# Topology

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**Assignment-1**

## § Problem 1

A metric space is called **sequentially compact** if every sequence has a convergent subsequence. Call a metric space **totally bounded** if for every $\epsilon > 0$, the metric space can be covered by finitely many open balls of radius $\epsilon$. Prove that TFAE for a metric space $(X, d)$ :

1. $X$ is compact.
2. $X$ has the Bolzano-Weirstrass property, in every infinite set has a limit point in $X$.
3. $X$ is sequentially compact.
4. $X$ is totally bounded and complete.

*Solution.* (Compact $\iff$ sequentially compact) Consider, $X$ is a comapct metric space , also assume $(x_n)$ is a sequence in $X$. Let, $A$ be the set of limit points of the sequence. If $A \cap X = \phi$ then for any $a \in X$, there is $\delta_a > 0$ such that $B(a, \delta_a)$ contains finitely many $x_n$. Notice that,$\{B(a, \delta_a) : a \in X\}$ is an open cover of $X$. So, there is finitely many $a_i \in X$ such that, $\bigcup_{i=1}^n B(a_i, \delta_{a_i})$ is covers $X$ but it contains finitely many of $(x_n) \rightarrow\leftarrow$.

Let, $X$ is sequentially compact also assume that, $\{G_\alpha\}_{\alpha\in\Lambda}$ is an open cover of $X$. Let, $\delta > 0$ be a number such that $B(\delta, x)$ is contained in some of $G_\alpha$, for every $x \in X$. Suppose,$\{G_\alpha\}$ do not have finite subcover. Let $x_0 = x$ (as mentioned above) and consider $B(x, \delta) = B_1$. Clearly this don't cover $X$. Take $x_1 \in X \setminus B_1$. Again take $B_2 = B(x_1, \delta)$. By our assumption $B_1 \cup B_2$ don't cover $X$. So take $x_3 \in X \setminus (B_1 \cup B_2)$.Continue this process. we will get an infinite sequence $(x_n)$ of centers of $B_i$'s. By seqential compactness, a subsequence $(x_{n_i}) \to \tilde{x} \in X$. But $d(x_{n_i}, x_{n_j}) \geq \delta$ for distinct $i$ and $j$. This contradicts the fact that the subsequence is convergent $\rightarrow\leftarrow$. So, Finite number of $B_i$ covers $X$. Each of $B_i$ is contained in some $G_i$ and hence every open cover has a finite subcover.

(sequentially comapct $\iff$ totally bounded and complete) We will start with a lemma.

• Lemma: A metric space is totally bounded if and only if every sequence has a Cauchy subsequence.

*Proof.* ($\Leftarrow$) We may assume $X \neq \emptyset$. Given $\epsilon > 0$, we inductively construct an $\epsilon$-net $\{p_1, \ldots, p_N\}$ as follows. Choose $p_1$ arbitrarily. Having chosen $p_1, \ldots, p_n$, if $\{p_1, \ldots, p_n\}$ is not yet an $\epsilon$-net, let $p_{n+1}$ be a point such that $d(p_m, p_{n+1}) \geq \epsilon$ for each $m = 1, \ldots, n$. We claim that this process must terminate. Indeed, if it didn't, we would obtain a sequence $\{p_n\}_{n=1}^{\infty}$ any two of whose points are at distance $\geq \epsilon$ from each other; but such a sequence clearly has no Cauchy subsequence.

($\Rightarrow$) Let $S_k$ be a $(1/2k)$-net for $k = 1, 2, 3, \ldots$ Given a sequence $\{p_n\}$ in $X$, extract a subsequence $\left\{q_n^{(1)}\right\}$ contained in a radius-$(1/2)$ neighborhood about one of the points in $S_1$. From $\left\{q_n^{(1)}\right\}$ extract a subsequence $\left\{q_n^{(2)}\right\}$ contained in a radius-$(1/4)$ neighborhood about one of the points in $S_2$. Keep going inductively: $\left\{q_n^{(k)}\right\}$ is a subsequence of $\left\{q_n^{(k-1)}\right\}$ contained in a radius- $(1/2k)$ neighborhood about one of the points in $S_k$. This is possible because for each $k$ the sequence $\left\{q_n^{(k-1)}\right\}$ is infinite while $S_k$ is finite. Note that $d\left(q_n^{(k)}, q_{n'}^{(k)}\right) < 1/k$ for all $k, n, n'$. Now consider the diagonal subsequence $\left\{q_k^{(k)}\right\}$. This is a subsequence of the original $\{p_n\}$; moreover, if $m, n \geq N$ then $q_m^{(m)}$ and $q_n^{(n)}$ are both contained in the subsequence $\left\{q_n^{(N)}\right\}$, and thus are at distance $< 1/N$. Thus $\left\{q_k^{(k)}\right\}$ is a Cauchy subsequence of $\{p_n\}$.$\square$

$$\begin{aligned}
X \text{ is totally bounded} &\iff \text{ every sequence has a cauchy subsequence} \\
X \text{ is totally bounded} + \text{ complete} &\iff \text{ every sequence has a cauchy subsequence which converges in } X \\
X \text{ is totally bounded} + \text{ complete} &\iff \text{X is compact sequentially}
\end{aligned}$$

( Bolzano-Weirstrass property $\iff$ X is sequentially compact.) Let, $X$ is sequentially compact. Consider an infinite subset $S$ of $X$ then take a countable subset $S$ call it $S'$. We can treat elements of $S'$ as an elements of sequence $(x_n)$. Since this sequence has a converging subsequence whose limit lies in $X$, there is a limit point of $S$ lies in $X$. So, $X$ satisfies Bolzano-Weirstrass property.

If $X$ satisfies Bolzano-Weirstrass propertythen every sequence of $x$ has a limit point in $X$ so, there is a converging subsequence of it. So,$X$ is sequentially compact.

We have shown $1 \iff 3 \iff 4, 2 \iff 3$.And hence our proof is complete. ■

## § Problem 2

> Given an example to show Heine-Borel theorem fails in metric spaces, ie. a set may be closed and bounded yet fail to be compact.

*Solution.* Consider $\mathbb{R}^2 \setminus \{(0,0)\}$ with the Euclidian metric restricted from $\mathbb{R}^2$. It is not hard to see that the set,

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$$

is closed in $\mathbb{R}^2 \setminus \{(0,0)\}$ because take any point in $D$ , any open disk around that has non trivial intersection with $D$. This set is clearly bounded. Take any sequence Sequences in $\mathbb{R}^2 \cap D$ which converges to $(0,0)$ in $\mathbb{R}^2$ don't have any limit in $D$. ■

## § Problem 3

> Prove that a totally bounded metric space is separable ie. contains a countable dense subset.

*Solution.* Let, $X$ be a totally bounded metric space. Let, $B_i^n$ be the 'open balls' of radius $\frac{1}{n}$ that covers all $X$ for, $1 \le i \le p_n$ and $n \in \mathbb{N}$. Here $p_n$ is the number of open balls required to cover the space. Let, $c_{ij}$ be the center of the ball $B_i^j$. Let us take the collection of all possible centers.

$$\mathcal{C} := \left\{ B_i^j : j \in \mathbb{N} \text{ and } 1 \le i \le p_j \right\}$$

Clearly, $\mathcal{C}$ is countable set. Now, consider any $x \in X$ and any $\epsilon > 0$. By 'archimedean property' there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. We know, $X$ can be covered by $B_i^N$. So, $x \in B_j^N$ for some $j$. Now, $d(x, c_{iN}) \le \frac{1}{N} < \epsilon$. Hence, $\mathcal{C}$ is dense in $X$. ■