AMATH 501: Homework 5 Sid Meka

1. Compute $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx$

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx =$$

$$2\int_{0}^{\infty} \frac{\cos x}{(x^2+1)^2} dx$$

we consider the complex valued function; $f(z) = \frac{e^{iz}}{(z^2+1)^2}$ and integrate $f(z)$ over a semicircular contour in the upper half plane consisting of the real line $(-\infty, \infty)$ and a semicircular arc of radius R denoted C_R. The poles of $f(z)$ are where $z^2+1=0$ and we have $z=i$ with pole order 2 and $z=-i$ with pole order 2. We are only integrating our contour in the upper half plane. Therefore, the relevant pole is $z=i$.

Discussing each part of the relevant contour:

Note: This is with respect to the original integral:

$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\, dx.$

The first part of two consists of the real line segment from $-R$ to R:

$$I_R = \int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2}\, dx$$

As $R \to \infty$, this becomes:

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2}\, dx$$

The second part of two deals with C_R:

Parametrization: $z = Re^{i\theta}$ with $\theta \in [0, \pi]$.

Integral over C_R: $\int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\, dz$

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\,dx =$$

$$2\pi i \operatorname{Res}((f(z), i) =$$

$$2\pi i \left(\frac{1}{1!} \lim_{z \to i} \left(\frac{d}{dz}((z-i)^2 f(z))\right)\right) =$$

$$2\pi i \lim_{z \to i} \left(\frac{d}{dz}((z-i)^2 f(z))\right) =$$

$$2\pi i \lim_{z \to i} \left(\frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right)\right) =$$

$$2\pi i \lim_{z \to i} \left(\frac{ie^{iz}}{(z+i)^2} - \frac{2e^{iz}}{(z+i)^3}\right) =$$

$$2\pi i \left(\frac{ie^{-1}}{(2i)^2} - \frac{2e^{-1}}{(2i)^3}\right) =$$

$$2\pi i \left(\frac{i}{4i^2 e} - \frac{2}{8i^3 e}\right) =$$

$$\frac{2\pi i^2}{4i^2 e} - \frac{4\pi i}{8i^3 e} =$$

$$\frac{\pi}{2e} - \frac{\pi}{2i^2 e} =$$

$$\frac{\pi}{2e} + \frac{\pi}{2e} =$$

$$\frac{2\pi}{2e} =$$

$$\frac{\pi}{e}$$

Note: I didn't end up using $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\,dx = 2\int_{0}^{\infty} \frac{\cos x}{(x^2+1)^2}\,dx$, but having that identity could be helpful or useful.

2. Prove that $\qquad$ This Integral is called I.

$$\int_0^{\infty} \frac{\sin x}{x(x^2+1)} dx = \frac{\pi}{2}\left(1-\frac{1}{e}\right)$$

$$\frac{1}{2i}\int_0^{\infty} \frac{e^{ix}-e^{-ix}}{x(x^2+1)} dx = \frac{\pi}{2}\left(1-\frac{1}{e}\right)$$

$$\frac{1}{2i}\left(\int_0^{\infty} \frac{e^{ix}}{x(x^2+1)} dx - \int_0^{\infty} \frac{e^{-ix}}{x(x^2+1)} dx\right) = \frac{\pi}{2}\left(1-\frac{1}{e}\right)$$

Let: $f(z) = \dfrac{e^{iz}}{z(z^2+1)}$

Consider the principal value integral:

$PV \int_{-\infty}^{\infty} f(z)\, dz$.

This principal value integral is justified because the singularity at $z=0$ is handled symmetrically at $z=0$ is handled symmetrically ensuring convergence.

We have the following components:

1. The real axis from $-R$ to R
2. C_R of radius R in the upper half plane

By Jordan's Lemma, the contribution of C_R vanishes as $R \to \infty$ since $|f(z)| \to 0$ exponentially fast for large $|z|$ when $Im(z) > 0$. Thus, we close the contour in the upper half plane.

