Topology

Assignment - 1

1. A metric space is called <u>sequentially compact</u> if every sequence has a convergent subsequence.

   + Call a metric space <u>totally bounded</u> if for every $\epsilon > 0$, the metric space can be covered by finitely many open balls of radius ϵ.

   Prove that TFAE for a metric space (X, d):

   (i) X is compact.

   (ii) X has the Bolzano-Weirstrass property, i.e. every infinite set has a limit point in X.

   (iii) X is sequentially compact.

   (iv) X is totally bounded and complete. (2)

2. Given an example to show Heine-Borel theorem fails in metric spaces, i.e. a set may be closed and bounded yet fail to be compact. (1)

3. Prove that a totally bounded metric space is <u>separable</u> i.e. contains a countable dense subset. (1).

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Homework & Assignment-2

1. Let X be a metric space, $x \in X$, $r > 0$. Is it true that $\overline{B(x,r)} = \{y \in X \mid d(x,y) \leq r\}$?

2. X be a metric space, $U \subseteq X$ open, $A \subseteq X$. Prove that $U \cap A = \phi \iff U \cap \bar{A} = \phi$.

3. X a metric space. Prove that $A \subseteq X$ is dense $\iff$ the only closed set ~~properly~~ $\supseteq A$ is X $\iff$ the only open set disjoint from A is ϕ $\iff$ A intersects every non empty open set $\iff$ A intersects every open ball.

4. Consider $X = \mathcal{C}([0,1], \mathbb{R})$ with the norm $\|f\| = \int_0^1 |f(x)| dx$ giving the metric $d(f,g) = \|f-g\|$. Define $f_n \in X$ by $f_n(x) = 1$ for $x \in [0, 1/2]$, $f_n(x) = -2^n(x - 1/2)$ for $x \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{2^n}]$ and $f_n(x) = 0$ for $x \in [1/2 + 1/2^n, 1]$. Prove that (f_n) is a Cauchy sequence in X which is not convergent.

5. Give an example to show that the diameter sequence converging to zero in Cantor's intersection theorem is essential, i.e. cannot be dropped from the hypothesis.

6. A closed subset of X is nowhere dense $\iff$ its complement is dense.

7. Prove that the Cantor set is nowhere dense.

8. Let (X,d) be a metric space, $x_0 \in X$. Show that $f_{x_0} : X \to \mathbb{R}$, $f_{x_0}(x) = d(x_0, x)$ is continuous. Is f_{x_0} uniformly continuous?

9. Let X, Y be metric spaces, $A \subseteq X$ non empty. Show that for $f, g : X \to Y$ continuous, $f(x) = g(x)$ $\forall x \in A$ $\Rightarrow f(x) = g(x)$ $\forall x \in \bar{A}$.

10. Let X be any nonempty set and $\mathcal{B}(X)$ be the $\mathbb{R}$-vector space of all bounded functions on X. Let $f \in \mathcal{B}$, $\|f\| := \sup\{|f(x)|, x \in X\}$. Prove that $(\mathcal{B}(X), d)$, for $d(f,g) := \|f-g\|$, is a complete metric space.

11. What the completion of (X, d) in Problem 4?

12. Let X = the set of positive integers with discrete metric. Let $\mathcal{C}(X,\mathbb{R})$ be the space of bounded (continuous) functions on X. Show that $\mathcal{C}(X,\mathbb{R})$ is not separable.

$\boxed{13}$. Replace X in (12) by an arbitrary discrete metric space. Prove that $\mathcal{C}(X,\mathbb{R})$ is separable $\Longleftrightarrow$ X is finite.

14. A function $f : X \times Y \to Z$ of topological spaces is jointly continuous in x & y if f is continuous; we say f is continuous in x, if for any $y \in Y$, the map : $X \to Z$ given by $x \mapsto f(x,y)$ is continuous, Similarly we define continuity of f in y.

