Topology

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Assignment-2

§ Problem 4

Consider $X = C([0,1], \mathbb{R})$ with the norm $\|f\| = \int_0^1 |f(x)|dx$ giving the metric $d(f,g) = \|f - g\|$. Define $f_n \in \dot{X}$ by $f_n(x) = 1$ for $x \in [0, 1/2]$, $f_n(x) = 1 - 2^n(x - 1/2)$ for $x \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2^n}\right]$ and $f_n(x) = 0$ for $x \in [1/2 + 1/2^n, 1]$. Prove that (f_n) is a Cauchy sequence in X Which is not convergent.

Solution. Given, $X = \mathcal{C}([0,1], \mathbb{R})$. Set of all Continuous function from $[0,1]$ to $\mathbb{R}$.We are giving this space a norm,

$$\|f\| = \int_0^1 |f| dx$$

Here, The metric is defined as, $d(f,g) = \|f - g\|$. The given function,

$$f_n(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}], \\ 1 - 2^n(x - \dfrac{1}{2}), & \text{if } x \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{2^n}] \\ 0, & \text{otherwise.} \end{cases},$$

Let $m > n$, then,

$$f_m(x) - f_n(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}], \\ (2^n - 2^m)(x - \dfrac{1}{2}), & \text{if } x \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{2^m}], \\ 1 - 2^n(x - \dfrac{1}{2}), & \text{if } x \in [\frac{1}{2} + \frac{1}{2^m}, \frac{1}{2} + \frac{1}{2^n}], \\ 0, & \text{otherwise} \end{cases}$$

Now,

$$\begin{aligned} d\left(f_n, f_m\right) &= \|f_n - f_m\| \\ &= \int_0^1 |f_n - f_m| dx \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2^m}} \left(2^m - 2^n\right)\left(x - \frac{1}{2}\right) dx + \int_{\frac{1}{2}+\frac{1}{2^m}}^{\frac{1}{2}+\frac{1}{2^n}} 1 - 2^n\left(x - \frac{1}{2}\right) dx \\ &= \frac{2^m - 2^n}{2}\left[\frac{1}{2^{2m}}\right] + \left[\frac{1}{2^n} - \frac{1}{2^m}\right] + \frac{2^n}{2}\left[-\frac{1}{2^{2n}} + \frac{1}{2^{2m}}\right] \\ &= \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}} \end{aligned}$$

For every $\epsilon > 0$ we must get $N \in \mathbb{N}$ such that, $\frac{1}{2^{n+1}} < \epsilon$ for all $n > N$. So, $d\left(f_n, f_m\right) < \epsilon$ for all $m, n > N$. Hence the Sequence is a Cauchy sequence.

If we assume, $\mathcal{C}([0,1], \mathbb{R})$ to be Complete metric space, then f_n must converge to a unique limit f, in $X = \mathcal{C}([0,1], \mathbb{R})$. But if we take limit for f_n, we will end up getting $f = \begin{cases} 0 & x\left[0, \frac{1}{2}\right] \\ 1 & x \in \left[\frac{1}{2}, 1\right] \end{cases}$ clearly this is not continuous function. so, $f \notin X$. And hence X is not Complete metric space. ■

§ Problem 13

> Let, X be an arbitrary discrete metric space. Prove that $C(X, R)$ is separable $\Leftrightarrow$ X is finite.

Solution. Let, X is a discrete metric space. Now, Assume $\mathcal{C}(x, \mathbb{R})$ is seperable. For contrary let X is not finite. Every function (that is bounded) in $\mathcal{C}(x, \mathbb{R})$ can be interept as sequence of points,

$$\prod_{\alpha \in \mathrm{X}} \left(x_\alpha\right) \text{ and } x_\alpha \in \mathbb{R}$$

From now we will represent any function $f \in C(X, \mathbb{R})$ as above, $f \equiv \prod_{\alpha \in x}\left(x_\alpha\right); x_\alpha \in \mathbb{R}$.Let, D be a Countable dense set of, $\mathcal{C}(X, \mathbb{R})$.Let,

$$D = \left\{\prod_{\alpha \in x}\left(x_\alpha^i\right) \mid i \in \mathbb{N}\right\}.$$

By defination $D = \mathcal{C}(X, \mathbb{R})$. Now we will prove a lemma.

> **The Sequence Lemma:** Let X be a topological Space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse is true for a metrizable space.

Proof. Suppose, $\{x_n\} \to x; \{x_n\} \subset A$. Then every. neighborhood of x Contains a point of A. So, $x \in \bar{A}$. Conversely let X is metrizable and $x \in \bar{A}$. Let, d be the metre for topology of x. For each $n \in \mathbb{N}$, the ball $B\left(x, \frac{1}{n}\right)$; choose x_n to be a point of $B\left(x, \frac{1}{n}\right) \cap A$. It's not hard to see that, this sequence x_n converges to x. □

Since, X is infinite set. We must get a Coutable subset of X. Let, $X' \subseteq X$ be that set. Let, $\{\beta_1, \beta_2, \ldots\}$ be the elements of X'. Let, D be a dense subset of X. we can write D as,

$$D = \left\{ \prod_{j \in \mathbb{N}} \left(x^i_{\beta_j} \right) \times \prod_{\alpha \in X \setminus X'} \left(x^i_\alpha \right) \mid i \in \mathbb{N} \right\}$$

Since, $\bar{D} = \mathcal{C}(x, \mathbb{R})$. For any $f \in \bar{D}$ We must get a seq. $\{f_n\} \subseteq D$ such that $\{f_n\} \to f$

Now we will assign each function in the subset to a point β_i (the same function might be assigned to multiple points, but each function must be assigned to at least one point). Suppose the function f_i is assigned to β_i.

Note that for all i, $f_i(\beta_i)$ is a distance of at least a from at least one of 0 and $2a$.(Here $a > 0$ is a fixed real).Choose y_i from $\{0, 2a\}$ such that $f_i(\beta_i)$ is a distance of at least a from y_i.

Then define the function $f : X \to \mathbb{R}$ by $f(\beta_i) = y_i$ for all i and $f(x) = 0$ for $x \notin \{\beta_1, \beta_2, \ldots\}$. f is continuous since X is discrete, and f is bounded since it only takes the values 0 and 2. Since $|f(\beta_i) - f_i(\beta_i)| \geq 1, \|f(x_i) - f_i(\beta_i)\| \geq a$ and thus f does not lie in the closure of $\{f_1, f_2, \ldots\}$.

Conversely, If X is finite then $\mathcal{C}(X, R)$ is same as $\mathbb{R}^n$. We know $\mathbb{R}^n$ is seperable so is $\mathcal{C}(X, \mathbb{R})$ ■