Lecture 2: Simple models and basic concepts

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Outline of the lecture

Simplest Models

White noise

We suppose that we observe $\{\varepsilon_t\}_{t\in\mathbb{Z}}$,

- **Weak:** a series of uncorrelated random variables
  Sometimes denoted as $\varepsilon_t \sim WN(0,\sigma_\varepsilon^2)$
- **Strong:** a series of iid random variables
  Sometimes denoted as $\varepsilon_t \sim IID(0,\sigma_\varepsilon^2)$
- **Gaussian:** a series of iid normal random variables
  Sometimes denoted as $\varepsilon_t \sim \mathscr{N}(0,\sigma_\varepsilon^2)$

Moving average and autoregression

Filtering

- **Moving average:** We have:

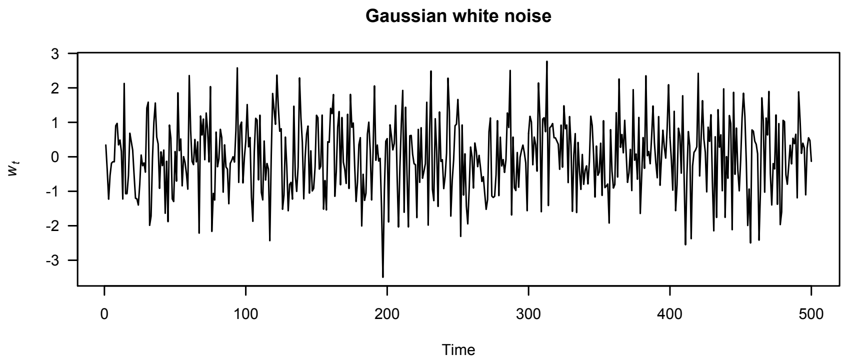
$$V_t = \frac{1}{3}(\varepsilon_{t-1} + \varepsilon_t + \varepsilon_{t+1})$$

- **Autoregression:** We have:

$$X_t = X_{t-1} - .9X_{t-2} + \varepsilon_t$$

A Gaussian Noise

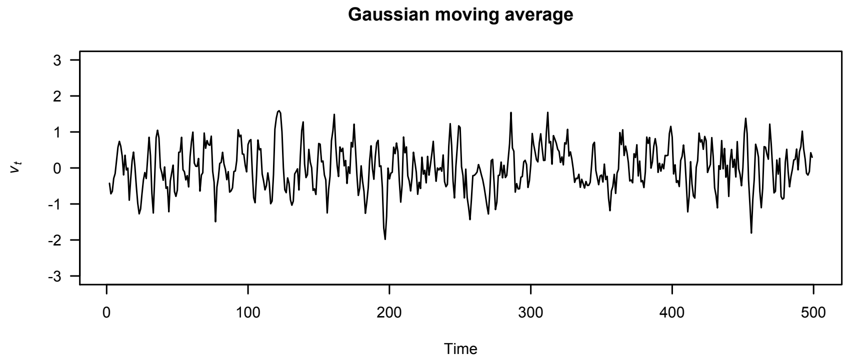
Computed in R



A Gaussian Noise ($\sigma^2 = 1,\ n = 500$)

Moving Average based on a Gaussian Noise

Computed in R



$$V_t = \frac{1}{3}\left(\varepsilon_{t-1} + \varepsilon_t + \varepsilon_{t+1}\right)$$

Moving Average based on a Gaussian Noise

Computed in R

$$\{\varepsilon_t\}_{t\in\mathbb{Z}} \longrightarrow \boxed{\mathcal{F}} \longrightarrow \{V_t\}_{t\in\mathbb{Z}}$$

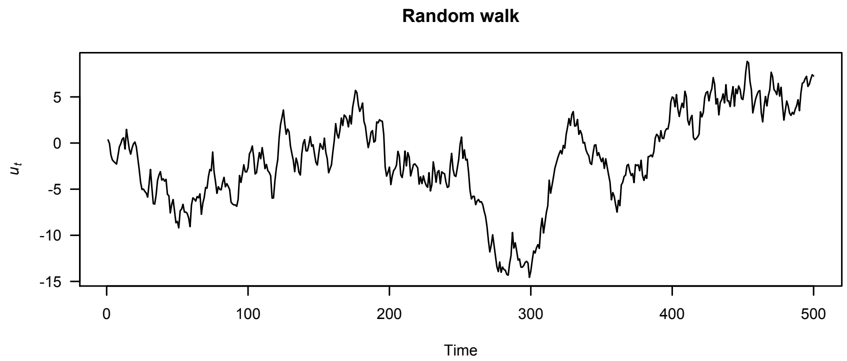
- Here $\mathcal{F}$ is a filter whose effect is given by

$$V_t = \frac{1}{3}\left(\varepsilon_{t-1} + \varepsilon_t + \varepsilon_{t+1}\right)$$

- Note that $\{\varepsilon_t\}$ and $\{V_t\}$ are defined $\forall t \in \mathbb{Z}$

Random Walk

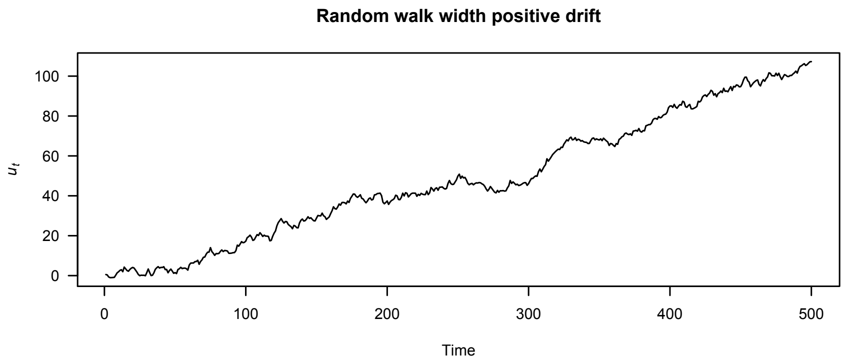
Computed in R



$$U_t = U_{t-1} + W_t, \quad U_0 = 0.$$

Random Walk

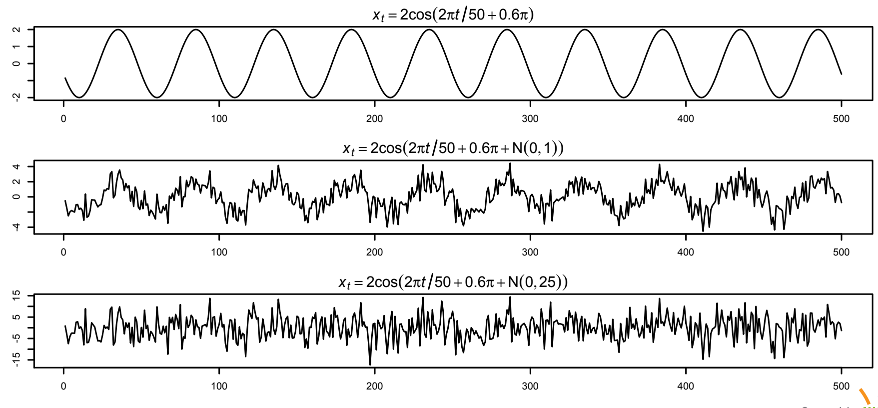
Computed in R



$$U_t = U_{t-1} + \delta + \varepsilon_t, \quad U_0 = 0.$$

Signal + Noise

Computed in R



$$U_t = U_{t-1} + \delta + \varepsilon_t, \quad U_0 = 0.$$

Mean and autocovariance function

Moving average

- We have:

$$V_t = \frac{1}{3}\left(\varepsilon_{t-1} + \varepsilon_t + \varepsilon_{t+1}\right)$$

$$\mathrm{E}[V_t] = \frac{1}{3}\left(\mathrm{E}[\varepsilon_{t-1}] + \mathrm{E}[\varepsilon_t] + \mathrm{E}[\varepsilon_{t+1}]\right) = 0$$

- and

$$\mathrm{Var}[V_t] = \frac{1}{9}\left(\mathrm{Var}[\varepsilon_{t-1}] + \mathrm{Var}[\varepsilon_t] + \mathrm{Var}[\varepsilon_{t+1}]\right) = \frac{\sigma_\varepsilon^2}{3}$$

Mean and autocovariance function

Moving average

- For the autocovariance we have:

$$V_t = \frac{1}{3}\left(\varepsilon_{t-1} + \varepsilon_t + \varepsilon_{t+1}\right)$$

$$V_{t+1} = \frac{1}{3}\left(\varepsilon_t + \varepsilon_{t+1} + \varepsilon_{t+2}\right)$$

- and

$$\begin{aligned}
&\mathrm{cov}[V_t, V_{t+1}] = \mathrm{E}[V_t \times V_{t+1}] \\
&= \frac{1}{9}\left(\mathrm{cov}[\varepsilon_{t-1}, \varepsilon_t] + \mathrm{cov}[\varepsilon_{t-1}, \varepsilon_{t+1}] + \mathrm{cov}[\varepsilon_{t-1}, \varepsilon_{t+2}]\right. \\
&+ \mathrm{cov}[\varepsilon_t, \varepsilon_t] + \mathrm{cov}[\varepsilon_t, \varepsilon_{t+1}] + \mathrm{cov}[\varepsilon_t, \varepsilon_{t+2}] \\
&\left.\mathrm{cov}[\varepsilon_{t+1}, \varepsilon_t] + \mathrm{cov}[\varepsilon_{t+1}, \varepsilon_{t+1}] + \mathrm{cov}[\varepsilon_{t+1}, \varepsilon_{t+2}]\right) \\
&= \frac{2}{9}\sigma_\varepsilon^2
\end{aligned}$$

Mean and autocovariance function

Moving average

It is easy to see that

$$\operatorname{cov}[V_t, V_{t+k}] = \begin{cases} \frac{\sigma_\varepsilon^2}{3}, & k = 0; \\ \frac{2}{9}\sigma_\varepsilon^2, & k = \pm 1; \\ \frac{1}{9}\sigma_\varepsilon^2, & k = \pm 2; \\ 0 & |k| > 2. \end{cases}$$

It does not depend on t. This process is stationary, weakly, and strictly (being Gaussian).