- Assume all X, Y, Z are metric spaces. Show that f is jointly continuous $\Longleftrightarrow$ $x_n \to x$, $y_n \to y$ implies $f(x_n, y_n) \to f(x,y)$.
- f is jointly continuous $\Rightarrow$ f is continuous in each variable separately. Show that converse of this is false.

Please submit solutions of Problems 4 & 13 as Assignment-2, both carry 2 points.

Homework - 3

1. For a metric space (X, d), the sphere with centre p & radius r is given by $S(p,r) = \{x \in X \mid d(p,x) = r\}$. When $X = \mathcal{C}([0,1], \mathbb{R})$, prove that for $f \in X$,
$$\partial B(f,r) = S(f,r).$$

2. i) Let X be a topological space, A, B subsets of X. Call A, B as <u>separated</u> if $\overline{A} \cap B = A \cap \overline{B} = \phi$. Prove that X is disconnected $\Leftrightarrow$ $\exists$ $A, B \subseteq X$, both nonempty & separated, $X = A \cup B$.

ii) Let (X, d) be a metric space, $p \in X$, $\delta > 0$. Let $A = B(p, \delta)$ and $B = \{x \in X \mid d(p,x) > \delta\}$. Prove that A and B are separated.

iii) Prove that a connected metric space with at least two points is necessarily uncountable.

3. If $A \subseteq X$ is connected then explore if A° is connected.

4. i) Call a subset E of a metric/topological space X as perfect if E is closed & every point of E is a limit point of E.
Let X be a separable metric space, $C \subseteq X$ closed. Prove that C is the union of a (possibly ϕ) perfect set and an atmost countable set.

ii) Every countable closed subset of $\mathbb{R}^n$ has isolated points.

iii) A point p of a metric space X is a condensation point of a set $E \subseteq X$ if every neighbourhood of p contains uncountably many points of E.

→ Let $E \subseteq \mathbb{R}^n$ be uncountable & P be the set of all condensation points of E. Prove that P is perfect & $P^c \cap E$ is at most countable.

[Hint: Let $\{V_i\}$ be a countable basis of $\mathbb{R}^n$ & W be the union of the V_i that have $V_i \cap E$ at most countable. Show that $P = W^c$.]

iv) Is it true that $D \subseteq \mathbb{R}^n$ is discrete then D is countable?

5. Prove the cardinality of the set of all connected components of a top. space, is an invariant of that space.

$\boxed{6}$. Consider $X = \mathcal{C}(S^n, \mathbb{R}^m)$ with the 'sup' metric. Prove that X is path connected.

$\boxed{7}$. i) Prove that: the Cantor set $\mathcal{C}_3$ is non empty!

ii) for $m, k \geq 1$, the intervals $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ are disjoint from $\mathcal{C}$.

iii) Any interval (a,b) contains a sub-interval of the form $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ for some $k, m \geq 1$.

iv) $\mathcal{C}$ contains no positive length inteval.

v) $\mathcal{C}$ is uncountable.

vi) Prove $\mathcal{C}$ is perfect following the steps:

let $x \in \mathcal{C}$ & J an inteval (open), $x \in J$ & I_n be that (closed) interval of F_n that contains x.

→ For large n, $I_n \subset J$. Let x_n be an end point of I_n such that $x_n \neq x$.

→ $x_n \in \mathcal{C}$

→ x is a limit point of $\mathcal{C}$.

Here $F_0 = [0,1]$, $F_1 = [0, 1/3] \cup [2/3, 1]$,

$F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

$\vdots$

F_{i+1} is obtained from F_i by removing middle third open interval of each interval present in F_i; So $F_0 \supset F_1 \supset \cdots$ &

$\mathcal{C} = \bigcap_i F_i$

Please submit Solutions of Problem 6 & Problem 7 as Assignment-3.

