Quotient Topology.

\# quotient topology $\quad f: X \text{ (top. Space)} \to Y$

$U \subseteq Y$, $\quad U$ as "open" $\Leftrightarrow f^{-1}(U)$

\# $X/\sim \quad = $ set of all equivalance class in X.

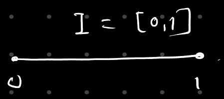
$\downarrow$

quotient space.

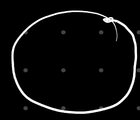
$\pi : X \longrightarrow X/\sim$

$U \subseteq X/\sim \;\to$ open iff $\pi^{-1}(U)$ is open X.

Ex.



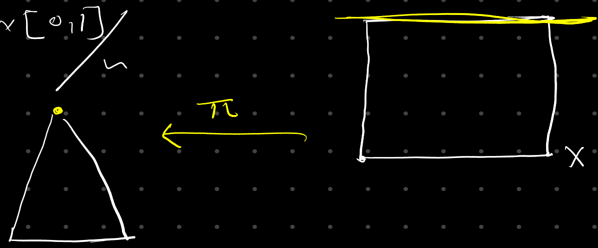
$\sim : 0 \sim 1$. $\quad I/\sim \;\longrightarrow\; ?$



$$I/\sim \;\cong S^1$$

\# Cone of a topo. $\quad X \times [0,1] \quad \sim : (x,1) \sim (y,1) \quad \forall\, x,y \in X.$

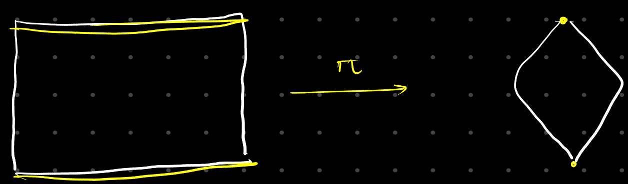
$$CX = X \times [0,1]/\sim$$



\# Suspension $\quad X \times [0,1] \quad \sim : (x,1) \sim (y,1) \quad \forall\, x,y \in X$

$(x,0) \sim (y,0) \quad \forall\, (x,y) \in X$

ΣX.



Thm. $X/\sim$ Hausdorff $\quad [x]$ is closed X.

Proof. $\quad [x]$ is not "closed" in X.

$y \in \partial[x] \hookrightarrow y \notin [x]$ –

$[x] \neq [y]$

$X/\sim \;\to\; [x] \quad [y] \qquad U \supseteq [y] \quad V \supseteq [x] \qquad U \cap V = \phi \qquad \pi^{-1}(U \cap V) = \phi$

Hausdorff

$\Rightarrow \pi^{-1}(U) \cap \pi^{-1}(V) = \phi$

$\uparrow$ open set

$[x]$ closed. ▨

$A \subseteq X$. $\quad X/A \to ?$: All the points of A are equivalent.

$A_1, \dots, A_r \subseteq X$. $\quad X/A_1, \dots, A_r$

$X/A_1, \dots, A_r \to$ Hausdorff $\iff A_1, \dots, A_r$ closed.

Thm.

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "f \circ q"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

f is Cont. $\iff f \circ q$ is Cont.

Proof. U open in $Z \Rightarrow f^{-1}(U)$ open $\iff q^{-1}(f^{-1}(U))$ open
$\iff (f \circ q)^{-1}(U)$ open

f cont $\iff f \circ q$ cont. $\boxtimes$

Thm. (Passing the quotient)

$$\begin{tikzcd} X \arrow[d, "q"'] \arrow[dr, "\tilde{f}"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

$\tilde{f}$ = this is Constant on "fibre" of q
$\tilde{f}(q^{-1}(y)) = \text{const}$.

$\tilde{f} = f \circ q$

f is unique Cont map. s.t. $\tilde{f} = f \circ q$.

Proof. $q(x) = y$
$\Rightarrow f(y) = \tilde{f}(x) \longrightarrow$ Conc.

Thm. (uniqueness):

$$\begin{tikzcd} X \arrow[d, "q_1"'] \arrow[dr, "q_2"] & \\ Y_1 \arrow[r, "\cong"'] & Y_2 \end{tikzcd}$$

$x, y \in X$
$q_1(x) = q_1(y)$
$\Updownarrow$
$q_2(x) = q_2(y)$.

