Topology

TRISHAN MONDAL

Assignment-3

PROBLEM 6

Q: Consider $X = \mathcal{C}(\mathbb{S}^n, \mathbb{R}^m)$ with "Sup Norm" metric. Prove that X is pathconnected.

Solution. We will show that every function $f \in X$ is path connected to 0("Zero function"). Consider, $\gamma_f(t, \mathbf{x}) = tf(\mathbf{x})$, where $x \in \mathbb{S}^n$ and $t \in [0,1]$. Since,$\mathbb{S}^n$ is compact $\sup \|f(\mathbf{x})\|$ is bounded. Let,$\|f(\mathbf{x})\| < K$. Then,for any $\varepsilon > 0$ we can get $\delta = \frac{\varepsilon}{2K}$ such that,

$$\|\gamma_f(t_1, \mathbf{x}) - \gamma_f(t_2, \mathbf{x})\| = \|f(x)\||t_2 - t_1| < 2K|t_2 - t_1| < \varepsilon$$

For any $t_1, t_2 \in [0,1]$ satisfying $|t_1 - t_2| < \delta$. So, γ_f is continuous(even uniformly continuous) and $\gamma_f(0, \mathbf{x}) = 0$, $\gamma_f(1, x) = f(\mathbf{x})$. Notice that $\gamma_f(1-t, \mathbf{x})$ is a path from 0 to f. So, for any function f, g we can construct a path between f amd g as following,

$$\gamma_{fg}(t, \mathbf{x}) = \begin{cases} (1-2t)f(\mathbf{x}) & \text{, if } t \in [0, \frac{1}{2}] \\ (2t-1)g(\mathbf{x}) & \text{, if } t \in [\frac{1}{2}, 1] \end{cases}$$

We can see the function is continuous as at $t = \frac{1}{2}$ both the part of the function is 0 and $\gamma_{fg}(0, \mathbf{x}) = f(\mathbf{x})$ and $\gamma_{fg}(1, \mathbf{x}) = g(\mathbf{x})$, so it is a path between f and g. And hence X is pathconnected. ■

PROBLEM 7

Q: Prove the following statements

1. The Cantor set $(\mathcal{C})$ is not Empty.
2. For $m, k \geq 1$, the intervals $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ are disjoint from $\mathcal{C}$.

Solution. Through out this Solution we will Consider the following construction, $F_0 = [0,1], F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $F_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup [\frac{8}{9}, 1]$. Here,$F_{i+1}$ is obtained from F_i by removing middle third of each interval present in F_i; So $F_0 \supset F_1 \supset \cdots$ and,

$$\mathcal{C} = \bigcap_i F_i$$

(1). Clearly the Cantor set contains $0, 1$ so it is non Empty.

(2). **Claim**: $F_n = \frac{F_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{F_{n-1}}{3}\right)$.

Proof: For $n = 2$.

$$F_2 = \frac{F_1}{3} \cup \left(\frac{2}{3} + \frac{F_1}{3}\right)$$

$$\Rightarrow F_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

Now we will assume our hypothesis is true for $n = k$. Now, when we are Constructing F_{k+1} from F_k We are removing, middle 3rd of each of the intervals. Notice, $F_k = \frac{F_{k-1}}{3} \cup \left(\frac{2}{3} + \frac{F_{k-1}}{3}\right)$ So, removing middle third of F_{k-1} give us F_k and hence,

$$F_{k+1} = \frac{F_k}{3} \cup \left(\frac{2}{3} + \frac{F_k}{3}\right).$$

Notice that,

$$\begin{aligned}
&F_0 = [0,1], F_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] = [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \\
&F_2 = F_1 \setminus \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{4}{9}, \frac{5}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \\
&= [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{4}{9}, \frac{5}{9}\right) \\
&F_{n+1} = \frac{F_n}{3} \cup \frac{1}{3}\left(F_n + 2\right) \\
&= \frac{1}{3}\left(F_n \cup (F_n + 2)\right) \\
&= \tfrac{F_{n-1}}{3} \cup \left(\tfrac{2}{3} + \tfrac{F_{n-1}}{3}\right) \setminus \textstyle\bigcup_{i=0}^{3^n-1} \left(\tfrac{3i+1}{3^{n+1}}, \tfrac{3i+2}{3^{n+1}}\right) \\
&= F_n \setminus \textstyle\bigcup_{i=0}^{3^n-1} \left(\tfrac{3i+1}{3^{n+1}}, \tfrac{3i+2}{3^{n+1}}\right) \\
&= [0,1] \setminus \bigcup_{i=0}^{n+1} \bigcup_{j=0}^{3^{i-1}-1} \left(\frac{3j+1}{3^{k+1}}, \frac{3j+2}{3^{i+1}}\right)
\end{aligned}$$

$$\therefore \mathcal{C} = [0,1] \setminus \bigcup_{i=0}^{\infty} \bigcup_{j=0}^{3^{i-1}-1} \left(\frac{3j+1}{3^{i+1}}, \frac{3j+2}{3^{i+1}}\right)$$

Which means the intervals $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ are disjoint from $\mathcal{C}$.

(3). We want to show that there is an integer k such that $a \leq \frac{3k+1}{3^n}$ and $\frac{3k+2}{3^n} \leq b$, so we want to have $\frac{a(3^n)-1}{3} \leq k \leq \frac{b(3^n)-2}{3}$. Such an integer k will exist if $\frac{b(3^n)-2}{3} - \frac{a(3^n)-1}{3} \geq 1$, so $(b-a)3^n \geq 4$ gives $3^{-n} \leq \frac{b-a}{4}$.

(4). If $\mathcal{C}$ contains some interval of positive length then, it must contains and open interval (a,b) but pervious result it must contains $\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$ for some, $m, k \geq 0$ but that is not possible. Hence $\mathcal{C}$ do not contains any interval of positive length.

(6). We want to show that every point of $\mathcal{C}$ is limit point. Let $x \in \mathcal{C}$ and $J = (c,d)$ an open interval, $x \in J$ and let I_n be that closed interval of F_n that contains x. Let, $a = \min\{|x-c|, |x-d|\}$, then we must get, m such that $3^m \geq \frac{1}{a}$. Thus, for every J we must get a m such that $I_m \subset J$. Take x_m be the end point of I_m. Since at every step we are removing middle third of interval of F_i, x_m will surrvive and $x_m \in \mathcal{C}$. So every deleted open set around x must intersects $\mathcal{C}$ and hence every point x is limit point. Hence $\mathcal{C}$ is perfect.

(5). From the construction of $\mathcal{C}$ we can see it is intersection of closed sets so it is closed. We know $\mathbb{R}$ with Euclidian metric is complete. And we know any closed subset of $\mathbb{R}$ is complete with respect to the restricted norm so is the Cantor set. If $\mathcal{C}$ is countable then we can write $\mathcal{C} = \{x_1, \cdots, x_n, \cdots\}$. Since each point of Cantor set is limit point we can say, $\mathcal{C} \setminus \{x_i\}$ is dense open set of the Cantor set for each $i \in \mathbb{N}$. We know in complete metric space intersection of countable open dense set is dense. Hence, $\bigcap_{i \in \mathbb{N}} \mathcal{C} \setminus \{x_i\}$ is dense but it is a null set. This gives a contradiction. Hence $\mathcal{C}$ is uncountable. ■