

Assignment-4

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§1. Problem 3.4

Without loss of generality, let $P = (0, 0)$ so that $F = F_2 + \dots + F_d$, where F_i is a form of degree i and $F_2 \neq 0$. By definition, P is a node iff $F_2 = L_1 L_2$ for distinct lines L_1, L_2 passing through P . Suppose that P is a node, and let $L_1 = uX + vY$ and $L_2 = pX + qY$. Then,

$$F_2 = upX^2 + (vp + uq)XY + cqY^2.$$

As L_1 and L_2 are distinct, we have $uq \neq vp$, and so $(vp - uq)^2 = (vp + uq)^2 - 4uvpq \neq 0$. But in this case we have $F_{XX}(P) = 2up$, $F_{YY}(P) = 2vq$, $F_{XY}(P) = vp + uq$. So, if P is a node, we get $F_{XX}(P)F_{YY}(P) \neq F_{XY}(P)^2$.

Now suppose $F_{XX}(P)F_{YY}(P) \neq F_{XY}(P)^2$. Let $2a = F_{XX}(P)$, $2c = F_{YY}(P)$ and $b = F_{XY}(P)$. Then the given condition translates to the equation $at^2 - bt + c = 0$ having two distinct roots α, β in k . Then,

$$F_2 = aX^2 + bXY + cY^2 = (X + \beta Y)(aX + a\alpha Y),$$

as $a\alpha + a\beta = b$ and $a\alpha\beta = c$. Therefore, the given condition implies that P is a node of F .

§2. Problem 3.6

§ Lemma – 1

The F and G be forms of degree r and $r + 1$ respectively with no common factors in $k[X_1, \dots, X_n]$, then $F + G$ is irreducible.

Proof (of Lemma). Suppose $F + G$ is reducible then there exists nonconstant polynomials $P, Q \in k[X_1, \dots, X_n]$ such that $F + G = PQ$. Now we consider the homogeneous both of these to get

$$X_{n+1}F + G = (F + G)^* = (PQ)^* = P^*Q^*.$$

But note that $X_{n+1}F + G \in k(X_1, \dots, X_n)[X_{n+1}]$ is irreducible, hence one of P^* or Q^* is in $k[X_1, \dots, X_n]$. Then by comparing degrees we can WLOG assume that $Q^* \in k[X_1, \dots, X_n]$, and let $P = X_{n+1}R + S$, where $R, S \in k[X_1, \dots, X_n]$. Then we get that

$$X_{n+1}F + G = X_{n+1}RQ^* + SQ^* \Rightarrow F = RQ^* \text{ and } G = SQ^*$$

But this contradicts the fact that F and G have no common factor, hence we get that $F + G$ is irreducible.

Now coming back to the main problem, suppose we are given tangent lines L_i with multiplicities r_i , and we want to find an irreducible curve F such that L_i is a tangent to F with multiplicity r_i . Note that $\prod_i L_i^{r_i}$ is a form of degree $m = \sum_i r_i$. Then we can find a homogeneous polynomial F_{m+1} of degree $m+1$ such that F_{m+1} is not divisible by any of the L_i (such a polynomial obviously exists). But then by the previous lemma $\prod_i L_i^{r_i} + F_{m+1}$ is irreducible, and clearly $F = \prod_i L_i^{r_i} + F_{m+1}$ satisfies the necessary conditions.

§3. Problem 3.8

Part (a). We will first prove it for the case when $P = Q = (0, 0)$. In this case the polynomial map $T : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ must look like (f, g) where f and g are polynomials vanishing at $(0, 0)$. In this case we can write $f = f_1 + \dots + f_t$, where $f_i \in k[x, y]$, $i \geq 1$ is homogeneous polynomial of degree i . Similarly, we can write for g (as both of them are vanishing at $(0, 0)$). If $m = m_P(F)$ then $F = F_m + F_{m+1} \dots$ again F_i are homogeneous polynomial of degree m . Now $F^T = F(f, g)$'s lowest degree will come from $F_m(f, g)$ since both f, g has at-least one degree term we can say, $m_Q(F^T) \geq m_P(F)$.

