

Assignment-4

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§1. Problem 3.6

§ Lemma – 1

The F and G be forms of degree r and $r + 1$ respectively with no common factors in $k[X_1, \dots, X_n]$, then $F + G$ is irreducible.

Proof (of Lemma). Suppose $F + G$ is reducible then there exists nonconstant polynomials $P, Q \in k[X_1, \dots, X_n]$ such that $F + G = PQ$. Now we consider the homogeneous both of these to get

$$X_{n+1}F + G = (F + G)^* = (PQ)^* = P^*Q^*.$$

But note that $X_{n+1}F + G \in k(X_1, \dots, X_n)[X_{n+1}]$ is irreducible, hence one of P^* or Q^* is in $k[X_1, \dots, X_n]$. Then by comparing degrees we can WLOG assume that $Q^* \in k[X_1, \dots, X_n]$, and let $P = X_{n+1}R + S$, where $R, S \in k[X_1, \dots, X_n]$. Then we get that

$$X_{n+1}F + G = X_{n+1}RQ^* + SQ^* \Rightarrow F = RQ^* \text{ and } G = SQ^*$$

But this contradicts the fact that F and G have no common factor, hence we get that $F + G$ is irreducible.

Now coming back to the main problem, suppose we are given tangent lines L_i with multiplicities r_i , and we want to find an irreducible curve F such that L_i is a tangent to F with multiplicity r_i . Note that $\prod_i L_i^{r_i}$ is a forms of degree $m = \sum_i r_i$. Then we can find a homogeneous polynomial F_{m+1} of degree $m + 1$ such that F_{m+1} is not divisible by any of the L_i (such a polynomial obvious exists). But then by the previous lemma $\prod_i L_i^{r_i} + F_{m+1}$ is irreducible, and clearly $F = \prod_i L_i^{r_i} + F_{m+1}$ satisfies the necessary conditions.

§2. Problem 3.8

Part (a). We will first prove it for the case when $P = Q = (0, 0)$. In this case the polynomial map $T : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ must look like (f, g) where f and g are polynomials vanishing at $(0, 0)$. In this case we can write $f = f_1 + \dots + f_t$, where $f_i \in k[x, y]$, $i \geq 1$ is homogeneous polynomial of degree i . Similarly, we can write for g (as both of them are vanishing at $(0, 0)$). If $m = m_P(F)$ then $F = F_m + F_{m+1} \dots$ again F_i are homogeneous polynomial of degree m . Now $F^T = F(f, g)$'s lowest degree will come from $F_m(f, g)$ since both f, g has at-least one degree term we can say, $m_Q(F^T) \geq m_P(F)$.

Now we will use the fact proved in page (33) to prove it for any P, Q . Let, $Q \neq (0, 0)$ or $Q = T(P) \neq (0, 0)$. Let $T_1 : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ be the affine transformation that maps $(0, 0)$ to Q

and T_2 be the affine map sends P to $(0, 0)$. Note that $T_1 \circ T \circ T_2$ is a polynomial map and it maps $(0, 0)$. So by the above calculation we can say,

$$\begin{aligned} m_P(F) &\leq m_{T_1 \circ T \circ T_2(0,0)}(F^{T_1 \circ T \circ T_2}) \\ &= m_{T_1 \circ T(Q)}(F^{T_1 \circ T}) \text{ (By result of page 33)} \\ &= m_{T(Q)}(F^T) \text{ (By result of page 33)} \end{aligned}$$

Part (b). Again we will prove it for $P = Q = (0, 0)$. Let $T = (f, g)$ and

$$J_Q T = \begin{pmatrix} \frac{\partial f}{\partial X}(Q) & \frac{\partial f}{\partial Y}(Q) \\ \frac{\partial g}{\partial X}(Q) & \frac{\partial g}{\partial Y}(Q) \end{pmatrix}.$$

