Homology with local coefficients

(In the context of Intersection Homology)

Trishan Mondal

In the case of ordinary Homology theory we have seen 'Homology with coefficients' $H_*(X, G)$, where the coefficients of simplex comes from the group (Abelian) G. During the study of homotopy theory of non-simply connected spaces, we consider the action of $\pi_1(X)$ on some Abelian groups. Local coefficient system are tools to organize this information. For Homology with local coefficients the 'coefficients' of simplex comes from a bundle of group. So, as we move around the space, it allows the coefficients to "twist"(change).

If we have a topological pseudo-manifold X with the following startification,

$$X = X_n \supseteq X_{n-2} \cdots \supset X_0 \supseteq \emptyset$$

then for any perversity p if X is a manifold, $IH_i^p(X) \simeq H_i(X)$, in this case if we deal with local system both the homology will also be same. But for the case of pseudo-manifold with singularity, if there is a local system $\mathcal{L}$ defined only on $X - \Sigma_X$ (non-singular part), which can't be extended to the whole X, we can talk about Intersection chains of X with local-coefficients but $H_*(X, \mathcal{L})$ do not make sense. **Recall** there are two ways in which we can define homology with local coefficients.

First way ($k[\pi]$-modules): Let, X be a locally connected topological with a simply connected universal covering $\tilde{X}$. Let, $p : \tilde{X} \to X$ be the covering and $\pi_1 := \pi_1(X)$ be the fundamental group of X. Consider the group ring $k[\pi]$. It is a non-commutative ring (as π may not be Abelian). Then make the following observations:

- Singular complex with integer coefficients $S_*(\tilde{X}; k)$ is a right $k[\pi]$ module; if we treat π as the group of deck transformations, every element in π will corresponds to a homoemorphism $\in \mathrm{Deck}(p)$. The composition of this homoemorphism with a simplex σ will give us a new simplex. We can also talk about the basis of $S_*(\tilde{X}; k)$ as a $k[\pi]$ module.
- Let, V be a vector-space (k-module). Then consider a representation of π,

  $$\rho : \pi \to GL(V)$$

  Thus it will give us an action, so that we can write V as a left $k[\pi]$ module. So, the tensor product $S_*(\tilde{X}; k) \otimes_{k[\pi]} V$ make sense.

Now we will just define $S_*(X; V) := S_*(\tilde{X}; k) \otimes_{k[\pi]} V$, there is a natural boundary oparetor. The homology corresponding to this complex is called homology with local coefficients. We write it like $H_*(X, V_\rho)$. This definition doesn't adopt easily for the case of Intersection homology. (Note that $H_*(X, V_\rho)$ is a module over k,) There is a more geometric (topological construction to it), which is easily adoptable for the case of Intersection chains.

1. Local systems and homology

Recall :(Consider X to be locally connected) If X is a topological space. The fundamental groupoid $\Pi(X)$ is the category with $\mathrm{Obj}(X)$ are elements of X and $\mathrm{Hom}(p, q)$ is the set of paths between p, q (upto homotopy).

Definition. 1.1 (Local system) A system of local coefficients is contravariant functor,

$$\mathcal{L} : \Pi(X) \to \mathrm{Vec}_k^V$$

(Vec_k^V is the k-vector space isomorphic to V).

Equivalently, $\mathcal{L}$ is a locally constant sheaf defined by a representation of $\pi_1(X)$. Consider, the Étale space for the sheaf $\mathcal{L}$, $E = \sqcup_{x\in X}\mathcal{L}(x)$. Thus we have a natural projection $\pi : E \to X$, such that fibre of the point x is $\mathcal{L}(x)$. Now let, $S_k(X; \mathcal{L})$ denote the set of all finite formal sum $\sum_{i=1}^m a_i\sigma_i$, where :

1. $\sigma_i : \Delta^k \to X$ is singular k-simplex and,
2. a_i is an element of the group $\mathcal{L}_{\sigma(e_0)}$. Where $e_0 = (1, 0, 0, 0 \cdots)$.

The obvious way to sum elements make sense and is well defined. To lessen the confusion one view $S_k(X; \mathcal{L})$ as a sub-space of $\oplus_{x\in X} S_k(X; \mathcal{L}(x))$. Now we will describe the differential $\partial : S_k(X; \mathcal{L}) \to S_{k-1}(X; \mathcal{L})$. **Recall:** there are face maps $f_m^k : \Delta^{k-1} \to \Delta^k$ defined by $f(t_0, t_1, \cdots, t_{k-1}) = (t_0 \cdots t_{m-1}, 0, t_m, \cdots)$.

Given a singular simplex $\sigma : \Delta^k \to X$ and $\gamma_\sigma : [0, 1] \to X$ be the path $\sigma(t, 1-t, 0 \ldots, 0)$. Then because $\pi : E \to X$ is a covering space (the fibre is discrete), the lift of the path γ_σ, gives us a isomorphism between $\mathcal{L}(\sigma(0, 1, \cdots)) \to \mathcal{L}(\sigma(1, 0, \cdots))$. We can define

$$\partial(a\sigma) = \tilde{\gamma}_\sigma(\sigma \circ f_0^k) + \sum_{m=1}^{k} (-1)^m a(\sigma \circ f_m^k)$$

It can be checked that it is a differential i.e. $\partial^2 = 0$. Thus we can define homology

$$H_*(X; \mathcal{L}) := H_*(S_\bullet(X; \mathcal{L}), \partial)$$

The following theorem will tell us the two definition are equivalent.

> **Theorem 1.1** The homology $H_k(X; \mathcal{L})$ is equals to $H_k(X; V_\rho)$. Where the representation ρ of $\pi_1(X)$ is determined by the local system $\pi : E \to X$.

It's easier to define Intersection homology with local coefficients with the second definition. Now suppose that X is a topological pseudomanifold with a fixed topological stratification

$$X = X_m \supseteq X_{m-2} \supseteq \cdots \supseteq X_0.$$

To make this procedure work for intersection homology we only need the local coefficient system $\mathcal{L}$ to be defined on the open subset $X - X_{m-2}$ of X, not on the whole of X. This is because the allowability conditions on intersection i-chains ξ mean that if the coefficient of ξ indexed by σ is non-zero then

$$\sigma^{-1}(X - X_{m-2}) \neq \emptyset$$

and similarly $\tau^{-1}(X - X_{m-2}) \neq \emptyset$ for any face τ of σ. Thus we can use this procedure to define the intersection homology groups $IH_i(X; \mathcal{L})$ of X with coefficients in $\mathcal{L}$ for any local coefficient system $\mathcal{L}$ on $X - X_{m-2}$.