Assignment-2

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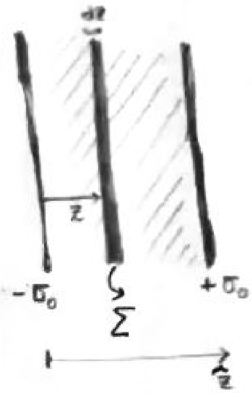
Problem 1



(i) At first of all set, the plate$(-\sigma_0)$ at $z=0$. At distance ~~from~~ of z from $(-\sigma_0)$ (Here $z>0$) Consider a strip of width dz. ~~at~~ At that distance charge density is fixed so, we can assume the strip as plate with charge density $\rho(z)dz$. Clearly due to this infinite plane Σ electric field at any point will be on the $\hat{z}$ direction. So, for all ρ over the given region, ~~charge~~ electric field will be on the $\hat{z}$ direction.

(ii) Charge density on the all space is defined as,

$$\tilde{\rho}(z) = \rho(z)\mathbb{1}[0<z<s] + \sigma_0\delta(z-s) - \sigma_0\delta(z).$$

Let $\vec{E}$ be the ~~te~~ electric field. We have shown the electric field depends only on z. and also, Electric field is only on z direction. So, $\nabla\cdot E = \frac{dE}{dz}$. So,

$$\frac{dE_z}{dz} = \frac{\tilde{\rho}(z)}{\epsilon_0}.$$

For, $z<0$ we can see that, $\tilde{\rho}(z)=0$. So electric field is constant there. It's given $\vec{E} = E_0\hat{e}_z$ So, for $0<z<s$, ($\epsilon>0$ very small)

$$E_z - E_0 = \int_{-\epsilon}^{z} \frac{\tilde{\rho}(z)dz}{\epsilon_0} = \int_{-\epsilon}^{z} \frac{2\rho_0 z\,dz}{s\epsilon_0} + \int_{-\epsilon}^{z} \frac{(-\sigma_0)}{\epsilon_0}\delta(\epsilon)dz.$$

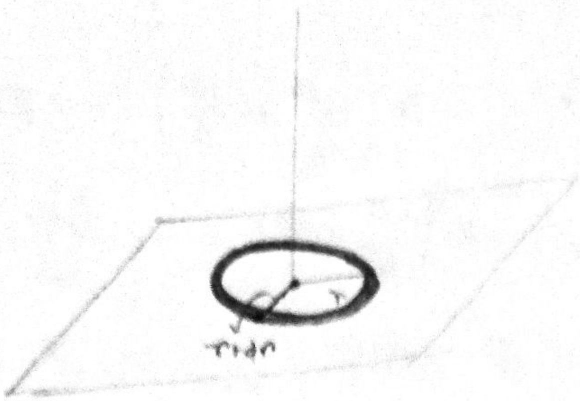
$$= \frac{\rho_0(z^2-\epsilon^2)}{s\epsilon_0} - \frac{\sigma_0}{\epsilon_0}.$$

Take, $\epsilon\to 0+$ to get,

$$E_z = \left(\frac{\rho_0(z^2)}{s\epsilon_0} - \frac{\sigma_0}{\epsilon_0} + E_0\right).$$

So, $E(z) = \left(\frac{\rho_0 z^2}{s\epsilon_0} - \frac{\sigma_0}{\epsilon_0} + E_0\right)$ for $0<s<z$.