Assume $J_Q T$ is invertible. Since $J_Q T$ is invertible, we can't have both $\frac{\partial f}{\partial X}(Q) = 0$ and $\frac{\partial f}{\partial Y}(Q) = 0$ or both $\frac{\partial g}{\partial X}(Q) = 0$ and $\frac{\partial g}{\partial Y}(Q) = 0$. Again by similar computation of part (a) we have, since $Q = (0, 0)$, this implies that the decomposition of f and g into homogeneous polynomials are $f = f_1 + \dots + f_m$ and $g = g_1 + \dots + g_n$. Thus,

$$F^T = F(f, g) = F_m(f, g) + F_{m+1}(f, g) + \dots$$

Since the lowest degree forms of f and g are of degree 1, we have that T does not decrease the degree of the form $F_m(f, g)$. Similarly, T does not decrease the degree of $F_{m+1}(f, g), \dots$. Therefore we have that $m_{(0,0)}(F^T) = m_{(0,0)}(F)$. Now assume that either $Q = (a_1, b_1) \neq (0, 0)$ or $P = (a_2, b_2) \neq (0, 0)$. Assume that $J_Q T$ is invertible. Let T_1 be the translation that takes $(0, 0)$ to Q and T_2 be the translation that takes P to $(0, 0)$. Then we have

$$d(T_1 \circ T \circ T_2) = d(T_1) \circ d(T) \circ d(T_2)$$

is also invertible. By the previous case $m_P(F) = m_{(0,0)}(F^{T_1 \circ T \circ T_2})$ and the similar computation of multiplicities we can say $m_P(F^T) = m_Q(F)$. And hence our proof is complete.

Part (c). If $F = Y - X^2$ and $T = (X^2, Y)$, $P = Q = (0, 0)$ we can see $m_P(F^T) = m_P(Y - X^4) = m_Q(F) = m_Q(Y - X^2)$. But the jacobian of T is not invertible at $(0, 0)$, as it is given by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. ■

§3. Problem 3.13

WLOG assume $P = (0, 0)$, then we know that

$$\dim_k(\mathcal{O}/\mathfrak{m}^n) = \dim_k(k[X, Y]/(F, I^n))$$

where $I = (X, Y) \subseteq k[X, Y]$. Now as multiplicity of F is $m_p(F)$, we have $F \in I^{m_p(F)}$ and hence we get that for $n \leq m_p(F)$, $F \in I^n$, thus $(F, I^n) = I^n$. But then we get that

$$\dim_k(k[X, Y]/(F, I^n)) = \dim_k(k[X, Y]/I^n) = \binom{n+1}{2}$$

Now from the exact sequence

$$0 \rightarrow \mathfrak{m}^n/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^{n+1} \rightarrow \mathcal{O}/\mathfrak{m}^n \rightarrow 0$$

we get that for $n \leq m_p(F)$.

$$\begin{aligned} \dim_k (\mathfrak{m}^n/\mathfrak{m}^{n+1}) &= \dim_k (\mathcal{O}/\mathfrak{m}^{n+1}) - \dim_k (\mathcal{O}/\mathfrak{m}^n) \\ &= \binom{n+2}{2} - \binom{n+1}{2} \\ &= n+1 \end{aligned}$$

In the proof of **Theorem 2, page 35 (Algebraic Curves, Fulton)**, we have already seen that

$$\dim_k (k[X, Y]/(F, I^n)) = nm - \frac{m(m-1)}{2},$$

where $m = m_P(F)$, hence we get that

$$\dim_k (\mathfrak{m}^n/\mathfrak{m}^{n+1}) = m$$

if $n \geq m_p(F)$. Now suppose P is not a simple point, then $m_P(F) \geq 2$, and hence $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$. Hence $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$ implies P is a simple point. Now if P is a simple point then $m_P(F) = 1$, and hence we get that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = m - \frac{m(m-1)}{2} = 1$, since $m = 1$. Thus we have shown that P is simple if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$, and otherwise we have $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 2$.

