Fundamental Groups

Homotopy

$\mathcal{C}(x,y)$ = Space of cts maps from $X \to Y$

Every map in this notes are Continuous

$f: X \to Y$
$g: X \to Y$

$G: [0,1] \times X \to Y$

$G(t,x)$ $\quad G(0,x) = f(x) \ \forall x \in X.$
$\quad G(1,x) = g(x) \ \forall x \in X.$

$f \overset{G}{\simeq} g$

$\mathcal{C}(X,Y) \to$ The Space of Cont. maps from $X \to Y$.

$\simeq \to$ equiv. on $\mathcal{C}(X,Y)$

- $f \simeq f$.
- $f \simeq g \quad G, \quad g \overset{H}{\simeq} f \qquad G(1-t) = H(t)$
- $f \simeq g, \ g \simeq h, \quad f \simeq h.$

- X and Y be two Spaces are said to be "Homotopy equiv".

  $f: X \to Y \quad \exists\, g: Y \to X$

  - $f \circ g = Id_Y$ and $g \circ f = Id_X$

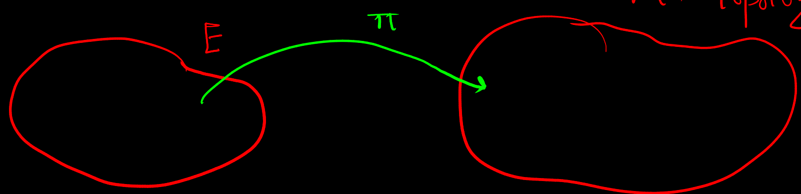
- A Space is Contractible if it is homotopic equiv to a point.
- Retract and deformation : $p: X \to A$ with $p|_A = Id_A$. if $p \simeq Id_X$
- Strong if h_t fix $A \quad \forall t \in [0,1]$
- Examples 1. $\mathbb{R}^n$ or $\mathbb{D}^n$ is Strong deformation Retract.

* for any topological Space, $X \times \{0\} \simeq X \times D^n$

* $S^1 \times D^2 \simeq S^1$

* E be a vector Bundle then $A \simeq E \simeq DE$ (Disk Bundle)

$M \to$ topological Manifold

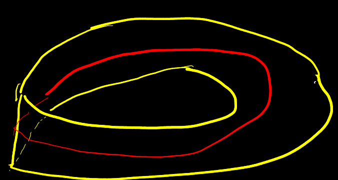


$\pi^{-1}(M) = E$
$\pi(E) \subseteq M$

(fibre) $\rightsquigarrow \pi^{-1}(x) \to$ Vector Space

E : " vector Bundle)

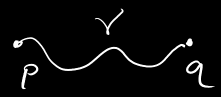
A = "Zero Section". $= \bigcup_{x \in M} \pi^{-1}(x)_0 \to$ Zero of $V.S\ \pi^{-1}(x)$



Mobius Strip $\simeq S$

What is Homotopy used for? → finding holes.

X be a topological group. $p, q \in X$. a path



$f : I \to \mathbb{C} \setminus \{0\}$ → hole at zero.

$$f(s) = \exp\{2\pi i s\}$$

$$H(t,s) = \exp\{2\pi i s t\}$$

$H(0,s) = 1$
$H(1,s) = f(s)$ → Every loop around "0" is homotopically eq. to "Constant loop".

This Don't help finding loops.

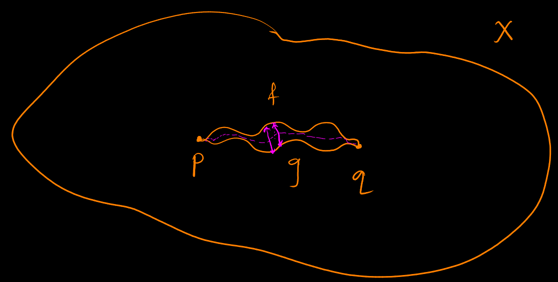
* We need to fix some points (Stationary Point)

X and Y be topological Space. $f, g \in [X, Y]$ $\quad f \overset{H}{\simeq} g$

$H : [0,1] \times X \to Y$

$$H(t,x) = f(x) \quad \forall x \in A,\ t \in [0,1].$$

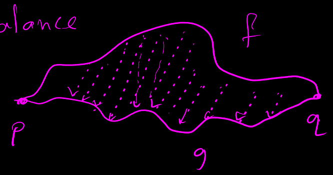
$A =$ Stationary points.



* f and g be paths from $p \to q$.