Proof.

$$\begin{tikzcd} X \arrow[d, "q_1"'] \arrow[dr, "q_2"] & \\ Y_1 \arrow[r, "\tilde{q}_1"'] & Y_2 \end{tikzcd}$$

$\tilde{q}_1 \circ q_1 = q_2$ —*

$$\begin{tikzcd} X \arrow[d, "q_2"'] \arrow[dr, "q_1"] & \\ Y_2 \arrow[r, "\tilde{q}_2"'] & Y_1 \end{tikzcd}$$

$\tilde{q}_2 \circ q_2 = q_1$

$\Rightarrow \tilde{q}_2 \circ \tilde{q}_1 \circ q_1 = q_1$

$$\begin{tikzcd} X \arrow[d, "q_1"'] \arrow[dr, "q_1"] & \\ Y_1 \arrow[r, "Id"'] & Y_1 \end{tikzcd}$$

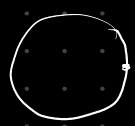
$\underline{\tilde{q}_2 \circ \tilde{q}_1} (q_1) = q_1$.

$q_1 \circ Id = q_1 \to$ ✓

$$\tilde{q}_2 \circ \tilde{q}_1 = Id$$
$$\tilde{q}_1 \circ \tilde{q}_2 = Id.$$

$$Y_1 \cong Y_2.$$

Ex. $I = [0,1]$, $0 \sim 1$. $I/\sim \;\cong S^1$



$f(x) = \exp\{2\pi x i\}$.

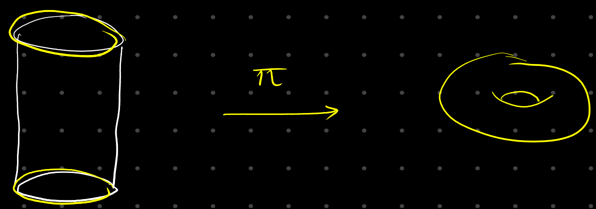
$f \to$ cts.

$f \to$ surj.

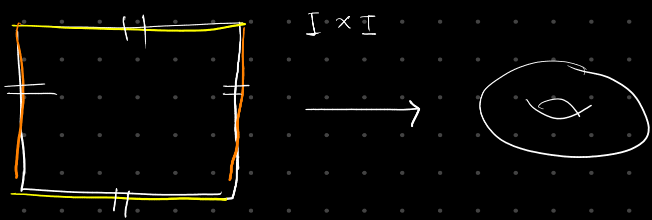
$f(0) = f(1) \to$ Satisfies Identification Prop.

$$\boxed{I/\sim \;\cong S^1}$$

Ex. (torus)



$= S^1 \times S^1$



$I \times I$, $\sim$: $(x,0) \sim (x,1)$
$(0,y) \sim (1,y)$

$I\times I/\sim \; = T \cong S^1 \times S^1$

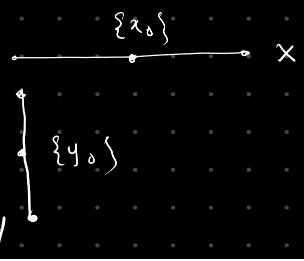
$f: I \times I \longrightarrow S^1 \times S^1$

$\hookrightarrow f(x,y) = \left(e^{2\pi x i}, e^{2\pi y i}\right)$

cont + surj + identified on equiv. class.

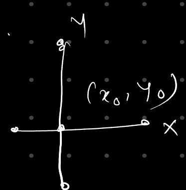
$$I\times I/\sim \;\cong S^1 \times S^1$$

Wedge Product.



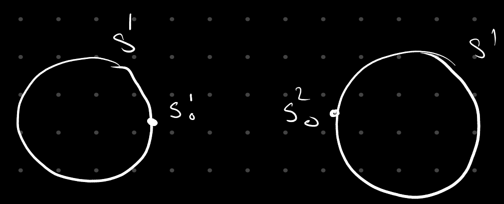
$X \sqcup Y$ $\sim$: $x_0 \sim y_0$.

