AMATH 501: Homework 2
Sid Meka

1. Show that vector field
$$\mathbf{F} = (\sin y + z)\mathbf{i} + (x\cos y - z)\mathbf{j} + (x - y)\mathbf{k}$$
is conservative and compute the value of its line integral from the origin to the point $(3, \pi/6, 2)$ along any path.
In order to figure this out, we check if the equations:

   - $\frac{\partial}{\partial x}(x\cos y - z) = \frac{\partial}{\partial y}(\sin y + z)$
   - $\frac{\partial}{\partial y}(x - y) = \frac{\partial}{\partial z}(x\cos y - z)$
   - $\frac{\partial}{\partial z}(\sin y + z) = \frac{\partial}{\partial x}(x - y)$

   hold true.

   - From $\frac{\partial}{\partial x}(x\cos y - z) = \frac{\partial}{\partial y}(\sin y + z)$, we find that $\cos y = \cos y$.
   - From $\frac{\partial}{\partial y}(x - y) = \frac{\partial}{\partial z}(x\cos y - z)$, we find that $-1 = -1$.
   - From $\frac{\partial}{\partial z}(\sin y + z) = \frac{\partial}{\partial x}(x - y)$, we find that $1 = 1$.

   Thus, all the three equations we were checking hold true meaning that $\mathbf{F}$ is a conservative vector field.

   Since $\mathbf{F}$ is a conservative vector field, we know that $\int_C F \cdot dr$ is path independent. Thus, we can say $\int_C \mathbf{F} \cdot dr = f(3, \frac{\pi}{6}, 2) - f(0, 0, 0)$. We first do need to figure out our potential function $f(x, y, z)$. Remember that $\mathbf{F} = \nabla f$, where $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$. So, that means
$\mathbf{F} = (\sin y + z)\mathbf{i} + (x\cos y - z)\mathbf{j} + (x - y)\mathbf{k}$ is the same thing as $\mathbf{F} = \nabla(x\sin y + xz - yz)$.
So, that means we have found $f(x, y, z) = x\sin y + xz - yz$.
Putting this all together, we need to find
$$f\left(3, \frac{\pi}{6}, 2\right) - f(0, 0, 0) =$$
$$\left(3\sin\left(\frac{\pi}{6}\right) + 3 \cdot 2 - \frac{\pi}{6} \cdot 2\right) - (0\sin 0 + (0)(0) - 0 \cdot 0) =$$
$$\frac{3}{2} + 6 - \frac{\pi}{3} =$$
$$\frac{15}{2} - \frac{\pi}{3}$$
Thus, the value of our desired line integral is $\frac{15}{2} - \frac{\pi}{3}$.

2. Consider the paraboloid $z = x^2 + y^2$. Find the surface area of the portion of this paraboloid that lies between the planes $z = 0$ and $z = 9$.

$$\text{Let's say } f(x,y) = x^2 + y^2$$

$$S = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA \text{ over some region } R$$

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

$$S = \iint_R \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

$$S = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dA$$

$$S = \iint_R \sqrt{1 + 4(x^2 + y^2)}\, dA$$

Remember that $z = x^2 + y^2$. So,

$$S = \iint_R \sqrt{1 + 4z}\, dA$$

Now, we have reached a point where we will convert to polar coordinates in order to finish this problem. We will set $x = r\cos\theta$ and $y = r\sin\theta$. We also have $z = r^2$ as $r^2 = x^2 + y^2$ and $z = x^2 + y^2$ and $dA = r\,dr\,d\theta$.
We have our bounds as $0 \leq r \leq 3$ as $0 \leq z \leq 9$ and θ ranges from 0 to 2π.
That leaves us with the integral:

$$S = \iint_R \sqrt{1 + 4z}\, dA = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

So, now we solve for:

$$\int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2}\, r\, dr\, d\theta =$$

$$2\pi \int_0^3 \sqrt{1 + 4r^2}\, r\, dr$$

We will now use u substitution.
We will set $u = 1 + 4r^2$.
That makes $du = 8r\,dr$ making $r\,dr = \frac{1}{8}du$.
When $r = 0$, $u = 1$.
When $r = 3$, $u = 37$.
That makes

$$2\pi \int_0^3 \sqrt{1 + 4r^2}\, r\, dr = 2\pi \int_1^{37} \frac{1}{8}\sqrt{u}\, du$$

So, we will continue by evaluating:

$$2\pi \int_1^{37} \frac{1}{8}\sqrt{u}\, du =$$

$$\frac{\pi}{4} \int_1^{37} \sqrt{u}\, du =$$

$$\frac{\pi}{4}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_{1}^{37} =$$

$$\frac{\pi}{4}\left(\frac{2}{3}(37)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}\right) =$$

$$\frac{2\pi}{12}((37)^{\frac{3}{2}} - 1) =$$

$$\frac{\pi}{6}(37^{\frac{3}{2}} - 1)$$

Thus, the surface area of the portion of this paraboloid that lies between the planes $z = 0$ and $z = 9$ is $\frac{\pi}{6}(37^{\frac{3}{2}} - 1)$.

