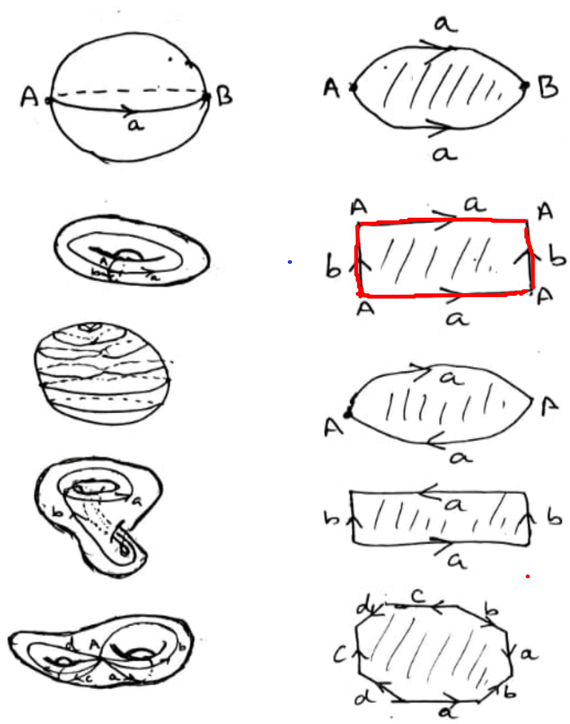
GOAL: Classification of all connected closed surfaces
(closed = compact & without boundary)

$\mathbb{D}^2$

$\mathbb{R}^2$

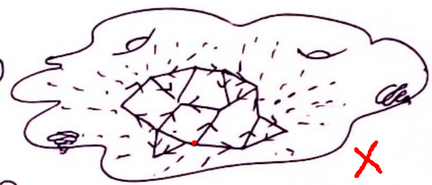
$\mathbb{Z} \times \mathbb{Z}$



$\mathbb{S}^2$	aa^{-1}
$\mathbb{T}^2$	$aba^{-1}b^{-1}$
$\mathbb{RP}^2$	aa
$\mathbb{K}$	$abab^{-1}$
$\mathbb{T}^2 \# \mathbb{T}^2$	$aba^{-1}b^{-1}cdc^{-1}d^{-1}$

① Step 1: Subdivide the surface with polygons (finitely many) and orient the sides to keep track of the gluing scheme. $P = \bigcup_{i=1}^{n} \{p_i\}$

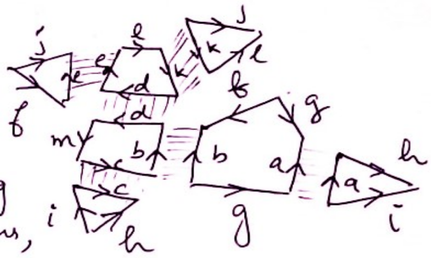
Compactness



X $a \in p_{i_j}$

Step 2: Lay these flat on a table and start gluing identified sides keeping the polygons flat by the following scheme: Note that if there are more than 1 polygons, $\forall i, j,\ \exists a \in p_{i_j}$ st $a \in p_{k_j}$ for some $k \neq i$.

Connectness



Glue $a \in p_{i_j}$ with $a \in p_{k_j}$ and go to $(j+1)^{th}$ stage.

Step 3: Clearly, this ends with a single polygon with each side identified with another (no boundary), so in total, there are even # of sides.

We now have our desired polygonal-representation which can be used to pick a word corresponding to the polygon (& thus the surface).

$\alpha, \beta, \gamma \cdots$ sequences of letters $\qquad a, b, c \cdots$ letters

(1) $\alpha a \sim a\alpha$ (Cyclicity)

(2) $\alpha \sim \alpha^{-1}$ (Inversion: $(\beta a)^{-1} = a^{-1}\beta^{-1}$)

(3) $\alpha a a^{-1} \sim \alpha$ (Cancelation: the empty word represents the sphere)

(4) $\alpha a \beta \beta' a^{-1} \sim \alpha a \beta' \beta a^{-1}$ (Rule for discord letters)

