Physics 457W Section 1

Chaotic Pendulum

Version 2

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Abstract

The purpose of this experiment is to study chaotic motion of double pendulum. The non-linear oscillator is built using point mass, drive motor, and damped magnet. The point mass makes the oscillation non-linear. The drive motor deliver drives force so the system can keep running with non-conservative force. The damped provides friction to the system. The equation used to describe the system is complicated, and it cannot be solved analytically. As a result, a phase plot and a Poincare plot are used to observe and confirm the chaotic system. In this experiment, a spring constant $k = 2.06 \pm 0.08 \frac{kg}{s^2}$ and a resonant frequency $\omega = 4.5 \pm 0.1\ rad/s$ were determined. The longest chaotic oscillations only lasted for 275 second. The phase plot and Poincare plot shows the system is chaotic, but it is not enough to prove the property of chaos. Unfortunately, a lot of data was lost due to software failure.

Introduction

Chaotic pendulum is a classical deterministic system. There is no randomness in this system. The motion of the oscillator can be predicted by differential equation with initial condition. This experiment observes a non-linear oscillation with drive and damped.

To study the chaotic system, base information of the system needs to be obtained first including spring constant k, natural frequency Ω_0, and resonance frequency ω. With the base information, a chaotic pendulum can be made. There is no analytical solution of the system. To analyze the system, phase plot and Poincare plot are used. Phase plot plots the angle and angular velocity of pendulum, and it can be used to distinguish between chaos and steady state. Poincare plot is also a plot in phase space, but the data is recorded base on driving period. The chaos data point does not repeat data point on Poincare plot. If chaos is achieved, and artistic looking graph can be obtained.

One important application of chaos in computer science is randomness. It has been addressed that chaos does not produce truly random information. The chaos motion is completely deterministic. However, it is nearly impossible to predict the chaotic system without knowing the differential equation. As a result, chaos system simulates randomness. Random numbers are very important for some computer algorithms. Nevertheless, a classical computer can never produce a true random number. The solution is to use a complicated equation that has no analytical solution. So, the output result of such equation can simulate random numbers.

Theoretical and background

Chaotic pendulum is a classical system, and there is no randomness in a classical system. Chaos is completely predictable and deterministic because the motion can be described by differential equations given an initial condition. However, the reason why the non-linear oscillation with damping and drive calls chaos is that the system is extremely sensitive to the initial condition. A slight difference in initial condition makes the pendulum travels in a different route. In contrast, a simple harmonic travels similar route if the initial condition is different. The system being studied here is a pendulum with point mass, drive, and damping.

The pendulum also connects to two spring. The force produces by a spring from equilibrium follow equation $F = -kx$. In the experiment, the spring acts on a disk with radius r, so the displacement can be written as $x = r\theta$. The torque from each spring is $\tau = r \times 2F = r \times -2kx = -2kr^2\theta$. The torque produces by gravity acting on point mass is $\tau = mgl\ sin(\theta)$ where point mass has distance l from the center of disk. The torque from drive does not directly acts on the disk but acts through a spring, so a spring constant k involves in the equation of driving torque $\tau = kr(Acos(\Omega_D t))$. There are two damping sources in this system. The first one is the damped cause by magnet that induces eddy current. The damp torque is positive correlate to the angular velocity $\tau = -b\omega$. The second damping source is the bearing at the center of the disk. The torque is given by $\tau = -b'(sign(\omega))$. The total torque is given by equation 1.

$$\tau_{total} = -b\omega - b'\big(sign(\omega)\big) - 2kr^2\theta + mgl\ sin(\theta) + kr\big(Acos(\Omega_D t)\big) \qquad (1)$$

The angular acceleration is denoted by $\tau = I\ \alpha$, so the angular acceleration of the system is given by equation 2.

$$\alpha = -\frac{b}{I}\omega - \frac{b'}{I}\big(sign(\omega)\big) - \frac{2kr^2}{I}\theta + \frac{mgl}{I}sin(\theta) + \frac{krA}{I}cos(\Omega_D t) \qquad (2)$$

Equation 2 can be rewrite as differential equation 3.

