Rings and Modules

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Assignment-3

Question 1.
(i) Consider a finitely generated R-module M. If $M_1 \subseteq M_2 \subseteq M$ be submodules such that $M/M_1 \cong M/M_2$ as R-modules, prove that $M_1 = M_2$. If M is not finitely generated, then is the above necessarily true?
(ii) If $M \cong N$ as R-modules, is it necessarily true that the annihilator ideals are the same; that is, $ann_R(M) = ann_R(N)$?

Solution.
(i). From the Isomorphism theorems we know that,

$$(M/M_1)/(M_2/M_1) \cong (M/M_2)$$

Which is isomorphic to M/M_1. So, $(M/M_1)/(M_2/M_1) \cong (M/M_1)$. Now we will use Problem 2.(i) to prove, $M_2/M_1 = (0)$. We know the natural surjection, $\pi : M/M_1 \to (M/M_1)/(M_2/M_1)$ induces an surjective map from M/M_1 to M/M_1. Since M is finitely generated M/M_1 is finitely generated. So, π induces an Isomorphism. And hence, $\ker(\pi) = M_2/M_1 = (0)$. Since $M_1 \subseteq M_2$ we can say $M_1 = M_2$.

If M is not finitely generated, then the statement is not necessarily true. For example, Consider $M = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ as a $\mathbb{Z}$ module. Now assume $M_1 = \mathbb{Z}/2\mathbb{Z}$ and $M_2 = \mathbb{Z}/2\mathbb{Z} \bigoplus \mathbb{Z}/2\mathbb{Z}$ are submodules of M and satisfying the given condition. We can see that, $M/M_1 \cong \bigoplus_{i\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z} \cong M/M_2$ but $M_2 \neq M_1$.

(ii). Let, φ be the module Isomorphism between M and N. Let, $r \in ann_R(M)$ then $rM = 0$ which means, $\varphi(rM) = r\varphi(M) = rN = 0$. So, $r \in ann_R(N)$. So, $ann_R(M) \subseteq ann_R(N)$. Similarly we can get a module homomorphism between N and M. We can show $ann_R(N) \subseteq ann_R(M)$. And hence $ann_R(M) = ann_R(N)$. ∎

Question 2.
(i) Let M be a finitely generated R-module, and let $\varphi : M \to M$ be a surjective R-module homomorphism. Prove that φ must be an isomorphism.
(ii) Recall that R is said to be Noetherian if all ideals are finitely generated. Equivalently, every ascending sequence of ideals $I_1 \subseteq I_2 \subseteq \cdots\cdots$ stabilizes; that is, there exists $n_0 \geq 1$ such that $I_n = I_{n_0}$ for all $n > n_0$. If R is Noetherian, and $\theta : R \to R$ is a surjective ring homomorphism, prove that θ must be an isomorphism.

Solution.

(i). M is a finitely generated R-module. So, M can be treated as finitely generated $R[t]$ module by defining $t \cdot m = \varphi(m)$. Since, φ is module homomorphism it's not hard to see this multiplication gives us, M is module over $R[t]$.

We can see that $t \cdot M = \varphi(M) = M$ since, φ is an surjective homomorphism. Similarly $t^n \cdot M = M$. From here we can say that $(t) \cdot M = M$. Now we will propose a lemma.

> **Lemma.** If M is finitely generated module R-module. If A is an ideal of R such that $M = AM$ then we will get $a \in A$ such that $(1 + a) \cdot M = 0$.

Proof. Let, $\{x_1, \cdots, x_n\}$ be generators of M. Since, $AM = M$ we can say,

$$\begin{aligned}
x_1 &= a_{11}x_1 + \cdots + a_{1n}x_n \\
x_2 &= a_{21}x_1 + \cdots + a_{2n}x_n \\
\vdots \quad & \quad \vdots \quad \vdots \qquad \quad \vdots \\
x_n &= a_{n1}x_1 + \cdots + a_{nn}x_n
\end{aligned}$$

$$\implies \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\implies \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 - a_{11} & \cdots & -a_{1n} \\ -a_{21} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots \\ -a_{n1} & \cdots & 1 - a_{nn} \end{pmatrix}}_{\text{call this matrix } A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

By multiplying $\text{adj}(A)$ from left to both side of the above equation we will get, $(\det A)x_i = 0$ for all $i = 1, \cdots, n$. Now, notice that the determinant of A is in form $1 + a$ for some $a \in A$ as all $a_{ij} \in A$. From here we can see that $(\det A)M = 0 = (1 + a)M$. □

