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PHYS 410 – Intro QUANTUM I
Homework for honor option: Intro to Quantum Computing

A. Theory: From classical state to quantum state
Let's consider a classical state system with 2 bits of two levels.

Define the term: Logic gates are the basic function in the circuit for a computer. There are gates such as NOT, AND, OR. In binary algebra, (not X = ~X), (X and Y = X*Y), (X or Y = X + Y). Remember that most calculations in binary algebra are the same as in decimal algebra except 1+1=1 in binary algebra.

Binary algebra in matrix notation
For one bit, $0 = |0\rangle = (1\ 0)$, $1 = |1\rangle = (0\ 1)$.
For two bits, $00 = |00\rangle = (1\ 0\ \ 0\ 0\ )$, $01 = |01\rangle = (0\ 1\ \ 0\ 0\ )$, $10 = |10\rangle = (0\ 0\ \ 1\ 0\ )$, $11 = |11\rangle = (0\ 0\ \ 0\ 1\ )$

A.1 Consider two classical states X and Y, draw the truth table for (not X), (X and Y), (X or Y). Now, let X and Y both be a 1x2 matric, and find the AND gate and OR gate matrix operator for the two bits two level classical systems. Hint: the NOT gate is $(0\ 1\ 1\ 0)$, and state X and state Y can be combined into a 1x4 matrix.

X	Y	~ X	X and Y	X or Y
0	0	1	0	0
0	1	1	0	1
1	0	0	0	1
1	1	0	1	1

Let X = $\left(x_1\ x_2\right)$ and Y = $\left(y_1\ y_2\right)$ and assume both X and Y are binary, that is if $x_1 = 1$, then $x_2 = 0$. If $x_1 = 0$, then $x_2 = 1$. Y follows the same pattern.

Define: $|0\rangle = (1\ 0)$, $|1\rangle = (0\ 1)$

$\left(x_1\ x_2\right)(0\ 1\ 1\ 0) = \left(x_2\ x_1\right)$

To implement a gate with two inputs, we need to define a
AND gate: $(1\ 1\ 0\ 0\ \ 1\ 0\ 0\ 1\ )\left(v_1\ v_2\ \ v_3\ v_4\ \right) = \left(v_1 + v_2 + v_3\ v_4\right)$, AND gate = $(1\ 1\ 0\ 0\ \ 1\ 0\ 0\ 1\ )$

OR gate: $(1\ 0\ 0\ 1\ \ 0\ 0\ 1\ 1\ )\left(v_1\ v_2\ \ v_3\ v_4\ \right) = \left(v_1\ v_2 + v_3 + v_4\right)$, OR gate = $(1\ 0\ 0\ 1\ \ 0\ 0\ 1\ 1\ )$

Check: 0 and 1 = 0 * 1 = 0

$|0\rangle\ and\ |1\rangle = AND|01\rangle = (1\ 1\ 0\ 0\ \ 1\ 0\ 0\ 1\ )(0\ 0\ \ 1\ 0\ ) = (0 + 0 + 1\ 0) = (1\ 0) = |0\rangle$

Check: 0 or 1 = 0 + 1 = 1

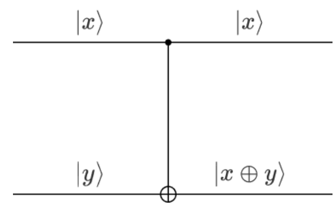
$|0\rangle\ or\ |1\rangle = OR|01\rangle = (1\ 0\ 0\ 1\ \ 0\ 0\ 1\ 1\ )(0\ 0\ \ 1\ 0\ ) = (0\ 0 + 1 + 0) = (0\ 1) = |1\rangle$

Let's now dig into the logic gate for a qubit.

Define the term: qubit is the bit system for quantum computers; qubits are no longer constrained to a discrete state like 0 and 1. Instead, a qubit can be a superposition of multiple levels. Here, we will only consider a two-level system.

Reversibility: In the quantum world, all operations that are not measurements are reversible and are represented by unitary matrices.

A.2 Controlled-Not gate: the controlled-not gate is like a xor gate in classical computing; however, the gate, like other quantum gates, has as many outputs as inputs. The gate also outputs the original input of the first state. Draw the truth table of the controlled-not gate, and find the matrix operator for it. Show that it's reversible.