$X \sqcup Y/\sim \; \longrightarrow X \vee Y$

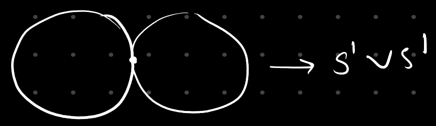


$X \vee Y \subseteq X \times Y$ $\qquad X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$

#

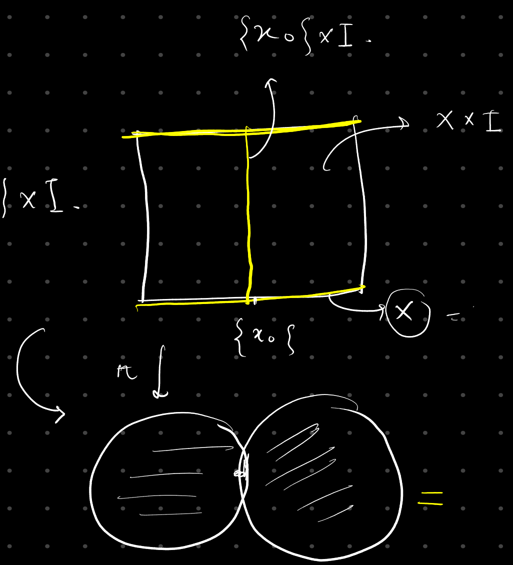


$S^1 \vee S^1 = ?$



(Reduced Suspension)

$$\Sigma X_+ = X \times I \big/ X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I.$$

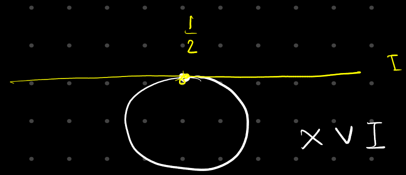


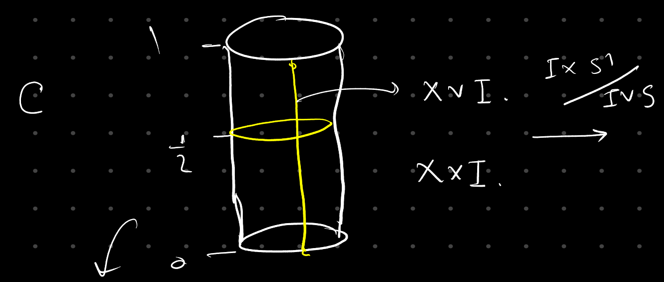
Smash Product: $X \times Y / X \vee Y = X \wedge Y$

Reduced Suspension of $X \cong X \wedge S^1$

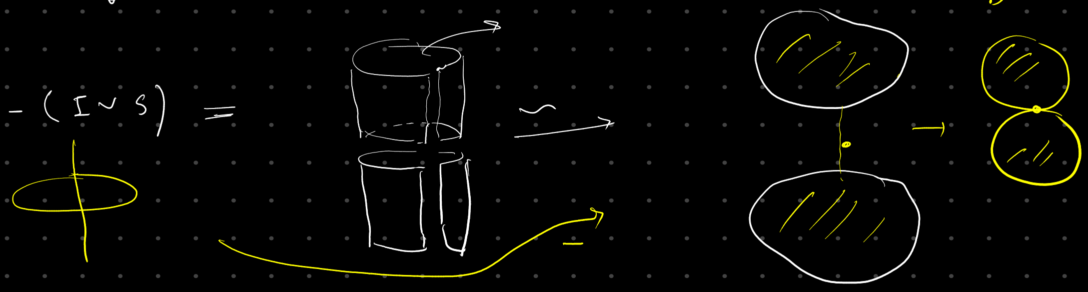
Ex. $X = I$. $\qquad I \wedge S^1$

$I \times S^1 / I \vee S^1$





$C - (I \vee S) =$



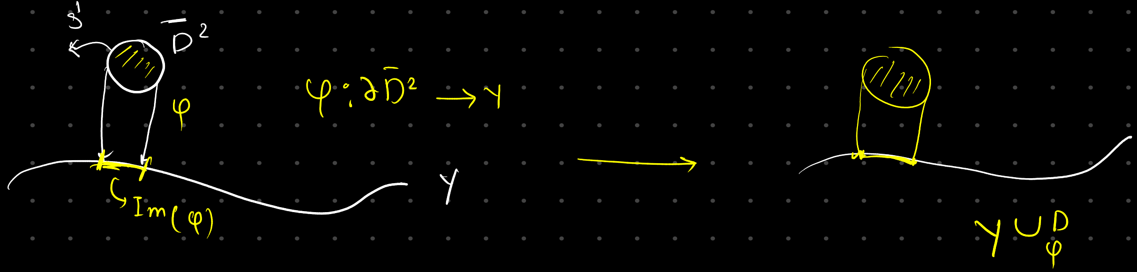
$*\quad \Sigma I_+ = I \wedge S^1$

Adjunction Spaces.

