Topology

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Assignment-4

Problem-1

> Let, X be a topological space. $\sim$ be an equivalance relation defined on i. $X/\sim$ be the quotient space, $p : X \to X/\sim$ be the quotient map. Then p is open(closed) iff $\bigcup_{u\in U}[u]$ is open (closed) for every open (closed) set $U \subseteq X$

Solution. If p is the quotient map which is open (closed), then for any open set $U \subseteq X$ we must have $p(U)$ is open (closed). We can see that $p^{-1}(p([U])) = \{[u] : u \in U\} = \bigcup_{u\in U}[u]$ which is open (closed) by defination.

Let, $A = \bigcup_{u\in U}[u]$ be open in X, Where $U \subseteq X$ is open(closed). We can see that $p^{-1}(p(U)) = \{[u] : u \in U\} = \bigcup_{u\in U}[u]$ which is open(closed) in X this is possible iff $p(U)$ is open(closed) in $X/\sim$. This is true foe any open(closed) sets. Hence p is an open(closed) map. ■

Problem-2

> X be regular and $A \subseteq X$ is closed then X/A is Hausdorff.

Solution. Let, $p : X \to X/A$ be the quotient map. Consider $x, y \in X/A$ be the points of the quotient space. If none of $x, y = [A]$ then look at the inverse image of x, y under p. i.e $p^{-1}(x), p^{-1}(y)$ are two distinct points of X. This is because p takes A to a point $[A]$ and other points to itself. (Moreover in Problem (i) of the given homework we have showed that $X \setminus A \cong X/A - \{[A]\}$). Since X is regular space, we can separate $p^{-1}(x), p^{-1}(y)$ by open sets. Let $p^{-1}(x) \in U_x, p^{-1}(y) \in U_y$ such that $U_x \cap U_y = \phi$. Notice that $p^{-1}(p(U_x)) = U_x$ is open and by defination of quotient topology we can say that $x \in p(U_x) = U$ is open in X/A. Similarly we can say $y \in p(U_y) = V$ is open in X/A. Also $U \cap V = \phi$.

Now, Consider that $x = [A]$ and y are two distinct point in X/A. Notice that $p^{-1}([A]) = A$ is closed in X. By the regularity of X we can separate A and y by open sets. By the similar argument as previous can we can get two open set U, V such that $[A] \in U$, $y \in V$ and $U \cap V = \phi$. Hence X is a Hausdorff space. ■