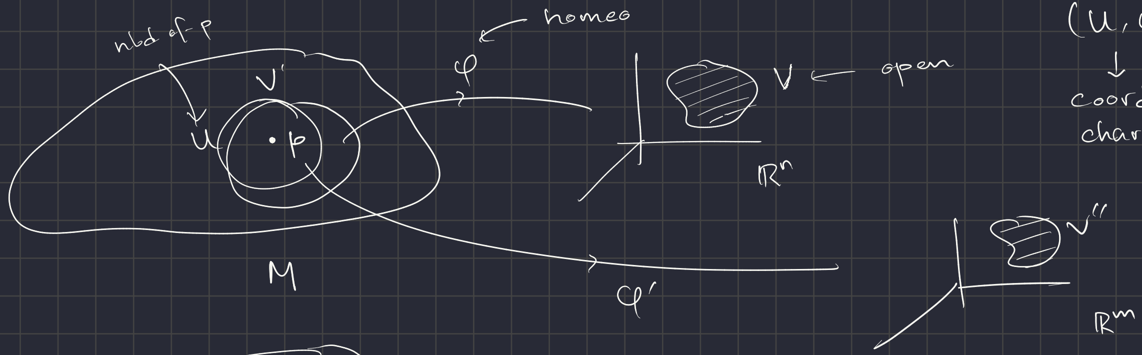
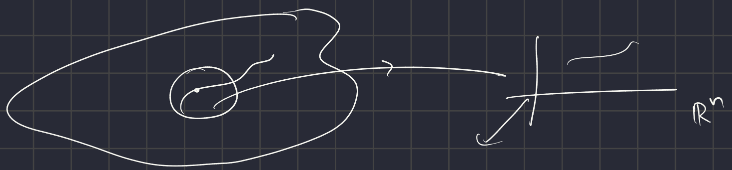
# Manifolds

topological space → ✓ locally $\underline{\underline{n}}$-dim euclidean
✓ Hausdorff
✓ Second countable

Second countable. A topo sp which admits a countable basis.




$(U, \varphi)$
↓
coordinate chart




Remarks.

Can a manifold be simul. a n-mani. and a m-mani. ? (Invariance of dimension)

$* \; U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ and $U \cong V \implies n = m$

Hausdorffness + locally euclidean $\overset{?}{\implies}$ locally compact

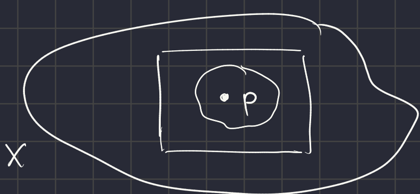
✓ locally compact
+
✓ Hausdorff
+
✓ second countable

$\Downarrow$

paracompact ?

paracompact + Hausdorff $\implies$ Existence of partitions of unity ?

**Locally compact.** X is locally compact if for any $p \in X$, $\exists$ compact subset of X which contains a nbd of p



# Paracompactness.

$\mathcal{A} \subseteq P(X)$ is said locally finite if each point $p \in X$, has some nbd which intersects finitely many elements (sets) in $\mathcal{A}$.

$\mathcal{A}$ is cover, you say another cover $\mathcal{B}$ is called refinement of $\mathcal{A}$ if for any $B \in \mathcal{B}$, $\exists$ $A \in \mathcal{A}$ such that $B \subseteq A$. Also if all sets in $\mathcal{B}$ are open, then it is called a open refinement.

Defn (paracompactness). $X$ is said paracompact if every cover of $X$ has a locally finite open refinement

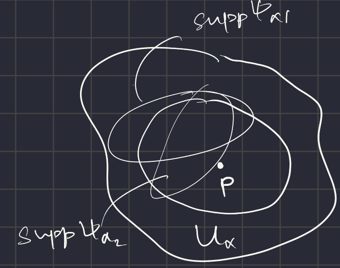
# Partitions of Unity

Defn. If $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ cover of $X$, a partition of unity subordinate to $\mathcal{U}$ it is a family cont. funcs

$$\{\psi_\alpha : X \longrightarrow [0,1]\}_{\alpha \in A}$$

which satisfies.

$V = U \cap U_\alpha$




① $\underline{\text{supp}}\ \psi_\alpha := \overline{\psi_\alpha^{-1}([0,1) \setminus \{0\})} \subseteq U_\alpha$

$\qquad\qquad\qquad \overline{\{x : \psi_\alpha(x) \neq 0\}}$

② The $\{\text{supp}\,\psi_\alpha\}_{\alpha \in A}$ has to be locally finite

③ $\underbrace{\sum_{\alpha \in A}}\psi_\alpha(p) = 1 \quad \forall p \in X$

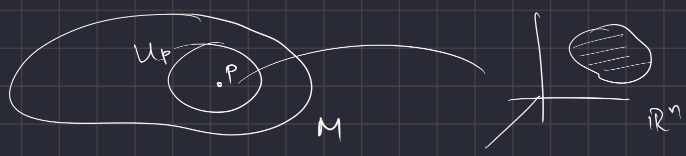
(condition ② says ③ is well defined).

