Electrodynamics

Assignment-III

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A.Z

4.1 (a) For this problem we will consider Cylindrical Co-ordinate system. Without Loss of generelity let, Ring is on the plane, $z=0$.

In this case charge density is given by,

$$\rho(r,\phi,z) = \lambda(\phi)\,\delta(r-R)\,\delta(z)$$

Assuming R is radius of the Ring. So the dipole moment is given by,

$$\vec{p} = \int \rho(\vec{r})\,\vec{r}\,d^3r$$

$$= \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dr \int_0^{2\pi} \lambda_0 \cos\phi\,(r\cos\phi, r\sin\phi, z)\, r\,\delta(r-R)\delta(z)\,d\phi$$

$$= \int_0^{2\pi} \lambda_0 R^2 \cos\phi\,(\cos\phi, \sin\phi, 0)\,d\phi$$

$$= \lambda_0 R^2 \pi\,\hat{x}$$

$$\boxed{\vec{p} = \lambda_0 \pi R^2\,\hat{x}}$$

(b) For this part we can carryout the same calculation. But in this case consider Spherical Co-ordinate system.

Charge Density is,

$$\rho(r,\theta,\phi) = \sigma_0 \cos\theta\,\delta(r-R)$$

The dipole moment is given by,

$$\vec{p} = \int \rho(\vec{r})\,\vec{r}\,d^3r$$

$$= \sigma_0 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta \int_0^{\infty} r^2 \cos\theta\,(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\,\delta(r-R)\,dr$$

$$= 2\pi R^3 \sigma_0 \int_0^{\pi} \sin\theta\cos^2\theta\,d\theta\;\hat{z}$$

$$= \frac{4\pi R^3 \sigma_0}{3}\,\hat{z}$$

$$\boxed{\vec{p} = \frac{4\pi}{3} R^3 \sigma_0\,\hat{z}}$$

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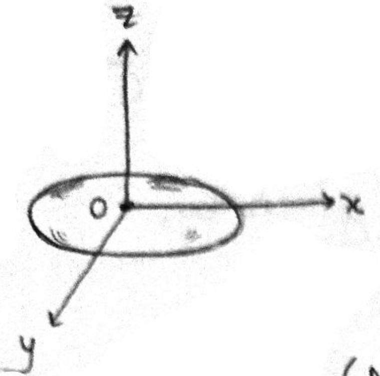
(a)



A disk of radius R is given. Since, Q is the total Charge, and it's surface charge density is σ.

Clearly, this case has azimuthal symmetry. We will use azimuthal symmetric spherical expansion. (Mainly Exterior expansion)

So, $$\varphi(r,\theta) = \sum_{l=0}^{\infty} \frac{A_l}{4\pi\epsilon_0 r^{l+1}} P_l(\cos\theta)$$ wher,

$$A_l = \int d^3r' \, r'^l \rho(r',\theta') P_l(\cos\theta')$$

In this case charge density is,

$$\rho(r',\theta') = \sigma \, \delta(\theta' - \pi/2) \frac{\Theta(R-r')}{r'}.$$

So, $$A_l = \sigma \int_0^{\pi} P_l(\cos\theta')\delta(\pi/2 - \theta') \sin\theta' d\theta' \int_0^{2\pi} d\phi' \int_0^R r'^{l+1} dr'$$

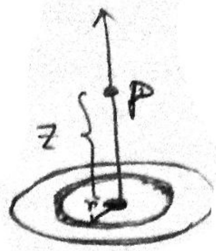
$$= \frac{\sigma (2\pi) R^{l+2}}{l+2} P_l(0)$$

$$= \frac{2Q R^l}{l+2} P_l(0) \left[\text{As } \sigma = \frac{Q}{\pi R^2}\right]$$

$$\therefore \varphi(r,\theta) = \sum_{l=0}^{\infty} \frac{2Q}{4\pi\epsilon_0} \left(\frac{R}{r}\right)^l \frac{P_l(0)}{r(l+2)} P_l(\cos\theta)$$

$$= \frac{Q}{4\pi\epsilon_0 r} \sum_{l=0}^{\infty} \left(\frac{R}{r}\right)^l \frac{2}{l+2} P_l(0) P_l(\cos\theta)$$

