Algorithms
Homework #2
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(1) $T(n)=2T(\lfloor n/2 \rfloor+17)+n=O(n\lg n)$

$T(n)=2T(\lfloor n/2 \rfloor+17)+n \le 2T(n/2+17)+n \le 2c(n/2+17)\lg(n/2+17)+n=$

$cn\lg(n/2+17)+34c\lg(n/2+17)+n=c(n+34)\lg(n/2+17)+n \le c(n+34)\lg(n/2+n/3)+n \ \forall n\ge 51$

$=c(n+34)\lg(5n/6)+n=c(n+34)(\lg n-\lg(6/5))+n=cn\lg n+34c\lg n-cn\lg(6/5)-34c\lg(6/5)+n$

$\fallingdotseq cn\lg n+34n\lg n-0.3cn-10.2c+n \le cn\lg n$

(2) $T(n)=4T(n/3)+n$

(master method)

$a=4, b=3, f(n)=n, n^{\log_b a}=n^{\log_3 4} \ge n^{1+\varepsilon}$ where $\varepsilon \fallingdotseq 0.26 \rightarrow f(n)=n=O(n^{\log_3 4})$

$T(n)=\Theta(n^{\log_3 4})$

(substitution proof)

assumption: $T(n) \le cn^{\log_3 4}$

$T(n)=4T(n/3)+n \le 4c(n/3)^{\log_3 4}+n=4c(n^{\log_3 4}/3^{\log_3 4})+n=4c(n^{\log_3 4}/4)+n=cn^{\log_3 4}+n \not< \neq cn^{\log_3 4}$

Let $T(n) \le cn^{\log_3 4}-3n$

$T(n)=4T(n/3)+n \le 4(c(n/3)^{\log_3 4}-3(n/3))+n=4(c(n^{\log_3 4}/3^{\log_3 4})-n)+n=4(c(n^{\log_3 4}/4)-n)+n$

$=cn^{\log_3 4}-4n+n=cn^{\log_3 4}-3n \le cn^{\log_3 4}$

$T(n)= O(n^{\log_3 4})$

(3)

(a) $T(n)=2T(n/2)+n^4$

$a=2, b=2, f(n)=n^4, n^{\log_b a}=n^{\log_2 2}=n^1=n \le n^{4-\varepsilon}$ where $\varepsilon=3 \rightarrow f(n)=n^4=\Omega(n)$

$af(n/b) \le cf(n), 2(n/2)^4=(1/8)n^4 \le cn^4$ where $c=1/8<1 \ \forall n$

$T(n)=\Theta(n^4)$

(b) $T(n)=T(7n/10)+n$

$a=1, b=10/7, f(n)=n, n^{\log_b a}=n^{\log_{10/7} 1}=n^0=1 \le n^{1-\varepsilon}$ where $\varepsilon=1 \rightarrow f(n)=n=\Omega(1)$

$af(n/b) \le cf(n), 7n/10=(7/10)n^4 \le cn^4$ where $c=7/10<1 \ \forall n$

$T(n)=\Theta(n)$

(c) $T(n)=16T(n/4)+n^2$

$a=16, b=4, f(n)=n^2, n^{\log_b a}=n^{\log_4 16}=n^2=n^2 \rightarrow f(n)=n^2=\Theta(n^2)$

$T(n)= \Theta(n^2\lg n)$

(d) $T(n)=7T(n/3)+n^2$

$a=7, b=3, f(n)=n^2, n^{\log_b a}=n^{\log_3 7} \le n^{2-\varepsilon}$ where $\varepsilon \fallingdotseq 0.3 \rightarrow f(n)=n^2=\Omega(n^{\log_3 7})$

$af(n/b) \le cf(n), 7(n/3)^2=(7/9)n^2 \le cn^4$ where $c=7/9<1 \ \forall n$

$T(n)=\Theta(n^2)$

(e) $T(n)=7T(n/2)+n^2$

$a=7, b=2, f(n)=n^2, n^{\log_b a}=n^{\log_2 7} \ge n^{2+\varepsilon}$ where $\varepsilon \fallingdotseq 0.8 \rightarrow f(n)=n^2=O(n^{\log_2 7})$

$T(n)=\Theta(n^{\log_2 7})$

(f) $T(n)=2T(n/4)+n^{1/2}$

$a=2$, $b=4$, $f(n)=n^{1/2}$, $n^{\log_b a}=n^{\log_4 2}=n^{1/2} \rightarrow f(n)=n^{1/2}=\Theta(n^{1/2})$

$T(n)=\Theta(n^{1/2}\lg n)$

(4)

(a) $T(n)=T(n/2+n^{1/2})+n$

Let $S(n) \le T(n) \le U(n)$

Let $U(n)=4U(n/2)+n$

$a=4$, $b=2$, $f(n)=n$, $n^{\log_b a}=n^{\log_2 4}=n^2 \rightarrow f(n)=n=O(n^2)$

$U(n)=\Theta(n^2)$

Let $S(n)=S(n/2)+n$

$a=1$, $b=2$, $f(n)=n$, $n^{\log_b a}=n^{\log_2 1}=n^0=1 \rightarrow f(n)=n=\Omega(1)$

