Statistics-III

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Assignment-4

§ Problem 1

If C is a generalized inverse of $X'X$ prove the following.

(a) C' is also a generalized inverse of $X'X$.

(b) A symmetric generalized inverse of $X'X$ exists.

(c) CX' is a generalized inverse of X.

(d) XCX' is unique.

(e) XCX' is symmetric and idempotent.

(f) Column spaces of XCX' and X are the same.

Solution.

(a) C is generalized inverse of $X'X$. So, $(X'X) \subset (X'X) = (X'X)$. Now take transpose of the both side in above equation. We will endup getting,

$$(X'X)\, C'\, (X'X) = (X'X)$$

So, C' is also a generalized inverse of $(X'X)$. $(X'X)$

(b) $X'X$ is symmetric matrix. Let, $\operatorname{Rank}(X'X) = p$. And $X'X$ has order n. Let, $X'X = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$; B_{11} has Rank p and it's also $p \times p$ matrix. The generalized inverse of $(X'X)^{-}$;

$$(X'X)^{-} = \begin{pmatrix} B_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly it's Symmetric.

(c) At first of all notice that, $P_\Omega = XCX'$ is projection matrix on to $\mathcal{M}_C(X)$. Now notice that,

$$XCX'X = X$$

As, XCX' maps each column of X to itself. So, CX' is generalized inverse of X.

(d) Let, $\Omega = \mathcal{M}_C(X)$, then $P\Omega$. defined by, $P_\Omega = XCX'$ is projection matrix of $\mathbb{R}^n$ onto Ω. so, P_Ω is unique.

(e) XCX' is projection matrix So it must be symmetric and idempotent.

(f) XCX' is projection onto $\Omega = \mathcal{M}_C(X)$ so, $\mathcal{M}_C(XCX') = \mathcal{M}_C(X)$. ■

§ Problem 2

Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & 1 & -3 \\ 1 & 2 & 0 \end{pmatrix}$.

(a) Find a generalized inverse $(A'A)^-$of $A'A$.

(b) Find a generalized inverse $(AA')^-$of AA'.

Solution.

(a) Given $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & 1 & -3 \\ 1 & 2 & 0 \end{pmatrix}$ Now, $A'A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & -3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & 1 & -3 \\ 1 & 2 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 7 & 6 & 8 \\ 6 & 10 & 2 \\ 8 & 2 & 14 \end{pmatrix}$$

Now, Rank $(A'A) = 2$. So, Let's take,

$$B_{11} = \begin{pmatrix} 7 & 6 \\ 6 & 10 \end{pmatrix}$$

The inverse of B_{11} will be,

$$B_{11}^{-1} = \tfrac{1}{34} \begin{pmatrix} 10 & -6 \\ 6 & 7 \end{pmatrix}.$$

$$So,\ \ (A'A)^- = \begin{pmatrix} \frac{10}{34} & -\frac{6}{34} & 0 \\ \frac{6}{34} & \frac{7}{34} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) In this case,

$$AA' = \begin{pmatrix} 3 & 6 & -3 & 3 \\ 6 & 12 & -6 & 6 \\ -3 & -6 & 11 & 1 \\ 3 & 6 & 1 & 5 \end{pmatrix}.$$

For Real matrix we know, Rank (A'A) $=$ Rank(AA'). So, Rank$(AA') = 2$. In this case $B_{11} = \begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix}$ which has determinant 0. So, take, $B_{22} = \begin{pmatrix} 11 & 1 \\ 1 & 5 \end{pmatrix}$ so,

$$B_{22}^{-1} = \frac{1}{54}\begin{pmatrix} 5 & -1 \\ 1 & 11 \end{pmatrix}.$$

$$\text{So, } (AA')^{-} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{54} & -\frac{1}{54} \\ 0 & 0 & \frac{1}{54} & \frac{11}{54} \end{pmatrix}$$

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§ Problem 3

(a) For all matrices $A_{m\times n}$, is it true that if B is a g-inverse of A, then A is a g-inverse of B ?
(b) Let $A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where B is $r_1 \times s_1$ and C is $r_2 \times s_2$. Let B^- and C^-be any g-inverses of B and C respectively. Show then that $G = \begin{pmatrix} B^- & 0 \\ 0 & C^- \end{pmatrix}$is a generalized inverse of A. Must all g-inverses of A have the form G ?
(c) Find a generalized inverse of $A = \begin{pmatrix} \mathbf{1}_3\mathbf{1}_3' & 0 \\ 0 & 2\mathbf{1}_2\mathbf{1}_2' \end{pmatrix}$, where $\mathbf{1}_k$ is the k-vector $(1, 1, \ldots, 1)'$.

Solution.

(a) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. In this case,

$$\begin{aligned} ABA &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A. \end{aligned}$$

So, B is generalized inverse of A. But,

$$\begin{aligned} BAB &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \neq B \end{aligned}$$

So, this is example where B is g-inverse of A but rot of $B.A$ is not g-inverse of B.

(b) Notice that,

$$\begin{aligned}
AGA &= \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} B^- & 0 \\ 0 & C^- \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \\
&= \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} B^-B & 0 \\ 0 & C^-C \end{pmatrix} \\
&= \begin{pmatrix} BB^-B & C \\ 0 & CC^-C \end{pmatrix} \\
&= \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \\
&= A.
\end{aligned}$$

So, G is generalized inverse of A. Now consider, $A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{pmatrix}; G = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$ is generalized inverse of A. So, It's not nessecery to A have all g-inverse in the given form.

(c)

$$A = \begin{pmatrix} 1_3 1_3' & 0 \\ 0 & 2 1_1 1_2' \end{pmatrix}$$

Let, $B = 1_3 1_3'$ and $C = 2 1_2 1_2'$. Here,

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Notice that, B has rank 1 . clearly,

$$B^- = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is g-inverse of B. Now, look at $C = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. It's not hard to see that Rank$(C) = 1$. So, we can write C^-, g-inverse of C which can be express as,

$$C^- = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$$

So, By the previous problem we Can say. $A^-$is g-inverse of A, where $A^-$is as following,

$$A^- = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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