Numerical Linear Algebra Homework 6

24.1. For each of the following statements, prove that it is true or give an example to show it is false. Throughout, $A \in \mathbb{C}^{m \times m}$ unless otherwise indicated, and ew stands for eigenvalue.

a. If λ is an ew of A and $\mu \in \mathbb{C}$, then $\lambda - \mu$ is an ew of $A - \mu I$. Let's start with having a vector v such that $Av = \lambda v$.

$$Av = \lambda v$$

$$Av - \mu I v = \lambda v - \mu I v$$

$$(A - \mu I)v = (\lambda - \mu)v$$

Thus showing if λ is an ew of A, then $\lambda - \mu$ is an ew of $A - \mu I$. This statement is true.

b. If A is real and λ is an ew of A, then so is $-\lambda$.
This statement is false.

Counterexample: For the Identity Matrix, I, all eigenvalues are 1, but -1 is not an eigenvalue of I.

c. If A is real and λ is an ew of A, then so is $\bar{\lambda}$.
Suppose λ is an ew of A with corresponding eigenvector x. Then, $Ax = \lambda x$. Taking the complex conjugate of this equation yields $A\bar{x} = \bar{\lambda}\bar{x}$. Note that $\bar{x}$ is nonzero since x is nonzero. Thus, $\bar{\lambda}$ is an ew of A with corresponding eigenvector $\bar{x}$. Therefore, this statement is true.

d. If λ is an ew of A and A is nonsingular, then λ^{-1} is an ew of A^{-1}. Note that because A is nonsingular, $\lambda \neq 0$. Let's start with having a vector v such that $Av = \lambda v$.

$$Av = \lambda v$$

$$A^{-1}Av = A^{-1}\lambda v$$

$$v = A^{-1}\lambda v$$

$$\frac{1}{\lambda}v = A^{-1}v$$

$$\lambda^{-1}v = A^{-1}v$$

Thus, we see that λ^{-1} is an ew of A^{-1}. This statement is true.

e. If all ew's of A are zero, then $A=0$.
This statement is false.

Counterexample:

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Notice that all ew's here are zero,

but A is not the zero matrix.

f. If A is Hermitian and λ is an ew of A, then $|\lambda|$ is a singular value of A. Recall Theorem 5.5: If $A=A^*$, then the singular values of A are the absolute values of the eigenvalues of A. Applying Theorem 5.5, we have that $|\lambda|$ is a singular value of A when A is Hermitian and λ is an ew of A. This statement is true.

9. If A is diagonalizable and all its ew's are equal, then A is diagonal.

Suppose we have an A where A is diagonalizable and its ew's are equal. So, we can express A such that $A = X\Lambda X^{-1}$ where X is invertible and Λ is diagonal. We know that the diagonal entries of Λ are the eigenvalues of A. Remember that our ew's of A are equal meaning that all diagonal entries of Λ are equal. Thus, $\Lambda = \lambda I$ for some λ. Thus,

$$
\begin{aligned}
&A = \\
&X\Lambda X^{-1} = \\
&X(\lambda I)X^{-1} = \\
&X\lambda I X^{-1} = \\
&\lambda X X^{-1} = \\
&\lambda I
\end{aligned}
$$

We see that A is diagonal when A is diagonalizable and all its ew's are equal. This statement is true.

A1. Find the LU decomposition with partial pivoting (identify matrices P, L, and U) for the following matrix by hand:

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

In order to find the LU Decomposition with partial pivoting for the matrix

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

we need to find matrices P, L, and U so that we have $PA = LU$, where A is our given matrix.