4.22 (b) $dq = \sigma(2\pi r)dr$



At point P, potential due to disk is,

$$\varphi(z,0) = \int_0^R \frac{2\cancel{\pi}\sigma}{4\cancel{\pi}\epsilon_0} \frac{r\,dr}{\sqrt{r^2+z^2}}$$

$$= \frac{\sigma}{2\epsilon_0} \int_0^R \frac{r\,dr}{\sqrt{r^2+z^2}}$$

$$= \frac{\sigma}{2\epsilon_0} \left[\sqrt{R^2+z^2} - z\right]$$

$$= \frac{\sigma z}{2\epsilon_0} \left[\sqrt{1+x^2} - 1\right] \quad \left[\text{Here } x = R/z\right]$$

$$= \frac{\sigma z}{2\epsilon_0} \int_0^x \frac{t}{\sqrt{1+t^2}} dt$$

$$= \frac{\sigma z}{2\epsilon_0} \int_0^x \sum_{l=0}^{\infty} P_l(0)\, t^{l+1}\, dt$$

$$= \frac{\sigma z}{2\epsilon_0} \sum_{l=0}^{\infty} P_l(0) \frac{x^{l+2}}{l+2}$$

$$= \frac{Q}{4\pi\epsilon_0} \frac{1}{z} \sum_{l=0}^{\infty} \left(\frac{R}{z}\right)^l \frac{2}{l+2} P_l(0) P_l(1) \quad \left[\text{As, } P_l(1)=1\right]$$

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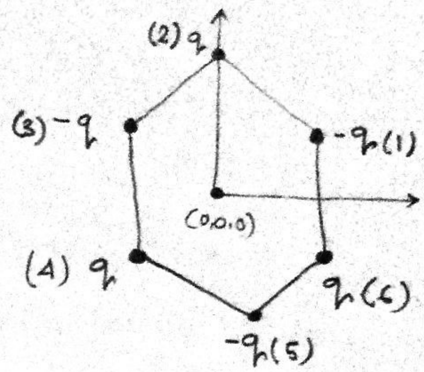

In this case, total chare, $Q = 0$.

Dipole moment,

$$\vec{p} = \int \rho(r)\vec{r}\, d^3r$$
$$= \sum_{i=1}^{6} q_i \vec{r_i}.$$

Here q_i is the point charges with co-ordinate $\vec{r_i}$.

Since all six points lies on vertices of a hexagone $\sum_{k=1}^{3} q_{2k-1}\vec{r}_{2k-1} = 0$ as ($\vec{r_1} + \vec{r_3} + \vec{r_4}$ is the Centroied of the equilateral triangle). By similer cal-culation, $\sum_{k=1}^{3} q_{2k}\vec{r}_{2k} = 0$. And Hence $\vec{p} = 0$.

Quadrupole moment,

$$Q_{ij} = \frac{1}{2}\int \rho(r)\, r^{(i)} r^{(j)}\, d^3r$$
$$= \frac{1}{2}\sum_{\alpha=1}^{6} q_\alpha r_\alpha^{(i)} r_\alpha^{(j)}$$

[Here, $r_\alpha^{(i)}$ and $r_\alpha^{(j)}$ are i^{th}, j^{th} co-ordinate of r_α]

Now, $Q_{ii} = \frac{1}{2}\sum_{\alpha=1}^{6} r_\alpha^{(i)2} q_\alpha$; $Q_{ij} = \frac{1}{2}\sum_{\alpha=1}^{6} r_\alpha^{(i)} r_\alpha^{(j)} q_\alpha$

$$Q_{11} = \frac{1}{2}\left(-\frac{3}{4} + \frac{3}{4} + \frac{3}{4} - \frac{3}{4}\right)a^2 q = 0.$$
$$Q_{22} = \frac{1}{2}\left(1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - 1\right)a^2 q = 0$$
$$Q_{31} = Q_{13} = Q_{23} = Q_{32} = 0$$
$$Q_{12} = Q_{21} = \frac{1}{2}\left[-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right]a^2 q = 0.$$

So the Quadrupole moment is also Zero.

Octapole moment,

$$O_{ijk} = \frac{1}{6}\int \rho(r)\, r^{(i)} r^{(j)} r^{(k)}\, d^3r$$
$$= \frac{1}{6}\sum_{\alpha=1}^{6} q_\alpha r_\alpha^{(i)} r_\alpha^{(j)} r_\alpha^{(k)}$$

Now, $O_{222} = \frac{1}{6}\sum_{\alpha=1}^{6} q_\alpha r_\alpha^{3(2)} = \frac{a^3 q}{4} \neq 0$ Hence Octapole moment is first non-zeromoment.