Now we will use the fact proved in page (33) to prove it for any P, Q . Let, $Q \neq (0, 0)$ or $Q = T(P) \neq (0, 0)$. Let $T_1 : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ be the affine transformation that maps $(0, 0)$ to Q and T_2 be the affine map sends P to $(0, 0)$. Note that $T_1 \circ T \circ T_2$ is a polynomial map and it maps $(0, 0)$. So by the above calculation we can say,

$$\begin{aligned} m_P(F) &\leq m_{T_1 \circ T \circ T_2(0,0)}(F^{T_1 \circ T \circ T_2}) \\ &= m_{T_1 \circ T(Q)}(F^{T_1 \circ T}) \quad (\text{By result of page 33}) \\ &= m_{T(Q)}(F^T) \quad (\text{By result of page 33}) \end{aligned}$$

Part (b). Again we will prove it for $P = Q = (0, 0)$. Let $T = (f, g)$ and

$$J_Q T = \begin{pmatrix} \frac{\partial f}{\partial X}(Q) & \frac{\partial f}{\partial Y}(Q) \\ \frac{\partial g}{\partial X}(Q) & \frac{\partial g}{\partial Y}(Q) \end{pmatrix}.$$

Assume $J_Q T$ is invertible. Since $J_Q T$ is invertible, we can't have both $\frac{\partial f}{\partial X}(Q) = 0$ and $\frac{\partial f}{\partial Y}(Q) = 0$ or both $\frac{\partial g}{\partial X}(Q) = 0$ and $\frac{\partial g}{\partial Y}(Q) = 0$. Again by similar computation of part (a) we have, since $Q = (0, 0)$, this implies that the decomposition of f and g into homogeneous polynomials are $f = f_1 + \dots + f_m$ and $g = g_1 + \dots + g_n$. Thus,

$$F^T = F(f, g) = F_m(f, g) + F_{m+1}(f, g) + \dots$$

Since the lowest degree forms of f and g are of degree 1, we have that T does not decrease the degree of the form $F_m(f, g)$. Similarly, T does not decrease the degree of $F_{m+1}(f, g), \dots$. Therefore we have that $m_{(0,0)}(F^T) = m_{(0,0)}(F)$. Now assume that either $Q = (a_1, b_1) \neq (0, 0)$ or $P = (a_2, b_2) \neq (0, 0)$. Assume that $J_Q T$ is invertible. Let T_1 be the translation that takes $(0, 0)$ to Q and T_2 be the translation that takes P to $(0, 0)$. Then we have

$$d(T_1 \circ T \circ T_2) = d(T_1) \circ d(T) \circ d(T_2)$$

is also invertible. By the previous case $m_P(F) = m_{(0,0)}(F^{T_1 \circ T \circ T_2})$ and the similar computation of multiplicities we can say $m_P(F^T) = m_Q(F)$. And hence our proof is complete.

Part (c). If $F = Y - X^2$ and $T = (X^2, Y)$, $P = Q = (0, 0)$ we can see $m_P(F^T) = m_P(Y - X^4) = m_Q(F) = m_Q(Y - X^2)$. But the jacobian of T is not invertible at $(0, 0)$, as it is given by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. ■

§4. Problem 3.12

- (a) We first note that $n \geq 1$, as $P \notin F$ for $n = 0$. For $n = 1$, F reduces to $Y = X$. As this is a line, it is its own tangent at P , and so $\text{ord}_P(L) = \text{ord}_P(F) = \infty$, because any curve has infinite valuation in the local ring at a simple point. Therefore, F has a higher flex at P for $n = 1$.

Now suppose $n \geq 2$. Then the tangent at P is $L : Y = 0$. Consider the non-tangent line $X = 0$. By the theorem on uniformizing parameters in $\mathcal{O}_P(F)$, x is a uniformizing parameter. By definition, $y = x^n$ in $\Gamma(F)$, and so $\text{ord}_P(L) = n$. Therefore, F has a flex at P iff $n \geq 3$, and the flex is ordinary iff $n = 3$.

