Números complexos

1. Sejam $z_1 = 1 + \sqrt{3}i$, $z_2 = 3 - 3i$ e $z_3 = \sqrt{2}e^{\frac{5\pi}{4}i}$ e calcule:

   (a) $z_1 \times z_2$ e represente o resultado no plano complexo

   (b) $\dfrac{z_2 + z_3}{iz_2 \times z_3}$

   (c) $z_2^7 \times z_3^4$

   (d) z_1^{-1}

   (e) as raízes cúbicas de z_1

   Resolução.

   (a)
$$\begin{aligned} z_1 \times z_2 &= (1+\sqrt{3}i)(3-3i) \\ &= (3+3\sqrt{3}) + (-3+3\sqrt{3})i \end{aligned}$$

   (b)
$$\begin{aligned} z_3 &= \sqrt{2}e^{\frac{5\pi}{4}i} = \sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right] \\ &= \sqrt{2}\left(-\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right) \\ &= \sqrt{2}\left(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i\right) \\ &= -1 - i \end{aligned}$$

$$\begin{aligned} \frac{z_2 + z_3}{iz_2 \times z_3} &= \frac{3-3i-1-i}{(3+3i)(-1-i)} = \frac{2-4i}{(-3+3)+(-3-3)i} = \frac{2-4i}{-6i} \\ &= \frac{i(2-4i)}{i(-6i)} = \frac{2i+4}{6} = \frac{4}{6} - \frac{2}{6}i = \frac{2}{3} - \frac{1}{3}i \end{aligned}$$

   (c) $\rho = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

   $\tan\theta = \frac{-3}{3} = -1 \Rightarrow_{\theta \in 4^{\text{o}}\text{Q}} \theta = \frac{-\pi}{4}$

$$\begin{aligned} z_2^7 \times z_3^4 &= \left(3\sqrt{2}e^{\frac{-\pi}{4}i}\right)^7 \times \left(\sqrt{2}e^{\frac{5\pi}{4}i}\right)^4 \\ &= 3^7\left(\sqrt{2}\right)^7 e^{\frac{-7\pi}{4}i} \times 2^2 e^{\frac{20\pi}{4}i} \\ &= 3^7 2^5 \sqrt{2} e^{\frac{13\pi}{4}i} \\ &= 3^7 2^5 \sqrt{2}\left(\cos\left(\tfrac{5\pi}{4}\right) + i\sin\left(\tfrac{5\pi}{4}\right)\right) \\ &= 3^7 2^5 \sqrt{2}\left(-\cos\left(\tfrac{\pi}{4}\right) - i\sin\left(\tfrac{\pi}{4}\right)\right) \\ &= 3^7 2^5 \sqrt{2}\left(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i\right) \\ &= 3^7 2^5 (-1-i) \end{aligned}$$

   (d) $z_1^{-1} = \dfrac{1}{1+\sqrt{3}i} = \dfrac{1-\sqrt{3}i}{(1+\sqrt{3}i)(1-\sqrt{3}i)} = \dfrac{1-\sqrt{3}i}{1+3} = \dfrac{1}{4} - \dfrac{\sqrt{3}}{4}i$

(e) $z_1 = 1 + \sqrt{3}i$

$$\begin{cases} \rho & = \sqrt{1+3} = 2 \\ \theta & = \arctan\left(\dfrac{\sqrt{3}}{1}\right) = \arctan\left(\sqrt{3}\right) \Rightarrow \Theta = \dfrac{\pi}{3} \end{cases}$$

$$z_1 = 2e^{\dfrac{\pi}{3}i}$$

$$\sqrt[3]{z_1} = \sqrt[3]{2e^{\dfrac{\pi}{3}i}} = \sqrt[3]{2}e^{\dfrac{\frac{\pi}{3}+2k\pi}{3}i} = \sqrt[3]{2}\text{cis}\left(\dfrac{\frac{\pi}{3}+2k\pi}{3}\right),\ k = 0, 1, 2$$

$$\begin{cases} k = 0 & \sqrt[3]{2} = \sqrt[3]{2}\text{cis}\left(\dfrac{\pi}{9}\right) \\ k = 1 & \sqrt[3]{2} = \sqrt[3]{2}\text{cis}\left(\dfrac{7\pi}{9}\right) \\ k = 2 & \sqrt[3]{2} = \sqrt[3]{2}\text{cis}\left(\dfrac{13\pi}{9}\right) \end{cases}$$

2. Escreva os seguintes números complexos na forma $z = \varrho\left[\cos(\theta) + i\sin(\theta)\right]$

(a) $z_2 = (1+i)^{16}$

(b) $z_3 = \left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right)^{-2}$

Resolução.

