Sistemas de equações lineares

0.1 Resolução de sistemas de equações lineares

1. Utilizando o método da condensação, classifique e resolva cada um dos seguintes sistemas de equações lineares.

a) $\begin{cases} x+y-z=1 \\ y=0 \\ x+z=-1 \end{cases}$ b) $\begin{cases} x+y-z=1 \\ x-3y+4z=-2 \\ x-3y+4z=-1 \end{cases}$

c) $\begin{cases} y+2z=-1 \\ 3x+4z=-1 \\ x-5y+4z=4 \\ y+6z=-1 \\ x-4y+6z=3 \\ 3x-y+2z=0 \end{cases}$ d) $\begin{cases} 2y-z+3t=-1 \\ x+y+2z-t=3 \\ x+3y+z+2t=2 \\ x+5y+5t=1 \end{cases}$

Resolução.

(a) $\begin{cases} x+y-z=1 \\ y=0 \\ x+z=-1 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{array}\right] \underset{L_3=L_3-L_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 \end{array}\right]_{L_3=L_3+L_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{array}\right] \Rightarrow \left| \begin{array}{l} \text{car}(\mathbf{A}) = \text{car}([\mathbf{A}\,|\,\mathbf{b}\,]) = 3 \\ \text{n}^{\text{o}} \text{ de incógnitas} = 3 \\ \text{grau de indeterminação} = 0 \end{array} \right. \Rightarrow \text{SPD}$$

$$\therefore \begin{cases} x+y-z=1 \\ y=0 \\ 2z=-2 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \\ z=-1 \end{cases} \quad (0,0,1)$$

(b) $\begin{cases} x+y-z=1 \\ x-3y+4z=-2 \\ x-3y+4z=-1 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -3 & 4 & -2 \\ 1 & -3 & 4 & -1 \end{array}\right] \underset{\substack{L_3=L_3-L_2 \\ L_2=L_2-L_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -4 & 5 & -3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\text{car}(\mathbf{A}) = 2 \neq \text{car}([\mathbf{A}\,|\,\mathbf{b}\,]) = 3 \Rightarrow \text{SI}$

(c) $\begin{cases} y+2z=-1 \\ 3x+4z=-1 \\ x-5y+4z=4 \\ y+6z=-1 \\ x-4y+6z=3 \\ 3x-y+2z=0 \end{cases}$

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & -1 \\ 3 & 0 & 4 & -1 \\ 1 & -5 & 4 & 4 \\ 0 & 1 & 6 & -1 \\ 1 & -4 & 6 & 3 \\ 3 & -1 & 2 & 0 \end{array}\right] \underset{L_3\longleftrightarrow L_1}{\sim} \left[\begin{array}{ccc|c} 1 & -5 & 4 & 4 \\ 3 & 0 & 4 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 6 & -1 \\ 1 & -4 & 6 & 3 \\ 3 & -1 & 2 & 0 \end{array}\right] \begin{array}{l} L_2=L_2-3L_1 \\ L_5=L_5-L_1 \\ L_6=L_6-3L_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -5 & 4 & 4 \\ 0 & 15 & -8 & -13 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 6 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 14 & -10 & -12 \end{array}\right] \underset{L_3\longleftrightarrow L_2}{\sim} \left[\begin{array}{ccc|c} 1 & -5 & 4 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 15 & -8 & -13 \\ 0 & 1 & 6 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 14 & -10 & -12 \end{array}\right] \begin{array}{l} L_3=L_3-15L_2 \\ L_4=L_4-L_2 \\ L_5=L_5-L_2 \\ L_6=L_6-14L_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -5 & 4 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -38 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -38 & 2 \end{array}\right] \underset{L_3\longleftrightarrow L_4}{\sim} \left[\begin{array}{ccc|c} 1 & -5 & 4 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & -38 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -38 & 2 \end{array}\right] \begin{array}{l} L_4=L_4+\frac{38}{4}L_3 \\ L_5=L_5-L_2 \\ L_6=L_6+\frac{38}{4}L_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -5 & 4 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right] \quad \left| \begin{array}{l} \operatorname{car}(\mathbf{A})=3 \\ \operatorname{car}([\mathbf{A}\,|\,\mathbf{b}\,])=4 \end{array} \right. \Rightarrow \text{Sistema impossível}$$

