AMATH 584: Numerical Linear Algebra: Exercise 6.1
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If P is an orthogonal projector, then $I - 2P$ is unitary. Prove this algebraically, and give a geometric interpretation.

Algebraic Interpretation:
In order to prove that $I - 2P$ is a unitary matrix, we will prove that both $(I - 2P)^*(I - 2P) = I$ and $(I - 2P)(I - 2P)^* = I$. Furthermore, we will show that $(I - 2P)^*(I - 2P) = (I - 2P)(I - 2P)^*$.

$$\begin{aligned}
(I - 2P)^*(I - 2P) &= \\
(I^* - 2P^*)(I - 2P) &= \\
(I - 2P)(I - 2P) &= && \text{(as } I^* = I \text{ and } P^* = P) \\
I - 2P - 2P + 4P^2 &= \\
I - 4P + 4P^2 &= \\
I - 4P + 4P &= && \text{(as } P^2 = P) \\
I &
\end{aligned}$$

$$\begin{aligned}
(I - 2P)(I - 2P)^* &= \\
(I - 2P)(I^* - 2P^*) &= \\
(I - 2P)(I - 2P) &= && \text{(as } I^* = I \text{ and } P^* = P) \\
I - 2P - 2P + 4P^2 &= \\
I - 4P + 4P^2 &= \\
I - 4P + 4P &= && \text{(as } P^2 = P) \\
I &
\end{aligned}$$

Thus, we have shown that $(I-2P)^*(I-2P) = I$ and $(I-2P)(I-2P)^* = I$. Also, because $(I-2P)^*(I-2P) = I$ and $(I - 2P)(I - 2P)^* = I$, it follows that $(I - 2P)^*(I - 2P) = (I - 2P)(I - 2P)^*$.

Geometric Interpretation:
For the geometric interpretation of $I - 2P$, we can think of this transformation as a reflection across the subspace onto which P projects.

1. **Orthogonal Projector** P: The matrix P is an orthogonal projection onto some subspace V. That means P takes any vector and projects it orthogonally onto V, leaving vectors in V unchanged and mapping vectors perpendicular to V, in $V^\perp$, to the zero vector, or $\vec{0}$.
2. **Reflection across** V: The matrix $I - 2P$ represents a reflection on the subspace V. This is because:
   - For any vector v in the subspace V, $Pv = v$. Applying $I - 2P$ to such a vector gives:

     $$(I - 2P)v = v - 2v = -v$$

     So, the transformation flips the vector, essentially reflecting it across the subspace.
   - For any vector $v^\perp$ in the orthogonal complement $V^\perp$, $Pv = 0$. Thus, applying $I - 2P$ to $v^\perp$ gives:

     $$(I - 2P)v^\perp = v^\perp - 0 = v^\perp$$

     This means vectors orthogonal to the subspace remain unchanged.

3. **Conclusion**: The matrix P is an orthogonal projection onto some subspace V. That means P takes any vector and projects it orthogonally onto V, leaving vectors in V unchanged and mapping vectors perpendicular to V meaning in $V^{\perp}$ to the zero vector, or $\vec{0}$. Furthermore, the transformation $I - 2P$ reflects vectors in V by flipping them, while leaving vectors in $V^{\perp}$ unchanged.

AMATH 584: Numerical Linear Algebra: Exercise 6.2
Sid Meka

Let E be the $m \times m$ matrix that extracts the even part of an m vector:

$Ex = (x + Fx)/2$, where F is the $m \times m$ matrix that flips $(x_1, \ldots, x_m)^*$ to $(x_m, \ldots, x_1)^*$. Is E an orthogonal projector, an oblique projector, or not a projector at all? What are its entries?

Let's analyze the matrix E that extracts the even part of a vector x. The matrix E is defined as: $Ex = \frac{1}{2}(x + Fx)$, where F is the $m \times m$ matrix that flips the components of x. Specifically, if $x = (x_1, x_2, \ldots, x_m)^*$, then $Fx = (x_m, x_{m-1}, \ldots, x_1)^*$. In other words, F is the matrix that reverses the order of the entries of x.

