Assignment-5

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Problem 5.5

Let $F(X,Y,Z) = \sum_{i=m}^{n} F_i(X,Z)Y^{n-i}$. Then for $P = [0:1:0]$

$$\begin{aligned} m_P(F) &= m_{\varphi(P)}(F_*) \\ &= m_{(0,0)}(\sum_{i=m}^{n} F_i(X,Z)) \\ &= m. \end{aligned}$$

A line L is tangent to F if and only if $I(P, F \cap L) > m_p(F)$, thus we must have

$$\begin{aligned} I(P, F \cap L) &= \dim_k \mathscr{O}_P(\mathbb{P}^2)/(F_*, L_*) \\ &= \dim_k \mathscr{O}_P(\mathbb{P}^2)/(F(X/Y, 1, Z/Y), L/Y) \\ &= \dim_k \mathscr{O}_{(0,0)}(\mathbb{A}^2)/(F(X,1,Z), L(X,1,Z))\mathscr{O}_{(0,0)}(\mathbb{A}^2) \\ &= I((0,0), F(X,1,Z) \cap L(X,1,Z)). \end{aligned}$$

Thus we get that $I((0,0), F(X,1,Z) \cap L(X,1,Z)) > m$, hence $L(X,1,Z)$ is tangent to $F(X,1,Z)$, thus it must be a factor of $F_m(X,Z)$ (by defnition of tangent for an affine curve). Therefore, the tangents to F are determined by the factors of $F_m(X,Z)$.

Problem 5.7

Let F and G be two plane curves with no common components. Let L be a line not contained in $V(FG) \subseteq \mathbb{P}^2$. Then by problem 12, we know that $F \cap L$ and $G \cap L$ are finite. Now there exists a projective transformation that takes the line L to Z. Then under this projective transformation we know that intersection numbers of F and G are preserved. And we have

$$F \cap G = (\underbrace{(F \cap U) \cap (G \cap U)}_{A}) \cup (\underbrace{(F \cap Z) \cup (G \cap Z)}_{B})$$

where $U = \{[x:y:z] \in \mathbb{P}^2 \mid z = 1\}$. Note that B is finite by the choice of the line L. Now $F \cap U$ and $G \cap U$ are affine curves given by $f = F(X,Y,1)$ and $g = G(X,Y,1)$. Now since F and G does not have any common component so does f and g (since otherwise we would have $hp = f$ and $hq = g$ for some $h, p, q \in k[X,Y]$, then $h^* p^* = F$ and $h^* q^* = G$, but then h^* is a common component of F and G, contradiction!). But we have previously shown that if two affine curves have no common component then $f \cap g$ is finite. Hence both A and B are finite, thus $F \cap G$ is finite.

Problem 5.12

Part (a). Let $P \in [0:1:0] \in F$ where F is a curve of degree of n. Let $F(X,Y,Z) = \sum_{i=0}^{n} F_i(Y,Z)X^i$ with F_i is a form of degree $n-i$ with $F_0 \neq 0$ and let $F_0(Y,Z) = \sum_{i=m}^{m+k} a_i Y^i Z^{n-i}$ (with $m,k \geq 0$ and $m+k \leq n-1$, there is no Y^n term as $P = [0:1:0] \in F$).

$$\begin{aligned}
\sum_{P\in\mathbb{P}^2} I(P, F\cap X) &= \sum_{P\in F_0\cap X} I(P, F_0 \cap X) \\
&= \sum_{P\in F_0\cap X\cap U_1} I(P, F_0\cap X) + I([0:0:1], F_0\cap X) \\
&= \sum_{t\in k} I([0:1:t], F_0\cap X) + I([0:0:1], F_0 \cap X) \\
&= \sum_{t\in k} \dim_k \left(\mathscr{O}_{[0:1:t]}(\mathbb{P}^2)/(F_{0*}\cap X_*)\right) + \dim_k \left(\mathscr{O}_{[0:0:1]}(\mathbb{P}^2)/(F_{0*}\cap X_*)\right) \\
&= \sum_{t\in k} \dim_k \left(\mathscr{O}_{(0,t)}(\mathbb{A}^2)/(F_0(1,Z), X)\mathscr{O}_{(0,t)}(\mathbb{A}^2)\right) + \dim_k \left(\mathscr{O}_{(0,0)}(\mathbb{P}^2)/(F_0(Y,1), X)\mathscr{O}_{(0,0)}(\mathbb{A}^2)\right) \\
&= \sum_{t\in k} I((0,t), F_0(1,Z)\cap X) + \operatorname{ord}^X_{(0,0)}(F_0(Y,1)) \\
&= \sum_{P\in F_0(1,Z)\cap X} I(P, F_0(1,Z)\cap X) + \operatorname{ord}^X_{(0,0)}(F_0(Y,1)) \\
&= \deg F_0(1,Z) \deg X + m \\
&= (n-m) + m = n.
\end{aligned}$$

Hence we have proved that $\sum_{P\in\mathbb{P}^2} I(P, F\cap X) = n$.