$$\frac{d^2\theta}{dt^2} = -\frac{b}{I}\frac{d\theta}{dt} - \frac{b'}{I}\left(sign\left(\frac{d\theta}{dt}\right)\right) - \frac{2kr^2}{I}\theta + \frac{mgl}{I}sin(\theta) + \frac{krA}{I}cos(\Omega_D t) \qquad (3)$$

The angle θ describe the position of the point mass on pendulum. The constants b and b' are the damped constant for damping magnet and friction on the bearing. The spring constant is k. The distance from disk center to point mass is l. The radius of the disk is r. A and Ω_D are the amplitude and drive frequency of driving motor. Finally, I is the total angular momentum of the system. A more detailed explanation and derivation is given in theoretical and background.

There is no analytical solution to this equation. However, in low amplitude, cosine and sine can be approximate using Tylor expansion. The system can act like simple harmonic at natural frequency Ω_0 if the amplitude is small.

$$\frac{d^2\theta'}{dt^2} = -\Omega_0^2\theta' - \frac{b}{I}\frac{d\theta'}{dt} + \frac{krA}{I}\cos\left(\Omega_D t\right)$$

If the amplitude is not small, phase plot and Poincare plot are the main tools to study the chaotic pendulum. Phase plot plots the angle versus the angular velocity of the pendulum. The phase plot of a steady system forms a circle; so, when the dots in the phase space does not form a circle, a chaotic system is found. The second tool is Poincare plot. Poincare plot is like phase plot, but only record one data point each driving period. The data points do not repeat the same position in phase space for a chaotic system.

Methods

The equipment of this experiment includes an pendulum stand, a disk mounted on a rotary sensor on the top of the pendulum stand, a point mass goes onto the disk, a motor that drive the pendulum, a photogate sensor, a rotary motion sensor, a universal interface that controls the motor and sensors, and a computer that runs software that manages the universal interface. A capstone software is used to manipulate driving motor and takes measurement from sensors. The setup details on pendulum stand is shown in the figure on the right. The pulley disk holds a string, and two identical springs are connected to each side of the string. The left spring connects driving motor by another string. The driving motor has an amplitude control bar. The control bar pass through a photogate each period, but the photogate is not shown on the figure.

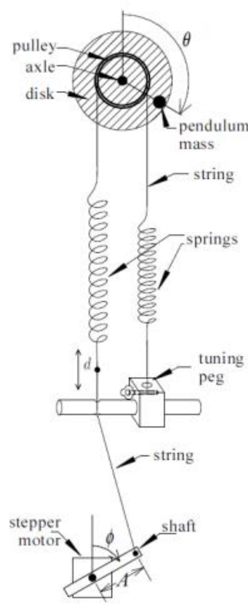

Figure 1 The double pendulum system for the experiment. It includes point mass mounted on a disk, springs, sensers, drive motor, and damped magnetic.

The first part of the experiment is to obtain the spring constant of the system, so a linear system is set. There is no driven force and point mass, and the magnetic drag is reduced to the lowest. A simple harmonic can be measured by giving an initial angular velocity to the disk. The second part of the experiment studies the kinetic energy and potential energy of the linear oscillation. The initial condition is for this part is the same as previous part. If the lowest potential energy of pendulum is defined as zero, the potential energy versus angle should follow equation $V(\theta) = kr^2\theta^2$. The third part of measures of experiment measures the resonance frequency with damping and driving force. The damping magnet is set to 5 mm from the disk, and the drive amplitude is set to about 1.5-2.5 cm.

The fourth part of the experiment is to find the equilibrium angle of disk with point mass. There is no driving or damping in this part. The goal is simply to measure the equilibrium angle of a non-linear oscillator while defining the angle with the highest potential as zero. The fifth part of this experiment has to do with the resonance frequency of non-linear oscillations. In other words, this part is the same as part three but with point mass attach to the disk. The purpose of part one to part five is to build up sufficient information for part six, chaos. In part six, the driving frequency should be set to the driving frequency that makes the resonance appear at part five. With minor modification on damped force, driving frequency, and driving frequency, a chaotic pendulum should occur where the point mass travel back and forth chaotically on both sides of disk.