- (b) We have $\frac{\partial F}{\partial Y} = 1$ and so P is a simple point. The line $X = 0$ passes through P and is not tangent to F , and so we take x as the uniformizing parameter. Following the proof of the theorem on uniformizing parameters, we get $F = YG - X^2H$, with $G = 1 + \dots \in k[X, Y]$ and $H = -a + \dots \in k[X]$. Hence, $y = x^2hg^{-1}$ in $\Gamma(F)$. Therefore, if $a = 0$, we get $\text{ord}_P(L) \geq 3$ and so F has a flex at P . Conversely, let F have a flex at P . Then, $y \in (x)^3$ and so h cannot have a constant term, i.e., $a = 0$. ■

We claim $\text{ord}_P(L) = \min \{i \mid H_i \neq 0\} + 2$. This is because, $\text{ord}_P(L) = d$ iff $y \in (x)^d \setminus (x)^{d+1}$, and this happens iff x^2h has first non-zero coefficient in degree d . Thus, $\text{ord}_P(L) = d$ iff H has first non-zero coefficient in degree $d - 2$.

§5. Problem 3.13

WLOG assume $P = (0, 0)$, then we know that

$$\dim_k(\mathcal{O}/\mathfrak{m}^n) = \dim_k(k[X, Y]/(F, I^n))$$

where $I = (X, Y) \subseteq k[X, Y]$. Now as multiplicity of F is $m_P(F)$, we have $F \in I^{m_P(F)}$ and hence we get that for $n \leq m_P(F)$, $F \in I^n$, thus $(F, I^n) = I^n$. But then we get that

$$\dim_k(k[X, Y]/(F, I^n)) = \dim_k(k[X, Y]/I^n) = \binom{n+1}{2}$$

Now from the exact sequence

$$0 \rightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^n \rightarrow 0$$

we get that for $n \leq m_p(F)$.

$$\begin{aligned}\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) &= \dim_k(\mathcal{O}/\mathfrak{m}^{n+1}) - \dim_k(\mathcal{O}/\mathfrak{m}^n) \\ &= \binom{n+2}{2} - \binom{n+1}{2} \\ &= n+1\end{aligned}$$

In the proof of **Theorem 2, page 35 (Algebraic Curves, Fulton)**, we have already seen that

$$\dim_k(k[X, Y]/(F, I^n)) = nm - \frac{m(m-1)}{2},$$

where $m = m_P(F)$, hence we get that

$$\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = m$$

if $n \geq m_P(F)$. Now suppose P is not a simple point, then $m_P(F) \geq 2$, and hence $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$. Hence $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$ implies P is a simple point. Now if P is a simple point then $m_P(F) = 1$, and hence we get that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = m - \frac{m(m-1)}{2} = 1$, since $m = 1$. Thus we have shown that P is simple if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$, and otherwise we have $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$.

§6. Problem 3.15

Part (a). With out loss of generality let, $P = (0, 0)$ and the corresponding maximal ideal in $k[x, y]$ is $\mathfrak{m}_p = (x, y)$ and extension it's image in $\mathcal{O}_p(\mathbb{A}^2)$ is $\mathfrak{m}_p(\mathbb{A}^2)$. Now we know,

$$k[x, y]/\mathfrak{m}_p^n \simeq k[x, y]_{\mathfrak{m}_p}/\mathfrak{m}_p^n k[x, y]_{\mathfrak{m}_p} \simeq \mathcal{O}_p/\mathfrak{m}_p(\mathbb{A}^2)^n$$

(it follows from the fact, residue field/ localization commutes with quotienting). Enough to calculate $\dim_k k[x, y]/\mathfrak{m}_p^n$. Now, $\mathfrak{m}_p^n = (x^n, x^{n-1}y, x^{n-2}y^2, \dots, y^n)$. The basis of $k[x, y]/\mathfrak{m}_p^n$ must be the standard i forms, with $i < n$. For each i there are such $i+1$ forms. And hence,

$$\chi(n) = \dim_k \mathcal{O}_p/\mathfrak{m}_p(\mathbb{A}^2)^n = \dim_k k[x, y]/\mathfrak{m}_p^n = \frac{n(n+1)}{2}$$