Homework-Assignment (4)

1. Let X be a topological space, $A \subseteq X$. Consider the quotient space X/A. Let $[A]$ be the point in X/A corresponding to A. If A is either open or closed, Show that $X - A$ is homeomorphic to $X/A - \{[A]\}$.
2. Let X be a top. space and $\sim$ be the equivalence relation on X corresponding to the connected components as its equivalence classes. Prove that the quotient $X/\sim$ is totally disconnected.
3. Let $p: X \to Y$ be a quotient map. Assume that all fibers of p are connected. Show that an open (or closed) subset $F \subseteq Y$ is connected $\Leftrightarrow$ $p^{-1}(F)$ is connected.
4. $p: X \to Y$ be a quotient map & X be locally connected. Prove that Y too is locally connected.
5. Let X be a top space, $\sim$ an equivalence on X, and let $X/\sim$ have the quotient topology, $p: X \to X/\sim$ the quotient map. Then p is open (closed) $\Leftrightarrow$ $\bigcup_{u \in U} [u]$ is open (closed) for every open (closed) subset $U \subseteq X$. (2)
6. Let X be regular and $A \subseteq X$ be closed then Prove that X/A is Hausdorff. (2)
7. Let X be normal, $A \subseteq X$ closed, then prove that X/A is normal.

* Problems 5 and 6 constitute Assignment-4. Submit by March-31.

Homework - 4

1. Recall that a top. space X is <u>locally</u> <u>connected</u> if every point has a local basis of connected open sets. Is it true that a connected subspace of a locally connected space is locally connected? Explain.
2. Give an example to show that local connectedness may not be preserved under continuous functions.
3. A finite non-empty product of locally connected spaces is locally connected, while an arbitrary non-empty product of loc. conn. space may not be.
4. Any arbitrary non-empty product of connected, locally conn. spaces is locally connected.
5. A top. space X is locally connected <u>if</u> the conn. components of every open subspace are open in X. Converse?
6. Prove Urysohn's lemma using Tietze's ext^n thm!
7. State and prove a generalization of Tietze's ext^n theorem which relates to functions taking values in $\mathbb{R}^n$.
8. Let X be completely regular, then the weak topology generated by $\mathcal{C}(X, \mathbb{R})$ equals the given ~~topology~~ on X.

9. Let $X \neq \phi$ a set & $\{X_i\}_i$ a non-empty class of top. spaces. For each i, $f_i : X \to X_i$ be given. Let τ be the weak topology on X gen. by the f_i's.

(a) Show that τ equals the topology on X gen. by the class of all inverse images in X of open sets in the X_i's.

(b) Let Y be a subspace of (X, τ), τ as in (a). Show that the subspace top. on Y is the weak top. gen. by the restrictions $f_i|_Y$.

10. Give a complete description of the weak topo. on $\mathbb{R}$ gen. by the family $\{f_i\}_i$ of <u>all</u> continuous maps : $\mathbb{R} \to \mathbb{R}$, wrt the usual top. on $\mathbb{R}$.

11. Locally path connected spaces are defined exactly analogously as (1). Prove the analogue of (5) in this context.
Prove that for a locally path conn. space connected components are same as path comps.

12. Let X be a connected normal space having more than one point. Show that X is uncountable.

13. A connected & regular space having at least two points is uncountable.

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Homework

1. Cone vs Geometric cone: Consider $X = \{(n,0) \mid n \in \mathbb{Z}^{>0}\} \subseteq \mathbb{R}^2$ and C be the subspace of $\mathbb{R}^2$ obtained by joining every $x \in X$ to $p = (0,1)$ by a line segment. Let $C(X) = (X \times [0,1]) / (X \times \{1\})$ be the cone over X.

→ Prove that there is a continuous bijection : $C(X) \to C$, but $C(X)$ is not homeomorphic to C.

In the above, X is not compact.

→ Let $X \subseteq \mathbb{R}^n$ be compact, $C(X)$ the topological cone over X and C the geometric cone : $\{(1-t)x + tp \mid x \in X, p \notin X \text{ fixed } \& \ t \in [0,1]\}$.

Prove then $C(X)$ is homeomorphic to C.

