Pre-Requisites

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1. Basics of Homology and Cohomology

Here we need to know about Simplicial and singular Homology as well as cohomology. We will assume these are known to readers. Rather we will start with "Homology with closed support" which is known as 'Borel-Moore' Homology (as it might not be part of normal Algebraic Topology texts). Recall, in the singular/simplicial homology(cohomology) theory we defined the chains as *finite* linear sum of simplices. It is also possible to work with infinite linear combinations. (Here we will only talk about triangulable spaces).

Let, $\mathcal{T}$ be a triangulation of the space X. With respect to this triangulation, $C_n^{BM}(\mathcal{T};\mathbb{F})$ be the space of all formal linear combinations, $\Psi = \sum \Psi_\sigma \sigma$ where the sum runs over the set of all n-simplices of $|\mathcal{T}|$. This sum need not to be finite. Here, the coefficients are from the field $\mathbb{F}$. Let's define 'Borel-Moore complex' of X as,

$$C_n^{BM}(X;\mathbb{F}) := \operatorname{colim} C_n^{BM}(\mathcal{T};\mathbb{F})$$

the colimit is taken with respect to the refinement of triangulation $\mathcal{T}$. We can define the boundary map ∂ for $C_n^{BM}(\mathcal{T};\mathbb{F}) \to C_{n-1}^{BM}(\mathcal{T};\mathbb{F})$. This can be extended to $C_n^{BM}(X;\mathbb{F}) \to C_{n-1}^{BM}(X;\mathbb{F})$ in a natural way. The homology groups corresponding to this chain complex is known as 'Borel-Moore' homology theory.

Remark : If X is a compact space then the formal sum of n-simplices can have finite sum only. In that case simplicial/singular homology are equivalent to the Borel-Moore homology.

2. Sheaf Cohomology, Cech Cohomology

§ 2.1 Category of Sheaves as Abelian Category

The Sheaves on a topological space X forms a Category $\mathbf{Sh}(X)$ where the objects are the 'sheaves' and maps are the 'sheaf morphisms' (remember we are talking about the sheaf $\mathscr{F}$ which we can treat as a functor $\mathscr{F}$: $\mathrm{Open}(X) \to \mathrm{Vec}_k$). Note that:

1. The maps $\mathrm{Hom}_{\mathbf{Sh}(X)}(\mathscr{F},\mathscr{G})$ form an Abelian group under the operation
$$(\psi+\phi)(U)(s) = \psi(U)(s) + \phi(U)(s)$$
and the composition of such morphisms are biadditive.
2. There is a zero-sheaf.
3. We can form the direct sum of sheaves just by defining,
$$(\mathscr{F}\oplus\mathscr{G})(U) = \mathscr{F}(U)\oplus\mathscr{G}(U)$$
4. By restricting on the stalks we can see kernal of the morphisms $\phi : \mathscr{F} \to \mathscr{G}$ satisfy the universal properties represented by certain diagrams.
5. The morphism $\phi : \mathscr{F} \to \mathscr{G}$ gives us two short exact sequence containing, $\ker\phi$, $\operatorname{Im}\phi$ and $\operatorname{coker}\phi$.

Thus $\mathbf{Sh}(X)$ is an Abelian Category. We can talk about left exact and right exact functors. If we are given a topological space X, Y with a continuous map $f : X \to Y$. It will induce a covariant (contravariant) functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)(f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ as well).

DEFINITION. 2.1 (Pushforward/ Pullback of sheaf)Let $f : X \to Y$ be a continuous map. If $\mathscr{F} \in \mathbf{Sh}(X)$, the **push-forward** of $\mathscr{F}$ under f is the sheaf on Y given by the section

$$\Gamma(U, f_*\mathscr{F}) = \Gamma(f^{-1}(U), \mathscr{F})$$

Similarly, **pullback** of a sheaf $\mathscr{G} \in \mathbf{Sh}(Y)$ associated to the sheaf with the section,

$$\Gamma(U, f^*\mathscr{G}) = \operatorname*{colim}_{f(U)\subset V} \Gamma(V, \mathscr{G})$$

Now we will look at some properties of 'push-forward'. If $s \in \Gamma(\mathscr{F}, U)$ then the **support**($|s|$) of this s is defined to be the clousre of $\{x \in U : s_x \neq 0\}$. We can also define push-forward with *proper support* as a sheaf given by the functor,

$$\Gamma(f_!\mathscr{F}, U) = \left\{s \in \Gamma(\mathscr{F}, f^{-1}(U)) | f : |s| \to Y \text{ is a proper map}\right\}$$

Prop 1: It's not hard to see the stalk $(f^*\mathscr{G})_x \cong \mathscr{G}_{f(x)}$. Using this we can say, pullback f^* is a exact functor.