$X_0 \subseteq X \qquad f : X_0 \to Y. \qquad X \sqcup Y/\sim$

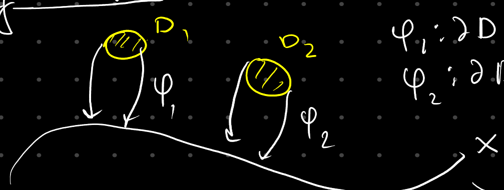
$\sim : \; x \sim f(x)$

$X \sqcup Y/\sim \; = Y \cup_f X.$

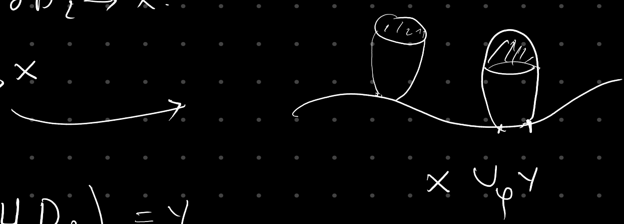


Cells. $X \to$ homeomorph. $\overline{B_r(x)}$ (X is called n- closed cell).
$X \to$ " $B_r(x)$ (X is called open n-cell).

Attaching Cells.



$\varphi_1 : \partial D_1 \to X$
$\varphi_2 : \partial D_2 \to X.$

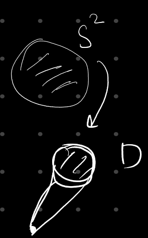


$(D_1 \sqcup D_2) = Y.$

$\varphi : \partial D_1 \sqcup \partial D_2 \longrightarrow X$

$\varphi|_{\partial D_1} \to \varphi_1$

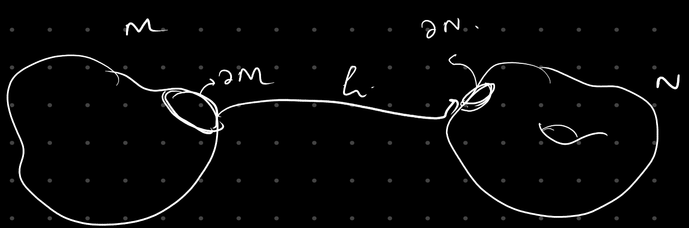
$\varphi|_{\partial D_2} \to \varphi_2$



$\cdot \quad \to$ Cell Complex.

Connected Sum.

Thm. (Inv. of Boundary). M and N two n-manifolds.
$\partial M \cong \partial N.$



$h : \partial M \to \partial N.$

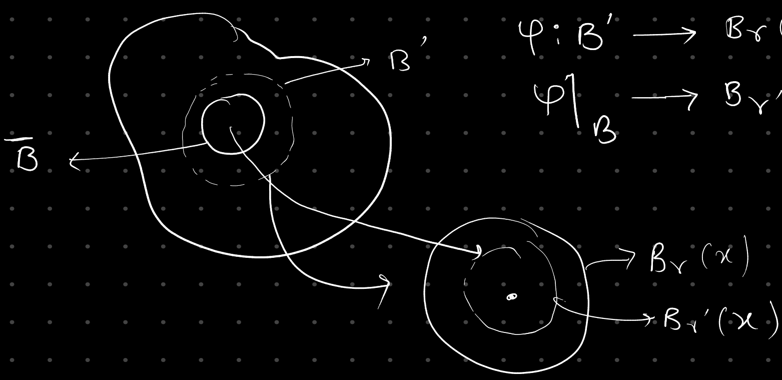
$N \cup_h M$ $\quad h : \partial M \to \partial N$

$(N \cup_h M) \to$ We have joined/attached M and N along their boundary.

Thm. (i) $N \cup_h M = n$-manifold $\to$ With no boundary.

* **Co-ordinate Ball;** $U \subseteq M$ $\quad \tilde{U} \xrightarrow[\text{homeo}]{} B_r(x) \subseteq \mathbb{R}^n$

* **Regular Coordinate Ball;**



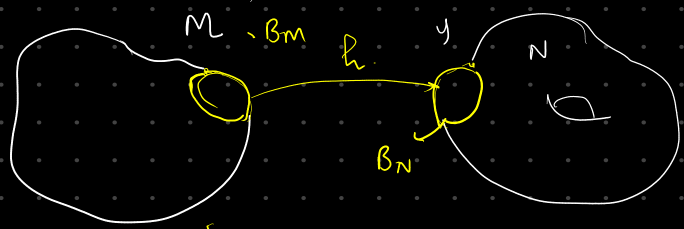
$B' \subseteq \bar{B}$

$\varphi : B' \to B_r(x)$

$\varphi|_B \to B_{r'}(x)$ $\quad r' < r$

$B \subseteq M$ said to be Regular Co-ord. ball iff $\exists B' \subseteq \bar{B}$ s.t. $\exists$ a homeomorph $\varphi : B' \to B_r(x)$ that is a homeomorph to $B_{r'}(x)$ when restricted to B.

