Assignment-3

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Problem 3.1. *(UAG 5.1) A rgular function on $\mathbb{P}^1$ is constant. Deduce that there are no non-constant morphisms $\mathbb{P}^1 \to \mathbb{A}^m$ for $m \geq 1$.*

Solution. Suppose $f \in k(\mathbb{P}^1)$ be a rational function, which is regular everywhere. If we restrict it to the affine piece $\mathbb{A}_{(0)}$, we get $f(x,1) = p(x) \in k[x]$ (as for the case of affine variety $\operatorname{dom} f = V$ iff $f \in k[V]$). Similarly, we can restrict f to another affine piece $\mathbb{A}_\infty$. We get, $f(1,y) = f(1/y,1) = p(1/y) \in k[y]$. It is possible iff p is constant.

Any morphisms $\mathbb{P}^1 \to \mathbb{A}^m$ can be given by $(f_1, \cdots, f_m)$ where f_i are regular on $\mathbb{P}^1$. Thus the function f is constant by the previous part. ■

Problem 3.2. *(The quadric surface in $\mathbb{P}^3$).*

(i) Show that the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ gives an isomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ with the quadric

$$S_{1,1} = Q : (X_0X_3 = X_1X_2) \subseteq \mathbb{P}^3.$$

(ii) What are the images in Q of the two families of lines $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$? Use this to find some disjoint lines in $\mathbb{P}^1 \times \mathbb{P}^1$, and conclude from this that $\mathbb{P}^1 \times \mathbb{P}^1 \ncong \mathbb{P}^2$.

(iii) Show that there are two lines of Q passing through the point $P = (1,0,0,0)$ and that the complement U of these two lines is the image of $\mathbb{A}^1 \times \mathbb{A}^1$ under the Segree embedding.

(iv) Show that under the projection $\pi|_Q : Q \dashrightarrow \mathbb{P}^2$, U maps isomorphically to a copy of $\mathbb{A}^2$, and the two lines through P are mapped to two points of $\mathbb{P}^2$.

(v) Find $\operatorname{dom}\pi$ and $\operatorname{dom}\varphi$, and give a geometric interpretation of the singularities of π and φ.

Solution.

(i) Let $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, $([X_0, X_1], [Y_0, Y_1]) \mapsto [X_0Y_0, X_0Y_1, X_1Y_0, X_1Y_1]$ be the Segree embedding. Then we clearly have $\operatorname{Im}\varphi = S_{1,1} \subseteq Q$. Since we know that the Segree embedding $S_{1,1} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Its enough to show that $Q \subseteq S_{1,1}$. Note that

$$\begin{aligned} Q &= \{[X_0, X_1, X_2, X_3] \in \mathbb{P}^3 \mid X_0X_3 - X_1X_2 = 0\} \\ &= \left\{[X_0, X_1, X_2, X_3] \in \mathbb{P}^3 \mid \det \begin{pmatrix} X_0 & X_1 \\ X_2 & X_3 \end{pmatrix} = 0\right\} \\ &= \left\{[X_0, X_1, X_2, X_3] \in \mathbb{P}^3 \mid \operatorname{rk} \begin{pmatrix} X_0 & X_1 \\ X_2 & X_3 \end{pmatrix} = 1\right\}, \end{aligned}$$

the rank can not be zero, as at least one of the entries X_0, X_1, X_2, X_3 is nonzero. Let $[X_0, X_1, X_2, X_3] \in Q$, and WLOG assume $X_0 \neq 0$, then we get there exists $\lambda, \mu \neq 0$ such that

$$\begin{pmatrix} X_0 \\ X_2 \end{pmatrix} = \lambda \begin{pmatrix} X_1 \\ X_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \mu \begin{pmatrix} X_2 \\ X_3 \end{pmatrix}$$

Thus we get that $X_1 = \frac{X_0}{\lambda}$, $X_2 = \frac{X_0}{\mu}$ and $X_3 = \frac{X_2}{\lambda} = \frac{X_0}{\mu\lambda}$, thus we get that

$$[X_0, X_1, X_2, X_3] = \left[X_0, \frac{X_0}{\lambda}, \frac{X_0}{\mu}, \frac{X_0}{\mu\lambda}\right] = [\mu\lambda, \mu, \lambda, 1] = \varphi([\mu, 1], [\lambda, 1]).$$

Therefore we have proved that $Q \subseteq S_{1,1}$, hence we get that φ induces an isomorphism of $S_{1,1}$ and Q.