Since the largest entry in the first column is in Row 3, we swap Row 1 and Row 3, giving the permutation matrix:

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Applying P_1 to A, we have:

$$P_1 A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

We do Gaussian Elimination to get L_1.

$$R_2 \leftarrow R_2 - 0.5 R_1 \qquad L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, we have

$$L_1 P_1 A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -\frac{1}{2} & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Now, we find another permutation matrix, where our largest value, in this case, 2 is the pivot. To do this, we swap Row 2 and Row 3:

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Applying P_2 to $L_1 P_1 A$, we have:

$$P_2 L_1 P_1 A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -\frac{1}{2} & 2 \end{bmatrix}$$

we do Gaussian Elimination to get L_2.

$$R_3 \leftarrow R_3 + 0.25 R_2 \qquad L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{4} & 1 \end{bmatrix}$$

Thus, we have

$$L_2 P_2 L_1 P_1 A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{9}{4} \end{bmatrix}$$

Let's set $U = L_2 P_2 L_1 P_1 A$. That means:

$$U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{9}{4} \end{bmatrix}$$

Note that $P = P_2 P_1$.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L_2 P_2 L_1 P_1 = L_2 P_2 L_1 I P_1$$

$$L_2 P_2 L_1 P_1 = L_2 P_2 L_1 P_2^{-1} P_2 P_1$$

$$U = L_2 P_2 L_1 P_1 A$$

$$U = L_2 P_2 L_1 I P_1 A$$

$$U = L_2 P_2 L_1 P_2^{-1} P_2 P_1 A$$

$$U = L_2 P_2 L_1 P_2^{-1} P A$$

$$U = L^{-1} P A$$

Notice how $U = L_2 P_2 L_1 P_2^{-1} PA$ and $U = L^{-1} PA$. This means $L_2 P_2 L_1 P_2^{-1} PA = L^{-1} PA$

$$L_2 P_2 L_1 P_2^{-1} PA = L^{-1} PA$$

$$L_2 P_2 L_1 P_2^{-1} = L^{-1}$$

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & 0.25 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.5 & -0.25 & 1 \end{bmatrix}$$

We have $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.5 & -0.25 & 1 \end{bmatrix}$, and

$U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{9}{4} \end{bmatrix}$

We have $PA = LU$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.5 & -0.25 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{9}{4} \end{bmatrix}$$

A2. Find the LU decomposition without pivoting (identify the matrices L and U) and Cholesky decomposition (identify the matrix R) of the following matrix by hand:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

In order to find the LU Decomposition for the matrix

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

we need to find matrices L and U so that we have $A = LU$, where A is our given matrix.

For L_1, we apply row operations:

$R_2 \leftarrow R_2 + \frac{1}{2}R_1$ and $R_3 \leftarrow R_3 + \frac{1}{2}R_1$ so that L_1 is:

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$$

This gives us:

$$L_1 A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 2.5 & -1.5 \\ 0 & -1.5 & 1.5 \end{bmatrix}$$

For L_2, we apply $R_3 \leftarrow R_3 + 0.6 R_2$

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.6 & 1 \end{bmatrix}$$

$$L_2 L_1 A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 2.5 & -1.5 \\ 0 & 0 & 0.6 \end{bmatrix}$$

Let's make our $U = L_2 L_1 A$.

Thus,

$$U = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 2.5 & -1.5 \\ 0 & 0 & 0.6 \end{bmatrix}$$

This means that our L should actually equal $(L_2 L_1)^{-1}$. Thus, $L = (L_2 L_1)^{-1}$.

$$(L_2 L_1)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -0.5 & -0.6 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -0.5 & -0.6 & 1 \end{bmatrix}$$

Thus, we have $A = LU$ or

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -0.5 & -0.6 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 2.5 & -1.5 \\ 0 & 0 & 0.6 \end{bmatrix}$$

For the Cholesky decomposition, we will state that $A = LDL^*$, where D is a diagonal matrix with the diagonal entries of U. This means

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}$$

Also, because $A = LDL^*$, that means $A = L\sqrt{D}\sqrt{D}L^*$.