Connected Sum (of Manifold)



$M \setminus B_M$ $\qquad$ $N \setminus B_N$

Thm. B is a regular Co-ord. Ball. $M \setminus B \to n$-manifold with boundary. $\partial(M \setminus B) \cong S^{n-1}$.

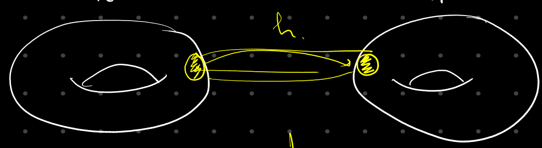
$N \cup_f M \qquad f : \partial(M/B) \xrightarrow{\sim} \partial(N/B).$

$M \# N = N \cup_f M.$

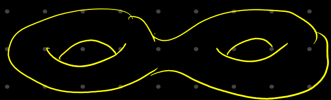
Ex.

$\mathbb{T} \qquad\qquad \mathbb{T}$

$h.$



$\downarrow$



$\mathbb{T} \# \mathbb{T}$

$\downarrow$

torus - 2 holes.

$\mathbb{T} \# \mathbb{T} \# \cdots \# \mathbb{T}(n) \to$ torus with n-holes.

Real Projective Space.

$\mathbb{R}^{n+1} - \{0\} \qquad (\sim:)\ \vec{x} \sim \vec{y} \iff \vec{x} = \lambda \vec{y} \qquad \lambda \in \mathbb{R} - \{0\}.$

$\quad \|$

$\mathbb{R}^{n+1}_* / \sim \;=\; \mathbb{P}\mathbb{R}^n \;\; (\mathbb{P}^n)$

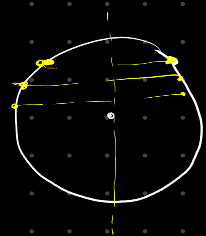
Another Rep. $\qquad S^n \qquad \sim : (x \sim y) \iff x = (-y).$

$? \quad S^n / \sim \;\cong\; \mathbb{P}^n$

$$S^n \xhookrightarrow{i} \mathbb{R}^{n+1}_* \xrightarrow{q} \mathbb{P}\mathbb{R}^n.$$

$q \circ i$

Ex.



$\bar{D}^2 \subseteq \mathbb{R}^2$

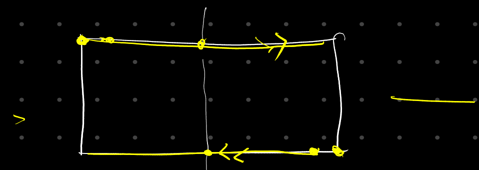
$(x,y) \sim (-x,y) \in \partial \bar{D}^2 = S^1$

$\bar{D}^2 / \sim \;\to ?\;\; S^2$

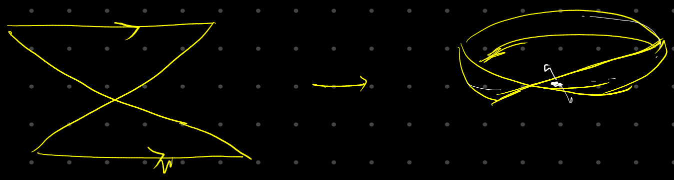
(Homework) $\longrightarrow \; \exists\, q : \bar{D}^2 \longrightarrow S^2$