Part (b). Now if L is not a line contained in F, we can find a projective transformation taking $P \in F \mapsto [0:1:0]$ and $L \mapsto X$, then by part (a), we get that

$$\sum_{P\in\mathbb{P}^2} I(P, F\cap L) = n.$$

Problem 5.14

We will begin with the assumption, the underlying field k is infinte and algebraically closed (according to contexts). The property of lines passing through points is a projective property. So we can take a suitable projective transformation so that $P_1 = [0:0:1]$. Thus, any line passing through this looks like $ax + by = 0$ where $a, b \in k$. The set of lines passing through P_1 is

$$A = \{x + my : m \in k\} \cup \{y = 0\}$$

Since, the field is infinite, there is infinitely many elements in A. Given two points in $\mathbb{P}^2$ there is a unique line passing through P_1 and that point. Thus the set of lines

$$L = \{\ell \text{ pass through } P_1 \text{ and } P_i : 2 \leq i \leq n\} \subset A$$

is finite. So there are only finitely many line in the above set. But in A there are infinitely elements. So, there are infinitely many elements in $A \setminus L$.

Since P_1 is a simple point of F, there is a tangent T at P so that the tangent T don't contained in $V(F)$ (or F). From the problem 5.12 we can say,

$$\sum I(P; F \cap T) = n$$

where $n = \deg F$. Thus, If we take $P_2, \cdots, P_m$ be the other intersection points (here $m \leq n$) of T and F, by the previous calculation we can say there exists infinitely many lines through P don't intersect F at P_i $(i > 1)$. These lines are transversal to F. ■

Problem 5.18

Let us consider the general equation of conic in $\mathbb{P}^2$, that is

$$Ax^2 + By^2 + Cz^2 + Exy + Fyz + Gzx = 0$$

Since the pont $[0 : 0 : 1]$ and $[0 : 1 : 0]$, $[1 : 0 : 0]$ passes through the above conic we can say, $A = B = C = 0$. Thus the equation of conic reduces to $Exy + Fyz + Gzx = 0$. Also the points $[1 : 1 : 1]$ and $[1 : 2 : 3]$ passes through the curve. So we have the following linear equations,

$$\begin{aligned} E + F + G &= 0 \\ 2E + 6F + 3G &= 0 \\ \implies \begin{pmatrix} 1 & 1 & 1 \\ 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} E \\ F \\ G \end{pmatrix} &= 0 \end{aligned}$$

Note that the rows of the aboe matrix are linearly independent. So the null space of it must have dimension 1. Note that $(3, -4, 1)^T$ is a solution to the above matrix equation. Since the dimension of null space is 1 we can say any other solution must be a scaler multiple of $(3, -4, 1)^T$. So the equation of conic passing through the five points is $\lambda(3xy - 4yz + zx) = 0$. This will represent a unique conic in $\mathbb{P}^2$. By contruction the conic is unique! ■

Problem 5.19

Let us consider an arbitrary cubic

$$aX^3 + bX^2Y + cX^2Z + dY^3 + eXY^2 + fY^2Z + gZ^3 + hXZ^2 + iYZ^2 + jXYZ$$

Now given that the cubic passes through the following points: $[0:0:1]$, $[0:1:1]$, $[1:0:1]$, $[1:1:1]$, $[0:2:1]$, $[2:0:1]$, $[1:2:1]$, $[2:1:1]$, and $[2:2:1]$ gives us

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 8 & 0 & 4 & 1 & 0 & 2 & 0 \\ 8 & 0 & 4 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 2 & 2 & 1 & 8 & 4 & 4 & 1 & 1 & 2 & 2 \\ 8 & 4 & 4 & 1 & 2 & 1 & 1 & 2 & 1 & 2 \\ 8 & 8 & 4 & 8 & 8 & 4 & 1 & 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \\ j \end{pmatrix} = \mathbf{0}.$$

The rank of the above matrix is 9, thus the dimension of the kernel is 1, hence there exists an unique cubic passing through all the points.