$$A =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1. & 0 \\ -0.5 & -0.6 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & -0.5 \\ 0 & 1 & -0.6 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -0.5 & -0.6 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2.5} & 0 \\ 0 & 0 & \sqrt{0.6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2.5} & 0 \\ 0 & 0 & \sqrt{0.6} \end{bmatrix} \begin{bmatrix} 1 & -0.5 & -0.5 \\ 0 & 1 & -0.6 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ -0.5\sqrt{2} & \sqrt{2.5} & 0 \\ -0.5\sqrt{2} & -0.6\sqrt{2.5} & \sqrt{0.6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -0.5\sqrt{2} & -0.5\sqrt{2} \\ 0 & \sqrt{2.5} & -0.6\sqrt{2.5} \\ 0 & 0 & \sqrt{0.6} \end{bmatrix}$$

Remember that above is R^*R. So,

$$R^* = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -0.5\sqrt{2} & \sqrt{2.5} & 0 \\ -0.5\sqrt{2} & -0.6\sqrt{2.5} & \sqrt{0.6} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{2} & -0.5\sqrt{2} & -0.5\sqrt{2.5} \\ 0 & \sqrt{2.5} & -0.6\sqrt{2.5} \\ 0 & 0 & \sqrt{0.6} \end{bmatrix}$$

Thus, we have identified R as desired.

We also have the decomposition $A = R^*R$ as needed.

A3. For $A \in \mathbb{R}^{m \times m}$, define the matrix exponential

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

a. Show that the matrix valued function $Y(t) = e^{tA}$ satisfies $Y'(t) = AY(t)$, $t > 0$, $Y(0) = I$.

We will show $Y(t) = e^{tA}$ satisfies the differential equation $Y'(t) = AY(t)$ with $Y(0) = I$.

The first step is to express $Y(t) = e^{tA}$ as a matrix exponential series:

$$Y(t) = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = I + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \dots$$

Now, to show $Y'(t)$:

$$Y'(t) = A + \frac{2(tA)A}{2!} + \frac{3(tA)^2 A}{3!} + \dots$$

Another way to write $Y'(t)$:

$$Y'(t) = A + tA(A) + \frac{(tA)^2 A}{2!} + \dots$$

we find that this means:

$$Y'(t) = A \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$$

Note that:

$$A \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = AY(t)$$

This means that we have $Y'(t) = AY(t)$.

For $t=0$, we evaluate $Y(0)$:

$$Y(0) = e^{0 \cdot A} = \sum_{k=0}^{\infty} \frac{(0 \cdot A)^k}{k!} = A^0 = I$$

Thus, $Y(0) = I$. We have shown that $Y'(t) = AY(t)$ for $t>0$ and $Y(0) = I$ in this setup.

b. Suppose A is diagonalizable: $A = X \Lambda X^{-1}$. Show that $e^{tA} = X e^{t\Lambda} X^{-1}$, where

$$e^{t\Lambda} = \begin{bmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_m} \end{bmatrix}$$

To show that $e^{tA} = X e^{t\Lambda} X^{-1}$ given that $A = X \Lambda X^{-1}$, we can proceed as follows:

The matrix exponential e^{tA} is defined by the power series:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$$

Since $A = X \Lambda X^{-1}$, we can say:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tX\Lambda X^{-1})^k}{k!}$$

Notice that we can say $(tX\Lambda X^{-1})^k = t^k (X\Lambda X^{-1})^k$.

