Overview Talk

Notes By: Trishan Mondal

Speaker: Suresh Nayak

Indian Statistical Institute, Bangalore

We will begin with 'cohomology of projective varieties' and we will see for smooth projective varieties many beautiful properties holds for de-Rahm cohomology, singular cohomology which don't get satiesfied for the case of 'singular projective varieties'. We will discuss those with examples in this talk.

Let $X \subseteq \mathbb{C}P^N$ be a projective variety of dimension n (in the sense of Krull dimension which will be same with the manifold dimension for the smooth case). X is given by zeroes of some homogeneous polynomial thus it a closed subspace of $\mathbb{C}P^N$ and hence it is compact. For the smooth case X is a 'smooth manifold'(complex manifold). Some properties of smooth X are described below,

- X is given by zeroes of $g_1, \cdots, g_{N-n}$ with the rank of the matrix $\left(\frac{\partial g_j}{\partial z_i}\right)_{ij}$ equal to $N-n$.
- Hermitian metric on Tangent space of X.
- X is an orientable manifold of dimension $2n$ admitting a Riemannian metric g and a 'complex structure' on it's Tangent space.
- There is also an alternating form ω (or Kähler differential).

1. Dualities

As a real manifold X has dimension $2n$. We can compute the singular(simplicial) homology(cohomology) for X with the coefficients in $\mathbb{R}$. Since X is compact orientable manifold we can talk about the cup product pairing as follows:

$$H^i(X;\mathbb{R}) \times H^{2n-i}(X;\mathbb{R}) \xrightarrow{\smile} H^{2n}(X;\mathbb{R}) \cong \mathbb{R}$$

is a 'non-degenerate' pairing. Thus we have **Poincare Duality**,

$$H^{2n-i}_{\text{Sing}}(X;\mathbb{R}) \cong H^i_{\text{Sing}}(X;\mathbb{R})^* \cong H_i^{\text{sing}}(X;\mathbb{R})$$

Since X is a smooth manifold we can talk about de-Rahm cohomology. In a sophisticated language 'de-Rahm cohomology is a cohomology of soft-resolution of constant sheaf'. In this case also we have the following as non-degenerate,

$$H^i_{DR}(X;\mathbb{R}) \times H^{2n-i}_{DR}(X;\mathbb{R}) \xrightarrow{\wedge} H^{2n}_{DR}(X;\mathbb{R}) \xrightarrow[\int -\text{vol-form}]{\sim} \mathbb{R}$$

Thus again we have the duality, $H^{2n-i}_{DR}(X;\mathbb{R}) \cong H^i_{DR}(X;\mathbb{R})^*$. Connecting de-Rahm cohomology and singular(simplicial) cohomology with coefficients in $\mathbb{R}$, there is a beautiful theorem by de-Rahm stated as follows,

> **Theorem 1.1 (De-Rahm's Theorem)** There is an isomorphism between the singular(simplicial) cohomology with coefficients in $\mathbb{R}$ and de-Rahm cohomology which is compatible with the product structure on both the V.S.

2. HODGE THEOREMS

There are two different versions of 'Hodge Theorem'. The metric (Riemannian) on X induces metric on de-Rahm complex $\Omega^\bullet(X)$. It is defined by,

$$(\omega, \eta) := \int_X \left(p \mapsto \langle \omega, \eta \rangle_p\right) \mathrm{Vol}_X$$

With respect to this inner product the exterior derivative d has an adjoint δ such that,

$$(d\omega_1, \omega_2) = (\omega_1, \delta\omega_2)$$

$*$ We can write down the adjoint explicitly, for any $\alpha \in \Omega^k(X)$, $\delta\alpha = (-1)^k(*)^{-1}d\alpha$ where $*$ is the 'Hodge star operator' $* : \wedge^k(T_pX)^* \to \wedge^{2n-k}(T_pX)^*$, given by $(\theta_1, \cdots, \theta_k) \mapsto (\theta_{k+1}, \cdots, \theta_{2n})$ where $\{\theta_j\}$ is an oriented orthonormal basis of $(T_pX)^*$. Set, $\Delta = \delta d + d\delta$ be the Laplacian. **Harmonic** forms are elements of $\Omega^\bullet(X)$ lies in the kernel of Δ. With this setup we are ready to state 'Hodge theorem 1'. This theorem gives us a decomposition of $\Omega^k(X)$.

Theorem 2.1 (HODGE THEOREM I) Every element of $H^k_{DR}(X;\mathbb{R})$ is uniquely represented by 'Harmonic forms' of degree k. Also Ω^k admits the following decomposition,

$$\Omega^k(X) \cong H^k_{DR}(X;\mathbb{R}) \oplus d(\Omega^{k+1}) \oplus \delta\left(\Omega^{k+1}\right)$$

We have perviously mentioned there is a 'complex structure' on the Tangent space of X. As of now we have not used this structure.

