INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2022-23
Statistics - III, Semesteral Examination, November 23, 2022
Time: $2\frac{1}{2}$ Hours **Total Marks: 50**

1. Suppose $\mathbf{Y} \sim N_p(\mu, \mathbf{\Sigma})$ where $\Sigma = \sigma^2\,(I_p + \rho\mathbf{1}\mathbf{1}')$, $0 < \rho < 1$.
(a) Show that $\sigma\,(I_p + \alpha\mathbf{1}\mathbf{1}') = \Sigma^{1/2}$ if $\alpha = (\sqrt{1+p\rho} - 1)/p$.
(b) Find the probability distribution of $\mathbf{Z} = \frac{1}{\sigma}\left(I_p - \frac{\alpha}{1+p\alpha}\mathbf{1}\mathbf{1}'\right)(\mathbf{Y} - \mu)$.
(c) Show that $\mathbf{Z}'\mathbf{Z} \sim \chi^2_r$. Find r.
(d) Find the partial correlation coefficient, $\rho_{13.2}$, between Y_1 and Y_3 given Y_2 ($\mathbf{Y} = (Y_1, Y_2, \ldots, Y_p)'$). [3+4+2+3]

2. Consider the model $\mathbf{Y} = \mathbf{X}\beta + \epsilon$, where $\mathbf{X}_{n\times p}$ has $\mathbf{1}$ as its first column and may not have full column rank; also $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$. Let $\hat\beta = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Y}$ and $RSS = (\mathbf{Y} - \mathbf{X}\hat\beta)'(\mathbf{Y} - \mathbf{X}\hat\beta)$, where $(\mathbf{X}'\mathbf{X})^{-}$ is any generalized inverse of $(\mathbf{X}'\mathbf{X})$.
(a) Find the joint distribution of $(\mathbf{1}'\mathbf{Y}, RSS)$.
(b) Suppose $p = 2$. When do we have that $\hat\beta_0$ and $\hat\beta_1$ are independently distributed? [6+6]

3. Consider the one-way model:

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij},\ 1 \le j \le 10;\ 1 \le i \le 4,$$

where ϵ_{ij} are i.i.d. $N(0, \sigma^2)$, with the standard identifiability constraints on α_i.
(a) Show that $\alpha_1 - \alpha_2$ is estimable.
(b) What is the Bonferroni inequality used for multiple comparisons?
(c) Construct a $100(1-\alpha)\%$ simultaneous confidence set for $(\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_4)$. [3+3+6]

4. Consider testing the hypothesis $H_0 : \beta_1 = \beta_2 = \cdots = \beta_{p-1} = 0$ under the model $\mathbf{Y}_{n\times 1} = X_{n\times p}\beta_{p\times 1} + \epsilon_{n\times 1}$, where $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$.
(a) Define the F-ratio statistic to test H_0. Find its expected value when H_0 is true, and again when H_0 is false. Explain why the expected value is larger when H_0 is false.
(b) Define the coefficient of determination, R^2. Find its probability distribution when H_0 is true, and show that it is a standard distribution. [9+5]

INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2021-22
Statistics - III, Semesteral Examination, January 5, 2022
Marks are shown in square brackets. **Total Marks: 50**
Time: $2\frac{1}{2}$ Hours; submission must be complete by 1 pm
e-mail: mohan.delampady@gmail.com

You may freely consult the lecture notes, but no other books or resources may be consulted. You may use any of the results stated and discussed in the lecture notes, by stating them explicitly. Results from the assignments may not be used without establishing them. Calculators may be used.

1. Let $Z_i, 1 \leq i \leq n$, $n \geq 5$, be independent $N(0, \sigma^2)$ random variables. Define $Y_1 = Z_1$, $Y_2 = Y_1 + Z_2$, $Y_3 = Z_3$, $Y_4 = Y_3 + Z_4$, and $Y_j = Y_{j-1} + Z_j$ for $5 \leq j \leq n$. Let $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4, \ldots, Y_n)'$.
(a) Find the probability distribution of $\mathbf{Y}$.
(b) Find the partial correlation coefficients $\rho_{12.3}$ and $\rho_{12.34}$ (between elements of $\mathbf{Y}$).
(c) Find the multiple correlation coefficient between Y_1 and (Y_2, Y_4). [2+4+3]

2. Consider the following model:
$y_1 = \alpha + \gamma + \epsilon_1$
$y_2 = \alpha + \delta + \epsilon_2$
$y_3 = \delta - \gamma + \epsilon_3$
$y_4 = \alpha - \gamma + 2\delta + \epsilon_4$
where α, γ and δ are unknown constants, and ϵ_i are uncorrelated random variables having mean 0 and variance σ^2.
(a) Is $\alpha - \gamma$ estimable? Justify. If it is estimable, find its BLUE.
(b) Is $\gamma - \delta$ estimable? Justify. If it is estimable, find its BLUE. [5+5]

3. Consider the model:

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij},$$

$1 \leq i \leq 4$, $j = 1, 2$, where ϵ_{ij} are i.i.d. $N(0, \sigma^2)$ and $\sum_{i=1}^{4} \alpha_i = 0$.
(a) Show that $\alpha_k - \alpha_l$, $1 \leq k < l \leq 4$ are estimable.
(b) Find best linear unbiased estimators of the above mentioned linear contrasts.
(c) Find simultaneous 95% confidence intervals for $\alpha_1 - \alpha_2$ and $\alpha_2 - \alpha_3$. [2+3+4]

4. Consider the model:

$$y_j = \mathbf{x}_j'\beta + \epsilon_j,\ E(\epsilon_j) = 0,\ Var(\epsilon_j) = j\sigma^2,\ j = 1, 2, \cdots, n; \quad \mathbf{x}_j,\ \beta \in R^p.$$

(a) Find a solution $\hat{\beta}$ for β by solving

$$\min_{\beta \in \mathbb{R}^p} \sum_{j=1}^{p} \frac{1}{j} \left(y_j - \mathbf{x}_j'\beta\right)^2.$$

(b) What is the condition on $\mathbf{a} \in R^p$ which makes the linear parametric function $\mathbf{a}'\beta$ estimable under this model?
(c) What is the BLUE of $\mathbf{a}'\beta$ if it is estimable under this model? [4+2+4]

5. Given below are two linear models under consideration:

$$\textbf{Model I:} \qquad y_i = \beta x_i + \epsilon_i, \quad 1 \le i \le n,$$

where ϵ_i are uncorrelated errors with mean 0 and common variance σ^2. Additionally, x_i's are not all equal to each other.

$$\textbf{Model II:} \qquad y_i = \alpha + \beta x_i + \epsilon_i, \quad 1 \le i \le n,$$

with the same assumptions on ϵ_i and x_i as given above.

