Dt: 24/02/2022

[120BM0014]

①

MATHEMATICAL METHODS

MID SEM

Q1) $u_{xx} + x u_{yy} = 0 \quad x > 0$

$$\Rightarrow \frac{\delta^2 u}{\delta x^2} + x \frac{\delta^2 u}{\delta y^2} = 0$$

$\Rightarrow a = 1$

$b = 0$

$c = x$

$b^2 - ac = 0^2 - 1 \times x = -x$

The characteristic curve when $x > 0$.

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - ac}}{a} = \frac{0 \pm \sqrt{-x}}{1} = -i\sqrt{x} \text{ or } i\sqrt{x}$$

So, $-\frac{dy}{dx} = D_\xi = D_\eta$

Now, $\frac{\xi_x}{\xi_y} = i\sqrt{x}$ $\qquad$ $\frac{\eta_x}{\eta_y} = -i\sqrt{x}$

$\Rightarrow \frac{\delta y}{\delta x} = i\sqrt{x}$ $\qquad$ $\Rightarrow \frac{\delta y}{\delta x} = -i\sqrt{x}$

$\Rightarrow y - ix^{3/2} = c$ $\qquad$ $\Rightarrow y + ix^{3/2} = c$

So, $\xi = y - ix^{3/2}$

$\eta = y + ix^{3/2}$

So, we introduce new variables,

$$\alpha = \frac{\xi + \eta}{2} \quad , \quad \beta = \frac{\xi - \eta}{2i}$$

$$\Rightarrow \frac{y - ix^{3/2} + y + ix^{3/2}}{2} = \frac{y - ix^{3/2} - (y + ix^{3/2})}{2i}$$

$$\Rightarrow y = -x^{3/2} \quad / \text{So, } \alpha = y \quad \beta = -x^{3/2}$$

So, $\frac{\delta^2 u}{\delta \alpha^2} - \frac{\delta^2 u}{\delta \beta^2} = 0$

$$\Rightarrow \boxed{U_{\alpha\alpha} - u_{\beta\beta} = 0} \text{ Reduced canonical form.}$$

Q4) 2π periodic func^n $[-\pi, \pi]$

let $f(x) = \sin x$

so, $f(-x) = \sin(-x)$
$= -\sin(x)$ odd.

$f(x) = \cos x$
$f(-x) = \cos(-x)$
$= \cos x$
even.

So, $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n\pi x \, dx$

$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos n\pi x \, dx$

$= \frac{2}{\pi} \left[f(x) \frac{\sin n\pi x}{n\pi} - f'(x) \left(\frac{-\cos n\pi x}{n^2\pi^2} \right) \right.$

$$F_n(x) = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x+t)\, S_n(t)\,dt + \int_{0}^{\pi} f(x+t)\, S_n(t)\,dt\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0}\left[f(x+t) - f(x-)\right]\frac{\sin((n+1/2)t)}{\sin(t/2)} + \frac{1}{2}f(x-)$$

$$+ \frac{1}{2}\int_{0}^{\pi}\left[f(x+t) - f(x+)\right]\frac{\sin((n+1/2)t)}{\sin(t/2)}\,dt + \frac{1}{2}f(x+)$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\left[g_-(t) + g_+(t)\right]\sin((n+1/2)t)\,dt$$
$$+ \frac{1}{2}f(x-) + \frac{1}{2}f(x+)$$

Q5) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \cancel{0\,dx} + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$a_0 = \pi/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left(\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right)$$

$$= \frac{1}{\pi}\left(\left\{\pi\frac{\cancel{\sin n\pi}^{0}}{n} - 0\right\} - \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2}\left[\cos nx\right]_0^{\pi}\right.$$

$$a_n = \frac{1}{\pi n^2}(\cos n\pi - \cos 0) = \frac{1}{\pi n^2}((-1)^n - 1)$$

$$\Rightarrow a_n = \begin{cases} 0, & n \text{ even} \\ -2/\pi n^2 & n \text{ odd.}\end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right.$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left(\left[-\frac{x\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos nx}{n}\right)dx\right)$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[x\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx \, dx\right)$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}(\pi\cos n\pi - 0) + \frac{1}{n}\left(\frac{\sin nx}{n}\right]_0^{\pi}\right)$$

$$= -\frac{1}{n}(-1)^n + \frac{1}{\pi n^2}\times 0$$

$$= -\frac{1}{n}\times(-1)^n$$

So, $b_n = -\frac{1}{n}$, when even

$+\frac{1}{n}$ when odd.

Now,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$ (decreasing function.)