(ii) We have $\varphi(\{p\} \times \mathbb{P}^1) = \{[aY_0, aY_1, bY_0, bY_1] \mid [Y_0, Y_1] \in \mathbb{P}^1\}$, which is equation of the line passing through $[a, 0, b, 0], [0, a, 0, b] \in \mathbb{P}^3$. Similarly image of $\mathbb{P}^1 \times \{p\}$ is again a line in $\mathbb{P}^3$. But then note that for $p \neq q \in \mathbb{P}^1$, we have $(\{p\} \times \mathbb{P}^1) \cap (\{q\} \times \mathbb{P}^1) = \emptyset$, hence their images are disjoint lines in Q. But we know that any two lines in $\mathbb{P}^2$ have a intersection, hence $\mathbb{P}^1 \times \mathbb{P}^1 \ncong \mathbb{P}^2$.

(iii) Let us consider the image of $\mathbb{A}^1 \times \mathbb{A}^1$ in $\mathbb{P}^3$ under the Segre embedding. We get

$$\varphi(\mathbb{A}^1 \times \mathbb{A}^1) = \{[ab, a, b, 1] \in \mathbb{P}^3 \mid a, b \in k\}.$$

Now consider the two lines $\ell_1 = \{[\mu, 0, \lambda, 0] \in \mathbb{P}^3 \mid [\mu, \lambda] \in \mathbb{P}^1\}$ and $\ell_2 = \{[\mu, \lambda, 0, 0] \in \mathbb{P}^3 \mid [\mu, \lambda] \in \mathbb{P}^1\}$ through $[1, 0, 0, 0]$ and contained in Q. We claim that the complement U of these two lines is $\varphi(\mathbb{A}^1 \times \mathbb{A}^1)$. Clearly we have $\varphi(\mathbb{A}^1 \times \mathbb{A}^1) \cap (\ell_1 \cup \ell_2) = \emptyset$. Conversely let $[X_0, X_1, X_2, X_3] \notin \varphi(\mathbb{A}^1 \times \mathbb{A}^1)$, then $[X_0, X_1, X_2, X_3] = \varphi([a, b], [1, 0]) = [a, 0, b, 0] \in \ell_1$ or $[X_0, X_1, X_2, X_3] = \varphi([1, 0], [c, d]) = [c, d, 0, 0] \in \ell_2$. Therefore we have shown that $U = \varphi(\mathbb{A}^1 \times \mathbb{A}^1)$.

(iv) Under the projection $\pi|_Q : Q \dashrightarrow \mathbb{P}^2$, $[X_0, X_1, X_2, X_3] \mapsto [X_1, X_2, X_3]$. Then

$$\pi(U) = \pi(\varphi(\mathbb{A}^1 \times \mathbb{A}^1)) = [a, b, 1] \in \mathbb{A}^2 \subseteq \mathbb{P}^2.$$

And the two lines ℓ_1 and ℓ_2 maps to the two points $[0, 1, 0]$ and $[1, 0, 0]$ respectively.

(v)

Problem 3.3. *Problem 5.3*

***Solution*.**

(a) The given map is a rational map. This is because it is well-defined for all $[x, y, z] \in \mathbb{P}^2 \setminus \{[0, 0, 1]\}$ and is a rational function in each coordinate of the image. We therefore have

$$\operatorname{dom} \varphi = [x, y, z] \in \mathbb{P}^2 \setminus \{[0, 0, 1]\}.$$

Further, this is a birational map, as it has the rational inverse given by the map in (c), $[x, y] \mapsto [x, y, 0]$.

