# Faculty of Computer and Information Sciences, Ain Shams University: Too Wrong to Pass Too Correct to Fail

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# Contents





# 1 Template

## 1.1 template

```cpp
#include <bits/stdc++.h>
#define IO ios_base::sync_with_stdio(0); cin.tie(0); cout.tie(0);
using namespace std;
mt19937 rng(chrono::steady_clock::now().time_since_epoch().count());

// Kactl defines
#define rep(i, a, b) for(int i = a; i < (b); ++i)
#define all(x) begin(x), end(x)
#define sz(x) (int)(x).size()
typedef long long ll;
typedef pair<int, int> pii;
typedef vector<int> vi;
typedef vector<double> vd;
```

# 2 Combinatorics

## 2.1 Burnside Lemma

```cpp
// |Classes|=sum (k ^C(pi))  / |G|
// C(pi)  the number of cycles in the permutation pi
// |G| the number of permutations
```

## 2.2 Catlan Numbers

```cpp
void init() {
    catalan[0] = catalan[1] = 1;
    for (int i=2; i<=n; i++) {
        catalan[i] = 0;
        for (int j=0; j < i; j++) {
            catalan[i] += (catalan[j] * catalan[i-j-1]) % MOD;
            if (catalan[i] >= MOD) {
                catalan[i] -= MOD;
            }
        }
    }
}
```

```
// 1- Number of correct bracket sequence consisting of n opening and n closing
    brackets.
// 2- The number of rooted full binary trees with n+1 leaves (vertices are not
    numbered).
// 3- The number of ways to completely parenthesize n+1 factors.
// 4- The number of triangulations of a convex polygon with n+2 sides
// 5- The number of ways to connect the 2n points on a circle to form n disjoint
    chords.
// 6- The number of non-isomorphic full binary trees with n internal nodes (i.e.
    nodes having at least one son).
// 7- The number of monotonic lattice paths from point (0,0) to point (n,n) in a
    square lattice of size nxn, which do not pass above the main diagonal (i.e
    . connecting (0,0) to (n,n)).
// 8- Number of permutations of length n that can be stack sorted (it can be
    shown that the rearrangement is stack sorted if and only if there is no
    such index i<j<k, such that ak<ai<aj).
// 9- The number of non-crossing partitions of a set of n elements.
// 10- The number of ways to cover the ladder 1..n using n rectangles (The
    ladder consists of n columns, where ith column has a height i).
```

# 3 Algebra

## 3.1 Gray Code

```
int g (int n) {
    return n ^ (n >> 1);
}
int rev_g (int g) {
  int n = 0;
  for (; g; g >>= 1)
    n ^= g;
  return n;
}
int calc(int x, int y) { ///2D Gray Code
    int a = g(x), b = g(y);
    int res = 0;
    f(i,0,LG) {
        int k1 = (a & (1 << i));
        int k2 = b & (1 << i);
        res |= k1 << (i + 1);
        res |= k2 << i;
    }
    return res;
}
```

## 3.2 Factorial modulo in p*log(n) (Wilson Theroem)

```
int factmod(int n, int p) {
    vector<int> f(p);
    f[0] = 1;
    for (int i = 1; i < p; i++)
        f[i] = f[i-1] * i % p;

    int res = 1;
    while (n > 1) {
        if ((n/p) % 2)
            res = p - res;
        res = res * f[n%p] % p;
        n /= p;
    }
    return res;
}
```

## 3.3 Iteration over submasks

```
int s = m;
while (s > 0) {
 s = (s-1) & m;
}
```

## 3.4 FFT

```
typedef complex<double> C;
typedef vector<double> vd;
void fft(vector<C>& a) {
    int n = sz(a), L = 31 - __builtin_clz(n);
    static vector<complex<long double>> R(2, 1);
    static vector<C> rt(2, 1); // (^ 10% fas te r i f double)
    for (static int k = 2; k < n; k *= 2) {
        R.resize(n);
        rt.resize(n);
        auto x = polar(1.0L, acos(-1.0L) / k);
        rep(i, k, 2 * k) rt[i] = R[i] = i & 1 ? R[i / 2] * x : R[i / 2];
    }
    vi rev(n);
    rep(i, 0, n) rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
    rep(i, 0, n) if (i < rev[i]) swap(a[i], a[rev[i]]);
    for (int k = 1; k < n; k *= 2)
        for (int i = 0; i < n; i += 2 * k) rep(j, 0, k) {
            C z = rt[j + k] * a[i + j + k]; //
            a[i + j + k] = a[i + j] - z;
            a[i + j] += z;
        }
}
vd conv(const vd& a, const vd& b) {
    if (a.empty() || b.empty()) return {};
    vd res(sz(a) + sz(b) - 1);
    int L = 32 - __builtin_clz(sz(res)), n = 1 << L;
    vector<C> in(n), out(n);
    copy(all(a), begin(in));
    rep(i, 0, sz(b)) in[i].imag(b[i]);
    fft(in);
    for (C& x : in) x *= x;
    rep(i, 0, n) out[i] = in[-i & (n - 1)] - conj(in[i]);
    fft(out);
    /// rep(i,0,sz(res)) res[i] = (MOD+(ll)round(imag(out[i]) / (4 * n))) % MOD;
         ///in case of mod
    rep(i, 0, sz(res)) res[i] = imag(out[i]) / (4 * n);
    return res;
}

//Applications
//1-All possible sums

//2-All possible scalar products
// We are given two arrays a[] and b[] of length n.
//We have to compute the products of a with every cyclic shift of b.
//We generate two new arrays of size 2n: We reverse a and append n zeros to it.
//And we just append b to itself. When we multiply these two arrays as
     polynomials,
//and look at the coefficients c[n-1], c[n], ..., c[2n-2] of the product c, we
     get:
//c[k]=sum i+j=k  a[i]b[j]

//3-Two stripes
//We are given two Boolean stripes (cyclic arrays of values 0 and 1) a and b.
//We want to find all ways to attach the first stripe to the second one,
//such that at no position we have a 1 of the first stripe next to a 1 of the
     second stripe.
```

## 3.5 FFT with mod

```
"FastFourierTransform.cpp"
typedef vector<ll> vl;
template<int M> vl convMod(const vl &a, const vl &b) {
    if (a.empty() || b.empty()) return {};
    vl res(sz(a) + sz(b) - 1);
    int B=32-__builtin_clz(sz(res)), n=1<<B, cut=int(sqrt(M));
    vector<C> L(n), R(n), outs(n), outl(n);
    rep(i,0,sz(a)) L[i] = C((int)a[i] / cut, (int)a[i] % cut);
    rep(i,0,sz(b)) R[i] = C((int)b[i] / cut, (int)b[i] % cut);
    fft(L), fft(R);
    rep(i,0,n) {
        int j = -i & (n - 1);
        outl[j] = (L[i] + conj(L[j])) * R[i] / (2.0 * n);
        outs[j] = (L[i] - conj(L[j])) * R[i] / (2.0 * n) / 1i;
    }
    fft(outl), fft(outs);
    rep(i,0,sz(res)) {
        ll av = ll(real(outl[i])+.5), cv = ll(imag(outs[i])+.5);
        ll bv = ll(imag(outl[i])+.5) + ll(real(outs[i])+.5);
        res[i] = ((av % M * cut + bv) % M * cut + cv) % M;
    }
    return res;
}
```

## 3.6 convolutions of AND-XOR-OR

```
// The size of a must be a power of two.
void FST(vi& a, bool inv) {
    for (int n = sz(a), step = 1; step < n; step *= 2) {
        for (int i = 0; i < n; i += 2 * step) rep(j,i,i+step) {
            int &u = a[j], &v = a[j + step]; tie(u, v) =
                inv ? pii(v - u, u) : pii(v, u + v); // AND
                // inv ? pii(v, u - v) : pii(u + v, u); // OR /// include-line
                // pii(u + v, u - v);                  // XOR /// include-line
        }
    }
    // if (inv) for (int& x : a) x /= sz(a); // XOR only /// include-line
}
vi conv(vi a, vi b) {
    FST(a, 0); FST(b, 0);
    rep(i,0,sz(a)) a[i] *= b[i];
    FST(a, 1); return a;
}
```

## 3.7 NTT of KACTL

```
const ll mod = (119 << 23) + 1, root = 62; // = 998244353
// For p < 2^30 there is also e.g. 5 << 25, 7 << 26, 479 << 21
// and 483 << 21 (same root). The last two are > 10^9.
typedef vector<ll> vl;
void ntt(vl &a) {
    int n = sz(a), L = 31 - __builtin_clz(n);
    static vl rt(2, 1);
    for (static int k = 2, s = 2; k < n; k *= 2, s++) {
        rt.resize(n);
        ll z[] = {1, modpow(root, mod >> s)};
        rep(i,k,2*k) rt[i] = rt[i / 2] * z[i & 1] % mod;
    }
    vi rev(n);
    rep(i,0,n) rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
    rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
    for (int k = 1; k < n; k *= 2)
        for (int i = 0; i < n; i += 2 * k) rep(j,0,k) {
            ll z = rt[j + k] * a[i + j + k] % mod, &ai = a[i + j];
            a[i + j + k] = ai - z + (z > ai ? mod : 0);
            ai += (ai + z >= mod ? z - mod : z);
        }
}
vl conv(const vl &a, const vl &b) {
    if (a.empty() || b.empty()) return {};
    int s = sz(a) + sz(b) - 1, B = 32 - __builtin_clz(s),
        n = 1 << B;
    int inv = modpow(n, mod - 2);
    vl L(a), R(b), out(n);
    L.resize(n), R.resize(n);
    ntt(L), ntt(R);
    rep(i,0,n)
        out[-i & (n - 1)] = (ll)L[i] * R[i] % mod * inv % mod;
    ntt(out);
    return {out.begin(), out.begin() + s};
}
```

## 3.8 Fibonacci

```
// F(n-1) * F(n+1) - F(n)^2 = (-1)^n
// F(n+k) = F(k) * F(n+1) + F(k-1) * F(n)
// F(2*n) = F(n) * (F(n+1) + F(n-1))
// GCD ( F(m) , F(n) ) = F(GCD(n,m))
```

## 3.9 Gauss Determinant

```
double det(vector<vector<double>>& a) {
    int n = sz(a); double res = 1;
    rep(i,0,n) {
        int b = i;
        rep(j,i+1,n) if (fabs(a[j][i]) > fabs(a[b][i])) b = j;
        if (i != b) swap(a[i], a[b]), res *= -1;
        res *= a[i][i];
        if (res == 0) return 0;
        rep(j,i+1,n) {
            double v = a[j][i] / a[i][i];
            if (v != 0) rep(k,i+1,n) a[j][k] -= v * a[i][k];
        }
    }
    return res;
}
// for integers
const ll mod = 12345;
ll det(vector<vector<ll>>& a) {
    int n = sz(a); ll ans = 1;
    rep(i,0,n) {
        rep(j,i+1,n) {
            while (a[j][i] != 0) { // gcd step
                ll t = a[i][i] / a[j][i];
                if (t) rep(k,i,n)
                a[i][k] = (a[i][k] - a[j][k] * t) % mod;
                swap(a[i], a[j]);
                ans *= -1;
            }
        }
        ans = ans * a[i][i] % mod;
        if (!ans) return 0;
    }
    return (ans + mod) % mod;
}
```

## 3.10 GAUSS SLAE

```
const double EPS = 1e-9;
const int INF = 2; // it doesn't actually have to be infinity or a big number

int gauss (vector < vector<double> > a, vector<double> & ans) {
    int n = (int) a.size();
    int m = (int) a[0].size() - 1;

    vector<int> where (m, -1);
    for (int col = 0, row = 0; col < m && row < n; ++col) {
        int sel = row;
        for (int i = row; i < n; ++i)
            if (abs (a[i][col]) > abs (a[sel][col]))
                sel = i;
        if (abs (a[sel][col]) < EPS)
            continue;
        for (int i = col; i <= m; ++i)
            swap (a[sel][i], a[row][i]);
        where[col] = row;

        for (int i = 0; i < n; ++i)
            if (i != row) {
                double c = a[i][col] / a[row][col];
                for (int j = col; j <= m; ++j)
                    a[i][j] -= a[row][j] * c;
            }
        ++row;
    }

    ans.assign (m, 0);
    for (int i = 0; i < m; ++i)
        if (where[i] != -1)
            ans[i] = a[where[i]][m] / a[where[i]][i];
    for (int i = 0; i < n; ++i) {
        double sum = 0;
        for (int j = 0; j < m; ++j)
            sum += ans[j] * a[i][j];
        if (abs (sum - a[i][m]) > EPS)
            return 0;
    }

    for (int i = 0; i < m; ++i)
        if (where[i] == -1)
            return INF;
    return 1;
}
```

