AMATH 501: Homework 1
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1.

(a) Consider a curve

$$r(t) = \frac{1}{3}\cos^3 t\mathbf{i} + \frac{1}{3}\sin^3 t\mathbf{j} + \sin^3 t\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors in the Cartesian coordinate. Then compute the arc length of the curve from $t = 0$ to $t = \frac{\pi}{2}$.
In order to find the arc length of the curve from $t = 0$ to $t = \frac{\pi}{2}$, we will calculate

$$\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{d}{dt}\left(\frac{1}{3}\cos^3 t\right)\right)^2 + \left(\frac{d}{dt}\left(\frac{1}{3}\sin^3 t\right)\right)^2 + \left(\frac{d}{dt}\left(\sin^3 t\right)\right)^2}\, dt =$$

$$\int_0^{\frac{\pi}{2}} \sqrt{(-\cos^2(t)\sin(t))^2 + \left(\cos(t)\sin^2(t)\right)^2 + \left(3\cos(t)\sin^2(t)\right)^2}\, dt =$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\left(\cos^4(t)\sin^2(t)\right) + \left(\cos^2(t)\sin^4(t)\right) + \left(9\cos^2(t)\sin^4(t)\right)}\, dt =$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\left(\cos^4(t)\sin^2(t)\right) + \left(10\cos^2(t)\sin^4(t)\right)}\, dt =$$

$$\int_0^{\frac{\pi}{2}} \sqrt{(\cos^2(t)\sin^2(t))(\cos^2(t) + 10\sin^2(t))}\, dt =$$

$$\int_0^{\frac{\pi}{2}} |\cos(t)\sin(t)|\sqrt{\cos^2(t) + 10\sin^2(t)}\, dt$$

Note that we are integrating on the interval $[0, \frac{\pi}{2}]$. Both $\cos(t)$ and $\sin(t)$ are non-negative on this interval. Therefore, because we are on interval $[0, \frac{\pi}{2}]$, we can state
$\int_0^{\frac{\pi}{2}} |\cos(t)\sin(t)| \sqrt{\cos^2(t) + 10\sin^2(t)}\, dt = \int_0^{\frac{\pi}{2}} \cos(t)\sin(t)\sqrt{\cos^2(t) + 10\sin^2(t)}\, dt$. Therefore, we will continue by integrating

$$\int_0^{\frac{\pi}{2}} \cos(t)\sin(t)\sqrt{\cos^2(t) + 10\sin^2(t)}\, dt\,.$$

We will use u substitution. We will set $u = \sin(t)$, which gives $du = \cos(t)\, dt\,$.
When $t = 0$, $u = 0$.
When $t = \frac{\pi}{2}$, $u = 1$.
This gives us

$$\int_0^1 u\sqrt{1 - u^2 + 10u^2}\, du =$$

$$\int_0^1 u\sqrt{1 + 9u^2}\, du$$

We will now use another substitution and call it v substitution. We will set $v = 1 + 9u^2$, which gives $dv = 18u\, du$ or $\frac{1}{18}dv = u\, du\,$.
When $u = 0$, $v = 1$.

When $u = 1$, $v = 10$.
This gives us

$$\frac{1}{18}\int_1^{10} \sqrt{v}\, dv =$$

$$\frac{1}{18}\left[\frac{2}{3}v^{\frac{3}{2}}\right]_1^{10} =$$

$$\frac{1}{18}\left(\frac{2}{3}\left(10^{\frac{3}{2}}\right) - \frac{2}{3}\left(1^{\frac{3}{2}}\right)\right) =$$

$$\frac{1}{18}\left(\frac{2}{3}\left(10^{\frac{3}{2}} - 1\right)\right) =$$

$$\frac{1}{27}\left(10\sqrt{10} - 1\right) =$$

$$\frac{10\sqrt{10} - 1}{27}$$

Thus, the arc length of $r(t)$ from $t = 0$ to $t = \frac{\pi}{2}$ is $\dfrac{10\sqrt{10} - 1}{27}$.