(b) The given map is not a rational map. This is because

$$\varphi([1, 0]) = [1, 0, 1] \neq [2, 0, 1] = \varphi([2, 0]),$$

but $[1, 0] \neq [2, 0]$.

(c) The given map is a rational map. This is because it is well-defined for all $[z, y] \in \mathbb{P}^1$ and is a rational function in each coordinate of the image. We therefore have

$$\operatorname{dom} \varphi = \mathbb{P}^1.$$

Further, this is a birational map, as it has the rational inverse given by the map in (a), $[x, y, z] \mapsto [x, y]$.

(d) The given map is a rational map. This is because it is well-defined for all $[x, y, z] \in \mathbb{P}^2$ with $xyz \neq 0$, and is a rational function in each coordinate of the image. We therefore have,

$$\operatorname{dom}\varphi = \{[x, y, z] \mid xyz \neq 0\}.$$

Further, φ^2 is the identity map on $\operatorname{dom}\varphi$, and so it is a birational map.

(e) The given map is a rational map. This is because it is well-defined for all $[x, y, z] \in \mathbb{P}^2$ with $z \neq 0$, and is a rational function in each coordinate of the image. We therefore have,

$$\operatorname{dom}\varphi = \{[x, y, z] \mid z \neq 0\}.$$

The map is not birational as the function fields of the domain and image are not isomorphic.

(f) The given map is a rational map. This is because it is well-defined for all $[x, y, z] \in \mathbb{P}^2$ with one of x, y non-zero, and is a rational function in each coordinate of the image. We therefore have,

$$\operatorname{dom}\varphi = \mathbb{P}^2 \setminus \{[0, 0, 1]\}.$$

The map is not birational as there is no rational inverse.

Problem 3.4. *Let $C \subseteq \mathbb{P}^3$ be an irreducible curve defined by $C = Q_1 \cap Q_2$, where $Q_1 : (TX = q_1)$, and $Q_2 : (TY = q_2)$, with q_1, q_2 quadratic forms in X, Y, Z. Show that the projection $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ defined by $(X, Y, Z, T) \mapsto (X, Y, Z)$ restricts to an isomorphism of C with the plane curve $D \subseteq \mathbb{P}^2$ given by $Xq_2 = Yq_1$.*

***Solution*.** Let us define the map $\varphi : D \dashrightarrow C$, as follows,

$$[X, Y, Z] \mapsto \begin{cases} [X, Y, Z, \frac{q_1}{X}] & \text{if } X \neq 0 \\ [X, Y, Z, \frac{q_2}{Y}] & \text{if } Y \neq 0 \end{cases}$$

Note that this is indeed a map from D to C, as if $[X, Y, Z] \in D$ with $X \neq 0$, then we get that $Xq_2 = Yq_1$, and hence, $TX = q_1$ and $TY = \frac{Yq_1}{X} = \frac{Xq_2}{X} = q_2$, thus $\varphi([X, Y, Z]) \in C$, and similarly for $Y \neq 0$, we have $[X, Y, Z, T] = \varphi([X, Y, Z]) \in C$. On the other hand restricting the projection onto C, we get that $\pi([X, Y, Z, T]) = [X, Y, Z]$, and since $TX = q_1$ and $TY = q_2$ we get that $Yq_1 = TXY = Xq_2$, thus we indeed have $[X, Y, Z] \in D$.

Finally note that $\pi|_C \circ \varphi = \mathrm{id}_D$ is obvious and

$$\varphi(\pi|_C([X, Y, Z, T])) = \varphi([X, Y, Z]) = \begin{cases} [X, Y, Z, \frac{q_1}{X}] & \text{if } X \neq 0 \\ [X, Y, Z, \frac{q_2}{Y}] & \text{if } Y \neq 0 \end{cases} = [X, Y, Z, T],$$

where the last equality follows from the fact that $TX = q_1$ and $TY = q_2$ for points in C. Thus we indeed have $\varphi \circ \pi|_C = \mathrm{id}_C$. Hence π restricted onto C induces an isomorphism of C with the plane curve D.

