Matrizes e Determinantes

0.1 Exercícios relativos a operações com matrizes

1. Considere a matriz identidade, $\mathbf{I}$, a matriz nula, $\mathbf{0}$, e as seguintes matrizes

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 1 & -2 \\ 0 & a \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 5 \\ 1 & 1 & 3 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 4 & 2 & 2 \\ -2 & 3 & 6 \\ 5 & 1 & -1 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 1 & 7 \\ 4 & 0 & 3 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ -3 & -1 & 1 & 0 \end{bmatrix}$$

(a) calcule, sempre que possível,

i. $2\mathbf{I} + 5\mathbf{J} - \mathbf{K}$ ii. $\mathbf{AQ}$ iii. $\mathbf{KLD} - \mathbf{F}$

iv. $(3\mathbf{I} - \mathbf{J})\mathbf{D}$ v. $\mathbf{G^0} + \mathbf{G^3}$

(b) encontre o valor de $a \in \mathbb{R}$ que satisfaz a igualdade $\mathbf{E}^2 = 3\mathbf{E} - 2\mathbf{I}$

Resolução.

(a) i.

$$\begin{aligned} 2\mathbf{I} + 5\mathbf{J} - \mathbf{K} &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 15 & 20 & 0 \\ -5 & 0 & 25 \\ 5 & 5 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 2 & 2 \\ -2 & 3 & 6 \\ 5 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 13 & 18 & -2 \\ -3 & -1 & 19 \\ 0 & 4 & 18 \end{bmatrix} \end{aligned}$$

$$\mathbf{AQ} = \begin{bmatrix} 2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & 2 & 0 & 0 \\ -3 & -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 13 & 7 & -3 & 0 \end{bmatrix}$$

iii.

$$\begin{aligned} \mathbf{KLD} - \mathbf{F} &= \begin{bmatrix} 22 & -6 & 24 \\ 21 & 7 & 37 \\ 12 & -9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 82 \\ 146 \\ 21 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 80 \\ 147 \\ 17 \end{bmatrix} \end{aligned}$$

iv.

$$
\begin{aligned}
(3\mathbf{I}-\mathbf{J})\mathbf{D} &= \left(3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 0 \\ -1 & 0 & 5 \\ 1 & 1 & 3 \end{bmatrix}\right) \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -4 & 0 \\ 1 & 3 & -5 \\ -1 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ -8 \\ -3 \end{bmatrix}
\end{aligned}
$$

v.

$$
\begin{aligned}
\mathbf{G^0}+\mathbf{G^3} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 25 & 26 \\ 39 & 38 \end{bmatrix} = \begin{bmatrix} 26 & 26 \\ 39 & 39 \end{bmatrix}
\end{aligned}
$$

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2. Considere a matriz identidade, $\mathbf{I}$, e a matriz nula, $\mathbf{0}$. Dê exemplos de matrizes reais do tipo 2×2 que satisfaça a igualdade $\mathbf{A}^2 = \mathbf{I}$

Resolução.

$$
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
\begin{cases} a^2+bc=1 \\ ab+bd=0 \\ ac+cd=0 \\ bc+d^2=1 \end{cases} \Leftrightarrow \begin{cases} bc=1-a^2 \\ b\,(a+d)=0 & \Leftrightarrow b=0 \vee a=-d \\ c\,(a+d)=0 & \Leftrightarrow c=0 \vee a=-d \\ d^2=1-bc \end{cases}
$$

Para $a=1$ e $b=3$, por exemplo, tem-se o sistema

$$
\begin{cases} 3c=0 \\ 3\,(1+d)=0 \\ c\,(1+d)=0 \\ d^2=1-3c \end{cases} \Leftrightarrow \begin{cases} c=0 \\ d=-1 \\ 0=0 \\ 1=1 \end{cases}.
$$

Logo, $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$

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3. Sabendo que $\mathbf{A}^k$ é o produto de uma matriz real quadrada $\mathbf{A}$ por si própria k vezes, por exemplo, $\mathbf{A}^2 = \mathbf{AA}$, deduza $\mathbf{A}^n$, para $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Resolução.

$$\mathbf{A^2} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \times 2 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A^3} = \mathbf{A^2 A} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \times 3 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A^4} = \mathbf{A^3 A} = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \times 4 \\ 0 & 1 \end{bmatrix}$$

Logo $\mathbf{A^n} = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$

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4. Verifique que duas quaisquer matrizes da forma $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ são permutáveis, onde a e b são dois números reais.

Resolução.

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \times \begin{bmatrix} x & y \\ -y & x \end{bmatrix} = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \times \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$\begin{bmatrix} (ax - by) & (ay + bx) \\ (-bx - ay) & (-by + ax) \end{bmatrix} = \begin{bmatrix} (ax - by) & (ay + bx) \\ (-bx - ay) & (-by + ax) \end{bmatrix}$$

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5. Considere $\mathbf{F} = \begin{bmatrix} 1 & x \\ y & 3 \end{bmatrix}$ e determine o valor dos escalares x e y, tais que $\mathbf{F}^2 + \mathbf{F} - \mathbf{I} = \begin{bmatrix} -1 & 5 \\ -10 & 9 \end{bmatrix}$.

Resolução.

$$\begin{bmatrix} 1 & x \\ y & 3 \end{bmatrix} \times \begin{bmatrix} 1 & x \\ y & 3 \end{bmatrix} + \begin{bmatrix} 1 & x \\ y & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ -10 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 + xy & 4x \\ 4y & xy + 9 \end{bmatrix} + \begin{bmatrix} 0 & x \\ y & 2 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ -10 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 + xy & 5x \\ 5y & xy + 11 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ -10 & 9 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} 1 + xy = -1 \\ 5x = 5 \\ 5y = -10 \\ xy + 11 = 9 \end{cases} \Leftrightarrow \begin{cases} xy = -2 \\ x = 1 \\ y = -2 \\ xy = -2 \end{cases} \Leftrightarrow x = 1 \wedge y = -2$$

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6. Considere $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ e encontre $f(\mathbf{A})$, sendo $f(x) = -3x^3 + 2x - 4$

Resolução.

$$\begin{aligned}
f(\mathbf{A}) &= -3\mathbf{A}^3 + 2\mathbf{A} - 4\mathbf{I} \\
&= -3\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= -3\begin{bmatrix} 0 & -4 \\ 4 & 8 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} \\
&= -3\begin{bmatrix} -4 & -12 \\ 12 & 20 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 12 & 36 \\ -36 & -60 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 34 \\ -34 & -58 \end{bmatrix}
\end{aligned}$$

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0.2 Exercícios relativos a operações sobre matrizes

1. Considere $\mathbf{G} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $\mathbf{P} = \begin{bmatrix} 5 & 0 \\ 1 & 1 \\ 0 & 4 \\ -1 & 0 \end{bmatrix}$ e $\mathbf{Q} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ -3 & -1 & 1 & 0 \end{bmatrix}$ e encontre a matriz $\mathbf{X}$ que satisfaz a igualdade $(\mathbf{QP})^T\,\mathbf{G}^2 = \mathbf{X} - 3\mathbf{I}$

   Resolução.

