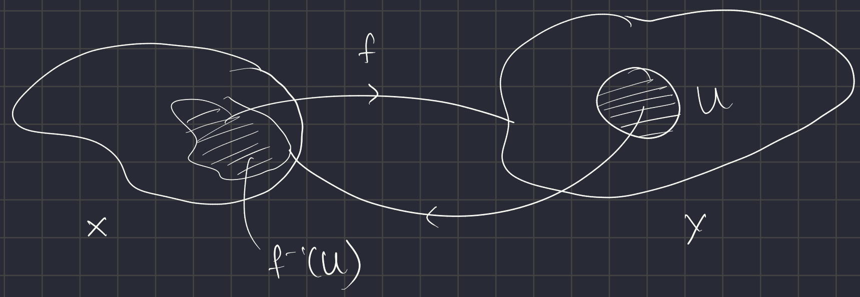
Continuous functions.



How do we define a topology?

① You define open sets which satisfies certain conditions.
(closed sets)

② You define $\{U_x\}_{x \in X}$ is a collection of neighborhoods
$\subseteq P(X)$ which are nbds of x

equivalent

③ You can define $cl : P(X) \to P(X)$ which satisfying Kuratowski's axioms.

Theorem (Characterisation of cont func). $f : X \to Y$ cont. TFAE

① V is open in $Y \Rightarrow f^{-1}(V)$ is open in X

② A is closed in $Y \Rightarrow f^{-1}(A)$ is closed in X

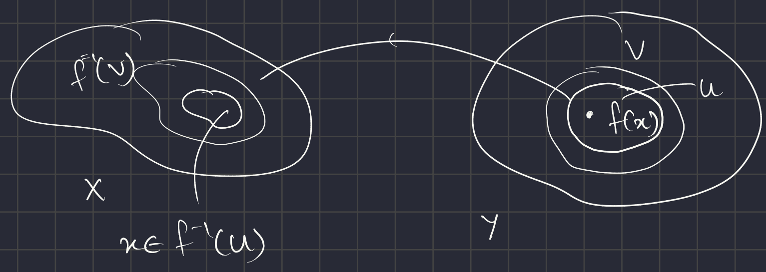
③ if V is any nbd of $f(x)$, then $f^{-1}(V)$ is a nbd of x. — f is cont at x

④ $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B}) \quad \forall B \subset Y$

$x \in f^{-1}(\bar{B})$
$\Leftrightarrow f(x) \in \bar{B}$
$\forall$ nbds of $f(x)$, say V
$V \cap B \neq \emptyset$

Pf. ② $\Rightarrow$ $Y \setminus A$ open $\Rightarrow f^{-1}(Y \setminus A)$ is open
$\Rightarrow f^{-1}(Y) \setminus f^{-1}(A)$
$\Rightarrow X \setminus f^{-1}(A)$ is open
$\Rightarrow f^{-1}(A)$ open.

③



Closure of a set B, closure is denoted by $\bar{B}$

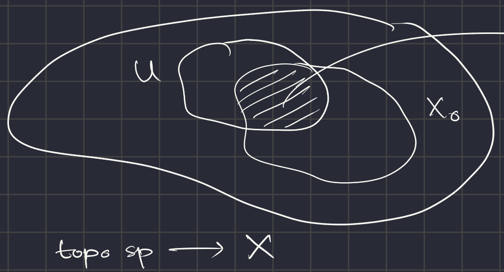
$x \in \bar{B}$ if for all nbds of x, U is nbd of x, $U \cap B \neq \emptyset$

Exc. $\bar{B}$ is closed! $\forall B \subset X$.

④ $x \in \overline{f^{-1}(B)}$, $\forall$ nbds of x, say U, we have $U \cap f^{-1}(B) \neq \emptyset$
$y \in U \cap f^{-1}(B)$
Suppose V is a nbd of $f(x)$, we need to show that $V \cap B \neq \emptyset$ (FTSOC say $V \cap B = \emptyset$)

then $f^{-1}(V)$ is a nbd of $x \Rightarrow f^{-1}(V) \cap f^{-1}(B) \neq \emptyset$
$\Rightarrow V \cap B \neq \emptyset$