Problem 3.5. *Problem 5.6*

***Solution*.**

(a) The affine pieces are:

(i) $(x = 1) : y^2 z = 1 + az^2 + bz^3$

(ii) $(y = 1) : z = x^3 + axz^2 + bz^3$

(iii) $(z = 1) : y^2 = x^3 + ax + b$

The intersections with the coordinate axes are:

(i) $(x = 0) : z(y^2 - bz^2) = 0$
(ii) $(y = 0) : x^3 + axz^2 + bz^3 = 0$
(iii) $(z = 0) : x^3 = 0$

(b) The affine pieces are:

(i) $(x = 1) : (y - z)^2 - 2yz(y + z) = 0$
(ii) $(y = 1) : (z - x)^2 - 2zx(z + x) = 0$
(iii) $(z = 1) : (x - y)^2 - 2xy(x + y) = 0$

The intersections with the coordinate axes are:

(i) $(x = 0) : y^2z^2 = 0$
(ii) $(y = 0) : z^2x^2 = 0$
(iii) $(z = 0) : x^2y^2 = 0$

(c) The affine pieces are:

(i) $(x = 1) : z^3 = (1 + z^2)y^2$
(ii) $(y = 1) : xz^3 = x^2 + z^2$
(iii) $(z = 1) : x = (x^2 + 1)y^2$

The intersections with the coordinate axes are:

(i) $(x = 0) : z^2y^2 = 0$
(ii) $(y = 0) : xz^3 = 0$
(iii) $(z = 0) : x^2y^2 = 0$

Problem 3.6. *(UAG 5.7) Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ be an isomorphism; identify graph of φ as subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. Now do the same if $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is given by map $(X, Y) \mapsto (X^2, Y^2)$.*

Solution. Consider the identity map $\mathrm{Id} : \mathbb{P}^1 \to \mathbb{P}^1$ and the given isomorphism, it will give us a map $\mathrm{Id} \times \varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ by $(x, y) \mapsto (x, \varphi(x))$. Under the identification of $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}^3$ we can say, $\mathrm{Id} \times \varphi$ is also a morphism of variety. In the variety $\mathbb{P}^1 \times \mathbb{P}^1$, the diagonal $\Delta = \{(x, x) : x \in \mathbb{P}^1\}$ is closed (simply because it is given by the vanishing of $x_0 - x_2$ and $x_1 - x_3$ where $[x_0 : x_1]$ and $[x_2 : x_3]$ are co-ordinates of two copies of $\mathbb{P}^1$). It's not hard to see the graph of φ is given by the inverse image of Δ under $\mathrm{Id} \times \varphi$.

$$\Gamma(\varphi) = (\mathrm{Id} \times \varphi)^{-1}(\Delta)$$

Since the graph is closed it's inverse image will also be closed. Thus the graph is a closed set and under zariski topology any closed set is given by vanishing of some set of polynomials. This will help us to identify $\Gamma(\varphi)$ as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1$. If φ is given by $[x : y] \to [f(x, y) : g(x, y)]$ then the graph can be given by the image of following vanishing set under segre embedding

$$\{[x_0 : x_1 : x_2 : x_3] : x_2 = f(x_0, x_1), x_3 = g(x_0, x_1)\}$$

If, φ given by $[x, y] \mapsto [x^2 : y^2]$ the image of $([x : y], [x^2, y^2])$ is $[x^3 : xy^2 : yx^2 : y^3]$(image under segre embedding). Which is rational curve $\mathbb{P}^1 \to \mathbb{P}^3$, a sub-variety of $\mathbb{P}^3$.

$$\Gamma(\varphi) \simeq \text{Rational curve in } \mathbb{P}^3$$

Problem 3.7. *(i) Prove that the product of two irreducible algebraic sets is again irreducible.*

(ii) Describe the closed sets of the topology on $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ which is the product of the Zariski topologies on the two factors; now find a closed subset of the Zariski topology of $\mathbb{A}^2$ not of this form.