   $\mathbf{X} = (\mathbf{QP})^T\,\mathbf{G}^2 + 3\mathbf{I}$

$$\begin{aligned}
(\mathbf{QP})^T &= \left(\begin{bmatrix} 2 & 2 & 0 & 0 \\ -3 & -1 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 5 & 0 \\ 1 & 1 \\ 0 & 4 \\ -1 & 0 \end{bmatrix}\right)^T = \begin{bmatrix} 12 & 2 \\ -16 & 3 \end{bmatrix}^T \\
\mathbf{G}^2 &= \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix} \\
\mathbf{X} &= \begin{bmatrix} 12 & -16 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -60 & -88 \\ 41 & 42 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -57 & -88 \\ 41 & 45 \end{bmatrix}
\end{aligned}$$

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2. Indique, justificando, o valor lógico da seguinte afirmação

   "$(\mathbf{ABC})^T$ representa uma matriz real 3×1, onde $\mathbf{A}$ é uma matriz real 3×3, $\mathbf{B}$ é uma matriz real 3×2 e $\mathbf{C}$ uma matriz real 2×1".

   Resolução.

Proposição Falsa

Atendendo à definição de produto de matrizes e de matriz transposta, tem-se

$$[\mathbf{A_{3\times3}B_{3\times2}C_{2\times1}}]^T$$

$$\left[(\mathbf{AB})_{\mathbf{3\times2}}\,\mathbf{C_{2\times1}}\right]^T$$

$$\left[(\mathbf{ABC})_{\mathbf{3\times1}}\right]^T$$

$$[\mathbf{ABC}]^T_{1\times3}$$

Logo $(\mathbf{ABC})^T$ representa uma matriz real 1×3 e não uma matriz real 3×1.

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3. Dadas as matrizes $\mathbf{A}=\begin{bmatrix}2 & 1\\ -1 & 3\\ 0 & 4\end{bmatrix}$ e $\mathbf{B}=\begin{bmatrix}a & 1 & b\\ 1 & c & -8\end{bmatrix}$, determine os valores de a, b, $c\in\mathbb{R}$ de modo que a matriz $\mathbf{AB}$ seja simétrica.

Resolução.

$\mathbf{AB}$ é simétrica se e só se $\mathbf{AB}=(\mathbf{AB})^T$

$$\mathbf{AB}=\begin{bmatrix}2 & 1\\ -1 & 3\\ 0 & 4\end{bmatrix}\times\begin{bmatrix}a & 1 & b\\ 1 & c & -8\end{bmatrix}=\begin{bmatrix}2a+1 & 2+c & 2b-8\\ -a+3 & -1+3c & -b-24\\ 4 & 4c & -32\end{bmatrix}$$

$$(\mathbf{AB})^T=\begin{bmatrix}2a+1 & -a+3 & 4\\ 2+c & -1+3c & 4c\\ 2b-8 & -b-24 & -32\end{bmatrix}$$

$$\begin{cases}-a+3=2+c\\ 4=2b-8\\ 4c=-b-24\end{cases}\Leftrightarrow b=6\Leftrightarrow\begin{cases}---\\ ---\\ c=-\dfrac{30}{4}=-\dfrac{15}{2}\end{cases}\Leftrightarrow\begin{cases}a=1+\dfrac{15}{2}=\dfrac{17}{2}\\ ---\\ ---\end{cases}$$

$$\Leftrightarrow\begin{cases}a=\dfrac{17}{2}\\ b=6\\ c=-\dfrac{15}{2}\end{cases}$$

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4. Considere $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -2 & 0 \end{bmatrix}$ e $\mathbf{B} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$. Encontre as matrizes reais $\mathbf{X}$ e $\mathbf{Y}$ que satisfaz o sistema $\begin{cases} \mathbf{Y}^T + \mathbf{ABA} = \mathbf{0} \\ (\mathbf{BA} - 2\mathbf{I})\,\mathbf{X} - \mathbf{YA} = \mathbf{0} \end{cases}$

Resolução.

$$\begin{cases} \mathbf{Y}^T + \mathbf{ABA} = \mathbf{0} \\ (\mathbf{BA} - 2\mathbf{I})\,\mathbf{X} - \mathbf{YA} = \mathbf{0} \end{cases}$$

$$\begin{cases} \mathbf{Y}^T = -\mathbf{ABA} \\ (\mathbf{BA} - 2\mathbf{I})\,\mathbf{X} = \mathbf{YA} \end{cases}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -2 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 2 & 0 \\ 1 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -3 \\ -1 & 0 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 7 & 1 \end{bmatrix}$$

$$\begin{cases} \mathbf{Y}^T = -\begin{bmatrix} 0 & 2 & -3 \\ -1 & 0 & 3 \\ 2 & -4 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -2 & 0 \end{bmatrix} = -\begin{bmatrix} 6 & -2 \\ -7 & -1 \\ 2 & 6 \end{bmatrix} \\ \left(\begin{bmatrix} -1 & -3 \\ 7 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right) \mathbf{X} = \mathbf{Y} \times \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -2 & 0 \end{bmatrix} \end{cases}$$

$$\begin{cases} \mathbf{Y}^T = \begin{bmatrix} -6 & 2 \\ 7 & 1 \\ -2 & -6 \end{bmatrix} \\ \begin{bmatrix} -3 & -3 \\ 7 & -1 \end{bmatrix} \mathbf{X} = \mathbf{Y} \times \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -2 & 0 \end{bmatrix} \end{cases}$$

$$\begin{cases} \mathbf{Y} = \begin{bmatrix} -6 & 7 & -2 \\ 2 & 1 & -6 \end{bmatrix} \\ \begin{bmatrix} -3 & -3 \\ 7 & -1 \end{bmatrix} \mathbf{X} = \begin{bmatrix} -6 & 7 & -2 \\ 2 & 1 & -6 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ -2 & 0 \end{bmatrix} \end{cases}$$