So, if we pick $x=0$

$$\Rightarrow 0 = \frac{\pi}{4} - \frac{2}{\pi}\left[\cos 0 + \frac{1}{3^2}\cos 0 + \frac{1}{5^2}\cos 0 + \frac{1}{7^2}\cos 0 + \cdots\right]$$

$$+ \sin 0 - \frac{\sin 0}{2} + \frac{\sin 0}{3} \cdots$$

$$\Rightarrow \frac{\pi}{4} = \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

$$\Rightarrow 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \boxed{\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}}$$

Q7 Fourier integral.

$$f(x) = \begin{cases} 0 & x<0 \quad (-\infty < x < 0) \\ e^{-x} & x>0 \quad (0<x<\infty) \\ 1/2 & x=0 \end{cases}$$

So, this is piece wise continuous.

$$f(0) = f(0^+) + f(0^-)$$
$$= \frac{1+0}{2}$$
$$= \frac{1}{2}$$

So, at $x=0$ $f(x) = \frac{1}{2}$

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty A(\omega)\cos(\omega x)\,d\omega + \int_0^\infty B(\omega)\sin(\omega x)\,d\omega\right.$$

$$\therefore A(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt$$

$$= \int_0^\infty e^{-t}\cdot\cos\omega t\,dt$$

$$= \left[e^{-t}\left[\left(\frac{-\cos\omega t + \omega\sin\omega t}{1+\omega^2}\right)\right]\right]_0^\infty$$

$$= \frac{1}{1+\omega^2}\left[-\frac{\cos\omega t}{e^t} + \cancel{\frac{\omega\sin\omega t}{e^t}}\right]_0^\infty$$

$$= \frac{1}{1+\omega^2}$$

$$B(\omega) = \int_{-\infty}^{\infty} f(t) \sin\omega t \, dt$$

$$= \int_{0}^{\infty} e^{-t} \sin\omega t \, dt$$

$$= \left[-e^{-t} \frac{(\sin\omega t - \omega\cos\omega t)}{1+\omega^2} \right]_0^{\infty}$$

$$= \frac{\omega}{1+\omega^2}$$

Hence,

$$f(x) = \frac{1}{\pi} \left[\int_0^{\infty} \frac{1}{1+\omega^2} \cos(\omega x) \, d\omega + \int_0^{\infty} \frac{\omega}{1+\omega^2} \cdot \sin(\omega x) \, d\omega \right]$$

Q8

$$f(x) = e^{-x}, \quad -\pi < x < \pi$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-x} e^{-\frac{in\pi x}{\pi}} \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-(1+in)x} \, dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{-(1+in)x}}{-(1+in)} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(1+in)} \left[e^{-(1+in)\pi} - e^{+(1+in)\pi} \right]$$

$$= \frac{-(1-in)}{2\pi(1+n^2)} \left[e^{-\pi} e^{-in\pi} - e^{\pi} e^{in\pi} \right]$$

Euler's formula

$$e^{in\pi} = \cos n\pi + i \sin n\pi$$
$$= (-1)^n$$
$$e^{-in\pi} = (-1)^n$$

So,

$$c_n = -\frac{(1-in)}{2\pi(1+n^2)}\left[e^{-\pi} - e^{\pi}\right](-1)^n$$
$$= \frac{(1-in)(-1)^n}{(1+n^2)\pi}$$

So

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty}\left(\frac{(1-in)(-1)^n}{(1+n^2)\pi}\sin h\pi\right)e^{inx}}$$
$$= e^{-x}$$

Complex form.

Q6

$$f(x) = 2 \quad -\infty < x < \infty$$

- Because, the function $f(x)$ has to be piecewise continuous on each interval $[0, l]$
- It should be integrable on real axis.
- If $f(x)$ has right and left hand derivatives at every $x \in (0, l)$. so, at every continuity it converges to $f(x)$ and discontinuity, it converges to $\frac{f(x^+) + f(x^-)}{2}$.

Q3

$y_{tt} - 16 y_{xx} = 0 \quad 0 < x < 2, t > 0$

$y(0,t) = y(2t) = 0$

$y(x,0) = 6\sin(\pi x)\ 3\sin(4\pi x) \quad 0 < x < 2$

$y_t(x,0) = 0$

$$\frac{\partial^2 y}{\partial t^2} - 16\frac{\partial^2 y}{\partial x^2} = 0$$

Let,

$y(x,t) = X(x) \cdot T(t)$

$y(0,t) = X(0), T(t)$

$\Rightarrow X'T = 16 XT'$

$\Rightarrow (X' - X)T = 16 XT'$

$\Rightarrow \frac{X' - X}{X} = 16\frac{T'}{T} = k$

$\Rightarrow \frac{X' - X}{X} = k \Rightarrow \frac{X'}{X} = k+1$

on integrating

$\log X = (k+1)x + \log C_1$

$\log X = (k+1)x \log e + \log C_1$

$\log X = \log e^{(k+1)x} + \log C_1$

Solving $\frac{T'}{T} = k$

$$\log T = \frac{k}{2}t + \log C_2$$

$$T = C_2 e^{kt/2}$$

$$\therefore \quad y(x,t) = XT = C_1 e^{(k+1)x} \times C_2 e^{kt/2} = C_1 C_2 e^{(k+1)x + kt/2}$$

when $t = 0$

$y(x, 0)$