- Composition is continuous



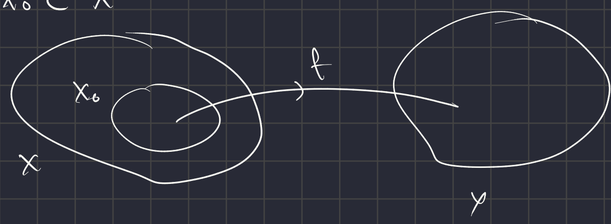
open sets in $X_0 := \{ X_0 \cap U \mid \text{where } U \text{ is open in } X \}$

subspace topology

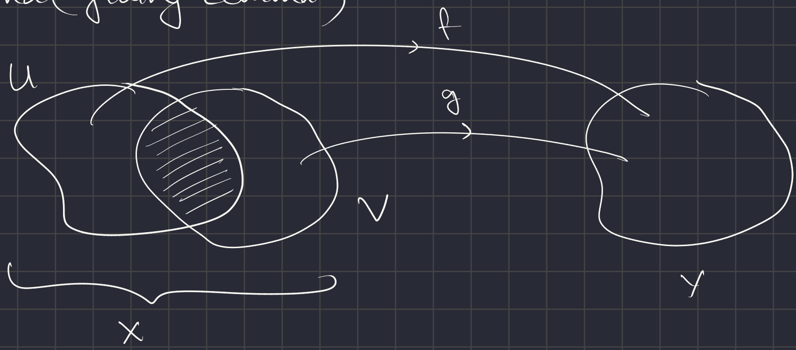
- $f : X \longrightarrow Y$ which is continuous say $X_0 \subset X$

$f|_{X_0} : X_0 \longrightarrow Y$

this continuous.



- Lemma (gluing Lemma)



$f : U \longrightarrow Y$ cont.
$g : V \longrightarrow Y$
such that

$f|_{U\cap V} \equiv g|_{U \cap V}$

$f(x) = g(x) \quad \forall x \in U \cap V$

$F : X \longrightarrow Y$

$$F(x) = \begin{cases} f(x) & \text{if } x \in U \\ g(x) & \text{if } x \in V \end{cases}$$ continuous function.

$F : X \to Y$

Example

$f(x) = x \quad \forall x \in \mathbb{R} \quad (f : \mathbb{R} \to \mathbb{R})$

Say X is a set, $\tau = \mathcal{P}(X) \longleftarrow$ disc topo

$\uparrow$ coll. of open sets

standard eucliden space

$f : \mathbb{R} \to \mathbb{R} \longleftarrow$ we if the discrete topo

$x \longmapsto x$

$\{x\} \longleftarrow \{x\}$

not open.

$g : X \longrightarrow Y$

$\uparrow$

disc topo

X set $\rightarrow$ topologies τ and τ', $\tau, \tau' \subseteq P(X)$

$\tau \subseteq \tau'$ or $\tau \supseteq \tau'$ or τ and τ' are not comparable.

we say τ' is finer than τ, τ is coarser than τ'

$X \longrightarrow$ finest topology = disc topo
coarsest topo = indisc topo $\rightarrow \{X, \emptyset\}$.

$\mathcal{S} \subseteq P(X)$, to answer: subbases topo generated by $\mathcal{S}$.

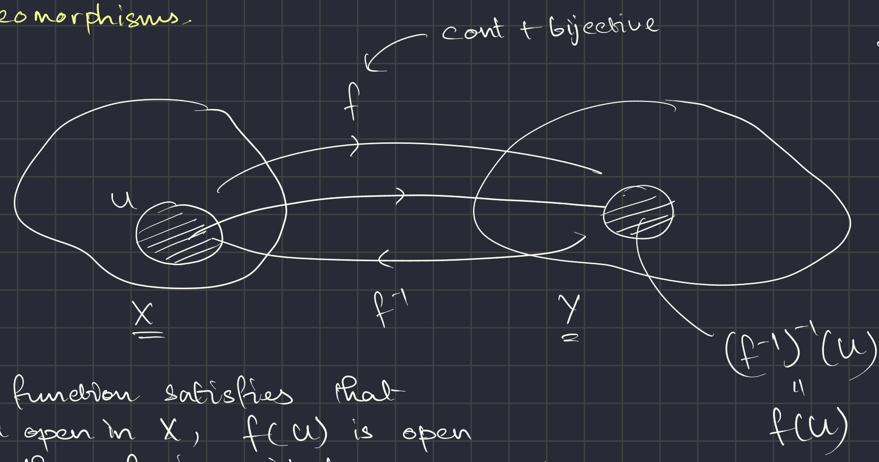
$\mathcal{B} = \{ \cap_I S \mid S \in \mathcal{S} \text{ and } |I| < \infty \} \leftarrow$ basis
$\Downarrow$
topology is called topo gen by subbases $\mathcal{S}$. $\equiv \tau_{\mathcal{S}}$

$\cap_\emptyset = X$ (notation).

τ any topo such $\mathcal{S} \subseteq \tau \Rightarrow \tau_{\mathcal{S}} \subseteq \tau$

coarsest topology.

Homeomorphisms.

