Assignment-4

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§1. Problem 3.6

§ **Lemma** – *1*

The F and G be forms of degree r and $r+1$ respectively with no common factors in $k[X_1, \dots, X_n]$, then $F + G$ is irreducible.

Proof (of Lemma). Suppose $F+G$ is reducible then there exists nonconstant polynomials $P, Q \in k[X_1, \dots, X_n]$ such that $F + G = PQ$. Now we consider the homogeneous both of these to get

$$X_{n+1}F + G = (F+G)^* = (PQ)^* = P^*Q^*.$$

But note that $X_{n+1}F + G \in k(X_1, \dots, X_n)[X_{n+1}]$ is irreducible, hence one of P^* or Q^* is in $k[X_1, \dots, X_n]$. Then by comparing degrees we can WLOG assume that $Q^* \in k[X_1, \dots, X_n]$, and let $P = X_{n+1}R + S$, where $R, S \in k[X_1, \dots, X_n]$. Then we get that

$$X_{n+1}F + G = X_{n+1}RQ^* + SQ^* \Rightarrow F = RQ^* \text{ and } G = SQ^*$$

But this contradicts the fact that F and G have no common factor, hence we get that $F+G$ is irreducible.

Now coming back to the main problem, suppose we are given tangent lines L_i with multiplicities r_i, and we want to find an irreducible curve F such that L_i is a tangent to F with multiplicity r_i. Note that $\prod_i L_i^{r_i}$ is a forms of degree $m = \sum_i r_i$. Then we can find a homogeneous polynomial F_{m+1} of degree $m+1$ such that F_{m+1} is not divisible by any of the L_i (such a polynomial obvious exists). But then by the previous lemma $\prod_i L_i^{r_i} + F_{m+1}$ is irreducible, and clearly $F = \prod_i L_i^{r_i} + F_{m+1}$ satisfies the necessary conditions.

§2. Problem 3.8

Part (a). We will first prove it for the case when $P = Q = (0,0)$. In this case the polynomial map $T : \mathbb{A}^2 \to \mathbb{A}^2$ must look like (f, g) where f and g are polynomials vanishing at $(0,0)$. In this case we can wite $f = f_i + \cdots + f_t$, where $f_i \in k[x,y], i \geq 1$ is homogeneous polynomial of degree i. Similarly, we can write for g (as both of them are vanishing at $(0,0)$). If $m = m_P(F)$ then $F = F_m + F_{m+1} \cdots$ again F_i are homogeneous polynomial of degree m. Now $F^T = F(f,g)$'s lowest degree will come from $F_m(f,g)$ since both f, g has at-least one degree term we can say, $m_Q(F^T) \geq m_p(F)$.

Now we will use the fact proved in page (33) to prove it for any P, Q. Let, $Q \neq (0,0)$ or $Q = T(P) \neq (0,0)$. Let $T_1 : \mathbb{A}^2 \to \mathbb{A}^2$ be the affine transformation that maps $(0,0)$ to Q

and T_2 be the affine map sends P to $(0,0)$. Note that $T_1 \circ T \circ T_2$ is a polynomial map and it maps $(0,0)$. So by the above calculation we can say,

$$\begin{aligned} m_P(F) &\leq m_{T_1 \circ T \circ T_2(0,0)}(F^{T_1 \circ T \circ T_2}) \\ &= m_{T_1 \circ T(Q)}(F^{T_1 \circ T}) \text{ (By result of page 33)} \\ &= m_{T(Q)}(F^T) \text{ (By result of page 33)} \end{aligned}$$

Part (b). Again we will prove it for $P = Q = (0,0)$. Let $T = (f,g)$ and

$$J_Q T = \begin{pmatrix} \dfrac{\partial f}{\partial X}(Q) & \dfrac{\partial f}{\partial Y}(Q) \\ \dfrac{\partial g}{\partial X}(Q) & \dfrac{\partial g}{\partial Y}(Q) \end{pmatrix}.$$