Part (b). Let, $\mathcal{O} = \mathcal{O}_p(\mathbb{A}^r)$ and $\mathfrak{m} = \mathfrak{m}_p(\mathbb{A}^r)$. Again let, $P = (0, \dots, 0)$ and $\mathfrak{m}_p = (x_1, \dots, x_r)$. Just by the similar past as above it is enough to calculate $\dim_k k[x_1, \dots, x_r]/\mathfrak{m}_p^n$. Now, \mathfrak{m}_p is generated by all standard forms of degree n . Thus, the basis of $k[x_1, \dots, x_r]/\mathfrak{m}_p^n$ must be the standard i forms, with $i < n$. Thus the basis set can be written as,

$$\mathcal{B} = \{x_1^{i_1} \cdots x_r^{i_r} : i_1 + \cdots + i_r \leq n-1\}$$

Now cardinality of the set is,

$$\begin{aligned}|\mathcal{B}| &= |\{x_1^{i_1} \cdots x_r^{i_r} : i_1 + \cdots + i_r \leq n-1\}| \\ &= |\{1^{i_0} x_1^{i_1} \cdots x_r^{i_r} : i_0 + i_1 + \cdots + i_r = n-1\}| \\ &= \binom{n+r-1}{r}\end{aligned}$$

So we must have,

$$\chi(n) = \dim_k \mathcal{O}/\mathfrak{m}^n = \dim_k k[x_1, \dots, x_r]/\mathfrak{m}_p^n = \binom{n+r-1}{r} = \frac{n(n+1) \cdots (n+r-1)}{r!}$$

Thus the leading coefficient is $1/r!$. ■

§7. Problem 3.16

In this problem we will try to trace the path of ‘Theorem 2’ in ‘page 35’. Let, $\mathcal{O} = \mathcal{O}_P(V(F))$ and $P = (0, 0, \dots)$ and $\mathfrak{m} = \mathfrak{m}_p(V(F))$. Consider the maximal ideal $\mathfrak{m}_p = (x_1, \dots, x_r)$ corresponding to the point P . Let, $R = k[x_1, \dots, x_r]$. Let, $m_P(F) = m$ (multiplicity of P w.r.t F). Then we have the follows SES(short exact sequence)

$$0 \longrightarrow R/\mathfrak{m}_p^{n-m} \xrightarrow{i} R/\mathfrak{m}_p^n \xrightarrow{\pi} R/(F, \mathfrak{m}_p^n) \longrightarrow 0$$

where i is the map $i(\bar{G}) = \overline{FG}$ and π the natural projection map. It’s an exact sequence. Thus by the previous problem we have,

$$\dim_k R/(F, \mathfrak{m}_p^n) = \dim_k R/\mathfrak{m}_p^n - \dim_k R/\mathfrak{m}_p^{n-m} = \binom{n+r-1}{r} - \binom{n+r-m-1}{r}$$

If we expand the above binomial coefficients it’s not hard to see the above is polynomial over n , which has degree $r-1$ and leading coefficient is $m/r!$. Now from a result stated in class * it follows,

$$R/(\mathfrak{m}_p^n, F) \simeq \mathcal{O}/\mathfrak{m}^n$$

Thus $\chi(n) = \dim_k \mathcal{O}/\mathfrak{m}^n$ is a polynomial of n of degree $(r-1)$ and leading coefficient is $m/r!$ as desired. ■

§8. Problem 3.19

From the definition of intersection number we can say, $I(P, F \cap G) \geq m_P(F)m_P(G)$ and the equality occurs if and only if F and G don’t have common tangent at the point P . If L is a tangent line to F we can say, $m_p(L) = 1$ and hence, $I(P, F \cap L) > m_p(F)$. Conversely, if L is a line that intersects F with $I(P, F \cap L) > m_p(F)$, we can say L and F have tangent line in common at P and hence L has to be tangent to F at P . ■

§9. Problem 3.22

- (a) We have $I(P, F \cap L) \geq m_P(F)m_P(L) \geq 2$, as P is a double point of F and L is a line. Further, equality does not hold as L is the common tangent to F and L at P , and hence we get $I(P, F \cap L) \geq 3$. ■
- (b) As $m_P(F) = 2$, $F = F_2 + F_3 + \dots$. We will repeatedly use the facts that intersection number depends only on the component passing through P , and also only on the image of one curve in the coordinate ring of the other.

