B.Math II-Ordinary Differential Equations

Assignment 2

From Simmons and Krantz Book, chapter 11: Out of the following 7 question, do any 5 question.

1. Page 408, Question 4 and 6.
2. Page 416, Question 2 and 3.
3. Page 425-426, Question 1 and 2.
4. Page 426, Question 3 and 5.
5. Page 431, Question 4 and 5.
6. Page 438, Question 1 and 3.
7. Page 451-452, Question 2 and 4.

2. Let (x_0, y_0) be a point in the phase plane. If $x_1(t), y_1(t)$ and $x_2(t), y_2(t)$ are solutions of Equation (11.6) such that $x_1(t_1) = x_0$, $y_1(t_1) = y_0$ and $x_2(t_2) = x_0$, $y_2(t_2) = y_0$ for suitable t_1, t_2, then show that there exists a constant c such that

$$x_1(t+c) = x_2(t) \quad \text{and} \quad y_1(t+c) = y_2(t).$$

3. Describe the relation between the phase portraits of the systems

$$\begin{cases} \dfrac{dx}{dt} = F(x, y) \\ \dfrac{dy}{dt} = G(x, y) \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{dx}{dt} = -F(x, y) \\ \dfrac{dy}{dt} = -G(x, y). \end{cases}$$

4. Sketch and describe the phase portrait of each of the following systems.

(a) $\begin{cases} \dfrac{dx}{dt} = 0 \\ \dfrac{dy}{dt} = 0 \end{cases}$ (c) $\begin{cases} \dfrac{dx}{dt} = 1 \\ \dfrac{dy}{dt} = 2 \end{cases}$

(b) $\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = 0 \end{cases}$ (d) $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -y \end{cases}$

5. The critical points and paths of Equation (11.4) are by definition those of the equivalent system (11.5). Use this rubric to find the critical points of Equations (11.1), (11.2), and (11.3).

6. Find the critical points of

(a) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} - (x^3 + x^2 - 2x) = 0$

(b) $\begin{cases} \dfrac{dx}{dt} = y^2 - 5x + 6 \\ \dfrac{dy}{dt} = x - y \end{cases}$

7. Find all solutions of the nonautonomous system

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = x + e^t \end{cases}$$

and sketch (in the x-y plane) some of the curves defined by these solutions.

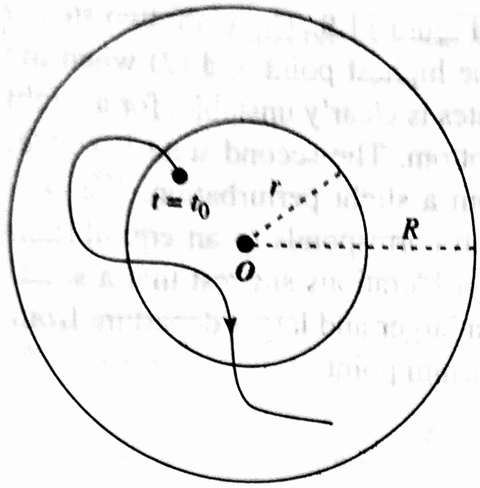

FIGURE 11.9

EXERCISES

1. For each of the following nonlinear systems, do the following:
   (i) Find the critical points;
   (ii) Find the differential equation of the paths;
   (iii) Solve this equation to find the paths;
   (iv) Sketch a few of the paths and show the direction of increasing t.

(a) $\begin{cases} \dfrac{dx}{dt} = y(x^2+1) \\ \dfrac{dy}{dt} = 2xy^2 \end{cases}$ (c) $\begin{cases} \dfrac{dx}{dt} = e^y \\ \dfrac{dy}{dt} = e^y \cos x \end{cases}$

(b) $\begin{cases} \dfrac{dx}{dt} = y(x^2+1) \\ \dfrac{dy}{dt} = -x(x^2+1) \end{cases}$ (d) $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = 2x^2y^2 \end{cases}$

2. Each of the following linear systems has the origin as an isolated critical point. For each system,
   (i) Find the general solution;
   (ii) Find the differential equation of the paths;
   (iii) Solve the equation found in part (ii) and sketch a few of the paths, showing the direction of increasing t;
   (iv) Discuss the stability of the critical points.

