Assignment-5

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Problem 5.5

Let $F(X,Y,Z) = \sum_{i=m}^{n} F_i(X,Z)Y^{n-i}$. Then for $P = [0:1:0]$

$$\begin{aligned}
m_P(F) &= m_{\varphi(P)}(F_*) \\
&= m_{(0,0)}(\sum_{i=m}^{n} F_i(X,Z)) \\
&= m.
\end{aligned}$$

A line L is tangent to F if and only if $I(P, F \cap L) > m_p(F)$, thus we must have

$$\begin{aligned}
I(P, F\cap L) &= \dim_k \mathscr{O}_P(\mathbb{P}^2)/(F_*, L_*) \\
&= \dim_k \mathscr{O}_P(\mathbb{P}^2)/(F(X/Y, 1, Z/Y), L/Y) \\
&= \dim_k \mathscr{O}_{(0,0)}(\mathbb{A}^2)/(F(X,1,Z), L(X,1,Z))\mathscr{O}_{(0,0)}(\mathbb{A}^2) \\
&= I((0,0), F(X,1,Z) \cap L(X,1,Z)).
\end{aligned}$$

Thus we get that $I((0,0), F(X,1,Z) \cap L(X,1,Z)) > m$, hence $L(X,1,Z)$ is tangent to $F(X,1,Z)$, thus it must be a factor of $F_m(X,Z)$ (by defnition of tangent for an affine curve). Therefore, the tangents to F are determined by the factors of $F_m(X,Z)$.

Problem 5.7

Let F and G be two plane curves with no common components. Let L be a line not contained in $V(FG) \subseteq \mathbb{P}^2$. Then by problem 12, we know that $F \cap L$ and $G \cap L$ are finite. Now there exists a projective transformation that takes the line L to Z. Then under this projective transformation we know that intersection numbers of F and G are preserved. And we have

$$F \cap G = \underbrace{((F\cap U) \cap (G \cap U))}_{A} \cup \underbrace{((F \cap Z) \cup (G \cap Z))}_{B}$$

where $U = \{[x:y:z] \in \mathbb{P}^2 \mid z = 1\}$. Note that B is finite by the choice of the line L. Now $F \cap U$ and $G \cap U$ are affine curves given by $f = F(X,Y,1)$ and $g = G(X,Y,1)$. Now since F and G does not have any common component so does f and g (since otherwise we would have $hp = f$ and $hq = g$ for some $h, p, q \in k[X,Y]$, then $h^*p^* = F$ and $h^*q^* = G$, but then h^* is a common component of F and G, contradiction!). But we have previously shown that if two affine curves have no common component then $f \cap g$ is finite. Hence both A and B are finite, thus $F \cap G$ is finite.

Problem 5.12

Part (a). Let $P \in [0:1:0] \in F$ where F is a curve of degree of n. Let $F(X,Y,Z) = \sum_{i=0}^{n} F_i(Y,Z)X^i$ with F_i is a form of degree $n-i$ with $F_0 \neq 0$ and let $F_0(Y,Z) = \sum_{i=m}^{m+k} a_i Y^i Z^{n-i}$ (with $m, k \geq 0$ and $m+k \leq n-1$, there is no Y^n term as $P = [0:1:0] \in F$).

$$\begin{aligned}
\sum_{P\in\mathbb{P}^2} I(P, F\cap X) &= \sum_{P\in F_0\cap X} I(P, F_0 \cap X) \\
&= \sum_{P\in F_0\cap X\cap U_1} I(P, F_0\cap X) + I([0:0:1], F_0\cap X) \\
&= \sum_{t\in k} I([0:1:t], F_0\cap X) + I([0:0:1], F_0 \cap X) \\
&= \sum_{t\in k} \dim_k \left(\mathscr{O}_{[0:1:t]}(\mathbb{P}^2)/(F_{0*}\cap X_*)\right) + \dim_k \left(\mathscr{O}_{[0:0:1]}(\mathbb{P}^2)/(F_{0*}\cap X_*)\right) \\
&= \sum_{t\in k} \dim_k \left(\mathscr{O}_{(0,t)}(\mathbb{A}^2)/(F_0(1,Z), X)\mathscr{O}_{(0,t)}(\mathbb{A}^2)\right) + \dim_k \left(\mathscr{O}_{(0,0)}(\mathbb{P}^2)/(F_0(Y,1), X)\mathscr{O}_{(0,0)}(\mathbb{A}^2)\right) \\
&= \sum_{t\in k} I((0,t), F_0(1,Z)\cap X) + \operatorname{ord}^X_{(0,0)}(F_0(Y,1)) \\
&= \sum_{P\in F_0(1,Z)\cap X} I(P, F_0(1,Z)\cap X) + \operatorname{ord}^X_{(0,0)}(F_0(Y,1)) \\
&= \deg F_0(1,Z) \deg X + m \\
&= (n-m) + m = n.
\end{aligned}$$

Hence we have proved that $\sum_{P\in\mathbb{P}^2} I(P, F\cap X) = n$.