Autocovariance function

Remark

In the previous example we see that:

$$\operatorname{cov}[V_s, V_t] = \gamma(s,t) = \gamma(|t-s|)$$

It does not depend on t, nor on s. This process is weakly stationary.

Autocorrelation function

Definition

The **autocorrelation function** (ACF) is defined as

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}$$

In the stationary case it depends only on $|t-s|\ldots$

Cross-covariance and Cross-correlation functions

Definition

The **cross-covariance function** (ACF) is defined as

$$\gamma_{XY}(s,t) = \operatorname{cov}[X_s, Y_t] = \mathrm{E}[(X_s - \mu_{X_s})(Y_t - \mu_{Y_t})]$$

The **cross-correlation function** (ACF) is defined as

$$\rho_{XY}(s,t) = \frac{\gamma_{XY}(s,t)}{\sqrt{\gamma_X(s,s)\gamma_Y(t,t)}}$$

Joint stationarity

Definition

- Two time series, say, $\{X_t\}$ and $\{Y_t\}$, are **jointly stationary** if they are each stationary, and the cross-covariance function

$$\gamma_{XY}(k) = \mathrm{cov}[X_{t+k}, Y_t] = \mathrm{E}[(X_{t+k} - \mu_X)(Y_t - \mu_Y)]$$

  depens only on k, not t.
- In that case, the cross-correlation is

$$\rho_{XY}(k) = \frac{\gamma_{XY}(k)}{\sqrt{\gamma_X(0)}\sqrt{\gamma_Y(0)}}$$

Join stationarity

Example

- Let $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ $WN(0, \sigma_\varepsilon^2)$
- Define

$$X_t = \varepsilon_t + \varepsilon_{t-1} \quad \text{and} \quad Y_t = \varepsilon_t - \varepsilon_{t-1}$$

- We easily see that both processes are of mean 0;
  - $\gamma_X(0) = \gamma_Y(0) = 2\sigma_\varepsilon^2$;
  - $\gamma_X(1) = \gamma_X(-1) = \sigma_\varepsilon^2$;
  - $\gamma_Y(1) = \gamma_Y(-1) = -\sigma_\varepsilon^2$;

  other autocovariances are 0.
- Cross-covariance:

$$\gamma_{X,Y}(1) = \mathrm{cov}[X_{t+1}, Y(t)] = \mathrm{cov}[\varepsilon_{t+1} + \varepsilon_t, \varepsilon_t - \varepsilon_{t-1}] = \sigma_\varepsilon^2$$

Join stationarity

Example

- in a similar way:
  - $\gamma_{X,Y}(0) = 0$
  - $\gamma_{X,Y}(-1) = -\sigma_\varepsilon^2$

  So

$$\rho_{XY}(k) = \begin{cases} 0, & h = 0 \\ \frac{1}{2} & k = 1 \\ -\frac{1}{2} & k = -1 \\ 0 & |k| \geqslant 2. \end{cases}$$

R script

For the noise, moving average etc...

```
set.seed(131016)
w = rnorm(500,0,1)  # 500 N(0,1) variates
v = filter(w, sides=2, rep(1/3,3))  # moving average
u <- cumsum(w)
par(mfrow=c(1,1))
plot.ts(w, main="Gaussian␣white␣noise",las=1,ylab=expression(↘
→italic(w[t])))
plot.ts(v, ylim=c(-3,3), main="Gaussian␣moving␣average",las↘
→=1,ylab=expression(italic(v[t])))
plot.ts(u, main="Random␣walk",las=1,ylab=expression(italic(u[↘
→t])))
delta <- 0.2
u <- cumsum(w+ delta)
plot.ts(u, main="Random␣walk␣width␣positive␣drift",las=1,ylab↘
→=expression(italic(u[t])))
```

R script

Signal + noise example

```
cs = 2*cos(2*pi*(1:500)/50 + .6*pi)
w = rnorm(500,0,1)
#
quartz(width=6, height=2.75,pointsize=7.5,title="Noisy signal")
set.seed(131016)
par(mfrow=c(3,1), mar=c(3,2,2,1), cex.main=1.5)    # help(par) for info
plot.ts(cs, main = expression(italic(x[t])==2*cos(2*italic(pi*t/50+.6*pi))))
plot.ts(cs + w, main = expression(italic(x[t])==2*cos(italic(2*pi*t/50+.6*pi)+N(0,1))))
plot.ts(cs + 5*w, main = expression(italic(x[t])==2*cos(italic(2*pi*t/50+.6*pi)+N(0,25))))
```


Outline of the lecture

Stationarity required!

Lapointe (1998)

- Modeling starts with stationary series
- Non-stationary time series need to be transformed
- Basic tools: differences (working with increments)

Operators I

Backward shift operator

Definition

- Let $Y_1, \ \ldots, \ Y_t$, a time series.
- The *backward shift operator* applied to Y_t gives Y_{t-1}.
- We write $B(Y_t) = B\,Y_t = Y_{t-1}$.

B operates on Y_t to shift it backward one point in time.

Example

$$X_1, \ \ldots, \ X_{10} = \{0.3, \ 9.1, \ 4.2, \ 2.7, \ 5.1, \ 2.2, \ 3.4, \ 0.7, \ 5.3, \ 8.0\}$$
$$B(X_1, \ \ldots, \ X_{10}) = \{\mathrm{NA}, \ 0.3, \ 9.1, \ 4.2, \ 2.7, \ 5.1, \ 2.2, \ 3.4, \ 0.7, \ 5.3\}$$

Note that $BY_1 = Y_0$ is unknown here indicated by “NA”.

Operators II

Backward shift operator

- It can be applied more than ounce:

$$B^2 Y_t = B(B(Y_t)) = B(Y_{t-1}) = Y_{t-2}$$

- In general:

$$B^k Y_t = B(B^{k-1}(Y_t)) = \cdots = Y_{t-k}$$

- Note that

$$B^0 Y_t = \mathbb{1}(Y_t) = Y_t$$

- We can consider algebraic expressions:

$$(1 - B)Y_t = Y_t - BY_t = Y_t - Y_{t-1}$$

giving the increment from Y_{t-1} to Y_t.

Representation in R I

```
> (x<-ts(c(0.3,9.1,4.2,2.7,5.1,2.2,3.4,0.7,5.3,8.0)))
Time Series:
Start = 1
End = 10
Frequency = 1
 [1] 0.3 9.1 4.2 2.7 5.1 2.2 3.4 0.7 5.3 8.0
> (back.x <- lag(x,-1))
Time Series:
Start = 2
End = 11
Frequency = 1
 [1] 0.3 9.1 4.2 2.7 5.1 2.2 3.4 0.7 5.3 8.0
```


Representation in R II

```
> (x-back.x)
Time Series:
Start = 2
End = 10
Frequency = 1
[1]  8.8 -4.9 -1.5  2.4 -2.9  1.2 -2.7  4.6  2.7
> (diff(x))
Time Series:
Start = 2
End = 10
Frequency = 1
[1]  8.8 -4.9 -1.5  2.4 -2.9  1.2 -2.7  4.6  2.7
#
#
```


Representation in R III

```
x<-ts(c(0.3,9.1,4.2,2.7,5.1,2.2,3.4,0.7,5.3,8.0))
Bx <- lag(x,-1)
Fx <- lag(x,1)
Delta.x <- x-Bx
tab <- cbind(x,Bx,Fx,Delta.x)
```

Representation in R IV

t	$\{X_t\}$	$B\{X_t\}$	$F\{X_t\}$	$\nabla\{X_t\}$
0			0.3	
1	0.3		9.1	
2	9.1	0.3	4.2	8.8
3	4.2	9.1	2.7	-4.9
4	2.7	4.2	5.1	-1.5
5	5.1	2.7	2.2	2.4
6	2.2	5.1	3.4	-2.9
7	3.4	2.2	0.7	1.2
8	0.7	3.4	5.3	-2.7
9	5.3	0.7	8.0	4.6
10	8.0	5.3		2.7
11		8.0		

Operators

regular difference operator

Definition (∇Y_t)

- The regular difference operator is often used to stabilize time series with "linear" trend
- We have a special and largely used notation for this operator:

$$\nabla Y_t = (1 - B)Y_t = Y_t - Y_{t-1}$$

N.b. The symbol ∇ is named *"nabla"*.