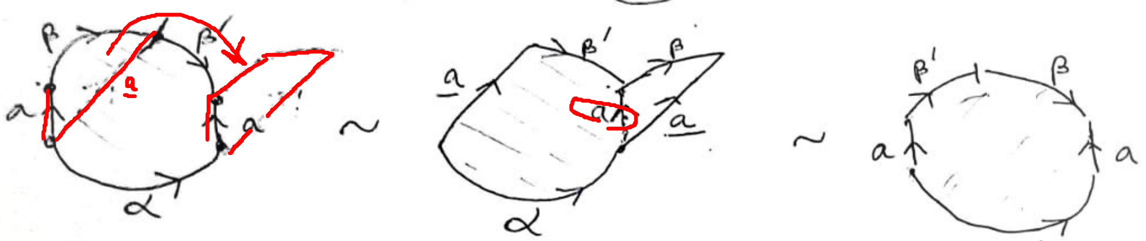
(5) $\alpha a \beta a \sim \alpha \beta^{-1} a a$ (Rule for concord letters)

(Can think of these relations defined in a free group)

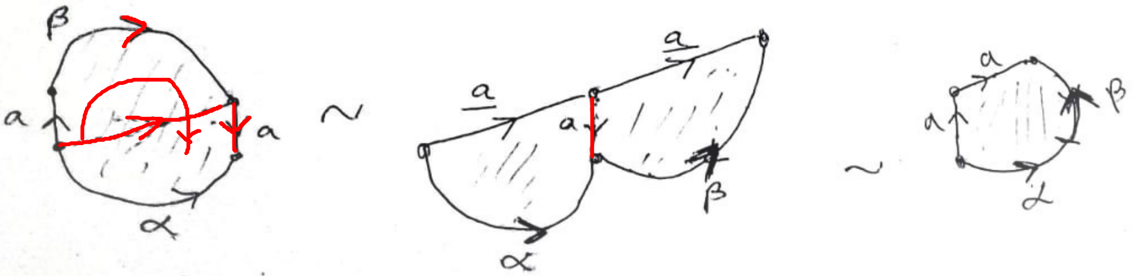
(1) & (2) are trivial to show:

(3)

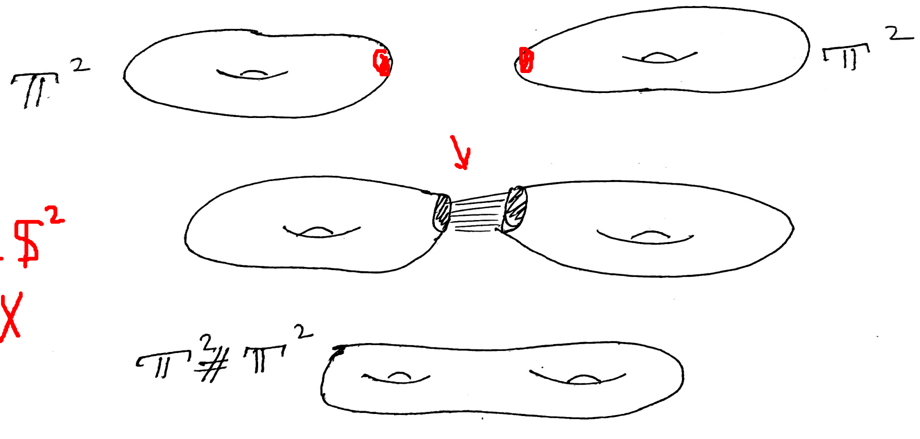
(4)



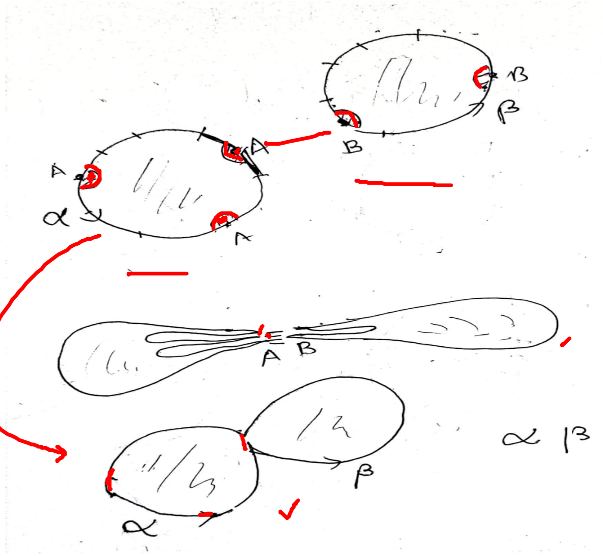
(5)