$\lvert x_{in}\rangle$	$\lvert y\rangle$	$\lvert x_{out}\rangle$	$\lvert x\oplus y\rangle$
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

Controlled-Not gate:

$(1\ 0\ 0\ 1\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0\ 1\ 1\ 0\ )\left(v_1\ v_2\ \ v_3\ v_4\ \right) = \left(v_1\ v_2\ \ v_4\ v_3\ \right),\ CN\ =\ (1\ 0\ 0\ 1\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0$

$CN|00\rangle\ = (1\ 0\ 0\ 1\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0\ 1\ 1\ 0\ )(1\ 0\ \ 0\ 0\ ) = (1\ 0\ \ 0\ 0\ ) = |00\rangle$

$CN|01\rangle\ = (1\ 0\ 0\ 1\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0\ 1\ 1\ 0\ )(0\ 1\ \ 0\ 0\ ) = (0\ 1\ \ 0\ 0\ ) = |01\rangle$

$CN|10\rangle\ = (1\ 0\ 0\ 1\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0\ 1\ 1\ 0\ )(0\ 0\ \ 1\ 0\ ) = (0\ 0\ \ 0\ 1\ ) = |11\rangle$

$CN|11\rangle\ = (1\ 0\ 0\ 1\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0\ 1\ 1\ 0\ )(0\ 0\ \ 0\ 1\ ) = (0\ 0\ \ 1\ 0\ ) = |10\rangle$

To find reverse matrix of 4x4, we can use the augmented matrix

$(1\ 0\ 0\ 1\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 1\ 0\ 0\ 1\ )\sim(1\ 0\ 0\ 1\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0\ 1\ 1\ 0\ )$

Notice that CN operator is its own reversed matrix. That means $|x\oplus x\oplus y\rangle\ =\ |y\rangle$.

Instead of shifting the state into a classical discrete state, it's more interesting to shift the state into a state of superposition.

A.3 Show that the state $|\psi\rangle\ =\ c_0|0\rangle\ +\ c_1|1\rangle$ can be represented as

$|\psi\rangle = cos(\theta)|0\rangle + e^{i\phi} sin(\theta)|1\rangle$. (Remember that the state is normalized.)

$$\left|c_0\right|^2 + \left|c_1\right|^2 = 1$$

Rewrite the coefficients in complex polar form:

Let $c_0 = r_0 e^{i\phi_0}$

And $c_1 = r_1 e^{i\phi_1}$

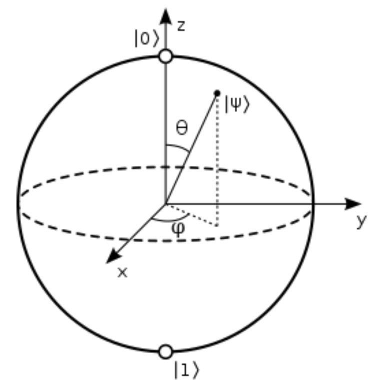
$$|\psi\rangle = r_0 e^{i\phi_0}|0\rangle + r_1 e^{i\phi_1}|1\rangle$$

A quantum physical state does not change if we multiply its corresponding vector by an arbitrary complex number.

$$e^{-i\phi_0}|\psi\rangle = r_0|0\rangle + r_1 e^{i\phi_1 - i\phi_0}|1\rangle$$

Let $e^{i\phi} = e^{i\phi_1 - i\phi_0}$

$$|\psi\rangle = r_0|0\rangle + r_1 e^{i\phi}|1\rangle$$



And we know that $\left|r_0\right|^2 + \left|r_1 e^{i\phi}\right|^2 = \left|r_0\right|^2 + \left|r_1\right|^2 = 1$

Let $r_0 = cos(\theta)$

And $r_1 = sin(\theta)$

Therefore, $|\psi\rangle = \cos cos\,(\theta)\,|0\rangle + e^{i\phi} \sin sin\,(\theta)\,|1\rangle$.
This is a state of Bloch sphere.