(b) Define a position vector $\mathbf{r} = \mathbf{r}(t)$ and $\dot{\mathbf{r}}$, $\ddot{\mathbf{r}}$ and $\dddot{\mathbf{r}}$ denote $\frac{d}{dt}\mathbf{r}$, $\frac{d^2}{dt^2}\mathbf{r}$ and $\frac{d^3}{dt^3}\mathbf{r}$. Compute and simplify $\frac{d}{dt}[(r \times \dot{r}) \cdot \ddot{r}]$.

$$\frac{d}{dt}[(r \times \dot{r}) \cdot \ddot{r}] =$$

$$\left(\frac{d}{dt}(r \times \dot{r})\right) \cdot \ddot{r} + (r \times \dot{r}) \cdot \frac{d}{dt}\ddot{r} =$$

$$((\dot{r} \times \dot{r}) + (r \times \ddot{r})) \cdot \ddot{r} + (r \times \dot{r}) \cdot \dddot{r} =$$

$$(r \times \ddot{r}) \cdot \ddot{r} + (r \times \dot{r}) \cdot \dddot{r} =$$

$$(r \times \dot{r}) \cdot \dddot{r}$$

(c) Assume that there is a differentiable position vector $\mathbf{r} = \mathbf{r}(t)$ and it has a constant length. Show that $\frac{d}{dt}\mathbf{r}$ is perpendicular to $\mathbf{r}$ and interpret this geometrically.

We know that $\mathbf{r}(t)$ is a differentiable position vector of constant length. The length of a vector is the magnitude, or norm, $|\mathbf{r}(t)|$, and since the length of $\mathbf{r}(t)$ is constant, we know that $|\mathbf{r}(t)|$ is some constant.

The magnitude, or norm, of a vector is given by:

$$|\mathbf{r}(t)| = \sqrt{\mathbf{r}(t) \cdot \mathbf{r}(t)}$$

Since $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$, and we know this is constant, we can differentiate both sides with respect to t:
$\frac{d}{dt}\left(\mathbf{r}(t) \cdot \mathbf{r}(t)\right)$ is equal to the derivative of some constant, which would give us 0.
Now we apply product rule to differentiate $\mathbf{r}(t) \cdot \mathbf{r}(t)$.

$$2\mathbf{r}(t) \cdot \frac{d}{dt}\mathbf{r}(t) = 0$$

Dividing both sides by 2 we get:

$$\mathbf{r}(t) \cdot \frac{d}{dt}\mathbf{r}(t) = 0$$

This shows that the dot product of $\mathbf{r}(t)$ with its derivative $\frac{d}{dt}\mathbf{r}(t)$ is 0, meaning that the vector $\frac{d}{dt}\mathbf{r}(t)$ is perpendicular to $\mathbf{r}(t)$.

2.

(a) In 3D, define **i**, **j** and **k** are unit vectors in the Cartesian coordinate. Given $\mathbf{F}_1 = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ and $\mathbf{F}_2 = \frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}}$, compute div$\mathbf{F}_i$ and curl$\mathbf{F}_i$, $i = 1, 2$.

For $\mathbf{F}_1$:

$$\mathbf{F}_{1_x} = \frac{x}{\sqrt{x^2+y^2}},\ \mathbf{F}_{1_y} = \frac{y}{\sqrt{x^2+y^2}},\ \text{and } \mathbf{F}_{1_z} = 0$$

$$\text{div}\mathbf{F}_1 = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial z}0$$

$$\text{div}\mathbf{F}_1 = \frac{\sqrt{x^2+y^2}\cdot 1 - x\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x}{x^2+y^2} + \frac{\sqrt{x^2+y^2}\cdot 1 - y\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y}{x^2+y^2} + 0$$

$$\text{div}\mathbf{F}_1 = \frac{\sqrt{x^2+y^2} - \frac{2x^2}{2\sqrt{x^2+y^2}}}{x^2+y^2} + \frac{\sqrt{x^2+y^2} - \frac{2y^2}{2\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\text{div}\mathbf{F}_1 = \frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} + \frac{\sqrt{x^2+y^2} - \frac{y^2}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\text{div}\mathbf{F}_1 = \frac{2\sqrt{x^2+y^2} - \frac{x^2+y^2}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

