1. Find and classify all singularities of the following functions:

a. $f(z) = \dfrac{1}{z^3(z^2+1)}$

In order to find the singularities of $f(z)$ here, we need to find where $f(z)$ is not analytic. This happens when the denominator is equal to 0.

$$z^3(z^2+1)=0$$

$$z=0 \quad \text{or} \quad z^2+1=0$$

$$z^2=-1$$

$$z=\pm i$$

Our singularities are: $z=0$ with pole order 3, $z=i$ as a simple pole, and $z=-i$ as a simple pole.

b. $f(z) = \frac{z+2}{z \sin z}$

In order to find all singularities of $f(z)$ here, we need to find where $f(z)$ is not analytic. This happens when the denominator is equal to 0.

$$z \sin z = 0$$

$$z = 0 \quad \text{or} \quad \sin z = 0$$

$$z = n\pi,\ n \in \mathbb{Z}$$

Our singularities are: $z = n\pi$, $n \in \mathbb{Z}$. Note that $z = 0$ has a pole order of 2, whereas our other singularities have a pole order of 1.

2. Evaluate $I = \int_C \frac{5z+3}{z(z^2+2z-3)} dz$

$$I = \int_C \frac{5z+3}{z(z+3)(z-1)} dz$$

The singularities of our integrand are: $z=0$, $z=-3$, and $z=1$. Note that $z=0$ and $z=1$ are inside $|z|=2$; whereas $z=-3$ is outside $|z|=2$. Thus, the relevant singularities are $z=0$ and $z=1$ as they lie within our contour.

$$I = 2\pi i \left(\text{Res}\left(\frac{5z+3}{z(z+3)(z-1)}, z=0\right) + \text{Res}\left(\frac{5z+3}{z(z+3)(z-1)}, z=1\right)\right)$$

$$I = 2\pi i \left(\lim_{z\to 0} \left(z \cdot \frac{5z+3}{z(z+3)(z-1)}\right) + \lim_{z\to 1} \left((z-1) \cdot \frac{5z+3}{z(z+3)(z-1)}\right)\right)$$

$$I = 2\pi i \left(\lim_{z\to 0} \left(\frac{5z+3}{(z+3)(z-1)}\right) + \lim_{z\to 1} \left(\frac{5z+3}{z(z+3)}\right)\right)$$

$$I = 2\pi i \left(\frac{3}{3 \cdot -1} + \frac{8}{4} \right)$$

$$I = 2\pi i(-1+2)$$

$$I = 2\pi i$$

3. Evaluate $I = \int_{-\infty}^{\infty} \frac{4}{(x^2+2x+3)^2}\,dx$

Rewrite in terms of z: $I = \int_{-\infty}^{\infty} \frac{4}{(z^2+2z+3)^2}\,dz$

Solve $z^2+2z+3=0$

$$z = \frac{-2 \pm \sqrt{4-12}}{2}$$

$$z = \frac{-2 \pm \sqrt{-8}}{2}$$

$$z = \frac{-2 \pm 2\sqrt{-2}}{2}$$

$$z = -1 \pm \sqrt{-2}$$

$$z = -1 \pm i\sqrt{2}$$

Thus, we have the roots or poles $-1+i\sqrt{2}$ and $-1-i\sqrt{2}$. Note that because our denominator was $(z^2+2z+3)^2$ for our integrand, that means both our poles have order 2.

Notice how $-1+i\sqrt{2}$ is inside our contour, the unit circle, whereas $-1-i\sqrt{2}$ is outside our contour, the unit circle. Thus:

$$I = 2\pi i\left(\operatorname{Res}\left(\frac{4}{(z^2+2z+3)^2},\ z=-1+i\sqrt{2}\right)\right)$$

$$I = 2\pi i\left(\lim_{z\to -1+i\sqrt{2}}\left(\frac{d}{dz}\left(\frac{4}{(z^2+2z+3)^2}\cdot(z-(-1+i\sqrt{2}))^2\right)\right)\right)$$

$$I = 2\pi i\left(\lim_{z\to -1+i\sqrt{2}}\left(\frac{d}{dz}\left(4(z^2+2z+3)^{-2}\cdot(z-(-1+i\sqrt{2}))^2\right)\right)\right)$$

$$I = 2\pi i\left(\lim_{z\to -1+i\sqrt{2}}\left(\frac{d}{dz}\left(\frac{4(z-(-1+i\sqrt{2}))^2}{(z^2+2z+3)^2}\right)\right)\right)$$

$$I = 2\pi i\left(\lim_{z\to -1+i\sqrt{2}}\left(\frac{d}{dz}\left(\frac{4(z+1-i\sqrt{2})^2}{(z^2+2z+3)^2}\right)\right)\right)$$

$$I = 2\pi i\left(\lim_{z\to -1+i\sqrt{2}}\left(\frac{d}{dz}\left(\frac{4(z+1-i\sqrt{2})^2}{(z+1-i\sqrt{2})^2(z+1+i\sqrt{2})^2}\right)\right)\right)$$

$$I = 2\pi i\left(\lim_{z\to -1+i\sqrt{2}}\left(\frac{d}{dz}\left(\frac{4}{(z+1+i\sqrt{2})^2}\right)\right)\right)$$