Now we will proceed to the main problem. As we have already noted $(t) \cdots M = M$ there must exists a polynomial $g \in (t)$ such that, $(1 + g).M = 0$. If $g = \sum_{i=1}^{n} a_i t^i$ we can say, $1_M = -\sum a_i \varphi^i$ which means $\psi \circ \varphi = 1_M$ as a module homomorphism where $\psi = -\sum_{i=1}^{n} a_i \varphi^{i-1}$. So, φ needs to be injective hence φ is an isomorphism. ■

(ii). $\theta : R \to R$ is a Surjective map. we can see that, $\ker(\theta) \subseteq \text{Ker}\,(\theta^2)$. So, we can get a chain of ideals $\ker(\theta) \subseteq \ker(\theta^2) \subseteq \cdots \subseteq \ker(\theta^n) \subseteq \ldots$. Since, R is noetherian, we can Say the above chain terminates. Let say, $\ker(\theta^n) = \ker(\theta^{n_0})\, \forall n > n_0$.

Claim: $\ker(\theta^{n_0}) = \{0\}$

Proof: If $f : R \to R$ is a surjective homomorphism with $\ker(f) = \ker(f^2)$, then if $x \in \ker f$ We have, $f(x) = 0$ we can say, $f(y) = x$ for some $y \in R$ as f is Surjective. so, $f^2(y) = 0 \Rightarrow y \in \ker(f^2)$ So, $y \in \ker(f) = \ker(f^2)$ which means, $f(y) = x = 0$. So, $\text{Ker}(f) = 0$. If we take $f = \theta^{n_0}$ then we get $\ker \theta^{n_0} = \{0\}$. □

We can use the above claim to get $\ker\theta$ is $\{0\}$. which readily gives us θ is an isomorphism. ■

Question 3.
(i) If M is a finitely generated R-module where R is Noetherian, prove that all submodules of M are necessarily finitely generated.
(ii) If $N_1 \subseteq N_2$ are R-modules such that N_1 and N_2/N_1 are finitely generated as R-modules, then N_2 must be finitely generated as well.

Solution.
(i). We will proceed by induction on the number of generators of M. If M is $\langle v\rangle$, then $\varphi\colon R \to M$ defined by $r \mapsto rv$ is an epimorphism, and so $M \cong R/\ker(\phi)$, and is Noetherian as the quotient of a Noetherian ring. So every submodule of M is finitely generated. Suppose the statement holds for all R-modules with k generators, for $k \leq n$. Let $M = \langle v_1, \dots, v_{n+1}\rangle$. Let $M' = \langle v_1, \dots, v_n\rangle$. Consider N be a submodule of M. Now, assume $T = N \cap M'$. So T is a submodule of M', hence finitely generated. Now by Isomorphism theorems of module we get,

$$N/T = N/(N \cap M') \cong (N + M')/M'.$$

Now, $(N + M')/M'$ is a M/M' submodule. Since M/M' is cyclic we can say $(N + M')/M'$ is finitely generated (from $n = 1$ case). So, both T and N/T is finitely generated and by Problem 3.(ii) we can say N is finitely generated.

(ii) Given $N_1 \subseteq N_2$. N_1 and N_2/N_1 are finitely generated R-module We can Say. there is a surjective homomorphism $\varphi_1 : R^n \to N_1$ and. $\varphi_2 : R^m \to N_2$ for some, $n, m \in M$. Consider the following exact sequence.

$$0 \longrightarrow N_1 \xhookrightarrow{i} N_2 \xrightarrow{\pi} N_2/N_1 \longrightarrow 0$$

We claim that the above exact sequence can be extended to the following commuatative diagram with both rows being exact.

$$\begin{tikzcd}
0 \arrow[r] & N_1 \arrow[r, hook, "i"] & N_2 \arrow[r, "\pi"] & N_2/N_1 \arrow[r] & 0 \\
0 \arrow[r] & R^n \arrow[u, "\varphi_1"] \arrow[r, dashed, "i'"] & R^{m+n} \arrow[u, dashed, "\varphi"] \arrow[r, dashed, "\pi'"] & R^n \arrow[u, "\varphi_2"'] \arrow[r] & 0
\end{tikzcd}$$

Let, $\{e_1, e_2, \dots, e_n\}$ be generating set of R^n and $\{e_1, \dots e_{n+m}\}$ be generating set of R^{m+n}. Consider, $i'(e_j) = e_j$ for $j = 1, 2, \dots n$ and take. $\pi'(e_j) = e_j\mathbb{I}[j > n]$. This construction allows the bottom row to be exact. $i(\varphi_1(e_1)), \dots i(\varphi_1(e_n))$ and, $\{e_{n+1}, \dots e_{m+n}\}$ maps to $\pi^{-1}(\varphi_2(e_{m+1})), \dots, \pi^{-1}(\varphi_2(e_{m+n}))$.