## 3.11 Matrix Inverse

```
#define ld long double
vector < vector<ld> > gauss (vector < vector<ld> > a) {

    int n = (int) a.size();
    vector<vector<ld> > ans(n, vector<ld>(n, 0));

    for(int i = 0; i < n; i++)
        ans[i][i] = 1;
    for(int i = 0; i < n; i++) {
        for(int j = i + 1; j < n; j++)
            if(a[j][i] > a[i][i]) {
                a[j].swap(a[i]);
                ans[j].swap(ans[i]);
            }
        ld val = a[i][i];
        for(int j = 0; j < n; j++) {
            a[i][j] /= val;
            ans[i][j] /= val;
        }
        for(int j = 0; j < n; j++) {
            if(j == i)continue;
            val = a[j][i];
            for(int k = 0; k < n; k++) {
                a[j][k] -= val * a[i][k];
                ans[j][k] -= val * ans[i][k];
            }
        }
    }
    return ans;
}
```

# 4 Data Structures

## 4.1 UnionFindRollback

```
struct RollbackUF {
    vi e; vector<pii> st;
    RollbackUF(int n) : e(n, -1) {}
    int size(int x) { return -e[find(x)]; }
    int find(int x) { return e[x] < 0 ? x : find(e[x]); }
    int time() { return sz(st); }
    void rollback(int t) {
        for (int i = time(); i --> t;)
            e[st[i].first] = st[i].second;
```

```
        st.resize(t);
    }
    bool join(int a, int b) {
        a = find(a), b = find(b);
        if (a == b) return false;
        if (e[a] > e[b]) swap(a, b);
        st.push_back({a, e[a]});
        st.push_back({b, e[b]});
        e[a] += e[b]; e[b] = a;
        return true;
    }
};
```

## 4.2 2D BIT

```
void upd(int x, int y, int val) {
    for(int i = x; i <= n; i += i & -i)
    for(int j = y; j <= m; j += j & -j)
    bit[i][j] += val;
}
int get(int x, int y) {
    int ans = 0;
    for(int i = x; i; i -= i & -i)
    for(int j = y; j; j -= j & -j)
    ans += bit[i][j];
}
```

## 4.3 2D Sparse table

```
const int N = 505, LG = 10;
int st[N][N][LG][LG];
int a[N][N], lg2[N];
int yo(int x1, int y1, int x2, int y2) {
  x2++;
  y2++;
  int a = lg2[x2 - x1], b = lg2[y2 - y1];
  return max(
         max(st[x1][y1][a][b], st[x2 - (1 << a)][y1][a][b]),
         max(st[x1][y2 - (1 << b)][a][b], st[x2 - (1 << a)][y2 - (1 << b)][a][b])
       );
}
void build(int n, int m) { // 0 indexed
  for (int i = 2; i < N; i++) lg2[i] = lg2[i >> 1] + 1;
  for (int i = 0; i < n; i++) {
    for (int j = 0; j < m; j++) {
      st[i][j][0][0] = a[i][j];
    }
  }
  for (int a = 0; a < LG; a++) {
    for (int b = 0; b < LG; b++) {
      if (a + b == 0) continue;
      for (int i = 0; i + (1 << a) <= n; i++) {
        for (int j = 0; j + (1 << b) <= m; j++) {
          if (!a) {
            st[i][j][a][b] = max(st[i][j][a][b - 1], st[i][j + (1 << (b - 1))][a][b - 1]);
          } else {
            st[i][j][a][b] = max(st[i][j][a - 1][b], st[i + (1 << (a - 1))][j][a - 1][b]);
          }
        }
      }
    }
  }
}
```

## 4.4 Mo With Updates

```
///O(N^5/3) note that the block size is not a standard size
/// O(2SQ + N^2 / S + Q * N^2 / S^2) = O(Q * N^(2/3)) if S = n^(2/3)
/// fact: S = (2 * n * n)^(1/3) give the best complexity
const int block_size = 2000;
struct Query{
    int l, r, t, idx;
    Query(int l,int r,int t,int idx) : l(l),r(r),t(t),idx(idx) {}
    bool operator < (Query o) const{
        if(l / block_size != o.l / block_size) return l < o.l;
        if(r / block_size != o.r/block_size) return r < o.r;
        return t < o.t;
    }
};
int L = 0, R = -1, K = -1;
while(L < Q[i].l)del(a[L++]);
while(L > Q[i].l)add(a[--L]);
while(R < Q[i].r)add(a[++R]);
while(R > Q[i].r)del(a[R--]);
while(K < Q[i].t)upd(++K);
while(K > Q[i].t)err(K--);
```

## 4.5 Ordered Set

```
#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>
using namespace __gnu_pbds;
#define ordered_set tree<int, null_type,less<int>, rb_tree_tag,
       tree_order_statistics_node_update>
//order_of_key(k): returns the number of elements strictly less than k
//find_by_order(k): returns an iterator to the k-th element (0-based)
```

## 4.6 Persistent Seg Tree

```

int val[ N * 60 ], L[ N * 60 ], R[ N * 60 ], ptr, tree[N]; /// N * lgN
int upd(int root, int s, int e, int idx) {
    int ret = ++ptr;
    val[ret] = L[ret] = R[ret] = 0;
    if (s == e) {
        val[ret] = val[root] + 1;
        return ret;
    }
    int md = (s + e) >> 1;
    if (idx <= md) {
        L[ret] = upd(L[root], s, md, idx), R[ret] = R[root];
    } else {
        R[ret] = upd(R[root], md + 1, e, idx), L[ret] = L[root];
    }
    val[ret] = max(val[L[ret]], val[R[ret]]);
    return ret;
}
int qry(int node, int s, int e, int l, int r){
  if(r < s || e < l || !node)return 0; //Punishment Value
  if(l <= s && e <= r){
    return val[node];
  }
  int md = (s+e)>>1;
  return max(qry(L[node], s, md, l, r), qry(R[node],md+1,e,l,r));
}
int merge(int x, int y, int s, int e) {
    if(!x||!y)return x | y;
    if(s == e) {
        val[x] += val[y];
        return x;
    }
    int md = (s + e) >> 1;
    L[x] = merge(L[x], L[y], s, md);
    R[x] = merge(R[x], R[y], md+1,e);
    val[x] = val[L[x]] + val[R[x]];
    return x;
}
```

## 4.7 Treap

```

mt19937_64 mrand(chrono::steady_clock::now().time_since_epoch().count());
struct Node {
    int key, pri = mrand(), sz = 1;
    int lz = 0;
    int idx;
    array<Node*, 2> c = {NULL,NULL};
    Node(int key, int idx) : key(key), idx(idx) {}
};
int getsz(Node* t){
    return t ? t->sz : 0;
}
Node* calc(Node* t) {
    t->sz = 1 + getsz(t->c[0]) + getsz(t->c[1]);
    return t;
}
void prop(Node* cur) {
    if(!cur || !cur->lz)
        return;
    cur->key += cur->lz;
    if(cur->c[0])
        cur->c[0]->lz += cur->lz;
    if(cur->c[1])
        cur->c[1]->lz += cur->lz;
    cur->lz = 0;
}
array<Node*, 2> split(Node* t, int k) {
    prop(t);
    if(!t)
        return {t, t};
    if(getsz(t->c[0]) >= k) {   ///answer is in left node
        auto ret = split(t->c[0], k);
        t->c[0] = ret[1];
        return {ret[0], calc(t)};
    } else {  ///k > t->c[0]
```

```
        auto ret = split(t->c[1], k - 1 - getsz(t->c[0]));
        t->c[1] = ret[0];
        return {calc(t), ret[1]};
    }
}
Node* merge(Node* u, Node* v) {
    prop(u);
    prop(v);
    if(!u || !v)
        return u ? u : v;
    if(u->pri>v->pri) {
        u->c[1] = merge(u->c[1], v);
        return calc(u);
    } else {
        v->c[0] = merge(u, v->c[0]);
        return calc(v);
    }
}
int cnt(Node* cur, int x) {
    prop(cur);
    if(!cur)
        return 0;
    if(cur->key <= x)
        return getsz(cur->c[0]) + 1 + cnt(cur->c[1], x);
    return cnt(cur->c[0], x);
}
Node* ins(Node* root, int val, int idx, int pos) {
    auto splitted = split(root, pos);
    root = merge(splitted[0], new Node(val, idx));
    return merge(root, splitted[1]);
}
```

## 4.8 Wavelet Tree

```
// remember your array and values must be 1-based
struct wavelet_tree {
    int lo, hi;
    wavelet_tree *l, *r;
    vector<int> b;

    //nos are in range [x,y]
    //array indices are [from, to)
    wavelet_tree(int *from, int *to, int x, int y) {
        lo = x, hi = y;
        if (lo == hi or from >= to)
            return;
        int mid = (lo + hi) / 2;
        auto f = [mid](int x) {
            return x <= mid;
        };
        b.reserve(to - from + 1);
        b.pb(0);
        for (auto it = from; it != to; it++)
            b.pb(b.back() + f(*it));
        //see how lambda function is used here
        auto pivot = stable_partition(from, to, f);
        l = new wavelet_tree(from, pivot, lo, mid);
        r = new wavelet_tree(pivot, to, mid + 1, hi);
    }

    //kth smallest element in [l, r]
    int kth(int l, int r, int k) {
        if (l > r)
            return 0;
        if (lo == hi)
            return lo;
        int inLeft = b[r] - b[l - 1];
        int lb = b[l - 1]; //amt of nos in first (l-1) nos that go in left
        int rb = b[r]; //amt of nos in first (r) nos that go in left
        if (k <= inLeft)
            return this->l->kth(lb + 1, rb, k);
        return this->r->kth(l - lb, r - rb, k - inLeft);
    }

    //count of nos in [l, r] Less than or equal to k
    int LTE(int l, int r, int k) {
        if (l > r or k < lo)
            return 0;
        if (hi <= k)
            return r - l + 1;
        int lb = b[l - 1], rb = b[r];
        return this->l->LTE(lb + 1, rb, k) + this->r->LTE(l - lb, r - rb, k);
    }

    //count of nos in [l, r] equal to k
    int count(int l, int r, int k) {
        if (l > r or k < lo or k > hi)
            return 0;
        if (lo == hi)
            return r - l + 1;
        int lb = b[l - 1], rb = b[r], mid = (lo + hi) / 2;
        if (k <= mid)
            return this->l->count(lb + 1, rb, k);
        return this->r->count(l - lb, r - rb, k);
    }
};
```

## 4.9 SparseTable

```
int S[N];
for(int i = 2; i < N; i++) S[i] = S[i >> 1] + 1;
for (int i = 1; i <= K; i++)
    for (int j = 0; j + (1 << i) <= N; j++)
        st[i][j] = f(st[i - 1][j], st[i - 1][j + (1 << (i - 1))]);

int query(int l, int r) {
    int k = S[r - l + 1];
    return mrg(st[k][l], st[k][r-(1<<k)+1]);
}
```

# 5 DP

## 5.1 CHT Line Container

```
struct Line {
    mutable ll m, b, p;
    bool operator<(const Line &o) const { return m < o.m; }
    bool operator<(ll x) const { return p < x; }
};
struct LineContainer : multiset<Line, less<>> {
    // (for doubles, use inf = 1/.0, div(a,b) = a/b)
    static const ll inf = LLONG_MAX;
    ll div(ll db, ll dm) { // floored division
        return db / dm - ((db ^ dm) < 0 && db % dm);
    }
    bool isect(iterator x, iterator y) {
        if (y == end()) {
            x->p = inf;
            return false;
        }
        if (x->m == y->m)
            x->p = x->b > y->b ? inf : -inf;
        else
            x->p = div(y->b - x->b, x->m - y->m);
        return x->p >= y->p;
    }
    void add(ll m, ll b) {
        auto z = insert({m, b, 0}), y = z++, x = y;
        while (isect(y, z))
            z = erase(z);
        if (x != begin() && isect(--x, y))
            isect(x, y = erase(y));
        while ((y = x) != begin() && (--x)->p >= y->p)
            isect(x, erase(y));
    }
    ll query(ll x) {
        assert(!empty());
        auto l = *lower_bound(x);
        return l.m * x + l.b;
    }
};
```