Assume $J_Q T$ is invertible. Since $J_Q T$ is invertible, we can't have both $\dfrac{\partial f}{\partial X}(Q) = 0$ and $\dfrac{\partial f}{\partial Y}(Q) = 0$ or both $\dfrac{\partial g}{\partial X}(Q) = 0$ and $\dfrac{\partial g}{\partial Y}(Q) = 0$. Again by similar computation of part (a) we have, since $Q = (0,0)$, this implies that the decomposition of f and g into homogeneous polynomials are $f = f_1 + \cdots + f_m$ and $g = g_1 + \cdots + g_n$. Thus,

$$F^T = F(f,g) = F_m(f,g) + F_{m+1}(f,g) + \dots$$

Since the lowest degree forms of f and g are of degree 1, we have that T does not decrease the degree of the form $F_m(f,g)$. Similarly, T does not decrease the degree of $F_{m+1}(f,g), \cdots$. Therefore we have that $m_{(0,0)}(F^T) = m_{(0,0)}(F)$. Now assume that either $Q = (a_1, b_1) \neq (0,0)$ or $P = (a_2, b_2) \neq (0,0)$. Assume that $J_Q T$ is invertible. Let T_1 be the translation that takes $(0,0)$ to Q and T_2 be the translation that takes P to $(0,0)$. Then we have

$$d(T_1 \circ T \circ T_2) = d(T_1) \circ d(T) \circ d(T_2)$$

is also invertible. By the previous case $m_P(F) = m_{(0,0)}(F^{T_1 \circ T \circ T_2})$and the similar computation of multiplicities we can say $m_P(F^T) = m_Q(F)$. And hence our proof is complete.

Part (c). If $F = Y - X^2$ and $T = (X^2, Y)$, $P = Q = (0,0)$ we can see $m_P(F^T) = m_P(Y - X^4) = m_Q(F) = m_Q(Y - X^2)$. But the jacobian of T is not invertible at $(0,0)$,as it is given by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. ■

§3. Problem 3.13

WLOG assume $P = (0,0)$, then we know that

$$\dim_k(\mathscr{O}/\mathfrak{m}^n) = \dim_k(k[X,Y]/(F, I^n))$$

where $I = (X,Y) \subseteq k[X,Y]$. Now as multiplicity of F is $m_p(F)$, we have $F \in I^{m_P(F)}$ and hence we get that for $n \leq m_p(F)$, $F \in I^n$, thus $(F, I^n) = I^n$. But then we get that

$$\dim_k(k[X,Y]/(F, I^n)) = \dim_k(k[X,Y]/I^n) = \binom{n+1}{2}$$

Now from the exact sequence

$$0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to \mathscr{O}/\mathfrak{m}^{n+1} \to \mathscr{O}/\mathfrak{m}^n \to 0$$

we get that for $n \leq m_p(F)$.

$$\begin{aligned}\dim_k\left(\mathfrak{m}^n/\mathfrak{m}^{n+1}\right) &= \dim_k\left(\mathscr{O}/\mathfrak{m}^{n+1}\right) - \dim_k(\mathscr{O}/\mathfrak{m}^n) \\ &= \binom{n+2}{2} - \binom{n+1}{2} \\ &= n+1\end{aligned}$$

§4. Problem 3.15

Part (a). With out loss of generality let, $P = (0,0)$ and the corresponding maximal ideal in $k[x,y]$ is $\mathfrak{m}_p = (x,y)$ and extension it's image in $\mathscr{O}_p(\mathbb{A}^2)$ is $\mathfrak{m}_p(\mathbb{A}^2)$. Now we know,

$$k[x,y]/\mathfrak{m}_p^n \simeq k[x,y]_{\mathfrak{m}_p}/\mathfrak{m}_p^n k[x,y]_{\mathfrak{m}_p} \simeq \mathscr{O}_p/\mathfrak{m}_p(\mathbb{A}^2)^n$$