Problem 5.25

Since the polynomial $F = F_1F_2$ have $c \geq 1$ simple component, the polynomial may not be irreducible. Let, $F = F_1F_2$ and at every point P, $m_P(F) = m_P(F_1) + m_P(F_2)$. Thus,

$$\begin{aligned} \sum_P \frac{m_P(F)(m_P(F)-1)}{2} &= \sum_P \frac{(m_P(F_1)+m_P(F_2))(m_P(F_1)+m_P(F_2)-1)}{2} \\ &= \sum_P \frac{m_P(F_1)(m_P(F_1)-1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2)-1)}{2} \\ &\quad + \sum_P m_P(F_1)m_P(F_2) \end{aligned}$$

Let, $p = \deg F_1$ and $q = \deg F_2$. If F_1 and F_2 were irreducible then we must have

$$\begin{aligned} \sum_P \frac{m_P(F)(m_P(F)-1)}{2} &= \sum_P \frac{m_P(F_1)(m_P(F_1)-1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2)-1)}{2} \\ &\quad + \sum_P m_P(F_1)m_P(F_2) \\ &\overset{*}{\leq} \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + pq \\ &= \frac{(p+q-1)(p+q-2)}{2} + 1 \\ &= \frac{(n-1)(n-2)}{2} + 1 \end{aligned}$$

here, $*$ comes from the corollary 1 of Bézout's theorem and theorem of section 5.4. In this case we had $c = 2$. Now we will proceed using induction. Assume the result is true for some curve with $c-1$ simple components. Again assume $F = F1F_2$ with the degrees mentioned above and F_1 has $c-1$-simple components and F_2 is irreducible. Thus using induction we have,

$$\begin{aligned}\sum_P \frac{m_P(F)(m_P(F)-1)}{2} &= \sum_P \frac{m_P(F_1)(m_P(F_1)-1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2)-1)}{2} \\ &\quad + \sum_P m_P(F_1)m_P(F_2) \\ &\leq \underbrace{\frac{(p-1)(p-2)}{2} + c - 2}_{\text{induction step}} + \frac{(q-1)(q-2)}{2} + pq \\ &= \frac{(p+q-1)(p+q-2)}{2} + c - 1 = \frac{(n-1)(n-2)}{2} + c - 1\end{aligned}$$

Thus our induction step is complete. It's not hard to note that a polynomial of degree n can have at most n linear factor, i.e atmost n simple components. Thus $c \leq n$ and hence the final term in the above calculation is bounded above by $n(n-1)/2$. ■

Problem 5.28

Let L be a line through P. As $P = [0,1,0]$, L must be of the form $aX + bZ = 0$. If $b = 0$, that is, L is the line $X = 0$, then $L \cap F$ consists of $[0,1,0]$ and $[0,0,1]$.

Now suppose that $b \neq 0$. Then, any point on L satisfies $Z = \frac{-a}{b}X$. Putting this in the polynomial defining F we get,

$$X^{p+1} - Y^p\left(\frac{-a}{b}\right)X = X(bX^p + aY^b) = bX(X - \lambda Y)^p,$$

where $\lambda^p = \frac{-a}{b}$ and we use the fact that the field is of characteristic p. Hence, either $X = 0$ or $X = \lambda Y$. This gives either $Z = 0$ or $Z = \lambda^{p+1}Y$. So, if L is not the line $X = 0$, $L \cap F$ consists of the points $[\lambda y, y, \lambda^{p+1}y], y \in k$, where $\lambda^p = \frac{-a}{b}$.

We have,

$$\frac{\partial F}{\partial X} = (p+1)X^p = X^p, \quad \frac{\partial F}{\partial Y} = -pY^{p-1}Z = 0, \quad \frac{\partial F}{\partial Z} = -Y^p.$$

Hence, $[x,y,z]$ is a simple point of F iff $x^{p+1} = y^p z$ and one of x, y is non-zero. The tangent to F at such a point is then given by $x^p X - y^p Z = 0$, which clearly passes through the point $[0,1,0]$ as required. ■

Problem 5.31

Part (a). Applying the Pascal's theorem with $P_1 = P_2$, $P_3 = P_4$ and $P_5 = P_6$ we get that, for any triangle $P_1P_3P_5$ inscribed on a cubic, the intersection of the tangent at each vertex with the opposite side of the triangle are collinear. In the given figure $P_1P_3P_5$ is the triangle inscribed on a cubic, and the tangent at P_1 intersects the opposite side P_3P_5 at D, we similarly define E and F, then D, E and F are collinear.