Assume that **Model I** is the correct model (from which y_is arise). However, suppose one computes the least squares estimate of β using the incorrect model (i.e., **Model II**).
(a) Compute the mean and variance of this estimate of β under the correct model.
(b) Compare results in (a) above with those of the *best linear unbiased* estimate of β under the correct model. [6+6]

INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2020-21
Statistics - III, Semesteral Examination, December 21, 2020
Marks are shown in square brackets. Total Marks: 50
Time: $2\frac{1}{2}$ Hours; submission must be complete by 1 pm
e-mail: mohan.delampady@gmail.com

You may freely consult the lecture notes, but no other books or resources may be consulted. You may use any of the results stated and discussed in the lecture notes, by stating them explicitly. Results from the assignments may not be used without establishing them. Calculators may be used. Statistical tables are provided in a separate file, Ftable.pdf

1. Suppose $\mathbf{Y} \sim N_p(\mathbf{0}, \mathbf{\Sigma})$ where Rank$(\Sigma) = r \leq p$ and let B and C be any real symmetric matrices with trace$(B\Sigma) = r_1$ and trace$(C\Sigma) = r_2$.
(a) Show that $\mathbf{Y}'B\mathbf{Y}$ and $\mathbf{Y}'C\mathbf{Y}$ are independent χ^2 random variables if and only if

$$\Sigma B \Sigma B \Sigma = \Sigma B \Sigma,\ \Sigma C \Sigma C \Sigma = \Sigma C \Sigma,\ \Sigma B \Sigma C \Sigma = \mathbf{0}.$$

(b) Show that the degrees of freedom of these χ^2 distributions can be expressed in terms of r_1 and r_2. [14+6]

2. Let $Z_i, 1 \leq i \leq 4$ be independent $N(0, \sigma^2)$ random variables. Define $Y_1 = Z_1$, $Y_2 = Y_1 + Z_2$, $Y_3 = Y_1 - Z_3$ and $Y_4 = Z_4$. Let $\mathbf{Y} = (Y_1, Y_2, Y_3, Y_4)'$.
(a) Find the probability distribution of $\mathbf{Y}$.
(b) Find the partial correlation coefficients $\rho_{12.3}$, $\rho_{12.4}$ and $\rho_{12.34}$ (between elements of $\mathbf{Y}$).
(c) Find the multiple correlation coefficient between Y_1 and (Y_2, Y_4).
[5+6+4]

3. A multiple linear regression analysis of a data set containing $n = 30$ observations on the variables X_1 = Year, 1 denoting 1930, X_2 = Pre-season precipitation, X_3 = June rain, X_4 = June temperature, X_5 = July rain, and Y = Corn Yield, for the state of Iowa, USA, yielded the following results. Here Y is taken as the response and $X_1, \ldots, X_5$ as the regressors. The regression matrix has full column rank here.

$\sum_{i=1}^{n} y_i^2 = 72551.25$, $\bar{y} = 47.81677$, $R^2 = 0.605$.

(a) Construct the ANOVA table for regression.
(b) Conduct a test at a reasonable level of significance for the usefulness of the regressors, $X_1, \ldots, X_5$, in explaining the variability of Y.
(c) What assumptions are needed to justify this analysis? [6+6+3]

Table entry for p is the critical value F^* with probability p lying to its right.

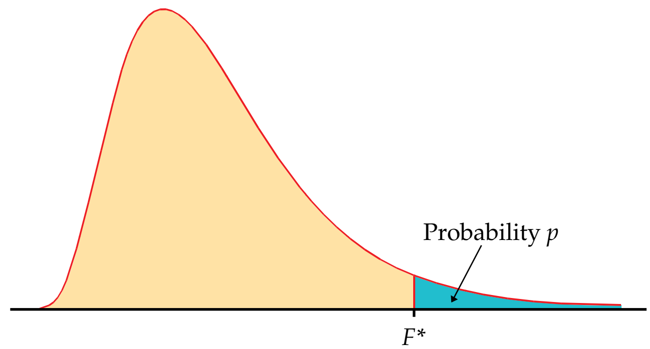

TABLE E

F critical values

Degrees of freedom in the denominator	p	1	2	3	4	5	6	7	8	9
		Degrees of freedom in the numerator								
1	.100	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86
	.050	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
	.025	647.79	799.50	864.16	899.58	921.85	937.11	948.22	956.66	963.28
	.010	4052.2	4999.5	5403.4	5624.6	5763.6	5859.0	5928.4	5981.1	6022.5
	.001	405284	500000	540379	562500	576405	585937	592873	598144	602284
2	.100	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38
	.050	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
	.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39
	.010	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
	.001	998.50	999.00	999.17	999.25	999.30	999.33	999.36	999.37	999.39
3	.100	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24
	.050	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
	.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47
	.010	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
	.001	167.03	148.50	141.11	137.10	134.58	132.85	131.58	130.62	129.86
4	.100	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94
	.050	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
	.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90
	.010	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66
	.001	74.14	61.25	56.18	53.44	51.71	50.53	49.66	49.00	48.47
5	.100	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32
	.050	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
	.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68
	.010	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
	.001	47.18	37.12	33.20	31.09	29.75	28.83	28.16	27.65	27.24
6	.100	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96
	.050	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
	.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52
	.010	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
	.001	35.51	27.00	23.70	21.92	20.80	20.03	19.46	19.03	18.69
7	.100	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72
	.050	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
	.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82
	.010	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
	.001	29.25	21.69	18.77	17.20	16.21	15.52	15.02	14.63	14.33

Table entry for p is the critical value F^* with probability p lying to its right.

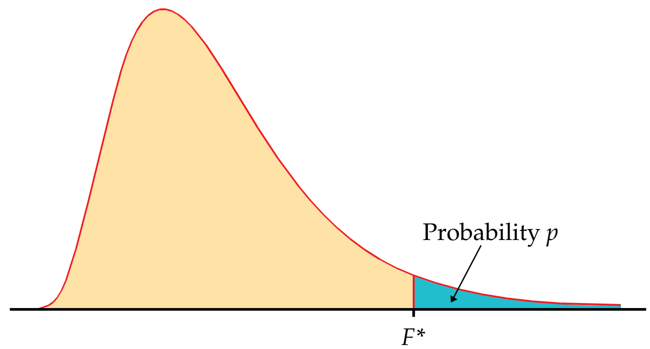

TABLE E

F critical values (continued)