Operators

Obey simple algebra

One has

$$\nabla^2 Y_t = (1 - B)^2 Y_t = (1 - 2B + B^2)Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$$

and it makes sense:

$$\begin{aligned}(1 - B)^2 Y_t &= (1 - B)((1 - B)Y_t) \\ &= (1 - B)(Y_t - Y_{t-1}) \\ &= (1 - B)Y_t - (1 - B)Y_{t-1} \\ &= Y_t - Y_{t-1} - Y_{t-1} + Y_{t-2} \\ &= Y_t - 2Y_{t-1} + Y_{t-2}\end{aligned}$$

Regular difference

How does it work We have linear trend?

Suppose

$$\mu_t = \beta_0 + \beta_1 t$$

We observe

$$Y_t = \beta_0 + \beta_1 t + \varepsilon_t$$

with $\mathrm{E}[\varepsilon_t] = 0$. Let us apply the regular difference operator

$$\begin{aligned}(1 - B)Y_t &= Y_t - Y_{t-1} \\ &= (\beta_0 + \beta_1 t + \varepsilon_t) - (\beta_0 + \beta_1(t - 1) + \varepsilon_{t-1}) \\ &= \beta_1 + \varepsilon_t - \varepsilon_{t-1}\end{aligned}$$

It results in a time series of mean β_1.

Regular difference

How does it work We have quadratic trend?

Suppose

$$\mu_t = \beta_0 + \beta_1 t + \beta_2 t^2$$

We observe

$$Y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \varepsilon_t$$

with $\mathrm{E}[\varepsilon_t] = 0$. Let us apply the regular difference operator twice

$$\begin{aligned}(1-B)^2 Y_t &= Y_t - 2Y_{t-1} + Y_{t-2} \\ &= (\beta_0 + \beta_1 t + \varepsilon_t) - 2(\beta_0 + \beta_1(t-1) + \varepsilon_{t-1}) \\ &\quad + \beta_0 + \beta_1(t-2) + \beta_2(t-2)^2 + \varepsilon_{t-2} \\ &= 2\beta_2 + \varepsilon_t - 2\,\varepsilon_{t-1} + \varepsilon_{t-2}\end{aligned}$$

It results in a time series of mean $2\beta_2$.

Regular difference

How does it work We have polynomial trend of degree p?

Suppose

$$\mu_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \cdots + \beta_p t^p$$

We observe

$$Y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \cdots + \beta_p t^p + \varepsilon_t$$

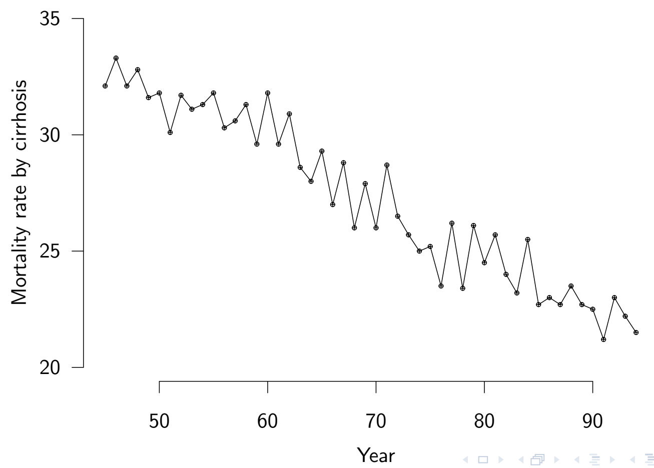
with $\mathrm{E}[\varepsilon_t] = 0$. Let us apply the regular difference operator p times:

$$(1-B)^p Y_t = p\beta_p + (1-B)^p \varepsilon_t$$

It results in a time series of mean $p\beta_p$.

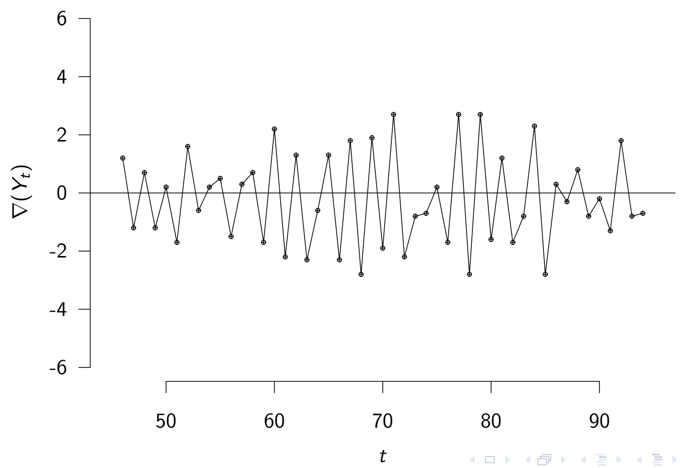
Example

On the cirrhosis data



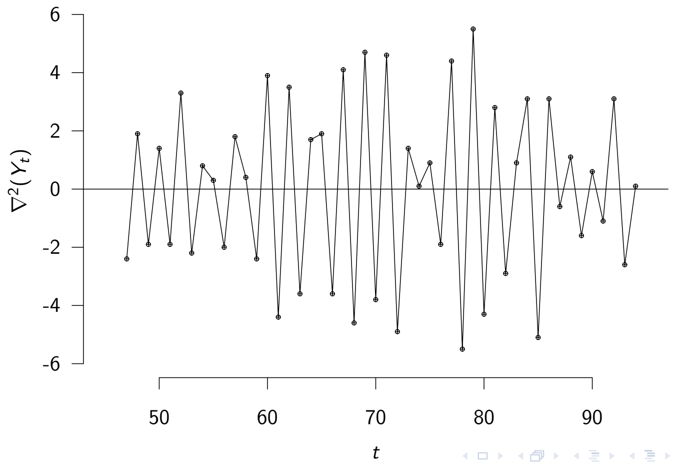
Example

Graphic of ∇Y_t



Example

Graphic of $\nabla^2 Y_t$



Remark

Overdifferencing

- Increases the variance
- Leads to more complex model...
- It should be avoided

Outline of the lecture

1. Time Series Models
2. Modeling and Stationarity
3. General Linear process
4. Moving Average Process
5. Autoregressive Processes
6. Autoregressive moving average process
7. Invertibility

General Linear proces

Definition

- A (causal) linear process, $\{Y_t\}$, is one that can be represented as a weighted linear combination of present and past white noise terms as

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots \tag{1}$$

- Conditions must be set on the ψ-weights for the right-hand side to be well defined in L^2.
  It suffices to assume that

$$\sum_{i=1}^{\infty} \psi_i^2 < \infty \tag{2}$$

- With $\psi_0 = 1$, we can write:

$$Y_t = \sum_{k=0}^{\infty} \psi_k e_{t-k}, \quad t \in \mathbb{Z}$$

One-sided Infinite moving average

$$\{e_t\}_{t\in\mathbb{Z}} \longrightarrow \boxed{\;\mathcal{F}\;} \longrightarrow \{V_t\}_{t\in\mathbb{Z}}$$

- Here $\mathcal{F}$ is a filter whose effect is given by

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$$

- Note that $\{\varepsilon_t\}$ and $\{V_t\}$ are defined $\forall t \in \mathbb{Z}$

General Linear proces

First two moments

- Since $\sum_{k=1}^{\infty} \psi_k^2 < \infty$:

$$\begin{aligned}
\mathrm{E}[Y_t] &= \sum_{k=0}^{\infty} \psi_k \mathrm{E}[e_{t-k}] = 0, \quad t \in \mathbb{Z} \\
\mathrm{cov}[Y_t, Y_{t-k}] &= \mathrm{E}[Y_t Y_{t-k}] \\
&= \mathrm{E}\left[\sum_{i=0}^{\infty} \psi_i e_{t-i} \sum_{j=0}^{\infty} \psi_j e_{t-j-k}\right] \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \psi_i \psi_j \mathrm{E}[e_{t-i} e_{t-j-k}] \\
&= \sigma_e^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \qquad (\text{for } t-i = t-j-k)
\end{aligned}$$

General Linear proces

An example

- Let $\psi_j = \phi^j$ where $|\phi| < 1$.
- Then

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots$$

For this example,

$$\mathrm{E}[Y_t] = \mathrm{E}\left[e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots\right] = 0$$

so that $\{Y_t\}$ has a constant mean of zero. Also,

$$\begin{aligned}
\mathrm{Var}[Y_t] &= \mathrm{Var}\left[e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots\right] \\
&= \mathrm{Var}[e_t] + \phi^2 \mathrm{Var}[e_{t-1}] + \phi^4 \mathrm{Var}[e_{t-2}] + \cdots \\
&= \sigma_e^2 (1 + \phi^2 + \phi^4 + \cdots) \\
&= \frac{\sigma_e^2}{1-\phi^2} \quad \text{(by summing a geometric series)}
\end{aligned}$$