2. Let X, Y be topological spaces, $X \xrightarrow{f} Y$ a continuous map, the $C(f): C(X) \to C(Y)$, $[(x,t)] \longmapsto [(f(x), t)]$ is continuous.

Group actions: We say a group G acts on a topological space Y evenly if any $y \in Y$ has an open neighbourhood U such that $\forall$ $g \neq h \in G$, $g \cdot U \cap h \cdot U = \emptyset$.

(i) • The group μ_n of all n^{th} roots of unity in $\mathbb{C}$ acts on $\mathbb{C}$ by left multiplication.

→ Show that this action is not even.

→ The same action of μ_n on $\mathbb{C}^{\times} = \mathbb{C} - \{0\}$ is even.

(ii) • Let G act evenly on Y. Consider the orbits of points in Y under this action. Prove that they are all discrete.

(iii) • Let G be the subgroup of the group of all self homeomorphisms of $\mathbb{R}^2$ generated by the translation $(x, y) \mapsto (x+1, y)$ and the map $(x, y) \mapsto (-x, y+1)$.

Prove that this is an even action of G on $\mathbb{R}^2$. Also show that $\mathbb{R}^2/G$ is the Klein bottle.

(iv) • Let G be a finite group action fixed point freely on a Hausdorff Space Y, i.e. $g.y = y$ for some $g \in G$ & $y \in Y \Rightarrow g = e$.

Prove that such an action is even.

(v) • We discussed an action of μ_n on the Complex sphere $S^{m-1}_{\mathbb{C}} := \{(z_1, \cdots, z_m) \mid |z_1|^2 + \cdots + |z_m|^2 = 1\}$, $\zeta \cdot (z_1, \cdots, z_m) := (\zeta z_1, \cdots, \zeta z_m)$. Prove this is an even action.

* Probems (i), (iii), (iv), (v) constitute Assignment-5: Submit by 15^{th} April.

Homework

1. Let $V_{n,k}$ be the set of k-frames in $\mathbb{R}^n$; a k-frame is an orthonormal set of k vectors in $\mathbb{R}^n$. Prove that $O(n)$ acts transitively on $V_{n,k}$. Hence identify $O(n)/O(n-1) \simeq V_{n,k}$.

   ($V_{n,k}$ is called the Stiefel manifold of k-frames).

2. Prove in detail $SU(n)/SU(n-1) \simeq S^{2n-1}$, $n \geq 2$.

3. $\phi: G \to H$ be a surjective continuous homomorphism of topological groups. Let $K := \mathrm{Ker}(\phi)$. Show that K is closed and normal (assume these groups are T_1). If G is compact, prove that G/K is homeomorphic to H by a group isomorphism.

4. Show that $Sp(1) \xrightarrow{\sim} S^3$.

5. Prove $SO(n)$ is connected.

6. Prove $U(n)/SU(n) \xrightarrow{\sim} S^1$.

7. Prove that $\mathbb{R}P^n \xrightarrow{\sim} O(n+1)/(O(n) \times O(1))$,

   $\mathbb{C}P^n \xrightarrow{\sim} U(n+1)/(U(n) \times U(1))$, $\mathbb{H}P^n \xrightarrow{\sim} ??$

8. Prove analogues of (1) for other groups.

Homework

1. Give an example of a topological group G, $A, B \subseteq G$, both closed, but AB not closed.
   - Show that if A is closed & B a compact subset of a topological group G, then AB is closed.
2. Let $G = \mathbb{R}$, $\alpha \in \mathbb{R}$ irrational. Prove that, for $A = \mathbb{Z}$, $B = \alpha\mathbb{Z}$, $A + B$ is not closed.
3. G be a topological group and $N \trianglelefteq G$ be a normal subgroup. Prove that G/N is discrete if and only if N is open.
4. The topological groups $\mathbb{C}^*$ and $(\mathbb{R}^{>0}, \cdot) \times S^1$ are isomorphic as topological groups (i.e. via a group isom, which is a homeomorphism)
5. The topological group $S^1 \times S^1$ is a topological quotient group of the topological group $\mathbb{C}^\times$.
6. Let $D \subset GL_n(\mathbb{R})$ be the subgroup of all scalar matrices. Prove that $GL_n(\mathbb{R})/D$ is locally compact and Hausdorff top. group.
7. $f: G_1 \to G_2$ be a surjective continuous homomorphism of topological groups. Assume G_1 is compact & G_2 is Hausdorff. Prove that f is an open map.
8. Give an example $f: K \to Y$ continuous, K compact top space, Y Hausdorff, f onto, but not open.