(d) $\begin{cases} 2y-z+3t=-1 \\ x+y+2z-t=3 \\ x+3y+z+2t=2 \\ x+5y+5t=1 \end{cases}$

$$\left[\begin{array}{cccc|c} 0 & 2 & -1 & 3 & -1 \\ 1 & 1 & 2 & -1 & 3 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 5 & 0 & 5 & 1 \end{array}\right] \underset{L_1\leftrightarrow L_2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 3 \\ 0 & 2 & -1 & 3 & -1 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 5 & 0 & 5 & 1 \end{array}\right] \begin{array}{l} L_3=L_3-L_1 \\ L_4=L_4-L_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 3 \\ 0 & 2 & -1 & 3 & -1 \\ 0 & 2 & -1 & 3 & -1 \\ 0 & 4 & -2 & 6 & -2 \end{array}\right] \underset{\begin{array}{l} L_3=L_3-L_2 \\ L_4=L_4-2L_2 \end{array}}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 3 \\ 0 & 2 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left| \begin{array}{l} \operatorname{car}(\mathbf{A})=\operatorname{car}([\mathbf{A}\,|\,\mathbf{b}\,])=2 \\ \text{n}^{\underline{\text{o}}} \text{ de incógnitas} = 4 \\ \text{grau de indeterminação} = 2 \end{array} \right. \Rightarrow \text{SP2xI}$$

$$\therefore \left\{ \begin{array}{l} x+y+2z-t=3 \\ 2y-z+3t=-1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x=3-y-2z+t \\ y=\dfrac{z-3t-1}{2} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x=\frac{1}{2}(7-5z+5t) \\ y=\frac{1}{2}(z-3t-1) \end{array} \right.$$

$\left\{ \left(\frac{7-5z+5t}{2}, \frac{z-3t-1}{2}, z, t \right), z, t \in \mathbb{R} \right\}$

2. Considere as matrizes $\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ a & 0 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ e $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

   e determine:

   (a) o valor do parâmetro real, a, d.e modo que o sistema $\mathbf{Ax} = \mathbf{b}$ seja de Cramer;

   (b) as soluções do sistema nas condições da alínea anterior, utilizando a regra de Cramer.

   Resolução.

   (a) $\det(\mathbf{A}) = \begin{vmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ a & 0 & 1 \end{vmatrix}_{L_3=L_3+L_1} = \begin{vmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ a & 1 & 0 \end{vmatrix} = -2+3a$

   $\therefore$ o sistema é de Cramer se e só se $a \neq \frac{2}{3}$

   (b) as soluções do sistema nas condições da alínea anterior, utilizando a regra de Cramer.

$$x = \frac{\begin{vmatrix} 1 & 1 & -1 \\ -1 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix}_{L_3=L_3+L_1}}{3a-2} = \frac{\begin{vmatrix} 1 & 1 & -1 \\ -1 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix}_{TL\,L_1}}{3a-2} = \frac{10}{3a-2}$$

$$y = \frac{\begin{vmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ a & 2 & 1 \end{vmatrix}_{C_2=C_2+C_3}}{3a-2} = \frac{\begin{vmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ a & 3 & 1 \end{vmatrix}_{TL\,L_1}}{3a-2} = \frac{-6-a}{3a-2}$$

$$z = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ a & 0 & 2 \end{vmatrix}_{C_2=C_2-C_3}}{3a-2} = \frac{\begin{vmatrix} 0 & 0 & 1 \\ 2 & 4 & -1 \\ a & -2 & 2 \end{vmatrix}_{TL\,L_1}}{3a-2} = \frac{-4-4a}{3a-2}$$

■

3. Comente, exemplificando, as seguintes afirmações

   (a) "*um sistema de 4 equações a 3 incógnitas é um sistema determinado*"

   (b) "*um sistema de 3 equações a 3 incógnitas é um sistema determinado*"

   (c) "*um sistema de 2 equações a 3 incógnitas é um sistema determinado*"

   (d) "*um sistema de 2 equações a 3 incógnitas é um sistema impossível*"

   Resolução.

