QUESITON 3

Ans 3 (a)

$$x^T a = \sum_{i=1}^{n} x_i a_i$$

$$a^T x = \sum_{i=1}^{n} a_i x_i$$

$$\Rightarrow \quad x^T a = a^T x$$

$$\Rightarrow \quad \nabla_x (x^T a) = \nabla_x (a^T x)$$

$$\nabla_x (x^T a) = \begin{bmatrix} \dfrac{\partial \sum_{i=1}^{n} a_i x_i}{\partial x_1} \\ \dfrac{\partial \sum_{i=1}^{n} a_i x_i}{\partial x_2} \\ \vdots \\ \dfrac{\partial \sum_{i=1}^{n} a_i x_i}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$= a$$

$$\therefore \boxed{\nabla_x (x^T a) = \nabla_x (a^T x) = a}$$

Hence proved

(b) $$x^T A x = \begin{bmatrix} x_1 \dots x_n \end{bmatrix} \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} x_i A_{i1} & \dots & \sum_{i=1}^{n} x_i A_{in} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} x_i A_{i1} x_1 & \dots & \sum_{i=1}^{n} x_i A_{in} x_n \end{bmatrix}$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} x_i A_{ij} x_j$$

$$\Rightarrow \boxed{x^T A x = \sum_{j=1}^{n} \sum_{j=1}^{n} x_i A_{ij} x_j}$$

$$\nabla_x (x^T A x) = \begin{bmatrix} \frac{\partial}{\partial x_1} \sum_{j=1}^{n} \sum_{i=1}^{n} x_i A_{ij} x_j \\ \vdots \\ \frac{\partial}{\partial x_n} \sum_{j=1}^{n} \sum_{i=1}^{n} x_i A_{ij} x_j \end{bmatrix}$$

Let $n = 3$

$$\underline{\underline{x^T A x}} = x_1 A_{11} x_1 + x_1 A_{12} x_2 + x_1 A_{13} x_3 +$$
$$x_2 A_{21} x_1 + x_2 A_{22} x_2 + x_2 A_{23} x_3 +$$
$$x_3 A_{31} x_1 + x_3 A_{32} x_2 + x_3 A_{33} x_3$$

$$\nabla_x (x^T A x) = \begin{bmatrix} \sum_{j=1}^{n} A_{1j} x_j + \sum_{i=1}^{n} A_{i1} x_i \\ \vdots \\ \sum_{j=1}^{n} A_{nj} x_j + \sum_{i=1}^{n} A_{in} x_i \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j=1}^{n} A_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} A_{nj} x_j \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n} A_{i1} x_i \\ \vdots \\ \sum_{i=1} A_{in} x_i \end{bmatrix}$$

$$= Ax + A^T x$$

$$\Rightarrow \boxed{\nabla_x (x^T A x) = A + A^T x}$$

Hence proved!

(c) $$\boxed{\nabla_x (x^T A x) = 2Ax}$$

$$\nabla_x (x^T A x) = (A + A^T)x \quad [\text{from 3b}]$$

$$A = A^T \quad \because \text{ symm. matrix}$$

$$\Rightarrow \nabla_x (x^T A x) = 2Ax$$

Hence proved!

(d)

Since A = symmetric matrix

$$\boxed{A = A^T}$$

$$[(Ax+b)^T(Ax+b)] = (x^TA^T + b^T)(Ax+b)$$

$$= x^TA^T.Ax + x^TA^Tb + b^TAx + b^Tb$$

$$\nabla_x[(Ax+b)^T(Ax+b)] = \nabla_x(x^TAAx) + \nabla_x(x^TA^Tb) + \nabla_x(b^TAx) + \nabla_x(b^Tb)$$

$\nabla_x(b^Tb) = 0$ $\because$ constant vector

$\nabla_x(x^TA^Tb) = A^Tb$ [using eq. from 3a]

$\Rightarrow \nabla_x[x^T(A^Tb)] = A^Tb$ $\because$ A^Tb = col vec

$\nabla_x(b^TAx) = A^Tb$ [using eq. from 3a]

$\Rightarrow \nabla_x[(b^TA)x] = (b^TA)^T = A^Tb$

$\nabla_x(x^TA^2x) = 2A^2x$ [using eq. from 3c]

$= 2A^TAx$

$$\nabla_x[(Ax+b)^T(Ax+b)] = 2A^TAx + A^Tb + A^Tb$$

$$\boxed{\nabla_x[(Ax+b)^T(Ax+b)] = 2A^T(Ax+b)}$$

Hence proved!