1. (a) Suppose X and Y are i.i.d $N(0,1)$. Using the Jacobian method, show that $\frac{X}{Y}$ has the Cauchy distribution.
(b) If X and Y are independent continuous random variables which are symmetric about 0, then $\frac{X}{Y}$ and $\frac{X}{|Y|}$ have the same distribution. Therefore, if X and Y are i.i.d $N(0,1)$, $\frac{X}{|Y|}$ also has the Cauchy distribution.

(a) Consider the transformation,

$$g(x,y) = (x/y, y).$$ g is defined on $\mathbb{R} \times \mathbb{R}/\{0\}$

Clearly $g^{-1}(u,v) = (uv, v)$. Jacobian of this map is given by,

$$J_{g^{-1}}(u,v) = \det \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}$$

$$= v.$$

So, $f_{U,V}(u,v) = f_{X,Y}(g^{-1}(u,v))\,|v|$. $[U = X/Y, V = Y]$

$$= f_{X,Y}(uv, v)|v|$$

$$= \frac{1}{2\pi} e^{-\left(\frac{u^2+1}{2}\right)v^2} |v|$$

Now we will look onto the marginal pdf of

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}\, dv = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{u^2+1}{2}\right)v^2} |v|\,dv$$

$$= \frac{1}{\pi}\int_0^{\infty} e^{-\left(\frac{u^2+1}{2}\right)v^2} v\,dv.$$

$$= \frac{1}{\pi}\frac{1}{u^2+1} \qquad u \in (-\infty,\infty)$$

So, $f_U(u)$ follows Cauchy distribution.

$$
\begin{aligned}
F_{X/Y}(u) &= \mathbb{P}\left(X/Y \leq u\right) \\
&= \mathbb{P}(X \leq uY, Y>0) + \mathbb{P}(X \geq uY, Y<0) \\
&= \mathbb{P}(X \leq uY, Y > 0) + \mathbb{P}(-X \leq -uY, Y<0) \\
&= \mathbb{P}(X \leq uY, Y>0) + \mathbb{P}(-X \leq u|Y|, Y<0) \\
&= \mathbb{P}(X \leq u|Y|, Y>0) + \mathbb{P}(-X \leq u|Y|, Y<0) \\
&= \mathbb{P}(X \leq u|Y|, Y>0) + \mathbb{P}(X \leq u|Y|, Y<0) \quad [\text{Symmetric in } X] \\
&= \mathbb{P}\left(X/|Y| \leq u\right) \\
&= F_{X/|Y|}(u).
\end{aligned}
$$

$\blacksquare$

Rest follows from Here.

2. Suppose X and Y are i.i.d $N(0,1)$. Consider the transformation $(X,Y) \to (R,\Theta)$ where $X = R\cos\Theta$ and $Y = R\sin\Theta$. Find the joint distribution of (R,Θ).

$X, Y \overset{iid}{\sim} N(0,1)$

Let, $g(x,y) = (g_1(x,y),\ g_2(x,y))$ (where $g_i(x,y)$ are Scaler functions.)

Let, $g_1(x,y) = R$

$g_2(x,y) = \theta$.

The Inverse function of g is defined as,

$$g^{-1}(R,\theta) = (R\cos\theta,\ R\sin\theta).$$

Here, $R \in (0,\infty)$ and $\theta \in [0,2\pi)$

So, $g : \mathbb{R}^2 \longrightarrow (0,\infty)\times[0,2\pi]$ If we take $g_1(x,y)$ and $g_2(x,y)$ Suitably; $g_1(x,y) = \sqrt{x^2+y^2}$; both are Continuous and partial derivatives are Continuous.