# 6 Geometry

## 6.1 Convex Hull

```
struct point {
    ll x,  y;
    point(ll x, ll y) : x(x), y(y) {}
    point operator -(point other) {
        return point(x - other.x, y - other.y);
    }
    bool operator <(const point  &other) const {
        return x != other.x ? x < other.x : y < other.y;
    }
};
ll cross(point a, point b) {
    return a.x * b.y - a.y * b.x;
}
ll dot(point a, point b) {
    return a.x * b.x + a.y * b.y;
}
struct sortCCW {
    point center;

```

```cpp
    sortCCW(point center) : center(center) {}

    bool operator()(point a, point b) {
        ll res = cross(a - center, b - center);
        if(res)
            return res > 0;
        return dot(a - center, a - center) < dot(b - center, b - center);
    }
};
vector<point> hull(vector<point> v) {
    sort(v.begin(), v.end());
    sort(v.begin() + 1, v.end(), sortCCW(v[0]));
    v.push_back(v[0]);
    vector<point> ans ;
    for(auto i : v) {
        int sz = ans.size();
        while(sz > 1 && cross(i - ans[sz - 1], ans[sz - 2] - ans[sz - 1]) <= 0)
            ans.pop_back(), sz--;
        ans.push_back(i);
    }
    ans.pop_back();
    return ans;
}
```

## 6.2 Geometry Template

```cpp
using  ptype =  double edit this first ;
double EPS = 1e-9;
struct point {
    ptype x,  y;
    point(ptype x, ptype y) : x(x), y(y) {}
    point operator -(const point & other)const { return point(x - other.x, y - other.y);}
    point operator +(const point & other)const { return point(x + other.x, y + other.y);}
    point operator *(ptype c) const { return point(x * c, y * c); }
    point operator /(ptype c) const { return point(x / c, y / c); }
    point prep() { return point(-y, x); }
};
ptype cross(point a, point b) { return a.x * b.y - a.y * b.x;}
ptype dot(point a, point b) {return a.x * b.x + a.y * b.y;}
double abs(point a) {return sqrt(dot(a, a));}

double angle (point a, point b) { // angle between [0 , pi]
    return acos(dot(a, b) / abs(a) / abs(b));
}
// a : point in Line, d : Line direction
point LineLineIntersect(point a1, point d1, point a2, point d2) {
    return a1 + d1 * cross(a2 - a1, d2) / cross(d1, d2);
}
// Line a---b, point C
point ProjectPointLine(point a, point b, point c) {
    return a + (b - a) * 1.0 * dot(c - a, b - a) / dot(b - a, b - a);
}
// segment a---b, point C
point ProjectPointSegment(point a, point b, point c) {
    double r = dot(c - a, b - a) / dot(b - a, b - a);
    if(r < 0)
        return a;
    if(r > 1)
        return b;
    return a + (b - a) * r ;
}
// Line a---b, point p
point reflectAroundLine(point a, point b, point p) {
    return ProjectPointLine(a, b, p) * 2 - p;// (proj-p) *2 + p
}
// Around origin
point RotateCCW(point p, double t) {
    return point(p.x * cos(t) - p.y * sin(t),
                 p.x * sin(t) + p.y * cos(t));
}
// Line a---b
vector<point> CircleLineIntersect(point a, point b, point center, double r) {
    a = a - center;
    b = b - center;
    point p = ProjectPointLine(a, b, point(0, 0)); // project point from center to the Line
    if(dot(p, p) > r * r)
        return {};
    double len = sqrt(r * r - dot(p, p));
    if(len < EPS)
        return {center + p};

    point d = (a - b) / abs(a - b);
    return {center + p + d * len, center + p - d * len};
}

vector<point> CircleCircleIntersect(point c1, ld r1, point c2, ld r2) {
    if (r1 < r2) {
        swap(r1, r2);
        swap(c1, c2);
    }
    ld d = abs(c2 - c1); // distance between c1,c2
    if (d > r1 + r2 || d < r1 - r2 || d < EPS)  // zero or infinite solutions
        return {};
    ld angle = acos(min((d * d + r1 * r1 - r2 * r2) / (2 * r1 * d), (ld) 1.0));
    point p = (c2 - c1) / d * r1;

    if (angle < EPS)
        return {c1 + p};

    return {c1 + RotateCCW(p, angle), c1 + RotateCCW(p, -angle)};

}
point circumcircle(point p1, point p2, point p3) {
    return LineLineIntersect((p1 + p2) / 2, (p1 - p2).prep(),
                             (p1 + p3) / 2, (p1 - p3).prep() );
}
//I : number points with integer coordinates lying strictly inside the polygon.
//B : number of points lying on polygon sides by B.
//Area = I + B/2 - 1
```

## 6.3 Half Plane Intersection

```cpp
// Redefine epsilon and infinity as necessary. Be mindful of precision errors.
#define ld long double
const ld eps = 1e-9, inf = 1e9;

// Basic point/vector struct.
struct Point {

    ld x, y;
    explicit Point(ld x = 0, ld y = 0) : x(x), y(y) {}

    // Addition, substraction, multiply by constant, cross product.
    friend Point operator + (const Point& p, const Point& q) {
        return Point(p.x + q.x, p.y + q.y);
    }
    friend Point operator - (const Point& p, const Point& q) {
        return Point(p.x - q.x, p.y - q.y);
    }
    friend Point operator * (const Point& p, const ld& k) {
        return Point(p.x * k, p.y * k);
    }
    friend ld cross(const Point& p, const Point& q) {
        return p.x * q.y - p.y * q.x;
    }
};

// Basic half-plane struct.
struct Halfplane {
    // 'p' is a passing point of the line and 'pq' is the direction vector of the line.
    Point p, pq;
    ld angle;

    Halfplane() {}
    Halfplane(const Point& a, const Point& b) : p(a), pq(b - a) {
        angle = atan2l(pq.y, pq.x);
    }
    // Check if point 'r' is outside this half-plane.
    // Every half-plane allows the region to the LEFT of its line.
    bool out(const Point& r) {
        return cross(pq, r - p) < -eps;
    }
    // Comparator for sorting.
    // If the angle of both half-planes is equal, the leftmost one should go first.
    bool operator < (const Halfplane& e) const {
        if (fabsl(angle - e.angle) < eps) return cross(pq, e.p - p) < 0;
        return angle < e.angle;
    }
    // We use equal comparator for std::unique to easily remove parallel half-planes.
    bool operator == (const Halfplane& e) const {
        return fabsl(angle - e.angle) < eps;
    }
    // Intersection point of the lines of two half-planes. It is assumed they're never parallel.
    friend Point inter(const Halfplane& s, const Halfplane& t) {
        ld alpha = cross((t.p - s.p), t.pq) / cross(s.pq, t.pq);
        return s.p + (s.pq * alpha);
    }
};
// Actual algorithm
vector<Point> hp_intersect(vector<Halfplane>& H) {
    Point box[4] = {  // Bounding box in CCW order
        Point(inf, inf),
        Point(-inf, inf),
```

```cpp
        Point(-inf, -inf),
        Point(inf, -inf)
    };

    for(int i = 0; i<4; i++) { // Add bounding box half-planes.
        Halfplane aux(box[i], box[(i+1) % 4]);
        H.push_back(aux);
    }
    // Sort and remove duplicates
    sort(H.begin(), H.end());
    H.erase(unique(H.begin(), H.end()), H.end());

    deque<Halfplane> dq;
    int len = 0;
    for(int i = 0; i < int(H.size()); i++) {
        // Remove from the back of the deque while last half-plane is redundant
        while (len > 1 && H[i].out(inter(dq[len-1], dq[len-2]))) {
            dq.pop_back();
            --len;
        }
        // Remove from the front of the deque while first half-plane is redundant
        while (len > 1 && H[i].out(inter(dq[0], dq[1]))) {
            dq.pop_front();
            --len;
        }
        // Add new half-plane
        dq.push_back(H[i]);
        ++len;
    }

    // Final cleanup: Check half-planes at the front against the back and vice-versa
    while (len > 2 && dq[0].out(inter(dq[len-1], dq[len-2]))) {
        dq.pop_back();
        --len;
    }
    while (len > 2 && dq[len-1].out(inter(dq[0], dq[1]))) {
        dq.pop_front();
        --len;
    }
    // Report empty intersection if necessary
    if (len < 3) return vector<Point>();

    // Reconstruct the convex polygon from the remaining half-planes.
    vector<Point> ret(len);
    for(int i = 0; i+1 < len; i++) {
        ret[i] = inter(dq[i], dq[i+1]);
    }
    ret.back() = inter(dq[len-1], dq[0]);
    return ret;
}
```

## 6.4 Segments Intersection

```cpp
const double EPS = 1E-9;

struct pt {
    double x, y;
};

struct seg {
    pt p, q;
    int id;

    double get_y(double x) const {
        if (abs(p.x - q.x) < EPS)
            return p.y;
        return p.y + (q.y - p.y) * (x - p.x) / (q.x - p.x);
    }
};

bool intersect1d(double l1, double r1, double l2, double r2) {
    if (l1 > r1)
        swap(l1, r1);
    if (l2 > r2)
        swap(l2, r2);
    return max(l1, l2) <= min(r1, r2) + EPS;
}

int vec(const pt& a, const pt& b, const pt& c) {
    double s = (b.x - a.x) * (c.y - a.y) - (b.y - a.y) * (c.x - a.x);
    return abs(s) < EPS ? 0 : s > 0 ? +1 : -1;
}

bool intersect(const seg& a, const seg& b)
{
    return intersect1d(a.p.x, a.q.x, b.p.x, b.q.x) &&
           intersect1d(a.p.y, a.q.y, b.p.y, b.q.y) &&
               vec(a.p, a.q, b.p) * vec(a.p, a.q, b.q) <= 0 &&
               vec(b.p, b.q, a.p) * vec(b.p, b.q, a.q) <= 0;
}

bool operator<(const seg& a, const seg& b)
{
    double x = max(min(a.p.x, a.q.x), min(b.p.x, b.q.x));
    return a.get_y(x) < b.get_y(x) - EPS;
}

struct event {
    double x;
    int tp, id;

    event() {}
    event(double x, int tp, int id) : x(x), tp(tp), id(id) {}

    bool operator<(const event& e) const {
        if (abs(x - e.x) > EPS)
            return x < e.x;
        return tp > e.tp;
    }
};

set<seg> s;
vector<set<seg>::iterator> where;

set<seg>::iterator prev(set<seg>::iterator it) {
    return it == s.begin() ? s.end() : --it;
}

set<seg>::iterator next(set<seg>::iterator it) {
    return ++it;
}

pair<int, int> solve(const vector<seg>& a) {
    int n = (int)a.size();
    vector<event> e;
    for (int i = 0; i < n; ++i) {
        e.push_back(event(min(a[i].p.x, a[i].q.x), +1, i));
        e.push_back(event(max(a[i].p.x, a[i].q.x), -1, i));
    }
    sort(e.begin(), e.end());

    s.clear();
    where.resize(a.size());
    for (size_t i = 0; i < e.size(); ++i) {
        int id = e[i].id;
        if (e[i].tp == +1) {
            set<seg>::iterator nxt = s.lower_bound(a[id]), prv = prev(nxt);
            if (nxt != s.end() && intersect(*nxt, a[id]))
                return make_pair(nxt->id, id);
            if (prv != s.end() && intersect(*prv, a[id]))
                return make_pair(prv->id, id);
            where[id] = s.insert(nxt, a[id]);
        } else {
            set<seg>::iterator nxt = next(where[id]), prv = prev(where[id]);
            if (nxt != s.end() && prv != s.end() && intersect(*nxt, *prv))
                return make_pair(prv->id, nxt->id);
            s.erase(where[id]);
        }
    }

    return make_pair(-1, -1);
}
```