Connected sum



$X \# S^2$
$= X$



Also,

if $X = \alpha\beta$

where every second occurence of a letter in β is in β itself,

then $(X) = (\alpha)\#(\beta)$

and α and β can be manipulated freely according to (1) to (5)

i.e, if α, β themselves represent a surface

then $\beta \sim \beta'$ & $\alpha \sim \alpha' \Rightarrow \alpha\beta \sim \alpha'\beta'$

Ex:

(i) $\mathbb{RP}^2 \# \mathbb{RP}^2 = aabb \overset{(5)}{=} aba^{-1}b = \mathbb{K}$

(ii) $\mathbb{RP}^2 \# \mathbb{T}^2 = aabcb^{-1}c^{-1} \overset{(1)}{=} (bc)(cb)^{-1}aa \overset{(5)}{=} bcacba$

$\overset{(1)}{=} abcacb \overset{(5)}{=} (ac^{-1}a^{-1}c^{-1})bb \overset{(2)}{=} caca^{-1}bb = \mathbb{K} \# \mathbb{RP}^2$

$\overset{(1)}{=} \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$.

The Classification Thm of Connected Closed Surfaces

* Every connected closed surface can be reduced to one of the following:

(i) $\mathbb{S}^2$ (ii) $\mathbb{T}^2 \# \mathbb{T}^2 \# \mathbb{T}^2 \cdots \# \mathbb{T}^2$

(iii) $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$

$\alpha\, aa\beta\gamma = \alpha\beta aa\gamma$

Proof: Step 1: Hive off every concord pair (if any exist) to the right.

Say $X = \alpha a \beta a \gamma \overset{(1)}{=} \gamma\alpha a\beta a \overset{(5)}{=} \gamma\alpha\beta^{-1}aa = (X_1) \# \mathbb{RP}^2$

Similarly, if X_1 has a concord pair, $X = X_2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

Repeat until Y has only discord pairs, and

$$X = Y \# (\mathbb{RP}^2)^{\# n}.$$

Step 2: Hive off every pair of discord letters separated by another pair to the left. (Again, if any exist)

Say $Y = \alpha a \beta b \gamma a^{-1} \delta b^{-1} \varepsilon \overset{(4)}{=} \alpha a b \gamma \beta a^{-1} \delta b^{-1} \varepsilon \overset{(4)}{=} \alpha a b a^{-1} \delta \gamma \beta b^{-1} \varepsilon$

$= aba^{-1}b^{-1}\varepsilon\alpha\delta\gamma\beta = \mathbb{T}^2 \# Y_1$

Repeat as before to get $Y = (\mathbb{T}^2)^{\# m} \# Z$ where every separated discord pair is separated by a word (in Z) representing a surface. Repeating this inductively yields $Z = \mathbb{S}^2$.

$a \quad b \quad c \quad c^{-1} b^{-1} \quad a^{-1}$

So far $X = (\mathbb{T}^2)^{\#m} \# S^2 \# (\mathbb{RP}^2)^{\#n} = (\mathbb{T}^2)^{\#m} \# (\mathbb{RP}^2)^{\#n}$

If $m = n = 0$, $X = S^2$

If $m \neq 0$, $n = 0$, $X = (\mathbb{T}^2)^{\# m}$

If $m \neq 0$, $n \neq 0$,

$X = (\mathbb{T}^2)^{\#(m-1)} \# \mathbb{T}^2 \# \mathbb{RP}^2 \# (\mathbb{RP}^2)^{\#(n-1)}$

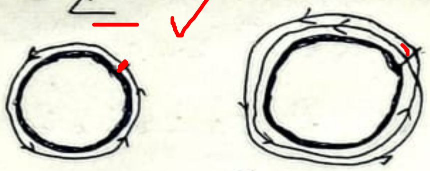
$= (\mathbb{T}^2)^{\#(m-1)} \# (\mathbb{RP}^2)^{\#(n+2)}$ (using Ex ii)

(Repeat this inductively

$= (\mathbb{RP}^2)^{\#(n+2m)}$. ∎

(m = 0, n ≠ 0, trivial)

$\pi_1(S^2) = \mathbb{Z}$ ✓



$0 \leq s \leq 1$: $\gamma_0(s) = e^{2\pi i s}$ $\gamma_1(s) = e^{2\pi i (2s)}$

Fundamental Theorem of Algebra

* Every non-constant polynomial with complex coeff. has a root in $\mathbb{C}$.