Now we will show that φ is surjective. For any $n \in N_2$ take $\varphi_2^{-1}(\pi(n))$ which is in R^n (as φ_2 is surjective). Now by our construction π' is surjective homomorphism. So, there is a pre image of n in R^{m+n} that is $\pi'^{-1}(\varphi_2^{-1}(\pi(n)))$.

So, we can write $N_2 \cong R^{m+n}/\ker\varphi$. Since, R^{m+n} is finitely generated quotient of it is also finitely generated. So, N_2 is finitely generated.

Question 4.
Prove that every R-module is free if, and only of, R is a field.

Solution. Let's assume every R module is free. Consider $x \neq 0$ is an element of R. Clearly, $R/(x)$ is also a R-module and by defination it must be free. So it must have a R-basis. If B is a basis then, $x \cdot B = \{0\}$. Hence we must get a linear combination of basis elements with each coefficient being $x(\neq 0)$. So, there can not have any such basis. Only possibility is $B = \{0\}$. But then $R \cong Rx$ which means we must get an non zero elemet y such that $yx = 1$ (Since R is commuatative ring and contains 1). So each element of R has inverse and hence R is a Field.

If R is a field then any R-module is a vector space over R and we know every vector space has a basis and every element of the vector space can be written as linear combination of finite basis elements. So every R-module is free. ■

Question 5.(Generalization of a theorem of Cohen due to Jothilingam)
If M is a finitely generated R-module. If PM is a finitely generated submodule of M for each prime ideal P of R, show that all submodules of M are finitely generated as well.

Solution. Let, $\Sigma = \{$Set of all submodules of M which are not finitely generated$\}$. We can give this set partial order $\subseteq$, $I \subseteq J$ iff I is contained in J. Let, $M_1 \subseteq M_2 \subseteq M_3 \cdots$ be a chain of submodules of M. Consider the union of these modules $\mathfrak{M} = \cup M_i$. If the union was finitely generated then all the generators of $\mathfrak{M}$ must belong to some M_n. Then any elements of M_n can be generated by these finite generators which is a contradiction. We know every partially orderd sets which has upper bound has a unique maximal element N in Σ. Let, N be the maximal submodule of M which is not finitely generated.

Claim : $ann_R(M/N)$ is a prime ideal of R.

Proof. Call the ideal $ann_R(M/N) = I$. Define, $N(s) = \{x \in M : xs \in N\}$.Clearly this is a R-Module. If I is not prime then there is $ab \in I$ such that $a, b \notin I$. We can see that $N + aM$ and $N + bM$ strictly contains N as both a, b don't annihilate M/N. By maximality of N we can say that both $N + aM, N + bM$ are finitely generated. Notice that $N \subseteq N(s)$. We can see $N(a), N(b)$ is strictly contains N. Because $N \subset bM$ as. $abM \subseteq N$, $bM \subseteq N(a)$ and hence $N \subset N + bM \subseteq N(a)$. So we can say that $N(a)$ is also finitely generated.

Since $N+aM$ is finitely generated, let $\{n_i+am_i\}$ this set generates $N+aM$. Let, $a \in N \subset N+aM$, we can write, $a = \sum a_i(n_i + am_i) = \sum a_i n_i + a \sum a_i m_i$ Which means $a \sum a_i \in N$ and hence $a \sum a_i \in N(a)$. So, $a \in Rn_1 + \cdots + Rn_k + aN(a)$ Which means $N \subseteq Rn_1 + \cdots + Rn_k + aN(a) \subseteq N$ and hence N is finitely generated which is a contradiction. □

Take an element $x \in M \backslash N$ and put $J = (N : x)$. There are two possibilities, $I = J$ or $I \subset J$.

