

# Overview Talk

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We will begin with ‘cohomology of projective varieties’ and we will see for smooth projective varieties many beautiful properties holds for de-Rahm cohomology, singular cohomology which don’t get satisfied for the case of ‘singular projective varieties’. We will discuss those with examples in this talk.

Let  $X \subseteq \mathbb{C}P^N$  be a projective variety of dimension  $n$  (in the sense of Krull dimension which will be same with the manifold dimension for the smooth case).  $X$  is given by zeroes of some homogeneous polynomial thus it a closed subspace of  $\mathbb{C}P^N$  and hence it is compact. For the smooth case  $X$  is a ‘smooth manifold’ (complex manifold). Some properties of smooth  $X$  are described below,

- $X$  is given by zeroes of  $g_1, \dots, g_{N-n}$  with the rank of the matrix  $\left(\frac{\partial g_j}{\partial z_i}\right)_{ij}$  equal to  $N - n$ .
- Hermitian metric on Tangent space of  $X$ .
- $X$  is an orientable manifold of dimension  $2n$  admitting a Riemannian metric  $g$  and a ‘complex structure’ on it’s Tangent space.
- There is also an alternating form  $\omega$  (or Kähler differential).

## 1. DUALITIES

As a real manifold  $X$  has dimension  $2n$ . We can compute the singular(simplicial) homology(cohomology) for  $X$  with the coefficients in  $\mathbb{R}$ . Since  $X$  is compact orientable manifold we can talk about the cup product pairing as follows:

$$H^i(X; \mathbb{R}) \times H^{2n-i}(X; \mathbb{R}) \xrightarrow{\smile} H^{2n}(X; \mathbb{R}) \cong \mathbb{R}$$

is a ‘non-degenerate’ pairing. Thus we have **Poincare Duality**,

$$H_{\text{Sing}}^{2n-i}(X; \mathbb{R}) \cong H_{\text{Sing}}^i(X; \mathbb{R})^* \cong H_i^{\text{sing}}(X; \mathbb{R})$$

Since  $X$  is a smooth manifold we can talk about de-Rahm cohomology. In a sophisticated language ‘de-Rahm cohomology is a cohomology of soft-resolution of constant sheaf’. In this case also we have the following as non-degenerate,

$$H_{DR}^i(X; \mathbb{R}) \times H_{DR}^{2n-i}(X; \mathbb{R}) \xrightarrow{\wedge} H_{DR}^{2n}(X; \mathbb{R}) \xrightarrow[\int - \text{vol-form}]{\sim} \mathbb{R}$$

Thus again we have the duality,  $H_{DR}^{2n-i}(X; \mathbb{R}) \cong H_{DR}^i(X; \mathbb{R})^*$ . Connecting de-Rahm cohomology and singular(simplicial) cohomology with coefficients in  $\mathbb{R}$ , there is a beautiful theorem by de-Rahm stated as follows,

**Theorem 1.1 (DE-RAHM’S THEOREM)** There is an isomorphism between the singular(simplicial) cohomology with coefficients in  $\mathbb{R}$  and de-Rahm cohomology which is compatible with the product structure on both the V.S.

## 2. HODGE THEOREMS

There are two different versions of ‘Hodge Theorem’. The metric (Riemannian) on  $X$  induces metric on de-Rahm complex  $\Omega^\bullet(X)$ . It is defined by,

$$(\omega, \eta) := \int_X \left( p \mapsto \langle \omega, \eta \rangle_p \right) \text{Vol}_X$$

With respect to this inner product the exterior derivative  $d$  has an adjoint  $\delta$  such that,

$$(d\omega_1, \omega_2) = (\omega_1, \delta\omega_2)$$

\* We can write down the adjoint explicitly, for any  $\alpha \in \Omega^k(X)$ ,  $\delta\alpha = (-1)^k (*)^{-1} d\alpha$  where  $*$  is the ‘Hodge star operator’  $*$  :  $\wedge^k(T_p X)^* \rightarrow \wedge^{2n-k}(T_p X)^*$ , given by  $(\theta_1, \dots, \theta_k) \mapsto (\theta_{k+1}, \dots, \theta_{2n})$  where  $\{\theta_j\}$  is an oriented orthonormal basis of  $(T_p X)^*$ . Set,  $\Delta = \delta d + d\delta$  be the Laplacian. **Harmonic** forms are elements of  $\Omega^\bullet(X)$  lies in the kernel of  $\Delta$ . With this setup we are ready to state ‘Hodge theorem 1’. This theorem gives us a decomposition of  $\Omega^k(X)$ .

**Theorem 2.1 (HODGE THEOREM I)** Every element of  $H_{DR}^k(X; \mathbb{R})$  is uniquely represented by ‘Harmonic forms’ of degree  $k$ . Also  $\Omega^k$  admits the following decomposition,

$$\Omega^k(X) \cong H_{DR}^k(X; \mathbb{R}) \oplus d(\Omega^{k+1}) \oplus \delta(\Omega^{k+1})$$

We have perviously mentioned there is a ‘complex structure’ on the Tangent space of  $X$ . As of now we have not used this structure.

