Assignment-4

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§1. Problem 3.8

Part (a). We will first prove it for the case when $P = Q = (0,0)$. In this case the polynomial map $T : \mathbb{A}^2 \to \mathbb{A}^2$ must look like (f, g) where f and g are polynomials vanishing at $(0,0)$. In this case we can wite $f = f_i + \cdots + f_t$, where $f_i \in k[x, y], i \geq 1$ is homogeneous polynomial of degree i. Similarly, we can write for g (as both of them are vanishing at $(0,0)$). If $m = m_P(F)$ then $F = F_m + F_{m+1} \cdots$ again F_i are homogeneous polynomial of degree m. Now $F^T = F(f, g)$'s lowest degree will come from $F_m(f, g)$ since both f, g has at-least one degree term we can say, $m_Q(F^T) \geq m_p(F)$.

Now we will use the fact proved in page (33) to prove it for any P, Q. Let, $Q \neq (0,0)$ or $Q = T(P) \neq (0,0)$. Let $T_1 : \mathbb{A}^2 \to \mathbb{A}^2$ be the affine transformation that maps $(0,0)$ to Q and T_2 be the affine map sends P to $(0,0)$. Note that $T_1 \circ T \circ T_2$ is a polynomial map and it maps $(0,0)$. So by the above calculation we can say,

$$\begin{aligned} m_P(F) &\leq m_{T_1 \circ T \circ T_2(0,0)}(F^{T_1 \circ T \circ T_2}) \\ &= m_{T_1 \circ T(Q)}(F^{T_1 \circ T}) \text{ (By result of page 33)} \\ &= m_{T(Q)}(F^T) \text{ (By result of page 33)} \end{aligned}$$

Part (b). Again we will prove it for $P = Q = (0,0)$. Let $T = (f, g)$ and

$$J_Q T = \begin{pmatrix} \frac{\partial f}{\partial X}(Q) & \frac{\partial f}{\partial Y}(Q) \\ \frac{\partial g}{\partial X}(Q) & \frac{\partial g}{\partial Y}(Q) \end{pmatrix}.$$

Assume $J_Q T$ is invertible. Since $J_Q T$ is invertible, we can't have both $\frac{\partial f}{\partial X}(Q) = 0$ and $\frac{\partial f}{\partial Y}(Q) = 0$ or both $\frac{\partial g}{\partial X}(Q) = 0$ and $\frac{\partial g}{\partial Y}(Q) = 0$. Again by similar computation of part (a) we have, since $Q = (0,0)$, this implies that the decomposition of f and g into homogeneous polynomials are $f = f_1 + \cdots + f_m$ and $g = g_1 + \cdots + g_n$. Thus,

$$F^T = F(f, g) = F_m(f, g) + F_{m+1}(f, g) + \ldots$$

Since the lowest degree forms of f and g are of degree 1, we have that T does not decrease the degree of the form $F_m(f, g)$. Similarly, T does not decrease the degree of $F_{m+1}(f, g), \cdots$. Therefore we have that $m_{(0,0)}(F^T) = m_{(0,0)}(F)$. Now assume that either $Q = (a_1, b_1) \neq (0,0)$ or $P = (a_2, b_2) \neq (0,0)$. Assume that $J_Q T$ is invertible. Let T_1 be the translation that takes $(0,0)$ to Q and T_2 be the translation that takes P to $(0,0)$. Then we have

$$d(T_1 \circ T \circ T_2) = d(T_1) \circ d(T) \circ d(T_2)$$

is also invertible. By the previous case $m_P(F) = m_{(0,0)}(F^{T_1 \circ T \circ T_2})$and the similar computation of multiplicities we can say $m_P(F^T) = m_Q(F)$. And hence our proof is complete.

Part (c). If $F = Y - X^2$ and $T = (X^2, Y)$, $P = Q = (0,0)$ we can see $m_P(F^T) = m_P(Y - X^4) = m_Q(F) = m_Q(Y - X^2)$. But the jacobian of T is not invertible at $(0,0)$,as it is given by the matrix$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. ■

§2. Problem 3.15

Part (a). With out loss of generality let, $P = (0,0)$ and the corresponding maximal ideal in $k[x,y]$ is $\mathfrak{m}_p = (x,y)$ and extension it's image in $\mathscr{O}_p(\mathbb{A}^2)$ is $\mathfrak{m}_p(\mathbb{A}^2)$. Now we know,

$$k[x,y]/\mathfrak{m}_p^n \simeq k[x,y]_{\mathfrak{m}_p}/\mathfrak{m}_p^n k[x,y]_{\mathfrak{m}_p} \simeq \mathscr{O}_p/\mathfrak{m}_p(\mathbb{A}^2)^n$$