For every (R,θ) we must get unique $(x,y) \in \mathbb{R}^2$ s.t, $g_1(x,y) = R$ and g_2 measures angle of the point in with positive x direction

Jacobian of $g^{-1}(R,\theta)$ is,

$$J_{g^{-1}}(R,\theta) = \det\begin{pmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{pmatrix}$$

$= R \neq 0$ So g is continuous

So, after applying change of density formula we must get,

$$f_{R,\theta}(r,\theta) = f_{X,Y}(g^{-1}(r,\theta))\, r$$

$$= \frac{1}{2\pi}\exp\left\{-\frac{1}{2}r^2(\cos^2\theta+\sin^2\theta)\right\} r$$

$$= \frac{r}{2\pi}\exp\left\{-\frac{r^2}{2}\right\}.$$

So, $f_{R,\theta}(r,\theta) = \left(re^{-r^2/2}\right)\cdot\left(\frac{1}{2\pi}\right)$ $\quad r \in [0,\infty)$, $\theta \in [0,2\pi)$.

3. Let $Y_1, \ldots, Y_n$ be independent random variables with unit variance, and let $X_1 = Y_1$, $X_i = Y_i - Y_{i-1}$ for $1 < i \le n$. Find the covariance matrix of $\mathbf{X} = (X_1, X_2, \ldots, X_n)'$.

Let, $Y_1, \ldots, Y_n$ be Independent random Variables with Variance 1.

Take, $\underset{\sim}{Y} = (Y_1, \ldots, Y_n)^T$ be the Random Vector.

$$\text{Cov}(\underset{\sim}{Y}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = I_{n\times n} = \Sigma_Y \text{ (Say)}.$$

Now,

$$X_1 = Y_1$$
$$X_2 = Y_2 - Y_1$$
$$\vdots$$
$$X_i = Y_i - Y_{i-1}$$

So,

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & & 0 \\ -1 & 1 & \cdots & & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & & \cdots & -1 & 1 \end{pmatrix}}_{\text{(Call it } A.)} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

Then. $\text{Cov}(\underset{\sim}{X}) = A\,\text{Cov}(\underset{\sim}{Y})A^T$
$= AIA^T = AA^T$

Inner product of the i^{th} Row$_A$ $\langle R_i, R_i \rangle = 2$.

and $\langle R_{i+1}, R_i \rangle = \left\langle \begin{pmatrix} 0 \\ \vdots \\ -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \right\rangle = -1$.

Aso, $\langle R_{i+1}, R_i \rangle = (-1)$

So, $(AA^T)_{ij} = \begin{cases} -1 & \text{if } |i-j| = 1 \\ 2 & \text{if } i = j \neq 1. \\ 1 & \text{if } i = j = 1 \\ 0 & \text{o.w} \end{cases}$

$$\text{So, } \Sigma_X = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & -1 & 2 & \ddots & \\ & & & & -1 \\ & & & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & \cdots & & \\ -1 & 2 & -1 & & \\ 0 & -1 & 2 & -1 & \\ & & & & -1 \\ 0 & & \cdots & -1 & 2 \end{pmatrix}$$

The above is the Co-variance Matrix.

4. Suppose $\Sigma =$ Cov $(X) = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$. Show that $-1/2 \leq \rho \leq 1$.

$$\Sigma = \text{Cov}(X) = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}.$$

This is a Covariance matrix. So it is positive and Semi definite.

So, $\det(\Sigma) \geqslant 0$ — ⓘ

Now, $|\Sigma_{ij}| \leqslant \sqrt{\Sigma_{ii}\Sigma_{jj}}$ (By Cauchy Swerz)

$\Rightarrow |\rho| \leqslant 1.$ — (ii)