* If there is $H : I \times I \to X$
$\begin{cases} H(t,0) = f(t) \\ H(t,1) = g(t) \end{cases}$

* $f \sim g$ → $\sim$ is equivalance Reation.



Set of all loop who has p as base point → $\Omega(X, p)$

$c_p \in \Omega(X, P)$

Constant Loop.

* Homotopy is equivalance Relation in $\Omega(X, p)$.

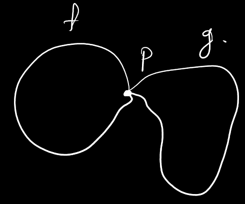
↓ turn this to a Group

$\pi_1(X, P)$ → Fundamental Group

- $c_p \in \pi_1(X, P)$ is identity element.
- $f \in \pi_1$ then $f(1-t) = \bar{f}$ → inverse.
- $f \cdot g$ → as $\quad f \cdot g = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(1-2t) & \frac{1}{2} \leq t \leq 1 \end{cases} \quad \in \pi_1(X, P)$

Homotopy is invariant after multiplication:

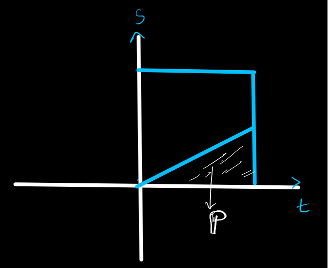
$$f \overset{H}{\simeq} f_0$$
$$g \overset{G}{\simeq} g_0$$



T given By,

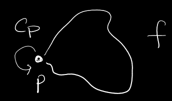
$$T_t(x) = \begin{cases} H(2x,t) & 0 \le x \le \frac{1}{2} \\ G(1-2x,t) & \frac{1}{2} \le x \le 1 \end{cases} \qquad f.g \overset{T}{\simeq} f_0 \cdot g_0$$

Re-Review the Def^n of Fund. Grp. Using Path classes :-



$$[f]\cdot[g] = [f.g]$$

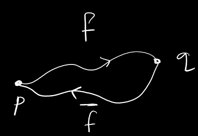
\# $[c_p]\cdot[f] = [f]$



Pf. $c_p . f \overset{H}{\simeq} f$

$$H(s,t) = \begin{cases} p & t \geqslant 2s \\ f\left(\frac{2s-t}{2-t}\right) & t \le 2s. \end{cases}$$

$$H(s,0) = f(s)$$
$$H(s,1) = c_p \cdot f$$



\# $[f]\cdot[\bar{f}] = c_p$

$$f \cdot \bar{f} \sim c_p.$$

$$H(s,t) = \begin{cases} f(2s) & 0 \le s \le t/2 \\ f(t) & t/2 \le s \le 1-t/2 \\ f(2-2s) & 1-t/2 \le s \le 1. \end{cases}$$

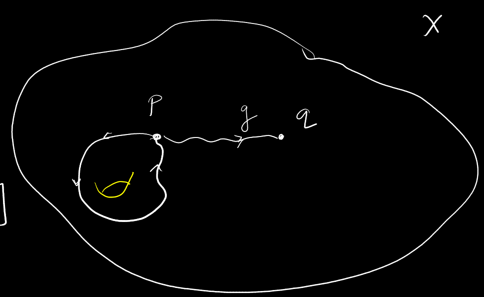
$$H(s,0) = c_p$$
$$H(s,1) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ f(2-2s) & \frac{1}{2} \le s \le 1 \end{cases}$$

Change of Base point:-

X be path Connected. $p, q \in X$ and $g:[0,1] \to X$ be a path b/w p, q. The map

$$\Phi_g : \pi_1(X,p) \longrightarrow \pi_1(X,q)$$
$$\Phi_g[f] = [\bar{g}]\cdot[f]\cdot[g] \quad \text{isomorphism.}$$

Pf.



$$\phi[f_1]\cdot\phi[f_2]$$
$$= [\bar{g}][f_1][g]\cdot[\bar{g}][f_2][g]$$
$$= [\bar{g}][f_1]\cdot\underbrace{[c_p]\cdot[f_2]}[g]$$
$$= [\bar{g}][f_1]\cdot[f_2][g]$$
$$= \phi([f_1]\cdot[f_2]).$$

$$\boxed{\pi_1(X,p) \cong \pi_1(X,q)}$$

Notice, $\Phi_{\bar{g}} : \pi_1(X,q) \to \pi_1(X,p)$

$$\Phi_g[\Phi_{\bar{g}}(x)] = [\bar{g}][g](x)[\bar{g}][g]$$
$$= x$$

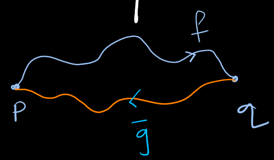
→ Inverse exist So mustbe Iso.

