Topology

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Assignment-5

§ Question 2

Definition 2.1 ▶ Even action

We say a group G acts on a topological space X **evenly** if for any $x \in X$ has an neighbourhood U such that $\forall h \neq g \in G$,

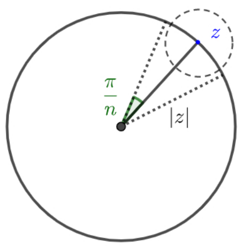
$$gU \cap hU = \phi$$

(i). The group μ_n of all n^{th} roots of unity in $\mathbb{C}$ acts on $\mathbb{C}$ by left multiplication

- Show that this action is not even.
- The same action of μ_n on $\mathbb{C}^\times = \mathbb{C} - \{0\}$ is even.

Solution. If we act μ_n on any neighbourhood, U around 0 we can see that $\{0\} \in g \cdot U$ forall $g \in \mu_n$, which means $\{0\} \in gU \cap hU$ for any neighbourhood U (of 0) and $g \neq h \in \mu_n$.So, this action is not even.

• Consider, z be a point in the complex plane which is not zero. Take a circle of radius $|z|$ centerd at origin. when we are acting μ_n on z, it's rotating along the circle of given radius with some multiple of $\frac{2\pi}{n}$.



Now we will look at the arc of the circle which makes angle $\frac{2\pi}{n}$ at center and keeps z at the middle of the arc. We will construct a circle centerd at z and touching the radii which are $\frac{2\pi}{n}$ angle apart from each other. we can see that radius of such circle wil be $|z| \sin \frac{\pi}{n}$.

Now consider an open disk centerd at z with radius $\frac{|z|}{2} \sin \frac{\pi}{n}$. Call this disk D_z. When we translate D_z by the group action of μ_n , the disk will move in such a way that it's center stay at the circle and the center will move $\frac{2\pi k}{n}$ angle, where $k \in \{0, \cdots, n-1\}$. we can clearly see that $D_z \cap g \cdot D_z = \phi$, $e \neq g \in \mu_n$. Now by symmetry we can say that $g.D_z \cap h.D_z = \phi$ forall $g \neq h \in \mu_n$.

(iii). Let G be the subgroup of the group of all self homeomorphisms of $\mathbb{R}^2$ generated by the translation $(x, y) \mapsto (x + 1, y)$ and the map $(x, y) \mapsto (-x, y + 1)$. Prove that this is an even action of G on $\mathbb{R}^2$. Also show that $\mathbb{R}^2/G$ is the Klein bottle.

Solution. Let, φ_1, φ_2 be the homeomorphisms corresponding $(x, y) \mapsto (x + 1, y)$ and $(x, y) \mapsto (-x, y + 1)$ respectively. The group generated by these two homeomorphisms is given as following,

$$G = \langle \varphi_1, \varphi_2 \rangle$$

We can see that, $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1^{-1}$ and $\varphi_2 \circ \varphi_1 = \varphi_1^{-1} \circ \varphi_2$. So any element in the group G can be written as $\varphi_1^m \circ \varphi_2^n$ for some $m, n \in \mathbb{Z}$. Generators of the group are distance preseving homeomorphisms. So and element of the group is distance preseving homeomorphism. For any point $(x, y) \in \mathbb{R}^2$ take an open disk centerd at that point with diameter $d < 1$. Call this disk $D_{(x,y)}$, we will show, $g(D_{(x,y)}) \cap h(D_{(x,y)}) = \emptyset$. Which means the group action is even.

Let, g is an element in G then $g = \varphi_1^m \circ \varphi_2^n$. So, $g.D_{(x,y)} = \{((-1)^n + u + m, v + n) : (u, v) \in D_{(x,y)}\}$. If there is a point (x', y') the intersection of $D_{(x,y)}$ and $g.D_{(x,y)}$ then distance between (x', y') and $((-1)^n x' + m, y' + n)$ is $< d$. which means ,

$$\sqrt{(((-1)^n - 1)x' + m)^2 + n^2} \leq d < 1$$

since n is an integer we must have $n = 0$ and then $m^2 \leq d < 1$ which means $m = 0$ i.e $g = e$. If g is not identiy then $g(D_{(x,y)}) \cap (D_{(x,y)}) = \emptyset$. We can see that $\varphi_1(x, y), \varphi_2(x, y)$ are atleast 1-unit distance apart from (x, y). By the similar calculation as sbove, for any two distinct element $g, h \in G$ we can say that $g(x, y)$ and $h(x, y)$ are atleast 1-unit apart from each other $\square$