Degrees of freedom in the numerator										
10	12	15	20	25	30	40	50	60	120	1000
60.19	60.71	61.22	61.74	62.05	62.26	62.53	62.69	62.79	63.06	63.30
241.88	243.91	245.95	248.01	249.26	250.10	251.14	251.77	252.20	253.25	254.19
968.63	976.71	984.87	993.10	998.08	1001.4	1005.6	1008.1	1009.8	1014.0	1017.7
6055.8	6106.3	6157.3	6208.7	6239.8	6260.6	6286.8	6302.5	6313.0	6339.4	6362.7
605621	610668	615764	620908	624017	626099	628712	630285	631337	633972	636301
9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.47	9.48	9.49
19.40	19.41	19.43	19.45	19.46	19.46	19.47	19.48	19.48	19.49	19.49
39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.48	39.49	39.50
99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.48	99.49	99.50
999.40	999.42	999.43	999.45	999.46	999.47	999.47	999.48	999.48	999.49	999.50
5.23	5.22	5.20	5.18	5.17	5.17	5.16	5.15	5.15	5.14	5.13
8.79	8.74	8.70	8.66	8.63	8.62	8.59	8.58	8.57	8.55	8.53
14.42	14.34	14.25	14.17	14.12	14.08	14.04	14.01	13.99	13.95	13.91
27.23	27.05	26.87	26.69	26.58	26.50	26.41	26.35	26.32	26.22	26.14
129.25	128.32	127.37	126.42	125.84	125.45	124.96	124.66	124.47	123.97	123.53
3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.80	3.79	3.78	3.76
5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.70	5.69	5.66	5.63
8.84	8.75	8.66	8.56	8.50	8.46	8.41	8.38	8.36	8.31	8.26
14.55	14.37	14.20	14.02	13.91	13.84	13.75	13.69	13.65	13.56	13.47
48.05	47.41	46.76	46.10	45.70	45.43	45.09	44.88	44.75	44.40	44.09
3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.15	3.14	3.12	3.11
4.74	4.68	4.62	4.56	4.52	4.50	4.46	4.44	4.43	4.40	4.37
6.62	6.52	6.43	6.33	6.27	6.23	6.18	6.14	6.12	6.07	6.02
10.05	9.89	9.72	9.55	9.45	9.38	9.29	9.24	9.20	9.11	9.03
26.92	26.42	25.91	25.39	25.08	24.87	24.60	24.44	24.33	24.06	23.82
2.94	2.90	2.87	2.84	2.81	2.80	2.78	2.77	2.76	2.74	2.72
4.06	4.00	3.94	3.87	3.83	3.81	3.77	3.75	3.74	3.70	3.67
5.46	5.37	5.27	5.17	5.11	5.07	5.01	4.98	4.96	4.90	4.86
7.87	7.72	7.56	7.40	7.30	7.23	7.14	7.09	7.06	6.97	6.89
18.41	17.99	17.56	17.12	16.85	16.67	16.44	16.31	16.21	15.98	15.77
2.70	2.67	2.63	2.59	2.57	2.56	2.54	2.52	2.51	2.49	2.47
3.64	3.57	3.51	3.44	3.40	3.38	3.34	3.32	3.30	3.27	3.23
4.76	4.67	4.57	4.47	4.40	4.36	4.31	4.28	4.25	4.20	4.15
6.62	6.47	6.31	6.16	6.06	5.99	5.91	5.86	5.82	5.74	5.66
14.08	13.71	13.32	12.93	12.69	12.53	12.33	12.20	12.12	11.91	11.72

(*Continued*)

TABLE E
F critical values (continued)

Degrees of freedom in the denominator	*p*	1	2	3	4	5	6	7	8	9
		Degrees of freedom in the numerator								
8	.100	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56
	.050	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
	.025	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36
	.010	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
	.001	25.41	18.49	15.83	14.39	13.48	12.86	12.40	12.05	11.77
9	.100	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44
	.050	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
	.025	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03
	.010	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
	.001	22.86	16.39	13.90	12.56	11.71	11.13	10.70	10.37	10.11
10	.100	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35
	.050	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
	.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78
	.010	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
	.001	21.04	14.91	12.55	11.28	10.48	9.93	9.52	9.20	8.96
11	.100	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27
	.050	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
	.025	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59
	.010	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
	.001	19.69	13.81	11.56	10.35	9.58	9.05	8.66	8.35	8.12
12	.100	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21
	.050	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
	.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44
	.010	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
	.001	18.64	12.97	10.80	9.63	8.89	8.38	8.00	7.71	7.48
13	.100	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16
	.050	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
	.025	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31
	.010	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
	.001	17.82	12.31	10.21	9.07	8.35	7.86	7.49	7.21	6.98
14	.100	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12
	.050	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
	.025	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21
	.010	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
	.001	17.14	11.78	9.73	8.62	7.92	7.44	7.08	6.80	6.58
15	.100	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09
	.050	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
	.025	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12
	.010	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
	.001	16.59	11.34	9.34	8.25	7.57	7.09	6.74	6.47	6.26
16	.100	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06
	.050	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
	.025	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05
	.010	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
	.001	16.12	10.97	9.01	7.94	7.27	6.80	6.46	6.19	5.98
17	.100	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03
	.050	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
	.025	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98
	.010	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68
	.001	15.72	10.66	8.73	7.68	7.02	6.56	6.22	5.96	5.75

TABLE E

F critical values (continued)