Part (b). Now if L is not a line contained in F, we can find a projective transformation taking $P \in F \mapsto [0:1:0]$ and $L \mapsto X$, then by part (a), we get that

$$\sum_{P\in\mathbb{P}^2} I(P, F\cap L) = n.$$

Problem 5.14

We will begin with the assumption, the underlying field k is infinte and algebraically closed (according to contexts). The property of lines passing through points is a projective property. So we can take a suitable projective transformation so that $P_1 = [0:0:1]$. Thus, any line passing through this looks like $ax + by = 0$ where $a, b \in k$. The set of lines passing through P_1 is

$$A = \{x + my : m \in k\} \cup \{y = 0\}$$

Since, the field is infinite, there is infinitely many elements in A. Given two points in $\mathbb{P}^2$ there is a unique line passing through P_1 and that point. Thus the set of lines

$$L = \{\ell \text{ pass through } P_1 \text{ and } P_i : 2 \leq i \leq n\} \subset A$$

is finite. So there are only finitely many line in the above set. But in A there are infinitely elements. So, there are infinitely many elements in $A \setminus L$.

Since P_1 is a simple point of F, there is a tangent T at P so that the tangent T don't contained in $V(F)$ (or F). From the problem 5.12 we can say,

$$\sum I(P; F \cap T) = n$$

where $n = \deg F$. Thus, If we take $P_2, \cdots, P_m$ be the other intersection points (here $m \leq n$) of T and F, by the previous calculation we can say there exists infinitely many lines through P don't intersect F at P_i $(i > 1)$. These lines are transversal to F. ■

Problem 5.18

Let us consider the general equation of conic in $\mathbb{P}^2$, that is

$$Ax^2 + By^2 + Cz^2 + Exy + Fyz + Gzx = 0$$

Since the pont $[0 : 0 : 1]$ and $[0 : 1 : 0]$, $[1 : 0 : 0]$ passes through the above conic we can say, $A = B = C = 0$. Thus the equation of conic reduces to $Exy + Fyz + Gzx = 0$. Also the points $[1 : 1 : 1]$ and $[1 : 2 : 3]$ passes through the curve. So we have the following linear equations,

$$\begin{aligned} E + F + G &= 0 \\ 2E + 6F + 3G &= 0 \\ \implies \begin{pmatrix} 1 & 1 & 1 \\ 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} E \\ F \\ G \end{pmatrix} &= 0 \end{aligned}$$

Note that the rows of the aboe matrix are linearly independent. So the null space of it must have dimension 1. Note that $(3, -4, 1)^T$ is a solution to the above matrix equation. Since the dimension of null space is 1 we can say any other solution must be a scaler multiple of $(3, -4, 1)^T$. So the equation of conic passing through the five points is $\lambda(3xy - 4yz + zx) = 0$. This will represent a unique conic in $\mathbb{P}^2$. By contruction the conic is unique! ■

Problem 5.25

Since the polynomial $F = F_1F_2$ have $c \geq 1$ simple component, the polynomial may not be irreducible. Let, $F = F_1F_2$ and at every point P, $m_P(F) = m_P(F_1) + m_P(F_2)$. Thus,

$$\begin{aligned} \sum_P \frac{m_P(F)(m_P(F) - 1)}{2} &= \sum_P \frac{(m_P(F_1) + m_P(F_2))(m_P(F_1) + m_P(F_2) - 1)}{2} \\ &= \sum_P \frac{m_P(F_1)(m_P(F_1) - 1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2) - 1)}{2} \\ &+ \sum_P m_P(F_1)m_P(F_2) \end{aligned}$$

Let, $p = \deg F_1$ and $q = \deg F_2$. If F_1 and F_2 were irreducible then we must have

$$\begin{aligned}
\sum_P \frac{m_P(F)(m_P(F)-1)}{2} &= \sum_P \frac{m_P(F_1)(m_P(F_1)-1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2)-1)}{2} \\
&\quad + \sum_P m_P(F_1)m_P(F_2) \\
&\overset{*}{\leq} \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + pq \\
&= \frac{(p+q-1)(p+q-2)}{2} + 1 \\
&= \frac{(n-1)(n-2)}{2} + 1
\end{aligned}$$

here, $*$ comes from the corollary 1 of Bézout's theorem and theorem of section 5.4. In this case we had $c = 2$. Now we will proceed using induction. Assume the result is true for some curve with $c-1$ simple components. Again assume $F = F1F_2$ with the degrees mentioned above and F_1 has $c-1$-simple components and F_2 is irreducible. Thus using induction we have,

$$\begin{aligned}
\sum_P \frac{m_P(F)(m_P(F)-1)}{2} &= \sum_P \frac{m_P(F_1)(m_P(F_1)-1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2)-1)}{2} \\
&\quad + \sum_P m_P(F_1)m_P(F_2) \\
&\leq \underbrace{\frac{(p-1)(p-2)}{2} + c - 2}_{\text{induction step}} + \frac{(q-1)(q-2)}{2} + pq \\
&= \frac{(p+q-1)(p+q-2)}{2} + c - 1 = \frac{(n-1)(n-2)}{2} + c - 1
\end{aligned}$$

Thus our induction step is complete. It's not hard to note that a polynomial of degree n can have at most n linear factor, i.e atmost n simple components. Thus $c \leq n$ and hence the final term in the above calculation is bounded above by $n(n-1)/2$. ∎