If (x, y) lies in $\mathbb{R}^2$, by applying the homeomorphism φ_1^m for some appropriate integer m to (x, y), we can convert it to a point (a, y) where $a \in [0, 1)$ (this is like taking fractinal part). Then by applying the homeomorphism φ_2^n for some appropriate integer v to (a, y), we get the point $((-1)^n a, b)$ where $b \in [0, 1]$. If v is even, we get a point lying in $[0, 1]^2$ lying in the same equivalence class as (x, y) in $\mathbb{R}^2/G$. Otherwise another application of g gives us such a point lying in $[0, 1)^2$. Moreover no two points in $[0, 1]^2$ lie in the same equivalence class of $\mathbb{R}^2/G$. So $\mathbb{R}^2/G$ can be identified with the space $[0, 1]^2$ with the quotient topology induced.

Consider the unit square $\mathcal{S} = [0, 1] \times [0, 1]$ We can see that any orbit of the given action has a representative on $\mathcal{S}$. If we look at the point interior of the square, they are representative of themself. This is because any $g \in G$ must take a point atleast 1-distance apart from itself by translation. We will look on the boundary of the square where, the points of the form $(0, y)$ are representative with $(1, y)$ (by φ_1) and the points of the form $(x, 1)$ representative with $(1 - x, 0)$ (by $\varphi_1 \circ \varphi_2^{-1}$). We can also see all four vertex belong to same orbit. $(0, y)$ and $(x, 1)$ can't be representative to eachother if $0 < x, y < 1$ this is clearly because the distance in y-coordinate is greater than 0 but less than 1.Similarly we can show $(0, y)$,$(1, y)$ can t be representative with $(x, 0)$ and $(x, 1)$ in any means.

Since, any orbit of the action has a representative in $\mathcal{S}$ we can say that the orbit can be written as $[(x, y)]$. Where $(x, y) \in \mathcal{S}$. Let, $q : \mathbb{R}^2 \to \mathbb{R}^2/G$ be the quotient map.The restriction $q_S = q|_S$ will induce a quotient map to the orbit space on $\mathcal{S}$, which is basically a map send (x, y) to $[(x, y)]$. i.e $q_S : \mathcal{S} \to$ orbit space on $\mathcal{S}$ (this is because q_S is a continuous surjective map from comapact to Hausdorff space. It was shown in class that X/G is Hausdorff when a group G is acting on a Hausdorff topological space X.). Let, $\pi : \mathcal{S} \to K$ be the odentification map from $\mathcal{S}$ to klein bottle K. Define a map ϕ from the orbit space to K as each orbit $[(x, y)] \mapsto \overline{(x, y)}$. Here $\overline{(x, y)}$ is the point in K correspopnding to the point in (x, y).

$$\begin{tikzcd} \mathcal{S} \arrow[d, "\pi"'] \arrow[dr, "q_s"] & \\ K & \text{orbit space on } \mathcal{S} \arrow[l, "\psi"] \end{tikzcd}$$

For the ψ we have constructed the above diagramm commutes. Now, It's not hard to see that ψ is one one and onto i.e. bijective. Here, $\psi \circ q_s = \pi$ since q_s is continuous and π is quotient map we can say ψ is a continuous (**done in class**) and bijective map. Since q_s is also a quotient map from comapact set, orbit space on $\mathcal{S}$ is also comapact and we know Klein bottle is a surface without boundary (**discussed in class**) it is Hausdorff. So, by closed map lemma (**done in class**) a continuous bijective map from a comapact space to Hausdorff space is homeomorphism. Hence, orbit space on $\mathcal{S}$ which is equal to orbit space on $\mathbb{R}^2$ is homeomorphic to Klein bottle K.