§4. Problem 3.15

Part (a). With out loss of generality let, $P = (0, 0)$ and the corresponding maximal ideal in $k[x, y]$ is $\mathfrak{m}_p = (x, y)$ and extension it's image in $\mathcal{O}_p(\mathbb{A}^2)$ is $\mathfrak{m}_p(\mathbb{A}^2)$. Now we know,

$$k[x, y]/\mathfrak{m}_p^n \simeq k[x, y]_{\mathfrak{m}_p}/\mathfrak{m}_p^n k[x, y]_{\mathfrak{m}_p} \simeq \mathcal{O}_p/\mathfrak{m}_p(\mathbb{A}^2)^n$$

(it follows from the fact, residue field/ localization commutes with quotienting). Enough to commute $\dim_k k[x, y]/\mathfrak{m}_p^n$. Now, $\mathfrak{m}_p^n = (x^n, x^{n-1}y, x^{n-2}y^2, \dots, y^n)$. The basis of $k[x, y]/\mathfrak{m}_p^n$ must be the standard i forms, with $i < n$. For each i there are such $i+1$ forms. And hence,

$$\chi(n) = \dim_k \mathcal{O}_p/\mathfrak{m}_p(\mathbb{A}^2)^n = \dim_k k[x, y]/\mathfrak{m}_p^n = \frac{n(n+1)}{2}$$

Part (b). Let, $\mathcal{O} = \mathcal{O}_p(\mathbb{A}^r)$ and $\mathfrak{m} = \mathfrak{m}_p(\mathbb{A}^r)$. Again let, $P = (0, \dots, 0)$ and $\mathfrak{m}_p = (x_1, \dots, x_r)$. Just by the similar past as above it is enough to calculate $\dim_k k[x_1, \dots, x_r]/\mathfrak{m}_p^n$. Now, \mathfrak{m}_p is generated by all standard forms of degree n . Thus, the basis of $k[x_1, \dots, x_r]/\mathfrak{m}_p^n$ must be the standard i forms, with $i < n$. Thus the basis set can be written as,

$$\mathcal{B} = \{x_1^{i_1} \cdots x_r^{i_r} : i_1 + \cdots + i_r \leq n-1\}$$

Now cardinality of the set is,

$$\begin{aligned}
|\mathcal{B}| &= |\{x_1^{i_1} \cdots x_r^{i_r} : i_1 + \cdots + i_r \leq n-1\}| \\
&= |\{1^{i_0} x_1^{i_1} \cdots x_r^{i_r} : i_0 + i_1 + \cdots + i_r = n-1\}| \\
&= \binom{n+r-1}{r}
\end{aligned}$$

So we must have,

$$\chi(n) = \dim_k \mathcal{O}/\mathfrak{m}^n = \dim_k k[x_1, \dots, x_r]/\mathfrak{m}_p^n = \binom{n+r-1}{r} = \frac{n(n+1) \cdots (n+r-1)}{r!}$$

Thus the leading coefficient is $1/r!$. ■

§5. Problem 3.16

In this problem we will try to trace the path of ‘Theorem 2’ in ‘page 35’. Let, $\mathcal{O} = \mathcal{O}_P(V(F))$ and $P = (0, 0, \dots)$ and $\mathfrak{m} = \mathfrak{m}_p(V(F))$. Consider the maximal ideal $\mathfrak{m}_p = (x_1, \dots, x_r)$ corresponding to the point P . Let, $R = k[x_1, \dots, x_r]$. Let, $m_P(F) = m$ (multiplicity of P w.r.t F). Then we have the follows SES(short exact sequence)

$$0 \longrightarrow R/\mathfrak{m}_p^{n-m} \xrightarrow{i} R/\mathfrak{m}_p^n \xrightarrow{\pi} R/(F, \mathfrak{m}_p^n) \longrightarrow 0$$

where i is the map $i(\bar{G}) = \overline{FG}$ and π the natural projection map. It’s an exact sequence. Thus by the previous problem we have,

$$\dim_k R/(F, \mathfrak{m}_p^n) = \dim_k R/\mathfrak{m}_p^n - \dim_k R/\mathfrak{m}_p^{n-m} = \binom{n+r-1}{r} - \binom{n+r-m-1}{r}$$

If we expand the above binomial coefficients it’s not hard to see the above is polynomial over n , which has degree $r-1$ and leading coefficient is $m/r!$. Now from a result stated in class * it follows,

$$R/(\mathfrak{m}_p^n, F) \simeq \mathcal{O}/\mathfrak{m}^n$$

Thus $\chi(n) = \dim_k \mathcal{O}/\mathfrak{m}^n$ is a polynomial of n of degree $(r-1)$ and leading coefficient is $m/r!$ as desired. ■