Suppose P is a cusp. If $F_{XX}(P) = F_{XXX}(P) = 0$, we get

$$I(P, F \cap L) = I(P, Y \cap (aX^4 + bX^5 + \cdots)) \geq m_P(aX^4 + bX^5 + \cdots) = 4,$$

which contradicts the assumption that P is a cusp. If $F_{XX}(P) \neq 0$, we will have

$$I(P, F \cap L) = I(P, Y \cap (X^2(1 + bX^3 + \cdots))) = I(P, Y \cap X^2) = 2,$$

which is again a contradiction to the assumption that P is a cusp. Therefore, we get if P is a cusp, we must have $F_{XXX}(P) \neq 0$.

Conversely, assume that $F_{XXX}(P) \neq 0$. By (a), $I(P, F \cap L) \geq 3$. It cannot happen that $F_{XX}(P) \neq 0$, as then we would get the intersection number is 2 as above. Hence, we get

$$I(P, F \cap L) = I(P, Y \cap (X^3(a + bX^4 + \cdots))) = I(P, Y \cap X^3) = 3,$$

which shows that P is a cusp. ■

- (c) Let $P = (0, 0)$ and $L = Y$ without loss of generality. Suppose F has the components F_1, \dots, F_k passing through P . Then, $I(P, F \cap L) = \sum_{i=1}^k I(P, F_i \cap L)$. But, for each i , L is a common tangent of F_i and itself at P , so that $I(P, F_i \cap L) > 1$. Hence,

$$I(P, F \cap L) \geq 2k,$$

and as $I(P, F \cap L) = 3$, we get $k = 1$. Therefore, F has a unique component passing through P . ■

§10. Problem 3.23

We mimic the proofs in 3.22 to get the following generalization. Let $m = m_P(F) \geq 2$ and without loss of generality, assume $P = (0, 0)$ and $L = Y$ is the unique tangent at P to F .

- (i) We claim that P is a hypercusp iff $\frac{\partial F}{\partial X^{m+1}} \neq 0$. We know $F = F_m + \cdots$.

Suppose P is a hypercusp. If $F_{X^m}(P) = F_{X^{m+1}}(P) = 0$, we get

$$I(P, F \cap L) = I(P, Y \cap (aX^{m+2} + \cdots)) \geq m_P(aX^{m+2} + \cdots) = m + 2,$$

which contradicts the assumption that P is a hypercusp. If $F_{X^m}(P) \neq 0$, we will have

$$I(P, F \cap L) = I(P, Y \cap (X^m(1 + bX^{m+1} + \cdots))) = I(P, Y \cap X^m) = m,$$

which is again a contradiction to the assumption that P is a hypercusp. Therefore, we get if P is a hypercusp, we must have $F_{X^{m+1}}(P) \neq 0$.

Conversely, assume that $F_{X^{m+1}}(P) \neq 0$. We have $I(P, F \cap L) > m$ as L is a common tangent to F and itself at P . It cannot happen that $F_{X^m}(P) \neq 0$, as then we would get the intersection number is m as above. Hence, we get

$$I(P, F \cap L) = I(P, Y \cap (X^{m+1}(a + \cdots))) = I(P, Y \cap X^{m+1}) = m + 1,$$

which shows that P is a hypercusp. ■

- (ii) We claim that if P is a hypercusp, then F has at most $\lfloor \frac{m+1}{2} \rfloor$ components passing through P .