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(b) From expansion of $\frac{1}{|r-r'|}$ (mainly exterior expansion)

$$\varphi(r) = \left(\int \frac{\rho(r')}{|r-r'|} d^3r' \right) (4\pi\epsilon_0)^{-1}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \mathcal{O}_{ijk} \, f_{ijk}(r) + \cdots$$

$$f_{ijk}(r) \propto \frac{1}{r^4}.$$

And hence for $r \gg r'$; $\varphi \propto \frac{1}{r^4}$.

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(a) Exterior expansion of potential,

$$\varphi(r) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} \frac{Y_{lm}(\Omega)}{r^{l+1}}$$

At the boundary of the sphere potential is,

$$\varphi(R,\Omega) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{A_{lm}}{R^{l+1}} Y_{lm}(\Omega).$$

From orthogonality of $Y_{lm}(\Omega)$ we can say,

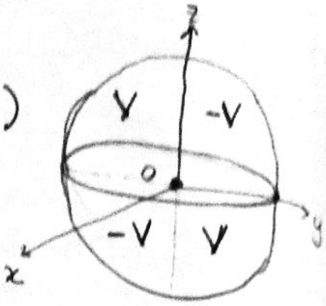
$$\frac{1}{4\pi\epsilon_0} \frac{A_{lm}}{R^{l+1}} = \int \varphi(R,\hat{\Omega}) Y^*_{lm}(\hat{\Omega})\, d\hat{\Omega}$$

$$\text{So, } \varphi(r,\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm} \left(\frac{R}{r}\right)^{l+1} \int \varphi(R,\hat{\Omega}) Y^*_{lm}(\hat{\Omega})\, d\hat{\Omega}.$$

(b)

Recall,

$$\varphi(R,\Omega) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{A_{lm}}{R^{l+1}} Y_{lm}(\Omega)$$



If we fix θ then changing ϕ by integral multiple of $\pi/2$, $\varphi(R,\Omega)$ changes sign.

$$(*) \quad Y_{lm}(\Omega) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \, P_l^m(\cos\theta)\, e^{im\phi}$$

(*) Is described in "Classical Electrodynamics" – Jackson.
$P_l^{(m)}(x)$ is generalized Legendre polynomial.

$$\varphi(R,\theta,\phi) = -\varphi(R,\theta,\phi+\pi/2).$$

$$\Rightarrow e^{im\phi} = -e^{im\phi + m\pi i/2}$$

$$\Rightarrow e^{m\pi/2\, i} = -1$$

$$\Rightarrow m \in 4\mathbb{Z}+2.$$

So, $|m| \geq 2$ and hence, $l \geq 2$. For potential we will look on to least possible value of l. If $l=2$, Then,

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta\, e^{2i\phi}$$

If we fix ϕ then changing θ by integral multiple of $\pi/2$ will change sign of $\varphi(R,\Omega)$. So, $A_{22}=0$ as, $\sin^2\theta$ is always +ve.

We will look onto $l=3$, and Y_{32}, Y_{3-2} will contribute.

Recall,

$$\varphi(r,\Omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}\left(\frac{R}{r}\right)^{l+1} \int \varphi(R,\tilde{\Omega})\, Y_{lm}^*(\tilde{\Omega})\, d\tilde{\Omega}.$$

Notice,

$$Y_{32} = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \sin^2\theta\cos\theta\, e^{2i\phi}$$

$$Y_{3-2} = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \sin^2\theta\cos\theta\, e^{-2i\phi}$$

We will compute

$$\int \varphi(R,\tilde{\Omega})\, Y^*_{32}(\tilde{\Omega})\, d\tilde{\Omega}$$

$$= k\int_0^{\pi} \sin^3\theta' \cos\theta' d\theta' \int_0^{2\pi} e^{-2i\phi'} \varphi(R,\tilde{\Omega})\, d\phi .$$

Here, $k = \frac{1}{4}\sqrt{\frac{105}{2\pi}}$.

