Rings And Modules

Assignment-0

Q 1. If R is a ring in which $x^2 - x$ belongs to the center $Z(R)$ for each $x \in R$. Prove that R must be commutative.

$(x+y)^2 - (x+y) \in Z(R)$

$\Rightarrow (x^2-x) + (y^2-y) + (xy+yx) \in Z(R)$

$\Rightarrow \quad xy + yx \in Z(R)$.

So, $y(xy+yx) = (xy+yx)y$

$\Rightarrow y^2x = xy^2$.

So, every square element commutes.

$\Rightarrow (y^2-y)x + yx = xy + x(y^2-y)$

$\Rightarrow yx = xy \quad [y^2-y \in Z(R)]$.

Q 2. Let R be a ring in which $x^2 = 0$ implies $x = 0$. Show that any idempotent e of R must be in the center $Z(R)$. In particular, in a ring R in which the only nilpotent element is 0, each idempotent element is contained in $Z(R)$.

\# Let, e be an idempotent element of R.

$(ye - eye)^2$

$= (yeye - ye(eye) - (eye)(ye) + eyeeye$.

$= (yeye - yeye - eyeye + eyeye)$

$= 0$

$\Rightarrow ye = eye$.

Similarly,

$(ey - eye)^2 = 0$

$\Rightarrow ey - eye = 0$

$\Rightarrow ey = ye \quad \forall y \in R$

$\Rightarrow e \in Z(R)$.

Q 3. Let R be a commutative ring with unity. If u is unit, and a is nilpotent, prove that $u + a$ is a unit.

a is nilpotent element $\Rightarrow$ $a \in$ nilrad $(R) \subseteq J$.

So, $1+a$ is unit.

if $1+a$ is not unit then $1+a \in J$ (Jacobson Radical,

Let, $a \in m$ (a maximum ideal)

$\Rightarrow 1 \in m \quad \rightarrow\leftarrow$

$\Rightarrow$ $1+a$ is unit.

u is a unit. Let, $u^{-1}.u = 1 = u.u^{-1}$

$1+u^{-1}a$ is a unit.

product of two unit is unit.

$\Rightarrow u(1+u^{-1}a) = u+a$ is a unit.

Q 4. Give an example of a ring and nilpotent elements a, b in it such that $a + b$ is not nilpotent. Can this happen in a commutative ring? Why or why not?

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Q 5. Let d be a square-free positive integer. Consider the rings $\mathbb{Z}[\sqrt{d}] := \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$ and $\mathbb{Z}[\sqrt{-d}] := \{a + b\sqrt{-d} : a, b \in \mathbb{Z}\}$ under the usual addition and multiplication of complex numbers. Show that the first ring has infinitely many units. Further, find all the units in the second ring.

Let, $a, b \in \mathbb{Z}$. If we can get solution to the diophentine equation $a^2 - db^2 = 1$. So, there are infinitely many units of $\mathbb{Z}(\sqrt{d})$

Q 6. Let R be any ring in which the equation $ax = b$ has solutions for any $a \neq 0$ and $b \in R$. Prove:

(i) R has no (left or right) zero-divisors other than 0,

(ii) R has a unity,

(iii) R is a division ring or a field.

(i) $a \neq 0$; $za = 0$ for some $z \neq 0$.

$zt = b$ $(b \neq 0)$ (exist such t)

$ax = t$ (has solution) $(a \neq 0)(t \neq 0)$

$\Rightarrow zax = zt$

$\Rightarrow b = 0 \rightarrow\leftarrow$ (No Right inverse)

$az = 0$ for some $z \neq 0$,
$at = b \neq 0$ has solution $t \neq 0$

$zx = t$ has solution

$0 = azx = at = b \neq 0$

$\Rightarrow \rightarrow\leftarrow$

(ii)

$ae = a$
$aex = a^2$

$\Rightarrow ax = a^2$

$\Rightarrow a(x-a) = 0$

$\Rightarrow x - a = 0$

$x = a$

$\Rightarrow ae = ea = a \Rightarrow e =$ unity of R.

