Continued Fraction Expansion, Convergents, and the Best Rational Approximations

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It's well known that irrational numbers have an infinite decimal expansion that does not follow any periodic pattern. So, deriving a concrete numerical value for irrational numbers is impossible. However, the closest mathematicians can get is by approximating irrational numbers using rational numbers. Naturally, since the rational numbers are so dense in the real numbers, using rational numbers as an approximation proves efficient. Based on the theorem in real analysis that says between any two rational numbers there always exists an irrational number, and between any two irrational numbers, there is a rational number. So, using this information, if we can find the rational closest to the irrational number in question, we have found the best approximation of that irrational. This is much easier said than done and requires more than just estimations of the closest rational numbers. Mathematicians have found that there is a more refined set of numbers that can exist as the exact closest rational number to an irrational number. These are called the convergents of said irrational number derived by finding the continued fraction expansion of the irrational number.

First, any irrational number can be expressed as a sum of its integer part and its fractional part. Furthermore, its fractional part can be defined as a reciprocal iterative process. To simplify, A continued fraction expansion of an irrational number is defined as a sequence of integers that represent a finite or infinite iterative process of writing an irrational number as the sum of its integer part and reciprocal part. There have been many theories to derive the continued fraction expansion, such as the BRAIN approach explained by Sikorav or the standard algorithm used. There are many versions and varieties of continued fraction expansions. These vary by being generalized or having periodic properties. Here, we will focus on the standard algorithm and the standard continued fraction expansion. This allows us to utilise the uniqueness of the expansions, keeping the numerator of the reciprocal parts 1 and allowing only the integer part to be negative for simplification and convention. The standard algorithm displays the continued fraction expansion as a sequence of integers where each input; ak, is drawn from the integer part at each step k of the expansion.

I.e: form represents the expansion is represented by form [a0, a1,...,ak,....].

$$* = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

We express the irrational as the k part of the iterative process, noting the value of k can be infinite.

Directly from this algorithmic continued fraction expansion, the convergents of the irrational number α can be derived by isolating each k step of the iterative process. We define the set of convergents by [c0, c1,..., ck,...], where each ck is in Q (the rational numbers).

$$c_0 = a_0$$

$$c_1 = a_0 + \frac{1}{a_1}$$

$$\vdots$$

$$c_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{ect \ldots + \cfrac{1}{a_n}} \ldots}}$$

$$\vdots$$

If α is irrational, and has the continued fraction expansion [a0, a1,...,ak,....], then the following theorems hold about the convergents of α.

Theorem 1: α is the limit of the sequence of convergents $\frac{p_n}{q_n}$

Theorem 2: The nth convergent (cn) satisfies $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n-1}}$

Theorem 3: Each convergent (cn) is closer to α than the previous convergent (cn-1)

$\left|\alpha - \frac{p_{n+1}}{q_n+1}\right| < \left|\alpha - \frac{p_n}{q_n}\right|$

Therefore, for the sequence [c0,c1,..,ck,...] to be a valid sequence of convergents for an irrational number α, the three theorems must hold.

Subsequently, the best rational approximation for an irrational number α is one element in the sequence [c0,c1,..,ck,...]. This is proven in the theorem below.

Theorem 4: If $\frac{p_n}{q_n}$ is one of the convergents of α, then the best rational approximation to α is $\frac{p_n}{q_n}$ s.t the denominator is at most qn.

Ie. The best rational approximations are drawn from continued fractions.

Notice that despite showing that all convergents for α are the best rational approximations, they are not the only ones.

Consider α = $\frac{1}{\phi}$. The golden ratio is known to have the continued fraction expansion given by the Fibonacci numbers. More specifically, the convergent cn = $\frac{F_{n+1}}{F_n}$.

Therefore the continued fraction expansion of $\frac{1}{\phi}$ is $\frac{1}{c_n}$of the golden ratio.

There for given the set of convergents [c0, c1,...,ck,..] are the convergents for the golden ratio, the set of convergents for α is now defined by [$\frac{1}{c_0}, \frac{1}{c_1}, \ldots, \frac{1}{c_k}$ ....]. So, we now have that $\frac{1}{c_n} = \frac{F_n}{F_{n+1}}$.

All the theorems listed above have been proved in lecture. Thus we may conclude that $\frac{1}{c_n}$ is the nth convergent of $\frac{1}{\phi}$, and it is also the best rational approximation for $\frac{1}{\phi}$

Furthermore, These conclusions can be expanded to include all best rational approximations for some irrational number α.

This says that for all intervals I that's a subset of the Torus and all x in the Torus, we have that..//

$$\lim_{N \to inf} \frac{|I \cap \{\{x + n *\}|0 \leq n \leq N\}|}{N} = length\ (I).$$

This gives you the limit as n approaches infinity of the countable number of points in the first N rotations of x by α that are in the uncountable interval I is equivalent to the length of interval I.

In summary, the length of an interval I along the Torus; [0,1), can be found by calculating the limit of the number of points in the first N rotations of x by an irrational number that can be approximated rationally using convergents of continued fraction expansions. The best rational approximations to an irrational number α lie among the convergents of that irrational number. The biggest issue that mathematicians have faced when working on Tarski's circle-squaring problem, is approximating pi. If being able to approximate pi was so easy, then we would be able to congruently decompose a polygon into a circle and vice versa, or just simply use Euclidean geometry. However, as many mathematicians have realized, these approaches are not possible. This best rational approximation allows for a concrete method to find the length of I. This allows us to define Gα by producing limits and interval interactions to provide another step in solving Tarski's circle squaring problem.

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