(it follows from the fact, residue field/ localization commutes with quotienting). Enough to commute $\dim_k k[x,y]/\mathfrak{m}_p^n$. Now, $\mathfrak{m}_p^n = (x^n, x^{n-1}y, x^{n-2}y^2, \cdots, y^n)$. The basis of $k[x,y]/\mathfrak{m}_p^n$ must be the standard i forms, with $i < n$. For each i there are such $i+1$ forms. And hence,

$$\chi(n) = \dim_k \mathscr{O}_p/\mathfrak{m}_p(\mathbb{A}^2)^n = \dim_k k[x,y]/\mathfrak{m}_p^n = \frac{n(n+1)}{2}$$

Part (b). Let, $\mathscr{O} = \mathscr{O}_p(\mathbb{A}^r)$ and $\mathfrak{m} = \mathfrak{m}_p(\mathbb{A}^r)$. Again let, $P = (0, \cdots, 0)$ and $\mathfrak{m}_p = (x_1, \cdots, x_r)$. Just by the similar past as above it is enough to calculate $\dim_k k[x_1, \cdots, x_r]/\mathfrak{m}_p^n$. Now, $\mathfrak{m}_p$ is generated by all standard forms of degree n. Thus,the basis of $k[x_1, \cdots, x_2]/\mathfrak{m}_p^n$ must be the standard i forms, with $i < n$. Thus the basis set can be written as,

$$\mathcal{B} = \left\{ x_1^{i_1} \cdots x_r^{i_r} : i_1 + \cdots + i_r \leq n-1 \right\}$$

Now cardinality of the set is,

$$\begin{aligned} |\mathcal{B}| &= \left| \left\{ x_1^{i_1} \cdots x_r^{i_r} : i_1 + \cdots + i_r \leq n-1 \right\} \right| \\ &= \left| \left\{ 1^{i_0} x_1^{i_1} \cdots x_r^{i_r} : i_0 + i_1 + \cdots + i_r = n-1 \right\} \right| \\ &= \binom{n+r-1}{r} \end{aligned}$$

So we must have,

$$\chi(n) = \dim_k \mathscr{O}/\mathfrak{m}^n = \dim_k k[x_1, \cdots, x_r]/\mathfrak{m}_p^n = \binom{n+r-1}{r} = \frac{n(n+1)\cdots(n+r-1)}{r!}$$

Thus the leading coefficient is $1/r!$. ■

§3. Problem 3.16

In this problem we will try to trace the path of 'Theorem 2' in 'page 35'. Let, $\mathscr{O} = \mathscr{O}_P(V(F))$ and $P = (0, 0, \cdots)$ and $\mathfrak{m} = \mathfrak{m}_p(V(F))$. Consider the maximal ideal $\mathfrak{m}_p = (x_1, \cdots, x_r)$

corresponding to the point P. Let, $R = k[x_1, \cdots, x_r]$. Let, $m_P(F) = m$ (multiplicity of P w.r.t F). Then we have the follows SES(short exact sequence)

$$0 \longrightarrow R/\mathfrak{m}_p^{n-m} \xrightarrow{i} R/\mathfrak{m}_p^n \xrightarrow{\pi} R/(F, \mathfrak{m}_p^n) \longrightarrow 0$$

where i is the map $i(\bar{G}) = \overline{FG}$ and π the natural projection map. It's an exact sequence. Thus by the previous problem we have,

$$\dim_k R/(F, \mathfrak{m}_p^n) = \dim_k R/\mathfrak{m}_p^n - \dim_k R/\mathfrak{m}_p^{n-m} = \binom{n+r-1}{r} - \binom{n+r-m-1}{r}$$

If we expand the above binomal coefficients it's not hard to see the above is polynomial over n, which has degree $r-1$ and leading coefficient is $m/r!$. Now from a rsult stated in class it follows,

$$R/(\mathfrak{m}_p^n, F) \simeq \mathscr{O}/\mathfrak{m}^n$$

Thus $\chi(n) = \dim_k \mathscr{O}/\mathfrak{m}^n$ is a polynomial of n of degree $(r-1)$ and leading coefficient is $m/r!$ as desired. ■

§4. Problem 3.23 and 3.24