## 6.5 Rectangles Union

```cpp
#include<bits/stdc++.h>
#define P(x,y) make_pair(x,y)
using namespace std;
class Rectangle {
public:
    int x1, y1, x2, y2;
    static Rectangle empt;
    Rectangle() {
        x1 = y1 = x2 = y2 = 0;
    }
    Rectangle(int X1, int Y1, int X2, int Y2) {
        x1 = X1;
        y1 = Y1;
        x2 = X2;
        y2 = Y2;
    }
};
struct Event {
    int x, y1, y2, type;
    Event() {}
    Event(int x, int y1, int y2, int type): x(x), y1(y1), y2(y2), type(type) {}
};
bool operator < (const Event&A, const Event&B) {
//if(A.x != B.x)
    return A.x < B.x;
```

```
    //if(A.y1 != B.y1) return A.y1 < B.y1;
    //if(A.y2 != B.y2()) A.y2 < B.y2;
}
const int MX = (1 << 17);
struct Node {
    int prob, sum, ans;
    Node() {}
    Node(int prob, int sum, int ans): prob(prob), sum(sum), ans(ans) {}
};
Node tree[MX * 4];
int interval[MX];
void build(int x, int a, int b) {
    tree[x] = Node(0, 0, 0);
    if(a == b) {
        tree[x].sum += interval[a];
        return;
    }
    build(x * 2, a, (a + b) / 2);
    build(x * 2 + 1, (a + b) / 2 + 1, b);
    tree[x].sum = tree[x * 2].sum + tree[x * 2 + 1].sum;
}
int ask(int x) {
    if(tree[x].prob)
        return tree[x].sum;
    return tree[x].ans;
}
int st, en, V;
void update(int x, int a, int b) {
    if(st > b || en < a)
        return;
    if(a >= st && b <= en) {
        tree[x].prob += V;
        return;
    }
    update(x * 2, a, (a + b) / 2);
    update(x * 2 + 1, (a + b) / 2 + 1, b);
    tree[x].ans = ask(x * 2) + ask(x * 2 + 1);
}
Rectangle Rectangle::empt = Rectangle();
vector < Rectangle > Rect;
vector < int > sorted;
vector < Event > sweep;
void compressncalc() {
    sweep.clear();
    sorted.clear();
    for(auto R : Rect) {
        sorted.push_back(R.y1);
        sorted.push_back(R.y2);
    }
    sort(sorted.begin(), sorted.end());
    sorted.erase(unique(sorted.begin(), sorted.end()), sorted.end());
    int sz = sorted.size();
    for(int j = 0; j < sorted.size() - 1; j++)
        interval[j + 1] = sorted[j + 1] - sorted[j];
    for(auto R : Rect) {
        sweep.push_back(Event(R.x1, R.y1, R.y2, 1));
        sweep.push_back(Event(R.x2, R.y1, R.y2, -1));
    }
    sort(sweep.begin(), sweep.end());
    build(1, 1, sz - 1);
}
long long ans;
void Sweep() {
    ans = 0;
    if(sorted.empty() || sweep.empty())
        return;
    int last = 0, sz_ = sorted.size();
    for(int j = 0; j < sweep.size(); j++) {
        ans += 1ll * (sweep[j].x - last) * ask(1);
        last = sweep[j].x;
        V = sweep[j].type;
        st = lower_bound(sorted.begin(), sorted.end(), sweep[j].y1) - sorted.begin() + 1;
        en = lower_bound(sorted.begin(), sorted.end(), sweep[j].y2) - sorted.begin();
        update(1, 1, sz_ - 1);
    }
}
int main() {
//    freopen("in.in","r",stdin);
    int n;
    scanf("%d", &n);
    for(int j = 1; j <= n; j++) {
        int a, b, c, d;
        scanf("%d %d %d %d", &a, &b, &c, &d);
        Rect.push_back(Rectangle(a, b, c, d));
    }
    compressncalc();
    Sweep();
    cout << ans << endl;
}
```

# 7 Graphs

## 7.1 2 SAD

```
/**
 * Description: Calculates a valid assignment to boolean variables a, b, c,... to a 2-SAT problem, so that an expression of the type $(a\|\|b)\&\&(!a\|\|c)\&\&(d\|\|!b)\&\&...$ becomes true, or reports that it is unsatisfiable.
 * Negated variables are represented by bit-inversions (\texttt{\tilde{}x}).
 * Usage:
 *  TwoSat ts(number of boolean variables);
 *  ts.either(0, \tilde3); // Var 0 is true or var 3 is false
 *  ts.setValue(2); // Var 2 is true
 *  ts.atMostOne({0,\tilde1,2}); // <= 1 of vars 0, \tilde1 and 2 are true
 *  ts.solve(); // Returns true iff it is solvable
 *  ts.values[0..N-1] holds the assigned values to the vars
 * Time: O(N+E), where N is the number of boolean variables, and E is the number of clauses.
 */
struct TwoSat {
    int N;
    vector<vi> gr;
    vi values; // 0 = false, 1 = true

    TwoSat(int n = 0) : N(n), gr(2*n) {}

    int addVar() { // (optional)
        gr.emplace_back();
        gr.emplace_back();
        return N++;
    }

    void either(int f, int j) {
        f = max(2*f, -1-2*f);
        j = max(2*j, -1-2*j);
        gr[f].push_back(j^1);
        gr[j].push_back(f^1);
    }
    void setValue(int x) { either(x, x); }

    void atMostOne(const vi& li) { // (optional)
        if (sz(li) <= 1) return;
        int cur = ~li[0];
        rep(i,2,sz(li)) {
            int next = addVar();
            either(cur, ~li[i]);
            either(cur, next);
            either(~li[i], next);
            cur = ~next;
        }
        either(cur, ~li[1]);
    }

    vi val, comp, z; int time = 0;
    int dfs(int i) {
        int low = val[i] = ++time, x; z.push_back(i);
        for(int e : gr[i]) if (!comp[e])
            low = min(low, val[e] ?: dfs(e));
        if (low == val[i]) do {
            x = z.back(); z.pop_back();
            comp[x] = low;
            if (values[x>>1] == -1)
                values[x>>1] = x&1;
        } while (x != i);
        return val[i] = low;
    }

    bool solve() {
        values.assign(N, -1);
        val.assign(2*N, 0); comp = val;
        rep(i,0,2*N) if (!comp[i]) dfs(i);
        rep(i,0,N) if (comp[2*i] == comp[2*i+1]) return 0;
        return 1;
    }
};
```

## 7.2 Ariculation Point

```
vector<int> adj[N];
int dfsn[N], low[N], instack[N], ar_point[N], timer;
stack<int> st;

void dfs(int node, int par){
    dfsn[node] = low[node] = ++timer;
    int kam = 0;
    for(auto i: adj[node]){
```

```cpp
        if(i == par) continue;
        if(dfsn[i] == 0){
            kam++;
            dfs(i, node);
            low[node] = min(low[node], low[i]);
            if(dfsn[node] <= low[i] && par != 0) ar_point[node] = 1;
        }
        else low[node] = min(low[node], dfsn[i]);
    }
    if(par == 0 && kam > 1) ar_point[node] = 1;
}
int main(){
    // Input
    for(int i = 1; i <= n; i++){
        if(dfsn[i] == 0) dfs(i, 0);
    }
    int c = 0;
    for(int i = 1; i <= n; i++){
        if(ar_point[i]) c++;
    }
    cout << c << '\n';
}
```

## 7.3 Bridges Tree and Diameter

```cpp
#include <bits/stdc++.h>
#define ll long long
using namespace std;
const int N = 3e5 + 5, mod = 1e9 + 7;

vector<int> adj[N], bridge_tree[N];
int dfsn[N], low[N], cost[N], timer, cnt, comp_id[N], kam[N], ans;
stack<int> st;


void dfs(int node, int par){
    dfsn[node] = low[node] = ++timer;
    st.push(node);
    for(auto i: adj[node]){
        if(i == par) continue;
        if(dfsn[i] == 0){
            dfs(i, node);
            low[node] = min(low[node], low[i]);
        }
        else low[node] = min(low[node], dfsn[i]);
    }
    if(dfsn[node] == low[node]){
        cnt++;
        while(1){
            int cur = st.top();
            st.pop();
            comp_id[cur] = cnt;
            if(cur == node) break;
        }
    }
}

void dfs2(int node, int par){
    kam[node] = 0;
    int mx = 0, second_mx = 0;
    for(auto i: bridge_tree[node]){
        if(i == par) continue;
        dfs2(i, node);
        kam[node] = max(kam[node], 1 + kam[i]);
        if(kam[i] > mx){
            second_mx = mx;
            mx = kam[i];
        }
        else second_mx = max(second_mx, kam[i]);
    }
    ans = max(ans, kam[node]);
    if(second_mx) ans = max(ans, 2 + mx + second_mx);
}

int main(){
    ios_base::sync_with_stdio(0);cin.tie(0);cout.tie(0);
    int n, m;
    cin >> n >> m;
    while(m--){
        int u, v;
        cin >> u >> v;
        adj[u].push_back(v);
        adj[v].push_back(u);
    }
    dfs(1, 0);
    for(int i = 1; i <= n; i++){
        for(auto j: adj[i]){
            if(comp_id[i] != comp_id[j]){
                bridge_tree[comp_id[i]].push_back(comp_id[j]);
            }
        }
    }
    dfs2(1, 0);
    cout << ans;

    return 0;
}
```

## 7.4 Dinic With Scalling

```cpp
///O(ElgFlow) on Bipratite Graphs and O(EVlgFlow) on other graphs (I think)
struct Dinic {
    #define vi vector<int>
    #define rep(i,a,b)  f(i,a,b)
    struct Edge {
        int to, rev;
        ll c, oc;
        int id;
        ll flow() { return max(oc - c, 0LL); } // if you need flows
    };
    vi lvl, ptr, q;
    vector<vector<Edge>> adj;
    Dinic(int n) : lvl(n), ptr(n), q(n), adj(n) {}
    void addEdge(int a, int b, ll c, int id, ll rcap = 0) {
        adj[a].push_back({b, sz(adj[b]), c, c, id});
        adj[b].push_back({a, sz(adj[a]) - 1, rcap, rcap,id});
    }
    ll dfs(int v, int t, ll f) {
        if (v == t || !f) return f;
        for (int& i = ptr[v]; i < sz(adj[v]); i++) {
            Edge& e = adj[v][i];
            if (lvl[e.to] == lvl[v] + 1)
                if (ll p = dfs(e.to, t, min(f, e.c))) {
                    e.c -= p, adj[e.to][e.rev].c += p;
                    return p;
                }
        }
        return 0;
    }
    ll calc(int s, int t) {
        ll flow = 0; q[0] = s;
        rep(L,0,31) do { // 'int L=30' maybe faster for random data
            lvl = ptr = vi(sz(q));
            int qi = 0, qe = lvl[s] = 1;
            while (qi < qe && !lvl[t]) {
                int v = q[qi++];
                for (Edge e : adj[v])
                    if (!lvl[e.to] && e.c >> (30 - L))
                        q[qe++] = e.to, lvl[e.to] = lvl[v] + 1;
            }
            while (ll p = dfs(s, t, LLONG_MAX)) flow += p;
        } while (lvl[t]);
        return flow;
    }
    bool leftOfMinCut(int a) { return lvl[a] != 0; }
};
```

## 7.5 Gomory Hu

```cpp
/**
 * Author: chilli, Takanori MAEHARA
 * Date: 2020-04-03
 * License: CC0
 * Source: https://github.com/spaghetti-source/algorithm/blob/master/graph/
     gomory_hu_tree.cc#L102
 * Description: Given a list of edges representing an undirected flow graph,
 * returns edges of the Gomory-Hu tree. The max flow between any pair of
 * vertices is given by minimum edge weight along the Gomory-Hu tree path.
 * Time: $O(V)$ Flow Computations
 * Status: Tested on CERC 2015 J, stress-tested
 *
 * Details: The implementation used here is not actually the original
 * Gomory-Hu, but Gusfield's simplified version: "Very simple methods for all
 * pairs network flow analysis". PushRelabel is used here, but any flow
 * implementation that supports `leftOfMinCut' also works.
 */
#pragma once

#include "PushRelabel.h"

typedef array<ll, 3> Edge;
vector<Edge> gomoryHu(int N, vector<Edge> ed) {
    vector<Edge> tree;
    vi par(N);
    rep(i,1,N) {
        PushRelabel D(N); // Dinic also works
        for (Edge t : ed) D.addEdge(t[0], t[1], t[2], t[2]);
        tree.push_back({i, par[i], D.calc(i, par[i])});
        rep(j,i+1,N)
            if (par[j] == par[i] && D.leftOfMinCut(j)) par[j] = i;
```

```
    }
    return tree;
}
```