**Theorem.** If $M$ is a n-dim manifold then $\exists$ embedding $i : M \hookrightarrow \mathbb{R}^{2n+1}$.
If $M$ is a smooth n-dim manifold, $\exists$ " $i : M \hookrightarrow \mathbb{R}^{2n}$

**Theorem.** Every compact manifold is homeomorphic a subset of some euclidean space.

Pf. Suppose comp n-dim mani $M$.
$\exists$ a finite open cover say $\underbrace{U_1, \dots, U_k}_{\mathcal{U}}$ which are homeomorphic a subset of $\mathbb{R}^n$.

$\exists$ a partition of unity subordinate to $\mathcal{U}$, $\psi_i : M \longrightarrow \mathbb{R}$ $(\psi_i \in [0,1])$




further $\varphi_i : U_i \hookrightarrow \mathbb{R}^n$

$\{U_p\}_{p \in M}$

$F_i : M \longrightarrow \mathbb{R}^n$

supp $\psi_i \subseteq U_i$

$$F_i(p) = \begin{cases} \underset{\mathbb{R}}{\psi_i(p)}\ \underset{\mathbb{R}^n}{\varphi_i(p)} & p \in U_i \\ 0 & \text{if } p \in M \setminus \text{supp}\,\psi_i \end{cases}$$

Now define $F : M \longrightarrow \mathbb{R}^{nk+k}$

$$F(p) = \underbrace{(\underset{\mathbb{R}^n}{F_1(p)}, \dots, F_k(p), \underset{\mathbb{R}}{\psi_1(p)}, \dots, \psi_k(p))}_{\in\ \mathbb{R}^{nk+k}}$$

Take $F(x) = F(y)$, $\quad \sum_{i=1}^{k} \psi_i(x) = \sum_{i=1}^{k} \psi_i(y) = 1$ (by ③)

$\hookrightarrow \psi_i(x) = \psi_i(y)$ $\quad \hookrightarrow$ some $\psi_i(x) > 0$

$\Rightarrow x \in \operatorname{supp}\psi_i \subseteq U_i$

$\Rightarrow \psi_i(y) > 0$

$\Rightarrow y \in \operatorname{supp}\psi_i \subseteq U_i$

$\Rightarrow F_i(x) = F_i(y) \xRightarrow{\because x, y \in U_i} \psi_i(x)\varphi_i(x) = \psi_i(y)\varphi_i(y)$

$\Rightarrow \varphi_i(x) = \varphi_i(y)$

$\Rightarrow x = y$

Thus $F$ is inj, $F(M) \cong M$ (closed map Lemma)

$F : X \longrightarrow Y$ then

$X$ compact, $Y$ Hausdorff

① If $F$ is inj $\Rightarrow$ embedding

② If $F$ is bij $\Rightarrow$ homeomorphism.

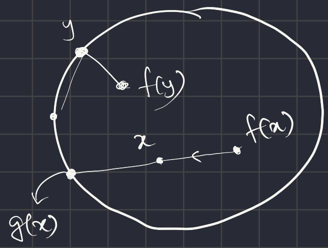
# Brouwer's FPT

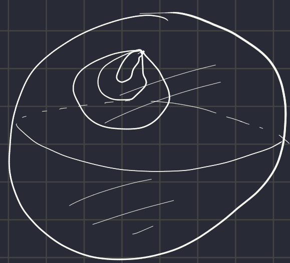
Theorem. $f : D^n \longrightarrow D^n$ continuous, then it admits a fixed point, i.e $f(x) = x$ for some $x$.

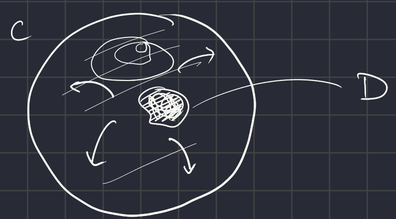
$D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$

$n = 1$. (trivial)

Say that $f$ doesnot any fixed point $f(x) \neq x \ \forall x \in D^n$




$g : D^n \longrightarrow S^{n-1}$ continuous

$D^n \subset \mathbb{R}^n$, $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ it is surj (?)

$g|_{S^{n-1}} = id_{S^{n-1}}$

thus $\nexists$ any continuous function $g$

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$n = 2$, $g : D^2 \longrightarrow S^1$ such $g|_{S^1} \equiv id_{S^1}$

$g_* : \pi_1(D^2) \longrightarrow \pi_1(S^1)$

$\{1\} \longrightarrow \mathbb{Z}$

cannot happen

$\pi_1 : \text{Top} \longrightarrow \text{Grp}$

$(\pi_1, \pi_1) : X \longmapsto \pi_1(X) \leftarrow$ fundamental grp

$f : X \longrightarrow Y$

induces $\hookrightarrow$ $f_* : \pi_1(X) \longrightarrow \pi_1(Y)$