3.

(a) Find the flux of the field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + (x^2 + y^2)\mathbf{k}$ outwards across the outer surface of the hemisphere

$$z = \sqrt{1 - x^2 - y^2}\ , \quad z \geq 0\ .$$

To find the flux of the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + (x^2 + y^2)\mathbf{k}$ across the outer surface of the hemisphere $z = \sqrt{1 - x^2 - y^2}$, $z \geq 0$, we can apply the Divergence Theorem, which relates the flux of a vector field across a closed surface to the volume integral of the divergence of the field inside the region. The divergence of $\mathbf{F}$ is given by

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

For $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + (x^2 + y^2)\mathbf{k}$, we calculate each partial derivative:

$$\begin{aligned}
\frac{\partial F_x}{\partial x} &= \frac{\partial y}{\partial x} = 0, \\
\frac{\partial F_y}{\partial y} &= \frac{\partial(-x)}{\partial y} = 0, \\
\frac{\partial F_z}{\partial z} &= \frac{\partial(x^2 + y^2)}{\partial z} = 0.
\end{aligned}$$

Thus, the divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = 0 + 0 + 0 = 0.$$

Since $\nabla \cdot \mathbf{F} = 0$ throughout the volume of the hemisphere, the Divergence Theorem implies that the flux through the closed surface, which includes the hemisphere and the flat circular base, is zero:

$$\iint_S \mathbf{F} \cdot d\mathbf{A} = 0.$$

Here, S is the closed surface consisting of the hemisphere S_1 (the outer curved surface) and the disk S_2 (the flat base in the xy plane). Thus, we have

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{A} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{A} = 0,$$

which we can rearrange to solve for the flux through the hemisphere:

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{A} = -\iint_{S_2} \mathbf{F} \cdot d\mathbf{A}.$$

On the disk S_2, located in the xy-plane where $z = 0$, the normal vector $d\mathbf{A}$ points in the $-\mathbf{k}$ direction. So, $d\mathbf{A} = -dA\,\mathbf{k}$, where dA is the area element in the xy plane.
The field $\mathbf{F}$ on S_2 is given by

$$\mathbf{F} = y\mathbf{i} - x\mathbf{j} + (x^2 + y^2)\mathbf{k}.$$

Thus,

$$\mathbf{F} \cdot d\mathbf{A} = \mathbf{F} \cdot (-\mathbf{k})\, dA = -(x^2 + y^2)\, dA.$$

The flux through S_2 is therefore

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{A} = -\iint_{S_2} (x^2 + y^2)\, dA.$$

In polar coordinates, where $x = r\cos\theta$ and $y = r\sin\theta$, we have $x^2 + y^2 = r^2$ and $dA = r\,dr\,d\theta$. The limits for r are 0 to 1, and for θ are 0 to 2π. Thus,

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{A} = -\int_0^{2\pi}\int_0^1 r^2 \cdot r\,dr\,d\theta = -\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta.$$

Evaluating the inner integral:

$$\int_0^1 r^3\,dr = \left[\frac{r^4}{4}\right]_0^1 = \frac{1}{4}.$$

Now, integrating with respect to θ:

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{A} = -\int_0^{2\pi} \frac{1}{4}\,d\theta = -\frac{1}{4} \cdot 2\pi = -\frac{\pi}{2}.$$

The flux of $\mathbf{F}$ outwards across the outer surface of the hemisphere S_1 is

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{A} = -\iint_{S_2} \mathbf{F} \cdot d\mathbf{A} = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

(b) Find the flux of field $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the surface of the cylinder $x^2 + y^2 = a^2$ bounded by $z = 0$ and $z = b$, where $a, b > 0$ are given constants. Then compute the divergence of $\mathbf{F}_1$ and use the divergence theorem to compute the flux again.

We will find the Divergence of $\mathbf{F}_1$.

$$\nabla \cdot \mathbf{F}_1 = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$
$$\nabla \cdot \mathbf{F}_1 = 1 + 1 + 1$$
$$\nabla \cdot \mathbf{F}_1 = 3$$

We now set up the Divergence Theorem:

$$\iint_{\partial V} \mathbf{F}_1 \cdot d\mathbf{A} = \iiint_V (\nabla \cdot \mathbf{F}_1)\,dV$$

$$\iint_{\partial V} \mathbf{F}_1 \cdot d\mathbf{A} = 3V \text{ , where } V \text{ is the volume of the Cylinder}$$

We have that $V = \pi a^2 b$ from the definition of volume of a Cylinder. Thus, we have that

$$\iint_{\partial V} \mathbf{F}_1 \cdot d\mathbf{A} = 3\pi a^2 b$$