§6. Problem 3.23 and 3.24

§Exercises in chapter 2 needed for proving theorems in chapter 3

2.22 We know given a map $f : V \rightarrow W$ between affine varieties, it extends to a ring homomorphism $f^* : \mathcal{O}_{f(P)}(W) \rightarrow \mathcal{O}_P(V)$. Now if we have an affine transformation $T : \mathbb{A}^n \rightarrow \mathbb{A}^n$ it will have inverse affine map T^{-1} . By the functoriality of pullback we can say they will induce T^* and T^{-1*} in the corresponding local ring of regular functions. We can also note $T^* \circ T^{-1*}$ and $T^{-1*} \circ T^*$ is identity and hence T^* is isomorphism. Thus $T^* : \mathcal{O}_{T(P)}(\mathbb{A}^n) \rightarrow \mathcal{O}_P(\mathbb{A}^n)$ is an isomorphism. If we restrict T to $V \subset \mathbb{A}^k$ on that case T will map V to an isomorphic (as subvariety) copy $V^T \subset \mathbb{A}^n$. Again by the same computation we can say, $\mathcal{O}_P(V) \simeq \mathcal{O}_{T(P)}(V^T)$ are isomorphic.

2.34 In this case if $F + G$ was reducible then we could write $F + G = fg$. Now if we homogenize the polynomial we will get,

$$(F + G)^* = x_{n+1}F + G = f^*g^*$$

here treat $(F + G)^*$ as linear a polynomial over the ring $k[x_1, \dots, x_n]$, which is UFD and hence by Gauss lemma $k[x_1, \dots, x_n][x_{n+1}]$ is also UFD. But it can't have any non-constant factor over $k[x_1, \dots, x_n][x_{n+1}]$. So, $F + G$ is irreducible.

2.35(c), 2.36 is done in the computation step of **3.15** part (b). So not doing it again.

2.44* (* marked in previous section) At first we will define a map $\psi : \mathcal{O}_P(\mathbb{A}^n) \rightarrow \mathcal{O}_P(V)/J'\mathcal{O}_P(V)$. Firstly, we have the map $\mathcal{O}_P(\mathbb{A}^n) \rightarrow \mathcal{O}_P(V)$, which takes f/g (such that $g(P) \neq 0$) to \bar{f}/\bar{g} where \bar{f}, \bar{g} are f, g modulo $I = I(V)$. It's not hard to see $g \notin I$ so $\bar{g}(P) \neq 0$. Thus the map is well defined. J is an ideal containing I and J' is the image in local ring, then there is a natural projection map $\mathcal{O}_P(V)/J'\mathcal{O}_P(V)$. Composition of this two map will be ψ .

Now it's not hard to see ψ is a surjective homomorphism. We will compute the kernel of ψ . Let, $f/g \in \mathcal{O}_P(\mathbb{A}^n)$ such that $\bar{f}/\bar{g} \in J'\mathcal{O}_P(V)$. We can write

$$\bar{f}/\bar{g} = \sum \frac{j_i}{g'_i}$$

where $j_i \in J'$ and g'_i are polynomial corresponding g_i (that don't vanish at P), i.e $g'_i = g_i \pmod{I}$. So, $\bar{f} \times (\prod g'_i) \in J'\mathcal{O}_P(V)$. Thus we can say, $f \times (\prod g_i) \in J\mathcal{O}_P(\mathbb{A}^n)$. Since g_i are invertible we can say $f \in J\mathcal{O}_P(\mathbb{A}^n)$. So, $\ker \psi \subseteq J\mathcal{O}_P(\mathbb{A}^n)$. It's not hard to see $J\mathcal{O}_P(\mathbb{A}^n) \subseteq \ker \psi$ thus we get, $\ker \psi = J\mathcal{O}_P(\mathbb{A}^n)$. And thus we have a natural isomorphism

$$\bar{\psi} : \mathcal{O}_P(\mathbb{A}^n)/J\mathcal{O}_P(\mathbb{A}^n) \rightarrow \mathcal{O}_P(V)/J'\mathcal{O}_P(V)$$

If $J = I$ then the right side is just $\mathcal{O}_P(V)$ and thus $\mathcal{O}_P(V) \simeq \mathcal{O}_P(\mathbb{A}^n)/I\mathcal{O}_P(\mathbb{A}^n)$.