A.4 The phase shift gates:

The phase shift gate is defined as $R(\theta) = \left(1\ 0\ 0\ e^{\theta}\right)$
This gate performs the following:

$$\left(1\ 0\ 0\ e^{\theta}\right)\left(cos(\theta')\ e^{i\phi} \sin sin\,(\theta')\ \right) = \left(cos(\theta')\ e^{\theta} e^{i\phi} \sin sin\,(\theta')\ \right)$$

If we want to rotate an axis

$$R_x(\theta) = \cos cos\left(\frac{\theta}{2}\right) I - i \sin sin\left(\frac{\theta}{2}\right) X = \left(\cos cos\left(\frac{\theta}{2}\right) \quad - i \sin sin\left(\frac{\theta}{2}\right) \quad - i \sin sin\left(\frac{\theta}{2}\right) \quad \cos cos\left(\frac{\theta}{2}\right)\right)$$

$$R_y(\theta) = \cos cos\left(\frac{\theta}{2}\right) I - i \sin sin\left(\frac{\theta}{2}\right) Y = \left(\cos cos\left(\frac{\theta}{2}\right) \quad - \sin sin\left(\frac{\theta}{2}\right) \quad \sin sin\left(\frac{\theta}{2}\right) \quad \cos cos\left(\frac{\theta}{2}\right)\right)$$

$$R_z(\theta) = \cos cos\left(\frac{\theta}{2}\right) I - i \sin sin\left(\frac{\theta}{2}\right) Z = \left(e^{-i\frac{\theta}{2}} \; 0 \; 0 \; e^{i\frac{\theta}{2}}\right)$$

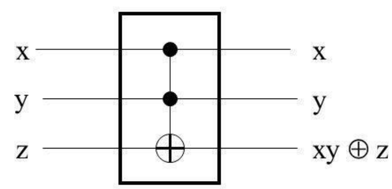
B. Exercise:

B.1 Draw the truth table for NAND gate and find it's matrix operator.

X	Y	X and Y	X nand Y
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

NOT = (0 1 1 0), AND = (1 1 0 0 1 0 0 1)
NAND = NOT * AND = (0 1 1 0)(1 1 0 0 1 0 0 1) = (0 0 1 1 0 1 1 0)
Check: 1 and 1 = 0
$|1\rangle\ nand\ |1\rangle = NAND|11\rangle = (0\ 0\ 1\ 1\ \ 0\ 1\ 1\ 0\ )(0\ 0\ \ 0\ 1\ ) = (1\ 0\ ) = |0\rangle$

B.2 Toffoli gate is defined as the showing picture. Find the matrix representation of Toffoli gate. Is it reversible?



Toffoli gate is:

$T = (1\ 0\ 0\ 1\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 1\ 0\ 0\ 1\ \ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ \ \ 1\ 0\ 0$

Notice that $T\ *\ T\ =\ I\ and\ T\ =\ T^{-1}$, Toffoli gate's reversed gate is Toffoli gate.
Yes, it's reversible.

B.3 Square root of one bit NOT gate. What is it? What is the reverse of it? What does the square of the square root not physically represent?

The square root of NOT gate is defined as:
$\sqrt{NOT} = \frac{1}{\sqrt{2}}(1\ \ -\ 1\ 1\ 1\ )$

To fine $\sqrt{NOT}^{-1}$, we can use the 2x2 inverse matrix formula
$(a\ b\ c\ d\ )^{-1} = \frac{1}{ad-bc}(d\ \ -\ b\ \ -\ c\ a\ )$

$\sqrt{NOT}^{-1} = \frac{1}{\sqrt{2}}(1\ 1\ \ -\ 1\ 1\ )$

Notice that $\sqrt{NOT} \neq \sqrt{NOT}^{-1}$

$\sqrt{NOT} * \sqrt{NOT} = (0\ \ -\ 1\ 1\ 0\ )$

$$\sqrt{NOT}^2|0\rangle = (0 \ - \ 1\ 1\ 0\ )(1\ 0\ ) = (0\ 1\ ) = |1\rangle$$
$$\sqrt{NOT}^2|1 = (0 \ - \ 1\ 1\ 0\ )(0\ 1\ ) = (-\ 1\ 0\ ) = (1\ 0\ ) = |0\rangle$$

Remember that $(-\ 1\ 0\ ) = (1\ 0\ )$ in state representation.