Now,

$$\int_0^{\pi} \sin^3\theta' \cos\theta' d\theta \int_0^{2\pi} e^{-2i\phi'} \varphi(R,\tilde{\Omega})\, d\phi'$$

$$= \sum_{n=0}^{1}\sum_{m=0}^{3} V\,(-1)^{m+n} \int_{\frac{n\pi}{2}}^{\frac{n+1}{2}\pi} \int_{\frac{m\pi}{2}}^{\frac{m+1}{2}\pi} \sin^3\theta' \cos\theta d\theta' \, e^{-2i\phi'} d\phi$$

$$= V\sum_{n=0}^{1}\sum_{m=0}^{3} (-1)^{m+n} \frac{\left[\cos^4\left(\frac{n\pi}{2}\right) - \sin^4\left(\frac{n\pi}{2}\right)\right]}{4} \frac{e^{-m\pi i}}{i}$$

$$= \frac{V}{4i}\sum_{m=0}^{3}\left[(-1)^{m+0} + (-1)^{m+2}\right] e^{-m\pi i}$$

$$= \frac{V}{2i}\sum_{m=0}^{3} (-1)^m e^{-m\pi i}$$

$$= \frac{V}{2i}\sum_{m=0}^{3} (-1)^{2m}$$

$$= \frac{4V}{2i} = \frac{2V}{i}$$

Now, $\int \varphi(R,\tilde{\Omega})\, Y^*_{3-2}(\Omega)\, d\hat{\Omega}$

$= \frac{k}{i}(-2V)$. (By similar calculation; also by taking conjugate).

So, $\varphi(r) \approx \frac{2kVR^4}{r^4} \frac{(Y_{32} - Y_{3-2})}{i} = \frac{4kVR^4}{r^4} \sin^2\theta \cos\theta \sin 2\phi$

So, $\varphi(r) \simeq \frac{105}{2\pi} R^4 V \frac{\sin^2\theta \cos\theta \sin 2\phi}{r^4}.$

Taking $r \to \infty$ must give us $\varphi(r) \longrightarrow 0$.

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4.15

(a) The charge density,

$$\rho(x,y,z) = q\delta(z)\{\delta(x-a)\delta(y-a)+\delta(x+a)\delta(y+a)\}$$
$$-q\delta(z)\{\delta(x-a)\delta(y+a)+\delta(x-a)\delta(y+a)\}$$

$$Q_{ij} = \frac{1}{2}\int d^3r\, r_i r_j\, \rho(\vec{r})$$

$Q_{xx}=0,\quad Q_{yy}=0,\quad Q_{zz}=0$

$Q_{xz}=Q_{zx}=Q_{yz}=Q_{zy}=0$ and,

$$Q_{xy}=Q_{yx}=\frac{q}{2}(a^2+a^2+a^2+a^2)=2qa^2$$

$$Q = 2qa^2\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\varphi(\vec{r}) = \frac{Q_{ij}}{4\pi\epsilon_0}\frac{3r_ir_j-\delta_{ij}r^2}{r^5} = \frac{12qa^2}{4\pi\epsilon_0}\frac{xy}{(x^2+y^2)^{5/2}}$$

$$= \frac{3qa^2}{\pi\epsilon_0}\frac{xy}{(x^2+y^2)^{5/2}}$$

(b) $\overleftrightarrow{Q} = \hat{i}\,Q_{ij}\,\hat{j} = 2qa^2(\hat{x}\hat{y}+\hat{y}\hat{x})$.

(c) We can write,

$$\hat{x} = \sin\theta\cos\phi\,\hat{r}+\cos\theta\cos\phi\,\hat{\theta}-\sin\phi\,\hat{\phi}$$
$$\hat{y} = \sin\theta\sin\phi\,\hat{r}+\cos\theta\sin\phi\,\hat{\theta}+\cos\phi\,\hat{\phi}$$
$$\hat{z} = \cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}.$$

Now put this in eq^n (b) to get,

$Q_{rr} = 4qa^2\sin^2\theta\cos\phi\sin\phi$, $Q_{\theta\theta} = 4qa^2\cos^2\theta\cos\phi\sin\phi$.

$Q_{\phi\phi} = -4qa^2\sin\phi\cos\phi$, $Q_{r\theta}=Q_{\theta r} = 4qa^2\sin\theta\cos\theta\sin\phi\cos\phi$.