The general $m \times m$ matrix F is:

$$F = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

The Matrix F has 1 on its antidiagonal and 0 elsewhere. This matrix can be expressed as having elements defined by:

$$F_{ij} = \begin{cases} 1 & \text{if } i + j = m + 1 \\ 0 & \text{otherwise} \end{cases}$$

Also note that $F^2 = I$ because swapping the columns back and forth is the identity operator, where I is the Identity Matrix.

Now, let's calculate E:

$$Ex = \frac{1}{2}(x + Fx)$$

$$Ex = \frac{1}{2}(Ix + Fx)$$

$$Ex = \frac{1}{2}(I + F)x$$

$$E = \frac{1}{2}(I + F)$$

Thus, we have that $E = \frac{1}{2}(I + F)$.

Checking if $E^2 = E$:

$$E^2 = \left(\frac{1}{2}(I + F)\right)\left(\frac{1}{2}(I + F)\right)$$

$$E^2 = \frac{1}{4}(I + F)(I + F)$$

$$E^2 = \frac{1}{4}(I^2 + IF + FI + F^2)$$

Note that

- $I^2 = I$ as the Identity Matrix multiplied by the Identity Matrix is still the Identity Matrix
- $IF = F$
- $FI = F$

- $F^2 = I$ as swapping the columns back and forth is the identity operator as previously indicated

This allows us to simplify E^2 further:

$$E^2 = \frac{1}{4}(I + F + F + I)$$

$$E^2 = \frac{1}{4}(2I + 2F)$$

$$E^2 = \frac{1}{2}(I + F)$$

Note that previously we have indicated that $E = \frac{1}{2}(I + F)$. We now have that $E^2 = \frac{1}{2}(I + F)$. Thus, we have that

$$E^2 = E$$

Because $E^2 = E$, we have that E is a projector.

Now, let's compute E^*:

$$E^* = \left(\frac{1}{2}(I + F)\right)^*$$

$$E^* = \frac{1}{2}(I^* + F^*)$$

Because I and F are both Symmetric Matrices, we have that $I^* = I$ and $F^* = F$. Thus,

$$E^* = \frac{1}{2}(I + F)$$

Once again, $E = \frac{1}{2}(I + F)$. Thus,

$$E^* = E$$

Because $E^2 = E$ and $E^* = E$, we have that E is an orthogonal projector.

The matrix E is defined as:

$$E = \frac{1}{2}(I + F)$$

where I is the $m \times m$ identity matrix and F is previously defined in $m \times m$. For clarity:

$$F = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Thus, the entries of E are given by

$$E_{ij} = \frac{1}{2}(I_{ij} + F_{ij})$$

Case 1: m is odd:
If m is odd, we have that the diagonal and the antidiagonal intersect at what we will refer to as the central

element. This intersection entry or central element has the value 1 while the other elements on the diagonal and antidiagonal have the value $\frac{1}{2}$. All other entries have the value 0. Here's E in this case:

$$E = \begin{bmatrix} \frac{1}{2} & 0 & \cdots & 0 & \cdots & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdots & 0 & \cdots & \frac{1}{2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \iddots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{1}{2} & \cdots & 0 & \cdots & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \cdots & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix}$$

Case 2: m is even:
If m is even, the diagonal and antidiagonal do not intersect. Therefore, there is no central element or intersection entry like we had for Case 1. That means all elements on the diagonal and antidiagonal have the value $\frac{1}{2}$ while all other entries have the value 0. Thus, we have the following for E:

$$E = \begin{bmatrix} \frac{1}{2} & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \cdots & 0 & 0 & \cdots & \frac{1}{2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{1}{2} & \cdots & 0 & 0 & \cdots & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix}$$

Conclusion for the entries of E:

$$E_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } i = \frac{m+1}{2} \text{ (applies for odd } m \text{ only)} \\ \frac{1}{2} & \text{if } (i = j \text{ or } i + j = m + 1) \text{ and } (m \text{ is odd and } (i, j) \neq (\frac{m+1}{2}, \frac{m+1}{2})) \\ \frac{1}{2} & \text{if } (i = j \text{ or } i + j = m + 1) \text{ and } (m \text{ is even}) \\ 0 & \text{otherwise} \end{cases}$$

Note that for the cases where $E_{ij} = \frac{1}{2}$, we have used parentheses for logical grouping. First evaluate the expressions inside the parentheses, and then apply the word and to combine the conditions logically.