"Simply Connected" → X path - Connected and $\pi_1(X,p) = \{c_p\}$.
⇓
for any point $p \in X$, $\pi_1(x,p) \cong \pi_1(x,q)$ $\forall q \in X$
⇒ Just write $\pi_1(x)$

i.e. X with $\pi_1(x) = \{c_p\}$ and X being Path Connected

X is simply connected if and only if two paths b/w any two points in X are path Homotopic.

Pf. (⇒)

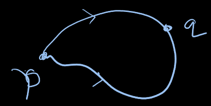


$f \cdot \bar{g}$ → Behaves as a "loop"

$\therefore f \cdot \bar{g} \in \pi_1(x,p) \cong \{c_p\}$

$\Rightarrow f \cdot \bar{g} \sim c_p$.
$\Rightarrow f \sim g$.

(⇐) $f \sim g \Rightarrow f \cdot \bar{g} \sim c_p$.
Let $r \in \pi_1(x,p)$



take $q \in ran(r)$ and $f_1 = r|_{p\to q}$
$f_2 = r|_{q \to p}$ ⇒ $f_1 \cdot f_2 \cong c_p$ as $f_1 \sim \bar{f}_2$

Every Convex Set of $\mathbb{R}^n$ is simply Connected: For any two Path f, g

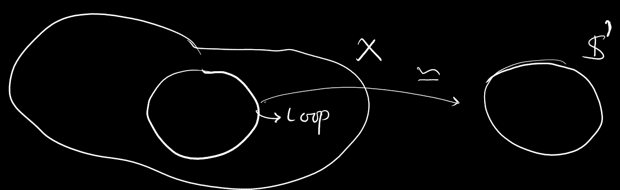
$H(s,t) = t f(s) + (1-t) g(s)$
• $f \underset{H}{\sim} g$ → Apply Prev. Lemma.

$\omega : I \to S^1$ defined by $\omega(s) = e^{2\pi i s}$

f be a loop in topological Space X. $f: I \to X$.
$\exists!$ $\tilde{f} : S^1 \to X$ Such that $f = \tilde{f} \circ \omega$

$$\begin{tikzcd} I \arrow[r] \arrow[d] & S^1 \arrow[dl, "\exists ! \tilde{f}"] \\ X & \end{tikzcd}$$

(By universal Prop.)



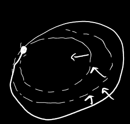
TFAE → $f : I \to X$ be a loop with base point $p \in X$. and $\tilde{f} : S^1 \to X$.

i) * f is null homotopic (i.e $\sim c_p$)
ii) * $\tilde{f}$ is freely Homotopic to a Constant map.
iii) * $\tilde{f}$ extends a Continuous map from $D \to X$!

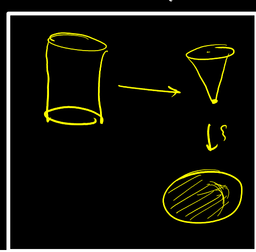
Proof. (i → ii) $f \sim c_p$ Let H be the homotopy $H : I \times I \to X$
$\tilde{H} : S^1 \times I \to X$ is Cont. map. With
$\tilde{H}(x,0) = \tilde{f}$
$\tilde{H}(x,1) = c_p$.
$\Rightarrow \tilde{f} \overset{\tilde{H}}{\sim}$ Constant map

$$\begin{tikzcd} I \times I \arrow[d, "\tilde{\omega} = \omega \times Id"'] \arrow[dr, "H"] & \\ S \times I \arrow[r, "\exists ! \tilde{H}"'] & X \end{tikzcd}$$

$H = \tilde{H}(\tilde{\omega})$



(ii → iii) $K : S^1 \to X$ be a Constant map. $\tilde{f} \overset{H}{\sim} K$
$H_0 = k$, $H_1 = f$ $CS^1 = S^1 \times I / S^1 \times \{0\} \cong \bar{B}^2$



• H takes $S^1 \times \{0\}$ to a single point

$$\begin{tikzcd} S^1 \times I \arrow[r, "H"] \arrow[d, "q"] & X \\ \bar{B}^2 \cong CS^1 \arrow[ur] & \end{tikzcd}$$

$!\tilde{H}: D \to X$

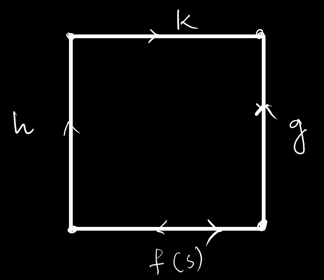
$\hat{H}\Big|_{S^1} = \hat{f}$

$\uparrow$ closed Disk D

(iii → i) $\hat{f}$ extends to a Cont. map $F: \bar{D} \to X$. $F\Big|_{S^1} = \tilde{f}$

$\bar{D}$ is Convex and Symply Connected → Finishes the proof.