(**iv**). Let G be a finite group action fixed point freely on a Hausdorff space Y, ie. $g \cdot y = y$ for some $g \in G, y \in Y \Rightarrow g = e$. Prove that such an action is even.

Solution. Let, $\{g_0(=1), g_1, g_2, \cdots, g_n\}$ are element of G. Now consider $\{x, g_1.x, \cdots, g_n.x\}$ where $x \in Y$, all of these are distinct as the action is fixed point free. Since there are finitely many point in the above set we can separate them by disjoint open sets(**This was done in class**)**.i.e there are open sets $U_0, \cdots U_n$ with $g_i.x \in U_i$ and $g_j.x \in U_j$ and $U_i \cap U_j$ for $i \neq j$.

Consider $W_i = U_0 \cap g_i^{-1} U_i$ clearly this is open set since thi is finite intersection of opensets and $g^{-1}U_i$ open as the map $x \mapsto g.x$ is continuous. So, W_i are open neighbourhood of x. Define, $U = \cap_{i=1}^{n} W_i$. This is also finite intersection of opensets which means it is open. We have $g_i U \subset U_i$. So, $g_i U \cap U = \emptyset$. And hence the action is free. ■

** **Lemma :** There exist pairwise disjoint open neighbourhoods of $g_1 \cdot y, \ldots, g_n.\ y$.
Proof. Induction on n. For $n = 1, 2$, the statement is clear. Now suppose $n > 2$, and we have disjoint open neighbourhoods $U_1', \ldots, U_{n-1}'$ of $g_1 \cdot y, \ldots, g_{n-1} \cdot y$ respectively. Then since Y is Hausdorff, and $g_1 \cdot y, \ldots, g_n \cdot y$ are all distinct points, there exist open sets $V_1', \ldots, V_{n-1}'$ and open sets $W_1', \ldots, W_{n-1}'$ such that $g_i \cdot y \in V_i'$ for all $1 \leq i \leq n-1$ and $g_n \cdot y \in W_i'$ for all i, and $V_i' \cap W_i' = \emptyset$ for all $1 \leq i \leq n-1$. Then $U_1' \cap V_1', \ldots, U_{n-1}' \cap V_{n-1}', \bigcap_{i=1}^{n-1} W_i'$ are disjoint open neighbourhoods of $g_1 \cdot y, \ldots, g_{n-1} \cdot y, g_n \cdot y$ respectively. □

(**v**). We discussed an action of μ_n on the complex sphere

$$S_{\mathbb{C}}^{m-1} := \Big\{ (z_1, \cdots, z_m) : |z_1|^2 + \cdots + |z_m|^2 = 1 \Big\}$$

$\zeta \cdot (z_1, \ldots, z_m) := (\zeta z_1, \ldots, \zeta z_m)$. Prove this is an even action.

Solution. Let $(z_1, \ldots, z_m)$ be an arbitrary point in $S_{\mathbb{C}}^{m-1}$. Then at least one of $z_1, \ldots, z_m$ is not zero; assume W.L.O.G. that $z_1 \neq 0$. Then by part (i) of this assignment, there exists an open neighbourhood U of z_1 such that $g, h \in \mu_n$ and $g \neq h$ implies that $g(U) \cap h(U) = \emptyset$. Then $U \times \prod_{i=1}^{n-1} \mathbb{C}$ is an open neighbourhood of $(z_1, \ldots, z_m)$ in $\mathbb{C}^{m-1}$. But then $\left(U \times \prod_{i=1}^{n-1} \mathbb{C}\right) \cap S_{\mathbb{C}}^{m-1}$ is an open neighbourhood of $(z_1, \ldots, z_m)$ in $S_{\mathbb{C}}^{m-1}$, and $g, h \in \mu_n$ and $g \neq h$ implies that

$$g_i \left(\left(U \times \prod_{i=1}^{n-1} \mathbb{C} \right) \cap S_{\mathbb{C}}^{m-1} \right) \cap g_j \left(\left(U \times \prod_{i=1}^{n-1} \mathbb{C} \right) \cap S_{\mathbb{C}}^{m-1} \right) = \emptyset$$

And hence We are done. ∎