Proof. Let $p(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$ be the given polynomial

Assume $p(z) \neq 0 \ \forall z$ so $f_r(s) = \dfrac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$ is well defined.

$$f_r(0) = 1 = f_r(1)$$

For small r $(r = 0)$

$f_r(s) = 1 \quad \forall\, 0 \leq s \leq 1$

For large r $(r > \max(1, |a_1| + |a_2| + \cdots |a_n|))$

$f_r(s) \simeq \dfrac{r^n e^{2\pi i (ns)}/r^n}{1}$

$= e^{2\pi i (ns)}$

if $|z| = R > \max(1, |a_1| + |a_2| + \cdots |a_n|)$

$|z^n| = R^n = R R^{n-1} > (|a_1| + |a_2| + \cdots |a_n|) R^{n-1} \geq |a_1 z^{n-1} + \cdots + a_n|$

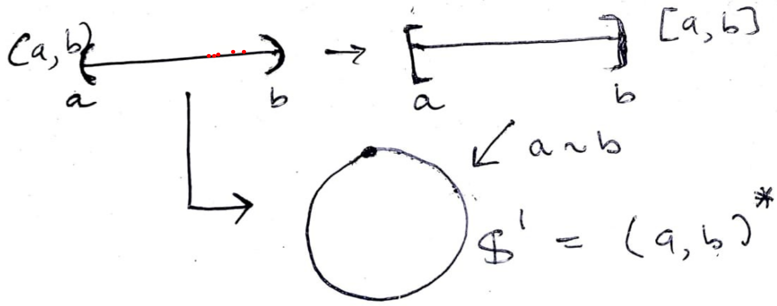
Hence $P_t(z) = z^n + t(a_1 z^{n-1} + \cdots a_n) \neq 0 \quad \forall\ 0 \leq t \leq 1$ & $|z| = R$.

$$g_R(t,s) = \frac{P_t(Re^{2\pi i s})/P_t(R)}{|P_t(Re^{2\pi i s})/P_t(R)|}$$ ✓

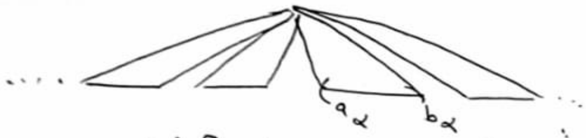
doesn't move $\leftarrow f_0(\cdot) \simeq f_R(\cdot) = g_R(1,\cdot) \simeq g_R(0,\cdot) \rightarrow$ goes around n times

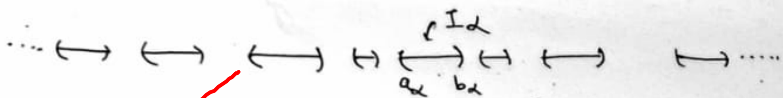
$n = 0$ $\rightarrow\leftarrow$

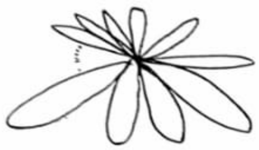
One point-compactification



$X = \bigcup_{\alpha=1}^{\infty} I_\alpha$ $(a_\alpha, b_\alpha) = I_\alpha \subseteq \mathbb{R}$ open intervals, $I_\alpha \cap I_{\alpha'} = \phi$ $(\forall \alpha \neq \alpha')$; $\boxed{l = \inf_\alpha (b_\alpha - a_\alpha)}$

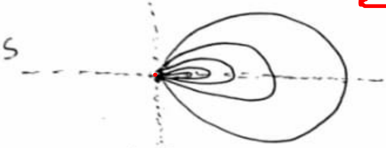






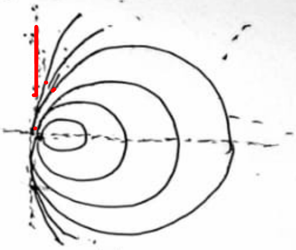
$S^2 \wedge S^1 \wedge S^1 \ldots\ldots$

vs



Hawaiian earring
$\text{radius}_n = 1/n$

vs



$\text{radius}_n = n$

$\prod_{n=1}^{\infty} [0, 1/n] = [0,1]^{\infty}$ → $h(a)_n = n \cdot a_n$

Hilbert's cube