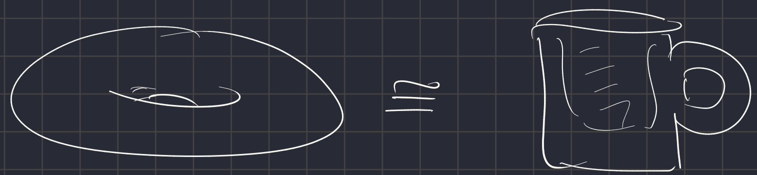


Defn. $f: X \rightarrow Y$ which is cont and bijective, further f^{-1} is also continuous then f is homeo.

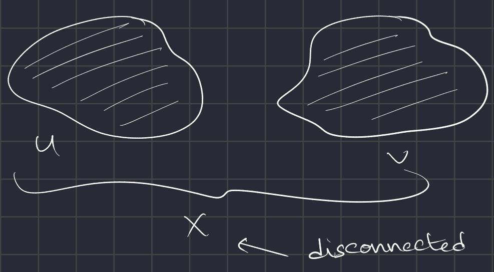
$X \cong Y$

X is homeomorphic to Y.

When function satisfies that for U open in X, $f(U)$ is open in Y, then f is said to a open map.

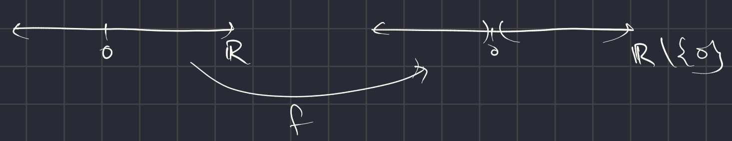


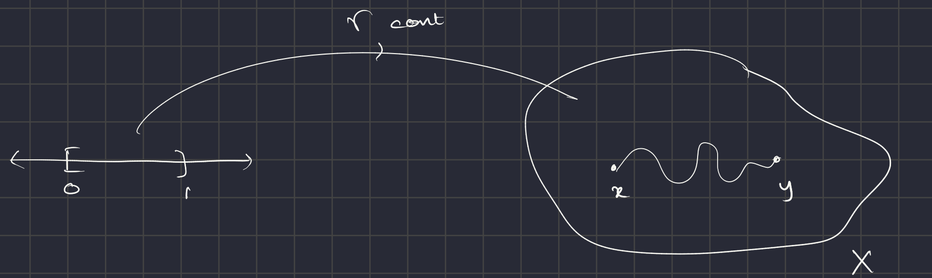
Connectedness and pathconnectedness.



Defn. A space X is connected if $\nexists$ any open subsets such that $X = U \cup V$ and $U \cap V = \emptyset$ and $U, V \neq \emptyset$.

Theorem. Connectedness is a topo invariant. Any continuous image of conn space is conn.





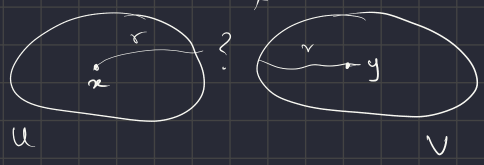
$\gamma(0)=x$ and $\gamma(1)=y$

So if such a γ exists you say x and y pathconnected

Defn. If any two points are path conn. then X is pathconn.

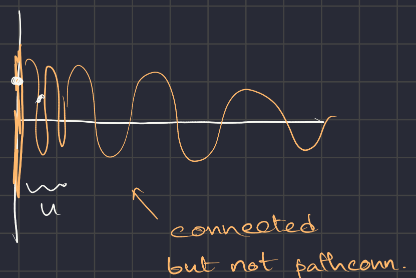
Theorem. Pathconnectedness is topo inv.

pathconnectedness $\Rightarrow$ connectedness.
$\not\Leftarrow$



$[0,1]$ is connected (why?)

$= \gamma^{-1}(U) \cup \gamma^{-1}(V)$



$A = \{(x, \sin(\frac{1}{x})) : 1 \geq x > 0\}$

$X = \{0\} \times [-1,1] \cup A$

Defn. A component is a maximal conn. subset of X
A pathcomponent " " path conn. subset of X } defined using equivalence reln.

$x \sim_c y$ if $\exists$ a conn. subspace of X, say A such that $x, y \in A$

- reflexive $\rightarrow \{x\}$ conn
- symmetric
- transitive

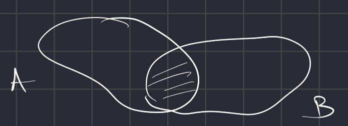
$x \sim_c y \quad y \sim_c z$

$\Downarrow \quad \Downarrow$

$\underbrace{x, y \in A \quad y, z \in B}$

$x, z \in A \cup B$ is connected $\Rightarrow x \sim_c z$

Lemma. So A, B conn such $A \cap B \neq \emptyset \Rightarrow A \cup B$ is conn.



equivalence classes $(\sim_c) \rightarrow$ maximal conn. subset $\equiv$ components.

that components form a partition of X. Components are always closed. (Exc).