(a) $\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -y \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = -2y \end{cases}$ (c) $\begin{cases} \dfrac{dx}{dt} = 4y \\ \dfrac{dy}{dt} = -x \end{cases}$

3. Sketch the phase portrait of the equation $d^2x/dt^2 = 2x^3$, and show that it has an unstable isolated critical point at the origin.

Thus, above the parabola $p^2 - 4q = 0$, we have $p^2 - 4q < 0$ so that m_1, m_2 are conjugate complex numbers; these are pure imaginary if and only if $p = 0$. We have just described Cases C and E for the spirals and the centers.

Below the p-axis we have $q < 0$, which means that m_1, m_2 are real, distinct, and have opposite signs. This yields the saddle points of Case B. And, finally, the zone between these two regions (including the parabola but excluding the p-axis) is characterized by the relations $p^2 - 4q \geq 0$ and $q > 0$. Thus m_1, m_2 are real and of the same sign. Here we have the nodes coming from Cases C and D.

Furthermore, it is clear that there is precisely one region of asymptotic stability: the first quadrant. We state the result formally as follows:

Theorem 11.2

The critical point $(0, 0)$ of the linear system in Equation (11.22) is asymptotically stable if and only if the coefficients $p = -(a_1 + b_2)$ and $q = a_1b_2 - a_2b_1$ of the auxiliary equation (11.24) are both positive.

Finally, it should be stressed that we have studied the paths of our linear system near a critical point by analyzing explicit solutions of the system. In the next two sections we enter more fully into the spirit of the subject and the technique by investigating similar problems for nonlinear systems that in general cannot be solved explicitly.

EXERCISES

1. Determine the nature and stability properties of the critical point $(0, 0)$ for each of the following linear autonomous system:

(a) $\begin{cases} \dfrac{dx}{dt} = 2x \\ \dfrac{dy}{dt} = 3y \end{cases}$

(b) $\begin{cases} \dfrac{dx}{dt} = -x - 2y \\ \dfrac{dy}{dt} = 4x - 5y \end{cases}$

(c) $\begin{cases} \dfrac{dx}{dt} = -3x + 4y \\ \dfrac{dy}{dt} = -2x + 3y \end{cases}$

(d) $\begin{cases} \dfrac{dx}{dt} = 5x + 2y \\ \dfrac{dy}{dt} = -17x - 5y \end{cases}$

(e) $\begin{cases} \dfrac{dx}{dt} = -4x - y \\ \dfrac{dy}{dt} = x - 2y \end{cases}$

(f) $\begin{cases} \dfrac{dx}{dt} = 4x - 3y \\ \dfrac{dy}{dt} = 8x - 6y \end{cases}$

(g) $\begin{cases} \dfrac{dx}{dt} = 4x - 2y \\ \dfrac{dy}{dt} = 5x + 2y \end{cases}$

2. If $a_1b_2 - a_2b_1 = 0$, then show that the system in Equation (11.22) has infinitely many critical points, none of which are isolated.

3. (a) If $a_1b_2 - a_2b_1 \neq 0$, then show that the system

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y + c_1 \\ \dfrac{dy}{dt} = a_2x + b_2y + c_2 \end{cases}$$

has a single isolated critical point (x_0, y_0).

(b) Show that the system in (a) can be written in the form of Equation (11.22) by means of the change of variables $\overline{x} = x - x_0$, $\overline{y} = y - y_0$.

(c) Find the critical point of the system

$$\begin{cases} \dfrac{dx}{dt} = 2x - 2y + 10 \\ \dfrac{dy}{dt} = 11x - 8y + 49 \, . \end{cases}$$

Write the system in the form of Equation (11.22) by changing variables, and determine the nature and stability properties of the critical point.