9. Let $K \leq G$ be a compact subgroup of a top. group G. Show that the quotient map $G \to G/K$ is closed.

- Is the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \equiv S^1$ a closed map?

10. Prove that $O(n)$ is homeomorphic to $SO(n) \times \mathbb{Z}/2$. Are these isomorphic as top. grps?

11. Let G be a top. Hausdorff grp, $A, B \subseteq G$ compact subsets. Prove that AB is compact.

12. Prove that every nontrivial discrete subgrp of $(\mathbb{R}, +)$ is cyclic.

13. Let $A, B \in O(2)$, $\det A = 1$, $\det B = -1$. Show that $B^2 = I$ & $BAB^{-1} = A^{-1}$. Deduce that every discrete subgroup of $O(2)$ is cyclic or dihedral.

14. Show that the group of topological group automorphisms of S^1 is $\simeq \mathbb{Z}/2$.

15. T be an autom. of the top grp $(\mathbb{R}, +)$. Prove that $T(x) = x\,T(1)$ $\forall\, x \in \mathbb{R}$. Deduce that $\mathrm{Aut}_{\text{TOP-Gr}}((\mathbb{R}, +)) \simeq \mathbb{R} \times \mathbb{Z}/2$.

Misc. Problems

1. Prove that S^n is path connected for $n \geq 1$.
2. Prove, by giving explicit maps, $D^n/_{S^{n-1}}$ is homeomorphic to S^n, where
   D^n = Closed unit ball in $\mathbb{R}^n$ & $S^{n-1} = \partial D^n$.
3. Recognize the space obtained as $\mathcal{M}/_{S^1}$, where $\mathcal{M}$ = Möbius band & S^1 is its boundary circle.
4. Let $f: X \to Y$ be a quotient map, $A \subseteq X$ a subspace & let $f(A)$ have the induced topology from Y. Is $f|_A : A \to f(A)$ a quotient map?
5. Let $X \subseteq \mathbb{R}^2$ be the subspace given by

$$X = \bigcup_{n=1}^{\infty} \{(x,y) \in \mathbb{R}^2 \mid (x - 1/n)^2 + y^2 = 1/n^2\},$$

the Hawaiian earring. Let $Y = X \cup X^{op}$, where

$X^{op} \subseteq \mathbb{R}^2$ is the subspace $\{(x,y) \mid (x + 1/n)^2 + y^2 = 1/n^2\}$.

Are X and Y homeomorphic? Explain.

6. Let Y be a fixed top. space, Λ an indexing set with cardinality at least $\aleph_0$. Let $Y_\alpha = Y$ $\forall \alpha \in \Lambda$ & let $Z = \prod_\alpha Y_\alpha$. Then for any indexing set Γ with cardinality $\leq$ cardinality of Λ & $Z_\beta = Z$ $\forall \beta \in \Gamma$, $\prod_{\beta \in \Gamma} Z_\beta$ is homeomorphic to Z.
7. Prove that the cube $\overbrace{I \times \cdots \times I}^{n}$ is a quotient of

the Cantor set.

8. Let $X_1 \subset X_2 \subset \cdots$ be a sequence of top spaces, where X_i is a closed subspace of X_{i+1} $\forall i$. Let $X = \bigcup_i X_i$ and define $U \subseteq X$ to be open in X if $X \cap U_i$ is open in X_i $\forall i$.