General Linear proces

An example

- Furthermore,

$$\begin{aligned}
\mathrm{cov}[Y_t, Y_{t-1}] &= \mathrm{cov}\left[e_t + \phi e_{t-1} + \phi 2 e_{t-2} + \cdots,\right. \\
&\qquad \left. e_{t-1} + \phi e_{t-2} + \phi 2 e_t{-}3 + \cdots\right] \\
&= \mathrm{cov}[\phi e_{t-1}, e_{t-1}] + \mathrm{cov}\left[\phi^2 e_{t-2}, \phi e_{t-2}\right] + \cdots \\
&= \phi\sigma_e^2 + \phi^3\sigma_e^2 + \phi^5\sigma_e^2 + \cdots \\
&= \phi\sigma_e^2 (1 + \phi^2 + \phi^4 + \cdots) \\
&= \frac{\phi\sigma_e^2}{1-\phi^2} \quad \text{(again summing a geometric series)}
\end{aligned}$$

- Thus

$$\mathrm{Corr}[Y_t, Y_{t-1}] = \left[\frac{\phi\sigma_e^2}{1-\phi^2}\right] \Big/ \left[\frac{\sigma_e^2}{1-\phi^2}\right] = \phi$$

General Linear process
An example

- In a similar manner, we can find $\mathrm{cov}[Y_t, Y_{t-k}] = \dfrac{\phi^k \sigma_e^2}{1-\phi^2}$
- Thus
$$\mathrm{Corr}[Y_t, Y_{t-k}] = \phi^k \tag{3}$$
- The process is stationary: the autocovariance structure depends only on time lag and not We havebsolute time

Outline of the lecture

Moving Average Process
Definition

- Only a finite number of the ψ-weights are nonzero, we have what is called a moving average process:
$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \tag{4}$$
- More precisely, we have a **moving average of order** q denoted by MA(q).

Moving Average Process
MA(1)

- Clearly $\mathrm{E}[Y_t] = 0$ and $\mathrm{Var}[Y_t] = \sigma_e^2(1+\theta^2)$. Now
$$\begin{aligned}\mathrm{cov}[Y_t, Y_{t-1}] &= \mathrm{cov}[e_t - \theta e_{t-1}, e_{t-1} - \theta e_{t-2}] \\ &= \mathrm{cov}[-\theta e_{t-1}, e_{t-1}] = -\theta\sigma_e^2\end{aligned}$$
and
$$\begin{aligned}\mathrm{cov}[Y_t, Y_{t-2}] &= \mathrm{cov}[e_t - \theta e_{t-1}, e_{t-2} - \theta e_{t-3}] \\ &= 0\end{aligned}$$
since there are no e's with subscripts in common between Y_t and Y_{t-2}.

Moving Average Process

MA(1)

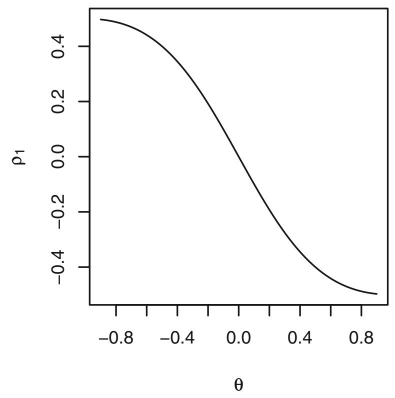
- In summary, for an MA(1) model $Y_t = e_t - \theta e_{t-1}$,

$$\left.\begin{aligned} \mathrm{E}[Y_t] &= 0 \\ \gamma_0 &= \mathrm{Var}[Y_t] = \sigma_e^2(1+\theta^2) \\ \gamma_1 &= -\theta\sigma_e^2 \\ \rho_1 &= -\theta/(1+\theta^2) \\ \gamma_k &= \rho_k = 0, \quad k \geqslant 2 \end{aligned}\right\} \qquad (5)$$

Moving Average Process

MA(1)

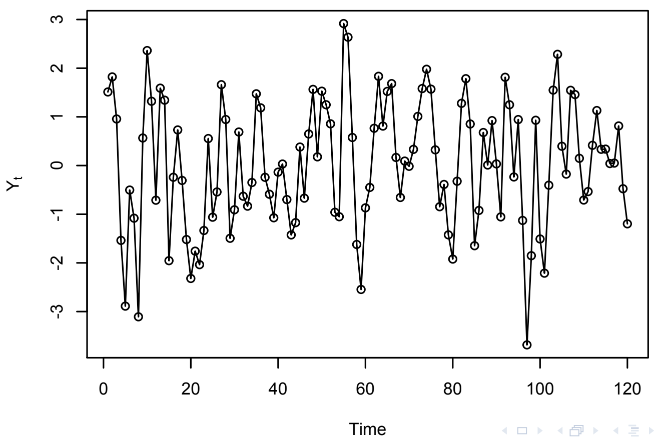
Lag 1 Autocorrelation of an MA(1) Process for Different θ



A Simulated MA(1) process

Using R

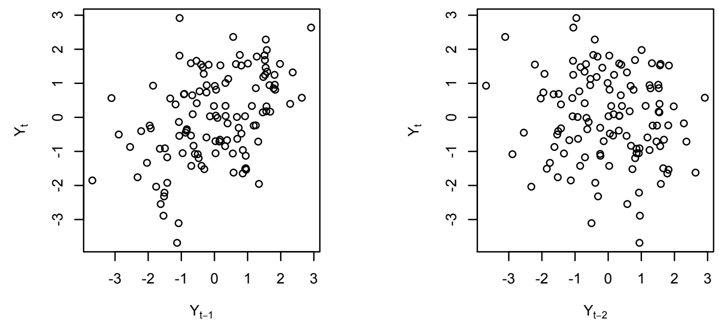
Time Plot of an MA(1) Process with $\theta = -0.9$



A Simulated MA(1) process

Using R

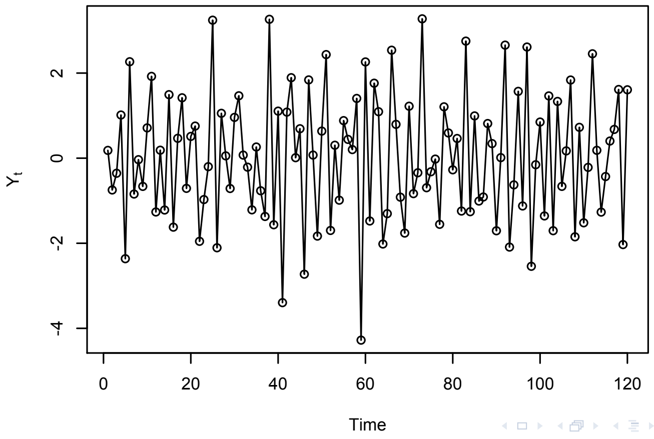
Plot of Y_t versus Y_{t-k} for MA(1) Series in Exhibit 54



A Simulated MA(1) process

Using R

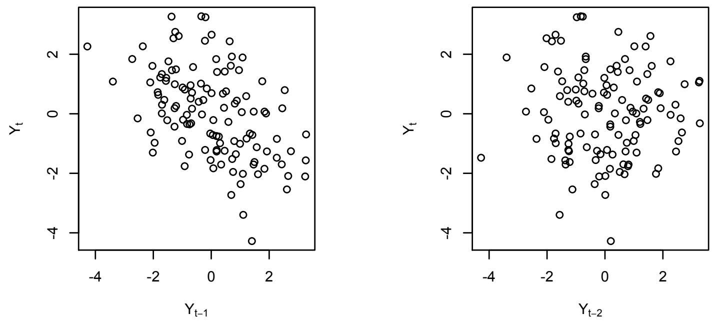
Time Plot of an MA(1) Process with $\theta = +0.9$



A Simulated MA(1) process

Using R

Plot of Y_t versus Y_{t-k} for MA(1) Series in Exhibit 54



The Second Order Moving Average

Ma(2)

Consider the moving average process of order 2:

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

Here

$$\begin{aligned}
\gamma_0 &= \mathrm{Var}[Y_t] = \mathrm{Var}[e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}] = (1 + \theta_1^2 + \theta_2^2)\sigma_e^2 \\
\gamma_1 &= \mathrm{cov}[Y_t, Y_{t-1}] = \mathrm{cov}[e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3}] \\
&= \mathrm{cov}[-\theta_1 e_{t-1}, e_{t-1}] + \mathrm{cov}[-\theta_1 e_{t-2}, -\theta_2 e_{t-2}] \\
&= [-\theta_1 + (-\theta_1)(-\theta_2)]\sigma_e^2 \\
&= (-\theta_1 + \theta_1\theta_2)\sigma_e^2
\end{aligned}$$

The Second Order Moving Average

Ma(2)

and

$$\begin{aligned}
\gamma_2 &= \mathrm{cov}[Y_t, Y_{t-2}] = \mathrm{cov}[e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4}] \\
&= \mathrm{cov}[-\theta_2 e_{t-2}, e_{t-2}] \\
&= -\theta_2 \sigma_e^2
\end{aligned}$$

