AMATH 501 Homework 3 Sid Meka

1. Express in Polar Exponential Form, Find $|z|$, and Arg z:

$-2i$: Modulus: $|z| = |-2i| = 2$

$\text{Arg}(z) = \frac{\pi}{4} + 2\pi n, \quad n \in \mathbb{Z}$

Polar Form: $2e^{-\frac{i\pi}{2}}$

$2+3i$: Modulus: $|2+3i| = \sqrt{2^2+3^2} = \sqrt{13}$

$\text{Arg}(z) = \tan^{-1}\left(\frac{3}{2}\right) + 2\pi n, \quad n \in \mathbb{Z}$

Polar Form: $\sqrt{13}\, e^{i\tan^{-1}\left(\frac{3}{2}\right)}$

$\frac{1}{2} + \frac{\sqrt{3}}{2}i$: Modulus: $\left|\frac{1}{2} + \frac{\sqrt{3}}{2}i\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

$\text{Arg}(z) = \frac{\pi}{3} + 2\pi n, \quad n \in \mathbb{Z}$

Polar Form: $e^{\frac{\pi i}{3}}$

$(1+i)^{50}$: Modulus: $|(1+i)^{50}| = \left(\sqrt{1+1}\right)^{50} = \left(\sqrt{2}\right)^{50} = 2^{25}$

$\text{Arg}(z) = \frac{\pi}{2} + 2\pi n, \quad n \in \mathbb{Z}$

Polar Form: $(1+i)^{50} = \left(\sqrt{2}\, e^{\frac{i\pi}{4}}\right)^{50} = \sqrt{2}\, e^{\frac{i\pi}{2}}$

2. Solve for Roots:

a. $z^4+16=0$

$z^4=-16$

$z^2=\pm 4i$

$z=\pm\sqrt{2}\pm i\sqrt{2}$

$z=\{\sqrt{2}+i\sqrt{2}, -\sqrt{2}+i\sqrt{2}, -\sqrt{2}-i\sqrt{2}, \sqrt{2}-i\sqrt{2}\}$

b. $z^3=4$

Modulus: $\sqrt[3]{4}$

$Arg(z)=\frac{2\pi k}{3}$, $k=\{0,1,2\}$

$z_1=\sqrt[3]{4}$

$z_2=\sqrt[3]{4}e^{\frac{2\pi i}{3}}$ $\quad z_2=\sqrt[3]{4}\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)=-\frac{\sqrt[3]{4}}{2}+i\frac{\sqrt{3}\sqrt[3]{4}}{2}$

$z_3=\sqrt[3]{4}e^{\frac{4\pi i}{3}}$ $\quad z_3=\sqrt[3]{4}\left(\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right)=-\frac{\sqrt[3]{4}}{2}-i\frac{\sqrt{3}\sqrt[3]{4}}{2}$

3. Find the derivative $f'(z)$ where it exists and state where f is analytic.

a. $f = \frac{1}{2}x + y + i(\sin x + \cos y)$

$f(x,y) = u(x,y) + iv(x,y)$

$u(x,y) = \frac{1}{2}x + y$

$v(x,y) = \sin x + \cos y$

Check where $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{1}{2} = -\sin y$ and $1 = -\cos x$

$\cos x = -1$ and $\sin y = -\frac{1}{2}$

$x = \pi + 2\pi k,\ k \in \mathbb{Z}$ and $(y = -\frac{\pi}{6} + 2\pi n$ or $y = -\frac{5\pi}{6} + 2\pi n),\ n \in \mathbb{Z}$

The function, f, is differentiable on the set of points for x and y noted above.

$f' = \frac{1}{2} + i\cos x$

The function, f, is nowhere analytic.

b. $f = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$

$f(x,y) = u(x,y) + iv(x,y)$

$u(x,y) = x^3 + 3xy^2 - 3x$

$v(x,y) = y^3 + 3x^2y - 3y$

Check where $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial}{\partial x}(x^3 + 3xy^2 - 3x) = \frac{\partial}{\partial y}(y^3 + 3x^2y - 3y)$ and

$\frac{\partial}{\partial y}(x^3 + 3xy^2 - 3x) = -\frac{\partial}{\partial x}(y^3 + 3x^2y - 3y)$

$3x^2 + 3y^2 - 3 = 3y^2 + 3x^2 - 3$ and

$6xy = -6xy$

The function, f, is differentiable on $(x,y) = (0,0)$.

$f' = 3x^2 + 3y^2 - 3 + 6xyi$

The function, f, is nowhere analytic.

c. $f = \frac{1}{z^2+3iz-2}$

The function is differentiable and analytic everywhere except when $z^2+3iz-2=0$

$z=-i$ and $z=2i$

The function is differentiable and analytic everywhere except when $z=-i$ and $z=2i$.

$$f'(z) = -\frac{2z+3i}{z^2+3iz-2}$$

4. Evaluate $\int_C f(z)\,dz$ on a unit circle C centered at the origin.

a. $\int_{|z|=1} e^{iz}\,dz = 0$ by Cauchy's Integral Theorem

b. $\int_{|z|=1} \frac{1}{z-\frac{1}{3}}\,dz = 2\pi i \cdot 1 = 2\pi i$ by Cauchy's Integral Theorem.

c. $\int_{|z|=1} \frac{1}{3z^2+1}\,dz = 2\pi i\left(-\frac{i\sqrt{3}}{6} + \frac{i\sqrt{3}}{6}\right) = 0$

Singularities: $z = \pm\frac{i}{\sqrt{3}}$

$\text{Res}\left(\frac{1}{3z^2+1}, \frac{i}{\sqrt{3}}\right) = \lim_{z \to \frac{i}{\sqrt{3}}} \frac{1}{3z^2+1} \cdot \left(z - \frac{i}{\sqrt{3}}\right)$

$\text{Res}\left(\frac{1}{3z^2+1}, \frac{i}{\sqrt{3}}\right) = -\frac{i\sqrt{3}}{6}$

$\text{Res}\left(\frac{1}{3z^2+1}, -\frac{i}{\sqrt{3}}\right) = \lim_{z \to -\frac{i}{\sqrt{3}}} \frac{1}{3z^2+1} \cdot \left(z + \frac{i}{\sqrt{3}}\right)$

$\text{Res}\left(\frac{1}{3z^2+1}, -\frac{i}{\sqrt{3}}\right) = \frac{i\sqrt{3}}{6}$

d. $\int_{|z|=1} \frac{1}{z^2-4}\, dz = 0$ by Cauchy's Integral Theorem