Prop 2: The push-forward functor (or 'push-forward functor with prope support') is left-exact but not exact.
Proof. If we have an exact sequence of sheaves, $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$. We know the section functor $\Gamma(U, -)$ is left exact. Thus we have the following exact sequence,

$$0 \to \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{G}) \to \Gamma(U, \mathscr{H})$$

Thus for V a open set in Y we can say the following is exact (By putting $U = f^{-1}U$),

$$0 \to \Gamma(U, f_*\mathscr{F}) \to \Gamma(U, f_*\mathscr{G}) \to \Gamma(U, f_*\mathscr{H})$$

Theorem 2.1 The push-forward f_* and pullback f^* is adjoint functor. In other words there is an isomorphism,

$$\mathrm{Hom}_{\mathbf{Sh}(X)}(f^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathbf{Sh}(Y)}(\mathscr{G}, f_*\mathscr{F})$$

§ 2.2 Injective Resolution and Sheaf Cohomology

Let, $\mathscr{F}$ be a sheaf on the topological space X. We will construct a sheaf $\mathcal{I}^0(\mathscr{F})$ such that $\mathscr{F} \to \mathcal{I}^0(\mathscr{F})$ is injective. Recall the construction of Etale space $\pi : E_{\mathscr{F}} \to X$ corresponding the sheaf $\mathscr{F}$. For every open set $U \subset X$, $\Gamma(U, \mathscr{F})$ is given by the set of continuous section of the map π restricted on the set U. If we define the sheaf $\mathcal{I}^0(\mathscr{F})$ with the sections,

$$\Gamma(U, \mathcal{I}^0(\mathscr{F})) = \{\text{sections (continuous or discontinuous) of } \pi|_U\} = \prod_{x\in U} \mathscr{F}_x$$

It's not hard to see that, $\mathcal{I}^0\mathscr{F}$ is a sheaf and there is a natural injective map $\mathscr{F} \to \mathcal{I}^0(\mathscr{F})$. This will give rise to an exact sequence of sheaves,

$$0 \to \mathscr{F} \to \mathcal{I}^0\mathscr{F} \to \mathcal{Q}^0 \to 0$$

Here, $\mathcal{Q}$ is the quotient sheaf, with sections, $\Gamma(U, \mathcal{Q}^0) = \Gamma\Big(U, \frac{\Gamma(U,\mathscr{F})}{\Gamma(U,\mathcal{I}^0\mathscr{F})}\Big)$. We can apply the same construction on $\mathbb{Q}$ to get the SES,

$$0 \to \mathcal{Q}^0 \to \mathcal{I}^0(\mathcal{Q}^0) \to \mathcal{Q}^1 \to 0$$

Just for notational purpose write $\mathcal{I}^0(\mathcal{Q}^0) = \mathcal{I}^1\mathscr{F}$. We can carry out the same construction to get banch of SES, which combining will give us a long exact sequence as follows:

$$0 \to \mathscr{F} \to \mathcal{I}^0\mathscr{F} \to \mathcal{I}^1\mathscr{F} \to \cdots$$

Let's apply the left-exact functor $\Gamma(X, -)$ to the above LES to get the following complex,

$$0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathcal{I}^0\mathscr{F}) \to \Gamma(X, \mathcal{I}^1\mathscr{F}) \to \cdots$$

We can define the sheaf cohomology of X as,

$$H^i(X;\mathscr{F}) := \frac{\ker\left(\Gamma(X,\mathcal{I}^i\mathscr{F}) \to \Gamma(X,\mathcal{I}^{i+1}\mathscr{F})\right)}{\operatorname{Im}\left(\Gamma(X,\mathcal{I}^{i-1}\mathscr{F}) \to \Gamma(X,\mathcal{I}^{i}\mathscr{F})\right)}$$

The above construction of injective resolution for a sheaf $\mathscr{F}$ is called **Godement resolution**. Now note that, $H^0(X,\mathscr{F})$ is $\ker\left(\Gamma(X,\mathcal{I}^0\mathscr{F}) \to \Gamma(U,\mathcal{I}^1\mathscr{F})\right)$ by definition. This map is given by the composition $\mathcal{I}^0\mathscr{F} \to \mathcal{Q}^0 \hookrightarrow \mathcal{I}^1\mathscr{F}$, applying global section functor we get, $\Gamma(X,\mathcal{Q}^0) \to \Gamma(X,\mathcal{I}^1\mathscr{F})$ is injective and hence:

$$\begin{aligned}
\ker\left(\Gamma(X,\mathcal{I}^0\mathscr{F}) \to \Gamma(X,\mathcal{I}^1\mathscr{F})\right) &= \ker\left(\Gamma(X,\mathcal{I}^0\mathscr{F}) \to \Gamma(X,\mathcal{Q}^0)\right)\\
&= \operatorname{Im}\left(\Gamma(X,\mathscr{F}) \to \Gamma(X,\mathcal{I}^0\mathscr{F})\right)\\
&\cong \Gamma(X,\mathscr{F})
\end{aligned}$$

Thus the 0-th sheaf cohomology is ismorphic to the global section of the sheaf $\mathscr{F}$.

DEFINITION. 2.2 (Flasque Sheaf)A sheaf $\mathscr{F}$ is a Flasque sheaf if for every open set $U \subset X$ the natural map $\Gamma(X,\mathscr{F}) \to \Gamma(U,\mathscr{F})$ is surjective.

In the above Godement resolution $\mathcal{I}^0\mathscr{F}$ is a flasque sheaf (clear from the definition). By induction we can say, $I^k\mathscr{F}$ is flasque.

Some properties of flasque sheaf:

1. If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is a exact sequence of shevaes with $\mathscr{F}$ being flasque, we can say the following is an exact sequence,
$$0 \to \Gamma(U,\mathscr{F}) \to \Gamma(U,\mathscr{G}) \to \Gamma(U,\mathscr{H}) \to 0$$
2. $\mathcal{I}^k(-)$ is an exact functor.
3. $\Gamma(X,\mathcal{I}^k(-))$ is an exact functor.

4 *. If $\mathscr{F}$ is a flasque sheaf, the sheaf cohomology $H^i(X,\mathscr{F})$ is trivial for $i > 0$.

§ 2.3 Čech Cohomology