Degrees of freedom in the numerator										
10	12	15	20	25	30	40	50	60	120	1000
2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.35	2.34	2.32	2.30
3.35	3.28	3.22	3.15	3.11	3.08	3.04	3.02	3.01	2.97	2.93
4.30	4.20	4.10	4.00	3.94	3.89	3.84	3.81	3.78	3.73	3.68
5.81	5.67	5.52	5.36	5.26	5.20	5.12	5.07	5.03	4.95	4.87
11.54	11.19	10.84	10.48	10.26	10.11	9.92	9.80	9.73	9.53	9.36
2.42	2.38	2.34	2.30	2.27	2.25	2.23	2.22	2.21	2.18	2.16
3.14	3.07	3.01	2.94	2.89	2.86	2.83	2.80	2.79	2.75	2.71
3.96	3.87	3.77	3.67	3.60	3.56	3.51	3.47	3.45	3.39	3.34
5.26	5.11	4.96	4.81	4.71	4.65	4.57	4.52	4.48	4.40	4.32
9.89	9.57	9.24	8.90	8.69	8.55	8.37	8.26	8.19	8.00	7.84
2.32	2.28	2.24	2.20	2.17	2.16	2.13	2.12	2.11	2.08	2.06
2.98	2.91	2.85	2.77	2.73	2.70	2.66	2.64	2.62	2.58	2.54
3.72	3.62	3.52	3.42	3.35	3.31	3.26	3.22	3.20	3.14	3.09
4.85	4.71	4.56	4.41	4.31	4.25	4.17	4.12	4.08	4.00	3.92
8.75	8.45	8.13	7.80	7.60	7.47	7.30	7.19	7.12	6.94	6.78
2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.04	2.03	2.00	1.98
2.85	2.79	2.72	2.65	2.60	2.57	2.53	2.51	2.49	2.45	2.41
3.53	3.43	3.33	3.23	3.16	3.12	3.06	3.03	3.00	2.94	2.89
4.54	4.40	4.25	4.10	4.01	3.94	3.86	3.81	3.78	3.69	3.61
7.92	7.63	7.32	7.01	6.81	6.68	6.52	6.42	6.35	6.18	6.02
2.19	2.15	2.10	2.06	2.03	2.01	1.99	1.97	1.96	1.93	1.91
2.75	2.69	2.62	2.54	2.50	2.47	2.43	2.40	2.38	2.34	2.30
3.37	3.28	3.18	3.07	3.01	2.96	2.91	2.87	2.85	2.79	2.73
4.30	4.16	4.01	3.86	3.76	3.70	3.62	3.57	3.54	3.45	3.37
7.29	7.00	6.71	6.40	6.22	6.09	5.93	5.83	5.76	5.59	5.44
2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.92	1.90	1.88	1.85
2.67	2.60	2.53	2.46	2.41	2.38	2.34	2.31	2.30	2.25	2.21
3.25	3.15	3.05	2.95	2.88	2.84	2.78	2.74	2.72	2.66	2.60
4.10	3.96	3.82	3.66	3.57	3.51	3.43	3.38	3.34	3.25	3.18
6.80	6.52	6.23	5.93	5.75	5.63	5.47	5.37	5.30	5.14	4.99
2.10	2.05	2.01	1.96	1.93	1.91	1.89	1.87	1.86	1.83	1.80
2.60	2.53	2.46	2.39	2.34	2.31	2.27	2.24	2.22	2.18	2.14
3.15	3.05	2.95	2.84	2.78	2.73	2.67	2.64	2.61	2.55	2.50
3.94	3.80	3.66	3.51	3.41	3.35	3.27	3.22	3.18	3.09	3.02
6.40	6.13	5.85	5.56	5.38	5.25	5.10	5.00	4.94	4.77	4.62
2.06	2.02	1.97	1.92	1.89	1.87	1.85	1.83	1.82	1.79	1.76
2.54	2.48	2.40	2.33	2.28	2.25	2.20	2.18	2.16	2.11	2.07
3.06	2.96	2.86	2.76	2.69	2.64	2.59	2.55	2.52	2.46	2.40
3.80	3.67	3.52	3.37	3.28	3.21	3.13	3.08	3.05	2.96	2.88
6.08	5.81	5.54	5.25	5.07	4.95	4.80	4.70	4.64	4.47	4.33
2.03	1.99	1.94	1.89	1.86	1.84	1.81	1.79	1.78	1.75	1.72
2.49	2.42	2.35	2.28	2.23	2.19	2.15	2.12	2.11	2.06	2.02
2.99	2.89	2.79	2.68	2.61	2.57	2.51	2.47	2.45	2.38	2.32
3.69	3.55	3.41	3.26	3.16	3.10	3.02	2.97	2.93	2.84	2.76
5.81	5.55	5.27	4.99	4.82	4.70	4.54	4.45	4.39	4.23	4.08
2.00	1.96	1.91	1.86	1.83	1.81	1.78	1.76	1.75	1.72	1.69
2.45	2.38	2.31	2.23	2.18	2.15	2.10	2.08	2.06	2.01	1.97
2.92	2.82	2.72	2.62	2.55	2.50	2.44	2.41	2.38	2.32	2.26
3.59	3.46	3.31	3.16	3.07	3.00	2.92	2.87	2.83	2.75	2.66
5.58	5.32	5.05	4.78	4.60	4.48	4.33	4.24	4.18	4.02	3.87

(Continued)

TABLE E

F critical values (continued)

		Degrees of freedom in the numerator								
Degrees of freedom in the denominator	*p*	1	2	3	4	5	6	7	8	9
	.100	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00
	.050	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
18	.025	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93
	.010	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
	.001	15.38	10.39	8.49	7.46	6.81	6.35	6.02	5.76	5.56
	.100	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98
	.050	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
19	.025	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88
	.010	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
	.001	15.08	10.16	8.28	7.27	6.62	6.18	5.85	5.59	5.39
	.100	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96
	.050	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
20	.025	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84
	.010	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
	.001	14.82	9.95	8.10	7.10	6.46	6.02	5.69	5.44	5.24
	.100	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95
	.050	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
21	.025	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80
	.010	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
	.001	14.59	9.77	7.94	6.95	6.32	5.88	5.56	5.31	5.11
	.100	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93
	.050	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
22	.025	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76
	.010	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
	.001	14.38	9.61	7.80	6.81	6.19	5.76	5.44	5.19	4.99
	.100	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92
	.050	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
23	.025	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73
	.010	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
	.001	14.20	9.47	7.67	6.70	6.08	5.65	5.33	5.09	4.89
	.100	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91
	.050	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
24	.025	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70
	.010	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
	.001	14.03	9.34	7.55	6.59	5.98	5.55	5.23	4.99	4.80
	.100	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89
	.050	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
25	.025	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68
	.010	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
	.001	13.88	9.22	7.45	6.49	5.89	5.46	5.15	4.91	4.71
	.100	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88
	.050	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
26	.025	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65
	.010	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
	.001	13.74	9.12	7.36	6.41	5.80	5.38	5.07	4.83	4.64
	.100	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87
	.050	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
27	.025	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63
	.010	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
	.001	13.61	9.02	7.27	6.33	5.73	5.31	5.00	4.76	4.57

TABLE E

F critical values (continued)