***Solution*.**

(i) Suppose that $X \times Y = Q_1 \cup Q_2$, with each Q_i a closed subset of $X \times Y$. For each $x \in X$, the closed set $\{x\} \times Y$ is isomorphic to Y, and is therefore irreducible. Since $\{x\} \times Y = ((\{x\} \times Y) \cap Q_1) \cup ((\{x\} \times Y) \cap Q_2)$ either $\{x\} \times Y \subseteq Q_1$ or else $\{x\} \times Y \subseteq Q_2$.

The subset $X_1 \subseteq X$ consisting of those $x \in X$ with $\{x\} \times Y \subseteq Q_1$ is a closed subset, to see this note that $X_1 = \cap_{y \in Y} X_y$, where X_y is the collection of points $x \in X$ such that $\{x\} \times \{y\} \in Q_1$. Since $X_y \times \{y\} = (X \times \{y\}) \cap Q_1$, X_y and hence X_1 is closed. Similarly we can define the closed subset X_2.

Since $X = X_1 \cup X_2$ and X is irreducible, we either have $X = X_1$ or $X = X_2$. But $X = X_i$ implies $X \times Y = Q_i$, contradicting the fact the both of the Q_i's are nonempty. Therefore $X \times Y$ is irreducible.

(ii) We know that the closed subsets of $\mathbb{A}^1$ under the Zariski topology are finite subsets of $\mathbb{A}^1$ and the whole set $\mathbb{A}^1$. Thus under the product topology on $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ closed subsets are once again finite subsets of $\mathbb{A}^1 \times \mathbb{A}^1$, $\{x_1, \ldots, x_n\} \times \mathbb{A}^1$, $\mathbb{A}^1 \times \{y_1, \ldots, y_m\}$ and $\mathbb{A}^1 \times \mathbb{A}^1$.

Consider the closed subset $C = V(X - Y) = \{(a, a) \mid a \in k\} \subseteq \mathbb{A}^2$. If k is an infinite field, then C does not belong to any of the closed sets coming from the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$.

Problem 3.8. *Problem 5.12*

***Solution*.** The given curve is $C : (Y^2 Z = X^3 + aXZ^2 + bZ^3) \subset \mathbb{P}^2$. The affine pieces are

$$C_{(0)} : y^2 = x^3 + ax + b, \quad C_{(\infty)} : z' = x'^3 + ax'z'^2 + bz'^3$$

Let f be a regular function on C. Then, $\operatorname{dom} f \supset C_{(0)}$, and so, $f \in k[C_{(0)}] = k[x,y]/(y^2 - x^3 - ax - b)$. Hence, there is $q, r \in k[x]$ such that $f(x, y) \equiv q(x) + yr(x)$ in $k[C_{(0)}]$. Now, as $\operatorname{dom} f \supset C_{(\infty)}$, we get that

$$q\left(\frac{x'}{z'}\right) + \frac{1}{z'} r\left(\frac{x'}{z'}\right) \equiv p(x', z'),$$

for some polynomial p. Therefore, we can multiply out the denominators to get an expression

$$\widetilde{q}(x', z') + \widetilde{r}(x', z') = p(x', z') z'^m + A(x', z') g,$$

in $k[x', z']$, where $\widetilde{q}$ is homogeneous of degree m, $\widetilde{r}$ is homogeneous of degree $m - 1$, $g = x'^3 + ax'z'^2 + bz'^3 - z'$. We now write $p = p_1 + p_2$ and $A = A_1 + A_2$, where p_1, A_1 consist of the odd degree terms and p_2, A_2 consist of the even degree terms. Then, assuming m is odd, we get