$Q_{r\phi}=Q_{\phi r} = 2qa^2\sin\theta(\cos^2\phi-\sin^2\phi)$

$Q_{\theta\phi}=Q_{\phi\theta} = 2qa^2\cos\theta(\cos^2\phi-\sin^2\phi)$

(d) In Spherical Co-ordinate, all term involving r_θ and r_ϕ will vanish,

$$\varphi_Q(\vec{r}) = Q_{ij}\,\frac{3r_ir_j - r^2\delta_{ij}}{r^5}$$

$$= \sum_{i=1}^{3} Q_{ii}\,\frac{3r_i^2 - r^2}{r^5} + \underbrace{\sum_{i\neq j} Q_{ij}\,\frac{3r_ir_j - r^2\delta_{ij}}{r^5}}_{0}$$

0 [as, $r_ir_j = 0$, $\delta_{ij} = 0$]

$$= Q_{rr}\,\frac{2r^2}{r^5} - \frac{Q_{\theta\theta}r^2}{r^5} - Q_{\phi\phi}\frac{r^2}{r^5}$$

$$= \frac{2Q_{rr} - Q_{\theta\theta} - Q_{\phi\phi}}{r^3}.$$

$$= \frac{8qa^2\sin^2\theta\cos\phi\sin\phi + 4qa^2\sin\phi\cos\phi\sin^2\theta}{r^3}$$

$$= \frac{12qa^2}{r^3}\sin^2\theta\cos\phi\sin\phi.$$

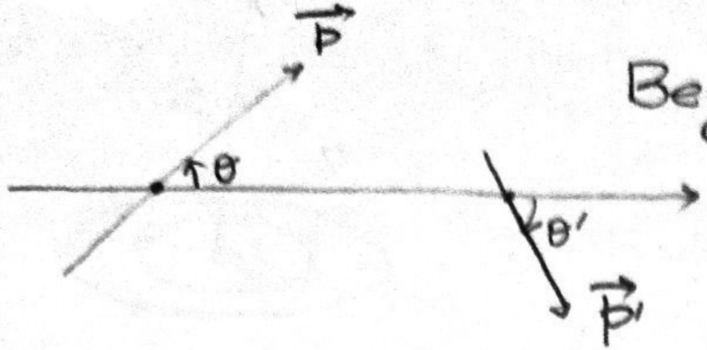
(e) If we put, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$.
Then, we will get,

$$\varphi_Q(\vec{r}) = 12qa^2\,\frac{xy}{(x^2+y^2)^{5/2}}$$

This is same as we computed in part (I)

(f) The expression of Q_{ij}; we derived from Cartesian expansion of $\frac{1}{|r-r'|}$. We can't use that formula for calculating $\{Q\}$. in Spherical Co-ordinate.

4.8



Begin with Polar Co-ordinate System.
electric field due to dipole is given by.

$$\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

The potential energy of $\vec{p}'$ due to $\vec{p}$ is,

$$V = -\vec{p}'\cdot\vec{E}$$

$$= \frac{-pp'}{4\pi\epsilon_0 r^3}\left[\sin\theta'\sin\theta - 2\cos\theta\cos\theta'\right]$$

Now, $\frac{\partial V}{\partial\theta'} = 0$ In order to minimize V. So,

$$2\sin\theta'\cos\theta = \cos\theta'\sin\theta$$

$$\boxed{\tan\theta = 2\tan\theta'}$$

Checking of θ' gives minimum.

$$\frac{\partial^2 V}{\partial \theta^2} = \frac{p p'}{4\pi\epsilon_0 r^3}\left[\sin\theta \sin\theta' + 2\cos\theta \cos\theta'\right]$$

$$\cos\theta'\left[\sin\theta \tan\theta' + 2\cos\theta\right]$$

$$= \cos\theta'\left[\frac{\sin\theta \tan\theta}{2} + 2\cos\theta\right]$$

$$= \cos\theta' \frac{\left[\sin^2\theta + 4\cos^2\theta\right]}{2\cos\theta}$$

$$= (2\cos\theta)^{-1} \frac{\left[\sin^2\theta + 4\cos^2\theta\right]}{\sqrt{\sin^2\theta + 4\cos^2\theta}} \cos\theta.$$

$$= \frac{1}{2}\sqrt{\sin^2\theta + 4\cos^2\theta}. > 0.$$