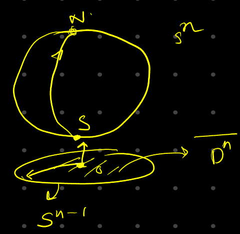
Start with an Example.

$\overline{D}^n / S^{n-1} \cong S^n$.

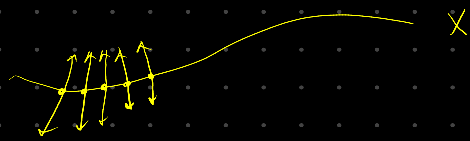
$q : \overline{D}^n \to S^n$

$q(x) = \left(2\sqrt{1-|x|^2}\,x \;,\; 2|x|-1\right)$

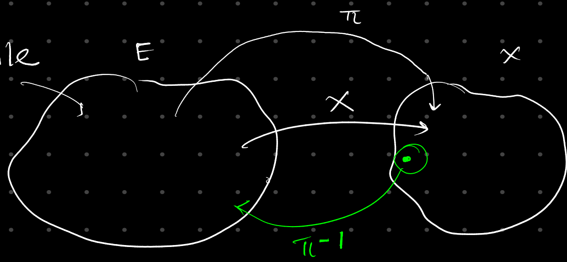


- $0 \to S$
- Radius $(\overline{D}_n) \to$ Semi great Circle (S^{n-1}).

Vector Bundle.



$(X, E, \pi) \longleftrightarrow$ Vector Bundle



π is a Surjective map

↳ $\pi : E \to X$

$\pi^{-1}(x) \to$ vector space.

(finite dim)

open nbd that contain p.

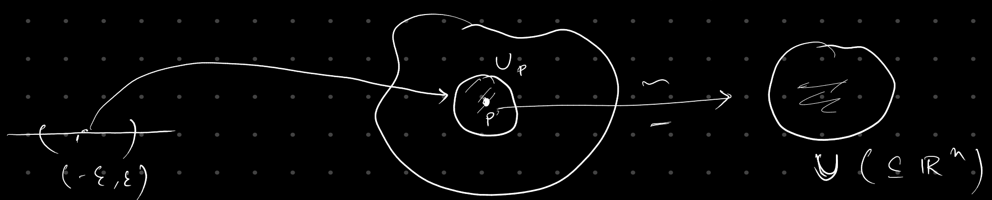
$\forall p \in X. \quad U_p \quad : \quad \underline{U_p \times \mathbb{R}^k} \cong \pi^{-1}(U_p)$

Call the (π, E, X)

Tangent Bundle. (Manifold).

"Smooth Curve"

M (n-manifold)



Smooth Curve. $\gamma : (-\varepsilon, \varepsilon) \longrightarrow U_p \xrightarrow{\varphi} \underline{U} \qquad \gamma(0) = p$

$\underline{\varphi \circ \gamma} : (-\varepsilon, \varepsilon) \longrightarrow U \; (\subseteq \mathbb{R}^n)$

diff.

$\frac{d}{dt}(\varphi \circ \gamma)\Big|_{t=0} \longrightarrow$ traces a vector in $\mathbb{R}^n$

↳ Called the tangent vector along γ.

V.S $(T_pM) \longrightarrow$ the tangent Space of the Manifold M at at point p.

{collection of all tangent vector} = $T_pM \Longrightarrow$ forms a V.S.

$$\begin{tikzcd} & M \times TM \overset{\parallel}{} E \arrow[dl] \arrow[d] & \\ \pi(x, T_x M) = x & & \end{tikzcd}$$

$$TM = \bigsqcup_{p \in M} T_p M.$$

Tangent Bundle.

Thom Space

$\hookrightarrow E$ (vector Bundle) $\to$ define a metric on this
(tensor metric)

Example with; $\mathbb{R}^n$: $\langle x, y \rangle = y^t x.$ $\langle \, , \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}.$

$\langle x, x \rangle = \|x\| \geq 0$ $\langle x, x \rangle \geq 0$

(Abuse of Cross product / But something like this)

metric tensor. $g : E \times E \to E \times \mathbb{R}$

$\boxed{g(x,x) \geq 0} \longrightarrow$ Riemann tensor.

$g(x,x)$

$DE = \{ x \in E \mid \|x\| \leq 1 \}.$

$SE = \{ x \in E \mid \|x\| = 1 \}.$

Zero section

$$Th(E) = DE / SE.$$

Trivial Bundle. $E = X \times \mathbb{R}^n$

$(x, \mathbb{R}^n)$

$\pi(x, v) = x \quad \forall v \in \mathbb{R}^n.$

$DE = X \times D_1 \cong X \times \bar{D}^n$

$D_1 = \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \} \equiv \bar{D}^n$

$SE = X \times S^{n-1}$

The Thom Space of trivial Bundle, $Th(E) = \dfrac{X \times \bar{D}^n}{X \times S^{n-1}}$

(yet to prove). $\dfrac{X\times \bar{D}^n}{X\times S^{n-1}} \cong X \wedge \left(\bar{D}^n / S^{n-1}\right) \cong \underline{X\wedge S^n}$

$\boxed{\text{Recall } \; X\times Y / X\vee Y \cong X\wedge Y}$

Recall, ΣX_+ (Reduced Susp) $\cong X\wedge S^1$

$\Sigma^n X_n$ (n^{th} Reduced Susp.) $\cong X\wedge S^n$

Thom Space of a Trivial Bundle is n^{th} Reduced Suspension of X.

$$\begin{tikzcd}
X\times \bar{D}^n \arrow[rr, "f(x{,}s)\to(x{,}q(s))", "f"'] \arrow[d, "\pi"'] \arrow[drr, "\text{* } g\circ f \text{ a quotient map}"] & & X\times \bar{D}^n/S^{n-1} \arrow[d, "g"] \\
\dfrac{X\times\bar{D}^n}{X\times S^{n-1}} \arrow[rr, "\cong ?"] & & \left(X\times \bar{D}^n/S^{n-1}\right) \Big/ X\times S^{n-1}/S^{n-1}
\end{tikzcd}$$

g is a quotient map

$\left(X\times \bar{D}^n/S^{n-1}\right) \Big/ X\times S^{n-1}/S^{n-1} \cong X\wedge \bar{D}^n/S^{n-1}$ (Def^n of Smash product)

$$\begin{tikzcd}
* \; X\times S^{n-1} \arrow[d, "f"] \\
X\times\{N\} \arrow[d, "g"] \\
X\times\{N\}/X\times\{N\} \equiv \text{point.}
\end{tikzcd}$$