## 7.6 Kosaraju

```
/*
  g :  Adjacency List of the original graph
  rg : Reversed Adjacency List
  vis : A bitset to mark visited nodes
  adj : Adjacency List of the super graph
  stk : holds dfs ordered elements
  cmp[i] : holds the component of node i
  go[i] : holds the nodes inside the strongly connected component i
*/

#define FOR(i,a,b) for(int i = a; i < b; i++)
#define pb push_back

const int N = 1e5+5;

vector<vector<int>>g, rg;
vector<vector<int>>go;
bitset<N>vis;
vector<vector<int>>adj;
stack<int>stk;
int n, m, cmp[N];
void add_edge(int u, int v){
  g[u].push_back(v);
  rg[v].push_back(u);
}
void dfs(int u){
  vis[u]=1;
  for(auto v : g[u])if(!vis[v])dfs(v);
  stk.push(u);
}
void rdfs(int u,int c){
  vis[u] = 1;
  cmp[u] = c;
  go[c].push_back(u);
  for(auto v : rg[u])if(!vis[v])rdfs(v,c);
}
int scc(){
  vis.reset();
  for(int i = 0; i < n; i++)if(!vis[i])
    dfs(i);
  vis.reset();
  int c = 0;
  while(stk.size()){
    auto cur = stk.top();
    stk.pop();
    if(!vis[cur])
      rdfs(cur,c++);

  }
  return c;
}
```

## 7.7 Maximum Clique

```
///Complexity O(3 ^ (N/3)) i.e works for 50
///you can change it to maximum independent set by flipping the edges 0->1, 1->0
///if you want to extract the nodes they are 1-bits in R
int g[60][60];
int res;
long long edges[60];
void BronKerbosch(int n, long long R, long long P, long long X) {
  if (P == 0LL && X == 0LL) { //here we will find all possible maximal cliques (not maximum) i.e. there is no node which can be included in this set
    int t = __builtin_popcountll(R);
    res = max(res, t);
    return;
  }
  int u = 0;
  while (!((1LL << u) & (P | X))) u ++;
  for (int v = 0; v < n; v++) {
    if (((1LL << v) & P) && !((1LL << v) & edges[u])) {
      BronKerbosch(n, R | (1LL << v), P & edges[v], X & edges[v]);
      P -= (1LL << v);
      X |= (1LL << v);
    }
  }
}
int max_clique (int n) {
  res = 0;
  for (int i = 1; i <= n; i++) {
    edges[i - 1] = 0;
    for (int j = 1; j <= n; j++)  if (g[i][j]) edges[i - 1] |= ( 1LL << (j - 1) );
  }
  BronKerbosch(n, 0, (1LL << n) - 1, 0);
  return res;
}
```

## 7.8 HopcraftKarp matching (Bipartite)

```
// Hopcroft-Karp's (1-based)
// Complexity: O(m * sqrt{n})
struct graph {
    int L, R;
    vector<vector<int> > adj;

    graph(int l, int r) : L(l), R(r), adj(l + 1) {}

    void add_edge(int u, int v) {
        adj[u].push_back(v + L);
    }

    int maximum_matching() {
        vector<int> mate(L + R + 1, -1), level(L + 1);
        function<bool(void)> levelize = [&]() {
            queue<int> q;
            for (int i = 1; i <= L; i++) {
                level[i] = -1;
                if (mate[i] < 0)
                    q.push(i), level[i] = 0;
            }
            while (!q.empty()) {
                int node = q.front();
                q.pop();
                for (auto i: adj[node]) {
                    int v = mate[i];
                    if (v < 0)
                        return true;
                    if (level[v] < 0) {
                        level[v] = level[node] + 1;
                        q.push(v);
                    }
                }
            }
            return false;
        };
        function<bool(int)> augment = [&](int node) {
            for (auto i: adj[node]) {
                int v = mate[i];
                if (v < 0 || (level[v] > level[node] && augment(v))) {
                    mate[node] = i;
                    mate[i] = node;
                    return true;
                }
            }
            return false;
        };
        int match = 0;
        while (levelize())
            for (int i = 1; i <= L; i++)
                if (mate[i] < 0 && augment(i))
                    match++;
        return match;
    }
};
```

## 7.9 Hungarian Weighted matching (Bipartite)

```
// Weighted Bipartite matching N^2 * M
// note that n must be <= m so in case in your problem n >= m, just swap
// also note this void set(int x, int y, ll v){a[x+1][y+1]=v;}
// the algorithim assumes you're using 0-index but it's using 1-based
struct Hungarian {
    const ll INF = 1000000000000000000; ///10^18
    int n,m;
    vector<vector<ll> > a;
    vector<ll> u,v;vector<int> p,way;
    Hungarian(int n, int m):
    n(n),m(m),a(n+1,vector<ll>(m+1,INF-1)),u(n+1),v(m+1),p(m+1),way(m+1){}
    void set(int x, int y, ll v){a[x+1][y+1]=v;}
    ll assign(){
        for(int i = 1; i <= n; i++){
            int j0=0;p[0]=i;
            vector<ll> minv(m+1,INF);
            vector<char> used(m+1,false);
            do {
                used[j0]=true;
                int i0=p[j0],j1;ll delta=INF;
                for(int j = 1; j <= m; j++)if(!used[j]){
                    ll cur=a[i0][j]-u[i0]-v[j];
                    if(cur<minv[j])minv[j]=cur,way[j]=j0;
                    if(minv[j]<delta)delta=minv[j],j1=j;
```

```
                }
                for(int j = 0; j <= m; j++)
                    if(used[j])u[p[j]]+=delta,v[j]-=delta;
                    else minv[j]-=delta;
                j0=j1;
            } while(p[j0]);
            do {
                int j1=way[j0];p[j0]=p[j1];j0=j1;
            } while(j0);
        }
        return -v[0];
    }
    vector<int> restoreAnswer() {   ///run it after assign
        vector<int> ans (n+1);
        for (int j=1; j<=m; ++j)
            ans[p[j]] = j;
        return ans;
    }
};
```

## 7.10 MinCostMaxFlow

```
/*
    Notes:
        make sure you notice the #define int ll
        focus on the data types of the max flow everythign inside is integer
        addEdge(u,v,cap,cost)
        note that for min cost max flow the cost is sum of cost * flow over all
            edges
*/
struct Edge {
    int to;
    int cost;
    int cap, flow, backEdge;
};
struct MCMF {
    const int inf = 1000000010;
    int n;
    vector<vector<Edge>> g;
    MCMF(int _n) {
        n = _n + 1;
        g.resize(n);
    }
    void addEdge(int u, int v, int cap, int cost) {
        Edge e1 = {v, cost, cap, 0, (int) g[v].size()};
        Edge e2 = {u, -cost, 0, 0, (int) g[u].size()};
        g[u].push_back(e1);
        g[v].push_back(e2);
    }
    pair<int, int> minCostMaxFlow(int s, int t) {
        int flow = 0;
        int cost = 0;
        vector<int> state(n), from(n), from_edge(n);
        vector<int> d(n);
        deque<int> q;
        while (true) {
            for (int i = 0; i < n; i++)
                state[i] = 2, d[i] = inf, from[i] = -1;
            state[s] = 1;
            q.clear();
            q.push_back(s);
            d[s] = 0;
            while (!q.empty()) {
                int v = q.front();
                q.pop_front();
                state[v] = 0;
                for (int i = 0; i < (int) g[v].size(); i++) {
                    Edge e = g[v][i];
                    if (e.flow >= e.cap || (d[e.to] <= d[v] + e.cost))
                        continue;
                    int to = e.to;
                    d[to] = d[v] + e.cost;
                    from[to] = v;
                    from_edge[to] = i;
                    if (state[to] == 1) continue;
                    if (!state[to] || (!q.empty() && d[q.front()] > d[to]))
                        q.push_front(to);
                    else q.push_back(to);
                    state[to] = 1;
                }
            }
            if (d[t] == inf) break;
            int it = t, addflow = inf;
            while (it != s) {
                addflow = min(addflow,
                              g[from[it]][from_edge[it]].cap
                              - g[from[it]][from_edge[it]].flow);
                it = from[it];
            }
            it = t;
            while (it != s) {
                g[from[it]][from_edge[it]].flow += addflow;
                g[it][g[from[it]][from_edge[it]].backEdge].flow -= addflow;
                cost += g[from[it]][from_edge[it]].cost * addflow;
                it = from[it];
            }
            flow += addflow;
        }
        return {cost, flow};
    }
};
```

## 7.11 Push Relabel Max Flow

```
struct edge {
    int from, to, cap, flow, index;
    edge(int from, int to, int cap, int flow, int index) :
            from(from), to(to), cap(cap), flow(flow), index(index) {}
};

struct PushRelabel {
    int n;
    vector <vector<edge>> g;
    vector<long long> excess;
    vector<int> height, active, count;
    queue<int> Q;

    PushRelabel(int n) :
            n(n), g(n), excess(n), height(n), active(n), count(2 * n) {}

    void addEdge(int from, int to, int cap) {
        g[from].push_back(edge(from, to, cap, 0, g[to].size()));
        if (from == to)
            g[from].back().index++;
        g[to].push_back(edge(to, from, 0, 0, g[from].size() - 1));
    }
    void enqueue(int v) {
        if (!active[v] && excess[v] > 0) {
            active[v] = true;
            Q.push(v);
        }
    }
    void push(edge &e) {
        int amt = (int) min(excess[e.from], (long long) e.cap - e.flow);
        if (height[e.from] <= height[e.to] || amt == 0)
            return;
        e.flow += amt;
        g[e.to][e.index].flow -= amt;
        excess[e.to] += amt;
        excess[e.from] -= amt;
        enqueue(e.to);
    }
    void relabel(int v) {
        count[height[v]]--;
        int d = 2 * n;
        for (auto &it: g[v]) {
            if (it.cap - it.flow > 0)
                d = min(d, height[it.to] + 1);
        }
        height[v] = d;
        count[height[v]]++;
        enqueue(v);
    }
    void gap(int k) {
        for (int v = 0; v < n; v++) {
            if (height[v] < k)
                continue;
            count[height[v]]--;
            height[v] = max(height[v], n + 1);
            count[height[v]]++;
            enqueue(v);
        }
    }
    void discharge(int v) {
        for (int i = 0; excess[v] > 0 && i < g[v].size(); i++)
            push(g[v][i]);
        if (excess[v] > 0) {
            if (count[height[v]] == 1)
                gap(height[v]);
            else
                relabel(v);
        }
    }
    long long max_flow(int source, int dest) {
        count[0] = n - 1;
        count[n] = 1;
        height[source] = n;
        active[source] = active[dest] = 1;
        for (auto &it: g[source]) {
```


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```
            excess[source] += it.cap;
            push(it);
        }
        while (!Q.empty()) {
            int v = Q.front();
            Q.pop();
            active[v] = false;
            discharge(v);
        }
        long long max_flow = 0;
        for (auto &e: g[source])
            max_flow += e.flow;

        return max_flow;
    }
};
```

## 7.12 Minmimum Vertex Cover (Bipartite)

```
int myrandom (int i) { return std::rand()%i;}

struct MinimumVertexCover {
    int n, id;
    vector<vector<int> > g;
    vector<int> color, m, seen;
    vector<int> comp[2];
    MinimumVertexCover() {}
    MinimumVertexCover(int n, vector<vector<int> > g) {
        this->n = n;
        this->g = g;
        color = m = vector<int>(n, -1);
        seen = vector<int>(n, 0);
        makeBipartite();
    }

    void dfsBipartite(int node, int col) {
        if (color[node] != -1) {
            assert(color[node] == col); /* MSH BIPARTITE YA BASHMOHANDES */
            return;
        }
        color[node] = col;
        comp[col].push_back(node);
        for (int i = 0; i < int(g[node].size()); i++)
            dfsBipartite(g[node][i], 1 - col);
    }

    void makeBipartite() {
        for (int i = 0; i < n; i++)
            if (color[i] == -1)
                dfsBipartite(i, 0);
    }