$af(n/b) \le cf(n)$, $(n/2)=(1/2)n \le cn$ where $c=1/2<1$ $\forall n$

$S(n)=\Theta(n)$

guess $T(n) \le cn\lg n$

$T(n)=T(n/2+n^{1/2})+n \le c(n/2+n^{1/2})\lg(n/2+n^{1/2})+n=(1/2)cn\lg(n/2+n^{1/2})+cn^{1/2}\lg(n/2+n^{1/2})+n$

$\le cn\lg(n/2+n^{1/2})+n \le cn\lg n+n \not\le\neq cn\lg n$

Let $T(n) \le cn\lg n-2n$

$T(n)=T(n/2+n^{1/2})+n \le c(n/2+n^{1/2})\lg(n/2+n^{1/2})-2(n/2+n^{1/2})+n$

$=(1/2)cn\lg(n/2+n^{1/2})+cn^{1/2}\lg(n/2+n^{1/2})-n-2n^{1/2}+n \le cn\lg(n/2+n^{1/2})-2n^{1/2}$

$\le cn\lg n-2n^{1/2} \le cn\lg n$

$T(n)=O(n\lg n)$

(b) $T(n)=T(n/2)+T(n^{1/2})+n$

$T(n/2) > T(n^{1/2})$ for sufficient large n

$S(n)=2T(n^{1/2})+n \le T(n)=T(n/2)+T(n^{1/2})+n \le 2T(n/2)+n=U(n)$

$a=2$, $b=2$, $f(n)=n$, $n^{\log_b a}=n^{\log_2 2}=n^1 \rightarrow f(n)=n=\Theta(n)$

$U(n)= \Theta(n\lg n)$ guess $T(n) \le cn\lg n$

$T(n)=T(n/2)+T(n^{1/2})+n \le c(n/2)\lg(n/2)+cn^{1/2}\lg n^{1/2}+n=(cn/2)(\lg n-\lg 2)+cn^{1/2}(1/2)\lg n+n$

$=(cn/2)(\lg n-1)+cn^{1/2}(1/2)\lg n+n=(c/2)n\lg n-(c/2)n+(c/2)n^{1/2}\lg n+n \le cn\lg n-(c/2)n+n=cn\lg n+(c/2)n \not\le\neq cn\lg n$

Let $T(n) \le cn\lg n-2n$

$T(n)=T(n/2)+T(n^{1/2})+n \le c(n/2)\lg(n/2)-2(n/2)+cn^{1/2}\lg n^{1/2}-2n^{1/2}+n$

$=(cn/2)(\lg n-\lg 2)-n+cn^{1/2}(1/2)\lg n-2n^{1/2}+n=(c/2)n(\lg n-1)-n+(c/2)n^{1/2}\lg n-2n^{1/2}+n$

$=(c/2)n\lg n+(c/2)n^{1/2}\lg n-(c/2)n-n-2n^{1/2}+n \le cn\lg n-(c/2)n-n-2n^{1/2}+n=cn\lg n-(c/2)n-2n^{1/2}$

$=cn\lg n-d$ where $d=(c/2)n-2n^{1/2}>0 \le cn\lg n$

$T(n)=O(n\lg n)$

(5)

(a)

(using binary-search)

```
FIND-INDEX(A,1,n)
FIND-INDEX(A,l,h)
    while(l<=h)
        i=(l+h)/2
        if(A[i]==i)
            return i
        else if(A[i]<i)
            FIND-INDEX(A,i+1,h)
        else if(A[i]>i)
            FIND-INDEX(A,l,i-1)
        return 0
```

(b)

$T(n)=T(n/2)+\Theta(1)$

(c) $T(n)=T(n/2)+\Theta(1)$

a=1, b=2, $f(n)=\Theta(1)$, $n^{\log_b a}=n^{\log_2 1}=n^0=1 \rightarrow f(n)=\Theta(1)=\Theta(1)$

$T(n)=\Theta(\lg n)$

(6)

(a)

priority queue insertion & deletion performance on $n^{1/2}$ elements $O(\lg n^{1/2})$

$T(n)=nO(\lg n^{1/2})=O(n\lg n^{1/2})$

(b)

$T(n)=n^{1/2}T(n^{1/2})+O(n\lg n^{1/2})$

$T(n)/n=T(n^{1/2})/n^{1/2}+O(n\lg n^{1/2})/n$

Let $S(n)=T(n)/n$

$S(n)=S(n^{1/2})+n\lg n^{1/2}/n=S(n^{1/2})+\lg n^{1/2}$

Let $m=\lg n$

$S(2^m)=S(2^{m/2})+\lg 2^{m/2}= S(2^{m/2})+m/2$

Let $S'(m)=S(2^m)$

$S'(m)=S'(m/2)+m-1$

a=1, b=2, $f(m)=m/2$, $m^{\log_b a}=m^{\log_2 1}=m^0=1 \le m^{1-\varepsilon}$ where $\varepsilon \fallingdotseq 1 \rightarrow f(m)=m/2=\Omega(1)$

$af(m/b) \le cf(m)$, $(m/2)/2=(1/2)(m/2) \le c(m/2)$ where $c=1/2<1\ \forall m$

$S'(m)=\Theta(m/2)=\Theta(m)$

$T(n)=nS(n)=nS'(\lg n)=nO(\lg n)=O(n\lg n)$