$$\begin{cases} --- \\ \begin{bmatrix} -3 & -3 \\ 7 & -1 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & -13 \\ 14 & 1 \end{bmatrix} \end{cases}$$

$$\begin{cases} --- \\ \begin{bmatrix} -3a-3c & -3b-3d \\ 7a-c & 7b-d \end{bmatrix} = \begin{bmatrix} -2 & -13 \\ 14 & 1 \end{bmatrix} \end{cases}$$

$$\begin{cases} -3a-3c=-2 \\ -3b-3d=-13 \\ 7a-c=14 \\ 7b-d=1 \end{cases} \Leftrightarrow \begin{cases} -3a-21a+42=-2 \\ -3b-21b+3=-13 \\ c=7a-14 \\ d=7b-1 \end{cases}$$

$$\begin{cases} a=\frac{-44}{-24}=\frac{11}{6} \\ b=\frac{-16}{-24}=\frac{2}{3} \\ c=\frac{77}{6}-14 \\ d=\frac{14}{3}-1 \end{cases} \Leftrightarrow \begin{cases} a=\frac{11}{6} \\ b=\frac{2}{3} \\ c=-\frac{7}{6} \\ d=\frac{11}{3} \end{cases}$$

$$\begin{cases} \mathbf{Y} = \begin{bmatrix} -6 & 7 & -2 \\ 2 & 1 & -6 \end{bmatrix} \\ \mathbf{X} = \begin{bmatrix} \frac{11}{6} & \frac{2}{3} \\ -\frac{7}{6} & \frac{11}{3} \end{bmatrix} \end{cases}$$

■

5. Utilizando o método da condensação, determine a característica de cada uma das seguintes matrizes

(a) $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 4 \\ -2 & -4 & 2 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -3 & 2 & -4 & -1 \\ -1 & 0 & 0 & -1 \\ 2 & -4 & 0 & 2 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} -1 & -1 & 2 & 0 & -1 \\ 0 & 1 & -3 & 5 & 3 \\ 2 & 4 & 2 & -10 & -6 \end{bmatrix}$

Resolução.

(a) $$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 4 \\ -2 & -4 & 2 \end{bmatrix} \underset{\substack{L_2=L_2-L_1 \\ L_3=L_3+2L_1}}{\sim} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{car}(\mathbf{A})=2$$

(b)

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -3 & 2 & -4 & -1 \\ -1 & 0 & 0 & -1 \\ 2 & -4 & 0 & 2 \end{bmatrix} \underset{\substack{L_2=L_2+3L_1 \\ L_3=L_3+L_1 \\ L_4=L_4-2L_1}}{}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -2 & -4 & 2 \end{bmatrix} \underset{\substack{L_3=L_3-L_2 \\ L_4=L_4-2L_2}}{}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -8 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \operatorname{car}(\mathbf{A}) = 3$$

(c)

$$\mathbf{A} = \begin{bmatrix} -1 & -1 & 2 & 0 & -1 \\ 0 & 1 & -3 & 5 & 3 \\ 2 & 4 & 2 & -10 & -6 \end{bmatrix} \underset{L_3=L_3+2L_1}{}$$

$$\sim \begin{bmatrix} -1 & -1 & 2 & 0 & -1 \\ 0 & 1 & -3 & 5 & 3 \\ 0 & 2 & 6 & -10 & -8 \end{bmatrix} \underset{L_3=L_3-2L_2}{}$$

$$\sim \begin{bmatrix} -1 & -1 & 2 & 0 & -1 \\ 0 & 1 & -3 & 5 & 3 \\ 0 & 0 & 12 & -20 & -14 \end{bmatrix} \Rightarrow \operatorname{car}(\mathbf{A}) = 3$$

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6. Considere as seguintes matrizes
$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ e $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}$ e determine

a característica da matriz $\mathbf{M}$, tal que $\mathbf{M} = 2\mathbf{A} - \mathbf{B} + \mathbf{C}^T\mathbf{D}^T + 2\mathbf{I}$

Resolução.

$$\mathbf{C^T D^T} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 2 & 4 & 2 \\ -2 & 0 & 2 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -1 & 1 \end{bmatrix} + \cdots$$

$$\cdots + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 0 & 2 \\ 3 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 4 \\ 0 & 0 & 2 \\ 3 & 0 & 5 \end{bmatrix}_{C_1 \leftrightarrow C_2} \sim \begin{bmatrix} 2 & 4 & 4 \\ 0 & 0 & 2 \\ 0 & 3 & 5 \end{bmatrix}_{L_2 \leftrightarrow L_3} \sim \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$\Rightarrow \mathrm{car}(\mathbf{A}) = 3$

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7. Utilizando o método da condensação, discuta, em função dos parâmetros reais a e b, o valor da característica de cada uma das seguintes matrizes

(a) $\mathbf{A} = \begin{bmatrix} 3 & 2 & a \\ 3 & 1 & 1 \\ -6 & 2 & -1 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & a & 2 \\ 1 & 2 & b \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 1 & 1 & b & a \\ 1 & a & b & c \\ 1 & a & 1 & 1 \end{bmatrix}$

Resolução.

(a)

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & a \\ 3 & 1 & 1 \\ -6 & 2 & -1 \end{bmatrix}_{\substack{L_2 = L_2 - L_1 \\ L_3 = L_3 + 2L_1}}$$

$$\sim \begin{bmatrix} 3 & 2 & a \\ 0 & -1 & 1-a \\ 0 & 6 & (-1+2a) \end{bmatrix}_{L_3 = L_3 + 6L_2}$$

$$\sim \begin{bmatrix} 3 & 2 & a \\ 0 & -1 & 1-a \\ 0 & 0 & (-4a+5) \end{bmatrix}$$

$$\Rightarrow \operatorname{car}(\mathbf{A}) = \begin{cases} 3 & \Leftarrow -4a+5 \neq 0 \Leftrightarrow a \neq \dfrac{5}{4} \\ 2 & \Leftarrow -4a+5 = 0 \Leftrightarrow a = \dfrac{5}{4} \end{cases}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & a & 2 \\ 1 & 2 & b \end{bmatrix}_{\substack{L_2=L_2+L_1 \\ L_3=L_3-L_1}} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & (a+2) & 3 \\ 0 & 0 & (b-1) \end{bmatrix}$$

$$\Rightarrow \operatorname{car}(\mathbf{A}) = 3 \; sse \; a+2 \neq 0 \wedge \Leftrightarrow b-1 \neq 0$$

se $a = -2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & (b-1) \end{bmatrix}_{C_2 \leftrightarrow C_3} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & (b-1) & 0 \end{bmatrix}_{L_3=L_3+\frac{1-b}{3}L_2}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \operatorname{car}(\mathbf{A}) = 2$$

se $b = 1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & (a+2) & 3 \\ 0 & 0 & 0 \end{bmatrix}_{C_2 \leftrightarrow C_3} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & (a+2) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \operatorname{car}(\mathbf{A}) = 2$$

$$\text{Conclusão: } \operatorname{car}(\mathbf{A}) = \begin{cases} 3 & \Leftarrow a \neq -2 \wedge b \neq 1 \\ 2 & \Leftarrow a = -2 \vee b = 1 \end{cases}$$