Suppose F has the components F_1, \dots, F_k passing through P . Then, $I(P, F \cap L) = \sum_{i=1}^k I(P, F_i \cap L)$. But, for each i , L is a common tangent of F_i and itself at P , so that $I(P, F_i \cap L) > 1$. Hence,

$$I(P, F \cap L) \geq 2k,$$

and as $I(P, F \cap L) = m + 1$, we get $k \leq \lfloor \frac{m+1}{2} \rfloor$. ■

§11. Problem 3.24

- (a) By Problem 3.13, the vector space $\mathfrak{m}/\mathfrak{m}^2$ is of dimension 2 as P is not a simple point. The vector space consisting of all degree 1 forms also has dimension 2, and so we only need to show that the map $aX + bY \mapsto ax + by$ is an injective linear map to show that the spaces are isomorphic, and in fact this map is an isomorphism. Linearity is clear from the definition of the map. Because \mathfrak{m}^2 is generated by $\bar{x}^2, \bar{x}\bar{y}$ and \bar{y}^2 , we also get that $aX + bY$ is in the kernel iff $a = b = 0$, and so we are done. ■
- (b) For each i , L_i is a common tangent to F and itself at P , and hence, $I(P, F \cap L_i) > m_P(F) = m$. Further, for $i \neq j$, L_i and L_j are distinct linear forms, i.e., $L_i \neq \lambda L_j$ for any $\lambda \in k$. By (a), their images in $\mathfrak{m}/\mathfrak{m}^2$ must also be linearly independent and hence $\bar{l}_i \neq \lambda \bar{l}_j$ for any $\lambda \in k$.
- (c) Let L_i be the linear part of G_i for each i . Then, as $\bar{l}_i = \bar{g}_i \neq 0$, we get $\bar{l}_i \neq \lambda \bar{l}_j$ for any $\lambda \in k$ if $i \neq j$. We also note that as $\bar{g}_i \neq 0$, $m_P(G_i) = 1$. Now, as $I(P, F \cap G_i) \geq m \cdot m_P(G_i)$ and we are given $I(P, F \cap G_i) > m$, each G_i must have a common tangent with F at P . Hence, we get F has m distinct tangents L_1, \dots, L_m at P and so P is an ordinary multiple point. ■
- (d) We first note the following fact: $g \in \mathfrak{m}$ satisfies $\dim \mathcal{O}_P(F)/(g) = I(P, F \cap G)$, where g is the image of G in the coordinate ring. This is because, Problem 2.44 gives $\mathcal{O}_P(\mathbb{A}^2)/(F, G) \simeq \mathcal{O}_P(F)/(g)$ and so

$$I(P, F \cap G) = \dim \mathcal{O}_P(\mathbb{A}^2)/(F, G) = \dim \mathcal{O}_P(F)/(g).$$

If P is an ordinary multiple point with tangents L_1, \dots, L_m , we can take $g_i = l_i$, where l_i is the image of the tangent L_i in \mathfrak{m} . These satisfy the properties that $\bar{l}_i \neq \lambda \bar{l}_j$ for all $\lambda \in k$, if $i \neq j$, and that $I(P, F \cap L_i) > m$, and so by the fact above we are done.

Conversely, assume that there are $g_1, \dots, g_m \in \mathfrak{m}$ such that $\bar{g}_i \neq \lambda \bar{g}_j$ for all $\lambda \in k$ if $i \neq j$ and $\dim \mathcal{O}_P(F)/(g_i) > m$. By (a), there is a unique degree 1 form $L_i \in k[X, Y]$ such that $\bar{l}_i = \bar{g}_i$. These will then satisfy

$$I(P, F \cap L_i) = I(P, F \cap l_i) > m = m_P(F) \cdot m_P(L_i)$$

by the fact above. But this now shows that L_i must be tangent to F at P , as . Hence, we get m distinct tangents to F at P , and so P is an ordinary multiple point. ■

§Exercises in chapter 2 needed for proving theorems in chapter 3

2.15 Throughout this solution, let X_j denote the j^{th} coordinate of a point X in affine space. For example, $P_j = a_j$ for $P = (a_1, \dots, a_n)$.

