AMATH 584: Numerical Linear Algebra: Exercise 26.1
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Given what this exercise says, Show that the following conditions:

1. z is an eigenvalue of $A + \delta A$ for some δA with $||\delta A||_2 \leq \epsilon$.
2. There exists a vector $u \in \mathbb{C}^m$ with $||(A - zI)u||_2 \leq \epsilon$ and $||u||_2 = 1$.
3. $\sigma_m(zI - A) \leq \epsilon$ where σ_m denotes the smallest singular value.
4. $||(zI - A)^{-1}||_2 \geq \epsilon^{-1}$ where $(zI - A)^{-1}$ is known as the resolvent of A at z.

1. Statement 1 implies Statement 2:
   Suppose z is an eigenvalue of $A+\delta A$ for some δA with $||\delta A|| \leq \epsilon$. Let u be a corresponding eigenvector with $||u|| = 1$. Then, $(A+\delta a)u = zu$. That gives us $Au - zu = (A - zI)u$. Note that $(A - zI) = -\delta Au$. Taking the norm of each side, we have:

   $$||(A - zI)u||_2 = || - \delta Au|| \leq ||\delta A||\,||u|| \leq \epsilon$$

   This proves that Statement 1 implies Statement 2.

2. Statement 2 implies Statement 3:
   We first prove that

   $$\sigma_m(A) = \min_{||u||_2=1} ||Au||_2$$

   Let $A = U\Sigma V^*$, $v = V^*u$ with $||u||_2 = ||v||_2 = 1$. Then, we have the equations:

   $$||Au||_2 = ||\Sigma v||_2$$

   $$||\Sigma v||_2^2 = \sum_{i=1}^{m} \sigma_i^2 v_i^2 \quad \text{, where } \sum_{i=1}^{m} v_i^2 = 1$$

   To minimize $||\Sigma v||_2$, set $v_m = 1$, $v_i = 0$ for $i \neq m$. Thus, we have:

   $$\min_{||u||_2=1} ||Au||_2 = \sigma_m$$

   This means that:

   $$\sigma_m(A - zI) = \min_{||u||_2=1} ||(A - zI)u||_2$$

   So, if there exists a vector $u \in \mathbb{C}^m$ with $||(A - zI)u||_2 \leq \epsilon$ and $||u||_2 = 1$, then it follows that

   $$\sigma_m(A - zI) = \min_{||u||_2=1} ||(A - zI)u||_2 \leq ||(A - zI)u||_2 \leq \epsilon$$

   Thus we have shown that if such a vector u exists, then $\sigma_m(A - zI) \leq \epsilon$.
   This proves that Statement 2 implies Statement 3.

3. Statement 3 implies Statement 4:
   Recall the largest singular value of a matrix is the reciprocal of the smallest singular value of its inverse. We then have:

   $$||(zI - A)^{-1}||_2 = \sigma_{\max}((zI - A)^{-1}) = \sigma_{\min}(zI - A)^{-1} \geq \epsilon^{-1}$$

   This proves that Statement 3 implies Statement 4.

4. Statement 4 implies Statement 1:
   Let u, v be the first left and right singular vectors of $(zI-A)^{-1}$ with singular value $\sigma^{-1} = ||(zI-A)^{-1}||$. Then, $||u|| = 1$ and $||v|| = 1$ meaning that $||u|| = ||v||$ and $\sigma \leq \epsilon$. Furthermore, by definition, $(zI - A)^{-1}v = \sigma^{-1}u$. Rearranging, we have $\sigma v = (zI - A)u$ so

   $$(A - \sigma vu^*)u = Au - \sigma vu^*u = Au - \sigma v = zu$$

   This means z is an eigenvector of $A - \sigma vu^*$ and $||ovu^*|| = \sigma\,||v||\,||u^*|| = \sigma \leq \epsilon$.
   This proves that Statement 4 implies Statement 1.

Thus, we have that our four statements are equivalent.

AMATH 584: Numerical Linear Algebra: Exercise 27.1
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Let $A \in \mathbb{C}^{m\times m}$ be given, not necessarily hermitian. Show that a number $z \in \mathbb{C}$ is a Rayleigh quotient of A if and only if it is a diagonal entry of Q^*AQ for some unitary matrix Q. Thus Rayleigh quotients are just diagonal entries of matrices, once you transform orthogonally to the right coordinate system.