§ The presence of complex structure I

The complex structure gives rise to Eigen decomposition of complexified tangent/co-tangent bundles on X. Thus we have,

$$\boldsymbol{\Omega}^1_{X,\mathbb{C}} := \boldsymbol{\Omega}^1_X \otimes \mathbb{C} \cong \boldsymbol{\Omega}^{1,0}_X \oplus \boldsymbol{\Omega}^{0,1}_X$$

Then, $\boldsymbol{\Omega}^k_{X,\mathbb{C}} = \wedge^k \boldsymbol{\Omega}^1_{X,\mathbb{C}} \cong \bigoplus_{p+q=k} \Omega^{p,q}_X$. Here, $\boldsymbol{\Omega}^{p,q}_X = \wedge^p \boldsymbol{\Omega}^{1,0}_X \otimes \wedge^q \boldsymbol{\Omega}^{0,1}$. Thus, $\boldsymbol{\Omega}^{p,q}_X$ is the V.S of the smooth (p,q)-forms $dz_1 \wedge \cdots \wedge dz_p \wedge d\bar{z}_{p+1} \cdots \wedge d\bar{z}_{p+q}$. we can note $\boldsymbol{\Omega}^{\bar{\mathbf{p}},\mathbf{q}} = \boldsymbol{\Omega}^{q,p}$. With this setup we are ready to note the Hodge theorem II.

Theorem 2.2 (HODGE THEOREM II) Every Harmonic form in $H^k(X;\mathbb{C})$ decomposes as a sum of harmonic (p,q)-forms of a bi-degree, where $p+q=k$. Thus,

$$H^k_{DR}(X,\mathbb{C}) \cong \oplus_{p+q=k} H^{p,q}(X;\mathbb{C})$$

COROLLARY. *If k is odd then $\dim_{\mathbb{C}}(H^k_{DR}(X;\mathbb{C}))$ even.*

3. HARD LEFSCHETZ THEOREM

The hermitian metric $\mathfrak{h}$ on TX via its decomposition gives rise to an alternating 2-forms can be shown to be a $(1,1)$-form, call it ω. Using this we get a linear map,

$$L : H^k_{DR}(X;\mathbb{R}) \xrightarrow{\text{product with } \wedge\omega} H^{k+2}(X;\mathbb{R})$$

Hard Lefschetz Theorem – The map, $L^{n-k} : H^k_{DR}(X,\mathbb{R}) \to H^{2n}_{DR}(X;\mathbb{R})$ induces an isomorphism for $k \leq n$.

COROLLARY. *L is injctive for $k < n$. Thus the odd degree Betti number $h^i := \dim_{\mathbb{R}} H^k(X;\mathbb{R})$ increases upto the middle degree and then decreases there after.*

Thus we havethe following HODGE DIAGRAM.

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{n,0} & & & & & & h^{0,n} \\
 & & & h^{n,n} & & &
\end{array}
$$

In the above diagram i-th row sums up to give i-th Betti numbers of X. One more interesting result due to 'Lefschetz' is 'Hyperplane Theorem'.

> **Theorem 3.1**If $\mathcal{H}$ is a generic Hyperplane in $\mathbb{C}P^N$ then the natural map $H^i(X;\mathbb{C}) \to H^i(X \cap \mathcal{H};\mathbb{C})$ is isomorphism for $i < n$ and for $i = n$ it is injection.

4. ALL RESULTS STATED ABOVE FAILS FOR SINGULAR VARIETIES

Example 1: For, $X = v(yz)$ the 'Poincare duality' fails.

We can write $X = V(y) \cap V(z)$ and $P = [1,0,0]$ is the point of intersection of $V(y)$ and $V(z)$. It's not hard to see $V(y)$ and $V(z)$ are $\mathbb{C}P^1$ thus the space X is wedge of two $\mathbb{C}P^1$ in other words it's homeomorphic to $\mathbb{S}^2 \vee \mathbb{S}^2$. We can note that,

$$
H_i(X;\mathbb{C}) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i = 1 \\ \mathbb{C} \oplus \mathbb{C} & i = 2 \end{cases}
$$

clearly, the 'Poincare duality' fails in this case. But if we normalize the space X to get two disjoint union of $\mathbb{S}^2$ in this case the duality will hold. (We will see this normalizaion helps in more general case when we will deal with intersection homology).

Example 2: For $X = V(x^3 + y^3 - xyz)$. it doesn't admit the 'Hodge decomposition'.

By change of variable (kind of Grobner basis) we can see $X = V(y^2z - x^2(x+z))$. It can be shown there is a blow-up map $\pi : \mathbb{C}P^1 \to X$ serves as a 'quotient map' with $\pi^{-1}[0:0:1]$ is two point. So, X is a $\mathbb{S}^2$ with two points being pinched. Thus,

$$
H_i(X;\mathbb{C}) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i = 1, i > 2 \\ \mathbb{C} & i = 2 \end{cases}
$$

In this case, there is no 'Hodge decomposition' of X. As dimension of H^1 is 1(odd).

Example 3: For $X = V(x_i x_j : i \in \{0,1\}, j \in \{3,4\})$, 'Lefschetz intersection' theorem do not hold.

X is unioun of two copies of $\mathbb{C}P^2$. $X = \{x_0 = x_1 = 0\} \cup \{x_3 = x_4 = 0\}$, meets in single point $[0:0:1:0:0]$. Thus,

$$H_i(X;\mathbb{C}) = \begin{cases} \mathbb{C} & i = 0 \\ \mathbb{C} \oplus \mathbb{C} & i = 2,4 \\ 0 & \text{otherwise} \end{cases}$$

Take a generic hyperplane $\mathcal{H}$ in $\mathbb{C}P^4$. Then $X \cap \mathcal{H}$ is disjoint union of two $\mathbb{C}P^1$. The cohomology of $X \cap \mathcal{H}$ is $\mathbb{C} \oplus \mathbb{C}$ for $i = 0, 2$ and trivial for other indices. Thus 'Lefschetz intersection theorem' fails here.

Through-out this redaing seminar we will try to develop the notion of Intersection homology, $IH_*(X)$ so that the properties (mentioned above) can be generalized for 'singular projective varieties' with this homology theory.