6.4. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Answer the following questions by hand calculation.

Part a: What is the orthogonal projector P onto range(A), and what is the image under P of the vector $(1,2,3)^*$?

$$P = A(A^*A)^{-1}A^*$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}^*$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Find the image under P of the vector $(1,2,3)^*$:

$$P\begin{bmatrix}1\\2\\3\end{bmatrix} =$$

$$\begin{bmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 1 & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} =$$

$$\begin{bmatrix}\frac{1}{2}(1)+2(0)+3(\frac{1}{2})\\ 0(1)+2(1)+3(0)\\ \frac{1}{2}(1)+2(0)+3(\frac{1}{2})\end{bmatrix} =$$

$$\begin{bmatrix}2\\2\\2\end{bmatrix}$$

Image under P of the vector $(1,2,3)^*$: $\begin{bmatrix}2\\2\\2\end{bmatrix}$

Part b: Same questions for B

What is the orthogonal projector P onto range(B), and what is the image under P of the vector $(1,2,3)^*$?

$$P = B(B^*B)^{-1}B^*$$

$$P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}^*$$

$$P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\frac{1}{6}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}\right) \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{5}{6} & -\frac{2}{6} \\ -\frac{2}{6} & \frac{2}{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{5}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \\ \frac{5}{6} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{bmatrix}$$

Find the image under P of the vector $(1,2,3)^*$:

$$P\begin{bmatrix}1\\2\\3\end{bmatrix} =$$

$$\begin{bmatrix}\frac{5}{6} & \frac{1}{3} & \frac{1}{6}\\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}\\ \frac{1}{6} & -\frac{1}{3} & \frac{5}{6}\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} =$$

$$\begin{bmatrix}\frac{5}{6}(1)+2\left(\frac{1}{3}\right)+3\left(\frac{1}{6}\right)\\ \frac{1}{3}(1)+2\left(\frac{1}{3}\right)+3\left(-\frac{1}{3}\right)\\ \frac{1}{6}(1)+2\left(-\frac{1}{3}\right)+3\left(\frac{5}{6}\right)\end{bmatrix} =$$

$$\begin{bmatrix}\frac{5}{6}+\frac{2}{3}+\frac{1}{2}\\ \frac{1}{3}+\frac{2}{3}-1\\ \frac{1}{6}-\frac{2}{3}+\frac{15}{6}\end{bmatrix} =$$

$$\begin{bmatrix} \frac{5}{6} + \frac{4}{6} + \frac{3}{6} \\ 1 - 1 \\ \frac{16}{6} - \frac{4}{6} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

Image under P of the vector $(1, 2, 3)^*$:

$$\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

7.1. Consider again the matrices A and B of Exercise 6.4.

For clarity purposes, I'll restate matrices A and B:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Reduced QR Factorization for A with $\hat{Q}$ and $\hat{R}$:

1. Define the columns of A:

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

2. Compute q_i:

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\sqrt{1^2+0^2+1^2}} = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

3. Compute q_2:

$$\text{proj}_{q_1} a_2 = \left(\frac{q_1^* a_2}{q_1^* q_1}\right) q_1 = \frac{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{1} q_1 = 0 q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

4. Construct $\hat{Q}$: $\hat{Q} = [q_1 \ q_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

5. Compute $\hat{R}$:

$$\hat{R} = \hat{Q}^* A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$$

6. Giving the Reduced QR Factorization for A:

$$A = \hat{Q}\hat{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

Full QR Factorization for A with Q and R:

1. Construct the third orthonormal vector q_3: $q_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

2. This gives the full Q Matrix:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

3. Extend $\hat{R}$ to form R by adding a row of zeros:

$$R = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

4. Full QR Factorization:

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

1. Define the columns of B:

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

2. Compute q_1:

$$q_1 = \frac{b_1}{\|b_1\|} = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\sqrt{1^2+0^2+1^2}} = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\sqrt{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

3. Compute q_2:

$$\text{proj}_{q_1} b_2 = \left(\frac{q_1^* b_2}{q_1^* q_1}\right) q_1 = \frac{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}{1} q_1 =$$

$$\frac{2\cdot\frac{1}{\sqrt{2}}+0\cdot1+1\cdot0}{1} q_1 = \frac{\frac{2}{\sqrt{2}}}{1} q_1 = \sqrt{2}\, q_1 = \sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$u_2 = b_2 - \text{proj}_{q_1} b_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$q_2 = \frac{u_2}{\|u_2\|} = \frac{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}{\sqrt{1^2+1^2+(-1)^2}} = \frac{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}{\sqrt{3}} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