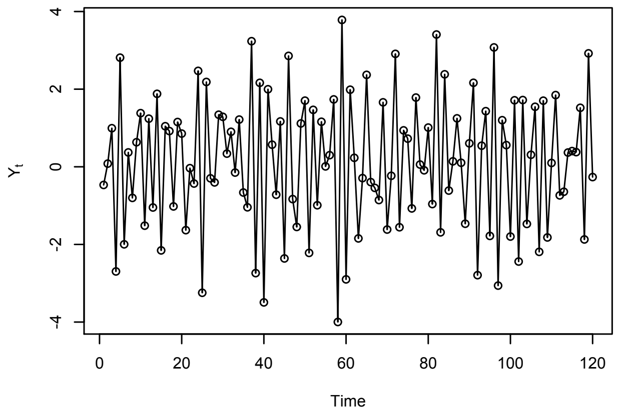
Thus, for an MA(2) process,

$$\begin{aligned}
\rho_1 &= \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} \\
\rho_2 &= \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} \\
\rho_k &= 0, \quad \text{for } k \geqslant 3.
\end{aligned}$$

A Simulated MA(1) process

Using R

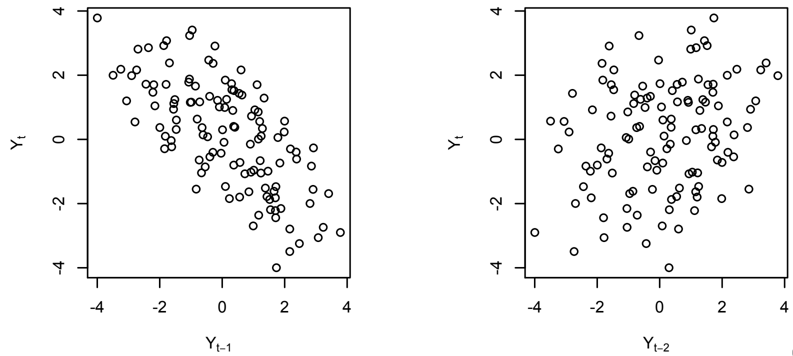
Time Plot of an MA(2) Process with $\theta_1 = 1$ and $\theta_2 = -0.6$



A Simulated MA(1) process

Strong negative r_1 with a weak positive r_2

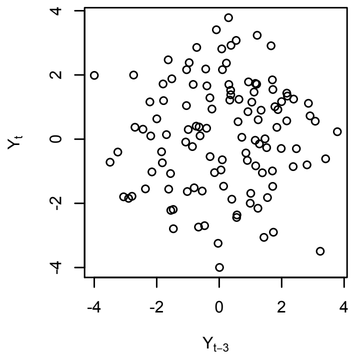
Plot of Y_t versus Y_{t-k} for MA(2) Series in Exhibit 57



A Simulated MA(1) process

Lack of autocorrelation from lag 3

Plot of Y_t versus Y_{t-k} for MA(2) Series in Exhibit 57



General MA(q) Process

Definition

For the more general MA(q) process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

similar calculations give

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2) \tag{6}$$

and

$$\rho_k = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} & \text{for } k = 1, 2, \ldots, q \\ 0 & \text{for } k > q \end{cases} \tag{7}$$

where the numerator of ρ_q is just $-\theta_q$.

Outline of the lecture

Autoregressive Processes

Definition

- Autoregressive processes are,as their name suggests, regressions on themselves. Specifically, a pth-order autoregressive process $\{Y_t\}$ satisfies the equation

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t, \quad \forall t \in \mathbb{Z}. \qquad (8)$$

The First-Order Autoregressive Process

Some easy computations

- Assume the series is stationary and satisfies

$$Y_t = \phi Y_{t-1} + e_t \qquad (9)$$

- Take variances of both sides of Equation (9) and obtain

$$\gamma_0 = \phi^2 \gamma_0 + \sigma_e^2$$

  Solving for γ_0 yields

$$\gamma_0 = \frac{\sigma_e^2}{1-\phi^2} \qquad (10)$$

- Immediate implication: $\phi^2 < 1$ or that $|\phi| < 1$.

The First-Order Autoregressive Process

Some easy computations

- Multiply both sides of (10) by Y_{t-k} ($k = 1, 2, \ldots$), and take expected values

$$\mathrm{E}[Y_{t-k} Y_t] = \phi \mathrm{E}[Y_{t-k} Y_{t-1}] + \mathrm{E}[Y_{t-k} e_t]$$

  or

$$\gamma_k = \phi \gamma_{k-1} + \mathrm{E}[Y_{t-k} e_t]$$

  Since $\mathrm{E}[e_t Y_{t-k}] = \mathrm{E}[e_t]\,\mathrm{E}[Y_{t-k}] = 0$

$$\gamma_k = \phi \gamma_{k-1}, \quad \text{for } k \geqslant 1 \qquad (11)$$

The First-Order Autoregressive Process

Some easy computations

- With $k = 1$, we get

$$\gamma_1 = \phi\gamma_0 = \phi\sigma_e^2/(1-\phi^2).$$

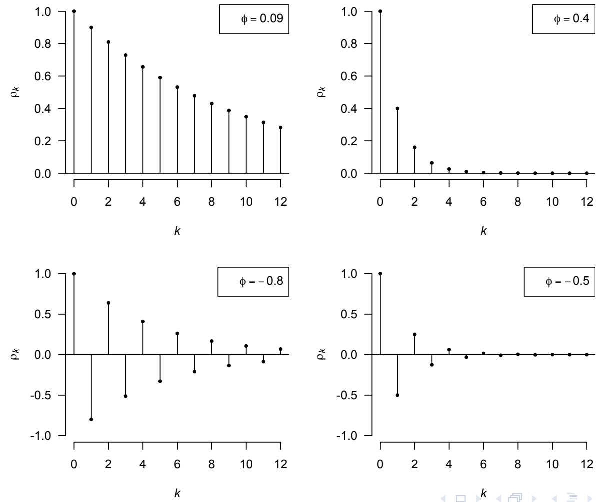
- With $k = 2$, we obtain

$$\gamma_2 = \phi^2\sigma_e^2/(1-\phi^2).$$

- In general

$$\gamma_k = \phi^k \frac{\sigma_e^2}{1-\phi^2} \qquad \rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k, \quad \text{for } k \geqslant 1 \tag{12}$$

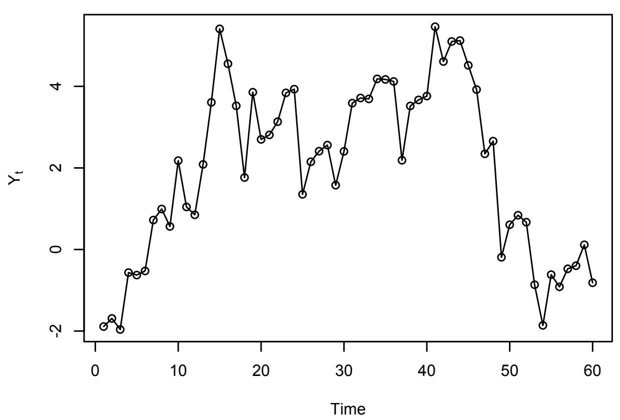
Autocorrelation Functions for Several AR(1) Models



A Simulated AR(1) process

Using R

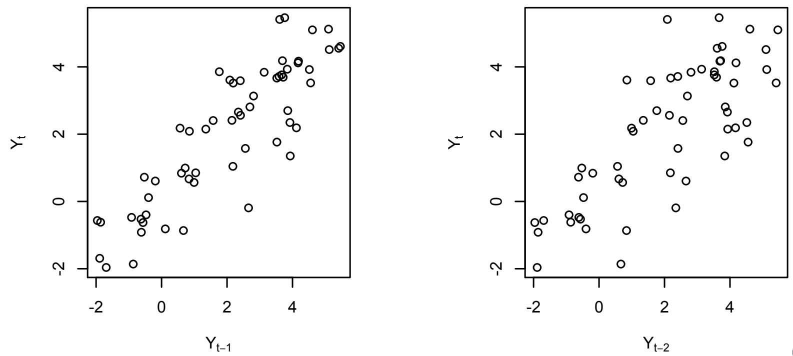
Time Plot of an AR(1 Process with $\phi = 0.9$



A Simulated AR(1) process

Strong positive r_1 and r_2

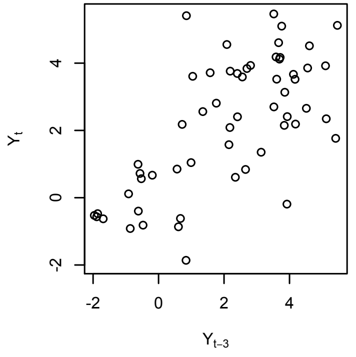
Plot of Y_t versus Y_{t-k}, $k = 1,\ 2$ for AR(1) Series in Exhibit 67



A Simulated AR(1) process

Weaker correlatiWe havet lag 3

Plot of Y_t versus Y_{t-3} for AR(1) Series in Exhibit 67



Autoregressive process

Of order 2

- Autoregressive equation:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t, \quad \forall t \in \mathbb{Z}.$$

with $\{e_t\}_{t\in\mathbb{Z}}$ WWN$(0;\sigma_e^2)$.

