The Menger Sponge and the Primary Diagonal Plane Hausdorff Dimensions; The Menger Slice

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In the lectures, there was a significant focus on the Cantor set and all its interesting causal sequences. From its zero length and zero area to its fractional or Hausdorff dimension, the Cantor set baffles many mathematicians to this day. To advance this idea, the exploration of the generalizations of the Cantor set in two dimensions, such as the Sierpinski carpet or the Cantor dust, carries ultimately just as many interesting repercussions. However, we can go even further than that and explore its 3-dimensional generalization, the Menger sponge.

<u>Definition 1:</u> The Menger Sponge is a 3D generalization of the Cantor set. It is constructed of six faces, each acting as a Sierpinski carpet.

If we cut the Menger sponge horizontally at any point, we arrive at some arrangement of squares. If we cut the Menger sponge vertically at any point parallel to an edge, we arrive at some arrangement of rectangles. However, what if there's a cut along some diagonal?

<u>Theorem 1</u>: The Menger sponge, when cut along its primary diagonal planes (the plane perpendicular to the cube's long diagonal), produces its own unique Hausdorff dimension.

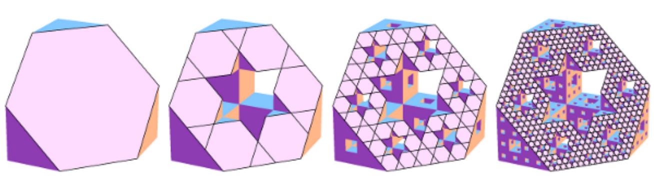
Supposing we cut the Sponge exactly in half and solve the Hausdorff dimension similarly to determining the Cantor set's, we arrive at an interesting consequence. When cutting the Menger sponge along any one of its primary diagonal planes, we arrive at some arrangement of triangles. Thus, along with the hexagonal fractal that's appearing, we also have a tri-fractal present.

For simplicity, denote the Menger sponge cut exactly on the mid-primary diagonal at time n by Mn, for any positive integer n. Suppose it has dimension d and volume h. Denote the Menger sponge cut exactly on the primary diagonal, producing the largest triangle at time n by Tn, for any positive integer n. Suppose it has dimension d and volume t.

At M0, we simply have a hexagonal face.
At T0, we simply have a triangular face.

At M1 we can see a center hexagram removed from the center of the hexagon. This produces 6 hexagonal surfaces and 6 triangle sub-faces along this cut.
At T1 we can see a center hexagram removed from the center of the triangle. This produces 1 hexagonal subface and 3 triangle sub faces along this cut.

Thus, we may conclude that along the hexagonal fractal has volume $h = \frac{6h}{3^d} + \frac{6t}{3^d}$, and we may conclude without loss of generality that on the triangle fractal $t = \frac{h}{3^d} + \frac{3t}{3^d}$.



Putting these all together, we arrive at $d = \frac{\log(\frac{9+\sqrt{33}}{2})}{\log(3)}$ ■

This can be compared to the Hausdorff dimension of the cantor set, and 2D generalizations of the cantor set. The interesting addition is the idea of an additional fractal dimension by cutting 3D fractals at different points. This is an extension of the knowledge from the lecture, and an interesting consequence of the fundamentals of the Cantor set provide.

Bibliography

Simons Foundation. (2014, May 22). *Mathematical impressions: The surprising menger sponge slice*. YouTube. https://www.youtube.com/watch?v=fWsmq9E4YC0