Assignment - 7

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Problem 1.

We know,

$$R^2 = \frac{SS_{reg}}{SST}$$

$$= \frac{SS_{reg}}{RSS + SS_{reg}}$$

$$= \frac{MS_{reg} \times (r-1)}{MSE \times (n-r) + (r-1) MS_{reg}}$$

$$= \frac{\frac{(r-1)}{(n-r)} F_{reg}}{1 + \left(\frac{r-1}{n-r}\right) F_{reg}}.$$

$F_{reg} \sim F_{r-1, n-r}$ under $H_0: \beta_1 = \cdots = \beta_{p-1} = 0$.

Now, $F_{r-1, n-r} \sim \dfrac{\chi^2_{r-1}/r-1}{\chi^2_{n-r}/n-r}$.

$$\therefore R^2 \sim \frac{\chi^2_{r-1}}{\chi^2_{n-r} + \chi^2_{r-1}}.$$

We know, $\chi^2_{r-1} \sim \text{Gamma}\left(\frac{r-1}{2}, \frac{1}{2}\right)$

$\chi^2_{n-r} \sim \text{Gamma}\left(\frac{n-r}{2}, \frac{1}{2}\right)$.

So, $R^2 \sim \text{Beta}\left(\frac{r-1}{2}, \frac{n-r}{2}\right)$. $\boxtimes$

Problem 2.

If, $X \sim N(\mu, \sigma^2)$ then,

$$\Phi\left(\frac{X-\mu}{\sigma}\right) \sim Unif(0,1)$$

Let, $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$ and I.I.D.

Now, $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$.

Let, $Y_i = \Phi\left(\frac{X_i-\mu}{\sigma}\right)$

Clearly, $Y_1, \ldots, Y_n \sim Unif(0,1)$.

Since, $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ we must have,

$$Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)}.$$

So, $\mathbb{E}\left(\Phi\left(\frac{X_{(i)}-\mu}{\sigma}\right)\right) = \mathbb{E}(Y_{(i)})$

Now, $f_{Y_{(i)}}(x) = \frac{n!}{(i-1)!\,(n-i)!}(1-x)^{n-i}x^{i-1}$

$$\mathbb{E}\left(\Phi\left(\frac{x_{(i)}-\mu}{\sigma}\right)\right) = \int_0^1 \frac{n!}{(i-1)!\,(n-i)!}(1-x)^{n-i}x^{i}\,dx$$

$$= \frac{n!}{(i-1)!\,(n-i)!} \cdot \frac{i!}{(n+1)!\,(n-i)!}$$

$$= \frac{i}{n+1}$$

$\blacksquare$

Problem 3.

Given $(X_1, \dots, X_n)^T \sim N\left((\mu_1, \dots, \mu_n)^T, \Sigma\right)$.

For Normal distribution,
We know,

$$\begin{pmatrix} X_i \\ X_j \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_i \\ \mu_j \end{pmatrix}, \begin{pmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ji} & \sigma_{jj} \end{pmatrix}\right)$$

$$\begin{pmatrix} X_k \\ X_l \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_k \\ \mu_l \end{pmatrix}, \begin{pmatrix} \sigma_{kk} & \sigma_{kl} \\ \sigma_{lk} & \sigma_{kk} \end{pmatrix}\right).$$

Now, $\mathrm{Var}\left((X_i, X_j)^t \mid (X_k, X_l)^t\right)$

$$= \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}'$$

Where, $\Sigma_{11} = \begin{pmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ji} & \sigma_{jj} \end{pmatrix}$

$$\Sigma_{12} = \begin{pmatrix} \sigma_{ik} & \sigma_{il} \\ \sigma_{jk} & \sigma_{jl} \end{pmatrix}$$

$$\Sigma_{22} = \begin{pmatrix} \sigma_{kk} & \sigma_{kl} \\ \sigma_{lk} & \sigma_{ll} \end{pmatrix}.$$

Now, $\Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}'$

$$= \frac{1}{\sigma_{kk}\sigma_{ll} - \sigma_{lk}^2} \begin{pmatrix} \sigma_{ik} & \sigma_{il} \\ \sigma_{jk} & \sigma_{jl} \end{pmatrix} \begin{pmatrix} \sigma_{ll} & -\sigma_{kl} \\ -\sigma_{kl} & \sigma_{ll} \end{pmatrix} \begin{pmatrix} \sigma_{ik} & \sigma_{jk} \\ \sigma_{il} & \sigma_{jl} \end{pmatrix}.$$

From this matrix only we can get, $\sigma_{ij\cdot kl}$.

$$\sigma_{ij\cdot kl} = \sigma_{ij} - \frac{1}{\sigma_{kk}\sigma_{ll} - \sigma_{kl}}\left[\sigma_{ik}\sigma_{jk}\sigma_{ll} - (\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk})\sigma_{lk} + \sigma_{il}\sigma_{jl}\sigma_{kk}\right]$$

$$= \left(\sigma_{ij} - \frac{\sigma_{ik}\sigma_{jk}}{\sigma_{kk}}\right) - \frac{\left(\sigma_{il} - \frac{\sigma_{ik}\sigma_{kl}}{\sigma_{kk}}\right)\left(\sigma_{jk} - \frac{\sigma_{jk}\sigma_{kl}}{\sigma_{ll}}\right)}{\left(\sigma_{kk} - \frac{\sigma_{kl}^2}{\sigma_{ll}}\right)}.$$

$$= \sigma_{ij\cdot k} - \frac{\sigma_{il\cdot k}\,\sigma_{jl\cdot k}}{\sigma_{ll\cdot k}}$$

$$\rho_{ij\cdot kl} = \frac{\sigma_{ij\cdot kl}}{\sqrt{\sigma_{ii\cdot kl}}\sqrt{\sigma_{jj\cdot kl}}}$$

$$= \frac{\sigma_{ij\cdot k} - \frac{\sigma_{il\cdot k}\,\sigma_{jl\cdot k}}{\sigma_{ll\cdot k}}}{\sqrt{\left(\sigma_{ii\cdot k} - \frac{\sigma_{il\cdot k}\,\sigma_{il\cdot k}}{\sigma_{ll\cdot k}}\right)\left(\sigma_{jj\cdot k} - \frac{\sigma_{jl\cdot k}^2}{\sigma_{ll\cdot k}}\right)}}$$

$$= \frac{\frac{\sigma_{ij\cdot k}}{\sqrt{\sigma_{ii\cdot k}\,\sigma_{jj\cdot k}}} - \frac{\sigma_{il\cdot k}\,\sigma_{jl\cdot k}}{\sigma_{ll}\sqrt{\sigma_{ii\cdot k}\,\sigma_{jj\cdot k}}}}{\sqrt{\left(1 - \frac{\sigma_{il\cdot k}^2}{\sigma_{ll\cdot k}\,\sigma_{ii\cdot k}}\right)\left(1 - \frac{\sigma_{jl}^2}{\sigma_{jj}\,\sigma_{ll}}\right)}}$$

$$= \frac{\rho_{ij\cdot k} - \rho_{il\cdot k}\,\rho_{jl\cdot k}}{\sqrt{(1-\rho_{il}^2)(1-\rho_{jl}^2)}}$$