From ⓘ we get,

$$\det(\Sigma) \geqslant 0$$

$$\Rightarrow (1-\rho^2) + \rho(\rho^2-\rho) + \rho(\rho^2-\rho) \geqslant 0$$

$$\Rightarrow 2\rho^2(\rho-1) + (1+\rho)(1-\rho) \geqslant 0$$

$$\Rightarrow (\rho-1)(2\rho^2-\rho-1) \geqslant 0$$

$$\Rightarrow (\rho-1)^2(2\rho+1) \geqslant 0$$

$$\Rightarrow 2\rho+1 \geqslant 0$$

$$\Rightarrow \rho \geqslant -\frac{1}{2} \quad \text{— (iii)}$$

Combining (ii) together with (iii) we get,

$$\boxed{-\frac{1}{2} \leqslant \rho \leqslant 1}.$$

5. Let $X_1, \ldots, X_n$ be i.i.d Exponential with mean 1. Define $Y_1 = nX_{(1)}$, $Y_2 = (n-1)(X_{(2)} - X_{(1)})$, $Y_i = (n-i+1)(X_{(i)} - X_{(i-1)})$ for $3 \leq i \leq n$, where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the order statistics. Show that $Y_1, \ldots, Y_n$ are i.i.d Exponential with mean 1.

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \mathrm{Exp}(1)$$

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$$

$$Y_1 = nX_{(1)}, \quad Y_2 = (n-1)(X_{(2)} - X_{(1)}).$$

$$\vdots$$

$$Y_i = (n+1-i)(X_{(i)} - X_{(i-1)}).$$

Now,

$$(Y_1, \ldots, Y_n)^T = \underbrace{\begin{pmatrix} n & 0 & 0 & \cdots & 0 \\ -(n-1) & (n-1) & 0 & \cdots & 0 \\ 0 & -(n-2) & (n-2) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}}_{\text{Call this matrix } A.} (X_{(1)}, \ldots, X_{(n)})^T$$

$$\det(A) = n! \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ 0 & & \ddots & \\ 0 & & -1 & 1 \end{pmatrix} = n! \neq 0$$

~~So~~ Using change of Density formula we get,

$$f_{Y_1, \ldots, Y_n}(y_1, \ldots, y_n) = f_{X_{(1)}, \ldots, X_{(n)}}(A^{-1}y) \det(A)^{-1}$$

$$= \frac{1}{n!} f_{X_{(1)}, \ldots, X_{(n)}}(A^{-1}(y)).$$

Let, $A^{-1}y = (y_1', \ldots, y_n')^T$ So,

$$f_{Y_1, \ldots, Y_n}(y_1, \ldots, y_n) = \frac{1}{n!} f_{X_{(1)}, \ldots, X_{(n)}}(y_1', \ldots, y_n')$$

$$= \frac{1}{n!}\left(n! \prod_{i=1}^{n} f(y_i')\right)$$

[f is pdf of X_i]

$$= \prod_{i=1}^{n} \exp(-y_i')$$

$$= \exp\left(-\sum_{i=1}^{n} y_i'\right).$$

Notice that,

$$\sum_{i=1}^{n} y_i' = (1,1,\dots,1)\,(y_1',\dots,y_n')$$
$$= (1,\dots,1)\,A^{-1}y$$
$$= (1,1,\dots,1)\,(x_{(1)},\dots,x_{(n)})^T$$
$$= \sum_{i=1}^{n} x_{(i)}$$

Notice that, $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} (n+1-i)(x_{(i)} - x_{(i-1)})$. (treat, $x_{(0)}=0$)

$$= \sum_{i=1}^{n} (n+1-i)x_{(i)} - (n+1-i-1)x_{(i-1)} + x_{(i-1)}$$
$$= \sum_{i=1}^{n} x_{(i)}$$

So, $f_{Y_1,\dots,Y_n}(y_1,\dots,y_n) = \exp\left(-\sum_{i=1} y_i\right)$.

So Look at the marginal pdf.

$$f_{Y_i}(y_i) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^{n} y_i\right) dy_1 \dots dy_{i-1}\, dy_{i+1} \dots dy_n$$
$$= \exp(-y_i)$$

So, $Y_i \sim \mathrm{Exp}(1)$. $\forall\, i = 1,\dots,n$.

Since for any Y_i and Y_j (with $i \neq j$)

$$f_{Y_i,Y_j}(y_i,y_j) = f_{Y_i}(y_i)\, f_{Y_j}(y_j).$$

So, $Y_i \perp\!\!\!\perp Y_j$ for all $i \neq j$. So,

$$Y_1,\dots,Y_n \overset{iid}{\sim} \mathrm{Exp}(1)$$