- Check this defines a topology on X, called the topology on X coherent with the X_i's.
- Each X_i has the subspace top from X.
- If each X_i is normal, show that X is normal.

9. Let X be metrizable. Prove TFAE

(i) X is bounded with respect to any metric d that gives the topology on X.

(ii) Every continuous $\varphi : X \to \mathbb{R}$ is bounded.

(iii) X is compact.

10. Write $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid z_0\bar{z}_0 + z_1\bar{z}_1 = 1\}$; p, q be relatively prime & $\mathbb{Z}/p = \langle g \rangle$. Define the lens space $L(p,q) := S^3/_{\mathbb{Z}/p}$, for the action $g \cdot (z_0, z_1) = (e^{2\pi i/p} z_0, e^{2\pi q i/p} z_1)$. Prove $L(2,1) \simeq \mathbb{R}P^3$. If $q - q'$ ~~divides~~ is $\equiv 0 \bmod p$, prove $L(p,q) \simeq L(p,q')$.

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Homework practice problems

1. Let p be a prime and n be a nonzero integer. Write $n = p^{\nu_p(n)}m$ where m is coprime with p. Extend the function ν_p to the set of nonzero rationals by the formula $\nu_p(\pm\frac{r}{s}) = \nu_p(r) - \nu_p(s)$. Finally, define $\rho(x, y) = p^{-\nu_p(x-y)}$ when $x \neq y$ and $\rho(x, x) = 0$ for arbitrary $x, y \in \mathbb{Q}$. Prove that ρ is well defined and is a metric on $\mathbb{Q}$. This is the p-adic metric on $\mathbb{Q}$.

2. Let $S^n = \{(a_0, \cdots, a_n) \in \mathbb{R}^{n+1} | a_0^2 + \cdots a_n^2 = 1\}$ be the n-sphere in $\mathbb{R}^{n+1}$. Let $S_+^{n+1} := \{(a_0, \cdots, a_n) \in S^n,\ a_n > 0\}$ Prove that S_+^n is homeomorphic to the open unit ball in $\mathbb{R}^n$, where we identify $\mathbb{R}^n$ with the subspace $\{(a_0, \cdots, a_{n-1}, 0),\ a_i \in \mathbb{R}\}$ of $\mathbb{R}^{n+1}$. ()

3. **(i)** Let X, Y be a topological spaces. Treat X as the indexing set for copies of Y, so $Y_x = Y$ for every $x \in X$. Consider $\Pi_{x \in X} Y_x$ with product topology. Let $C(X, Y)$ be the set of all continuous maps $X \to Y$. Identify $C(X, Y)$ with a subset of $\Pi_{x \in X} Y_x$. Prove that the subspace topology on $C(X, Y)$ from $\Pi_{x \in X} Y_x$ coincides with the topology of pointwise convergence, for which a subbasis is given by the sets of the form $\{x_i,\ U_i\}_{i=1}^k := \{f \in C(X, Y) | f(x_i) \in U_i, i = 1, \cdots, k\}$, where $x_1, \cdots, x_k \in X$ and $U_1, \cdots, U_k$ are open subsets of Y.
(ii) Consider a sequence (f_n) in $C(X, Y)$ that converges to f in the topology of pointwise convergence on $C(X, Y)$. Then the sequence $(f_n(x))$ converges to $f(x)$ for every x.

4. Let X and Y be topological spaces. Consider all possible sets of the form

$$[K, U] := \{f \in C(X, Y) :\ f(K) \subset U\},$$

where $K \subset X$ is compact and $U \subset Y$ is open. The topology generated by these sets as a subbasis of open sets, is called the compact-open topology on $C(X, Y)$. Prove that when X is compact and Y is a metric space, the compact-open topology coincides with the topology of uniform convergence on the metric space $(C(X, Y), d)$, $d(f, g) = sup_{x \in X} d(f(x), g(x))$.

5. Show that the set of all isolated points of a second countable topological space is empty or countable. Hence show that any uncountable subset A of such a space must have at least one point which is a limit point of A.