(c)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & b & a \\ 1 & a & b & c \\ 1 & a & 1 & 1 \end{bmatrix}_{\substack{L_2=L_2-L_1 \\ L_3=L_3-L_2}} \sim \begin{bmatrix} 1 & 1 & b & a \\ 0 & (a-1) & 0 & (c-a) \\ 0 & 0 & (1-b) & (1-c) \end{bmatrix}$$

$$\Rightarrow \operatorname{car}(\mathbf{A}) = 3 \; sse \; (a-1 \neq 0) \wedge (1-b \neq 0)$$

$$\text{se } a = 1 \begin{bmatrix} 1 & 1 & b & 1 \\ 0 & 0 & 0 & (c-1) \\ 0 & 0 & (1-b) & (1-c) \end{bmatrix}_{C_2 \leftrightarrow C_4} \sim \begin{bmatrix} 1 & 1 & b & 1 \\ 0 & (c-1) & 0 & 0 \\ 0 & (1-c) & (1-b) & 0 \end{bmatrix}_{L_3=L_3+L_2}$$

$$\sim \begin{bmatrix} 1 & 1 & b & 1 \\ 0 & (c-1) & 0 & 0 \\ 0 & 0 & (1-b) & 0 \end{bmatrix} \Rightarrow \operatorname{car}(\mathbf{A}) = 3 sse \, (c-1 \neq 0) \wedge (1-b \neq 0) \wedge$$

$(a=1)$

se $c=1\wedge (a=1) \begin{bmatrix} 1 & 1 & b & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (1-b) & 0 \end{bmatrix}_{L_3 \leftrightarrow L_2} \sim \begin{bmatrix} 1 & 1 & b & 1 \\ 0 & 0 & (1-b) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{C_3 \leftrightarrow C_2}$

$\sim \begin{bmatrix} 1 & b & 1 & 1 \\ 0 & (1-b) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \operatorname{car}(\mathbf{A}) = 2 \; sse \; (1-b \neq 0)$ e $\operatorname{car}(\mathbf{A}) = 1 \; sse \; (1-b=0)$

se $b=1\wedge (a=1) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & (c-1) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \operatorname{car}(\mathbf{A}) = 2 \; sse \; (c-1 \neq 0)$ e $\operatorname{car}(\mathbf{A}) = 1 \; sse \; (c-1=0)$

se $b=1 \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & (a-1) & 0 & (c-a) \\ 0 & 0 & 0 & (1-c) \end{bmatrix}_{C_2 \leftrightarrow C_3} \sim \begin{bmatrix} 1 & 1 & a & 1 \\ 0 & (a-1) & (c-a) & 0 \\ 0 & 0 & (1-c) & 0 \end{bmatrix}$

$\Rightarrow \operatorname{car}(\mathbf{A}) = 3 \; sse \; (a-1 \neq 0) \wedge (1-c \neq 0) \wedge (b=1)$

se $a=1\wedge (b=1)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & (c-1) & 0 \\ 0 & 0 & (1-c) & 0 \end{bmatrix}_{C_3 \leftrightarrow C_2} \sim \begin{bmatrix} 1 & 1 & 1 & b \\ 0 & (c-1) & 0 & 0 \\ 0 & (1-c) & 0 & 0 \end{bmatrix}_{L_3 = L_3 + L_2} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & (c-1) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \operatorname{car}(\mathbf{A}) = 2 \; sse \; (c-1 \neq 0) \wedge \operatorname{car}(\mathbf{A}) = 1 \; sse \; (c-1=0)$

se $c=1\wedge (b=1)$

$$\begin{bmatrix} 1 & 1 & a & b \\ 0 & (a-1) & (1-a) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \operatorname{car}(\mathbf{A}) = 2 \; sse \; (a-1 \neq 0) \wedge \operatorname{car}(\mathbf{A}) = 1 \; sse \; (a-1=0)$

$$\text{Conclusão: } \operatorname{car}(\mathbf{A}) = \begin{cases} 3 & \Leftarrow a \neq 1 \wedge b \neq 1, \forall c \\ 3 & \Leftarrow a \neq 1 \wedge c \neq 1, \forall b \\ 3 & \Leftarrow a = 1 \wedge b \neq 1 \wedge c \neq 1 \\ 2 & \Leftarrow a = 1 \wedge b \neq 1 \wedge c = 1 \\ 2 & \Leftarrow a = 1 \wedge b = 1 \wedge c \neq 1 \\ 2 & \Leftarrow a \neq 1 \wedge b = 1 \wedge c = 1 \\ 1 & \Leftarrow a = 1 \wedge b = 1 \wedge c = 1 \end{cases}$$

$\blacksquare$

0.3 Exercícios relativos a determinantes de uma matriz quadrada

1. Calcule

a) $\det(\mathbf{A}) = \begin{vmatrix} -2 & -1 \\ -2 & 3 \end{vmatrix}$ b) $\det(\mathbf{A}) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 1 & 7 \\ 3 & 2 & -4 \end{vmatrix}$

c) $\det(\mathbf{A}) = \begin{vmatrix} 2 & -1 & 0 & 1 & 3 \\ 3 & 0 & 1 & 5 & 3 \\ 6 & 2 & -1 & -5 & 2 \\ 2 & -1 & 0 & 1 & -3 \\ 1 & 0 & 1 & -1 & 1 \end{vmatrix}$ d) $\det(\mathbf{A}) = \begin{vmatrix} 1-i & -1 & 2i \\ i-1 & 2+2i & 1+i \\ -2 & -i & 1 \end{vmatrix}$

Resolução.