- (a) Let $T = (T_1, \dots, T_m) : \mathbb{A}^n \rightarrow \mathbb{A}^m$ be an affine change of coordinates, with $T_i(X) = \sum_{j=1}^n f_{i,j} X_j + f_i$. Let R be any point on the line PQ , so that $R_j = P_j + t(Q_j - P_j)$ for all j , for some fixed $t \in k$. Then,

$$T(R)_i = T_i(R) = \sum_{j=1}^n f_{i,j} R_j + f_i = \left(\sum_{j=1}^n f_{i,j} P_j + f_i \right) + t \left(\sum_{j=1}^n f_{i,j} Q_j - \sum_{j=1}^n f_{i,j} P_j \right) = T_i(P) + t(T_i(Q) - T_i(P))$$

and so, $T(R)_i = T(P)_i + t(T(Q)_i - T(P)_i)$ for all i . Hence, $T(R)$ is a point on the line joining $T(P)$ and $T(Q)$, i.e., $T(L)$ is the line through $T(P)$ and $T(Q)$. ■

- (b) Let L be the line through P and Q in \mathbb{A}^n . Then, $R \in L$ iff $R_j = P_j + t(Q_j - P_j)$ for all j , for some fixed $t \in k$. Without loss of generality, let $P_1 \neq Q_1$ and consider the polynomials (in $k[X_1, \dots, X_n]$) f_2, \dots, f_n defined as,

$$f_j(X) = X_j - P_j - \frac{Q_j - P_j}{Q_1 - P_1} (X_1 - P_1).$$

Then, $R \in L \iff f_j(R) = 0$ for all j . Hence, $L = V(f_2, \dots, f_n)$ is a linear subvariety of \mathbb{A}^n . It is of dimension 1, as the affine change of coordinates $T(X) = (X_1 - P_1, f_2(X), \dots, f_n(X))$ maps this linear subvariety to $V(X_2, \dots, X_n)$.

Conversely, let $V = V(X_2, \dots, X_n)$ be a linear subvariety of dimension 1. (We can assume that the variety is given by the vanishing of these coordinates by an affine change of coordinates.) Then, if P, Q are any two distinct points in V , we have $P = (p, 0, \dots, 0), Q = (q, 0, \dots, 0)$ for $p \neq q$ in k . Now, any point (x_1, \dots, x_n) is in V iff $x_2 = \dots = x_n = 0$, and this happens iff (x_1, \dots, x_n) is in the line through P and Q . Therefore, given any two distinct points in V , V is obtained as the line joining those points. ■

- (c) From (b), we get a line is a subvariety $V(f) \subseteq \mathbb{A}^2$, for f a linear polynomial in $k[X, Y]$. But this is exactly the definition of a hyperplane.
- (d) Let L_1 be parametrised as $t \mapsto P + t(Q - P)$, L_2 as $t \mapsto P + t(R - P)$, L_3 as $t \mapsto P' + t(Q' - P')$, L_4 as $t \mapsto P' + t(R' - P')$. As L_1, L_2 are distinct, the vectors $Q - P$ and $R - P$ in k^2 are linearly independent, and so there is a matrix M sending $Q - P$ to $Q' - P'$ and $R - P$ to $R' - P'$. The map $T(X) = M(X - P) + P'$ is an affine change of coordinates (being a composition of a translation and a linear map), maps P to P' and L_i to L'_i for $i = 1, 2$. ■

2.22 We know given a map $f : V \rightarrow W$ between affine varieties, it extends to a ring homomorphism $f^* : \mathcal{O}_{f(P)}(W) \rightarrow \mathcal{O}_P(V)$. Now if we have an affine transformation $T : \mathbb{A}^n \rightarrow \mathbb{A}^n$ it will have inverse affine map T^{-1} . By the functoriality of pullback we can say they will induce T^* and T^{-1*} in the corresponding local ring of regular functions. We can also note $T^* \circ T^{-1*}$ and $T^{-1*} \circ T^*$ is identity and hence T^* is isomorphism. Thus

$T^* : \mathcal{O}_{T(P)}(\mathbb{A}^n) \rightarrow \mathcal{O}_n(\mathbb{A}^n)$ is an isomorphism. If we restrict T to $V \subset \mathbb{A}^k$ on that case T will map V to an isomorphic (as subvariety) copy $V^T \subset \mathbb{A}^n$. Again by the same computation we can say, $\mathcal{O}_P(V) \simeq \mathcal{O}_{T(P)}(V^T)$ are isomorphic.