B.4 We prepare a state $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, we then use rotation operator to shift it. We shift it $R_x(\theta)$. What is the probability of finding the state in $|0\rangle$ afterward?

$$R_x(\theta)|\psi\rangle = \left(\cos cos\left(\frac{\theta}{2}\right) \ - i\sin sin\left(\frac{\theta}{2}\right) \ - i\sin sin\left(\frac{\theta}{2}\right) \ \cos cos\left(\frac{\theta}{2}\right)\ \right)\left(c_0\ c_1\right) = \left(c_0\cos cos\left(\frac{\theta}{2}\right) \ - c_1\right.$$

$$P(\ R_x(\theta)|\psi\rangle \to |0\rangle\ ) = \left|\langle R_x(\theta)\psi\rangle\right|^2 = \left|c_0\cos cos\left(\frac{\theta}{2}\right) \ - c_1 i\sin sin\left(\frac{\theta}{2}\right)\right|^2$$
$$= c_0^{\ 2}cos^2\left(\frac{\theta}{2}\right) + c_1^{\ 2}sin^2\left(\frac{\theta}{2}\right)$$

B.5 We now rotate the state by y-axis $R_y(\theta)$. What is the probability to find the state in $|0\rangle$ afterward?

$$R_y(\theta)|\psi\rangle = \left(\cos cos\left(\frac{\theta}{2}\right) \ - \sin sin\left(\frac{\theta}{2}\right) \ \sin sin\left(\frac{\theta}{2}\right) \ \cos cos\left(\frac{\theta}{2}\right)\ \right)\left(c_0\ c_1\right) = \left(c_0\cos cos\left(\frac{\theta}{2}\right) \ - c_1\sin si\right.$$

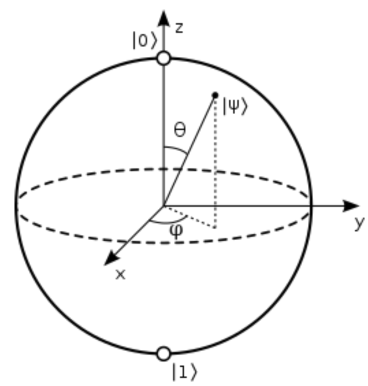
$$P(\ R_y(\theta)|\psi\rangle \to |0\rangle\ ) = \left|\langle R_y(\theta)\psi\rangle\right|^2 = \left|c_0\cos cos\left(\frac{\theta}{2}\right) \ - c_1\sin sin\left(\frac{\theta}{2}\right)\right|^2$$
$$= c_0^{\ 2}cos^2\left(\frac{\theta}{2}\right) + c_1^{\ 2}sin^2\left(\frac{\theta}{2}\right) - \ 2c_0c_1\cos cos\left(\frac{\theta}{2}\right)\sin sin\left(\frac{\theta}{2}\right)$$

B.6 We now rotate the state by y-axis $R_z(\theta)$. What is the probability of finding the state in $|0\rangle$ afterward?

$$R_z(\theta)|\psi\rangle = \left(e^{-i\frac{\theta}{2}}\ 0\ 0\ e^{i\frac{\theta}{2}}\right)\left(c_0\ c_1\right) = \left(c_0 e^{-i\frac{\theta}{2}}\ c_1 e^{i\frac{\theta}{2}}\right)$$

$$P(\ R_z(\theta)|\psi\rangle \rightarrow |0\rangle\ ) = \left|\langle R_z(\theta)\psi\rangle\right|^2 = \left|c_0 e^{-i\frac{\theta}{2}}\right|^2$$

$$= c_0^{\ 2}$$



Notice that rotating around z-axis does not affect the probability of the state. We can observe this property from the Bloch sphere.

B.7 Find the operator that can shift the state by rotating any arbitrary axis.

Define the axis as D.

$$R_D(\theta) = \cos cos\left(\frac{\theta}{2}\right) I\ -\ i \sin sin\left(\frac{\theta}{2}\right)\left(D_x X\ +\ D_y Y\ +\ D_z Z\right)$$

$$= \left(\cos cos\left(\frac{\theta}{2}\right)\ 0\ 0\ \cos cos\left(\frac{\theta}{2}\right)\ \right) + D_x\left(0\ \ -\ i \sin sin\left(\frac{\theta}{2}\right)\ \ -\ i \sin sin\left(\frac{\theta}{2}\right)\ 0\right)$$
$$+\ D_y\left(0\ \ -\ \sin sin\left(\frac{\theta}{2}\right)\ \ \sin sin\left(\frac{\theta}{2}\right)\ 0\right) + D_z\left(e^{-i\frac{\theta}{2}}\ -\ \cos cos\left(\frac{\theta}{2}\right)\ 0\ 0\ e^{i\frac{\theta}{2}}\ -\ \cos cos\left(\frac{\theta}{2}\right)\ \right)$$

C. Application:

Let's consider a problem. There is a function operated by an intelligent monkey. The input of the function is a state containing two bits of a two-level system $|x, y\rangle$. The intelligent monkey performs a secret algorithm and outputs a state containing two bits of a two-level system $|x', y'\rangle$. We can treat the secret algorithm as a function f. We do not know what the function f is, but we order the monkey to perform the function as many times as we want. Now, we are interested in if the function is constant or balanced.