    // match a node
    bool dfs(int node) {
      random_shuffle(g[node].begin(),g[node].end());
        for (int i = 0; i < g[node].size(); i++) {
            int child = g[node][i];
            if (m[child] == -1) {
                m[node] = child;
                m[child] = node;
                return true;
            }
            if (seen[child] == id)
                continue;
            seen[child] = id;
            int enemy = m[child];
            m[node] = child;
            m[child] = node;
            m[enemy] = -1;
            if (dfs(enemy))
                return true;
            m[node] = -1;
            m[child] = enemy;
            m[enemy] = child;
        }
        return false;
    }

    void makeMatching() {
    for(int j = 0; j < 5; j++)
      random_shuffle(comp[0].begin(),comp[0].end(),myrandom );
        for (int i = 0; i < int(comp[0].size()); i++) {
            id++;
            if(m[comp[0][i]] == -1)
                dfs(comp[0][i]);
        }
    }


    void recurse(int node, int x, vector<int> &minCover, vector<int> &done) {
        if (m[node] != -1)
            return;
        if (done[node])return;
        done[node] = 1;
        for (int i = 0; i < int(g[node].size()); i++) {
            int child = g[node][i];
            int newnode = m[child];
            if (done[child]) continue;
            if(newnode == -1) {
                continue;
            }
            done[child] = 2;
            minCover.push_back(child);
            m[newnode] = -1;
            recurse(newnode, x, minCover, done);
        }
    }

    vector<int> getAnswer() {
        vector<int> minCover, maxIndep;
        vector<int> done(n, 0);
        makeMatching();
        for (int x = 0; x < 2; x++)
            for (int i = 0; i < int(comp[x].size()); i++) {
                int node = comp[x][i];
                if (m[node] == -1)
                    recurse(node, x, minCover, done);
            }

        for (int i = 0; i < int(comp[0].size()); i++)
            if (!done[comp[0][i]]) {
                minCover.push_back(comp[0][i]);
            }
        return minCover;
    }
};
```

## 7.13 Prufer Code

```
const int N = 3e5 + 9;
/*
prufer code is a sequence of length n-2 to uniquely determine a labeled tree with n vertices
Each time take the leaf with the lowest number and add the node number the leaf is connected to
the sequence and remove the leaf. Then break the algo after n-2 iterations
*/
//0-indexed
int n;
vector<int> g[N];
int parent[N], degree[N];

void dfs (int v) {
  for (size_t i = 0; i < g[v].size(); ++i) {
    int to = g[v][i];
    if (to != parent[v]) {
      parent[to] = v;
      dfs (to);
    }
  }
}

vector<int> prufer_code() {
  parent[n - 1] = -1;
  dfs (n - 1);
  int ptr = -1;
  for (int i = 0; i < n; ++i) {
    degree[i] = (int) g[i].size();
    if (degree[i] == 1 && ptr == -1) ptr = i;
  }
  vector<int> result;
  int leaf = ptr;
  for (int iter = 0; iter < n - 2; ++iter) {
    int next = parent[leaf];
    result.push_back (next);
    --degree[next];
    if (degree[next] == 1 && next < ptr) leaf = next;
    else {
      ++ptr;
      while (ptr < n && degree[ptr] != 1) ++ptr;
      leaf = ptr;
    }
  }
  return result;
}
vector < pair<int, int> > prufer_to_tree(const vector<int> & prufer_code) {
  int n = (int) prufer_code.size() + 2;
  vector<int> degree (n, 1);
  for (int i = 0; i < n - 2; ++i) ++degree[prufer_code[i]];

  int ptr = 0;
  while (ptr < n && degree[ptr] != 1) ++ptr;
```

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```cpp
    int leaf = ptr;
    vector < pair<int, int> > result;
    for (int i = 0; i < n - 2; ++i) {
      int v = prufer_code[i];
      result.push_back (make_pair (leaf, v));
      --degree[leaf];
      if (--degree[v] == 1 && v < ptr) leaf = v;
      else {
        ++ptr;
        while (ptr < n && degree[ptr] != 1) ++ptr;
        leaf = ptr;
      }
    }
    for (int v = 0; v < n - 1; ++v) if (degree[v] == 1) result.push_back (
        make_pair (v, n - 1));
    return result;
  }
```

## 7.14 Tarjan Algo

```cpp
vector< vector<int> > scc;
vector<int> adj[N];
int dfsn[N], low[N], cost[N], timer, in_stack[N];
stack<int> st;

// to detect all the components (cycles) in a directed graph
void tarjan(int node){
    dfsn[node] = low[node] = ++timer;
    in_stack[node] = 1;
    st.push(node);
    for(auto i: adj[node]){
        if(dfsn[i] == 0){
            tarjan(i);
            low[node] = min(low[node], low[i]);
        }
        else if(in_stack[i]) low[node] = min(low[node], dfsn[i]);
    }
    if(dfsn[node] == low[node]){
        scc.push_back(vector<int>());
        while(1){
            int cur = st.top();
            st.pop();
            in_stack[cur] = 0;
            scc.back().push_back(cur);
            if(cur == node) break;
        }
    }
}
int main(){
    int m;
    cin >> m;
    while(m--){
        int u, v;
        cin >> u >> v;
        adj[u].push_back(v);
    }
    for(int i = 1; i <= n; i++){
        if(dfsn[i] == 0){
            tarjan(i);
        }
    }

    return 0;
}
```

# 8 NumberTheory

## 8.1 ModSum (Sum Of floored division)

```cpp
// log(m), with a large constant.
typedef unsigned long long ull;
ull sumsq(ull to) { return to / 2 * ((to-1) | 1); }

// return sum_{i=0}^{to-1} floor((ki + c) / m) (mod 2^64)
ull divsum(ull to, ull c, ull k, ull m) {
    ull res = k / m * sumsq(to) + c / m * to;
    k %= m; c %= m;
    if (!k) return res;
    ull to2 = (to * k + c) / m;
    return res + (to - 1) * to2 - divsum(to2, m-1 - c, m, k);
}
// return sum_{i=0}^{to-1} (ki+c) % m
ll modsum(ull to, ll c, ll k, ll m) {
    c = ((c % m) + m) % m;
    k = ((k % m) + m) % m;
    return to * c + k * sumsq(to) - m * divsum(to, c, k, m);
}
```

## 8.2 ModMulLL

```cpp
// Calculate a^b % c and a*b % c
typedef unsigned long long ull;
ull modmul(ull a, ull b, ull M) {
    ll ret = a * b - M * ull(1.L / M * a * b);
    return ret + M * (ret < 0) - M * (ret >= (ll)M);
}
ull modpow(ull b, ull e, ull mod) {
    ull ans = 1;
    for (; e; b = modmul(b, b, mod), e /= 2)
        if (e & 1) ans = modmul(ans, b, mod);
    return ans;
}
```

## 8.3 ModSqrt Finds x s.t $x^2 = a \mod p$

```cpp
// Description: Finds x s.t. x^2 = a mod p
// Time: O(log^2 p) worst case, O(log p) for most p
ll sqrt(ll a, ll p) {
    a %= p; if (a < 0) a += p;
    if (a == 0) return 0;
    assert(modpow(a, (p-1)/2, p) == 1); // else no solution
    if (p % 4 == 3) return modpow(a, (p+1)/4, p);
    // a^(n+3)/8 or 2^(n+3)/8 * 2^(n-1)/4 works if p % 8 == 5
    ll s = p - 1, n = 2;
    int r = 0, m;
    while (s % 2 == 0)
        ++r, s /= 2;
    /// find a non-square mod p
    while (modpow(n, (p - 1) / 2, p) != p - 1) ++n;
    ll x = modpow(a, (s + 1) / 2, p);
    ll b = modpow(a, s, p), g = modpow(n, s, p);
    for (;; r = m) {
        ll t = b;
        for (m = 0; m < r && t != 1; ++m)
            t = t * t % p;
        if (m == 0) return x;
        ll gs = modpow(g, 1LL << (r - m - 1), p);
        g = gs * gs % p;
        x = x * gs % p;
        b = b * g % p;
    }
}
```

## 8.4 MillerRabin Primality check

```cpp
#include "ModMulLL.h"
bool isPrime(ull n) {
    if (n < 2 || n % 6 % 4 != 1) return (n | 1) == 3;
    ull A[] = {2, 325, 9375, 28178, 450775, 9780504, 1795265022},
        s = __builtin_ctzll(n-1), d = n >> s;
    for (ull a : A) {   // ^ count trailing zeroes
        ull p = modpow(a%n, d, n), i = s;
        while (p != 1 && p != n - 1 && a % n && i--)
            p = modmul(p, p, n);
        if (p != n-1 && i != s) return 0;
    }
    return 1;
}
```

## 8.5 Pollard-rho randomized factorization algorithm $O(n^{1/4})$

```cpp
"ModMulLL.cpp", "MillerRabin.cpp"
ull pollard(ull n) {
    auto f = [n](ull x) { return modmul(x, x, n) + 1; };
    ull x = 0, y = 0, t = 30, prd = 2, i = 1, q;
    while (t++ % 40 || __gcd(prd, n) == 1) {
        if (x == y) x = ++i, y = f(x);
        if ((q = modmul(prd, max(x,y) - min(x,y), n))) prd = q;
        x = f(x), y = f(f(y));
    }
    return __gcd(prd, n);
}
vector<ull> factor(ull n) {
    if (n == 1) return {};
    if (isPrime(n)) return {n};
    ull x = pollard(n);
    auto l = factor(x), r = factor(n / x);
    l.insert(l.end(), all(r));
    return l;
}
```

## 8.6 Primitive Roots

```cpp
int primitive_root (int p) {
    vector<int> fact;
    int phi = p - 1,  n = phi;
    for (int i = 2; i * i <= n; ++i)
        if (n % i == 0) {
            fact.push_back (i);
            while (n % i == 0)
                n /= i;
        }
    if (n > 1)
        fact.push_back (n);

    for (int res = 2; res <= p; ++res) {
        bool ok = true;
        for (size_t i = 0; i < fact.size() && ok; ++i)
            ok &= powmod (res, phi / fact[i], p) != 1;
        if (ok)  return res;
    }
    return -1;
}
```

## 8.7 Discrete Logarithm minimum x for which $a^x = b\%m$

```cpp
// Returns the smallest x > 0 :  a^x = b mod m
ll modLog(ll a, ll b, ll m) {
    ll n = (ll) sqrt(m) + 1, e = 1, f = 1, j = 1;
    unordered_map<ll, ll> A;
    while (j <= n && (e = f = e * a % m) != b % m)
        A[e * b % m] = j++;
    if (e == b % m) return j;
    if ((__gcd(m, e) == (__gcd(m, b))
        rep(i, 2, n + 2) if (A.count(e = e * f % m))
            return n * i - A[e];
    return -1;
}
```

## 8.8 Discrete Root finds all numbers x such that $x^k = a\%n$

```cpp
// This program finds all numbers x such that x^k = a (mod n)
vector<int> discrete_root(int n, int k, int a) {
    if (a == 0)
        return {0};

    int g = primitive_root(n);
    // Baby-step giant-step discrete logarithm algorithm
    int sq = (int) sqrt(n + .0) + 1;
    vector<pair<int, int>> dec(sq);
    for (int i = 1; i <= sq; ++i)
        dec[i - 1] = {powmod(g, i * sq * k % (n - 1), n), i};
    sort(dec.begin(), dec.end());
    int any_ans = -1;
    for (int i = 0; i < sq; ++i) {
        int my = powmod(g, i * k % (n - 1), n) * a % n;
        auto it = lower_bound(dec.begin(), dec.end(), make_pair(my, 0));
        if (it != dec.end() && it->first == my) {
            any_ans = it->second * sq - i;
            break;
        }
    }
    if (any_ans == -1) return {};

    int delta = (n - 1) / __gcd(k, n - 1);
    vector<int> ans;
    for (int cur = any_ans % delta; cur < n - 1; cur += delta)
        ans.push_back(powmod(g, cur, n));
    sort(ans.begin(), ans.end());
    return ans;
}
```

## 8.9 Totient function

```cpp
void phi_1_to_n(int n) {
    for (int i = 0; i <= n; i++)
        phi[i] = i;
    for (int i = 2; i <= n; i++) {
        if (phi[i] == i) {
            for (int j = i; j <= n; j += i)
                phi[j] -= phi[j] / i;
        }
    }
}
```

## 8.10 CRT and EGCD

```cpp
ll extended(ll a, ll b, ll &x, ll &y) {
    if(b == 0) {
        x = 1;
        y = 0;
        return a;
    }
    ll x0, y0;
    ll g = extended(b, a % b, x0, y0);
    x = y0;
    y = x0 - a / b * y0;

    return g ;
}
ll de(ll a, ll b, ll c, ll &x, ll &y) {
    ll g = extended(abs(a), abs(b), x, y);
    if(c % g) return -1;
    x *= c / g;
    y *= c / g;
    if(a < 0)x = -x;
    if(b < 0)y = -y;
    return g;
}
pair<ll, ll> CRT(vector<ll> r, vector<ll> m) {
    ll r1 = r[0], m1 = m[0];
    for(int i = 1; i < r.size(); i++) {
        ll r2 = r[i], m2 = m[i];
        ll x0, y0;
        ll g = de(m1, -m2, r2 - r1, x0, y0);
        if(g == -1) return {-1, -1} ;
        x0 %= m2;
        ll nr = x0 * m1 + r1;
        ll nm = m1 / g * m2;
        r1 = (nr % nm + nm) % nm;
        m1 = nm;
    }
    return {r1, m1};
}
```