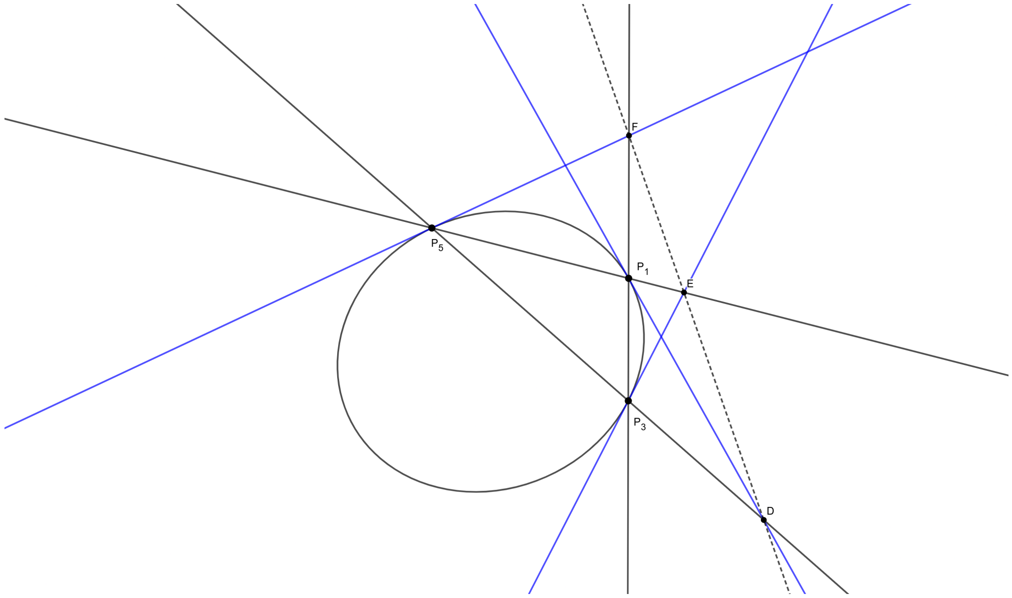

Figure 1: Sketch of Pascal's thoerem for $P_1 = P_2$, $P_3 = P_4$ and $P_5 = P_6$.

Part (b). Applying the Pascal Theorem with $P_1 = P_2$, we get that for any arbitrary five points P_1, P_3, P_4, P_5, P_6 on a cubic, let $E = P_1P_3 \cap P_5P_6$ and $F = P_1P_6 \cap P_3P_4$ and let $D = EF \cap P_4P_5$, then DA is the tangent at A to the given cubic.

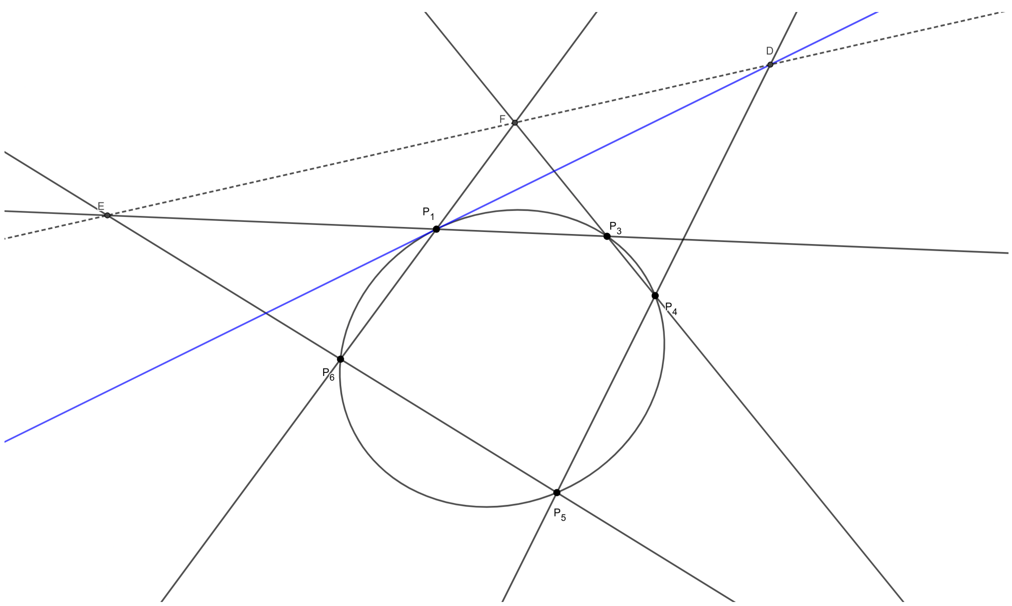

Figure 2: Sketch of Pascal's thoerem for $P_1 = P_2$.

Part (c). Using part (b), given any point P and a conic $\mathcal{C}$, we can construct a tangent at P to $\mathcal{C}$, as follows: let $P_1, \dots, P_4$ be four distinct points on the conic $\mathcal{C}$. Now let $E = PP_1 \cap P_3P_4$ and $F = PP_4 \cap P_1P_2$ and let $D = EF \cap P_2P_3$, then DP is the tangent at P to $\mathcal{C}$. Thus we can construct the tangent on a cubic, using only a straight-edge.