Results

The data showing here does not cover the entire experiment. Due to software failure and limited attempts to perform the experiment, some data is lost, and some experiment does not produce valuable data. The spring constants measured by simple harmonic with no point mass, drag, and damp is denoted by equation:

$$\Omega_0 = \sqrt{\frac{2kr^2}{I}}$$

Because there is no point mass here, only the disk contributes to the angular momentum of the system. The angular momentum of a disk is given by equation:

$$I = \frac{1}{2}MR^2$$

The variables measured in this part of experiment is the following:

$$\Omega_0 = 4.470 \pm 0.0025\, rad/s$$

$$R = 4.75 \pm 0.05cm$$

$$2r = 5.1 \pm 0.1cm$$

$$M = 118.64 \pm 0.1g$$

$$I = \frac{1}{2}MR^2 = 1.338 \times 10^{-4} \pm 3 \times 10^{-7} kg\, m^2$$

$$k = \frac{\Omega_0^2\, I}{2\, r^2} = 2.06 \pm 0.08 \frac{kg}{s^2}$$

Although the second to fifth parts of experiment was performed, the data cannot be obtained due to software bugs. As a result, the few result can be shown here includes the resonance frequency of damped and driven oscillation without point mass and the equilibrium angles with point mass. The resonance frequency is $\omega = 4.5 \pm 0.1\, rad$. The left equilibrium angle is $\theta = -2.164\, rad$. The right equilibrium angle is $\theta = 1.658\, rad$.

The longest chaos we were able to observe in this experiment only last for 275 second. The phase plot and the Poincare plot are the following:

The figure on the right, the pendulum chaotically oscillates in both left and right. The right side of disk is the has a positive angle, and the left side of disk has a negative angle. Due to its chaotic nature, the phase plot dot covers two complete ovals on both positive angle and negative angle.

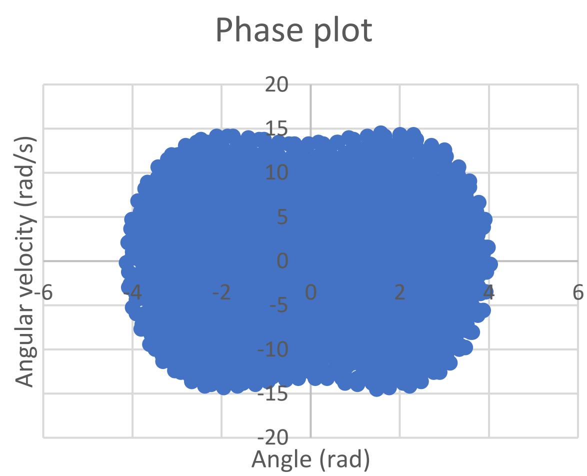

Figure 2 Phase plot graphs the pendulum's angle versus angular velocity. It shows the cycling behavior of the system. The dots cover the entire area which indicates the system is chaotic.

The figure below shows the Poincare plot of the chaotic system. Poincare plot is a phase plot but only record the data every driven period. The special part of Poincare plot is the dot never repeats its location on phase space if the system is chaotic. Because the chaos system only last for 275 second, an artistic looking Poincare plot was not obtained to further prove that the dot does not repeat its location in a longer period.