## 8.11 Xor With Gauss

```cpp
void insertVector(int mask) {
    for (int i = d - 1; i >= 0; i--) {
        if ((mask & 1 << i) == 0) continue;
        if (!basis[i]) {
            basis[i] = mask;
            return;
        }
        mask ^= basis[i];
    }
}
```

## 8.12 Josephus

```cpp
// n = total person
// will kill every kth person, if k = 2, 2,4,6,...
// returns the mth killed person
ll josephus(ll n, ll k, ll m) {
  m = n - m;
  if (k <= 1)return n - m;
  ll i = m;
  while (i < n) {
    ll r = (i - m + k - 2) / (k - 1);
    if ((i + r) > n) r = n - i;
    else if (!r) r = 1;
    i += r;
    m = (m + (r * k)) % i;
  } return m + 1;
}
```

# 9 Strings

## 9.1 Aho-Corasick Mostafa

```cpp
struct AC_FSM {
#define ALPHABET_SIZE 26

    struct Node {
        int child[ALPHABET_SIZE], failure = 0, match_parent = -1;
        vector<int> match;

        Node() {
            for (int i = 0; i < ALPHABET_SIZE; ++i)child[i] = -1;
        }
    };

    vector<Node> a;

    AC_FSM() {
        a.push_back(Node());
    }

    void construct_automaton(vector<string> &words) {
        for (int w = 0, n = 0; w < words.size(); ++w, n = 0) {
```

```
            for (int i = 0; i < words[w].size(); ++i) {
                if (a[n].child[words[w][i] - 'a'] == -1) {
                    a[n].child[words[w][i] - 'a'] = a.size();
                    a.push_back(Node());
                }
                n = a[n].child[words[w][i] - 'a'];
            }
            a[n].match.push_back(w);
        }
        queue<int> q;
        for (int k = 0; k < ALPHABET_SIZE; ++k) {
            if (a[0].child[k] == -1) a[0].child[k] = 0;
            else if (a[0].child[k] > 0) {
                a[a[0].child[k]].failure = 0;
                q.push(a[0].child[k]);
            }
        }
        while (!q.empty()) {
            int r = q.front();
            q.pop();
            for (int k = 0, arck; k < ALPHABET_SIZE; ++k) {
                if ((arck = a[r].child[k]) != -1) {
                    q.push(arck);
                    int v = a[r].failure;
                    while (a[v].child[k] == -1) v = a[v].failure;
                    a[arck].failure = a[v].child[k];
                    a[arck].match_parent = a[v].child[k];
                    while (a[arck].match_parent != -1 &&
                           a[a[arck].match_parent].match.empty())
                        a[arck].match_parent =
                                a[a[arck].match_parent].match_parent;
                }
            }
        }
    }

    void aho_corasick(string &sentence, vector<string> &words,
                      vector<vector<int> > &matches) {
        matches.assign(words.size(), vector<int>());
        int state = 0, ss = 0;
        for (int i = 0; i < sentence.length(); ++i, ss = state) {
            while (a[ss].child[sentence[i] - 'a'] == -1)
                ss = a[ss].failure;
            state = a[state].child[sentence[i] - 'a'] = a[ss].child[sentence[i]
                - 'a'];
            for (ss = state; ss != -1; ss = a[ss].match_parent)
                for (int w: a[ss].match)
                    matches[w].push_back(i + 1 - words[w].length());
        }
    }
};
```

## 9.2 KMP Anany

```
vector<int> fail(string s) {
    int n = s.size();
    vector<int> pi(n);
    for(int i = 1; i < n; i++) {
        int g = pi[i-1];
        while(g && s[i] != s[g])
            g = pi[g-1];
        g += s[i] == s[g];
        pi[i] = g;
    }
    return pi;
}
vector<int> KMP(string s, string t) {
    vector<int> pi = fail(t);
    vector<int> ret;
    for(int i = 0, g = 0; i < s.size(); i++) {
        while (g && s[i] != t[g])
            g = pi[g-1];
        g += s[i] == t[g];
        if(g == t.size()) { ///occurrence found
            ret.push_back(i-t.size()+1);
            g = pi[g-1];
        }
    }
    return ret;
}
```

## 9.3 Manacher Kactl

```
// If the size of palindrome centered at i is x, then d1[i] stores (x+1)/2.

vector<int> d1(n);
for (int i = 0, l = 0, r = -1; i < n; i++) {
    int k = (i > r) ? 1 : min(d1[l + r - i], r - i + 1);
    while (0 <= i - k && i + k < n && s[i - k] == s[i + k]) {
        k++;
    }
    d1[i] = k--;
    if (i + k > r) {
        l = i - k;
        r = i + k;
    }
}

// If the size of palindrome centered at i is x, then d2[i] stores x/2
vector<int> d2(n);
for (int i = 0, l = 0, r = -1; i < n; i++) {
    int k = (i > r) ? 0 : min(d2[l + r - i + 1], r - i + 1);
    while (0 <= i - k - 1 && i + k < n && s[i - k - 1] == s[i + k]) {
        k++;
    }
    d2[i] = k--;
    if (i + k > r) {
        l = i - k - 1;
        r = i + k ;
    }
}
```

## 9.4 Suffix Array Kactl

```
struct SuffixArray {
    using vi = vector<int>;
    #define rep(i,a,b)  for(int i = a; i < b; i++)
    #define all(x) begin(x), end(x)
    /*
        Note this code is considers also the empty suffix
        so hear sa[0] = n and sa[1] is the smallest non empty suffix
        and sa[n] is the largest non empty suffix
        also LCP[i] = LCP(sa[i-1], sa[i]), meanining LCP[0] = LCP[1] = 0
        if you want to get LCP(i..j) you need to build a mapping between
        sa[i] and i, and build a min sparse table to calculate the minimum
        note that this minimum should consider sa[i+1...j] since you don't want
        to consider LCP(sa[i], sa[i-1])

        you should also print the suffix array and lcp at the beginning of the
            contest
        to clarify this stuff
    */
    vi sa, lcp;
    SuffixArray(string& s, int lim=256) { // or basic_string<int>
        int n = sz(s) + 1, k = 0, a, b;
        vi x(all(s)+1), y(n), ws(max(n, lim)), rank(n);
        sa = lcp = y, iota(all(sa), 0);
        for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim = p) {
            p = j, iota(all(y), n - j);
            rep(i,0,n) if (sa[i] >= j) y[p++] = sa[i] - j;
            fill(all(ws), 0);
            rep(i,0,n) ws[x[i]]++;
            rep(i,1,lim) ws[i] += ws[i - 1];
            for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
            swap(x, y), p = 1, x[sa[0]] = 0;
            rep(i,1,n) a = sa[i - 1], b = sa[i], x[b] =
                (y[a] == y[b] && y[a + j] == y[b + j]) ? p - 1 : p++;
        }
        rep(i,1,n) rank[sa[i]] = i;
        for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
            for (k && k--, j = sa[rank[i] - 1];
                    s[i + k] == s[j + k]; k++);
    }
};
```

## 9.5 Suffix Automaton Mostafa

```
struct SA {
    struct node {
        int to[26];
        int link, len, co = 0;

        node() {
            memset(to, 0, sizeof to);
            co = 0, link = 0, len = 0;
        }
    };

    int last, sz;
    vector<node> v;

    SA() {
        v = vector<node>(1);
        last = 0, sz = 1;
    }

    void add_letter(int c) {
        int p = last;
        last = sz++;
        v.push_back({});
```

```
            v[last].len = v[p].len + 1;
            v[last].co = 1;
            for (; v[p].to[c] == 0; p = v[p].link)
                v[p].to[c] = last;
            if (v[p].to[c] == last) {
                v[last].link = 0;
                return;
            }
            int q = v[p].to[c];
            if (v[q].len == v[p].len + 1) {
                v[last].link = q;
                return;
            }
            int cl = sz++;
            v.push_back(v[q]);
            v.back().co = 0;
            v.back().len = v[p].len + 1;
            v[last].link = v[q].link = cl;

            for (; v[p].to[c] == q; p = v[p].link)
                v[p].to[c] = cl;
        }

        void build_co() {
            priority_queue<pair<int, int>> q;
            for (int i = sz - 1; i > 0; i--)
                q.push({v[i].len, i});
            while (q.size()) {
                int i = q.top().second;
                q.pop();
                v[v[i].link].co += v[i].co;
            }
        }
};
```

## 9.6 Zalgo Anany

```
int z[N], n;
void Zalgo(string s) {
    int L = 0, R = 0;
    for(int i = 1; i < n; i++) {
        if(i<=R&&z[i-L] < R - i + 1)z[i] = z[i-L];
        else {
            L = i;
            R = max(R,i);
            while(R < n && s[R-L] == s[R])R++;
            z[i] = R-L; --R;
        }
    }
}
```

## 9.7 lexicographically smallest rotation of a string

```
int minRotation(string s) {
    int a=0, N=sz(s); s += s;
    rep(b,0,N) rep(k,0,N) {
        if (a+k == b || s[a+k] < s[b+k]) {b += max(0, k-1); break;}
        if (s[a+k] > s[b+k]) { a = b; break; }
    }
    return a;
}
```

# 10 Trees

## 10.1 Centroid Decomposition

```
/*
    Properties:
        1. consider path(a,b) can be decomposed to path(a,lca(a,b)) and path(b,lca(a,b))
        where lca(a,b) is the lca on the centroid tree
        2. Each one of the n^2 paths is the concatenation of two paths in a set of O(n lg(n))
        paths from a node to all its ancestors in the centroid decomposition.
        3. Ancestor of a node in the original tree is either an ancestor in the CD tree or
        a descendadnt
*/
vector<int> adj[N]; ///adjacency list of original graph
int n;
int sz[N];
bool used[N];
int centPar[N]; ///parent in centroid
void init(int node, int par) {  ///initialize size
    sz[node] = 1;
    for(auto p : adj[node])
        if(p != par && !used[p]) {
            init(p, node);
            sz[node] += sz[p];
        }
}
int centroid(int node, int par, int limit) {    ///get centroid
    for(int p : adj[node])
        if(!used[p] && p != par && sz[p] * 2 > limit)
        return centroid(p, node, limit);
    return node;
}
int decompose(int node) {
    init(node,node);    ///calculate size
    int c = centroid(node, node, sz[node]); ///get centroid
    used[c] = true;
    for(auto p : adj[c])if(!used[p.F]) {    ///initialize parent for others and decompose
        centPar[decompose(p.F)] = c;
    }
    return c;
}
void update(int node, int distance, int col) {
    int centroid = node;
    while(centroid){
        ///solve
        centroid = centPar[centroid];
    }
}
int query(int node) {

    int ans = 0;

    int centroid = node;
    while(centroid) {
        ///solve
        centroid = centPar[centroid];
    }

    return ans;
}
```

## 10.2 Dsu On Trees

```
const int N = 1e5 + 9;
vector<int> adj[N];
int bigChild[N], sz[N];
void dfs(int node, int par) {
    for(auto v : adj[node]) if(v != par){
        dfs(v, node);
        sz[node] += sz[v];
        if(!bigChild[node] || sz[v] > sz[bigChild[node]]) {
            bigChild[node] = v;
        }
    }
}
void add(int node, int par, int bigChild, int delta) {

    ///modify node to data structure

    for(auto v : adj[node])
    if(v != par && v != bigChild)
        add(v, node, bigChild, delta);

}
void dfs2(int node, int par, bool keep) {
    for(auto v : adj[node])if(v != par && v != bigChild[node]) {
        dfs2(v, node, 0);
    }
    if(bigChild[node]) {
        dfs2(bigChild[node], node, true);
    }
    add(node, par, bigChild[node], 1);
    ///process queries
    if(!keep) {
        add(node, par, -1, -1);
    }
}
```