4. Construct $\hat{Q}$: $\hat{Q} = [q_1 \ \ q_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$

5. Compute $\hat{R}$:

$$\hat{R} = \hat{Q}^* B = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \sqrt{2} & \frac{2}{\sqrt{2}} \\ 0 & \sqrt{3} \end{bmatrix} =$$

$$\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

6. Giving the Reduced QR Factorization for B:

$$B = \hat{Q}\hat{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

Full QR Factorization for B with Q and R:

1. Construct the third orthonormal vector q_3: $q_3 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$

2. This gives the full Q Matrix:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

3. Extend $\hat{R}$ to form R by adding a row of zeros:

$$R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

4. Full QR Factorization:

$$B = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

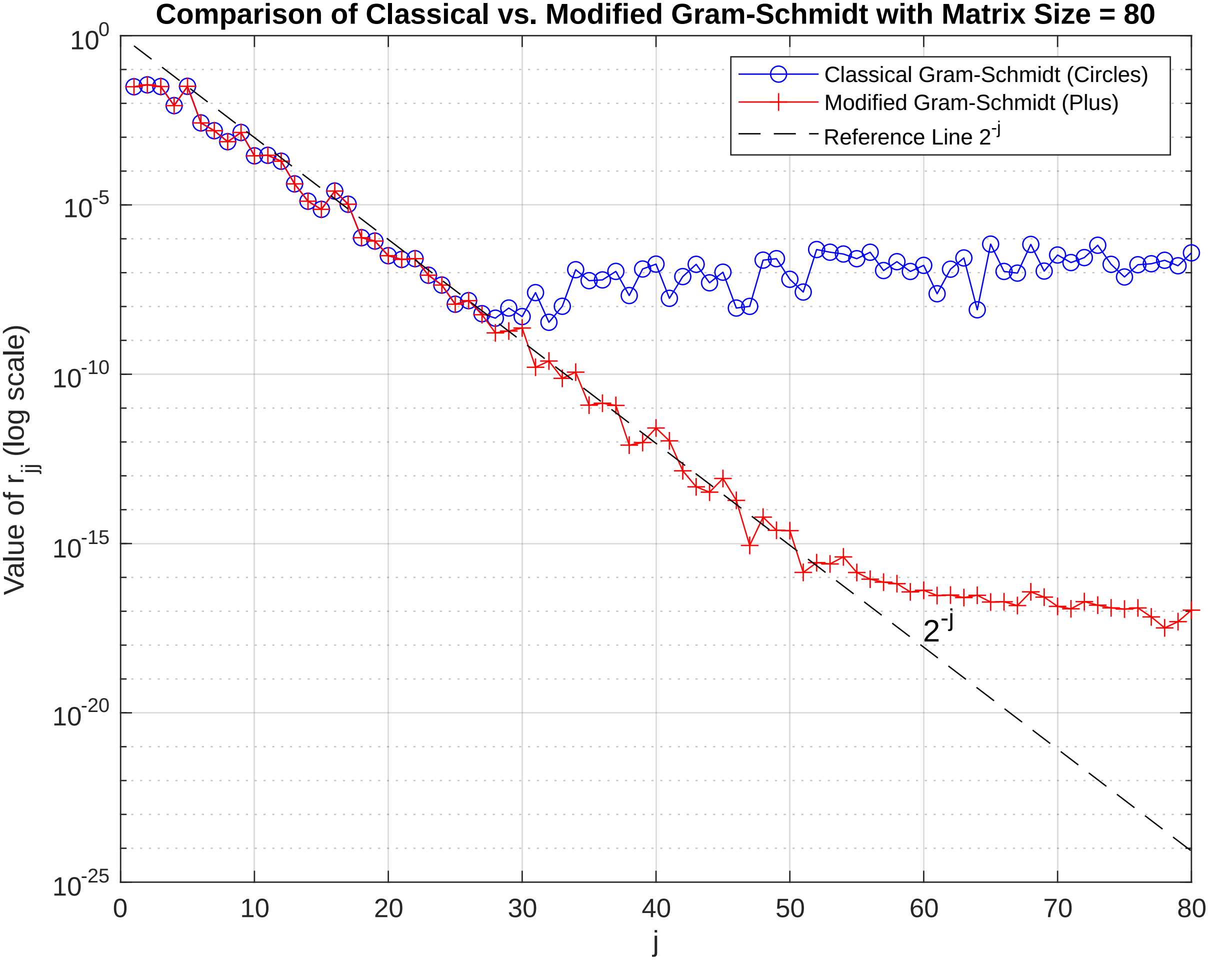
```
function gram_schmidt_experiment(matrix_size)
   if nargin < 1
       matrix_size = 80;
   end
   rng(42); % Set random seed for reproducibility
   % Exact MATLAB-style initialization of U, V, S, and A
   [U, ~] = qr(randn(matrix_size));  % U initialized as a random orthogonal
matrix
   [V, ~] = qr(randn(matrix_size));  % V initialized as a random orthogonal
matrix
   S = diag(2.^(-1:-1:-matrix_size));  % Create diagonal matrix with entries
2^-1 to 2^-matrix_size
   A = U * S * V';  % Construct matrix A
   % Apply Classical and Modified Gram-Schmidt
   [Q_classical, R_classical] = classical_gram_schmidt(A);
   [Q_modified, R_modified] = modified_gram_schmidt(A);
   % Extract the diagonal elements from R matrices
   diagonal_classical = diag(R_classical);
   diagonal_modified = diag(R_modified);
   % Create a reference line for 2^-j
   j_values = 1:matrix_size;
   reference_2_pow_neg_j = 2.^(-j_values);
   % Plot the diagonal elements and the reference line on a logarithmic scale