Problem



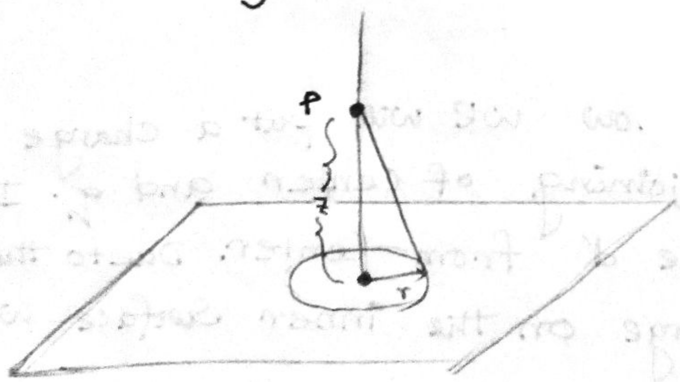
(a) Total charge on the plane is,

$$Q = \int \sigma (2\pi r)\, dr$$

$$= -qd \int_0^{\infty} \frac{r\, dr}{(r^2+s^2)^{3/2}}$$

$$= -\frac{qd}{s}$$

(b)



Due to a Circuler strip ~~or~~ of width "dr" potential at P is given by,

$$\frac{\sigma (2\pi r)\, dr}{4\pi\epsilon_0 \sqrt{z^2+r^2}}.$$

So the required potential is,

$$\varphi = -\frac{qd}{4\pi\epsilon_0} \int_0^{\infty} \frac{r \quad dr}{(z^2+r^2)^{1/2} (r^2+s^2)^{3/2}}.$$

Substitute, $y^2 = r^2 + z^2$

$y\,dr = r\,dr$

$$\therefore \int_{z}^{\infty} \frac{y\,dy}{y\,(y^2+s^2-z^2)^{3/2}} \left(\frac{-qd}{4\pi\epsilon_0}\right).$$

$$= \left(\frac{-qd}{4\pi\epsilon_0}\right) \int_{z}^{\infty} \frac{dy}{(y^2+a^2)^{3/2}} \qquad \left[\text{Here, } a^2 = s^2 - z^2\right].$$

Let, $y = a\tan\theta. \Rightarrow dy = a\sec^2\theta\, d\theta$. So,

$$\int_{\tan^{-1}\left(\frac{z}{a}\right)}^{\pi/2} \cos\theta\, d\theta = 1 - \cos\left(\tan^{-1}\left(\frac{z}{a}\right)\right).$$

$$= 1 - \frac{z}{\sqrt{a^2+z^2}}$$

$$= \frac{(s-z)}{s}$$

$$\therefore \varphi = \frac{-qd}{4\pi\epsilon_0} \cdot \frac{s-z}{s(s^2-z^2)} = \frac{-qd}{4\pi\epsilon_0 s\,(s+z)} = \frac{q}{4\pi\epsilon_0\,(s+z)}$$

This is same potential due to a paint charge q at $z = -s$. $\blacksquare$

Andrew Zangwill (2012 Ed. 21) (2013 Ed. 22)
Problem

(a) We will retrace the path we did for previous problem.

potential at Centere is,

$$\varphi = \int_0^\infty \frac{\sigma(2\pi r)}{4\pi\epsilon_0} f(r)dr$$

$$= \frac{\sigma}{2\epsilon_0} \int_0^\infty r f(r) dr.$$

(b) At $z = z$ we will Calculate the potential due to the infinite Sheet.

$$\varphi(z) = \frac{\sigma}{2\epsilon_0} \int_0^\infty r f\left((r^2+z^2)^{1/2}\right) dr$$

Substitute $y^2 = r^2 + z^2 \Rightarrow y\,dy = r\,dr$

$$\therefore \varphi(z) = \frac{\sigma}{2\epsilon_0} \int_z^\infty y f(y) dy$$

Clearly Electric field is,

$$\vec{E}(z) = -\vec{\nabla}\varphi(z) = \hat{z}\frac{\sigma}{2\epsilon_0}\frac{\partial}{\partial z}\int_\infty^z y\, f(y)\, dy.$$

$$= \lim_{a\to\infty} \frac{\sigma}{2\epsilon_0}\frac{\partial}{\partial z}\int_a^z y f(y)\, dy\,.\,\hat{z}$$

$$= \frac{\sigma}{2\epsilon_0} z f(z)\,\hat{z}.$$

$\blacksquare$

A.7 Problem (2013 Ed. 23) (2012 Ed. 24)