$$\widetilde{q} = p_2 z'^m + A_1 g, \quad \widetilde{r} = p_1 z_1^m + A_2 g.$$

A similar expression holds in case m is even, by switching p_1 with p_2 and A_1 with A_2. Now, $\widetilde{q}$ is homogeneous of degree m, and hence, $A_1 g$ must have degree at least m. Therefore, we get (as g has the term z') that $z' \mid \widetilde{q}$. Similarly, $z' \mid \widetilde{r}$. Hence, we can divide the entire expression by z', and get $\widetilde{q}$ homogeneous of degree $m - 1$ and $\widetilde{r}$ homogeneous of degree $m - 2$. Hence, assuming that m is the least possible we get $m = 0$, and so, $f \equiv c$ for some constant c. This shows that f must in fact be constant, as was required. ■

Problem 3.9. *(UAG 5.13) Study the embedding $\varphi : \mathbb{P}^2 \to \mathbb{P}^5$ given by $[x : y : z] \mapsto [x^2 : xy : xz : y^2 : yz : z^2]$ and prove that φ is an isomorphism. Prove that the lines of $\mathbb{P}^2$ go over the conics of $\mathbb{P}^5$ and the conics go over the twisted quartics of $\mathbb{P}^5$.*

For any line $\ell \subset \mathbb{P}^2$, write $\pi(\ell) \subseteq \mathbb{P}^5$ for the projective plane spanned by the conics $\varphi(\ell)$. Prove that union of $\pi(\ell)$ taken over all $\ell \subset \mathbb{P}^2$ is a cubic hypersurface $\Sigma \subseteq \mathbb{P}^5$.

Solution. Consider the following vanishing set on $\mathbb{P}^5$,

$$S = V(t_0t_3 - t_1^2, t_3t_5 - t_4^2, t_0t_5 - t_2^2, t_1t_2 - t_0t_4, t_1t_4 - t_3t_2, t_2t_4 - t_5t_1)$$

It's not hard to see $\operatorname{Im}\varphi \subset S$. Now note that the map φ gives us a surjective map between the following vector spaces,

$$\{\text{homogeneous quadratic polynomials in } t_0, \cdots, t_5\} \to \{\text{homogeneous quartics in } x, y, z\}$$

The first V.S is of dimension 21 and the later one has dimension 15. Thus the kernal has dimension 6. Now note that the polynomials defining S are linearly independent. So, $\operatorname{Im}\varphi = S$. Thus the image of φ is given by the variety S. Now take the map $\psi : S \to \mathbb{P}^3$ that maps $[t_0 : \cdots : t_5] \to [t_0 : t_1 : t_2]$ works as the inverse map of φ (it is defined except for $[0 : 0 : 0 : 0 : 0 : 1]$). So, φ is an isomorphism. Any line in $\mathbb{P}^2$ can be given by the set $\{[x : y : ax + by]\}$, image of this in S is intersection of conics which will be again a conic (it can be degenerate). Any conic in $\mathbb{P}^2$ can be re-parametrized so that it is given by $[u^2 : uv : v^2]$. It's image in S is twisted quardics.

To do the last part we can also identify S as the following set,

$$S = \left\{ [t_0 : t_1 : \cdots : t_5] \in \mathbb{P}^5 : \operatorname{rank} \begin{pmatrix} t_0 & t_1 & t_2 \\ t_1 & t_3 & t_4 \\ t_2 & t_4 & t_5 \end{pmatrix} \leq 1 \right\}$$

From the above identification of S we can say, $\cup_{\ell \subset \mathbb{P}^2} \pi(\ell)$ is given by $\det \begin{pmatrix} t_0 & t_1 & t_2 \\ t_1 & t_3 & t_4 \\ t_2 & t_4 & t_5 \end{pmatrix} = 0$. This clearly determines a hyper-surface in $\mathbb{P}^5$. ■