Degrees of freedom in the numerator										
10	12	15	20	25	30	40	50	60	120	1000
1.98	1.93	1.89	1.84	1.80	1.78	1.75	1.74	1.72	1.69	1.66
2.41	2.34	2.27	2.19	2.14	2.11	2.06	2.04	2.02	1.97	1.92
2.87	2.77	2.67	2.56	2.49	2.44	2.38	2.35	2.32	2.26	2.20
3.51	3.37	3.23	3.08	2.98	2.92	2.84	2.78	2.75	2.66	2.58
5.39	5.13	4.87	4.59	4.42	4.30	4.15	4.06	4.00	3.84	3.69
1.96	1.91	1.86	1.81	1.78	1.76	1.73	1.71	1.70	1.67	1.64
2.38	2.31	2.23	2.16	2.11	2.07	2.03	2.00	1.98	1.93	1.88
2.82	2.72	2.62	2.51	2.44	2.39	2.33	2.30	2.27	2.20	2.14
3.43	3.30	3.15	3.00	2.91	2.84	2.76	2.71	2.67	2.58	2.50
5.22	4.97	4.70	4.43	4.26	4.14	3.99	3.90	3.84	3.68	3.53
1.94	1.89	1.84	1.79	1.76	1.74	1.71	1.69	1.68	1.64	1.61
2.35	2.28	2.20	2.12	2.07	2.04	1.99	1.97	1.95	1.90	1.85
2.77	2.68	2.57	2.46	2.40	2.35	2.29	2.25	2.22	2.16	2.09
3.37	3.23	3.09	2.94	2.84	2.78	2.69	2.64	2.61	2.52	2.43
5.08	4.82	4.56	4.29	4.12	4.00	3.86	3.77	3.70	3.54	3.40
1.92	1.87	1.83	1.78	1.74	1.72	1.69	1.67	1.66	1.62	1.59
2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.94	1.92	1.87	1.82
2.73	2.64	2.53	2.42	2.36	2.31	2.25	2.21	2.18	2.11	2.05
3.31	3.17	3.03	2.88	2.79	2.72	2.64	2.58	2.55	2.46	2.37
4.95	4.70	4.44	4.17	4.00	3.88	3.74	3.64	3.58	3.42	3.28
1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.65	1.64	1.60	1.57
2.30	2.23	2.15	2.07	2.02	1.98	1.94	1.91	1.89	1.84	1.79
2.70	2.60	2.50	2.39	2.32	2.27	2.21	2.17	2.14	2.08	2.01
3.26	3.12	2.98	2.83	2.73	2.67	2.58	2.53	2.50	2.40	2.32
4.83	4.58	4.33	4.06	3.89	3.78	3.63	3.54	3.48	3.32	3.17
1.89	1.84	1.80	1.74	1.71	1.69	1.66	1.64	1.62	1.59	1.55
2.27	2.20	2.13	2.05	2.00	1.96	1.91	1.88	1.86	1.81	1.76
2.67	2.57	2.47	2.36	2.29	2.24	2.18	2.14	2.11	2.04	1.98
3.21	3.07	2.93	2.78	2.69	2.62	2.54	2.48	2.45	2.35	2.27
4.73	4.48	4.23	3.96	3.79	3.68	3.53	3.44	3.38	3.22	3.08
1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.62	1.61	1.57	1.54
2.25	2.18	2.11	2.03	1.97	1.94	1.89	1.86	1.84	1.79	1.74
2.64	2.54	2.44	2.33	2.26	2.21	2.15	2.11	2.08	2.01	1.94
3.17	3.03	2.89	2.74	2.64	2.58	2.49	2.44	2.40	2.31	2.22
4.64	4.39	4.14	3.87	3.71	3.59	3.45	3.36	3.29	3.14	2.99
1.87	1.82	1.77	1.72	1.68	1.66	1.63	1.61	1.59	1.56	1.52
2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.84	1.82	1.77	1.72
2.61	2.51	2.41	2.30	2.23	2.18	2.12	2.08	2.05	1.98	1.91
3.13	2.99	2.85	2.70	2.60	2.54	2.45	2.40	2.36	2.27	2.18
4.56	4.31	4.06	3.79	3.63	3.52	3.37	3.28	3.22	3.06	2.91
1.86	1.81	1.76	1.71	1.67	1.65	1.61	1.59	1.58	1.54	1.51
2.22	2.15	2.07	1.99	1.94	1.90	1.85	1.82	1.80	1.75	1.70
2.59	2.49	2.39	2.28	2.21	2.16	2.09	2.05	2.03	1.95	1.89
3.09	2.96	2.81	2.66	2.57	2.50	2.42	2.36	2.33	2.23	2.14
4.48	4.24	3.99	3.72	3.56	3.44	3.30	3.21	3.15	2.99	2.84
1.85	1.80	1.75	1.70	1.66	1.64	1.60	1.58	1.57	1.53	1.50
2.20	2.13	2.06	1.97	1.92	1.88	1.84	1.81	1.79	1.73	1.68
2.57	2.47	2.36	2.25	2.18	2.13	2.07	2.03	2.00	1.93	1.86
3.06	2.93	2.78	2.63	2.54	2.47	2.38	2.33	2.29	2.20	2.11
4.41	4.17	3.92	3.66	3.49	3.38	3.23	3.14	3.08	2.92	2.78

(*Continued*)

TABLE E

F critical values (continued)

		Degrees of freedom in the numerator								
Degrees of freedom in the denominator	*p*	1	2	3	4	5	6	7	8	9
	.100	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87
	.050	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
28	.025	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61
	.010	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
	.001	13.50	8.93	7.19	6.25	5.66	5.24	4.93	4.69	4.50
	.100	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86
	.050	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
29	.025	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59
	.010	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09
	.001	13.39	8.85	7.12	6.19	5.59	5.18	4.87	4.64	4.45
	.100	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85
	.050	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
30	.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57
	.010	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
	.001	13.29	8.77	7.05	6.12	5.53	5.12	4.82	4.58	4.39
	.100	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79
	.050	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
40	.025	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45
	.010	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
	.001	12.61	8.25	6.59	5.70	5.13	4.73	4.44	4.21	4.02
	.100	2.81	2.41	2.20	2.06	1.97	1.90	1.84	1.80	1.76
	.050	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07
50	.025	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38
	.010	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78
	.001	12.22	7.96	6.34	5.46	4.90	4.51	4.22	4.00	3.82
	.100	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74
	.050	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
60	.025	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33
	.010	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
	.001	11.97	7.77	6.17	5.31	4.76	4.37	4.09	3.86	3.69
	.100	2.76	2.36	2.14	2.00	1.91	1.83	1.78	1.73	1.69
	.050	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97
100	.025	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24
	.010	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59
	.001	11.50	7.41	5.86	5.02	4.48	4.11	3.83	3.61	3.44
	.100	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66
	.050	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93
200	.025	5.10	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18
	.010	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50
	.001	11.15	7.15	5.63	4.81	4.29	3.92	3.65	3.43	3.26
	.100	2.71	2.31	2.09	1.95	1.85	1.78	1.72	1.68	1.64
	.050	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89
1000	.025	5.04	3.70	3.13	2.80	2.58	2.42	2.30	2.20	2.13
	.010	6.66	4.63	3.80	3.34	3.04	2.82	2.66	2.53	2.43
	.001	10.89	6.96	5.46	4.65	4.14	3.78	3.51	3.30	3.13

TABLE E

F critical values (continued)