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$$\text{div}\mathbf{F}_1 = \frac{\sqrt{x^2+y^2}}{x^2+y^2}$$

$$\text{div}\mathbf{F}_1 = \frac{1}{\sqrt{x^2+y^2}}$$

$$\text{curl}\mathbf{F}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix}$$

$$\begin{aligned}\text{curl}\mathbf{F}_1 = &\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right)\mathbf{i} \\ &- \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\mathbf{j} \\ &+ \left(\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right)\mathbf{k}\end{aligned}$$

$$\text{curl}\mathbf{F}_1 = \left(\frac{\sqrt{x^2+y^2}\cdot 0 - y\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x}{x^2+y^2} - \frac{\sqrt{x^2+y^2}\cdot 0 - x\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y}{x^2+y^2}\right)\mathbf{k}$$

$$\text{curl}\mathbf{F}_1 = \left(\frac{-\frac{2xy}{2\sqrt{x^2+y^2}}}{x^2+y^2} - \frac{-\frac{2xy}{2\sqrt{x^2+y^2}}}{x^2+y^2} \right) \mathbf{k}$$

$$\text{curl}\mathbf{F}_1 = \left(\frac{-\frac{xy}{\sqrt{x^2+y^2}}}{x^2+y^2} + \frac{\frac{xy}{\sqrt{x^2+y^2}}}{x^2+y^2} \right) \mathbf{k}$$

$$\text{curl}\mathbf{F}_1 = \left(\frac{0}{x^2+y^2} \right) \mathbf{k}$$

$$\text{curl}\mathbf{F}_1 = 0$$

We have that $\text{div}\mathbf{F}_1 = \dfrac{1}{\sqrt{x^2+y^2}}$ and $\text{curl}\mathbf{F}_1 = 0$.

For $\mathbf{F}_2$:

$$\mathbf{F}_{2_x} = -\frac{y}{\sqrt{x^2+y^2}},\ \mathbf{F}_{2_y} = \frac{x}{\sqrt{x^2+y^2}},\ \text{and } \mathbf{F}_{2_z} = 0$$

$$\text{div}\mathbf{F}_2 = \frac{\partial}{\partial x}\left(-\frac{y}{\sqrt{x^2+y^2}} \right) + \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}} \right) + \frac{\partial}{\partial z} 0$$

$$\text{div}\mathbf{F}_2 = -\frac{\sqrt{x^2+y^2}\cdot 0 - y \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x}{x^2+y^2} + \frac{\sqrt{x^2+y^2}\cdot 0 - x \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y}{x^2+y^2} + 0$$

$$\text{div}\mathbf{F}_2 = -\frac{-\frac{2xy}{2\sqrt{x^2+y^2}}}{x^2+y^2} + \frac{-\frac{2xy}{2\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\text{div}\mathbf{F}_2 = \frac{\frac{xy}{\sqrt{x^2+y^2}}}{x^2+y^2} - \frac{\frac{xy}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\text{div}\mathbf{F}_2 = 0$$

$$\text{curl}\mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \end{vmatrix}$$

$$\begin{aligned} \text{curl}\mathbf{F}_2 = & \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2}} \right) \right) \mathbf{i} \\ & - \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}\left(-\frac{y}{\sqrt{x^2+y^2}} \right) \right) \mathbf{j} \\ & + \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}} \right) - \frac{\partial}{\partial y}\left(-\frac{y}{\sqrt{x^2+y^2}} \right) \right) \mathbf{k} \end{aligned}$$

$$\text{curl}\mathbf{F}_2 = \left(\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}} \right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}} \right) \right) \mathbf{k}$$

$$\text{curl}\mathbf{F}_2 = \left(\frac{\sqrt{x^2+y^2}\cdot 1 - x \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x}{x^2+y^2} + \frac{\sqrt{x^2+y^2}\cdot 1 - y \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y}{x^2+y^2} \right) \mathbf{k}$$

$$\mathrm{curl}\mathbf{F}_2 = \left(\frac{\sqrt{x^2+y^2} - \frac{2x^2}{2\sqrt{x^2+y^2}}}{x^2+y^2} + \frac{\sqrt{x^2+y^2} - \frac{2y^2}{2\sqrt{x^2+y^2}}}{x^2+y^2} \right) \mathbf{k}$$

$$\mathrm{curl}\mathbf{F}_2 = \frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2} - \frac{y^2}{\sqrt{x^2+y^2}}}{x^2+y^2} \mathbf{k}$$

$$\mathrm{curl}\mathbf{F}_2 = \frac{2\sqrt{x^2+y^2} - \frac{x^2+y^2}{\sqrt{x^2+y^2}}}{x^2+y^2} \mathbf{k}$$

$$\mathrm{curl}\mathbf{F}_2 = \frac{2\sqrt{x^2+y^2} - \sqrt{x^2+y^2}}{x^2+y^2} \mathbf{k}$$

$$\mathrm{curl}\mathbf{F}_2 = \frac{\sqrt{x^2+y^2}}{x^2+y^2} \mathbf{k}$$

$$\mathrm{curl}\mathbf{F}_2 = \frac{1}{\sqrt{x^2+y^2}} \mathbf{k}$$

We have that $\mathrm{div}\mathbf{F}_2 = 0$ and $\mathrm{curl}\mathbf{F}_2 = \dfrac{1}{\sqrt{x^2+y^2}}\mathbf{k}$.