(a) $\begin{vmatrix} -2 & -1 \\ -2 & 3 \end{vmatrix}_{L_2=L_2-L_1} = \begin{vmatrix} -2 & -1 \\ 0 & 4 \end{vmatrix} = (-2)\times 4 = -8$

(b)

$$\begin{vmatrix} 1 & 2 & -1 \\ -2 & 1 & 7 \\ 3 & 2 & -4 \end{vmatrix}_{\substack{L_2=L_2+2L_1 \\ L_3=L_3-3L_1}} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 5 & 5 \\ 0 & -4 & -1 \end{vmatrix}_{L_3=L_3+\frac{4}{5}L_2}$$

$$= \begin{vmatrix} 1 & 2 & -1 \\ 0 & 5 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1\times 5\times 3 = 15$$

(c)

$$\begin{vmatrix} 2 & -1 & 0 & 1 & 3 \\ 3 & 0 & 1 & 5 & 3 \\ 6 & 2 & -1 & -5 & 2 \\ 2 & -1 & 0 & 1 & -3 \\ 1 & 0 & 1 & -1 & 1 \end{vmatrix}_{L_1\leftrightarrow L_5}$$

$$= -\begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 3 & 0 & 1 & 5 & 3 \\ 6 & 2 & -1 & -5 & 2 \\ 2 & -1 & 0 & 1 & -3 \\ 2 & -1 & 0 & 1 & 3 \end{vmatrix}_{\substack{L_2=L_2-3L_1 \\ L_3=L_3-6L_1 \\ L_4=L_4-2L_1 \\ L_5=L_5-2L_1}}$$

$$= -\begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 8 & 0 \\ 0 & 2 & -7 & 1 & -4 \\ 0 & -1 & -2 & 3 & -5 \\ 0 & -1 & -2 & 3 & 1 \end{vmatrix}_{L_2\leftrightarrow L_5}$$

$$= \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & -1 & -2 & 3 & 1 \\ 0 & 2 & -7 & 1 & -4 \\ 0 & -1 & -2 & 3 & -5 \\ 0 & 0 & -2 & 8 & 0 \end{vmatrix}_{\substack{L_3=L_3+2L_2 \\ L_4=L_4-L_2}}$$

$$= \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & -1 & -2 & 3 & 1 \\ 0 & 0 & -11 & 7 & -2 \\ 0 & 0 & 0 & 0 & -6 \\ 0 & 0 & -2 & 8 & 0 \end{vmatrix}_{C_3 \leftrightarrow C_5}$$

$$= - \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 3 & -2 \\ 0 & 0 & -2 & 7 & -11 \\ 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 8 & -2 \end{vmatrix}_{L_4=L_4-3L_3}$$

$$= - \begin{vmatrix} 1 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 3 & -2 \\ 0 & 0 & -2 & 7 & -11 \\ 0 & 0 & 0 & -21 & 33 \\ 0 & 0 & 0 & 8 & -2 \end{vmatrix}_{C_4=C_4+4C_5}$$

$$= - \begin{vmatrix} 1 & 0 & 1 & 3 & 1 \\ 0 & -1 & 1 & -5 & -2 \\ 0 & 0 & -2 & -37 & -11 \\ 0 & 0 & 0 & 111 & 33 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix} = -1 \times (-1) \times (-2) \times 111 \times (-2) = 444$$

(d)

$$\begin{vmatrix} (1-i) & -1 & 2i \\ (i-1) & (2+2i) & (1+i) \\ -2 & -i & 1 \end{vmatrix}_{det(A)=det(A^T)}$$

$$= \begin{vmatrix} (1-i) & (i-1) & -2 \\ -1 & (2+2i) & -i \\ 2i & (1+i) & 1 \end{vmatrix}_{C_1 \leftrightarrow C_3} = - \begin{vmatrix} -2 & (i-1) & (1-i) \\ -i & (2+2i) & -1 \\ 1 & (1+i) & 2i \end{vmatrix}_{L_1 \leftrightarrow L_3}$$

$$= \begin{vmatrix} 1 & (1+i) & 2i \\ -i & (2+2i) & -1 \\ -2 & (i-1) & (1-i) \end{vmatrix}_{\substack{L_2=L_2+iL_1 \\ L_3=L_3+2L_1}}$$

$$= \begin{vmatrix} 1 & (1+i) & 2i \\ 0 & (1+3i) & -3 \\ 0 & (3i+1) & (1+3i) \end{vmatrix}_{L_3=L_3-L_2} = \begin{vmatrix} 1 & (1+i) & 2i \\ 0 & (1+3i) & -3 \\ 0 & 0 & (4+3i) \end{vmatrix}$$

$$= 1 \times (1+3i) \times (4+3i) = 4+3i+12i-9 = -5+15i$$

$\blacksquare$

2. Aplicando o Teorema de Laplace à fila mais conveniente, calcule $\det(\mathbf{A}) = \begin{vmatrix} 4 & 0 & 2 \\ 1 & -1 & -2 \\ 1 & 0 & -1 \end{vmatrix}$

   Resolução.

   $\det(\mathbf{A}) = \begin{vmatrix} 4 & 0 & 2 \\ 1 & -1 & -2 \\ 1 & 0 & -1 \end{vmatrix}$

   Aplicando o T.L. à 2ª coluna, vem:

   $$= 0 + (-1)A_{22} + 0 = (-1)\,(-1)^{2+2} \underbrace{\begin{vmatrix} 4 & 2 \\ 1 & -1 \end{vmatrix}}_{(-4)-(2)=-6} = (-1)\,(-6) = 6$$

$\blacksquare$

3. Considere $\det(\mathbf{A}) = \begin{vmatrix} 1 & -1 & 2 & 1 \\ 2 & 4 & 3 & 4 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$ e indique um determinante $\det(\mathbf{B})$, de **5ª** ordem e sem elementos nulos, cujo valor seja simétrico a $\det(\mathbf{A})$

   Resolução.

   $$\det(\mathbf{B}) = -\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 2 & 4 & 3 & 4 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}_{\substack{L_2=L_2+L_1 \\ L_3=L_3+L_1 \\ L_4=L_4+L_1 \\ L_5=L_5+L_1}}$$

   $$= -\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & 1 \\ 1 & 2 & 4 & 3 & 4 \\ 1 & 1 & -1 & 0 & -1 \\ 1 & 0 & 0 & 1 & 1 \end{vmatrix}_{\substack{L_1=L_1+L_2 \\ L_4=L_4+3L_2 \\ L_5=L_5+L_2}}$$

   $$= \begin{vmatrix} -2 & -1 & 1 & -2 & -1 \\ 1 & 1 & -1 & 2 & 1 \\ 1 & 2 & 4 & 3 & 4 \\ 4 & 4 & -4 & 6 & 2 \\ 2 & 1 & -1 & 3 & 2 \end{vmatrix}$$