Degrees of freedom in the numerator										
10	12	15	20	25	30	40	50	60	120	1000
1.84	1.79	1.74	1.69	1.65	1.63	1.59	1.57	1.56	1.52	1.48
2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.79	1.77	1.71	1.66
2.55	2.45	2.34	2.23	2.16	2.11	2.05	2.01	1.98	1.91	1.84
3.03	2.90	2.75	2.60	2.51	2.44	2.35	2.30	2.26	2.17	2.08
4.35	4.11	3.86	3.60	3.43	3.32	3.18	3.09	3.02	2.86	2.72
1.83	1.78	1.73	1.68	1.64	1.62	1.58	1.56	1.55	1.51	1.47
2.18	2.10	2.03	1.94	1.89	1.85	1.81	1.77	1.75	1.70	1.65
2.53	2.43	2.32	2.21	2.14	2.09	2.03	1.99	1.96	1.89	1.82
3.00	2.87	2.73	2.57	2.48	2.41	2.33	2.27	2.23	2.14	2.05
4.29	4.05	3.80	3.54	3.38	3.27	3.12	3.03	2.97	2.81	2.66
1.82	1.77	1.72	1.67	1.63	1.61	1.57	1.55	1.54	1.50	1.46
2.16	2.09	2.01	1.93	1.88	1.84	1.79	1.76	1.74	1.68	1.63
2.51	2.41	2.31	2.20	2.12	2.07	2.01	1.97	1.94	1.87	1.80
2.98	2.84	2.70	2.55	2.45	2.39	2.30	2.25	2.21	2.11	2.02
4.24	4.00	3.75	3.49	3.33	3.22	3.07	2.98	2.92	2.76	2.61
1.76	1.71	1.66	1.61	1.57	1.54	1.51	1.48	1.47	1.42	1.38
2.08	2.00	1.92	1.84	1.78	1.74	1.69	1.66	1.64	1.58	1.52
2.39	2.29	2.18	2.07	1.99	1.94	1.88	1.83	1.80	1.72	1.65
2.80	2.66	2.52	2.37	2.27	2.20	2.11	2.06	2.02	1.92	1.82
3.87	3.64	3.40	3.14	2.98	2.87	2.73	2.64	2.57	2.41	2.25
1.73	1.68	1.63	1.57	1.53	1.50	1.46	1.44	1.42	1.38	1.33
2.03	1.95	1.87	1.78	1.73	1.69	1.63	1.60	1.58	1.51	1.45
2.32	2.22	2.11	1.99	1.92	1.87	1.80	1.75	1.72	1.64	1.56
2.70	2.56	2.42	2.27	2.17	2.10	2.01	1.95	1.91	1.80	1.70
3.67	3.44	3.20	2.95	2.79	2.68	2.53	2.44	2.38	2.21	2.05
1.71	1.66	1.60	1.54	1.50	1.48	1.44	1.41	1.40	1.35	1.30
1.99	1.92	1.84	1.75	1.69	1.65	1.59	1.56	1.53	1.47	1.40
2.27	2.17	2.06	1.94	1.87	1.82	1.74	1.70	1.67	1.58	1.49
2.63	2.50	2.35	2.20	2.10	2.03	1.94	1.88	1.84	1.73	1.62
3.54	3.32	3.08	2.83	2.67	2.55	2.41	2.32	2.25	2.08	1.92
1.66	1.61	1.56	1.49	1.45	1.42	1.38	1.35	1.34	1.28	1.22
1.93	1.85	1.77	1.68	1.62	1.57	1.52	1.48	1.45	1.38	1.30
2.18	2.08	1.97	1.85	1.77	1.71	1.64	1.59	1.56	1.46	1.36
2.50	2.37	2.22	2.07	1.97	1.89	1.80	1.74	1.69	1.57	1.45
3.30	3.07	2.84	2.59	2.43	2.32	2.17	2.08	2.01	1.83	1.64
1.63	1.58	1.52	1.46	1.41	1.38	1.34	1.31	1.29	1.23	1.16
1.88	1.80	1.72	1.62	1.56	1.52	1.46	1.41	1.39	1.30	1.21
2.11	2.01	1.90	1.78	1.70	1.64	1.56	1.51	1.47	1.37	1.25
2.41	2.27	2.13	1.97	1.87	1.79	1.69	1.63	1.58	1.45	1.30
3.12	2.90	2.67	2.42	2.26	2.15	2.00	1.90	1.83	1.64	1.43
1.61	1.55	1.49	1.43	1.38	1.35	1.30	1.27	1.25	1.18	1.08
1.84	1.76	1.68	1.58	1.52	1.47	1.41	1.36	1.33	1.24	1.11
2.06	1.96	1.85	1.72	1.64	1.58	1.50	1.45	1.41	1.29	1.13
2.34	2.20	2.06	1.90	1.79	1.72	1.61	1.54	1.50	1.35	1.16
2.99	2.77	2.54	2.30	2.14	2.02	1.87	1.77	1.69	1.49	1.22

INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2019-20
Statistics - III, Semesteral Examination, November 18, 2019
Marks are shown in square brackets. Total Marks: 50

1. Let $Z_i, 1 \le i \le 4$ be independent $N(\mu, \sigma^2)$ random variables. Define $X_1 = Z_1$, $X_2 = Z_1 + Z_2$, $X_3 = Z_1 + Z_3$ and $X_4 = Z_2 + Z_4$. Let $\mathbf{X} = (X_1, \ldots, X_4)'$.
(a) Find the probability distribution of $\mathbf{X}$.
(b) Find the partial correlation coefficients $\rho_{12.3}$ and $\rho_{12.34}$ (between elements of $\mathbf{X}$). [10]

2. Consider the model:
$y_1 = \theta + \gamma + \epsilon_1$
$y_2 = \theta - \phi + \epsilon_2$
$y_3 = 2\theta - \phi + \gamma + \epsilon_3$
$y_4 = \phi + \gamma + \epsilon_4$,
where ϵ_i are uncorrelated having mean 0 and variance σ^2.
(a) Show that $\theta + 2\gamma + \phi$ is estimable. What is its BLUE?
(b) What is the degrees of freedom of the residual sum of squares? [14]

3. (a) Define the multiple correlation coefficient and the coefficient of determination. Explain how they are related to each other.
(b) What is a q-q plot or a normal probability plot? How is it useful in linear regression?
(c) How is Bonferroni inequality used for multiple comparisons in the one-way classification model? [12]

4. Consider the Gauss-Markov model, $\mathbf{Y} = X\beta + \epsilon$, where $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$ and $X_{n\times p}$ has rank $r \le p$. Consider testing $H_0 : A\beta = 0$, where $A_{q\times p}$ has rank q and $A\beta$ is a collection of estimable linear functions of β. Let $\hat\beta$ be a least squares solution of β and $(X'X)^-$ be a generalized inverse of $X'X$. Let $\hat\beta_0$ be a least squares solution of β under the constraints given by H_0.
(a) Show that $A(X'X)^-X'X(X'X)^-A' = A(X'X)^-A'$.
(b) Find the probability distribution of $A\hat\beta$.
(c) Find $E((A\hat\beta)'(A(X'X)^-A')^{-1}A\hat\beta)$.
(d) Find the distribution of $(RSS_{H_0} - RSS)/RSS$ when H_0 is true. [14]

INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2018-19
Statistics - III, Semesteral Examination, November 19, 2018
Marks are shown in square brackets. **Total Marks: 50**