(b) Denote $h = 6 - 2x^2 - 4x - 2y^2 - 4y$ as the height of a mountain at the location (x, y) above the sea level. The positive x-axis points east and the positive y-axis points north. At $(1, 1)$, which direction is the steepest descent of h? What is the change rate of the elevation if you head northwest? Which point is the top of the mountain?

We have that $h(x, y) = 6 - 2x^2 - 4x - 2y^2 - 4y$.

$$h(x, y) = 6 - 2x^2 - 4x - 2y^2 - 4y$$

$$\nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)$$

$$\frac{\partial h}{\partial x} = -4x - 4 \text{ and } \frac{\partial h}{\partial y} = -4y - 4$$

$$\nabla h = (-4x - 4, -4y - 4)$$

$$\nabla h(1, 1) = (-4(1) - 4, -4(1) - 4)$$

$$\nabla h(1, 1) = (-4 - 4, -4 - 4)$$

$$\nabla h(1, 1) = (-8, -8)$$

The steepest descent is in the direction opposite to the gradient: $-\nabla h(1, 1) = (8, 8)$. Because our x and y components are both positive, that means our direction of our steepest descent is **northeast** as it is given that the positive x-axis points east and the positive y-axis points north.

The northwest direction corresponds to an angle of $135°$. Using what we know from Trigonometry, that angle gives the direction vector of:

$$\vec{d} = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$\vec{d} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

We now compute the directional derivative, $D_{\vec{d}}h$, by using the dot product:

$$D_{\vec{d}}h = \nabla h \cdot \vec{d} = (-8, -8) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{8}{\sqrt{2}} - \frac{8}{\sqrt{2}} = 0$$

The rate of change elevation heading northwest is 0. This means that at the point $(1, 1)$, the elevation does not change if you were to move in the northwest direction.

From analyzing the function, we can state that $h(x, y)$ is a downward facing paraboloid, which means that the critical point would indeed be a maximum. Once again, the partial derivatives are:

$$\frac{\partial h}{\partial x} = -4x - 4 \text{ and } \frac{\partial h}{\partial y} = -4y - 4$$

To find our critical point, in this case, a maximum, we set both partial derivatives equal to 0:

$$-4x - 4 = 0 \text{ and } -4y - 4 = 0$$

$$x = -1 \text{ and } y = -1$$

So, the critical point is $(-1, -1)$, and in this case, because we are working with a downward facing paraboloid, we have a maximum at $(x, y) = (-1, -1)$.

3. For a vector field $\mathbf{F} = (xz + xy)\mathbf{i} + \alpha(yz - xy)\mathbf{j} + \beta(yz + xz)\mathbf{k}$, we assume that there exists a vector $\mathbf{G}$ such that $\mathbf{F} = \text{curl}\mathbf{G}$.

(a) Determine constants α and β.
We know for a vector field to be a curl, it must be solenoidal, or $\nabla \cdot \mathbf{F} = 0$.
The Divergence of $\mathbf{F}$ is given by

$$\nabla \cdot \mathbf{F} = \frac{\partial \mathbf{F}_1}{\partial x} + \frac{\partial \mathbf{F}_2}{\partial y} + \frac{\partial \mathbf{F}_3}{\partial z}$$

where we define

$$\mathbf{F}_1 = xz + xy, \ \ \mathbf{F}_2 = \alpha(yz - xy), \text{ and } \mathbf{F}_3 = \beta(yz + xz)$$

$$\nabla \cdot \mathbf{F} = \frac{\partial(xz + xy)}{\partial x} + \frac{\partial \alpha(yz - xy)}{\partial y} + \frac{\partial \beta(yz + xz)}{\partial z}$$

$$\nabla \cdot \mathbf{F} = z + y + \alpha(z - x) + \beta(y + x)$$

Remember that for this problem, we have that $\nabla \cdot \mathbf{F} = 0$. So,

$$z + y + \alpha(z - x) + \beta(y + x) = 0$$

$$z + y + \alpha z - \alpha x + \beta y + \beta x = 0$$

Combining like terms:

$$(\beta - \alpha)x + (1 + \beta)y + (1 + \alpha)z = 0$$

We get the equations:

$$\beta - \alpha = 0$$

$$1 + \beta = 0$$

$$1 + \alpha = 0$$

$$\beta = \alpha$$

$$\beta = -1$$

$$\alpha = -1$$

Thus,

$$\boxed{\alpha = -1 \text{ and } \beta = -1}$$

(b) If $G = xyz\mathbf{i} - xyz\mathbf{j} + g(x, y, z)\mathbf{k}$. Find unknown function $g(x, y, z)$.
We are given the vector field $\mathbf{F}$ and assume it is the curl of some vector field $\mathbf{G} = xyz\mathbf{i} - xyz\mathbf{j} + g(x, y, z)$, where $g(x, y, z)$ is an unknown function.
The curl of a vector field $\mathbf{G} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is given by:

$$\nabla \times \mathbf{G} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

In this case, the components of $\mathbf{G}$ are:

$$P = xyz, \quad Q = -xyz, \quad R = g(x, y, z)$$

We now compute the curl $\nabla \times \mathbf{G}$ using these components:

The $\mathbf{i}$ component:

This is the difference between the derivative of R, which is $g(x, y, z)$ with respect to y and the derivative of Q with respect to z:

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = \frac{\partial g(x, y, z)}{\partial y} - \frac{\partial(-xyz)}{\partial z}$$

The derivative of $g(x, y, z)$ with respect to y is simply $\frac{\partial g}{\partial y}$, and the derivative of $-xyz$ with respect to z is $-xy$. Therefore:

$$\frac{\partial g}{\partial y} - (-xy) = \frac{\partial g}{\partial y} + xy$$

So, the $\mathbf{i}$ component of the curl is:

$$\frac{\partial g}{\partial y} + xy$$

The $\mathbf{j}$ component:

$$-\left(\frac{\partial g(x, y, z)}{\partial x} - \frac{\partial(xyz)}{\partial z}\right) = -\left(\frac{\partial g}{\partial x} - xy\right)$$

So, the $\mathbf{j}$ component of the curl is:

$$-\left(\frac{\partial g}{\partial x} - xy\right)$$

The $\mathbf{k}$ component of the curl is:

$$\frac{\partial(-xyz)}{\partial x} - \frac{\partial(xyz)}{\partial y} = -yz - xz$$

So, the $\mathbf{k}$ component of the curl is:

$$-yz - xz$$

Putting it together, the curl we just computed is:

$$\nabla \times \mathbf{G} = \left(\frac{\partial g}{\partial y} + xy\right)\mathbf{i} - \left(\frac{\partial g}{\partial x} - xy\right)\mathbf{j} + (-yz - xz)\mathbf{k}$$

We are told that $\mathbf{F} = (xz+xy)\mathbf{i} + \alpha(yz-xy)\mathbf{j} + \beta(yz+xz)\mathbf{k}$. Comparing this with the components of the curl:

- For the $\mathbf{i}$ component:

$$\frac{\partial g}{\partial y} + xy = xz + xy$$

$$\frac{\partial g}{\partial y} = xz$$

- For the $\mathbf{j}$ component:

$$-\left(\frac{\partial g}{\partial x} - xy\right) = \alpha(yz - xy)$$

$$\frac{\partial g}{\partial x} = xy - \alpha(yz - xy)$$

- For the $\mathbf{k}$ component:

$$-yz - xz = \beta(yz + xz)$$

$$\beta = -1$$

Note that we have $\beta = -1$ consistent with the first part of this exercise. Also, remember that $\alpha = -1$, which does indeed fit as well.

Finding $g(x, y, z)$:
We now solve the partial differential equations obtained for $g(x, y, z)$:

- For the **i** component:
$$\frac{\partial g}{\partial y} = xz$$
- For the **j** component:
$$\frac{\partial g}{\partial x} = xy - \alpha(yz - xy)$$
$$\frac{\partial g}{\partial x} = xy - (-1)(yz - xy)$$
$$\frac{\partial g}{\partial x} = xy + yz - xy$$
$$\frac{\partial g}{\partial x} = yz$$
- For the **k** component: we just have that $\beta = -1$.

So, $g(x, y, z) = \int xz\, dy$ and $g(x, y, z) = \int yz\, dx$. Doing both integrals gives us that

$$\boxed{g(x, y, z) = xyz + C}$$

where C is some constant of integration.