Mobius strip. $\qquad [-1,1] \times [0,1]$

$\sim : (x,0) \sim (-x,1)$



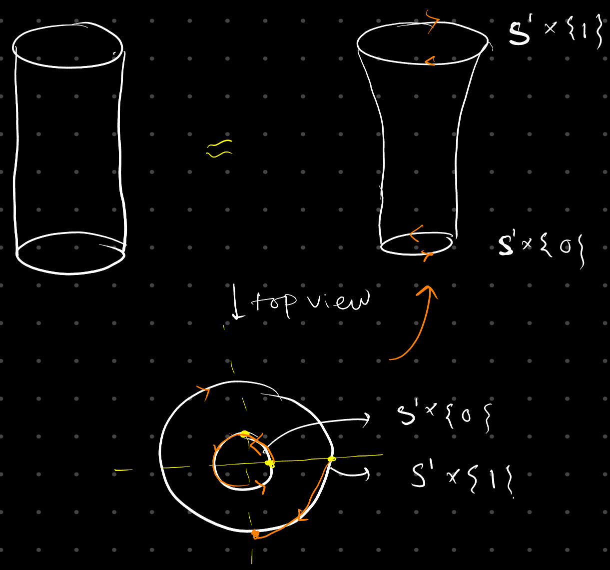
$\downarrow \pi$

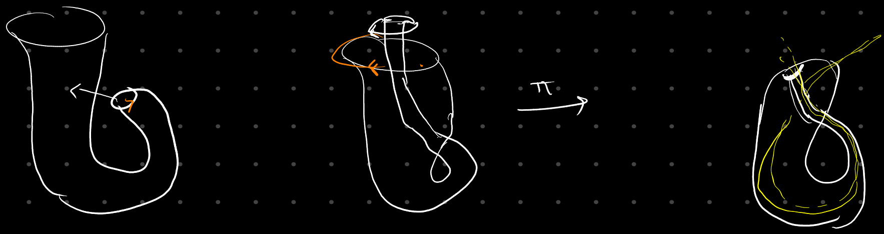


$= M.$

$f : [-1,1] \times [0,1] \longrightarrow M.$

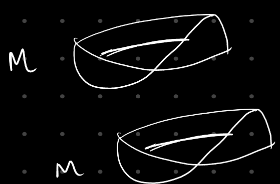
Klein Bottle. $S^1 \times I.$ $\sim : (z,0) \sim (\bar{z}, 1)$





$= K$

If you bisect Klein bottle (K) along the line (—) then you will end up getting two Mobius Strip.



$\partial M \xrightarrow{h} \partial M$

$M \cup_{h = Id} M \longrightarrow$ Attaching Mobius strip along Boundary.

$\cong$

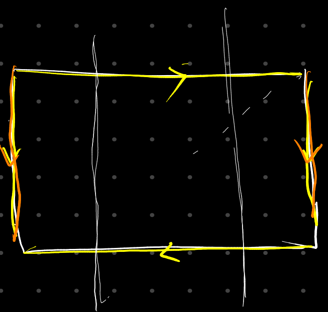
$K.$

Mobius strip is a 2-manifold.

$$\boxed{K = M \cup_{Id} M}$$

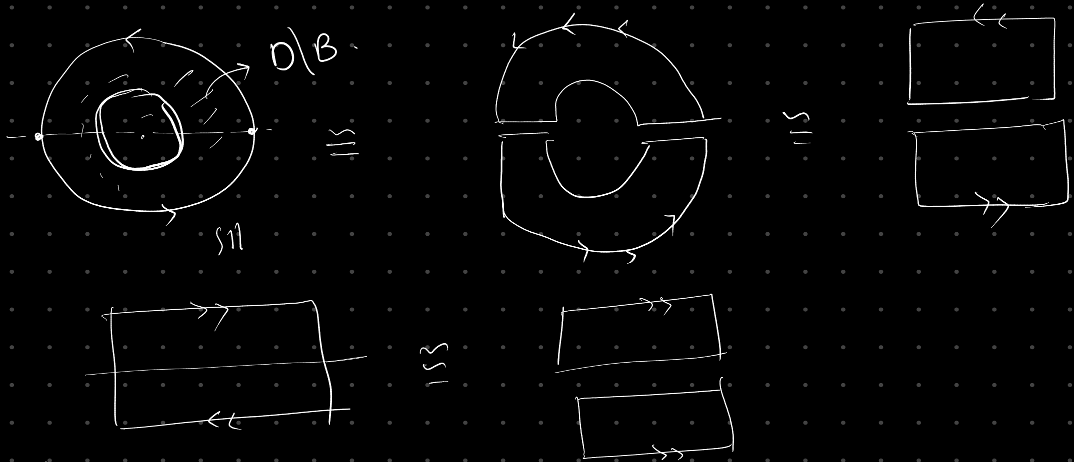
$M \cup_h N$ → n-manifold → with no boundary.

Klein Bottle is 2-manifold → with no boundary.



$\xrightarrow{\pi} K$.

Projective Plane: ($\mathbb{P}^2$)



$D\setminus B/\sim \longrightarrow$ Mobius.

$D/\sim$

$B/\sim \longrightarrow$ Disk.

$K \cong \mathbb{P}^2 \# \mathbb{P}^2$. (Important)