$$I = 2\pi i\left(\lim_{z\to -1+i\sqrt{2}}\left(\frac{d}{dz}\left(4(z+1+i\sqrt{2})^{-2}\right)\right)\right)$$

$$I = 2\pi i\left(\lim_{z\to -1+i\sqrt{2}}\left(-8(z+1+i\sqrt{2})^{-3}\right)\right)$$

$$I = 2\pi i\left(-8\left(-1+i\sqrt{2}+1+i\sqrt{2}\right)^{-3}\right)$$

$$I = 2\pi i\left(-8\left(2i\sqrt{2}\right)^{-3}\right)$$

$$I = -16\pi i \cdot \frac{1}{(2i\sqrt{2})^3}$$

$$I = -16\pi i \cdot \frac{1}{2^3 \cdot (\sqrt{2})^3 \cdot i^3}$$

$$I = -16\pi i \cdot \frac{1}{8 \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot i^3}$$

$$I = -16\pi i \cdot \frac{1}{8 \cdot 2 \cdot \sqrt{2} \cdot i^3}$$

$$I = -16\pi \cdot \frac{1}{16\sqrt{2}\, i^2}$$

$$I = -\pi \cdot -\frac{1}{\sqrt{2}}$$

$$I = \frac{\pi}{\sqrt{2}}$$

4. Evaluate $I = \int_0^{2\pi} \frac{2}{4\sin\theta - 7} \, d\theta$

$$z = e^{i\theta}, \; dz = ie^{i\theta} d\theta, \; d\theta = \frac{dz}{iz}$$

$$\sin\theta = \frac{z - z^{-1}}{2i}$$

$$I = \int_{|z|=1} \frac{2}{4\left(\frac{z-z^{-1}}{2i}\right) - 7} \cdot \frac{dz}{iz}$$

$$I = \int_{|z|=1} \frac{2}{2\left(\frac{z-z^{-1}}{i}\right) - 7} \cdot \frac{dz}{iz}$$

$$I = \int_{|z|=1} \frac{2}{2(z - z^{-1}) - 7i} \cdot \frac{dz}{z}$$

$$I = \int_{|z|=1} \frac{2}{2(z^2 - 1) - 7zi} \, dz$$

we now find our poles for this new integrand:

$$2(z^2-1)-7zi=0$$

$$2z^2-2-7zi=0$$

$$2z^2-7iz-2=0$$

$$z=\frac{7i\pm\sqrt{(7i)^2+16}}{4}$$

$$z=\frac{7i\pm\sqrt{-49+16}}{4}$$

$$z=\frac{7i\pm\sqrt{-33}}{4}$$

$$z=\frac{7i\pm i\sqrt{33}}{4}$$

$$z=\frac{(7\pm\sqrt{33})\,i}{4}$$

Note that $5<\sqrt{33}<6$. That means $\frac{(7+\sqrt{33})i}{4}$ is outside our contour while $\frac{(7-\sqrt{33})i}{4}$ is inside our contour.

$$I = 2\pi i \left(\operatorname{Res}\left(\frac{2}{2z^2 - 7iz - 2},\ z = \frac{(7-\sqrt{33})i}{4} \right)\right)$$

$$I = 2\pi i \left(\lim_{z \to \frac{(7-\sqrt{33})i}{4}} \left(\left(z - \frac{(7-\sqrt{33})i}{4}\right) \cdot \frac{2}{2(z^2-1) - 7zi} \right)\right)$$

$$I = 2\pi i \left(\lim_{z \to \frac{(7-\sqrt{33})i}{4}} \left(\frac{2z}{2(z^2-1) - 7zi} - \frac{2(7-\sqrt{33})i}{4(2(z^2-1) - 7zi)} \right)\right)$$

$$I = 2\pi i \left(\lim_{z \to \frac{(7-\sqrt{33})i}{4}} \left(\frac{2z}{2(z^2-1) - 7zi} - \frac{\frac{1}{2}(7-\sqrt{33})i}{2(z^2-1) - 7zi} \right)\right)$$

$$I = 2\pi i \left(\lim_{z \to \frac{(7-\sqrt{33})i}{4}} \left(\frac{2z - \frac{1}{2}(7-\sqrt{33})i}{2(z^2-1) - 7zi} \right)\right)$$

$$I = 2\pi i \left(\lim_{z \to \frac{(7-\sqrt{33})i}{4}} \left(\frac{2}{4z - 7i} \right)\right)$$

$$I = 2\pi i \left(\frac{2}{4\left(\frac{(7-\sqrt{33})i}{4}\right) - 7i} \right)$$

$$I = 2\pi i \left(\frac{2}{(7-\sqrt{33})i - 7i} \right)$$

$$I = 2\pi i \left(\frac{2}{7i - \sqrt{33}\,i - 7i} \right)$$

$$I = 2\pi i \left(\frac{2}{-i\sqrt{33}} \right)$$

$$I = \frac{4\pi i}{-i\sqrt{33}}$$

$$I = \frac{4\pi}{-\sqrt{33}}$$

$$I = -\frac{4\pi}{\sqrt{33}}$$

Specifying and Desribing the complex contour:

Notice that the contour C is to be inside the unit circle: $|z|=1$, which is oriented counterclockwise inside the complex plane. The substitution $z=e^{i\theta}$ maps the interval $[0, 2\pi]$ to this closed contour. We have already identified our relevant pole in this problem.