(a) Afirmação falsa. Por exemplo,

$$\begin{cases} x+y+z=1 \\ y+z=1 \\ z=1 \\ x+2y=2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{array}\right] \underset{L_4=L_4-L_1-L_2+3L_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$\therefore \quad \text{car}(\mathbf{A}) = 3 \neq \text{car}([\mathbf{A} \,|\, \mathbf{b}\,]) = 4 \Rightarrow \text{SI}$

(b) Afirmação falsa. Por exemplo,

$$\begin{cases} x+y-2z=1 \\ y+z=1 \\ x+2y-z=5 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & -1 & 5 \end{array}\right] \underset{L_3=L_3-L_1-L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$\therefore \quad \text{car}(\mathbf{A}) = 2 \neq \text{car}([\mathbf{A} \,|\, \mathbf{b}\,]) = 3 \Rightarrow \text{SI}$

(c) Afirmação falsa. Por exemplo,

$$\begin{cases} x+y+z=1 \\ y+z=1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \Rightarrow \left| \begin{array}{l} \text{car}(\mathbf{A}) = \text{car}([\mathbf{A} \,|\, \mathbf{b}\,]) = 2 \\ \text{n}^{\text{o}} \text{ de incógnitas} = 3 \\ \text{grau de indeterminação} = 1 \end{array} \right. \Rightarrow \text{SP1xI}$$

(d) Afirmação falsa. Ver exemplo anterior!

■

0.2 Discussão de sistemas de equações lineares

1. Discuta, por condensação, os seguintes sistemas homogéneos em função do parâmetro real a:

(a) $\begin{cases} x+ay+az=0 \\ ax+ay+az=0 \\ (a+1)\,x+(a+1)\,y+az=0 \end{cases}$

(b) $\begin{cases} 2x-(a+1)\,y-(a+1)\,z=0 \\ 2x+y=0 \\ x-ay-z=0 \end{cases}$

Resolução.

(a) $\begin{cases} x+ay+az=0 \\ ax+ay+az=0 \\ (a+1)\,x+(a+1)\,y+az=0 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & a & a & 0 \\ a & a & a & 0 \\ a+1 & a+1 & a & 0 \end{array}\right] \underset{C_1 \leftrightarrow C_3}{\sim} \left[\begin{array}{ccc|c} a & a & 1 & 0 \\ a & a & a & 0 \\ a & a+1 & a+1 & 0 \end{array}\right]_{\substack{L_2=L_2-L_1 \\ L_3=L_3-L_1}}$$

$$\sim \left[\begin{array}{ccc|c} a & a & 1 & 0 \\ 0 & 0 & a-1 & 0 \\ 0 & 1 & a & 0 \end{array}\right] \underset{L_2 \leftrightarrow L_3}{\sim} \left[\begin{array}{ccc|c} a & a & 1 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & a-1 & 0 \end{array}\right]$$

i. se $a \neq 0 \wedge a-1 \neq 0$

$$\left| \begin{array}{l} \operatorname{car}(\mathbf{A}) = \operatorname{car}([\mathbf{A} \mid \mathbf{b}\,]) = 3 \\ \text{n}^{\text{o}} \text{ de incógnitas} = 3 \\ \text{grau de indeterminação} = 0 \end{array} \right. \Rightarrow \text{SPD}$$

ii. se $a = 0 \vee a = 1$

- se $a = 0$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \underset{C_1 \leftrightarrow C_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right]_{L_3=L_3+L_1}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left| \begin{array}{l} \operatorname{car}(\mathbf{A}) = \operatorname{car}([\mathbf{A} \mid \mathbf{b}\,]) = 2 \\ \text{n}^{\text{o}} \text{ de incógnitas} = 3 \\ \text{grau de indeterminação} = 1 \end{array} \right. \Rightarrow \text{SP1xI}$$