(a)
$$\begin{aligned} z_2 &= (1+i)^{16} \\ &= \left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^{16} \\ &= 2^8 e^{4\pi i} \\ &= 256\left(\cos(0) + i\sin(0)\right) \end{aligned}$$

(b)
$$\begin{aligned} z_3 &= \left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right)^{-2} \\ &= \left(e^{\dfrac{\pi}{4}i}\right)^{-2} \\ &= e^{\dfrac{-\pi}{2}i} \\ &= \cos\left(-\dfrac{\pi}{2}\right) + i\sin\left(-\dfrac{\pi}{2}\right) \end{aligned}$$

3. Simplifique o seguinte número complexo $z_2 = \left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right)^3 + \left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i\right)^3$

Resolução.

$$z_2 = \left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right)^3 + \left(\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i\right)^3$$

$$\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i = \cos\left(\dfrac{\pi}{4}\right) + i\sin\left(\dfrac{\pi}{4}\right) = e^{i\dfrac{\pi}{4}}$$

$$\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i = \cos\left(\dfrac{-\pi}{4}\right) + i\sin\left(\dfrac{-\pi}{4}\right) = e^{i\left(\dfrac{-\pi}{4}\right)}$$

Logo,

$$z_2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^3 + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^3$$
$$= \left(e^{i\frac{\pi}{4}}\right)^3 + \left(e^{i\left(\frac{-\pi}{4}\right)}\right)^3 = e^{i\frac{3\pi}{4}} + e^{i\left(\frac{-3\pi}{4}\right)}$$
$$= \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)$$
$$= \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = -\sqrt{2}$$

∎

4. Determine todas as raízes das seguintes equações:

   (a) $x^4 - 2 = 0$

   (b) $z^3 - 3z^2 + 6z - 4 = 0$

   Resolução.

   (a) $x^4 - 2 = 0 \Leftrightarrow x^4 = 2 \Leftrightarrow x = \sqrt[4]{2}$

   Para $z = 2$, $\begin{cases} \rho & = \sqrt{4} = 2 \\ \theta & = \arctan\left(\frac{0}{2}\right) = \arctan(0) \Rightarrow \Theta = 0 \end{cases}$

   Logo, $z = 2e^{0i}$

   Pelo que, $\sqrt[4]{z} = \sqrt[4]{2e^{0i}} = \sqrt[4]{2}e^{\frac{0+2k\pi}{4}i} = \sqrt[4]{2}\text{cis}\left(\frac{k\pi}{2}\right)$, $k = 0, 1, 2, 3$

   $$\begin{cases} k = 0 & x_0 = \sqrt[4]{2}\text{cis}(0) = \sqrt[4]{2} \\ k = 1 & x_1 = \sqrt[4]{2}\text{cis}\left(\frac{\pi}{2}\right) = \sqrt[4]{2}(0 + i) = \sqrt[4]{2}i \\ k = 2 & x_2 = \sqrt[4]{2}\text{cis}(\pi) = \sqrt[4]{2}(-1 + 0i) = -\sqrt[4]{2} \\ k = 3 & x_3 = \sqrt[4]{2}\text{cis}\left(\frac{3\pi}{2}\right) = \sqrt[4]{2}(0 - i) = -\sqrt[4]{2}i \end{cases}$$

   Logo, as raízes são os elementos do conjunto $\left\{\sqrt[4]{2}, \sqrt[4]{2}\,i, -\sqrt[4]{2}i, -\sqrt[4]{2}\,i\right\}$

   (b) $z^3 - 3z^2 + 6z - 4 = 0$

   utilizando a regra de Rufini tem-se
   $$\begin{array}{c|cccc} & 1 & -3 & 6 & -4 \\ 1 & & 1 & -2 & 4 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

   $z^3 - 3z^2 + 6z - 4 = 0 \Longleftrightarrow (z - 1)\left(z^2 - 2z + 4\right) = 0$

   $z = 1 \vee z = \frac{2 \pm \sqrt{4 - 16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}\,i$

   Logo, as raízes são os elementos do conjunto $\left\{1, 1 - \sqrt{3}\,i, 1 + \sqrt{3}\,i\right\}$

5. Mostre que para todo o número complexo z se tem $e^{z+\pi i} = -e^z$.

   Resolução.

   $$e^{z+\pi i} = e^z \times e^{\pi i} = e^z \times (-1 + 0i) = -e^z$$