$\blacksquare$

4. Sendo a, b, c e d valores reais e aplicando as propriedades dos determinantes, calcule

(a) $\det(\mathbf{A}) = \begin{vmatrix} a & -1 & 0 \\ 2a & (b-2a) & -a \\ -a^2 & b & (b-a) \end{vmatrix}$

(b) $\det(\mathbf{A}) = \begin{vmatrix} 2b & b & b \\ (a+b) & a & b \\ (a+c) & a & b \end{vmatrix}$

(c) $\det(\mathbf{A}) = \begin{vmatrix} a & b & a \\ a & a & b \\ b & a & a \end{vmatrix}$

(d) $\det(\mathbf{A}) = \begin{vmatrix} \left(a^2+a\right) & 2a & 2 \\ a & a^2 & 1 \\ a^3 & -a & -1 \end{vmatrix}$

(e) $\det(\mathbf{A}) = \begin{vmatrix} a & (a+b) & c \\ b & a & (a+c) \\ 2a & b & c \end{vmatrix}$

Resolução.

(a) $$\begin{vmatrix} a & -1 & 0 \\ 2a & (b-2a) & -a \\ -a^2 & b & (b-a) \end{vmatrix}_{\substack{L_2=L_2-2L_1 \\ L_3=L_3+aL_1}}$$

$$= \begin{vmatrix} a & -1 & 0 \\ 0 & (b-2a+2) & -a \\ 0 & (b-a) & (b-a) \end{vmatrix}_{C_2=C_2-C_3}$$

$$= \begin{vmatrix} a & -1 & 0 \\ 0 & (b-a+2) & -a \\ 0 & 0 & (b-a) \end{vmatrix} = a\,(b-a+2)\,(b-a)$$

(b)

$$\begin{vmatrix} 2b & b & b \\ (a+b) & a & b \\ (a+c) & a & b \end{vmatrix}_{C_1 \leftrightarrow C_3}$$

$$= -\begin{vmatrix} b & b & 2b \\ b & a & (a+b) \\ b & a & (a+c) \end{vmatrix}_{\substack{L_2=L_2-L_1 \\ L_3=L_3-L_1}}$$

$$= -\begin{vmatrix} b & b & 2b \\ 0 & (a-b) & (a-b) \\ 0 & (a-b) & (a+c-2b) \end{vmatrix}_{L_3=L_3-L_2}$$

$$= -\begin{vmatrix} b & b & 2b \\ 0 & (a-b) & (a-b) \\ 0 & 0 & (c-b) \end{vmatrix} = -b\,(a-b)\,(c-b)$$

(c)

$$\begin{vmatrix} a & b & a \\ a & a & b \\ b & a & a \end{vmatrix}_{C_1=C_1+C_2+C_3}$$

$$= \begin{vmatrix} (2a+b) & b & a \\ (2a+b) & a & b \\ (2a+b) & a & a \end{vmatrix}_{\substack{L_2=L_2-L_1 \\ L_3=L_3-L_2}}$$

$$= \begin{vmatrix} (2a+b) & b & a \\ 0 & (a-b) & (b-a) \\ 0 & 0 & (a-b) \end{vmatrix} = (2a+b)(a-b)^2$$

(d)

$$\begin{vmatrix} \left(a^2+a\right) & 2a & 2 \\ a & a^2 & 1 \\ a^3 & -a & -1 \end{vmatrix}_{C_1 \leftrightarrow C_3}$$

$$= -\begin{vmatrix} 2 & 2a & \left(a^2+a\right) \\ 1 & a^2 & a \\ -1 & -a & a^3 \end{vmatrix}_{L_1 \leftrightarrow L_3}$$

$$= \begin{vmatrix} -1 & -a & a^3 \\ 1 & a^2 & a \\ 2 & 2a & \left(a^2+a\right) \end{vmatrix} = a \times a \begin{vmatrix} -1 & -1 & a^2 \\ 1 & a & 1 \\ 2 & 2 & (a+1) \end{vmatrix}_{\substack{L_2=L_2+L_1 \\ L_3=L_3+2L_1}}$$

$$= a^2 \begin{vmatrix} -1 & -1 & a^2 \\ 0 & (a-1) & \left(1+a^2\right) \\ 0 & 0 & \left(2a^2+a+1\right) \end{vmatrix} = -a^2 (a-1)\left(2a^2+a+1\right)$$

(e)

$$\begin{vmatrix} a & (a+b) & c \\ b & a & (a+c) \\ 2a & b & c \end{vmatrix}_{C_1=C_1+C_2+C_3}$$

$$= \begin{vmatrix} (2a+b+c) & (a+b) & c \\ (2a+b+c) & a & (a+c) \\ (2a+b+c) & b & c \end{vmatrix}_{\substack{L_2=L_2-L_1 \\ L_3=L_3-L_1}}$$

$$= \begin{vmatrix} (2a+b+c) & (a+b) & c \\ 0 & -b & a \\ 0 & -a & 0 \end{vmatrix}_{C_2 \leftrightarrow C_3}$$

$$= -\begin{vmatrix} (2a+b+c) & c & (a+b) \\ 0 & a & -b \\ 0 & 0 & -a \end{vmatrix} = a^2 (2a+b+c)$$

■

5. Sendo a, b, c e d valores reais e utilizando as propriedades dos determinantes, mostre que

(a) $\det(\mathbf{A}) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$

(b) $\det(\mathbf{A}) = \begin{vmatrix} (1-a) & (a-1) & -a \\ (1-b) & (b-1) & -b \\ -c & (c+1) & (c-1) \end{vmatrix} = b-a$

(c) $\det(\mathbf{A}) = \begin{vmatrix} 0 & 1 & a & 0 \\ 1 & 0 & 1 & -b \\ a & 1 & 0 & 1 \\ 0 & -b & 1 & 0 \end{vmatrix} = (1+ab)^2$

Resolução.