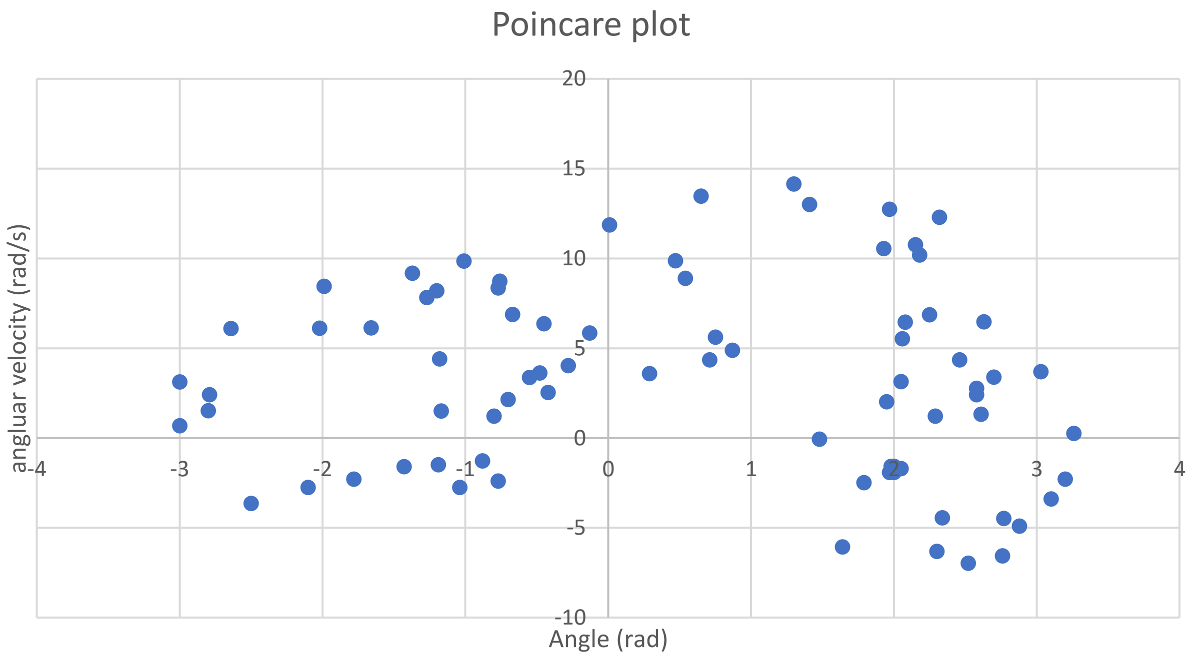

Figure 3 Poincare plot has the same axis as phase plot, but it's only dotted once in a drive period. The scattered distribution indicates the system is chaotic.

Error Analysis

To calculate the spring constant of the pendulum system, two equation are needed.

$$k = \frac{\Omega_0^2\, I}{2\, r^2}$$

$$I = \frac{1}{2} M R^2$$

Their uncertainty is derived from partial derivative.

$$\Delta I = \sqrt{\left(\frac{\partial I}{\partial M}\Delta M\right)^2 + \left(\frac{\partial I}{\partial R}\Delta R\right)^2} = \sqrt{\left(\frac{1}{2}R^2 \Delta M\right)^2 + (MR\Delta R)^2} = 3 \times 10^{-7} kg\ m^2$$

$$\Delta k = \sqrt{\left(\frac{\partial k}{\partial \Omega_0}\Delta\Omega_0\right)^2 + \left(\frac{\partial k}{\partial I}\Delta I\right)^2 + \left(\frac{\partial k}{\partial r}\Delta r\right)^2} = \sqrt{\left(\frac{\Omega_0\, I}{r^2}\Delta\Omega_0\right)^2 + \left(\frac{\Omega_0^2}{2\, r^2}\Delta I\right)^2 + \left(-\frac{\Omega_0^2\, I}{r^3}\Delta r\right)^2}$$

$$= 0.08 \frac{kg}{s^2}$$

Conclusion

In this experiment, a chaos system is built and observed. The chaotic pendulum has a disk that carries a point mass, and the point mass makes the system non-linear. The system also has a drive motor and a damped magnetic that produce drive torque and damped torque to the system. The chaotic pendulum is a non-linear oscillation with drive and damped modeled by equation $\frac{d^2\theta}{dt^2} = -\frac{b}{I}\frac{d\theta}{dt} - \frac{b'}{I}\left(sign\left(\frac{d\theta}{dt}\right)\right) - \frac{2kr^2}{I}\theta + \frac{mgl}{I}sin(\theta) + \frac{krA}{I}cos(\Omega_D t)$. The method starts by getting the basic information of the system. Unfortunately, a lot of data was not obtained due to software failure. The spring constant of the system was experimentally determined to be $k = 2.06 \pm 0.08 \frac{kg}{s^2}$. The resonance frequency was found to be $\omega = 4.5 \pm 0.1\ rad$. The longest chaos had been recorded only lasted for 275 second. The phase plot and Poincare plot shown in result hints the system was chaotic, but it is not enough to show the property of chaos by plotting the artistic looking Poincare plot.