Let X be a locally compact Hausdorff space. We say that X has a **one point compactification** if $\exists$ a topological space Y such that:

(i) Y is a compact Hausdorff space.
(ii) X is a open subspace of Y
(iii) $Y \setminus X$ contains only a single point.

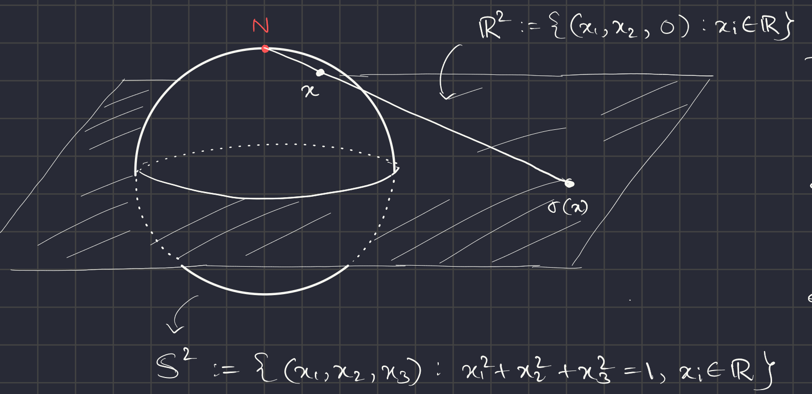
1. Let X be a locally compact Hausdorff space. A **one point compactification of X** is the topological space X^* defined as follows. Let ∞ be some object not in X, and let $X^* = X \sqcup \{\infty\}$ with the following topology:

$$\mathcal{T} = \{\text{open subsets of } X\} \cup \{\, U \subseteq X^* \mid X^* \setminus U \text{ is compact subset of } X \,\}$$

(a) Show that $\mathcal{T}$ is a topology on X^*.
(b) Show that X^* is a compact Hausdorff space.
(c) Show that X is open in X^* and has the subspace topology.
(d) Show that X is dense in X^* iff X is noncompact.

Exercise 1, shows that any locally compact Hausdorff space admits a one-point compactification. Further try to show that one-point compactification is unique upto homeomorphism, so as to say that if Y and Y' are two one-point compactification of X then $Y \cong Y'$. (Hint: construct the function $\varphi: Y \to Y'$ such that $\varphi|_X \equiv id_X$ and φ maps the point $Y \setminus X$ to the point $Y' \setminus X$. Show φ is a homeomorphism).

2. Show that a topological space is a locally compact Hausdorff space iff it is homeomorphic to an open subset of a compact Hausdorff space.



The stereographic projection of S^2 onto $\mathbb{R}^2$ is the function $\sigma : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ defined as follows, you take a point $x \in S^2 \setminus \{N\}$ then $\sigma(x)$ is the point of intersection of the line $\overline{Nx}$ with the plane $x_3 = 0$.

3. Show that σ extends to a homeomorphism of S^2 with the one-point compactification of $\mathbb{R}^2$.