   figure;
   plot(j_values, diagonal_classical, 'bo-', 'DisplayName', 'Classical
Gram-Schmidt (Circles)');
   hold on;
   plot(j_values, diagonal_modified, 'r+-', 'DisplayName', 'Modified
Gram-Schmidt (Plus)');
   plot(j_values, reference_2_pow_neg_j, 'k--', 'DisplayName', 'Reference Line
2^{-j}');
   % Add annotation for the reference line
   text(matrix_size * 0.75, 2^(-floor(matrix_size * 0.75)), '2^{-j}',
'FontSize', 12, 'VerticalAlignment', 'bottom', 'Color', 'black');
   % Configure the y-axis
   set(gca, 'YScale', 'log');
   ylim([10^-25, 10^0]);
   % Configure labels and title
   xlabel('j');
   ylabel('Value of r_{jj} (log scale)');
   title(sprintf('Comparison of Classical vs. Modified Gram-Schmidt with Matrix
Size = %d', matrix_size));
   % Add grid, legend, and show plot
   legend;
   grid on;
end
% Function for Classical Gram-Schmidt
function [Q, R] = classical_gram_schmidt(A)
   [m, n] = size(A);
```

```
    Q = zeros(m, n);
    R = zeros(n, n);

    for j = 1:n
        vj = A(:, j);
        for i = 1:j-1
            R(i, j) = Q(:, i)' * A(:, j);
            vj = vj - R(i, j) * Q(:, i);
        end
        R(j, j) = norm(vj);
        Q(:, j) = vj / R(j, j);
    end
end
% Function for Modified Gram-Schmidt
function [Q, R] = modified_gram_schmidt(A)
    [m, n] = size(A);
    Q = zeros(m, n);
    R = zeros(n, n);
    v = zeros(m, n);

    for i = 1:n
        v(:, i) = A(:, i);
    end

    for i = 1:n
        R(i, i) = norm(v(:, i));
        Q(:, i) = v(:, i) / R(i, i);
        for j = i+1:n
            R(i, j) = Q(:, i)' * v(:, j);
            v(:, j) = v(:, j) - R(i, j) * Q(:, i);
        end
    end
end
```


Comparison of Classical vs. Modified Gram-Schmidt with Matrix Size = 80
Classical Gram-Schmidt (Circles)
Modified Gram-Schmidt (Plus)
Reference Line 2^{-j}
2^{-j}
Value of r_{jj} (log scale)
j
10^{0}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
0
10
20
30
40
50
60
70
80