(it follows from the fact, residue field/ localization commutes with quotienting). Enough to commute $\dim_k k[x,y]/\mathfrak{m}_p^n$. Now, $\mathfrak{m}_p^n = (x^n, x^{n-1}y, x^{n-2}y^2, \cdots, y^n)$. The basis of $k[x,y]/\mathfrak{m}_p^n$ must be the standard i forms, with $i < n$. For each i there are such $i+1$ forms. And hence,

$$\chi(n) = \dim_k \mathscr{O}_p/\mathfrak{m}_p(\mathbb{A}^2)^n = \dim_k k[x,y]/\mathfrak{m}_p^n = \frac{n(n+1)}{2}$$

Part (b). Let, $\mathscr{O} = \mathscr{O}_p(\mathbb{A}^r)$ and $\mathfrak{m} = \mathfrak{m}_p(\mathbb{A}^r)$. Again let, $P = (0,\cdots,0)$ and $\mathfrak{m}_p = (x_1,\cdots,x_r)$. Just by the similar past as above it is enough to calculate $\dim_k k[x_1,\cdots,x_r]/\mathfrak{m}_p^n$. Now, $\mathfrak{m}_p$ is generated by all standard forms of degree n. Thus,the basis of $k[x_1,\cdots,x_2]/\mathfrak{m}_p^n$ must be the standard i forms, with $i < n$. Thus the basis set can be written as,

$$\mathcal{B} = \left\{x_1^{i_1}\cdots x_r^{i_r} : i_1 + \cdots + i_r \leq n-1\right\}$$

Now cardinality of the set is,

$$\begin{aligned}|\mathcal{B}| &= \left|\left\{x_1^{i_1}\cdots x_r^{i_r} : i_1 + \cdots + i_r \leq n-1\right\}\right| \\ &= \left|\left\{1^{i_0}x_1^{i_1}\cdots x_r^{i_r} : i_0 + i_1 + \cdots + i_r = n-1\right\}\right| \\ &= \binom{n+r-1}{r}\end{aligned}$$

So we must have,

$$\chi(n) = \dim_k \mathscr{O}/\mathfrak{m}^n = \dim_k k[x_1,\cdots,x_r]/\mathfrak{m}_p^n = \binom{n+r-1}{r} = \frac{n(n+1)\cdots(n+r-1)}{r!}$$

Thus the leading coefficient is $1/r!$. ■

§5. Problem 3.16

In this problem we will try to trace the path of 'Theorem 2' in 'page 35'. Let, $\mathscr{O} = \mathscr{O}_P(V(F))$ and $P = (0, 0, \cdots)$ and $\mathfrak{m} = \mathfrak{m}_p(V(F))$. Consider the maximal ideal $\mathfrak{m}_p = (x_1, \cdots, x_r)$ corresponding to the point P. Let, $R = k[x_1, \cdots, x_r]$. Let, $m_P(F) = m$ (multiplicity of P w.r.t F). Then we have the follows SES(short exact sequence)

$$0 \longrightarrow R/\mathfrak{m}_p^{n-m} \xrightarrow{i} R/\mathfrak{m}_p^n \xrightarrow{\pi} R/(F, \mathfrak{m}_p^n) \longrightarrow 0$$

where i is the map $i(\bar{G}) = \overline{FG}$ and π the natural projection map. It's an exact sequence. Thus by the previous problem we have,

$$\dim_k R/(F, \mathfrak{m}_p^n) = \dim_k R/\mathfrak{m}_p^n - \dim_k R/\mathfrak{m}_p^{n-m} = \binom{n+r-1}{r} - \binom{n+r-m-1}{r}$$

If we expand the above binomal coefficients it's not hard to see the above is polynomial over n, which has degree $r-1$ and leading coefficient is $m/r!$. Now from a rsult stated in class * it follows,

$$R/(\mathfrak{m}_p^n, F) \simeq \mathscr{O}/\mathfrak{m}^n$$

Thus $\chi(n) = \dim_k \mathscr{O}/\mathfrak{m}^n$ is a polynomial of n of degree $(r-1)$ and leading coefficient is $m/r!$ as desired. ∎