To prove that a number $z \in \mathbb{C}$ is a Rayleigh quotient of A if and only if it is a diagonal entry of Q^*AQ for some unitary matrix Q, we will use the Proof of the Biconditional.

First, we will prove that if a number $z \in \mathbb{C}$ is a Rayleigh quotient of A, then it is a diagonal entry of Q^*AQ for some unitary matrix Q.

Proof. Let $A \in \mathbb{C}^{m\times m}$ be given, not necessarily hermitian. Suppose $z \in \mathbb{C}$ is a Rayleigh quotient of A. By definition, there exists a nonzero vector $x \in \mathbb{C}^m$ such that:

$$z = \frac{x^*Ax}{x^*x}$$

1. Without loss of generality, we can assume $||x||_2 = 1$. In other words, we normalize x to a unit vector. Then, the Rayleigh quotient simplifies to: $z = x^*Ax$.
2. Now, extend x to an orthonormal basis of $\mathbb{C}^m$. Let $Q \in \mathbb{C}^{m\times m}$ be a unitary matrix whose first column is x. That is, $Q = [x|q_2|\cdots|q_m]$, where $q_2, \ldots, q_m$ are additional orthonormal vectors completing the basis.
3. Consider the matrix Q^*AQ, which is a unitary transformation of A. The $(1,1)$ entry of Q^*AQ is:

$$(Q^*AQ)_{(1,1)} = x^*Ax = z$$

Thus, we have shown that z is a diagonal entry of Q^*AQ for some unitary matrix Q. $\square$

Second, we will prove that if $z \in \mathbb{C}$ is a diagonal entry of Q^*AQ for some unitary matrix Q, then z is a Rayleigh quotient of A.

Proof. Suppose z is a diagonal entry of Q^*AQ for some unitary matrix Q.

1. Let $Q \in \mathbb{C}^{m\times m}$ be unitary, and suppose $(Q^*AQ)_{ii} = z$ for some $i \in \{1, 2, \ldots, m\}$.
2. Let e_i be the ith standard basis vector in $\mathbb{C}^m$. Then, it follows:

$$z = e_i^*(Q^*AQ)e_i = (Qe_i)^*A(Qe_i)$$

3. Define $x = Qe_i$, which is a unit vector since Q is unitary. Furthermore, we have $||x||_2 = ||Qe_i||_2 = 1$. Thus, if follows:

$$z = x^*Ax = \frac{x^*Ax}{x^*x}$$

   showing that z is a Rayleigh quotient of A.

$\square$

We have shown that if a number $z \in \mathbb{C}$ is a Rayleigh quotient of A, then it is a diagonal entry of Q^*AQ for some unitary matrix Q and if $z \in \mathbb{C}$ is a diagonal entry of Q^*AQ for some unitary matrix Q, then z is a Rayleigh quotient of A. Thus, we have shown that a number $z \in \mathbb{C}$ is a Rayleigh quotient of A if and only if it is a diagonal entry of Q^*AQ for some unitary matrix Q.

AMATH 584: Numerical Linear Algebra: Homework 7 A1
Sid Meka

Prove the *Bauer-Fike theorem*: suppose $A \in \mathbb{C}^m \times \mathbb{C}^m$ is diagonalizable with $A = V\Lambda V^{-1}$, and let $\delta A \in \mathbb{C}^m \times \mathbb{C}^m$ be arbitrary. If $\tilde{\lambda}_j$ is an eigenvalue of $A + \delta A$, show that there exists an eigenvalue λ_j of A such that

$$|\lambda_j - \tilde{\lambda}_j| \leq \kappa_2(V)||\delta A||_2$$

where $\kappa_2(V) = ||V||_2\,||V^{-1}||_2$ is the condition number of matrix V. In particular, when A is normal, show that

$$|\lambda_j - \tilde{\lambda}_j| \leq ||\delta A||_2$$

This theorem implies that the problem of computing eigenvalues of a normal matrix is well conditioned.