- A conn $\Rightarrow \bar{A}$ is conn.

$C \rightarrow$ component $\Rightarrow$ connected

$\Rightarrow \bar{C}$ conn. $\quad C \subseteq \bar{C} \Rightarrow C = \bar{C} \Rightarrow C$ is closed.

Defn. A path component is a maximal pathconnected subset in X

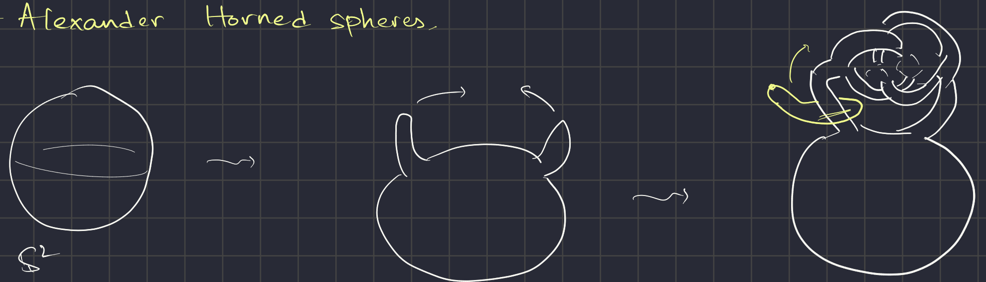
$\rightsquigarrow x \sim_p y \to x$ and y are pathconn.

So pathcomponents they form a partition of X.

pathcomponent is pathconn. $\Rightarrow$ path comp is conn. $\subseteq$ comp

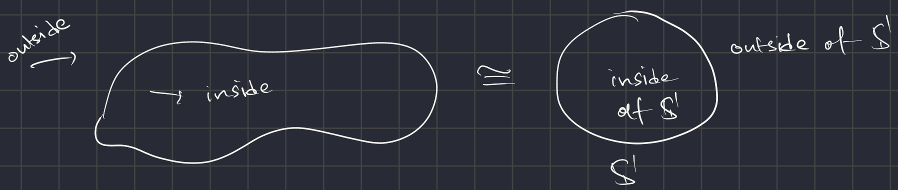
Any component is disjoint union of path comp.

Alexander Horned spheres.



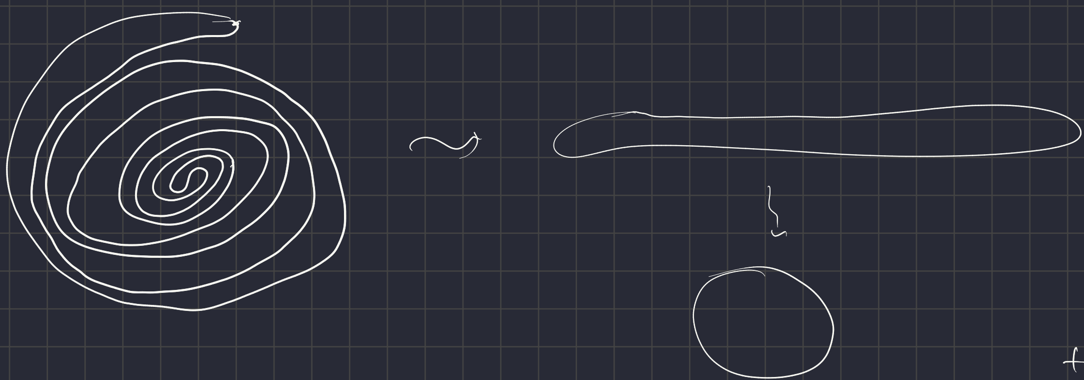
outside $\to$ not simply conn.

$\rightsquigarrow$ embedding of S^2 in $\mathbb{R}^3$



Jordan Curve thm. A simple loop breaks $\mathbb{R}^2$ into two comp.

Schoenflies. Inside is homeomorphic inside of S^1 and outside is homeo to outside of S^1. (true)



true GJCT

Schoenflies. He said for any embedding of S^2 in $\mathbb{R}^3$, it divides $\mathbb{R}^3$ into two components

false $\left\{\begin{array}{l}\text{but he further one comp} \cong \text{inside of } S^2 \text{ and} \\ \text{other comp} \cong \text{outside of } S^2\end{array}\right.$