Lecture 2: Fundamental concepts

Recall. . .

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Exercises

(Cryer and Chan, 2008, Chapter 2)

2.1 Suppose $E(X) = 2$, $Var(X) = 9$, $E(Y) = 0$, $Var(Y) = 4$, and $Corr(X,Y) = 0.25$. Find:
(a) $Var(X + Y)$.
(b) $Cov(X, X + Y)$.
(c) $Corr(X + Y, X - Y)$.

2.2 If X and Y are dependent but $Var(X) = Var(Y)$, find $Cov(X + Y, X - Y)$.

2.3 Let X have a distribution with mean μ and variance σ^2, and let $Y_t = X$ for all t.
(a) Show that $\{Y_t\}$ is strictly and weakly stationary.
(b) Find the autocovariance function for $\{Y_t\}$.
(c) Sketch a "typical" time plot of Y_t.

Exercises

2.4 Let $\{e_t\}$ be a zero mean white noise process. Suppose that the observed process is $Y_t = e_t + \theta e_{t-1}$, where θ is either 3 or 1/3.
(a) Find the autocorrelation function for $\{Y_t\}$ both when $\theta = 3$ and when $\theta = 1/3$.
(b) You should have discovered that the time series is stationary regardless of the value of θ and that the autocorrelation functions are the same for $\theta = 3$ and $\theta = 1/3$. For simplicity, suppose that the process mean is known to be zero and the variance of Y_t is known to be 1. You observe the series $\{Y_t\}$ for $t = 1, 2, \ldots, n$ and suppose that you can produce good estimates of the autocorrelations ρ_k. Do you think that you could determine which value of θ is correct (3 or 1/3) based on the estimate of ρ_k? Why or why not?

2.5 Suppose $Y_t = 5 + 2t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with autocovariance function γ_k.
(a) Find the mean function for $\{Y_t\}$.
(b) Find the autocovariance function for $\{Y_t\}$.
(c) Is $\{Y_t\}$ stationary? Why or why not?

$\{X$ for t odd

Exercises

(c) Is $\{Y_t\}$ stationary? Why or why not?

2.6 Let $\{X_t\}$ be a stationary time series, and define $Y_t = \begin{cases} X_t & \text{for } t \text{ odd} \\ X_t + 3 & \text{for } t \text{ even.} \end{cases}$
(a) Show that $Cov(Y_t, Y_{t-k})$ is free of t for all lags k.
(b) Is $\{Y_t\}$ stationary?

2.7 Suppose that $\{Y_t\}$ is stationary with autocovariance function γ_k.
(a) Show that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary by finding the mean and autocovariance function for $\{W_t\}$.
(b) Show that $U_t = \nabla^2 Y_t = \nabla[Y_t - Y_{t-1}] = Y_t - 2Y_{t-1} + Y_{t-2}$ is stationary. (You need not find the mean and autocovariance function for $\{U_t\}$.)

2.8 Suppose that $\{Y_t\}$ is stationary with autocovariance function γ_k. Show that for any fixed positive integer n and any constants $c_1, c_2, \ldots, c_n$, the process $\{W_t\}$ defined by $W_t = c_1 Y_t + c_2 Y_{t-1} + \cdots + c_n Y_{t-n+1}$ is stationary. (Note that Exercise 2.7 is a special case of this result.)

Exercises

2.9 Suppose $Y_t = \beta_0 + \beta_1 t + X_t$, where $\{X_t\}$ is a zero-mean stationary series with autocovariance function γ_k and β_0 and β_1 are constants.
(a) Show that $\{Y_t\}$ is not stationary but that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is stationary.
(b) In general, show that if $Y_t = \mu_t + X_t$, where $\{X_t\}$ is a zero-mean stationary series and μ_t is a polynomial in t of degree d, then $\nabla^m Y_t = \nabla(\nabla^{m-1} Y_t)$ is stationary for $m \geq d$ and nonstationary for $0 \leq m < d$.

2.10 Let $\{X_t\}$ be a zero-mean, unit-variance stationary process with autocorrelation function ρ_k. Suppose that μ_t is a nonconstant function and that σ_t is a positive-valued nonconstant function. The observed series is formed as $Y_t = \mu_t + \sigma_t X_t$.
(a) Find the mean and covariance function for the $\{Y_t\}$ process.
(b) Show that the autocorrelation function for the $\{Y_t\}$ process depends only on the time lag. Is the $\{Y_t\}$ process stationary?
(c) Is it possible to have a time series with a constant mean and with $Corr(Y_t, Y_{t-k})$ free of t but with $\{Y_t\}$ *not* stationary?

Exercises

2.11 Suppose $Cov(X_t, X_{t-k}) = \gamma_k$ is free of t but that $E(X_t) = 3t$.
(a) Is $\{X_t\}$ stationary?
(b) Let $Y_t = 7 - 3t + X_t$. Is $\{Y_t\}$ stationary?

2.12 Suppose that $Y_t = e_t - e_{t-12}$. Show that $\{Y_t\}$ is stationary and that, for $k > 0$, its autocorrelation function is nonzero only for lag $k = 12$.

2.13 Let $Y_t = e_t - \theta(e_{t-1})^2$. For this exercise, assume that the white noise series is normally distributed.
(a) Find the autocorrelation function for $\{Y_t\}$.
(b) Is $\{Y_t\}$ stationary?

2.14 Evaluate the mean and covariance function for each of the following processes. In each case, determine whether or not the process is stationary.
(a) $Y_t = \theta_0 + t e_t$.
(b) $W_t = \nabla Y_t$, where Y_t is as given in part (a).
(c) $Y_t = e_t e_{t-1}$. (You may assume that $\{e_t\}$ is normal white noise.)