- Autoregressive polynomial:

$$\phi(x) = 1 - \phi_1 B - \phi_2 B^2.$$

- Characteristic equation:

$$1 - \phi_1 x - \phi_2 x^2 = 0.$$

Autoregressive process AR(2)

Stationarity condition

- The roots of the characteristic equation:

$$1 - \phi_1 x - \phi_2 x^2 = 0.$$

ought to be outside the unit circle..

- The roots are

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

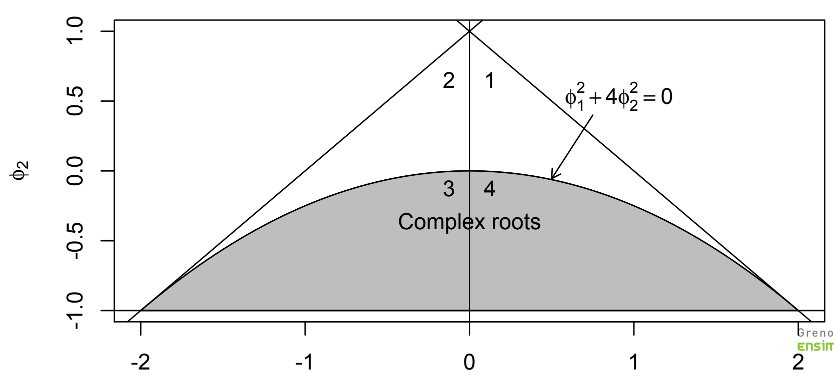
- They are outside the unit circle if, and only if

$$\begin{cases} \phi_1 + \phi_2 < 1 \\ \phi_2 - \phi_1 < 1 \\ |\phi_2| < 1 \end{cases}$$

Autoregressive process of order 2

Stationarity condition

(ϕ_1, ϕ_2) : Stationarity Region for AR(2) Process



Autoregressive process of order 2

Autocorrelation function

- We get a recursive formula:

$$
\begin{aligned}
Y_{t-k}Y_t &= \phi_1 Y_{t-k}Y_{t-1} + \phi_2 Y_{t-k}Y_{t-2} + Y_{t-k}e_t \\
\mathrm{E}[Y_{t-k}Y_t] &= \phi_1 \mathrm{E}[Y_{t-k}Y_{t-1}] + \phi_2 \mathrm{E}[Y_{t-k}Y_{t-2}] + \mathrm{E}[Y_{t-k}e_t] \\
\gamma_k &= \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}, \quad k = 1,\ 2,\ \ldots \\
&\text{or} \\
\rho_k &= \phi_1\rho_{k-1} + \phi_2\rho_{k-2}, \quad k = 1,\ 2,\ \ldots
\end{aligned}
$$

- With $k = 1$ and $k = 2$, these are the Yule-Walker equations.

Yule-Walker equations

$p = 2$

- Yule-Walker equations are

$$
\begin{aligned}
\rho_1 &= \phi_1 \ + \phi_2\rho_1 \\
\rho_2 &= \phi_1\rho_1 + \phi_2
\end{aligned}
$$

- Can be solved for ρ_1 and ρ_2

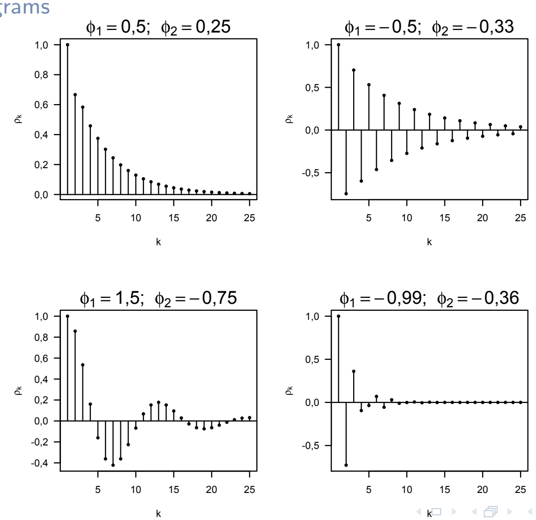
$$
\begin{aligned}
\rho_1 &= \frac{\phi_1}{1-\phi_2} \\
\rho_2 &= \phi_1\rho_1 + \phi_2 \\
&= \frac{\phi_2(1-\phi_2) - \phi_1^2}{1-\phi_2}
\end{aligned}
$$

- and for the others:

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}.$$

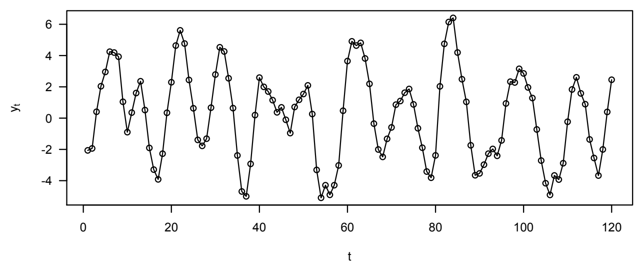
Autoregressive process of order 2

Possible correlograms



Autoregressive process of order 2

Trajectory with $\phi_1 = 1,5$ and $\phi_2 = -0,75$



Variance of an AR(2)

- On the one hand:

$$
\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t \\
\mathrm{Var}[Y_t] &= \phi_1^2 \,\mathrm{Var}[Y_{t-1}] + \phi_2^2 \,\mathrm{Var}[Y_{t-2}] + 2\phi_1\phi_2 \,\mathrm{cov}[Y_{t-1}; Y_{t-2}] + \mathrm{Var}[e_t] \\
\gamma_0 &= (\phi_1^2 + \phi^2)\gamma_0 + 2\phi_1\phi_2\gamma_1 + \sigma_e^2
\end{aligned}
$$

- On the other hand:

$$
\begin{aligned}
\gamma_k &= \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} \\
\gamma_1 &= \phi_1\gamma_0 + \phi_2\gamma_1
\end{aligned}
$$

- Two linear equations with 2 unknowns. For γ_0 we get:

$$
\gamma_0 = \frac{1-\phi_2}{1+\phi_2}\,\frac{\sigma_e^2}{(1-\phi_2)^2 - \phi_1^2}
$$

White noise representation

- Two representations:

$$
Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t \quad \text{and} \quad Y_t = \sum_{i=0}^{\infty} \psi_i e_{t-i}
$$

Backshift operator B

Definition

- Let $\{w_t\}$ a sequence of real numbers.
- The backshift operator gives a new sequence:

$$
B(\{w_t\}_{t\in\mathbb{Z}}) = \{w_{t-1}\}_{t\in\mathbb{Z}}
$$

- At time t, the value of $B(w_t)$ is w_{t-1}.

White noise representation

- We get two representations:

$$
Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} = e_t \Leftrightarrow (1 - \phi_1 B - \phi_2 B^2) Y_t = e_t
$$

and

$$
Y_t = \sum_{i=0}^{\infty} \psi_i e_{t-i} \Leftrightarrow Y_t = (1 + \psi_1 B + \psi_2 B^2 + \cdots) e_t
$$

- Two operators are involved:

$$
\phi(B) = 1 - \phi_1 B - \phi_2 B^2 \quad \text{and} \quad \psi(B) = \sum_{i=1}^{\infty} \psi_i B^i
$$

White noise representation

- We can think of an operator as being the inverse of the other one:

$$
\begin{aligned}
(1-\phi_1 B-\phi_2 B^2)\{Y_t\} &= \phi(B)\{Y_t\} = \{e_t\} \\
\sum_{i=1}^{\infty} \psi_i B^i \{e_t\} &= \psi(B)\{e_t\} = \{Y_t\}
\end{aligned}
$$

Formal computation

- We have:

$$
\begin{aligned}
\phi(B) Y_t &= e_t \\
Y_t &= \phi(B)^{-1} e_t = \psi(B) e_t
\end{aligned}
$$

- So,

$$
\begin{aligned}
\phi(B)^{-1} &= \psi(B) \\
\phi(B)\psi(B) &= \mathbf{1} = \mathbf{1} + 0 \times B + 0 \times B^2 + 0 \times B^3 + \cdots
\end{aligned}
$$

where $\mathbf{1}$ is the identity operator: $\mathbf{1}(\{w_t\}_{t\in\mathbb{Z}}) = \{w_t\}_{t\in\mathbb{Z}}$

Formal computation

continued

- Let us mutliply these two operators

$$
\begin{array}{l}
1+\psi_1 B+\psi_2 B^2+\psi_3 B^3+\cdots \\
\times\ 1-\phi_1 B-\phi_2 B^2 \\
\hline
1+\psi_1 B+\psi_2 B^2 \quad +\psi_3 B^3 \qquad +\psi_4 B^4 \quad +\cdots \\
\quad -\phi_1 B-\phi_1\psi_1 B^2-\phi_1\psi_2 B^3 \ -\phi_1\psi_3 B^4-\cdots \\
\qquad\qquad -\phi_2 B^2 \quad -\phi_2\psi_1 B^3 2-\phi_2\psi_2 B^4-\phi_3\psi_4 B^5-\cdots
\end{array}
$$

ought to give the identity $\mathbf{1}$.