§6. Problem 3.23 and 3.24

§Exercises in chapter 2 needed for proving theorems in chapter 3

2.22 We know given a map $f : V \to W$ between affine varieties, it extends to a ring homomorphism $f^* : \mathscr{O}_{f(P)}(W) \to \mathscr{O}_P(V)$. Now if we have an affine transformation $T : \mathbb{A}^n \to \mathbb{A}^n$ it will have inverse affine map T^{-1}. By the functoriality of pullback we can say they will induce T^* and T^{-1^*} in the corresponding local ring of regular functions. We can also note $T^* \circ T^{-1^*}$ and $T^{-1^*} \circ T^*$ is identity and hence T^* is isomorphism. Thus $T^* : \mathscr{O}_{T(P)}(\mathbb{A}^n) \to \mathscr{O}_n(\mathbb{A}^n)$ is an isomorphism. If we restrict T to $V \subset \mathbb{A}^k$ on that case T will map V to an isomorphic (as subvariety) copy $V^T \subset \mathbb{A}^n$. Again by the same computuation we can say, $\mathscr{O}_P(V) \simeq \mathscr{O}_{T(P)}(V^T)$ are isomorphic.

2.34 In this case if $F + G$ was reducible then we could write $F + G = fg$. Now if we homogenize the polynomial we will get,

$$(F + G)^* = x_{n+1}F + G = f^*g^*$$

here treat $(F + G)^*$ as linear a polynomial over the ring $k[x_1, \cdots, x_n]$, which is UFD and hence by Gauss lemma $k[x_1, \cdots, x_n][x_{n+1}]$ is also UFD. But it can't have any non-constant factor over $k[x_1, \cdots, x_n][x_{n+1}]$. So, $F + G$ is irreducilbe.

2.35(c),2.36 is done in the computation step of **3.15** part (b). So not doing it again.

2.44* (* marked in previous section) At first we will define a map $\psi : \mathscr{O}_P(\mathbb{A}^n) \to \mathscr{O}_P(V)/J'\mathscr{O}_P(V)$. Firtly, we have the map $\mathscr{O}_P(\mathbb{A}^n) \to \mathscr{O}_P(V)$, which takes f/g (such that $g(P) \neq 0$) to $\bar{f}/\bar{g}$ where $\bar{f}, \bar{g}$ are f, g modulo $I = I(V)$. It's not hard to see $g \notin I$ so $\bar{g}(P) \neq 0$. Thus the map is well defined. J is an ideal containing I and J' is the image in local ring, then there is a natural projection map $\mathscr{O}_P(V)/J'\mathscr{O}_p(V)$. Compositioon of this two map will be ψ.

Now it's not hard to see ψ is a surjective homomorphism. We will compute the kernal of it $\ker \psi$. Let, $f/g \in \mathscr{O}_p(\mathbb{A}^n)$ such that $\bar{f}/\bar{g} \in J'\mathscr{O}_p(V)$. We can write

$$\bar{f}/\bar{g} = \sum \frac{j_i}{g'_i}$$

where $j_i \in J'$ and g'_i are polynomial corresponding g_i(that don't vanish at P), i.e $g'_i = g_i \pmod{I}$. So, $\bar{f} \times (\prod g'_i) \in J'\mathscr{O}_p(V)$. Thus we can say, $f \times (\prod g_i) \in J\mathscr{O}_p(\mathbb{A}^n)$. Since g_i are invertible we can say $f \in J\mathscr{O}_p(\mathbb{A}^n)$. So, $\ker \psi \subseteq J\mathscr{O}_p(\mathbb{A}^n)$. It's not hard to see $J\mathscr{O}_p(\mathbb{A}^n) \subseteq \ker \psi$ thus we get, $\ker \psi = J\mathscr{O}_p(\mathbb{A}^n)$. And thus we have a natural isomorphism

$$\bar{\psi} : \mathscr{O}_p(\mathbb{A}^n)/J\mathscr{O}_p(\mathbb{A}^n) \to \mathscr{O}_p(V)/J'\mathscr{O}_p(V)$$

If $J = I$ then the right side is just $\mathscr{O}_p(V)$ and thus $\mathscr{O}_p(V) \simeq \mathscr{O}_p(\mathbb{A}^n)/I\mathscr{O}_p(\mathbb{A}^n)$.