Define the term: a function is balanced if the function is one-to-one. A function is constant if a binary input always denotes the same result (not one-to-one).

balanced: $f(0) = 0, f(1) = 1$

balanced: $f(0) = 1, f(1) = 0$

constant: $f(0) = 1, f(1) = 1,$

constant: $f(0) = 0, f(1) = 0,$

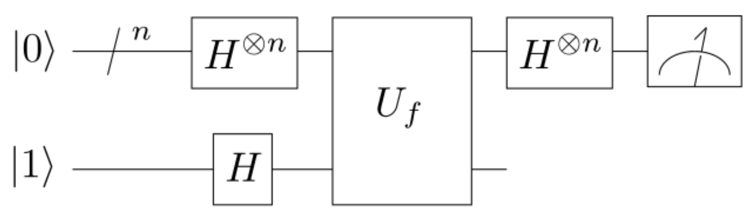
C.1 Find a classical way to find if the function f is balanced or constant.

We can input $|0, 0\rangle$ state into the function k times. Assuming k is large enough to decide if the function is balanced. If the function outputs the same result for k times, then the function is constant. If the function outputs at least one different result within these k of testing, then the function is balanced.

For a conventional deterministic algorithm, $k = 2^{n-1} + 1$ is needed to decide whether a function is balanced. So, in Computer Science, we say that the upper bound of the time complexity of this deterministic algorithm is $O(g) = 2^{n}$.

C.2 Use the Deutsch-Jozsa Quantum Algorithm to determine if the function f is balanced or constant.

This is Deutsch-Jozsa Algorithm.
U_f is the function f.
$H^{\otimes n}$ and H are both Hadamard operator.
n is for n bits. Here, n = 1.



$$H^{\otimes 1}|0\rangle = H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

The initial state is $|\psi_0\rangle = |0, 1\rangle = |0\rangle|1\rangle$

After performing two Hadamard operator to the state,

$$|\psi_1\rangle = (H|0\rangle)(H|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)$$

$$= \frac{1}{2}(+ |0,0\rangle - |0,1\rangle + |1,0\rangle - |1,1\rangle) = 00\ 01\ \ 10\ 11\ \ (+\ 1/2\ \ -\ 1/2\ \ +\ 1/2\ \ -\ 1/2\ )$$

Now, we multiply the state by f

$$|\psi_2\rangle = \left(\frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

If $f(0) = 0,\ f(1) = 0$, then $|\psi_2\rangle = \left(\frac{+|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

If $f(0) = 1,\ f(1) = 1$, then $|\psi_2\rangle = \left(\frac{-|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

If $f(0) = 0,\ f(1) = 1$, then $|\psi_2\rangle = \left(\frac{+|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

If $f(0) = 1,\ f(1) = 0$, then $|\psi_2\rangle = \left(\frac{-|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$

So, $|\psi_2\rangle = \pm\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$, if the function f is constant.

$|\psi_2\rangle = \pm\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$, if the function f is balanced.

Now, we apply the final Hadamard operator.

Remember that the inverse of Hadamard operator is itself. $H^{-1} = H$

$H\frac{|0\rangle + |1\rangle}{\sqrt{2}} = |0\rangle$, $H\frac{|0\rangle - |1\rangle}{\sqrt{2}} = |1\rangle$

Therefore, $|\psi_3\rangle = \pm|0\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$, if the function f is constant.

$|\psi_3\rangle = \pm|1\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$, if the function f is balanced.

As a result, by measuring the state of the top qubit, we can instantly find out if the function f is constant or balanced. With the technique of quantum computing, we don't need to repeat the experiment anymore. The time complexity is reduced to constant time $O(g) = c$.

Reference:

Yanofsky, Noson, and Mannucci, Mirco. "Chapter 5: Architecture and Chapter 6: Algorithm." *Quantum Computing for Computer Scientists,* Cambridge University Press.