(iii) follows from 2.

Q 7. Let R be a commutative ring and $f \in R[X]$ be a polynomial such that $fg = 0$ in $R[X]$ for some polynomial $g \neq 0$. Show that there exists $r \neq 0$ in R such that $rf = 0$.

$f(x) = a_n x^n + \cdots + a_0$
$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$
g be the polynomial with minimum degree

Such that, $fg = 0$ $\Rightarrow a_n b_m = 0$
$\Rightarrow a_n g = 0$ (as $\deg(a_n g) = 0$).

If, $f = a_0$ ($\deg f = 0$) $\Rightarrow b_m a_0 = b_m f = 0$.

Assume for $\deg \leq n-1$ we have shown $f'g = 0 \Rightarrow b_m f' = 0$

(Induction) Consider $g(f - a_n x^n) = 0$ (we can see it clearly).

$\Rightarrow g(f - a_n x^n) = gf' = 0$

$\Rightarrow b_m f' = 0 \Rightarrow \boxed{b_m f = 0}$

Q 8. Prove that for a commutative ring R, a polynomial $f = c_0 + c_1 X + \cdots + c_n X^n$ in $R[X]$ is nilpotent (as an element of $R[X]$) if, and only if, each $c_i \in R$ is nilpotent. Further, show that a polynomial $g = a_0 + a_1 X + \cdots + a_d X^d \in R[X]$ is a unit in $R[X]$ if and only if, $a_0 \in R$ is a unit and $a_1, \cdots, a_d \in R$ are nilpotent.

Nilpotency ($\Leftarrow$) $c_i \in \text{nilrad}(R) \rightarrow c_i x^i \in \text{nilrad}(R)$
$\Rightarrow \sum c_i x^i \in \text{nilrad}(R)$.

($\Rightarrow$) f is nilpotent $\Rightarrow f^n = 0$; $f = a_m x^m + \cdots + a_0 \Rightarrow a_m^n = 0$

$\Rightarrow (f - a_m x^m) \in \text{nilrad}(R_{[x]}) \Rightarrow a_{m-1} \in \text{nilrad}(R)$

$\vdots$

Iduct!!

unit: $g(x) = a_0 + a_1 x + \cdots + a_n x^n$

If g is unit in $R[x]$ then, $fg = 1 \Rightarrow a_0 b_0 = 1$ for some $b_0 \in R$

So, a_0 is unit in ~~$R[x]$~~ ~~or~~ R.

Compare terms of fg.

$$C_{m+n} = a_n b_m = 0$$

Claim: $a_n^r b_{m-r+1} = 0$

Induction, $a_n^k b_{m-k+1} = 0$

$$0 = C_{m+n-k} = b_{m-k} a_n + b_{m-k+1} a_{n-1} + \cdots + b_m a_{n-k}$$

$$\Rightarrow 0 = b_{m-k} a_n^{k+1} + (b_{m-k+1} a_n^k)(a_n a_{n-1}) + \cdots + (b_m a_n)(a_n^k a_{n-k})$$

$$\Rightarrow b_{m-k} a_n^{k+1} = 0$$

Now, $b_0 a_n^{m+1} = 0 \Rightarrow \boxed{a_n^{m+1} = 0}$

$\Rightarrow a_n x^n \in$ nilrad $(R[x])$

So, $g - a_n x^n$ is unit again. $\longrightarrow$ Induct.

Q 9. Let D be a division ring. Suppose the center $Z(D)$ is an infinite field. Then, show that each element $a \in D^*$ which has only finitely many conjugates under D^*, must lie in $Z(D)$.

Consider, group action $D^* \circlearrowright D$. by Conjugation

$$C(a) = \{x \in D^* \mid xa = ax\}$$

$D^*/C(a)$ = orbit(a) = finite = [Conjugacy class of a]

If, $D^*/C(a) \neq 1$.