- So,:

$$
\begin{cases}
\psi_0 &= 1 \\
\psi_1 &= \phi_1 \\
\psi_2 &= \phi_1\psi_1 + \phi_2 \\
\psi_j &= \phi_1\psi_{j-1} + \phi_2\psi_{j-2}, \quad j \geqslant 2.
\end{cases}
$$

Autoregressive process of order p

Definition

- Model equation

$$
Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \tag{13}
$$

or

$$
(1-\phi_1 B - \cdots - \phi_p B^p) Y_t = e_t
$$

- Charateristic polynomial:

$$
\phi(B) = 1 - \phi_1 x - \cdots - \phi_p x^p
$$

- Charateristic equation:

$$
1 - \phi_1 x - \cdots - \phi_p x^p = 0.
$$

Autoregressive process of order p

Stationarity condition

- e_t is not correlated with $Y_{t-1},\ Y_{t-2},\ Y_{t-3},\ \ldots$
- Equation (13) has a stationary solution if the roots of

$$1 - \phi_1 x - \cdots - \phi_p x^p = 0$$

are outside the unit circle.

Autoregressive process of order p

Yule-Walker equation

- Consider

$$Y_t Y_{t-k} = \phi_1 Y_{t-1} Y_{t-k} + \phi_2 Y_{t-2} Y_{t-k} + \cdots + \phi_p Y_{t-p} Y_{t-k} + e_t Y_{t-k}$$

- Taking the expected value on both sides:

$$\begin{aligned}\mathrm{E}[Y_t Y_{t-k}] = \phi_1 \mathrm{E}[Y_{t-1} Y_{t-k}] + \phi_2 \mathrm{E}[Y_{t-2} Y_{t-k}] + \cdots \\ + \phi_p \mathrm{E}[Y_{t-p} Y_{t-k}] + \mathrm{E}[e_t Y_{t-k}] \quad (14)\end{aligned}$$

giving

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \cdots + \phi_p \gamma_{k-p}$$

Autoregressive process of order p

Yule-Walker equation

- For $k = 1,\ \ldots.\ p$, dividing by γ_0 gives:

$$\begin{aligned}\rho_1 &= \phi_1 + \phi_2 \rho_1 + \cdots + \phi_p \rho_{p-1} \\ \rho_2 &= \phi_1 \rho_1 + \phi_2 + \cdots + \phi_p \rho_{p-2} \\ &\vdots \\ \rho_p &= \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + \cdots + \phi_p\end{aligned}$$

- For fixed values od $\phi_1,\ \ldots,\ \phi_p$, system can be solved and gives $\rho_1,\ \ldots,\ \rho_p$
- For fixed values od $\rho_1,\ \ldots,\ \rho_p$, system can be solved and gives $\phi_1,\ \ldots,\ \phi_p$

Autoregressive process of order p

Variance

- Remark:

$$\mathrm{E}[Y_t e_t] = \mathrm{E}[\phi_1 Y_{t-1} e_t + \phi_2 Y_{t-2} e_t + \cdots + \phi_p Y_{t-p} e_t] + \mathrm{E}[e_t e_t] \ = \sigma_e^2$$

with $k = 0$, (14) becomes:

$$\begin{aligned}\gamma_0 &= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \cdots + \phi_p \gamma_p + \sigma_e^2 \\ \gamma_0 &= \phi_1 \rho_1 \gamma_0 + \phi_2 \rho_2 \gamma_0 + \cdots + \phi_p \rho_p \gamma_0 + \sigma_e^2\end{aligned}$$

since $\rho_k = \gamma_k / \gamma_0$.

- Finally:

$$\gamma_0 = \frac{\sigma_e^2}{1 - \phi_1 \rho_1 - \phi_2 \rho_2 - \cdots - \phi_p \rho_p}$$

Autoregressive process of order p

White noise representation

- If $\{Y_t\}$; AR(p) is stationnary, we can also write:

$$Y_t = \sum_{j=0}^{\infty} \psi_j e_{t-j} = \psi(B).$$

- Explicite expression of ψ_j is not simple.
- Expand $(1 - \phi_1 B - \cdots - \phi_p B^p)^{-1}$ into power series.

Autoregressive process of order p

White noise representation

$$\begin{aligned}
(1-\phi_1 B-\phi^2 B^2)^{-1} &= 1+\phi_1\, B+\left(\phi_2+\phi_1^2\right)\, B^2+\left(2\,\phi_1\,\phi_2+\phi_1^3\right)\, B^3+\left(\phi_2^2+3\right. \\
&+ \left(3\,\phi_1\,\phi_2^2+4\,\phi_1^3\,\phi_2+\phi_1^5\right)\, B^5+\left(\phi_2^3+6\,\phi_1^2\,\phi_2^2+5\,\phi_1^4\,\phi_2+\phi_1^6\right)\, B^6 \\
&+ \left(4\,\phi_1\,\phi_2^3+10\,\phi_1^3\,\phi_2^2+6\,\phi_1^5\,\phi_2+\phi_1^7\right)\, B^7 \\
&+ \left(\phi_2^4+10\,\phi_1^2\,\phi_2^3+15\,\phi_1^4\,\phi_2^2+7\,\phi_1^6\,\phi_2+\phi_1^8\right)\, B^8 \\
&+ \left(5\,\phi_1\,\phi_2^4+20\,\phi_1^3\,\phi_2^3+21\,\phi_1^5\,\phi_2^2+8\,\phi_1^7\,\phi_2+\phi_1^9\right)\, B^9 \\
&+ \left(\phi_2^5+15\,\phi_1^2\,\phi_2^4+35\,\phi_1^4\,\phi_2^3+28\,\phi_1^6\,\phi_2^2+9\,\phi_1^8\,\phi_2+\phi_1^{10}\right)\, B^{10}+\cdots \qquad (15)
\end{aligned}$$

Outline of the lecture

Autoregressive moving-average process of order $(p; q)$

Definition

- Model equation

$$\begin{aligned}
Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} \\
+ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} + \cdots - \theta_q e_{t-q} \qquad (16)
\end{aligned}$$

where the polynomials

$$\phi(x) = 1 - \phi_1 x - \cdots - \phi_p x^p$$
$$\theta(x) = 1 - \theta_1 x - \cdots - \theta_q x^q$$

do not share common roots.

- Example ; Y_t : ARMA(1; 1)

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$$

Autoregressive moving-average process of order (1; 1)

Covariance structure

- Remark:

$$
\begin{aligned}
\mathrm{E}[e_t Y_t] &= \mathrm{E}[e_t(\phi Y_{t-1} + e_t - \theta e_{t-1})] \\
&= \sigma_e^2 \\
\mathrm{E}[e_{t-1} Y_t] &= \mathrm{E}[e_{t-1}(\phi Y_{t-1} + e_t - \theta e_{t-1})] \\
&= \phi\sigma_e^2 - \theta * \sigma_e^2 \\
&= (\phi - \theta)\sigma_e^2
\end{aligned}
$$

Autoregressive moving-average process of order (1; 1)

Covariance structure

- By

$$
\mathrm{E}[Y_{t-k} Y_t] = \mathrm{E}[Y_{t-k}(\phi Y_{t-1} + e_t - \theta e_{t-1})]
$$

we find

$$
\gamma_k = \begin{cases} \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2 & k = 0; \\ \phi\gamma_0 - \theta\sigma_e^2 & k = 1; \\ \phi\gamma_{k-1} & k \geqslant 2. \end{cases}
$$

Autoregressive moving-average process of order (1; 1)

Covariance structure

- We find:

$$
\begin{aligned}
\gamma_0 &= \frac{(1 - 2\phi\theta + \theta^2)}{1 - \phi^2}\sigma_e^2 \\
\gamma_1 &= \frac{(1 - \phi\theta)(\phi - \theta)}{1 - \phi^2}\sigma_e^2; \\
\gamma_k &= \phi\gamma_{k-1}, \quad k \geqslant 2
\end{aligned}
$$

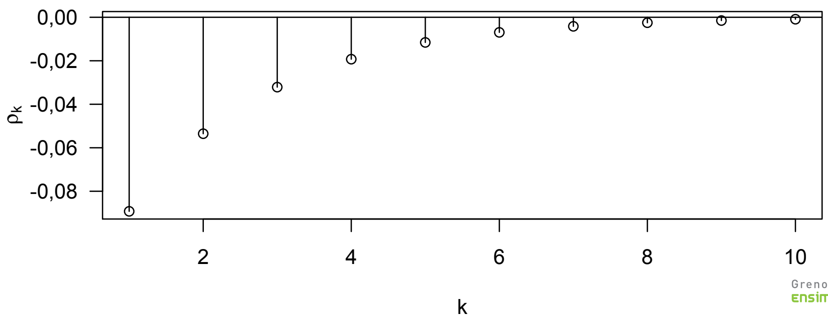
or

$$
\begin{aligned}
\rho_1 &= \frac{(1 - \phi\theta)(\phi - \theta)}{1 - 2\Phi\theta + \theta^2} \\
\rho_k &= \phi\rho_{k-1}, \quad k \geqslant 2
\end{aligned}
$$