(a)

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \underset{\substack{L_2=L_2-aL_1 \\ L_3=L_3-a^2L_1}}{}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & (b-a) & (c-a) \\ 0 & (b^2-a^2) & (c^2-a^2) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & (b-a) & (c-a) \\ 0 & (b-a)(b+a) & (c-a)(c+a) \end{vmatrix} \underset{L_3=L_3-(b+a)L_2}{}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & (b-a) & (c-a) \\ 0 & 0 & (c-a)(c+a)-(b+a)(c-a) \end{vmatrix}$$

$$= (b-a)\left[(c-a)(c+a)-(b+a)(c-a)\right]$$

$$= (b-a)(c-a)(c+a-b-a) = (b-a)(c-a)(c-b)$$

(b)

$$\begin{vmatrix} (1-a) & (a-1) & -a \\ (1-b) & (b-1) & -b \\ -c & (c+1) & (c-1) \end{vmatrix}_{\substack{C_1=C_1-C_3 \\ C_2=C_2+C_3}}$$

$$= \begin{vmatrix} 1 & -1 & -a \\ 1 & -1 & -b \\ (1-2c) & 2c & (c-1) \end{vmatrix}_{C_1=C_1+C_2}$$

$$= \begin{vmatrix} 0 & -1 & -a \\ 0 & -1 & -b \\ 1 & 2c & (c-1) \end{vmatrix}_{L_1 \leftrightarrow L_3}$$

$$= - \begin{vmatrix} 1 & 2c & (c-1) \\ 0 & -1 & -b \\ 0 & -1 & -a \end{vmatrix}_{L_3=L_3-L_2}$$

$$= - \begin{vmatrix} 1 & 2c & (c-1) \\ 0 & -1 & -b \\ 0 & 0 & (-a+b) \end{vmatrix} = -a+b$$

(c)

$$\begin{vmatrix} 0 & 1 & a & 0 \\ 1 & 0 & 1 & -b \\ a & 1 & 0 & 1 \\ 0 & -b & 1 & 0 \end{vmatrix}_{C_1 \leftrightarrow C_2}$$

$$= - \begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 1 & -b \\ 1 & a & 0 & 1 \\ -b & 0 & 1 & 0 \end{vmatrix}_{\substack{L_3=L_3-L_1 \\ L_4=L_4+bL_1}}$$

$$= - \begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 1 & -b \\ 0 & a & -a & 1 \\ 0 & 0 & (1+ab) & 0 \end{vmatrix}_{L_3=L_3-aL_2}$$

$$= - \begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 1 & -b \\ 0 & 0 & -2a & (1+ab) \\ 0 & 0 & (1+ab) & 0 \end{vmatrix}_{C_3 \leftrightarrow C_4}$$

$$= \begin{vmatrix} 1 & 0 & 0 & a \\ 0 & 1 & -b & 1 \\ 0 & 0 & (1+ab) & -2a \\ 0 & 0 & 0 & (1+ab) \end{vmatrix} = (1+ab)^2$$

■

6. Utilizando propriedades dos determinantes, resolva a seguints equações em $\mathbb{R}$ $\begin{vmatrix} x & -1 & 1 \\ 0 & -x & -1 \\ 2 & 1 & x \end{vmatrix} = 0$

Resolução.

$$\begin{vmatrix} x & -1 & 1 \\ 0 & -x & -1 \\ 2 & 1 & x \end{vmatrix}_{C_1 \leftrightarrow C_2} = 0 \Leftrightarrow -\begin{vmatrix} -1 & x & 1 \\ -x & 0 & -1 \\ 1 & 2 & x \end{vmatrix}_{\substack{L_2 = L_2 - xL_1 \\ L_3 = L_3 + L_1}} = 0 \Leftrightarrow$$

$$\Leftrightarrow -\begin{vmatrix} -1 & x & 1 \\ 0 & -x^2 & (-1-x) \\ 0 & (2+x) & (x+1) \end{vmatrix}_{L_3 = L_3 + L_2} = 0$$

$$\Leftrightarrow -\begin{vmatrix} -1 & x & 1 \\ 0 & -x^2 & (-1-x) \\ 0 & \left(2+x-x^2\right) & 0 \end{vmatrix}_{C_2 \leftrightarrow C_3} = 0$$

$$\Leftrightarrow \begin{vmatrix} -1 & 1 & x \\ 0 & (-1-x) & -x^2 \\ 0 & 0 & \left(2+x-x^2\right) \end{vmatrix} = 0$$

$$\Leftrightarrow (1+x)\left(2+x-x^2\right) = 0$$

$$\Leftrightarrow (1+x) = 0 \lor \left(2+x-x^2\right) = 0$$

$$\Leftrightarrow x = -1 \lor x = \frac{-1 \pm \sqrt{1+8}}{-2} = \frac{-1 \pm 3}{-2} \Leftrightarrow \underbrace{x = -1}_{Raíz\ dupla} \lor x = 2$$

■

7. Sendo a, b e c valores reais e considerando $\det(\mathbf{A}) = \begin{vmatrix} a & b & c \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1$, utilize as propriedades dos determinantes para obter o valor de $\det(\mathbf{B}) = \begin{vmatrix} a & 6 & (1+2a) \\ b & 0 & (1+2b) \\ c & 3 & (1+2c) \end{vmatrix}$, sem o calcular

Resolução.

$$\begin{vmatrix} a & 6 & (1+2a) \\ b & 0 & (1+2b) \\ c & 3 & (1+2c) \end{vmatrix}_{C_3 = C_3 - 2C_1} = \begin{vmatrix} a & 6 & 1 \\ b & 0 & 1 \\ c & 3 & 1 \end{vmatrix}_{\det(\mathbf{A}) = \det\left(\mathbf{A}^t\right)} =$$

$$= \begin{vmatrix} a & b & c \\ 6 & 0 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 3\begin{vmatrix} a & b & c \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3\det(\mathbf{A}) = 3$$

■

8. Sendo a, b, c e d valores reais e considerando $\det(\mathbf{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$,

prove a igualdade $\begin{vmatrix} 0 & a & d & b \\ 0 & c & d & d \\ b & b & a & b \\ b & b & b & b \end{vmatrix} = b(a-b)$

Resolução.

$$\begin{vmatrix} 0 & a & d & b \\ 0 & c & d & d \\ b & b & a & b \\ b & b & b & b \end{vmatrix}_{\substack{L_1 \leftrightarrow L_4 \\ L_2 \leftrightarrow L_3}} = \begin{vmatrix} b & b & b & b \\ b & b & a & b \\ 0 & c & d & d \\ 0 & a & d & b \end{vmatrix}_{L_2 = L_2 - L_1}$$

$$= \begin{vmatrix} b & b & b & b \\ 0 & 0 & (a-b) & 0 \\ 0 & c & d & d \\ 0 & a & d & b \end{vmatrix}_{T.L.\text{à}1^{\text{a}}\ \text{coluna}} = b(-1)^{1+1} \begin{vmatrix} 0 & (a-b) & 0 \\ c & d & d \\ a & d & b \end{vmatrix}_{T.L.\text{à}1^{\text{a}}\ \text{linha}}$$

$$= b(a-b)(-1)^{1+2} \begin{vmatrix} c & d \\ a & b \end{vmatrix}_{L_1 \leftrightarrow L_2} = +b(a-b) \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= b(a-b)\det(\mathbf{A}) = b(a-b)$$

■

9. Sabendo que $\det(\mathbf{A}_{4\text{x}4}) = 3$, determine $\det(2\mathbf{A})$

Resolução.