4. In Section 2.5, we studied the free vibrations of a mass attached to a spring by solving the equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + a^2x = 0 \, .$$

Here $b \geq 0$ and $a > 0$ are constants representing the viscosity of the medium and the stiffness of the spring, respectively. Consider the equivalent autonomous system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -a^2x - 2by \, . \end{cases} \tag{11.35}$$

Observe that (0, 0) is the only critical point of this system.

(a) Find the auxiliary equation of (11.35). What are p and q?

(b) In each of the following four cases, describe the nature and stability properties of the critical point, and give a brief physical interpretation of the corresponding motion of the mass:

(i) $b = 0$
(ii) $0 < b < a$
(iii) $b = a$
(iv) $b > a$

5. Solve under the hypotheses of Case E, and show that the result is a one-parameter family of ellipses surrounding the origin. *Hint*: Recall that if $Ax^2 + Bxy + Cy^2 = D$ is the equation of a real curve, then the curve is an ellipse if and only if the discriminant $B^2 - 4AC$ is negative (see [THO, p. 546]).

Math Nugget

Alexander Mikhailovich Liapunov (1857–1918) was a Russian mathematician and mechanical engineer. He performed the somewhat remarkable feat of producing a doctoral dissertation of lasting value. This classic work, still important today, was originally published in 1892 in Russian, but is now available in an English translation (*Stability of Motion*, Academic Press, New York, 1966). Liapunov died tragically, by violence, in Odessa; like many a middle-class intellectual of his time, he was a victim of the chaotic aftermath of the Russian revolution.

EXERCISES

1. Determine whether each of the following functions is of positive type, of negative type, or neither.
(a) $x^2 - xy - y^2$
(b) $2x^2 - 3xy + 3y^2$
(c) $-2x^2 + 3xy - y^2$
(d) $-x^2 - 4xy - 5y^2$

2. Show that a function of the form

$$ax^3 + bx^2y + cxy^2 + dy^3$$

cannot be either of positive type nor of negative type.

3. Show that (0, 0) is an asymptotically stable critical point for each of the following systems:

(a) $\begin{cases} \dfrac{dx}{dt} = -3x^3 - y \\ \dfrac{dy}{dt} = x^5 - 2y^3 \end{cases}$

(b) $\begin{cases} \dfrac{dx}{dt} = -2x + xy^3 \\ \dfrac{dy}{dt} = x^2y^2 - y^3 \end{cases}$

4. Show that the critical point (0, 0) of the system in Equation (11.36) is unstable if there exists a function $E(x, y)$ with the following properties:
(a) $E(x, y)$ is continuously differentiable in some region containing the origin;
(b) $E(0, 0) = 0$;
(c) Every circle centered at (0, 0) contains at least one point where $E(x, y)$ is positive;
(d) $(\partial E/\partial x)F + (\partial E/\partial y)G$ is of positive type.

5. Show that (0, 0) is an unstable critical point for the system

$$\begin{cases} \dfrac{dx}{dt} = 2xy + x^3 \\ \dfrac{dy}{dt} = -x^2 + y^5 \end{cases}$$

6. Assume that $f(x)$ satisfies $f(0) = 0$ and $xf(x) > 0$ for $x \neq 0$.

EXERCISES

1. Verify that (0, 0) is an asymptotically stable critical point of

$$\begin{cases} \dfrac{dx}{dt} = -y - x^3 \\ \dfrac{dy}{dt} = x - y^3 \end{cases}$$

but is an unstable critical point of

$$\begin{cases} \dfrac{dx}{dt} = -y + x^3 \\ \dfrac{dy}{dt} = x + y^3 . \end{cases}$$

How are these facts related to Exercise 6 below?

2. Sketch the family of curves whose polar equation is $r = a \sin 2\theta$ (see Figure 11.20). Express the differential equation of this family in the form $dy/dx = G(x, y)/F(x, y)$.

3. Verify that (0, 0) is a simple critical point for each of the following systems, and determine its nature and stability properties.

