

# Assignment-5

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## Problem 5.12

**Part (a).** Let  $P \in [0 : 1 : 0] \in F$  where  $F$  is a curve of degree  $n$ . Let  $F(X, Y, Z) = \sum_{i=0}^n F_i(Y, Z)X^i$  with  $F_i$  is a form of degree  $n-i$  with  $F_0 \neq 0$  and let  $F_0(Y, Z) = \sum_{i=m}^{m+k} a_i Y^i Z^{n-i}$  (with  $m, k \geq 0$  and  $m+k \leq n-1$ , there is no  $Y^n$  term as  $P = [0 : 1 : 0] \in F$ ).

$$\begin{aligned}
 \sum_{P \in \mathbb{P}^2} I(P, F \cap X) &= \sum_{P \in F_0 \cap X} I(P, F_0 \cap X) \\
 &= \sum_{P \in F_0 \cap X \cap U_1} I(P, F_0 \cap X) + I([0 : 0 : 1], F_0 \cap X) \\
 &= \sum_{t \in k} I([0 : 1 : t], F_0 \cap X) + I([0 : 0 : 1], F_0 \cap X) \\
 &= \sum_{t \in k} \dim_k (\mathcal{O}_{[0:1:t]}(\mathbb{P}^2)/(F_{0*} \cap X_*)) + \dim_k (\mathcal{O}_{[0:0:1]}(\mathbb{P}^2)/(F_{0*} \cap X_*)) \\
 &= \sum_{t \in k} \dim_k (\mathcal{O}_{(0,t)}(\mathbb{A}^2)/(F_0(1, Z), X) \mathcal{O}_{(0,t)}(\mathbb{A}^2)) + \dim_k (\mathcal{O}_{(0,0)}(\mathbb{P}^2)/(F_0(Y, 1), X) \mathcal{O}_{(0,0)}(\mathbb{A}^2)) \\
 &= \sum_{t \in k} I((0, t), F_0(1, Z) \cap X) + \text{ord}_{(0,0)}^X(F_0(Y, 1)) \\
 &= \sum_{P \in F_0(1, Z) \cap X} I(P, F_0(1, Z) \cap X) + \text{ord}_{(0,0)}^X(F_0(Y, 1)) \\
 &= \deg F_0(1, Z) \deg X + m \\
 &= (n - m) + m = n.
 \end{aligned}$$

Hence we have proved that  $\sum_{P \in \mathbb{P}^2} I(P, F \cap X) = n$ .

**Part (b).** Now if  $L$  is not a line contained in  $F$ , we can find a projective transformation taking  $P \in F \mapsto [0 : 1 : 0]$  and  $L \mapsto X$ , then by part (a), we get that

$$\sum_{P \in \mathbb{P}^2} I(P, F \cap L) = n.$$

## Problem 5.14

We will begin with the assumption, the underlying field  $k$  is infinite and algebraically closed (according to contexts). The property of lines passing through points is a projective property. So we can take a suitable projective transformation so that  $P_1 = [0 : 0 : 1]$ . Thus, any line

passing through this looks like  $ax + by = 0$  where  $a, b \in k$ . The set of lines passing through  $P_1$  is

$$A = \{x + my : m \in k\} \cup \{y = 0\}$$

Since, the field is infinite, there is infinitely many elements in  $A$ . Given two points in  $\mathbb{P}^2$  there is a unique line passing through  $P_1$  and that point. Thus the set of lines

$$L = \{\ell \text{ pass through } P_1 \text{ and } P_i : 2 \leq i \leq n\} \subset A$$

is finite. So there are only finitely many line in the above set. But in  $A$  there are infinitely elements. So, there are infinitely many elements in  $A \setminus L$ .