Now, we discuss the relevant poles:

- $z = 0$
- $z = i$

Both of which are simple poles

We move on to find the residues at each pole.

$$\text{Res}(f(z), 0) =$$

$$\lim_{z \to 0} z \cdot \frac{e^{iz}}{z(z^2+1)} =$$

$$\lim_{z \to 0} \frac{e^{iz}}{(z^2+1)} =$$

$$\frac{1}{1} =$$

$$1$$

$$\text{Res}(f(z), i) =$$

$$\lim_{z \to i} (z-i) \cdot \frac{e^{iz}}{z(z^2+1)} =$$

$$\lim_{z \to i} (z-i) \cdot \frac{e^{iz}}{z(z-i)(z+i)} =$$

$$\lim_{z \to i} \frac{e^{iz}}{z(z+i)} =$$

$$\frac{e^{-1}}{i(2i)} =$$

$$\frac{e^{-1}}{2i^2} =$$

$$\frac{e^{-1}}{-2} =$$

$$-\frac{1}{2e}$$

$$\int_{-\infty}^{\infty} f(z)\,dz = \pi i \,\mathrm{Res}(f(z),0) + 2\pi i \,\mathrm{Res}(f(z),i)$$

$$\int_{-\infty}^{\infty} f(z)\,dz = \pi i + 2\pi i \cdot -\frac{1}{2e}$$

$$\int_{-\infty}^{\infty} f(z)\,dz = \pi i - \frac{\pi i}{e}$$

Remember that $I = \frac{1}{2i} \int_{-\infty}^{\infty} f(z)\,dz$

$$I = \frac{1}{2i}\left(\pi i - \frac{\pi i}{e}\right)$$

$$I = \frac{\pi}{2} - \frac{\pi i}{2ie}$$

$$I = \frac{\pi}{2} - \frac{\pi}{2e}$$

$$I = \frac{\pi}{2}\left(1 - \frac{1}{e}\right)$$

Thus, we have proved

$$\int_0^{\infty} \frac{\sin x}{x(x^2+1)}\,dx = \frac{\pi}{2}\left(1 - \frac{1}{e}\right)$$

3. Find Fourier Transform $f(t) = \begin{cases} 1 & \text{for } -a<t<a \\ 0 & \text{otherwise} \end{cases}$ and then inverse transform

$$F(\lambda) = \int_{-a}^{a} e^{i\lambda t} \cdot 1 \, dt$$

$$F(\lambda) = \left(\frac{1}{i\lambda} e^{i\lambda t}\right)\Big|_{t=-a}^{t=a}$$

$$F(\lambda) = \frac{1}{i\lambda} \left(e^{i\lambda t}\right)\Big|_{t=-a}^{t=a}$$

$$F(\lambda) = \frac{1}{i\lambda} \left(e^{i\lambda a} - e^{-i\lambda a}\right)$$

$$F(\lambda) = \frac{1}{i\lambda} \cdot 2i\sin(\lambda a)$$

$$F(\lambda) = \frac{2\sin(\lambda a)}{\lambda}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} F(\lambda)\, d\lambda$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \frac{2\sin(\lambda a)}{\lambda}\, d\lambda$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \frac{2}{\lambda} \cdot \frac{e^{i\lambda a} - e^{-i\lambda a}}{2i}\, d\lambda$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} \cdot \frac{e^{i\lambda a} - e^{-i\lambda a}}{i\lambda}\, d\lambda$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda(a-t)} - e^{-i\lambda(a+t)}}{i\lambda}\, d\lambda$$

$$f(t) = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \frac{e^{i\lambda(a-t)}}{i\lambda}\, d\lambda - \frac{e^{-i\lambda(a+t)}}{i\lambda}\, d\lambda \right)$$

$$f(t) = \frac{1}{2\pi i}\left(\pi i \, Res(f(\lambda), 0) + \pi i \, Res(f(\lambda), 0)\right)$$

$$f(t) = \frac{1}{2\pi i}\left(\pi i \cdot sgn(a-t) + \pi i \cdot sgn(-(a+t))\right)$$

$$f(t) = \frac{1}{2\pi i}\left(\pi i \cdot sgn(a-t) + \pi i \cdot sgn(-a-t)\right)$$

$$f(t) = \frac{1}{2}\left(sgn(a-t) + sgn(-a-t)\right)$$

For $-a<t<a$: $sgn(a-t)=1$ and $sgn(-a-t)=-1$ so $f(t)=1$. Outside $-a<t<a$: The sign functions cancel so $f(t)=0$. Note that:

when $t>-a$, $-a-t<0$, so $sgn(-a-t)=-1$.

when $t<-a$, $-a-t>0$, so $sgn(-a-t)=1$.

The switching behavior is already there appropriately meaning that $f(t) \neq -1$ anywhere.

Thus, this gives us:

$$f(t) = \begin{cases} 1 & \text{for } -a < t < a \\ 0 & \text{otherwise} \end{cases}$$