- se $a = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left| \begin{array}{l} \operatorname{car}(\mathbf{A}) = \operatorname{car}([\mathbf{A} \mid \mathbf{b}\,]) = 2 \\ \text{n}^{\text{o}} \text{ de incógnitas} = 3 \\ \text{grau de indeterminação} = 1 \end{array} \right. \Rightarrow \text{SP1xI}$$

Conclusão: $\begin{cases} \text{SPD} \Leftarrow a \neq 0 \wedge a \neq 1 \\ \text{SP1xI} \Leftarrow a = 0 \vee a = 1 \end{cases}$

(b) $\begin{cases} 2x - (a+1)\,y - (a+1)\,z = 0 \\ 2x + y = 0 \\ x - ay - z = 0 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & -a-1 & -a-1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & -a & -1 & 0 \end{array}\right] \underset{L_1 \leftrightarrow L_3}{\sim} \left[\begin{array}{ccc|c} 1 & -a & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & -a-1 & -a-1 & 0 \end{array}\right]_{\substack{L_2=L_2-2L_1 \\ L_3=L_3-2L_1}}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -a & -1 & 0 \\ 0 & 1+2a & 2 & 0 \\ 0 & a-1 & 1-a & 0 \end{array}\right] \underset{C_2 \leftrightarrow C_3}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -a & 0 \\ 0 & 2 & 1+2a & 0 \\ 0 & 1-a & a-1 & 0 \end{array}\right]_{L_3=L_3+\frac{a-1}{2}L_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & -a & 0 \\ 0 & 2 & 1+2a & 0 \\ 0 & 0 & \frac{2a^2+a-3}{2} & 0 \end{array}\right]$$

i. se $2a^2 + a - 3 \neq 0 \Rightarrow \left| \begin{array}{l} \operatorname{car}(\mathbf{A}) = \operatorname{car}([\mathbf{A} \mid \mathbf{b}\,]) = 3 \\ \text{n}^{\text{o}} \text{ de incógnitas} = 3 \\ \text{grau de indeterminação} = 0 \end{array} \right. \Rightarrow \text{SPD}$

ii. se $2a^2 + a - 3 = 0 \Leftrightarrow a = 1 \vee a = -\frac{3}{2}$

- se $a = 1$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left| \begin{array}{l} \operatorname{car}(\mathbf{A}) = \operatorname{car}([\mathbf{A} \mid \mathbf{b}\,]) = 2 \\ \text{n}^{\text{o}} \text{ de incógnitas} = 3 \\ \text{grau de indeterminação} = 1 \end{array} \right. \Rightarrow \text{SP1xI}$$

- se $a = -\frac{3}{2}$

$$\left[\begin{array}{ccc|c} 1 & -1 & \frac{3}{2} & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left|\begin{array}{l} \text{car}(\mathbf{A}) = \text{car}([\mathbf{A} \,|\, \mathbf{b}\,]) = 2 \\ \text{n}^{\underline{\text{o}}} \text{ de incógnitas} = 3 \\ \text{grau de indeterminação} = 1 \end{array}\right. \quad \Rightarrow \text{SP1xI}$$

Conclusão: $\left\{\begin{array}{l} \text{SPD} \;\Leftarrow a \neq 1 \wedge a \neq -\frac{3}{2} \\ \text{SP1xI} \;\Leftarrow a = 1 \vee a = -\frac{3}{2} \end{array}\right.$

■

2. Considere o sistema $\left\{\begin{array}{l} 3x+y=ax \\ 5x+3y=-ay \end{array}\right.$ e calcule o parâmetro real a que permite obter um sistema com:

   (a) soluções não nulas;

   (b) uma única solução;

   (c) nenhuma solução.

   Resolução.