2.34 In this case if $F + G$ was reducible then we could write $F + G = fg$. Now if we homogenize the polynomial we will get,

$$(F + G)^* = x_{n+1}F + G = f^*g^*$$

here treat $(F + G)^*$ as linear a polynomial over the ring $k[x_1, \dots, x_n]$, which is UFD and hence by Gauss lemma $k[x_1, \dots, x_n][x_{n+1}]$ is also UFD. But it can't have any non-constant factor over $k[x_1, \dots, x_n][x_{n+1}]$. So, $F + G$ is irreducible.

2.35(c), 2.36 is done in the computation step of **3.15** part (b). So not doing it again.

2.44* (* marked in previous section) At first we will define a map $\psi : \mathcal{O}_P(\mathbb{A}^n) \rightarrow \mathcal{O}_P(V)/J'\mathcal{O}_P(V)$. Firstly, we have the map $\mathcal{O}_P(\mathbb{A}^n) \rightarrow \mathcal{O}_P(V)$, which takes f/g (such that $g(P) \neq 0$) to \bar{f}/\bar{g} where \bar{f}, \bar{g} are f, g modulo $I = I(V)$. It's not hard to see $g \notin I$ so $\bar{g}(P) \neq 0$. Thus the map is well defined. J is an ideal containing I and J' is the image in local ring, then there is a natural projection map $\mathcal{O}_P(V)/J'\mathcal{O}_P(V)$. Composition of this two map will be ψ .

Now it's not hard to see ψ is a surjective homomorphism. We will compute the kernel of it $\ker \psi$. Let, $f/g \in \mathcal{O}_P(\mathbb{A}^n)$ such that $\bar{f}/\bar{g} \in J'\mathcal{O}_P(V)$. We can write

$$\bar{f}/\bar{g} = \sum \frac{j_i}{g'_i}$$

where $j_i \in J'$ and g'_i are polynomial corresponding g_i (that don't vanish at P), i.e $g'_i = g_i \pmod{I}$. So, $\bar{f} \times (\prod g'_i) \in J'\mathcal{O}_P(V)$. Thus we can say, $f \times (\prod g_i) \in J\mathcal{O}_P(\mathbb{A}^n)$. Since g_i are invertible we can say $f \in J\mathcal{O}_P(\mathbb{A}^n)$. So, $\ker \psi \subseteq J\mathcal{O}_P(\mathbb{A}^n)$. It's not hard to see $J\mathcal{O}_P(\mathbb{A}^n) \subseteq \ker \psi$ thus we get, $\ker \psi = J\mathcal{O}_P(\mathbb{A}^n)$. And thus we have a natural isomorphism

$$\bar{\psi} : \mathcal{O}_P(\mathbb{A}^n)/J\mathcal{O}_P(\mathbb{A}^n) \rightarrow \mathcal{O}_P(V)/J'\mathcal{O}_P(V)$$

If $J = I$ then the right side is just $\mathcal{O}_P(V)$ and thus $\mathcal{O}_P(V) \simeq \mathcal{O}_P(\mathbb{A}^n)/I\mathcal{O}_P(\mathbb{A}^n)$.

2.42

(a) Consider the map $\varphi : R/I \rightarrow R/J$ defined as,

$$\varphi(x + I) = x + J.$$

This is a ring homomorphism, as

$$\begin{aligned} \varphi((x + I)(y + I) + (z + I)) &= \varphi((xy + z) + I) = (xy + z) + J \\ &= (x + J)(y + J) + (z + J) = \varphi(x + I)\varphi(y + I) + \varphi(z + I). \end{aligned}$$

This is surjective as given any $x + J \in R/J$, $x \in R$, we get $\varphi(x + I) = x + J$. We can do this because $I \subseteq J$ means $x \notin J \implies x \notin I$. ■

(b) Consider the map $\varphi : R/I \rightarrow S/IS$ defined as,

$$\varphi(x + I) = x + IS.$$

This is a ring homomorphism, as

$$\begin{aligned}\varphi((x+I)(y+I) + (z+I)) &= \varphi((xy+z) + I) = (xy+z) + IS \\ &= (x+IS)(y+IS) + (z+IS) = \varphi(x+I)\varphi(y+I) + \varphi(z+I).\end{aligned}$$

We can do this as for any ideal I of R , IS is an ideal of S if R is a subring of S . ■