Greek, Cyrillic, Armenian, Jewish, Arabic, Syrian e Maldivian
- 3 byte for other commonly used languages
- 4 byte very rare chars
- Recommended by IETF for e-mails

# Representing a text in ASCII format

- Characters codified in ASCII code

- Every line terminates with the “line terminator” character:
  - MS-DOS and Windows = CR + LF
  - UNIX = LF
  - MacOS = CR

- Pages sometimes separate by FF.

# Representing a text in ASCII format

- Do not confuse ASCII encoded text file with a formatted document (es. doc or pdf)
- The former contains only the characters of the text
- The latter contains many more additional information, covering formatting options, fonts, …