Statistics-III

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Assignment-3

§ Problem 1

> Let $y_1 = \theta + \epsilon_1, y_2 = 2\theta - \phi + \epsilon_2, y_3 = \theta + 2\phi + \epsilon_3$, where $E(\epsilon_i) = 0$, for $i = 1, 2, 3$. Find the least squares estimates of θ and ϕ.

Solution. Obtain X and β by writting the the equations in matrix from as following,

$$\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}}_{\text{call this } X} \begin{pmatrix} \theta \\ \phi \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}$$

$$Y = X\beta + \epsilon$$

Notice that, two coloumns of X are independent as their dot product is zero, which means $\text{Rank}(x) = 2$. We can use full rank case to calculate $\hat{\beta}_{LS}$. So,

$$\hat{\beta}_{LS} = (X'X)^{-1} X'Y$$

Now,

$$\begin{aligned} X'X &= \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \\ &= 6I_2 \end{aligned}$$

From here we can conclude, $(X'X)^{-1} = \frac{1}{6} I_2$ and $(X'X)^{-1} X' = \frac{1}{6} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}$.

So, the least square estimators of θ, ϕ is given by,

$$\therefore \hat{\beta}_{L_s} = \begin{pmatrix} \frac{y_1 + 2y_2 + y_3}{6} \\ \frac{2y_3 - y_2}{6} \end{pmatrix}$$

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§ Problem 2

If $P = X(X'X)^{-1}X'$, show that the column spaces of P and X are the same.

Solution. Recall,

$$\mathcal{M}_c(X) = \{a \in : a = Xb \in \mathbb{R}^p\}$$

This is column space of X. Notice,

$$\begin{aligned} P &= X(X'X)^{-1}X' \\ \Rightarrow P' &= X(X'X)^{-1}X' \end{aligned}$$

And,

$$\begin{aligned} P^2 &= X(X'X)^{-1}X^{-1}X(X'X)^{-1}X' \\ &= X(X'X)^{-1}X' \\ &= P \end{aligned}$$

i.e. P is symmmetic idempotent. Which means, P is a projection matrix. For any, $a \in \mathcal{M}_c(X), \exists b \in \mathbb{R}^p$ such that, $a = Xb$.

$$\begin{aligned} Pa &= X(X'X)^{-1}X'a \\ &= X(X'X)^{-1}X'Xb \\ &= Xb \\ &= a. \end{aligned}$$

So, P is projection matrix onto $\mathcal{M}_c(X)$.

THEOREM. Ω be a subspace of $\mathbb{R}^n$, Let P_Ω be projection matrix onto Ω. Then, $\mathcal{M}_c(P_\Omega) = \Omega$.

Using the above theorem We can Say, $\mathcal{M}_c(P) = \mathcal{M}_c(X)$. Which means column space of P and column space of X are same. ■

§ Problem 3

Prove the following statements.

(a). $B'B = 0$ iff $B = 0$.

(b). $LB'B = MB'B$ iff $LB' = MB'$.

Solution.

- (a) Let, $B = \begin{pmatrix} b_1 & b_2 \cdots b_n \end{pmatrix}$, where b_i are column vectors.

  Now,

$$B'B = \begin{pmatrix} b_1' \\ b_2' \\ \vdots \\ b_n' \end{pmatrix} (b_1 b_2 \cdots b_n)$$

$$= \begin{pmatrix} b_1 \cdot b_1 & \cdots & b_1 \cdot b_n \\ \vdots & \cdots & \vdots \\ b_n \cdot b_1 & \cdots & b_n \cdot b_n \end{pmatrix}$$

Here, $[B'B]_{ij} = b_i \cdot b_j$, inner product of i-th coloumn and j-th column. So, $B'B = 0 \Rightarrow \|b_i\|^2 = 0$ forall $i \in \{1, \ldots, n\}$. Which means elements of each coloumns consists zero only. i.e $B = 0$. Another direction is trivial ! ■

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$$\begin{aligned} & LB'B = MB'B \\ & \implies (L-M)B'B = 0 \\ & \implies B'B(L-M)^{prime} = 0 \\ & \implies (L-M)B'B(L-M) = 0 \\ & \implies (L-M)B' = 0 \\ & \implies LB' = MB' \end{aligned}$$

Another direction is again trivial ! ■