Proof. We can suppose $\tilde{\lambda}_j \notin \Lambda(A)$, otherwise take $\lambda_j = \tilde{\lambda}_j$ and the result in trivially true since $\kappa_2(V) \geq 1$. Since $\tilde{\lambda}_j$ is an eigenvalue of $A + \delta A$, we have $\det(A + \delta A - \tilde{\lambda}_j I) = 0$ and so

$$\begin{aligned}
0 &= \det(A + \delta A - \tilde{\lambda}_j I) \\
&= \det(V^{-1})\det(A + \delta A - \tilde{\lambda}_j I)\det(V) \\
&= \det(V^{-1}(A + \delta A - \tilde{\lambda}_j I)V) \\
&= \det(V^{-1}AV + V^{-1}\delta AV - V^{-1}\tilde{\lambda}_j IV) \\
&= \det(\Lambda + V^{-1}\delta AV - \tilde{\lambda}_j I) \\
&= \det((\Lambda - \tilde{\lambda}_j I) + V^{-1}\delta AV) \\
&= \det((\Lambda - \tilde{\lambda}_j I)((\Lambda - \tilde{\lambda}_j I)^{-1}V^{-1}\delta AV + I)) \\
&= \det(\Lambda - \tilde{\lambda}_j I)\det((\Lambda - \tilde{\lambda}_j I)^{-1}V^{-1}\delta AV + I)
\end{aligned}$$

Notice how we supposed that $\tilde{\lambda}_j \notin \Lambda(A)$. Therefore, we can say that $\det(\Lambda - \tilde{\lambda}_j I) \neq 0$ by definition. Thus, we can say:

$$\det((\Lambda - \tilde{\lambda}_j I)^{-1}V^{-1}\delta AV + I) = 0$$

We have -1 as an eigenvalue of $(\Lambda - \tilde{\lambda}_j I)^{-1}V^{-1}\delta AV$.
We know that $|\lambda_j| \leq ||A||_2$ where λ_j is an eigenvalue of A. In this instance, this gives us:

$$\begin{aligned}
&1 \\
&= |-1| \\
&\leq ||(\Lambda - \tilde{\lambda}_j I)^{-1}V^{-1}\delta AV||_2 \\
&\leq ||(\Lambda - \tilde{\lambda}_j I)^{-1}||_2\,||V^{-1}||_2\,||V||_2\,||\delta A||_2 \\
&= ||(\Lambda - \tilde{\lambda}_j I)^{-1}||_2\,\kappa_2(V)\,||\delta A||_2
\end{aligned}$$

Now, we find the 2-norm of $(\Lambda - \tilde{\lambda}_j I)^{-1}$: By definition, the induced 2-norm is:

$$||(\Lambda - \tilde{\lambda}_j I)^{-1}||_2 = \max_{||x||_2 \neq 0} \frac{||(\Lambda - \tilde{\lambda}_j I)^{-1}x||_2}{||x||_2}$$

Since Λ is diagonal with entries λ_j, the matrix $(\Lambda - \tilde{\lambda}_j I)$ is also diagonal with entries $\lambda_j - \tilde{\lambda}_j$. For a diagonal matrix, the 2-norm is the maximum of the reciprocals of its diagonal entries:

$$||(\Lambda - \tilde{\lambda}_j I)^{-1}||_2 = \max_{\lambda_j \in \Lambda(A)} \frac{1}{|\lambda_j - \tilde{\lambda}_j|}$$

This can also be written as the reciprocal of the minimum distance:

$$||(\Lambda - \tilde{\lambda}_j I)^{-1}||_2 = \frac{1}{\min\limits_{\lambda_j \in \Lambda(A)} |\lambda_j - \tilde{\lambda}_j|}$$

This means that we have:

$$\min_{\lambda_j \in \Lambda(A)} |\lambda_j - \tilde{\lambda}_j| \leq \kappa_2(V)\, ||\delta A||_2$$

Therefore, there exists an eigenvalue λ_j of A such that

$$|\lambda_j - \tilde{\lambda}_j| \leq \kappa_2(V)\, ||\delta A||_2$$

$\square$

When A is normal: Now if A is normal, that means V is unitary giving us $\kappa_2(V) = 1$. Note that $|\lambda_j - \tilde{\lambda}_j| \leq \kappa_2(V)\, ||\delta A||_2$ and with $\kappa_2(V) = 1$, that means $|\lambda_j - \tilde{\lambda}_j| \leq ||\delta A||_2$. Thus, when A is normal, $|\lambda_j - \tilde{\lambda}_j| \leq ||\delta A||_2$.