Square Lemma :→ Let $F: I \times I \to X$ be a Cont. map.



$$\left.\begin{aligned} F(s,0) &= f(s) \\ F(1,s) &= g(s) \\ F(0,s) &= h(s) \\ F(s,1) &= k(s) \end{aligned}\right\} \to \text{paths in } X$$

Then $h \cdot k \sim f \cdot g$.

Fundamental Group of Spheres.

• $\operatorname{diam}(S) = \sup\{d(x,y) \mid x, y \in S\}$.

$\uparrow$ Bounded Set S in a metric Space (X,d)

• Lebesgue Number → $\mathcal{U}$ be open Cover of a metric X. $\delta > 0$ be a number. s.t any $S \subseteq X$ with $\operatorname{diam}(S) = \delta$ $S \subseteq U \in \mathcal{U}$. δ is called Lebesgue Number.

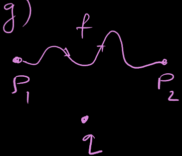
• **Lebesgue Number Lemma :→** Every open Cover of a Compact Subspace has a Lebesgue number.

Pf. $B(r(x), x) \to$ forms an open Cover.
$\downarrow$
take finitely many of them.

$\{B(r_1, x_1), \ldots, B(r_n, x_n)\} \to$ open Cover of X.

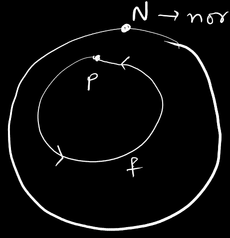
$\delta = \min_i r_i \to$ Lesbeque measure.

Lemma :- Suppose M is a manifold of dim $n \geq 2$. If a path in M (f) from P_1 to P_2. Let q be a point don't lie on Path(f) then $f \sim g$. ($\exists g$) $q \neq P_1, P_2$.



S^n is Simply Connected (Cor. of above Lemma).

$S^n = X$



$p \in S^n \setminus N$ $\quad f \in \pi_1(X, p) \subseteq \Omega(X, p)$

f is path homotopic to a loop in $S^n \setminus N$.

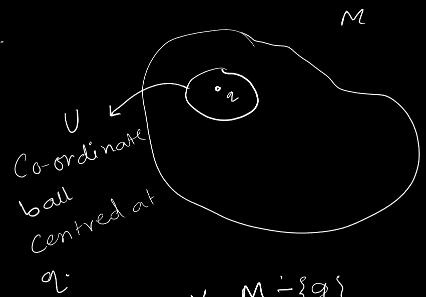
$S^n \setminus N \cong \mathbb{R}^n$.

So, $f \sim g\,(\in \pi_1(p, \mathbb{R}^n)) \sim c_p$.

$\therefore f$ is Connected and a loop $f \sim c_p \Rightarrow \pi_1(S^n) = \{c_p\}$.

$\boxed{\#\quad \pi_1(S^n) \text{ is trivial !!}}$

Pf (Lemma). $\{U, V\}$ be the open cover of M.

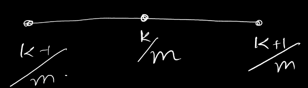


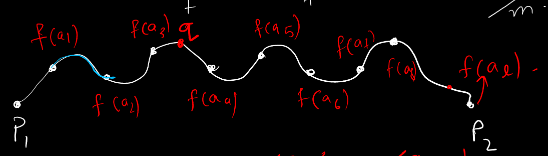
$f : I \to M$ be a path b/w p_1 and p_2.
$\{f^{-1}(U), f^{-1}(V)\}$ is open cover of I.

δ be the lebesgue number for this Cover. $m \in \mathbb{N}$ s.t $1/m < \delta$.