## 10.3 Heavy Light Decomposition (Along with Euler Tour)

```
/*
    Notes:
        1. 0-based
        2. solve function iterates over segments and handles them seperatly
        if you're gonna use it make sure you know what you're doing
        3. to update/query segment in[node], out[node]
        4. to update/query chain in[nxt[node]], in[node]
        nxt[node]: is the head of the chain so to go to the next chain node = par[nxt[node]]
*/
int sz[mxN], nxt[mxN];
```

```
int in[N], out[N], rin[N];
vector<int> g[mxN];
int par[mxN];

void dfs_sz(int v = 0, int p = -1) {
    sz[v] = 1;
    par[v] = p;
    for (auto &u : g[v]) {
        if (u == p) {
            swap(u, g[v].back());
        }
        if(u == p) continue;
        dfs_sz(u,v);
        sz[v] += sz[u];
        if (sz[u] > sz[g[v][0]])
            swap(u, g[v][0]);
    }
    if(v != 0)
        g[v].pop_back();
}

void dfs_hld(int v = 0) {
    in[v] = t++;
    rin[in[v]] = v;
    for (auto u : g[v]) {
        nxt[u] = (u == g[v][0] ? nxt[v] : u);
        dfs_hld(u);
    }
    out[v] = t;
}

int n;
bool isChild(int p, int u){
  return in[p] <= in[u] && out[u] <= out[p];
}
int solve(int u,int v) {
    vector<pair<int,int> > segu;
    vector<pair<int,int> > segv;
    if(isChild(u,v)){
      while(nxt[u] != nxt[v]){
        segv.push_back(make_pair(in[nxt[v]], in[v]));
        v = par[nxt[v]];
      }
      segv.push_back({in[u], in[v]});
    } else if(isChild(v,u)){
      while(nxt[u] != nxt[v]){
        segu.push_back(make_pair(in[nxt[u]], in[u]));
        u = par[nxt[u]];
      }
      segu.push_back({in[v], in[u]});
  } else {
      while(u  != v) {
        if(nxt[u] == nxt[v]) {
          if(in[u] < in[v]) segv.push_back({in[u],in[v]}), R.push_back({u+1,v+1});
          else segu.push_back({in[v],in[u]}), L.push_back({v+1,u+1});
          u = v;
          break;
        } else if(in[u] > in[v]) {
          segu.push_back({in[nxt[u]],in[u]}), L.push_back({nxt[u]+1, u+1});
          u = par[nxt[u]];
        } else {
          segv.push_back({in[nxt[v]],in[v]}), R.push_back({nxt[v]+1, v+1});
          v = par[nxt[v]];
        }
      }
    }
    reverse(segv.begin(),segv.end());
    int res = 0,state = 0;
    for(auto p : segu) {
        qry(1,1,0,n-1,p.first,p.second,state,res);
    }
    for(auto p : segv) {
        qry(0,1,0,n-1,p.first,p.second,state,res);
    }
    return res;
}
```

## 10.4 Mo on Trees

```
// Calculate the DFS order, {1, 2, 3, 3, 4, 4, 2, 5, 6, 6, 5, 1}.
// Let a query be (u, v), ST(u) <= ST(v), P = LCA(u,v)
// Case 1: P = u : the query range would be [ST(u), ST(v)]
// Case 2: P != u : range would be [EN(u), ST(v)] + [ST(P), ST(P)].
// the path will be the nodes that appears exactly once in that range
```

# 11 Numerical

## 11.1 Lagrange Polynomial

```
class LagrangePoly {
public:
    LagrangePoly(std::vector<long long> _a) {
        //f(i) = _a[i]
        //interpola o vetor em um polinomio de grau y.size() - 1
        y = _a;
        den.resize(y.size());
        int n = (int) y.size();
        for(int i = 0; i < n; i++) {
            y[i] = (y[i] % MOD + MOD) % MOD;
            den[i] = ifat[n - i - 1] * ifat[i] % MOD;
            if((n - i - 1) % 2 == 1) {
                den[i] = (MOD - den[i]) % MOD;
            }
        }
    }

    long long getVal(long long x) {
        int n = (int) y.size();
        x = (x % MOD + MOD) % MOD;
        if(x < n) {
            //return y[(int) x];
        }
        std::vector<long long> l, r;
        l.resize(n);
        l[0] = 1;
        for(int i = 1; i < n; i++) {
            l[i] = l[i - 1] * (x - (i - 1) + MOD) % MOD;
        }
        r.resize(n);
        r[n - 1] = 1;
        for(int i = n - 2; i >= 0; i--) {
            r[i] = r[i + 1] * (x - (i + 1) + MOD) % MOD;
        }
        long long ans = 0;
        for(int i = 0; i < n; i++) {
            long long coef = l[i] * r[i] % MOD;
            ans = (ans + coef * y[i] % MOD * den[i]) % MOD;
        }
        return ans;
    }

private:
    std::vector<long long> y, den;
};
```

## 11.2 Polynomials

```
struct Poly {
    vector<double> a;
    double operator()(double x) const {
        double val = 0;
        for (int i = sz(a); i--;) (val *= x) += a[i];
        return val;
    }
    void diff() {
        rep(i,1,sz(a)) a[i-1] = i*a[i];
        a.pop_back();
    }
    void divroot(double x0) {
        double b = a.back(), c; a.back() = 0;
        for(int i=sz(a)-1; i--;) c = a[i], a[i] = a[i+1]*x0+b, b=c;
        a.pop_back();
    }
};

// Finds the real roots to a polynomial
// O(n^2 log(1/e))
vector<double> polyRoots(Poly p, double xmin, double xmax) {
    if (sz(p.a) == 2) { return {-p.a[0]/p.a[1]}; }
    vector<double> ret;
    Poly der = p;
    der.diff();
    auto dr = polyRoots(der, xmin, xmax);
    dr.push_back(xmin-1);
    dr.push_back(xmax+1);
    sort(all(dr));
    rep(i,0,sz(dr)-1) {
        double l = dr[i], h = dr[i+1];
        bool sign = p(l) > 0;
        if (sign ^ (p(h) > 0)) {
            rep(it,0,60) { // while (h - l > 1e-8)
                double m = (l + h) / 2, f = p(m);
                if ((f <= 0) ^ sign) l = m;
                else h = m;
```

```
            }
            ret.push_back((l + h) / 2);
        }
    }
    return ret;
}

// Given n points (x[i], y[i]), computes an n-1-degree polynomial that passes
    through them.
// For numerical precision pick x[k] = c * cos(k / (n - 1) * pi).
// O(n^2)
typedef vector<double> vd;
vd interpolate(vd x, vd y, int n) {
    vd res(n), temp(n);
    rep(k,0,n-1) rep(i,k+1,n)
        y[i] = (y[i] - y[k]) / (x[i] - x[k]);
    double last = 0; temp[0] = 1;
    rep(k,0,n) rep(i,0,n) {
        res[i] += y[k] * temp[i];
        swap(last, temp[i]);
        temp[i] -= last * x[k];
    }
    return res;
}

// Recovers any n-order linear recurrence relation from the first 2n terms of
    the recurrence.
// Useful for guessing linear recurrences after bruteforcing the first terms.
// Should work on any field, but numerical stability for floats is not
    guaranteed.
// O (n^2)
vector<ll> berlekampMassey(vector<ll> s) {
    int n = sz(s), L = 0, m = 0;
    vector<ll> C(n), B(n), T;
    C[0] = B[0] = 1;

    ll b = 1;
    rep(i,0,n) { ++m;
        ll d = s[i] % mod;
        rep(j,1,L+1) d = (d + C[j] * s[i - j]) % mod;
        if (!d) continue;
        T = C; ll coef = d * modpow(b, mod-2) % mod;
        rep(j,m,n) C[j] = (C[j] - coef * B[j - m]) % mod;
        if (2 * L > i) continue;
        L = i + 1 - L; B = T; b = d; m = 0;
    }
    C.resize(L + 1); C.erase(C.begin());
    for (ll& x : C) x = (mod - x) % mod;
    return C;
}

// Generates the kth term of an n-order linear recurrence
// S[i] = S[i - j - 1]tr[j], given S[0..>= n - 1] and tr[0..n - 1]
// Useful together with Berlekamp-Massey.
// O(n^2 * log(k))
typedef vector<ll> Poly;
ll linearRec(Poly S, Poly tr, ll k) {
    int n = sz(tr);
    auto combine = [&](Poly a, Poly b) {
        Poly res(n * 2 + 1);
        rep(i,0,n+1) rep(j,0,n+1)
            res[i + j] = (res[i + j] + a[i] * b[j]) % mod;
        for (int i = 2 * n; i > n; --i) rep(j,0,n)
            res[i - 1 - j] = (res[i - 1 - j] + res[i] * tr[j]) % mod;
        res.resize(n + 1);
        return res;
    };
    Poly pol(n + 1), e(pol);
    pol[0] = e[1] = 1;
    for (++k; k; k /= 2) {
        if (k % 2) pol = combine(pol, e);
        e = combine(e, e);
    }
    ll res = 0;
    rep(i,0,n) res = (res + pol[i + 1] * S[i]) % mod;
    return res;
}
```

# 12 Guide

## 12.1 Strings

- Longest Common Substring is easier with suffix automaton
- Problems that tell you cound stuff that appears X times or count appearnces (Use suffixr links)
- Problems that tell you find the largest substring with some property (Use Suffix links)
- Remember suffix links are the same as aho corasic failure links (you can memoize them with dp)
- Problems that ask you to get the k-th string (can be either suffix automaton or array)
- Longest Common Prefix is mostly a (suffix automaton-array) thing
- try thinking bitsets

## 12.2 Volume

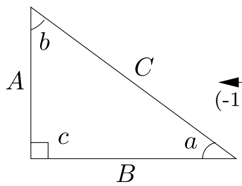
- Right circular cylinder $= \pi r^2 h$
- Pyramid $= \frac{Bh}{3}$
- Right circular cone $= \frac{\pi r^2 h}{3}$
- Sphere $= \frac{4}{3}\pi r^2 h$
- Sphere sector$= \frac{2}{3}\pi r^2 h = \frac{2}{3}\pi r^3(1 - cos(a))$
- Sphere cap $= \frac{\pi h^2(3r-h)}{3}$

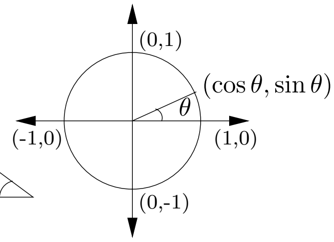
## 12.3 Graph Theory

- Euler formula: $v + f = e + 2$

## 12.4 Joseph problem

$$g(n,k) = \begin{cases} 0 & \text{if } n = 1 \\ (g(n-1,k)+k) \bmod n & \text{if } 1 < n < k \\ \left\lfloor \frac{k((g(n',k) - n \bmod k) \bmod n')}{k-1} \right\rfloor \text{ where } n' = n - \left\lfloor \frac{n}{k} \right\rfloor & \text{if } k \le n \end{cases}$$




Pythagorean theorem:
$$C^2 = A^2 + B^2.$$

Definitions:
$$\sin a = A/C, \quad \cos a = B/C,$$
$$\csc a = C/A, \quad \sec a = C/B,$$
$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:
$$\tfrac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:
$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$
$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$
$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$
$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$
$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$
$$\sin 2x = 2 \sin x \cos x, \qquad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$$
$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$
$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:
$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$



Multiplication:
$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff $A$ is non-singular.
$$\det A \cdot B = \det A \cdot \det B,$$
$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

$2 \times 2$ and $3 \times 3$ determinant:
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \begin{matrix} aei + bfg + cdh \\ - ceg - fha - ibd. \end{matrix}$$

Permanents:
$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

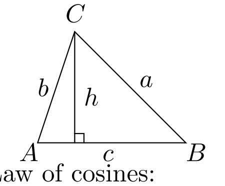
Hyperbolic Functions

Definitions:
$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

Identities:
$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$
$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$
$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$
$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

| $\theta$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $\frac{\pi}{2}$ | 1 | 0 | $\infty$ |

. . . in mathematics you don't understand things, you just get used to them.
– J. von Neumann




Law of cosines:
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:
$$A = \tfrac{1}{2}hc,$$
$$= \tfrac{1}{2}ab \sin C,$$
$$= \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Heron's formula:
$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$
$$s = \tfrac{1}{2}(a + b + c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

More identities:
$$\sin \tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$
$$\cos \tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$
$$\tan \tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x},$$
$$\cot \tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$
$$= \frac{1 + \cos x}{\sin x},$$
$$= \frac{\sin x}{1 - \cos x},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\tan x = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$
$$= -i \frac{e^{2ix} - 1}{e^{2ix} + 1},$$
$$\sin x = \frac{\sinh ix}{i},$$
$$\cos x = \cosh ix,$$
$$\tan x = \frac{\tanh ix}{i}.$$