A2. Consider the two matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_\varepsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varepsilon & 0 & 0 & 0 \end{bmatrix}$$

where ε is a small real number. What are the eigenvalues of A and A_ε? Would you say that the problem of computing the eigenvalues of the matrix A is well conditioned or ill conditioned?

Notice that matrix A is an upper triangular matrix with 0's on the diagonal. This means the eigenvalues of A are as follows:

$\lambda = 0, 0, 0, 0.$

The matrix A_ε is:

$$A_\varepsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varepsilon & 0 & 0 & 0 \end{bmatrix}$$

To find the eigenvalues, we solve $\det(A_\varepsilon - \lambda I) = 0$

Thus:

$$\det \begin{bmatrix} 0-\lambda & 1 & 0 & 0 \\ 0 & 0-\lambda & 1 & 0 \\ 0 & 0 & 0-\lambda & 1 \\ \varepsilon & 0 & 0 & 0-\lambda \end{bmatrix} = 0$$

More simply:

$$\det \begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ \varepsilon & 0 & 0 & -\lambda \end{bmatrix} = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ \varepsilon & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 & 0 \\ 0 & -\lambda & 1 \\ \varepsilon & 0 & -\lambda \end{vmatrix} + 0 - 0 = 0$$

$$(-\lambda)^4 - \left(0 - 1 \begin{vmatrix} 0 & 1 \\ \varepsilon & -\lambda \end{vmatrix} + 0\right) = 0$$

$$\lambda^4 - \varepsilon = 0$$

$$\lambda^4 = \varepsilon$$

$$\lambda^2 = \sqrt{\varepsilon} \text{ or } \lambda^2 = \sqrt{-\varepsilon}$$

$$\lambda = \sqrt[4]{\varepsilon} \text{ or } \lambda = -\sqrt[4]{\varepsilon} \text{ or } \lambda = i\sqrt[4]{\varepsilon} \text{ or } \lambda = -i\sqrt[4]{\varepsilon}$$

Eigenvalues of A_ε: $\sqrt[4]{\varepsilon}$, $-\sqrt[4]{\varepsilon}$, $i\sqrt[4]{\varepsilon}$, $-i\sqrt[4]{\varepsilon}$

Assessing the conditioning of the eigenvalue problem:

$$\hat{\kappa} = \sup_{\delta x} \frac{\|\delta f\|}{\|\delta x\|} = \left|\frac{\delta \lambda}{\epsilon}\right| = \frac{\epsilon^{\frac{1}{4}}}{\epsilon} = \epsilon^{-\frac{3}{4}}$$

Now to find $\hat{\kappa}_{lim}$, when ϵ approaches 0:

$$\hat{\kappa}_{lim} =$$

$$\lim_{\epsilon \to 0^+} \epsilon^{-\frac{3}{4}} =$$

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^{\frac{3}{4}}} =$$

$$\infty$$

We see that when ϵ approaches 0, $\hat{\kappa}_{lim}$ approaches ∞. This suggests that there are large changes when small perturbations in A occur. Thus, the problem of computing the eigenvalues of the matrix A is ill conditioned.

A3. Find the Householder reflector Q and an upper Hessenberg matrix H by hand such that $Q^* AQ = H$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

we find a vector, or subvector, $a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We now find $\|a_1\|_2 e_1 = \sqrt{1^2 + 1^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The Householder vector is computed as:

$$v = a_1 - \|a_1\|_2 e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - \sqrt{2} \\ 1 \end{bmatrix}.$$

The Householder reflector is:

$$\hat{Q}_1 = I - \frac{2vv^*}{v^* v} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Extend it to a 3x3 matrix to get Q:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Now, we apply to A:

$$Q^*A = \begin{bmatrix} 1 & 2 & 3 \\ \sqrt{2} & \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Rembember that $H = Q^*AQ$. Thus,

$$H = \begin{bmatrix} 1 & \frac{5}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \sqrt{2} & \frac{5}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

We have found our Q and H as desired.