Exercises

2.15 Suppose that X is a random variable with zero mean. Define a time series by $Y_t = (-1)^t X$.
(a) Find the mean function for $\{Y_t\}$.
(b) Find the covariance function for $\{Y_t\}$.
(c) Is $\{Y_t\}$ stationary?

2.16 Suppose $Y_t = A + X_t$, where $\{X_t\}$ is stationary and A is random but independent of $\{X_t\}$. Find the mean and covariance function for $\{Y_t\}$ in terms of the mean and autocovariance function for $\{X_t\}$ and the mean and variance of A.

2.17 Let $\{Y_t\}$ be stationary with autocovariance function γ_k. Let $\bar{Y} = \frac{1}{n}\sum_{t=1}^{n} Y_t$. Show that

$$Var(\bar{Y}) = \frac{\gamma_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\gamma_k$$

$$= \frac{1}{n}\sum_{k=-n+1}^{n-1}\left(1 - \frac{|k|}{n}\right)\gamma_k$$

Exercises

2.18 Let $\{Y_t\}$ be stationary with autocovariance function γ_k. Define the sample variance as $S^2 = \frac{1}{n-1}\sum_{t=1}^{n}(Y_t - \bar{Y})^2$.

(a) First show that $\sum_{t=1}^{n}(Y_t - \mu)^2 = \sum_{t=1}^{n}(Y_t - \bar{Y})^2 + n(\bar{Y} - \mu)^2$.

(b) Use part (a) to show that

(c) $E(S^2) = \frac{n}{n-1}\gamma_0 - \frac{n}{n-1}Var(\bar{Y}) = \gamma_0 - \frac{2}{n-1}\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\gamma_k$.

(Use the results of Exercise 2.17 for the last expression.)

(d) If $\{Y_t\}$ is a white noise process with variance γ_0, show that $E(S^2) = \gamma_0$.

Exercises

2.19 Let $Y_1 = \theta_0 + e_1$, and then for $t > 1$ define Y_t recursively by $Y_t = \theta_0 + Y_{t-1} + e_t$. Here θ_0 is a constant. The process $\{Y_t\}$ is called a **random walk with drift**.
(a) Show that Y_t may be rewritten as $Y_t = t\theta_0 + e_t + e_{t-1} + \cdots + e_1$.
(b) Find the mean function for Y_t.
(c) Find the autocovariance function for Y_t.

2.20 Consider the standard random walk model where $Y_t = Y_{t-1} + e_t$ with $Y_1 = e_1$.
(a) Use the representation of Y_t above to show that $\mu_t = \mu_{t-1}$ for $t > 1$ with initial condition $\mu_1 = E(e_1) = 0$. Hence show that $\mu_t = 0$ for all t.
(b) Similarly, show that $Var(Y_t) = Var(Y_{t-1}) + \sigma_e^2$ for $t > 1$ with $Var(Y_1) = \sigma_e^2$ and hence $Var(Y_t) = t\sigma_e^2$.
(c) For $0 \le t \le s$, use $Y_s = Y_t + e_{t+1} + e_{t+2} + \cdots + e_s$ to show that $Cov(Y_t, Y_s) = Var(Y_t)$ and, hence, that $Cov(Y_t, Y_s) = \min(t, s)\sigma_e^2$.

Exercises

2.21 For a random walk with random starting value, let $Y_t = Y_0 + e_t + e_{t-1} + \cdots + e_1$ for $t > 0$, where Y_0 has a distribution with mean μ_0 and variance σ_0^2. Suppose further that $Y_0, e_1, \ldots, e_t$ are independent.
(a) Show that $E(Y_t) = \mu_0$ for all t.
(b) Show that $Var(Y_t) = t\sigma_e^2 + \sigma_0^2$.
(c) Show that $Cov(Y_t, Y_s) = \min(t, s)\sigma_e^2 + \sigma_0^2$.
(d) Show that $Corr(Y_t, Y_s) = \sqrt{\dfrac{t\sigma_a^2 + \sigma_0^2}{s\sigma_a^2 + \sigma_0^2}}$ for $0 \le t \le s$.

Exercises

2.22 Let $\{e_t\}$ be a zero-mean white noise process, and let c be a constant with $|c| < 1$. Define Y_t recursively by $Y_t = cY_{t-1} + e_t$ with $Y_1 = e_1$.
(a) Show that $E(Y_t) = 0$.
(b) Show that $Var(Y_t) = \sigma_e^2(1 + c^2 + c^4 + \cdots + c^{2t-2})$. Is $\{Y_t\}$ stationary?
(c) Show that

$$Corr(Y_t, Y_{t-1}) = c\sqrt{\frac{Var(Y_{t-1})}{Var(Y_t)}} \text{ and, in general,}$$

$$Corr(Y_t, Y_{t-k}) = c^k\sqrt{\frac{Var(Y_{t-k})}{Var(Y_t)}} \quad \text{for } k > 0$$

Hint: Argue that Y_{t-1} is independent of e_t. Then use

$$Cov(Y_t, Y_{t-1}) = Cov(cY_{t-1} + e_t, Y_{t-1})$$

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(d) For large t, argue that

$$Var(Y_t) \approx \frac{\sigma_e^2}{1 - c^2} \quad \text{and} \quad Corr(Y_t, Y_{t-k}) \approx c^k \quad \text{for } k > 0$$

so that $\{Y_t\}$ could be called **asymptotically stationary**.
(e) Suppose now that we alter the initial condition and put $Y_1 = \dfrac{e_1}{\sqrt{1 - c^2}}$. Show that now $\{Y_t\}$ is stationary.

References

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