   (a) $\left\{\begin{array}{l} 3x+y=ax \\ 5x+3y=-ay \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} (3-a)\,x+y=0 \\ 5x+(3+a)\,y=0 \end{array}\right.$

   $$\left[\begin{array}{cc|c} 3-a & 1 & 0 \\ 5 & 3+a & 0 \end{array}\right] \underset{C_1 \leftrightarrow C_2}{\sim} \left[\begin{array}{cc|c} 1 & 3-a & 0 \\ 3+a & 5 & 0 \end{array}\right]_{L_2=L_2-(3+a)L_1}$$

   $$\sim \left[\begin{array}{cc|c} 1 & 3-a & 0 \\ 0 & a^2-4 & 0 \end{array}\right]$$

   i. se $a^2-4 \neq 0 \Rightarrow \left|\begin{array}{l} \mathrm{car}(\mathbf{A})=\mathrm{car}([\mathbf{A}\,|\,\mathbf{b}\,])=2 \\ \mathrm{n}^{\underline{\mathrm{o}}} \text{ de incógnitas} = 2 \\ \text{grau de indeterminação} = 0 \end{array}\right. \Rightarrow \mathrm{SPD}$

   ii. se $a^2-4=0 \Leftrightarrow a=2 \vee a=-2$

   - se $a=2$

     $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \left|\begin{array}{l} \mathrm{car}(\mathbf{A})=\mathrm{car}([\mathbf{A}\,|\,\mathbf{b}\,])=1 \\ \mathrm{n}^{\underline{\mathrm{o}}} \text{ de incógnitas} = 2 \\ \text{grau de indeterminação} = 1 \end{array}\right. \Rightarrow \mathrm{SP1xI}$

     – se $a=-2$

     $\left[\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \left|\begin{array}{l} \mathrm{car}(\mathbf{A})=\mathrm{car}([\mathbf{A}\,|\,\mathbf{b}\,])=1 \\ \mathrm{n}^{\underline{\mathrm{o}}} \text{ de incógnitas} = 2 \\ \text{grau de indeterminação} = 1 \end{array}\right. \Rightarrow \mathrm{SP1xI}$

   Conclusão: o sistema tem soluções não nulas se $a=2 \vee a=-2$.

   (b) O sistema terá uma única solução se $a \neq 2 \wedge a \neq -2$.

   (c) O sistema é homogéneo, pelo que terá sempre solução qualquer que seja o valor de a.

   ■

3. Discuta, por condensação, os seguintes sistemas com parâmetros reais a e b

   (a) $\left\{\begin{array}{l} 4x-ay+az=6 \\ x+(a+1)\,y=3 \\ x+2y-(a+1)\,z=-3 \end{array}\right.$

   (b) $\left\{\begin{array}{l} x-y+az=1 \\ ax+y-z=-a \\ (a+1)\,x+y-az=a+2 \end{array}\right.$

   (c) $\left\{\begin{array}{l} x+y+z+t=a \\ (a+2)\,x+(a+2)\,y+(2a+1)\,z+(b+2)\,t=a+1 \\ (a+1)\,x+2y+2az+(b+1)\,t=2a \end{array}\right.$

Resolução.

(a) $\begin{cases} 4x - ay + az = 6 \\ x + (a+1)\,y = 3 \\ x + 2y - (a+1)\,z = -3 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 2 & -a-1 & -3 \\ 1 & a+1 & 0 & 3 \\ 4 & -a & a & 6 \end{array}\right]_{\substack{L_2=L_2-L_1 \\ L_3=L_3-4L_1}}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -a-1 & -3 \\ 0 & a-1 & a+1 & 6 \\ 0 & -a-8 & 5a+4 & 18 \end{array}\right]_{L_2=L_2+L_3}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -a-1 & -3 \\ 0 & -9 & 6a+5 & 24 \\ 0 & -a-8 & 5a+4 & 18 \end{array}\right]_{L_3=L_3+\frac{(a+8)}{-9}L_2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -a-1 & -3 \\ 0 & -9 & 6a+5 & 24 \\ 0 & 0 & 6a^2+8a+4 & 24a+30 \end{array}\right]$$