Since  $P_1$  is a simple point of  $F$ , there is a tangent  $T$  at  $P$  so that the tangent  $T$  don't contained in  $V(F)$  (or  $F$ ). From the problem 5.12 we can say,

$$\sum I(P; F \cap T) = n$$

where  $n = \deg F$ . Thus, If we take  $P_2, \dots, P_m$  be the other intersection points (here  $m \leq n$ ) of  $T$  and  $F$ , by the previous calculation we can say there exists infinitely many lines through  $P$  don't intersect  $F$  at  $P_i$  ( $i > 1$ ). These lines are transversal to  $F$ . ■

### Problem 5.18

Let us consider the general equation of conic in  $\mathbb{P}^2$ , that is

$$Ax^2 + By^2 + Cz^2 + Exy + Fyz + Gzx = 0$$

Since the pont  $[0 : 0 : 1]$  and  $[0 : 1 : 0]$ ,  $[1 : 0 : 0]$  passes through the above conic we can say,  $A = B = C = 0$ . Thus the equation of conic reduces to  $Exy + Fyz + Gzx = 0$ . Also the points  $[1 : 1 : 1]$  and  $[1 : 2 : 3]$  passes through the curve. So we have the following linear equations,

$$\begin{aligned} E + F + G &= 0 \\ 2E + 6F + 3G &= 0 \\ \implies \begin{pmatrix} 1 & 1 & 1 \\ 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} E \\ F \\ G \end{pmatrix} &= 0 \end{aligned}$$

Note that the rows of the aboe matrix are linearly independent. So the null space of it must have dimension 1. Note that  $(3, -4, 1)^T$  is a solution to the above matrix equation. Since the dimension of null space is 1 we can say any other solution must be a scaler multiple of  $(3, -4, 1)^T$ . So the equation of conic passing through the five points is  $\lambda(3xy - 4yz + zx) = 0$ . This will represent a unique conic in  $\mathbb{P}^2$ . By contruction the conic is unique! ■

## Problem 5.25

Since the polynomial  $F = F_1 F_2$  have  $c \geq 1$  simple component, the polynomial may not be irreducible. Let,  $F = F_1 F_2$  and at every point  $P$ ,  $m_P(F) = m_P(F_1) + m_P(F_2)$ . Thus,

$$\begin{aligned} \sum_P \frac{m_P(F)(m_P(F) - 1)}{2} &= \sum_P \frac{(m_P(F_1) + m_P(F_2))(m_P(F_1) + m_P(F_2) - 1)}{2} \\ &= \sum_P \frac{m_P(F_1)(m_P(F_1) - 1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2) - 1)}{2} \\ &\quad + \sum_P m_P(F_1)m_P(F_2) \end{aligned}$$

Let,  $p = \deg F_1$  and  $q = \deg F_2$ . If  $F_1$  and  $F_2$  were irreducible then we must have

$$\begin{aligned} \sum_P \frac{m_P(F)(m_P(F) - 1)}{2} &= \sum_P \frac{m_P(F_1)(m_P(F_1) - 1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2) - 1)}{2} \\ &\quad + \sum_P m_P(F_1)m_P(F_2) \\ &\stackrel{*}{\leq} \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + pq \\ &= \frac{(p+q-1)(p+q-2)}{2} + 1 \\ &= \frac{(n-1)(n-2)}{2} + 1 \end{aligned}$$

here,  $*$  comes from the [corollary 1](#) of Bézout's theorem and theorem of [section 5.4](#). In this case we had  $c = 2$ . Now we will proceed using induction. Assume the result is true for some curve with  $c - 1$  simple components. Again assume  $F = F_1 F_2$  with the degrees mentioned above and  $F_1$  has  $c - 1$ -simple components and  $F_2$  is irreducible. Thus using induction we have,

$$\begin{aligned} \sum_P \frac{m_P(F)(m_P(F) - 1)}{2} &= \sum_P \frac{m_P(F_1)(m_P(F_1) - 1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2) - 1)}{2} \\ &\quad + \sum_P m_P(F_1)m_P(F_2) \\ &\leq \underbrace{\frac{(p-1)(p-2)}{2} + c - 2}_{\text{induction step}} + \frac{(q-1)(q-2)}{2} + pq \\ &= \frac{(p+q-1)(p+q-2)}{2} + c - 1 = \frac{(n-1)(n-2)}{2} + c - 1 \end{aligned}$$

Thus our induction step is complete. It's not hard to note that a polynomial of degree  $n$  can have at most  $n$  linear factor, i.e atmost  $n$  simple components. Thus  $c \leq n$  and hence the final term in the above calculation is bounded above by  $n(n-1)/2$ . ■