Assignment-5

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Problem 5.14

We will begin with the assumption, the underlying field k is infinte and algebraically closed (according to contexts). The property of lines passing through points is a projective property. So we can take a suitable projective transformation so that $P_1 = [0 : 0 : 1]$. Thus, any line passing through this looks like $ax + by = 0$ where $a, b \in k$. The set of lines passing through P_1 is

$$A = \{x + my : m \in k\} \cup \{y = 0\}$$

Since, the field is infinite, there is infinitely many elements in A. Given two points in $\mathbb{P}^2$ there is a unique line passing through P_1 and that point. Thus the set of lines

$$L = \{\ell \text{ pass through } P_1 \text{ and } P_i : 2 \leq i \leq n\} \subset A$$

is finite. So there are only finitely many line in the above set. But in A there are infinitely elements. So, there are infinitely many elements in $A \setminus L$.

Since P_1 is a simple point of F, there is a tangent T at P so that the tangent T don't contained in $V(F)$ (or F). From the problem 5.12 we can say,

$$\sum I(P; F \cap T) = n$$

where $n = \deg F$. Thus, If we take $P_2, \cdots, P_m$ be the other intersection points (here $m \leq n$) of T and F, by the previous calculation we can say there exists infinitely many lines through P don't intersect F at P_i $(i > 1)$. These lines are transversal to F. ■

Problem 5.18

Let us consider the general equation of conic in $\mathbb{P}^2$, that is

$$Ax^2 + By^2 + Cz^2 + Exy + Fyz + Gzx = 0$$

Since the pont $[0 : 0 : 1]$ and $[0 : 1 : 0]$, $[1 : 0 : 0]$ passes through the above conic we can say, $A = B = C = 0$. Thus the equation of conic reduces to $Exy + Fyz + Gzx = 0$. Also the points $[1 : 1 : 1]$ and $[1 : 2 : 3]$ passes through the curve. So we have the following linear equations,

$$\begin{aligned} E + F + G &= 0 \\ 2E + 6F + 3G &= 0 \end{aligned}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \\ 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} E \\ F \\ G \end{pmatrix} = 0$$

Note that the rows of the aboe matrix are linearly independent. So the null space of it must have dimension 1. Note that $(3,\,-4,\,1)^T$ is a solution to the above matrix equation. Since the dimension of null space is 1 we can say any other solution must be a scaler multiple of $(3,\,-4,\,1)^T$. So the equation of conic passing through the five points is $\lambda(3xy-4yz+zx)=0$. This will represent a unique conic in $\mathbb{P}^2$. By contruction the conic is unique! ■

Problem 5.25

Since the polynomial $F=F_1F_2$ have $c\geq 1$ simple component, the polynomial may not be irreducible. Let, $F=F_1F_2$ and at every point P, $m_P(F)=m_P(F_1)+m_P(F_2)$. Thus,

$$\begin{aligned}\sum_P \frac{m_P(F)(m_P(F)-1)}{2} &= \sum_P \frac{(m_P(F_1)+m_P(F_2))(m_P(F_1)+m_P(F_2)-1)}{2}\\ &= \sum_P \frac{m_P(F_1)(m_P(F_1)-1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2)-1)}{2}\\ &+ \sum_P m_P(F_1)m_P(F_2)\end{aligned}$$

Let, $p=\deg F_1$ and $q=\deg F_2$. If F_1 and F_2 were irreducible then we must have

$$\begin{aligned}\sum_P \frac{m_P(F)(m_P(F)-1)}{2} &= \sum_P \frac{m_P(F_1)(m_P(F_1)-1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2)-1)}{2}\\ &+ \sum_P m_P(F_1)m_P(F_2)\\ &\overset{*}{\leq} \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + pq\\ &= \frac{(p+q-1)(p+q-2)}{2} + 1\\ &= \frac{(n-1)(n-2)}{2} + 1\end{aligned}$$

here, $*$ comes from the corollary 1 of Bézout's theorem and theorem of section 5.4. In this case we had $c=2$. Now we will proceed using induction. Assume the result is true for some curve with $c-1$ simple components. Again assume $F=F1F_2$ with the degrees mentioned above and F_1 has $c-1$-simple components and F_2 is irreducible. Thus using induction we have,

$$\begin{aligned}\sum_P \frac{m_P(F)(m_P(F)-1)}{2} &= \sum_P \frac{m_P(F_1)(m_P(F_1)-1)}{2} + \sum_P \frac{m_P(F_2)(m_P(F_2)-1)}{2}\\ &+ \sum_P m_P(F_1)m_P(F_2)\\ &\leq \underbrace{\frac{(p-1)(p-2)}{2} + c - 2}_{\text{induction step}} + \frac{(q-1)(q-2)}{2} + pq\\ &= \frac{(p+q-1)(p+q-2)}{2} + c - 1 = \frac{(n-1)(n-2)}{2} + c - 1\end{aligned}$$

Thus our induction step is complete. It's not hard to note that a polynomial of degree n can have at most n linear factor, i.e atmost n simple components. Thus $c \leq n$ and hence the final term in the above calculation is bounded above by $n(n-1)/2$. ■