(a) $\begin{cases} \dfrac{dx}{dt} = x + y - 2xy \\ \dfrac{dy}{dt} = -2x + y + 3y^2 \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = -x - y - 3x^2y \\ \dfrac{dy}{dt} = -2x - 4y + y \sin x \end{cases}$

4. The van der Pol equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0$$

is equivalent to the system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x - \mu(x^2 - 1)y . \end{cases}$$

Investigate the stability properties of the critical point (0, 0) for the cases $\mu > 0$ and $\mu < 0$.

5. Show that if (0, 0) is a simple critical point of Equation (11.46), then it is necessarily isolated. *Hint*: Write conditions in Equation (11.48) in the form $f(x, y)/r = \epsilon_1 \to 0$ and $g(x, y)/r = \epsilon_2 \to 0$. In light of Equation (11.47), use polar coordinates to deduce a contradiction from the assumption that the right sides in Equation (11.46) both vanish at points arbitrarily close to the origin but different from it.

6. If (0, 0) is a simple critical point of Equation (11.46) and if $q = a_1b_2 - a_2b_1 < 0$, then Theorem 11.5 implies that (0, 0) is a saddle point of Equation (11.46) and

Math Nugget

Ivar Otto Bendixson (1861–1935) was a Swedish mathematician who published one important memoir in 1901; it served to supplement some of Poincaré's earlier work. Bendixson served as professor (and later as president) at the University of Stockholm. He was an energetic and long-time member of the Stockholm City Council.

EXERCISES

1. Verify that the nonlinear autonomous system

$$\begin{cases} \dfrac{dx}{dt} = 3x - y - xe^{x^2+y^2} \\ \dfrac{dy}{dt} = x + 3y - ye^{x^2+y^2} \end{cases}$$

has a periodic solution.

2. For each of the following differential equations, use a theorem of this section to determine whether or not the given differential equation has a periodic solution.

(a) $\dfrac{d^2x}{dt^2} + (5x^4 - 9x^2)\dfrac{dx}{dt} + x^5 = 0$

(b) $\dfrac{d^2x}{dt^2} - (x^2 + 1)\dfrac{dx}{dt} + x^5 = 0$

(c) $\dfrac{d^2x}{dt^2} - \left(\dfrac{dx}{dt}\right)^2 - (1 + x^2) = 0$

(d) $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + \left(\dfrac{dx}{dt}\right)^5 - 3x^3 = 0$

(e) $\dfrac{d^2x}{dt^2} + x^6\dfrac{dx}{dt} - x^2\dfrac{dx}{dt} + x = 0$

3. Show that any differential equation of the form

$$a\frac{d^2x}{dt^2} + b(x^2 - 1)\frac{dx}{dt} + cx = 0 \qquad (a, b, c > 0)$$

can be transformed into the van der Pol equation by a change of the independent variable.

4. Consider the nonlinear autonomous system

$$\begin{cases} \dfrac{dx}{dt} = 4x + 4y - x(x^2 + y^2) \\ \dfrac{dy}{dt} = -4x + 4y - y(x^2 + y^2). \end{cases}$$

(a) Transform the system into polar coordinate form.
(b) Apply the Poincaré-Bendixson theorem to show that there is a closed path between the circles $r = 1$ and $r = 3$.
(c) Find the general nonconstant solution $x = x(t)$, $y = y(t)$ of the original system, and use this solution to find a periodic solution corresponding to the closed path whose existence was established in (b).
(d) Sketch the closed path and at least two other paths in the phase plane.

Historical Note

Poincaré

Jules Henri Poincaré (1854–1912) was universally recognized at the beginning of the twentieth century as the greatest mathematician of his generation. Already a prodigy when he was quite young, he was watched with love and admiration by the entire country of France as he developed into one of the pre-eminent mathematicians of all time.

He began his academic career at Caen in 1879, but just two years later he was appointed to a professorship at the Sorbonne. He remained there for the rest of his life, lecturing on a different subject each year. His lectures were recorded, edited, and published by his students. In them, he treated with great originality and technical mastery all the known fields of pure and applied mathematics. Altogether, Poincaré produce