Case 1: $P = J$, we consider the submodule $N + xR$. By our maximality assumption on N, the submodule $N + xR$ is finitely generated. Let $\{n_i + xt_i : 1 \leq i \leq k\}$ be a set of generators for $N + xR$. Here $n_i \in N$ and $t_i \in R$ for all $1 \leq i \leq k$. Thus, each $a \in N$ can be expressed as

$$a = \sum_{i=1}^{k} (n_i + xt_i)\, r_i = \sum_{i=1}^{k} n_i r_i + x \left(\sum_{i=1}^{k} t_i r_i \right)$$

with $r_i \in R$ for each $1 \leq i \leq k$. This expression shows that $x\left(\sum_{i=1}^{k} t_i r_i\right) \in N$ and thus $\left(\sum_{i=1}^{k} t_i r_i\right) \in J = I$. Hence

$$N \subseteq n_1 R + n_2 R + \ldots + n_k R + xP.$$

On the other hand, we know that $xP \subset N$; therefore the last inclusion is actually an equality:

$$N = n_1 R + n_2 R + \ldots + n_k R + xI$$

Taking into consideration the fact that $xI \subseteq MP \subset N$, we obtain

$$\begin{aligned} N = & n_1 R + n_2 R + \ldots + n_k R + xI \\ & \subseteq n_1 R + n_2 R + \ldots + n_k R + IM \\ & \subseteq n_1 R + n_2 R + \ldots + n_k R + N \subseteq N. \end{aligned}$$

It follows that

$$N = n_1 R + n_2 R + \ldots + n_k R + IM$$

Since IM is finitely generated, the above identity shows that N is finitely generated; which is a contradiction.

Case 2: Now, let us consider the case $I \subset J$. In this case, there exists an element $s \in R \backslash I$ such that $xs \in N$. Thus N is properly contained in both $N(s)$ and $N + sM$. Thus N is finitely generated (we can trace the same proof we did for **Claim**). Which leads us to a contradiction. Hence $\Sigma = \phi$. So all submodule of M is finitely generated. ■

Question 6. Let,

$$0 \to M' \overset{\alpha}{\to} M \overset{\beta}{\to} M'' \to 0$$

be a short exact sequence of R-modules which splits. Recall this means α is $1-1, \beta$ is onto, $\operatorname{Ker}(\beta) = \operatorname{Image}(\alpha)$, and there exists a 'splitting' (an R module homomorphism) $s : M'' \to M$ such that $\beta \circ s = Id_{M''}$. Prove that the set of all splittings $s : M'' \to M$ is in bijection with the set $\operatorname{Hom}_R(M'', M')$ of all R-module homomorphisms from M'' to M'.

Solution. Consider $\operatorname{Hom}(M'', M)^*$ be the set of all homomorphism from M'' to M that is 'splitting' for the given exact sequence. Let, s is given as initial split for the exact sequence. We will give a set theoretic bijection $F : \operatorname{Hom}(M'', M') \to \operatorname{Hom}(M'', M)^*$ defined as $F : \varphi \mapsto s + \alpha \circ \varphi$.

Let, $s' \in \operatorname{Hom}(M'', M)^*$ clearly $\beta \circ (s - s') = 0$. From here we can say that, $\operatorname{Im}(s - s') \subseteq \ker \beta = \operatorname{Im} \alpha$. We can say, $s - s' = \alpha \circ \varphi$ for some $\varphi \in \operatorname{Hom}(M'', M')$. So, $F(\varphi') = s'$. Hence F is surjective.

Also, let, $s + \alpha \circ \varphi = s + \alpha \circ \psi$. We can say, $\alpha \circ \varphi(x) = \alpha \circ \psi(x)$ for all $x \in M''$. Since α is injective we can say that, $\varphi = \psi$.And hence F is injective. So, F is bijective. ■

Question 7. (Local-Global principle)
For a multiplicative subset S of R and an R-module M, define a relation on the set $M \times S$ by

$$(m_1, s_1) \sim (m_2, s_2)$$

if, and only if, there exists $s \in S$ such that $s\,(s_1 m_2 - s_2 m_1) = 0$. This is an equivalence relation (assume this - you can verify it but not show the calculation here). Denoting the equivalence class of $(m.s)$ by $\frac{m}{s}$, the equivalence classes form an $S^{-1}R$-module under the addition

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_1 m_2 + s_2 m_1}{s_1 s_2}$$

and the scalar multiplication

$$\frac{r}{s} \cdot \frac{m}{t} = \frac{rm}{st}$$

Assume this also - verify it for yourself or look up the routine proof.
Prove that if $S^{-1}M$ is the zero $S^{-1}R$-module with $S = R - \mathfrak{m}$ for each maximal ideal $\mathfrak{m}$, prove that $M = (0)$.

Solution. Let, there is a $0 \neq x \in M$. consider $\mathfrak{m}$ be the maximal ideal containing $ann_R(x)$. Now Consider $S = R - \mathfrak{m}$. If 0 is the only element in $S^{-1}M$ then, $\frac{x}{1} \sim 0$. which means there is $s \in S$ such that $sx = 0$. But then, $s \in ann_R(x)$ and $s \in R - \mathfrak{m}$. which is a contradiction. ■