Total energy Contained by Spherical Volume is,

$$U_E = \frac{1}{2}\int_V d^3r\, \rho(r)\varphi(r) \quad [\text{as } \rho \text{ is zero outside}]$$

V = Spherical Volume.

$$= \frac{\epsilon_0}{2}\int_V d^3r\, \varphi(r)\, \nabla\cdot E$$

$$U_E = \frac{\epsilon_0}{2}\int_V d^3r\, \nabla(\vec{E}\varphi) + \frac{\epsilon_0}{2}\int_V d^3r\, (-\nabla\cdot E)\cdot E$$

$$= \frac{\epsilon_0}{2}\int_V d^3r\, \nabla\cdot(\vec{E}\varphi) + \underbrace{\frac{\epsilon_0}{2}\int_V d^3r\, E\cdot E}_{\text{Total energy Contained inside the sphere}}$$

$$\therefore U_E(out) = \frac{\epsilon_0}{2}\int_V d^3r\, \nabla\cdot(\vec{E}\varphi)$$

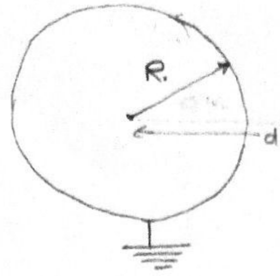
$$= \frac{\epsilon_0}{2}\int_S dS\cdot E\phi$$

S = Surface that Enclose V.

$$= \frac{\epsilon_0\varphi_0}{2}\int_S \vec{E}\cdot d\vec{S} = \frac{\epsilon_0\varphi_0 Q}{2\epsilon_0}$$

$$= \frac{\varphi_0 Q}{2}.$$

Problem 4



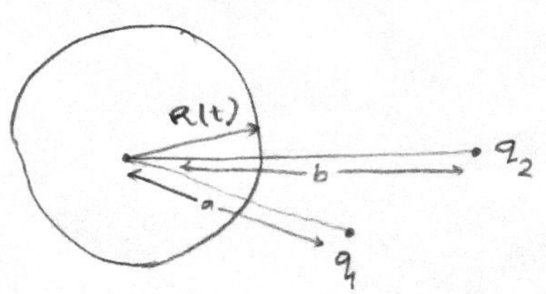
Consider the above Scenario. Let, the above is a Conducting Spherical Shell of radius R and a Charge q is at distance d from Center. Let, d' is the image of q under reflection with a Spherical Surface. From the inverson formulae, we know,

$$dd' = R^2$$
$$\therefore d' = R^2/d.$$

If $d > R$ then $d' < R$. Now we will put a charge ~~of~~ $-qR/d$ between line joining of center and q. It Should be at distance d' from Center. Due to this $-qR/d$ the total charge on the inner Surface will be, $-qR/d$. (This is basically the method of images).

Now get back to the problem.

(i) Consider, $t < a/v$. i.e. both q_1 and q_2 are outside of the Conducting Shell.



By the previous arguement we can see that due to q_1 and q_2, the charge on Shell (inside) is $-\frac{q_1 R}{a}, -\frac{q_2 R}{b}$ respectively.

So, total charge on inner surface of shell is,

$$Q = -\left(\frac{q_1}{a} + \frac{q_2}{b}\right) R.$$

$$\therefore \frac{dQ}{dt} = -\left(\frac{q_1}{a} + \frac{q_2}{b}\right) v.$$

(ii) If $a/v < t < b/v$. Then q_1 is inside and q_2 is outside. So total charge on shell is,

$$Q = -q_1 - \frac{q_2}{b} R.$$

$$\therefore \frac{dQ}{dt} = -\frac{q_2}{b} v.$$

(iii) If $b/v < t$ then both q_1 and q_2 are inside. So, total charge on shell is,

$$Q = -(q_1 + q_2)$$

$$\therefore \frac{dQ}{dt} = 0.$$