i. se $6a^2 + 8a + 4 \neq 0 \Rightarrow \left| \begin{array}{l} \text{car}(\mathbf{A}) = \text{car}([\mathbf{A} \,|\, \mathbf{b}\,]) = 3 \\ \text{n}^{\text{o}} \text{ de incógnitas} = 3 \\ \text{grau de indeterminação} = 0 \end{array} \right. \Rightarrow \text{SPD}$

ii. se $6a^2 + 8a + 4 = 0 \Leftrightarrow a = \frac{-4 \pm \sqrt{-8}}{6}$, equação impossível em $\mathbb{R}$.

Conclusão: SPD, $\forall a \in \mathbb{R}$.

(b) $\begin{cases} x - y + az = 1 \\ ax + y - z = -a \\ (a+1)\,x + y - az = a+2 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -1 & a & 1 \\ a & 1 & -1 & -a \\ a+1 & 1 & -a & a+2 \end{array}\right]_{C_1 \leftrightarrow C_2}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & a & 1 \\ 1 & a & -1 & -a \\ 1 & a+1 & -a & a+2 \end{array}\right]_{\substack{L_2=L_2+L_1 \\ L_3=L_3+L_1}}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & a & 1 \\ 0 & a+1 & -1+a & -a+1 \\ 0 & a+2 & 0 & a+3 \end{array}\right]_{L_2=L_2-L_3}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & a & 1 \\ 0 & -1 & -1+a & -2a-2 \\ 0 & a+2 & 0 & a+3 \end{array}\right]_{L_3=L_3+(a+2)L_2}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 1 & a & 1 \\ 0 & -1 & -1+a & -2a-2 \\ 0 & 0 & a^2+a-2 & -2a^2-5a-1 \end{array}\right]$$

i. se $a^2+a-2\neq 0 \Rightarrow \left|\begin{array}{l} \text{car}(\mathbf{A})=\text{car}([\mathbf{A}\,|\,\mathbf{b}\,])=3 \\ \text{n}^{\text{o}} \text{ de incógnitas}=3 \\ \text{grau de indeterminação}=0 \end{array}\right. \Rightarrow \text{SPD}$

ii. se $a^2+a-2=0 \Leftrightarrow a=-2 \vee a=1$

- se $a=-2$
$\left[\begin{array}{ccc|c} -1 & 1 & -2 & 1 \\ 0 & -1 & -3 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \text{car}(\mathbf{A})=2 \wedge \text{car}([\mathbf{A}\,|\,\mathbf{b}\,])=3 \Rightarrow$
SI
- se $a=1$
$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 0 & -8 \end{array}\right] \Rightarrow \text{car}(\mathbf{A})=2 \wedge \text{car}([\mathbf{A}\,|\,\mathbf{b}\,])=3 \Rightarrow$
SI

Conclusão: $\left\{\begin{array}{l} \text{SPD } \Leftarrow a\neq 1 \wedge a\neq -2 \\ \text{SI } \Leftarrow a=1 \vee a=-2 \end{array}\right.$

(c) $\left\{\begin{array}{l} x+y+z+t=a \\ (a+2)\,x+(a+2)\,y+(2a+1)\,z+(b+2)\,t=a+1 \\ (a+1)\,x+2y+2az+(b+1)\,t=2a \end{array}\right.$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ a+2 & a+2 & 2a+1 & b+2 & a+1 \\ a+1 & 2 & 2a & b+1 & 2a \end{array}\right] \begin{array}{l} {\scriptstyle L_2=L_2-2L_1} \\ {\scriptstyle L_3=L_3-L_1} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ a & a & 2a-1 & b & -a+1 \\ a & 1 & 2a-1 & b & a \end{array}\right] \begin{array}{l} {\scriptstyle L_2=L_2-aL_1} \\ {\scriptstyle L_3=L_3-L_2} \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 0 & a-1 & b-a & -a^2-a+1 \\ 0 & 1-a & 0 & 0 & 2a-1 \end{array}\right] {\scriptstyle L_2\leftrightarrow L_3}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 1-a & 0 & 0 & 2a-1 \\ 0 & 0 & a-1 & b-a & -a^2-a+1 \end{array}\right]$$