For $k=1$, we have $(X\Lambda X^{-1})^1 = X\Lambda X^{-1}$. We can use this as our base case for induction.
So now, for our inductive step, we need to show that if the statement holds for $k=n$, then the statement also holds for $k=n+1$.
The statement in question is to check if $(X\Lambda X^{-1})^k = X\Lambda^k X^{-1}$, and if this holds for $k=n$, this should hold for $k=n+1$. Assume that $(X\Lambda X^{-1})^n = X\Lambda^n X^{-1}$.
Consider: $(X\Lambda X^{-1})^{n+1} = (X\Lambda X^{-1})^n (X\Lambda X^{-1})$. Using the inductive hypothesis, we substitute $(X\Lambda X^{-1})^n = X\Lambda^n X^{-1}$, which gives us $(X\Lambda X^{-1})^{n+1} = (X\Lambda^n X^{-1})(X\Lambda X^{-1})$. We also have that $(X\Lambda X^{-1})^{n+1} = X\Lambda^n (X^{-1}X)\Lambda X^{-1}$ as matrix multiplication is associative. Thus, $(X\Lambda X^{-1})^{n+1} = X\Lambda^{n+1} X^{-1}$.

By induction, we have shown for all $k \geq 1$: $(X\Lambda X^{-1})^k = X\Lambda^k X^{-1}$. Thus, $t^k(X\Lambda X^{-1})^k = t^k X\Lambda^k X^{-1}$. This means $(tX\Lambda X^{-1})^k = t^k X\Lambda^k X^{-1}$. Thus,

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k X\Lambda^k X^{-1}}{k!}$$

Rearranging, we have

$$e^{tA} = \sum_{k=0}^{\infty} \frac{Xt^k\Lambda^k X^{-1}}{k!}$$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{X(t\Lambda)^k X^{-1}}{k!}$$

$$e^{tA} = X\left(\sum_{k=0}^{\infty} \frac{(t\Lambda)^k}{k!}\right)X^{-1}$$

Notice that $\sum_{k=0}^{\infty} \frac{(t\Lambda)^k}{k!}$ is the definition of $e^{t\Lambda}$.

Thus, $e^{tA} = X e^{t\Lambda} X^{-1}$.

A4. Prove Gerschgorin's Theorem: for any matrix $A \in \mathbb{C}^{m \times m}$, define the disk

$$G_i(A) = \{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{m} |a_{ij}| \}, \quad i = 1, \dots, m$$

then $\Lambda(A) \subset \bigcup_{i=1}^{m} G_i(A)$, where $\Lambda(A)$ is the set of eigenvalues of A. In other words, this theorem says that every eigenvalue of A lies in at least one of the m Gerschgorin disks of A in complex plane.

Let λ be any eigenvalue of A and x the corresponding eigenvector. Thus, we have

$$Ax = \lambda x$$

Let i be an index of $|x|$ such that $|x_i| = \|x\|_\infty$.
This gives us:

$$\sum_{j=1}^{m} a_{ij} x_j = \lambda x_i$$

Rewriting this equation, we get:

$$a_{ii} x_i + \sum_{j=1, j\neq i}^{m} a_{ij} x_j = \lambda x_i$$

$$a_{ii} + \frac{1}{x_i} \sum_{j=1, j\neq i}^{m} a_{ij} x_j = \lambda$$

Thus, we have $\lambda = a_{ii} + \frac{1}{x_i} \sum_{j=1, j\neq i}^{m} a_{ij} x_j$.

$$|\lambda - a_{ii}| = \left| \frac{1}{x_i} \sum_{j=1, j\neq i}^{m} a_{ij} x_j \right|$$

$$|\lambda - a_{ii}| = \left|\frac{1}{x_i} \sum_{j=1, j\neq i}^{m} a_{ij} x_j\right|$$

$$\leq \left|\frac{1}{x_i}\right| \sum_{j=1, j\neq i}^{m} |a_{ij} x_j|$$

$$\leq \sum_{j=1, j\neq i}^{m} |a_{ij}| \quad \text{because } |x_i| \geq |x_j|$$

Thus, we have:

$$|\lambda - a_{ii}| \leq \sum_{j=1, j\neq i}^{m} |a_{ij}|, \quad i=1,\dots,m$$

From this inequality, we conclude that $\lambda \in G_i(A)$. Since this is true for any eigenvalue λ of A, it follows that:

$\Lambda(A) \subset \bigcup_{i=1}^{m} G_i(A)$.