### § The presence of complex structure I

The complex structure gives rise to Eigen decomposition of complexified tangent/co-tangent bundles on  $X$ . Thus we have,

$$\Omega_{X, \mathbb{C}}^1 := \Omega_X^1 \otimes \mathbb{C} \cong \Omega_X^{1,0} \oplus \Omega_X^{0,1}$$

Then,  $\Omega_{X, \mathbb{C}}^k = \wedge^k \Omega_{X, \mathbb{C}}^1 \cong \bigoplus_{p+q=k} \Omega_X^{p,q}$ . Here,  $\Omega_X^{p,q} = \wedge^p \Omega_X^{1,0} \otimes \wedge^q \Omega_X^{0,1}$ . Thus,  $\Omega_X^{p,q}$  is the V.S of the smooth  $(p, q)$ -forms  $dz_1 \wedge \dots \wedge dz_p \wedge d\bar{z}_{p+1} \wedge \dots \wedge d\bar{z}_{p+q}$ . we can note  $\Omega_X^{p,q} = \Omega_X^{q,p}$ . With this setup we are ready to note the Hodge theorem II.

**Theorem 2.2 (HODGE THEOREM II)** Every Harmonic form in  $H^k(X; \mathbb{C})$  decomposes as a sum of harmonic  $(p, q)$ -forms of a bi-degree, where  $p + q = k$ . Thus,

$$H_{DR}^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X; \mathbb{C})$$

**COROLLARY.** If  $k$  is odd then  $\dim_{\mathbb{C}}(H_{DR}^k(X; \mathbb{C}))$  even.

## 3. HARD LEFSCHETZ THEOREM

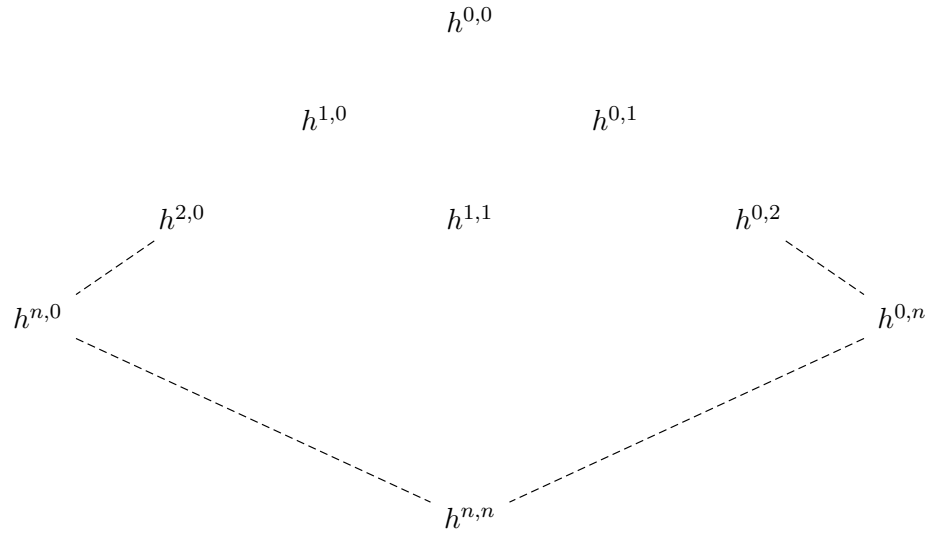
The hermitian metric  $\mathfrak{h}$  on  $TX$  via its decomposition gives rise to an alternating 2-forms can be shown to be a  $(1, 1)$ -form, call it  $\omega$ . Using this we get a linear map,

$$L : H_{DR}^k(X; \mathbb{R}) \xrightarrow{\text{product with } \wedge \omega} H^{k+2}(X; \mathbb{R})$$

**Hard Lefschetz Theorem** – The map,  $L^{n-k} : H_{DR}^k(X, \mathbb{R}) \rightarrow H_{DR}^{2n}(X; \mathbb{R})$  induces an isomorphism for  $k \leq n$ .

**COROLLARY.**  $L$  is injective for  $k < n$ . Thus the odd degree Betti number  $h^i := \dim_{\mathbb{R}} H^i(X; \mathbb{R})$  increases upto the middle degree and then decreases there after.

Thus we havethe following HODGE DIAGRAM.



In the above diagram  $i$ -th row sums up to give  $i$ -th Betti numbers of  $X$ . One more interesting result due to 'Lefschetz' is 'Hyperplane Theorem'.

**Theorem 3.1** If  $\mathcal{H}$  is a generic Hyperplane in  $\mathbb{C}P^N$  then the natural map  $H^i(X; \mathbb{C}) \rightarrow H^i(X \cap \mathcal{H}; \mathbb{C})$  is isomorphism for  $i < n$  and for  $i = n$  it is injection.

#### 4. ALL RESULTS STATED ABOVE FAILS FOR SINGULAR VARIETIES