$V = M - \{q\}$

- each Sub int. $[k/m, (k+1)/m]$ takes values either in U or V.
- $f(k/m) = q$ then $\left[\frac{k-1}{m}, \frac{k}{m}\right]$ and $\left[k/m, \frac{k+1}{m}\right] \xrightarrow{f} U \xleftarrow{f}$





$0 = a_0 < a_1 < \cdots < a_\ell = 1$
s.t $f(a_i) \neq q$.

$f|_{[a_i, a_{i+1}]} \subseteq V$ or U.

Each segment in U.

↳ Apply the Same Reason as the (Cor.) → U is simply Connect.

↳ $U \setminus \{q\} \xrightarrow{\sim} \mathbb{R}^n \setminus \{0\}$ → Path Connected $(n > 1)$.

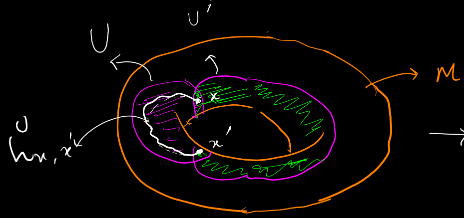
$\Rightarrow p_1, p_2$ has a path homotopic to the prev. $\not\ni q$.

And the Section in V ofc. miss $\{q\}$. ∎

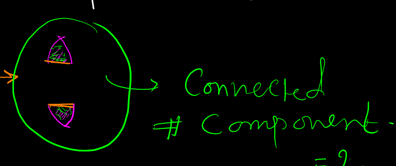
Fundamental Group of Manifolds

Theorem: Fundamental Group of Manifolds are Countable.

$\mathcal{U} :=$ open Cover of M. (Countable)



$\to U \cap U'$



obsv: $U \cap V' \to$ has Countable Component. ($U \cap U'$ has to be Covered by Countably element from $\mathcal{U}$)

Take Points from Countably many Compo.

$\{X\} \to$ Collection

Let, $x, x' \in X$ s.t $x, x' \in U$; $h^{U}_{x,x'} \to$ Path $x \to x'$ in U

A loop is Spacial if a loop based at P is Spacial if it is Product of paths of form $h^{U}_{xx'}$.

- There are only Countably many Special loops.
- f be any loop based at p. by "lebesge lemma" $\exists n$ s.t $\left[\frac{k-1}{n}, \frac{k}{n}\right] \xrightarrow{f} U_k \in \mathcal{U}$.

Call, $f_k = f|_{\left[\frac{k-1}{n}, \frac{k}{n}\right]}$ now, $[f] = [f_1] \cdots \cdots [f_n]$

$f(k/n) \in U_k \cap U_{k+1}$ $\exists x_k$ lie in the Same Component of $U_k \cap U_{k+1}$

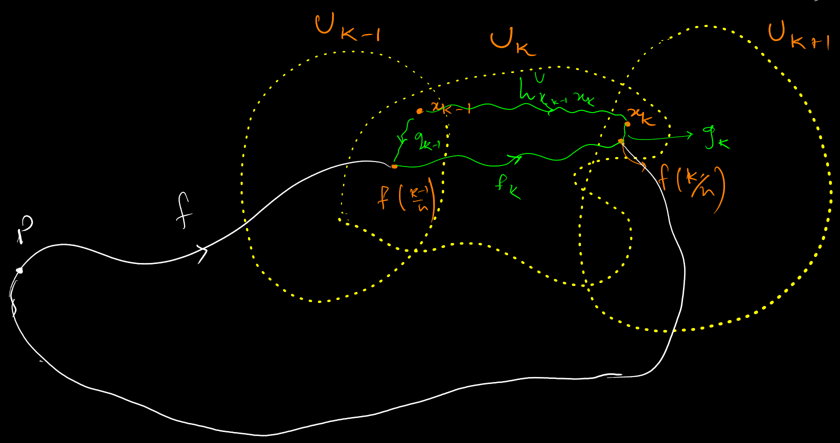
Choose path g_k in $U_k \cap U_{k+1}$. Set, $\tilde{f}_k = g_{k-1} \cdot f_k \cdot \bar{g}_k$

$[f] = [\tilde{f}_1] \cdots \cdots [\tilde{f}_n]$

$\hat{f}_k \to$ path from x_{k-1}, x_k in U_k as U_k is simply Connected.

$\tilde{f}_k =$ homotopic to Special loops.

$\{\text{loops}\} \xrightarrow{\text{Surjective}} \{\text{Special loops}\}$



$\tilde{f}_k = g_{k-1} f_k \bar{g}_k$

$\downarrow$

Path homotopic to $\underline{h^{U}_{x_{k-1} x_k}}$