Let, $x \in D^* \setminus C(a)$

$C_x = \{(c+x)a(c+x)^{-1}\} \mid c \in C(a)$

If, $(c_1+x)a(c_1+x)^{-1} = (c_2+x)a(c_2+x)^{-1} \Rightarrow (c_2+x)^{-1}(c_1+x) \in C(a)$

$\Rightarrow c_1 + x \in x C(a)$

$\Rightarrow c_1 + x = x c_2'$

$\Rightarrow c_1 = (1 - c_2')x$

$\Rightarrow x \in C(a)$

$\rightarrow (c+x)a(c+x)^{-1}$ are diff

$\Rightarrow |C_x| \nless \infty$ but $C_x \subseteq$ orbit (a) $\rightarrow\leftarrow$.

Q 10.

(a) If R is a ring in which $x^3 = x$ for all $x \in R$, then prove that R must be commutative.

(b) If R is a ring in which $x^4 = x$ for all $x \in R$, then prove that R must be commutative.

(c) Let R be a ring with unity 1. If $(xy)^r = xy$ is satisfied for three consecutive natural numbers r, and for all $x, y \in R$, prove that R must be commutative.

(a) (1) $\mathrm{ab} = 0 \;\Rightarrow\; \mathrm{ba} = 0$ via $\mathrm{ba} = (\mathrm{ba})^3 = \mathrm{b\,ab\,ab\,a} = 0$

(2) $\mathrm{c}^2 = \mathrm{c} \;\Rightarrow\; \mathrm{c}$ is central [which means that $\mathrm{xc} = \mathrm{cx}$ for all x]

Proof: $\mathrm{c(x - cx)} = 0 \;\Rightarrow\; \mathrm{(x - cx)c} = 0$ by (1), so $\mathrm{xc} = \mathrm{cxc}$
$\mathrm{(x - xc)c} = 0 \;\Rightarrow\; \mathrm{c(x - xc)} = 0$ by (1), so $\mathrm{cx} = \mathrm{cxc}$

(3) x^2 central via $\mathrm{c} = \mathrm{x}^2$ in (2)

(4) $\mathrm{c}^2 = 2\mathrm{c} \;\Rightarrow\; \mathrm{c}$ central. Proof: $\mathrm{c} = \mathrm{c}^3 = 2\mathrm{c}^2$ central by (3)

(5) $\mathrm{x} + \mathrm{x}^2$ central via $\mathrm{c} = \mathrm{x} + \mathrm{x}^2$ in (4)

(6) $\mathrm{x} = (\mathrm{x} + \mathrm{x}^2) - \mathrm{x}^2$ central via (3), (5) by centrals closed under subtraction. **QED**

(b) First, note $-x = (-x)^4 = x^4 = x$, so $x + x = 0$ for any x in R. Then $(x^2 + x)^2 = x^2 + x + x^3 + x^3 = x^2 + x$. Thus $x^2 + x$ is idempotent, and it is easy to see idempotent elements are central in this ring. [I give a proof of this at the end.]

Now let $x = a + b$, where a and b are arbitrary. From above, for any c in R, $c(x^2 + x) = (x^2 + x)c$, and expanding this out and cancelling terms we get $c(ab + ba) = (ab + ba)c$. Setting $c = a$, we get, after cancelling again, $a^2b = ba^2$. Thus, for any x in R, x^2 is central. Then of course $x = (x^2 + x) - x^2$ is central.

To prove that idempotents are central, first note that if $xy = 0$, then $yx = (yx)^4 = y(xy)(xy)(xy)x = 0$. So now if $z^2 = z$, then $z(y - zy) = 0$, so $(y - zy)z = 0$, or $yz = zyz$. Similarly, $(yz - y)z = 0$, so $z(yz - y) = 0$, or $zy = zyz$. Thus $yz = zy$.