1. Let $Z_i, 1 \leq i \leq 4$ be independent $N(0, \sigma^2)$ random variables. Define $X_1 = Z_1$ and $X_2 = Z_2 + Z_3$, $X_3 = Z_3 + Z_4$ and $X_4 = Z_1 + Z_2 + Z_3$. Let $\mathbf{X} = (X_1, \ldots, X_4)'$.
(a) Find the probability distribution of $\mathbf{X}$.
(b) Find the conditional distribution of (X_3, X_4) given X_2.
(c) Find the partial correlation coefficients $\rho_{12.3}$ and $\rho_{34.2}$ (between elements of $\mathbf{X}$). [12]

2. Consider the model:

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij},$$

$1 \leq i \leq 4$, $1 \leq j \leq 10$, $\sum_{i=1}^{4} \alpha_i = 0$, where ϵ_{ij} are i.i.d. $N(0, \sigma^2)$.
(a) Find the best linear unbiased estimators of μ and α_i.
(b) Construct 95% simultaneous confidence set for the vector $(\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_4, \alpha_4 - \alpha_1)$ using the method of Scheffe. Is it different from the one obtained using the method of Bonferroni? Justify. [15]

3. (a) Explain the concepts of randomization, blocks and confounding in the context of design of experiments.
(b) Suppose we want to compare two treatments. When will an experiment with matched pairs be superior to another with two independent samples? [11]

4. Consider the Gauss-Markov model, $\mathbf{Y} = X\beta + \epsilon$, where $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$ and $X_{n\times p}$ has rank $r \leq p$. Suppose $T_{n\times(p-1)}$, which is formed by the first $p-1$ columns of X, has rank r also. Let B denote any generalized inverse of $T'T$.
(a) Show that $\hat{\beta} = \begin{pmatrix} BT'\mathbf{Y} \\ 0 \end{pmatrix}$ minimizes $(\mathbf{Y} - X\beta)'(\mathbf{Y} - X\beta)$.
(b) Does BLUE of β exist? Find it if it exists. [12]

INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2017-18
Statistics - III, Semesteral Examination, November 20, 2017
Marks shown in square brackets. Time: 3 hours. Total Marks: 50

1. For $n \geq 4$, let $Z_j, 1 \leq j \leq n$ be i.i.d. $N(0, \sigma^2)$, $\sigma^2 > 0$. Define $X_1 = Z_1$, $X_2 = X_1 + Z_2$, $X_3 = X_1 + Z_3$ and $X_i = Z_i$ for $4 \leq i \leq n$.
Let $\mathbf{X} = (X_1, \ldots, X_n)'$.
(a) Find the probability distribution of $\mathbf{X}$.
(b) Find the partial correlation coefficient $\rho_{12.3}$ (between elements of $\mathbf{X}$).
(c) Without computation establish that here $\rho_{12.34} = \rho_{12.3}$. [13]

2. Consider the one-way model:

$$y_{ij} = \mu_i + \epsilon_{ij},\ 1 \leq j \leq n_i;\ 1 \leq i \leq k,$$

where ϵ_{ij} are i.i.d. $N(0, \sigma^2)$, $k \geq 4$ and $n_i > 1$ for all i.
(a) Show that $\mu_1 - \mu_2$ is estimable.
(b) What is the Bonferroni inequality used for multiple comparisons involving μ_i's?
(c) Construct a $100(1-\alpha)\%$ simultaneous confidence set for $(\mu_1 - \mu_2, \mu_2 - \mu_3, \mu_3 - \mu_4)$. [12]

3. Let Y be a response variable and $X_1, \ldots, X_k$ be covariates. Also, let ρ_i denote the correlation coefficient between Y and X_i, and let R denote the multiple correlation coefficient between Y and $X_1, \ldots, X_k$.
(a) Show that $R \geq \max\{|r_i|, 1 \leq i \leq k\}$.
(b) What is the exact relationship between R and r_i's when $k = 1$? [10]

4. Consider the linear model:

$$y_i = \beta_1 x_i + \beta_2 z_i + \epsilon_i,\ 1 \leq i \leq n,$$

where ϵ_i are uncorrelated errors with mean 0 and common variance σ^2; also x_i's are not proportional to z_i's. Suppose one computes the least squares estimate of β_1 using the incorrect model,

$$y_i = \beta_1 x_i + \epsilon_i,$$

with the same assumptions on ϵ_i as given above. Compute the mean and variance of this estimate of β_1, and compare them with those of the *best linear unbiased* estimate of β_1, under the correct model. [15]

INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2014-15
Statistics - III, Semesteral Examination, November 7, 2014
Marks are shown in square brackets. Total Marks: 50

1. For $n \geq 4$ let $Z_i, 1 \leq i \leq n$ be independent $N(0, \sigma^2)$ random variables. Consider $0 < \alpha < 1$. Define $X_1 = Z_1$ and $X_{i+1} = -\alpha X_i + \sqrt{1-\alpha^2} Z_{i+1}$ for $1 \leq i \leq n-1$. Let $\mathbf{X} = (X_1, \ldots, X_n)'$.
(a) Find the probability distribution of $\mathbf{X}$.
(b) Find the partial correlation coefficients $\rho_{12.3}$, $\rho_{13.2}$ and $\rho_{14.23}$ (between elements of $\mathbf{X}$). [11]

2. Consider the model $\mathbf{Y} = X\beta + \epsilon$, where $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$ and $X_{n\times p}$ has rank $r \leq p$ and its first column is $\mathbf{1}$.
(a) Define the coefficient of determination for this model. What does it measure?
(b) If $n = 14$, $p = 6$, $r = 4$ and $R^2 = 80\%$, compute the F-ratio for testing the usefulness of the regressors. [11]

3. Consider the model:

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij},$$

$1 \leq i \leq 4$, $1 \leq j \leq 10$, where ϵ_{ij} are i.i.d. $N(0, \sigma^2)$; also assume the usual constraints on the parameters for identifiability.
(a) Explain why constraints are needed for identifiability of parameters.
(b) Show that $\alpha_1 - \alpha_4$ is estimable, and find its BLUE.
(c) Provide a 95% confidence interval for $\alpha_1 - \alpha_4$.
(d) Find the maximum likelihood estimator of σ^2. Is it unbiased? [14]

4. Suppose $\mathbf{X} \sim N_p(\mathbf{0}, \mathbf{\Sigma})$ where $\text{Rank}(\Sigma) = r \leq p$ and let B and C be any symmetric matrices.
(a) Show that $\mathbf{X}'B\mathbf{X}$ and $\mathbf{X}'C\mathbf{X}$ are independent χ^2 random variables if and only if

$$\Sigma B \Sigma B \Sigma = \Sigma B \Sigma, \ \Sigma C \Sigma C \Sigma = \Sigma C \Sigma, \ \Sigma B \Sigma C \Sigma = \mathbf{0}.$$

(b) Find the degrees of freedom of these χ^2 distributions. [14]

INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2012-13
Statistics - III, Semesteral Examination, December 3, 2012
Marks are shown in square brackets. Total Marks: 50

1. Let $n \geq 4$ and let $Z_i, 1 \leq i \leq n$ be independent $N(0, \sigma^2)$ random variables; let $0 < \alpha < 1$. Define $X_1 = Z_1$ and $X_{i+1} = \alpha X_i + \sqrt{1-\alpha^2} Z_{i+1}$ for $1 \leq i \leq n-1$. Let $\mathbf{X} = (X_1, \ldots, X_n)'$.
(a) Find the probability distribution of $\mathbf{X}$.
(b) Find the partial correlation coefficients $\rho_{12.3}$ and $\rho_{12.34}$ (between elements of $\mathbf{X}$). [10]

2. Consider the model $\mathbf{Y} = X\beta + \epsilon$, where $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$ and $X_{n\times p}$ may not have full column rank but has $\mathbf{1}$ as its first column. Derive the joint distribution of $\bar{y}$ and the Residual Sum of Squares. [10]

3. Consider the problem of comparing $k \geq 2$ treatments. Suppose that under treatment i, the response $Y \sim N(\mu_i, \sigma^2)$, $1 \leq i \leq k$. If independent random samples of sizes $n_1, \ldots, n_k$, respectively, are available from groups of subjects who have undergone these treatments, describe the methodology for comparing the treatments. Show that the method reduces to a Student's t-test when $k = 2$. [10]

4. Consider the model:

$$y_{ijk} = \mu + \alpha_i + \tau_j + \delta_{ij} + \epsilon_{ijk},$$

$1 \leq i \leq I$, $1 \leq j \leq J$, $1 \leq k \leq K$, where ϵ_{ijk} are i.i.d. $N(0, \sigma^2)$ and with the usual constraints on the parameters for identifiability.
(a) Show that the least squares estimators of the parameters α_i, τ_j and δ_{ij} are also their maximum likelihood estimators.
(b) Find the maximum likelihood estimator of σ^2. Is it unbiased? [10]

5. Suppose $\mathbf{X} \sim N_p(\mu, \Sigma)$ where Rank$(\Sigma) = r \leq p$ and let B be any symmetric matrix such that $B\mu = \mathbf{0}$. Show that $\mathbf{X}'B\mathbf{X}$ has a χ^2 distribution if and only if

$$\Sigma B \Sigma B \Sigma = \Sigma B \Sigma.$$

Find the degrees of freedom of such a χ^2 distribution. [10]

INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, 2009-10
Statistics - III, Semesteral Examination, December 2, 2009
Marks are shown in square brackets. **Total Marks: 50**

1. Consider the model $\mathbf{Y} = \mathbf{X}\beta + \epsilon$, where $\mathbf{X}_{n\times p}$ has $\mathbf{1}$ as its first column and may not have full column rank; also $\epsilon \sim N_n(\mathbf{0}, \sigma^2 I_n)$. Let $\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Y}$ and $RSS = (\mathbf{Y} - \mathbf{X}\hat{\beta})'(\mathbf{Y} - \mathbf{X}\hat{\beta})$, where $(\mathbf{X}'\mathbf{X})^{-}$ is any generalized inverse of $(\mathbf{X}'\mathbf{X})$.
(a) Find the joint distribution of $(\mathbf{X}\hat{\beta}, RSS)$.
(b) Define the coefficient of determination and explain what it measures. [10]

2. Consider $Y_1, \ldots, Y_n$ i.i.d. $N(0, \sigma^2)$, $\sigma^2 > 0$, and let $X_i = \sum_{j=1}^{i} Y_j$ for $1 \le i \le n$.
(a) Find the covariance matrix of $\mathbf{X} = (X_1, X_2, \ldots, X_n)'$.
(b) Find the partial correlations $\rho_{12.3}$ and $\rho_{12.34}$ (between components of $\mathbf{X}$). [10]

3. A manufacturer of television sets is interested in the effect on tube conductivity of four different types of coating for colour picture tubes. The following conductivity data were obtained:

Coating type	Conductivity			
1	143	141	150	146
2	152	149	137	143
3	134	136	132	127
4	129	127	132	129

(a) Describe the methodology for determining whether conductivity for the different coating types significantly differ. Numerical computations are not needed.
(b) What is meant by a linear contrast in an experiment like this?
(c) What is the relation between the ANOVA null hypothesis and the hypotheses to check various linear contrasts? [10]

4. Consider the model:

$$y_{ij} = \mu + \alpha_i + \tau_j + \epsilon_{ij},$$

$1 \le i \le 4$, $j = 1, 2$, where ϵ_{ij} are i.i.d. $N(0, \sigma^2)$ and $0 = \tau_1 + \tau_2 = \sum_{i=1}^{4} \alpha_i$.
(a) Show that $\tau_1 - \tau_2$ and $\alpha_k - \alpha_l$, $1 \le k < l \le 4$ are estimable.
(b) Find the best linear unbiased estimators of the above mentioned linear contrasts.
(c) Find the variance of the estimators in (b) above and then provide an unbiased estimator for each of these variances. [10]

5. Describe the theory behind the normal probability plot (q-q plot). Why is it useful in linear regression? [10]

INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Third Year, First Semester, 2006-07
Statistics - III, Semesteral Examination, December 6, 2006

(8) 1. Consider the model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, $i = 1, \ldots, n$, where ϵ_i are i.i.d. $N(0, \sigma^2)$. Find the joint probability distribution of $\frac{1}{n}\sum_{i=1}^{n} y_i$ and residual sum of squares.

(12) 2. Let $Y_1, \ldots, Y_n$ be independent random variables with unit variance, and let $X_1 = Y_1$, $X_i = Y_i - Y_{i-1}$ for $1 < i \leq n$.
(a) Find the covariance matrix of $\mathbf{X} = (X_1, X_2, \ldots, X_n)'$.
(b) Find the partial correlations $\rho_{12.3}$ and $\rho_{12.34}$ (between components of $\mathbf{X}$).

(15) 4. The response time in milliseconds was determined for three different types of circuits that could be used in an automatic valve shutoff mechanism. The results were the following:

Circuit	Response time (ms)				
1	9	12	10	8	15
2	20	21	23	17	30
3	6	5	8	16	7

(a) Describe the methodology for determining whether the response times for the different circuit types significantly differ. Numerical computations are not needed.
(b) What is meant by a linear contrast in an experiment like this?
(c) What is the relation between the ANOVA null hypothesis and the hypotheses to check various linear contrasts?

(15) 5. Consider a completely randomized two-factor experiment.
(a) What do main effects and interactions mean in this context? Relate them to the means of cells formed by the levels of the factors.
(b) Describe how presence of these parameters can be checked.
(c) Provide an expression for the proportion of variability in the response variable explained by the two-factor model.