i. se $a\neq 1 \Rightarrow \left|\begin{array}{l} \text{car}(\mathbf{A})=\text{car}([\mathbf{A}\,|\,\mathbf{b}\,])=3 \\ \text{n}^{\text{o}} \text{ de incógnitas}=4 \\ \text{grau de indeterminação}=0 \end{array}\right. \Rightarrow \text{SP1xI}$

ii. se $a=1$
$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & b-1 & -1 \end{array}\right]$

- se $b\neq 1 \Rightarrow \text{car}(\mathbf{A})=2 \wedge \text{car}([\mathbf{A}\,|\,\mathbf{b}\,])=3 \Rightarrow \text{SI}$
- se $b=1 \Rightarrow \text{car}(\mathbf{A})=1 \wedge \text{car}([\mathbf{A}\,|\,\mathbf{b}\,])=2 \Rightarrow \text{SI}$

Conclusão: $\left\{\begin{array}{l} \text{SP1xI} \Leftarrow a\neq 1, \forall b \\ \text{SI } \Leftarrow a=1, \forall b \end{array}\right.$

$\blacksquare$

4. Considere o sistema de equações lineares nas incógnitas x, y e z

$$\begin{cases} x + y + a^2 z = 1 \\ x + a^2 y + z = 1 \\ a^2 x + y + z = a \end{cases} , \quad a \in \mathbb{R}.$$

(a) Discuta o sistema em função do parâmetro a, sabendo que $\det(\mathbf{A}) = -(a^2-1)^2(a^2+2)$

(b) Justifique a afirmação "Se $a = 0$, *a matriz dos coeficientes do sistema é invertível*".

(c) Determine $\mathbf{A}^{-1}$, fazendo $a = 0$;

(d) determine a solução do sistema para $a = 0$.

Resolução.

(a) Se

i. $\det(\mathbf{A}) = -(a^2-1)^2(a^2+2) \neq 0$ o sistema é de Cramer e, por isso, é um sistema SPD

ii. $-(a^2-1)^2(a^2+2) = 0 \Leftrightarrow a^2 - 1 = 0 \Leftrightarrow a = \pm 1$

- se $a = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \underset{\substack{L_2 = L_2 - L_1 \\ L_3 = L_3 - L_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \operatorname{car}(\mathbf{A}) = \operatorname{car}([\mathbf{A} \,|\, \mathbf{b}\,]) = 1 \Rightarrow$ SP2xI

- se $a = -1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right] \Rightarrow \operatorname{car}(\mathbf{A}) \neq \operatorname{car}([\mathbf{A} \,|\, \mathbf{b}\,]) \Rightarrow$$

SI

$$\text{Conclusão: } \begin{cases} \text{SPD } \Leftarrow a \neq 1 \wedge a \neq -1 \\ \text{SP2xI } \Leftarrow a = 1 \\ \text{SI } \Leftarrow a = -1 \end{cases}$$

(b) A afirmação é verdadeira, pois para $a = 0$ o determinante da matriz dos coeficientes das incógnitas é não nulo: $\det(\mathbf{A}) = -2$

(c) $$[\,\mathbf{A} \,|\, \mathbf{I}\,] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]_{L_2 = L_2 - L_1}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]_{\substack{L_3 = L_3 + L_2 \\ L_1 = L_1 + L_2}}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]_{\substack{L_3 = \frac{1}{2} L_3 \\ L_2 = -L_2}}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]_{\substack{L_2=L_2+L_3 \\ L_1=L_1-L_3}}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$$\therefore \qquad \mathbf{A}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix}$$

(d) $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \Leftrightarrow \mathbf{x} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

■