Autoregressive moving-average process of order (1; 1)

Correlation structure

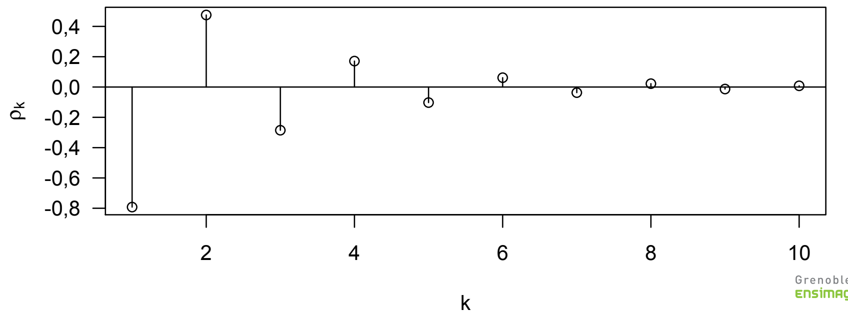
- $\phi = 0,6$ and $\theta = 0,7$



Autoregressive moving-average process of order (1; 1)

Correlation structure

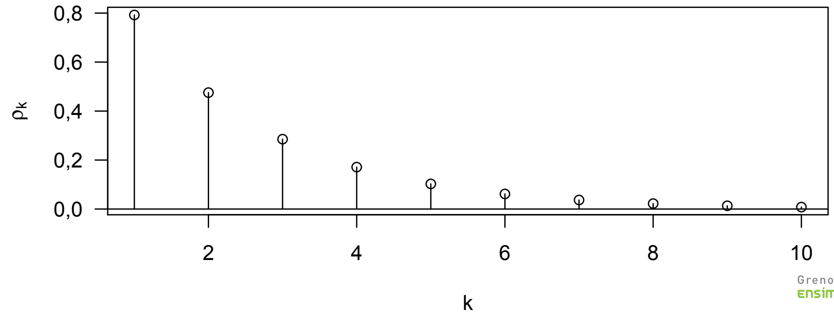
- $\phi = -0,6$ and $\theta = 0,7$



Autoregressive moving-average process of order (1; 1)

Correlation structure

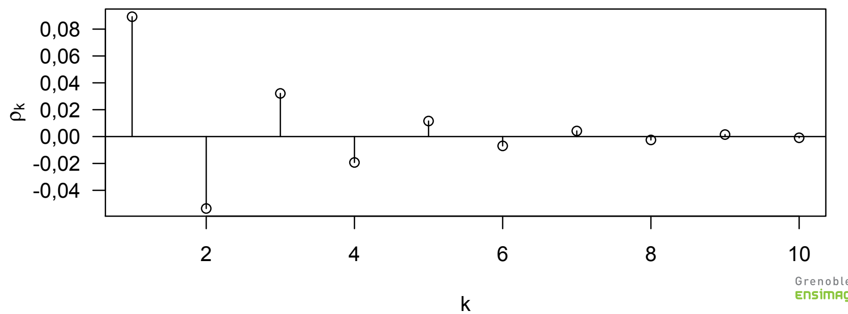
- $\phi = 0,6$ and $\theta = 0,7$



Autoregressive moving-average process of order (1; 1)

Correlation structure

- $\phi = -0,6$ and $\theta = -0,7$



Autoregressive moving-average process of order (1; 1)

White noise representation

- As on slide 83, we find:

$$Y_t = e_t + (\phi - \theta) \sum_{i=1}^{\infty} \phi^{i-1} e_{t-i}$$

that is,

$$\psi_i = (\phi - \theta)\phi^{i-1}$$

Autoregressive moving-average process of order $(p; q)$

White noise representation

- In fact proceeding as we did on slide 83, we find:

$$\begin{aligned}\psi_0 &= 1\\ \psi_1 &= -\theta_1 + \phi_1\\ \psi_2 &= -\theta_2 + \phi_2 + \phi_1\psi_1\\ &\vdots\\ \psi_j &= -\theta_j + \phi_p\psi_{j-p} + \phi_{p-1}\psi_{j-p+1} + \cdots + \phi_1\psi_{j-1}\end{aligned}$$

where $\psi_j = 0$ for $j < 0$ and $\theta_j = 0$ for $j > q$

Autoregressive moving-average process of order $(p; q)$

Autocorrelation function

- For an ARMA(p, q), we have:

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \phi_p\rho_{k-p} \quad k \geqslant \; q + 1.$$

- In R, the autocorrelation function is avaluated using `ARMAacf`

Outline of the lecture

Invertibility

Exemple MA(1) $\Leftrightarrow$ AR(∞)

- We have $Y_t = e_t - \theta e_{t-1}$ with $|\theta| < 1$
- Also,

$$\begin{aligned}e_t &= Y_t + \theta e_{t-1}\\ &= Y_t + \theta(Y_{t-1} + \theta e_{t-2})\\ &= Y_t + \theta Y_{t-1} + \theta^2 e_{t-2}\end{aligned}$$

- Continuing these substitutions we get:

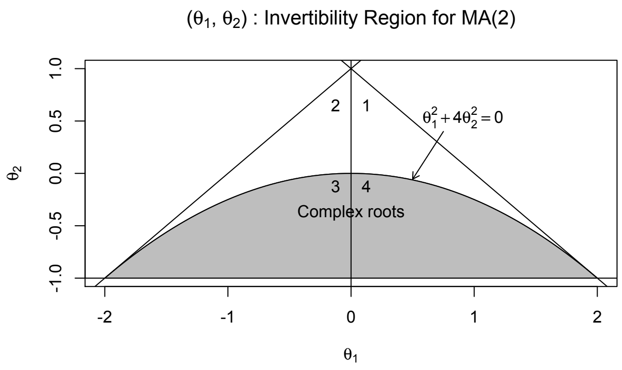
$$e_t = Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \theta^3 Y_{t-3} + \cdots$$

- where

$$Y_t = (-\theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} - \cdots) + e_t$$

Invertibility Region for MA(2) process

Invertibility conditions region for MA(2) process



Autoregressive process-moving average of order $(p; q)$

Invertibility

- The charateristic polynomial

$$\theta(x) = 1 - \theta_1 x - \cdots - \theta_q x^q$$

  has its roots outside the unit circle.
- There exists a real sequence $\{\pi_j\}_{j=1}^{\infty}$ such that

$$Y_t = \sum_{j=1}^{\infty} \pi_j Y_{t-j} + e_t.$$

- We always work with stationary and invertible ARMA$(p; q)$ processes.

Properties of ARMA processes

	AR(p)	**MA(q)**	**ARMA(p, q)**
In terms of past values	$\phi(B)Y_t = e_t$	$\theta^{-1}(B)Y_t = e_t$	$\theta^{-1}(B)\phi(B)Y_t = e_t$
In terms of white noise	$Y_t = \phi^{-1}(B)e_t$	$Y_t = \theta(B)e_t$	$Y_t = \phi^{-1}(B)\theta(B)e_t$
π weights	finite number	infinite number	infinite number
ψ weights	infinite number	finite number	infinite number
Stationarity condition	roots of $\phi(B) = 0$ lie outside the unit circle	always stationary	roots of $\phi(B) = 0$ lie outside the unit circle
Invertibility condition	always invertible	roots of $\theta(B) = 0$ lie outside the unit circle	roots of $\theta(B) = 0$ lie outside the unit circle
Autocorrelation function	infinite (damped exponentials and/or damped sine waves tails off	finite cuts off	infinite (damped exponentials and/or damped sine waves tails off
Partial autocorrelation function	finite cuts off	infinite (damped exponentials and/or damped sine waves tails off	infinite (damped exponentials and/or damped sine waves tails off

Ljung-Box statistics

$$Q_{LB} = n(n+2)\sum_{h=1}^{H} \frac{\rho_e^2(h)}{n-h} \sim \chi^2_{H-p-q}.$$

References

Cryer, J. D. and K.-S. Chan (2008). *Time Series Analysis: With Applications in R* (2 ed.). Springer texts in statistics. New York: Springer.

Lapointe, C. (1998). *Séries chronologiques en santé publique : initiation à la modélisaiton ARIMA*. Cap St-Ignace (Québec): Lepoint.

Shumway, R. H. and D. S. Stoffer (2011). *Time Series Analysis and Its Applications: With R Examples* (3 ed.). Springer texts in statistics. New York: Springer.

Shumway, R. H. and D. S. Stoffer (2016). *Time Series Analysis and Its Applications: With R Examples* (EZ ed.). Pittsburg: Free Dog Publishing.

Tsay, R. (2010). *Analysis of Financial Time Series* (3 ed.). John Wiley Sons Inc.