$\det(2\mathbf{A}) = 2^4 \det(\mathbf{A}) = 16 \times 3 = 48$

■

0.4 Exercícios que envolvem o cálculo da matriz inversa

1. Encontre todos os valores reais de a e b para os quais a matriz $\mathbf{B} = \begin{bmatrix} a & 1 & ab \\ 1 & a & b \\ -b & b & ab^2 \end{bmatrix}$ é invertível

Resolução.

$$\det(\mathbf{B}) = \begin{vmatrix} a & 1 & ab \\ 1 & a & b \\ -b & b & ab^2 \end{vmatrix}_{\mathrm{L_1=L_1-aL_2}} = \begin{vmatrix} 0 & \left(1-a^2\right) & 0 \\ 1 & a & b \\ -b & b & ab^2 \end{vmatrix}_{\text{T.L.à 1ª linha}}$$

$$= \left(1-\mathrm{a}^2\right)(-1)^{1+2} \begin{vmatrix} 1 & b \\ -b & ab^2 \end{vmatrix}$$

$$= \left(\mathrm{a}^2-1\right)\left(\mathrm{ab}^2+\mathrm{b}^2\right) = (a-1)(a+1)\,b^2\,(a+1)$$

Logo, $(a-1)(a+1)^2\,b^2 \neq 0 \Leftrightarrow a \neq 1 \wedge a \neq -1 \wedge b \neq 0$

■

2. Diga se as seguintes matrizes são singulares ou regulares e calcule, quando possível, a sua inversa pelo método da condensação

   (a) $\mathbf{C} = \begin{bmatrix} 2 & 1 & 12 \\ 1 & 0 & 3 \\ 3 & -1 & 4 \end{bmatrix}$

   (b) $\mathbf{G} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 2 \\ -1 & 3 & -9 \end{bmatrix}$

   Resolução.

   (a)

   $\det(\mathbf{C}) = -1 \neq 0$, logo a matriz é regular

$$[\,\mathbf{C}\,|\,\mathbf{I}\,] = \left[\begin{array}{ccc|ccc} 2 & 1 & 12 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & -1 & 4 & 0 & 0 & 1 \end{array}\right]_{L_1 \leftrightarrow L_2}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 1 & 12 & 1 & 0 & 0 \\ 3 & -1 & 4 & 0 & 0 & 1 \end{array}\right]_{\substack{L_2=L_2-2L_1 \\ L_3=L_3-3L_1}}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 6 & 1 & -2 & 0 \\ 0 & -1 & -5 & 0 & -3 & 1 \end{array}\right]_{L_3=L_3+L_2}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 6 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & -5 & 1 \end{array}\right]_{\substack{L_1=L_1-3L_3 \\ L_2=L_2-6L_3}}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 16 & -3 \\ 0 & 1 & 0 & -5 & 28 & -6 \\ 0 & 0 & 1 & 1 & -5 & 1 \end{array}\right]$$

$$\therefore \qquad \mathbf{C}^{-1} = \begin{bmatrix} -3 & 16 & -3 \\ -5 & 28 & -6 \\ 1 & -5 & 1 \end{bmatrix}$$

(b)

$$\det(\mathbf{G}) = -2 \neq 0, \text{ logo a matriz é regular}$$

$$[\,\mathbf{G}\,|\,\mathbf{I}\,] = \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 \\ -1 & 3 & -9 & 0 & 0 & 1 \end{array}\right]_{\substack{L_2=L_2-2L_1 \\ L_3=L_3+L_1}}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 4 & -11 & 1 & 0 & 1 \end{array}\right]_{\substack{L_3=L_3+2L_2 \\ L_2=-\frac{1}{2}L_2}}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{array}\right]_{\substack{L_1=L_1+2L_3 \\ L_2=L_2+3L_3}}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -5 & 4 & 2 \\ 0 & 1 & 0 & -8 & \frac{11}{2} & 3 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{array}\right]_{L_1=L_1-L_2}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -\frac{3}{2} & -1 \\ 0 & 1 & 0 & -8 & \frac{11}{2} & 3 \\ 0 & 0 & 1 & -3 & 2 & 1 \end{array}\right]$$

$$\therefore \quad \mathbf{G}^{-1} = \begin{bmatrix} 3 & -\frac{3}{2} & -1 \\ -8 & \frac{11}{2} & 3 \\ -3 & 2 & 1 \end{bmatrix}$$

3. Sendo $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}$ e $\mathbf{D} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, encontre a matriz $\mathbf{X}$ que satisfaz as seguintes igualdades, indicando as propriedades utilizadas em cada etapa da resolução

   (a) $\mathbf{AX} = \mathbf{B}^2 - 2\mathbf{C}$

   (b) $\mathbf{XA} = (\mathbf{B} - \mathbf{C})^2$

Resolução.

(a)

$$\mathbf{AX} = \mathbf{B}^2 - 2\mathbf{C} \Leftrightarrow \mathbf{X} = \mathbf{A}^{-1}\big(\mathbf{B}^2 - 2\mathbf{C}\big)$$

$$\mathbf{B}^2 - 2\mathbf{C} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 6 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -2 & -1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 4 & 9 \end{bmatrix}$$

(b)

$$\mathbf{XA} = (\mathbf{B} - \mathbf{C})^2 \Leftrightarrow \mathbf{X} = (\mathbf{B} - \mathbf{C})^2\,\mathbf{A^{-1}}$$

$$\begin{aligned}
\mathbf{B} - \mathbf{C} &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \\
(\mathbf{B} - \mathbf{C})^2 &= \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & -2 \end{bmatrix} \\
\mathbf{X} &= \begin{bmatrix} -1 & 2 \\ -1 & -2 \end{bmatrix} \times \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \times = \begin{bmatrix} 5 & -3 \\ -3 & 1 \end{bmatrix}
\end{aligned}$$

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