COMPLEX COBORDISM AND UNIVERSAL FORMAL GROUP LAW

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Abstract. This report presents a detailed study of complex oriented cohomology theories and their relationship with formal group laws, culminating in Quillen's identification of the complex cobordism spectrum MU as the universal complex oriented theory. We develop the algebraic and homotopical tools required to compute the complex cobordism ring MU_*, employing the Adams spectral sequence, the structure of the dual Steenrod algebra, and a change-of-rings argument. These methods yield Milnor's theorem that $MU_* \cong \mathbb{Z}[x_1, x_2, \ldots]$, a polynomial ring on even-degree generators. The final part of the report examines Quillen's theorem, establishing that this polynomial ring is isomorphic to the Lazard ring and that the formal group law associated with MU is the universal formal group law.

1. Introduction

The study of manifolds and how to classify them has been a central theme in topology since the field began. One of the most powerful tools for tackling this problem is *cobordism theory*, which assigns algebraic invariants that capture key geometric properties of manifolds. Among the different types of cobordism theories, *complex cobordism* has emerged as a universal framework, bridging algebraic topology with the rich structure of formal group laws.

The foundations of cobordism were laid by **René Thom** in the 1950s. Thom introduced cobordism as an equivalence relation on manifolds and showed that the resulting cobordism groups form a generalized homology theory [Tho54]. This groundbreaking work, which earned Thom the Fields Medal in 1958, transformed our understanding of manifolds by demonstrating that their global features could be studied using homotopy-theoretic methods.

Building on Thom's work, mathematicians developed various structured cobordism theories. In 1960, **John Milnor** introduced *complex cobordism* and computed the structure of its coefficient ring, showing that

$$MU_* \cong \mathbb{Z}[x_1, x_2, \ldots]$$

is a polynomial ring with generators in even degrees [Mil60]. Milnor's computation, carried out via the Adams spectral sequence, revealed a polynomial structure that would later prove essential in connecting complex cobordism to formal group laws.

Meanwhile, algebraists and algebraic geometers were investigating *formal group laws*—power series that encode group operations in a purely algebraic, "formal" setting. In 1955, **Michel Lazard** proved the existence of a *universal formal group law* classified by a graded ring, now known as the *Lazard ring* [Laz55]. Lazard showed that this ring also has a polynomial structure,

$$L \cong \mathbb{Z}[x_1, x_2, \ldots],$$

mirroring the structure Milnor had found for complex cobordism.

The deep connection between these two areas—complex cobordism and formal groups—was uncovered by **Daniel Quillen** in 1969. In [Qui69a], Quillen proved that the complex cobordism ring MU_* is canonically isomorphic to the Lazard ring L, and that the formal group law associated with complex cobordism is, in fact, the universal formal group law. This result, often described as "cryptic and insightful," fundamentally changed algebraic topology [Rav13], establishing complex cobordism as the universal complex-oriented cohomology theory.

In recent decades, the interplay between complex cobordism and formal groups has continued to drive major advances. A particularly striking development is the construction of *topological modular forms* (tmf), which can be seen as the next layer in the chromatic tower after complex cobordism and complex K-theory. The spectrum tmf was built by **Michael Hopkins** and **Haynes Miller** in the 1990s using the Goerss–Hopkins–Miller theorem [GH04], which ensures the existence of highly structured E_∞-ring spectra on the moduli stack of elliptic curves. This approach realizes tmf as the global sections of a sheaf of elliptic cohomology theories, with a coefficient ring closely connected to classical modular forms [Hop02].

Structure of this report. This report provides a detailed exposition of the classical theory of complex cobordism and its identification with the universal formal group law. We develop the necessary algebraic and homotopical machinery to understand both Milnor's computation of MU_* and Quillen's theorem.

In Section 2, we introduce complex-oriented cohomology theories and explore their basic properties, including the existence of Chern classes and the emergence of formal group laws from the tensor product of line bundles. We establish that complex cobordism MU serves as the universal example of such theories. Section 3 is dedicated to the computation of the complex cobordism ring. We employ the Adams spectral sequence, working with the dual Steenrod algebra and using change-of-rings arguments to reduce computations to manageable form. This yields Milnor's theorem that $MU_* \cong \mathbb{Z}[x_1, x_2, \ldots]$. In Section 4, we present Quillen's theorem, showing that the polynomial ring MU_* is canonically isomorphic to the Lazard ring and that the formal group law of complex cobordism is universal. The proof proceeds by analyzing the Hurewicz homomorphism from MU_* to $H_*(MU)$ and using the Adams spectral sequence to establish the necessary divisibility properties.

The appendices provide technical background on the dual Steenrod algebra, the Adams spectral sequence, and the change-of-rings theorem for Hopf algebras, which are essential tools for the main computations.

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2. Complex Oriented cohomology theory

A complex oriented cohomology theory means a generalized cohomology theory E, which is multiplicative and has a Thom class for every complex vector bundle. More explicitly, if $\xi \to X$ is a complex n-bundle there is a class $t(\xi) \in \tilde{E}^{2n}(X^{\xi})$ such that

(i) The image of $t(\xi)$ under the composition of following map goes to 1,
$$\tilde{E}^{2n}(X^{\xi}) \to \tilde{E}^{2n}(*^{\xi}) \to \tilde{E}^{2n}(S^{2n}) \to E^0(*)$$
(ii) The class $t(\xi)$ should be natural under pullbacks.
(iii) $t(\xi \oplus \eta) = t(\xi).t(\eta)$.

The following proposition from [Die11] would help us to define the complex oriented cohomology theory in a more subtle way,

Proposition 2.1. *Any class $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ that goes to 1 under the pullback of $i : \mathbb{C}P^1 \to \mathbb{C}P^\infty$ gives us a complex orientation on E.*

Remark 2.2. The ordinary cohomology theory given by $H\mathbb{Z}$ is a complex oriented cohomology theory. Complex K-theory is also complex orientable. Using Bott periodicity we get
$$\widetilde{KU}^2(\mathbb{C}P^1) = \widetilde{KU}^0(S^2) = \pi_2(BU \times \mathbb{Z}) \simeq \mathbb{Z}$$
The proposition 2.1 helps us to conclude that KU is a complex oriented cohomology theory.

The advantage of having Thom class $t(\xi)$ for $\xi \to X$ is that we have Thom-isomorphism by taking cup-product with the Thom class,
$$U_\xi : E^*(X) \xrightarrow{\simeq} \tilde{E}^{*+2n}(X^{\xi})$$
just like in the case of ordinary cohomology theory. It also gives rise to Chern classes $c_i(\xi) \in \tilde{E}^{2i}(X)$ satisfying

1. Naturality under pullbacks;
2. $c_n(\xi \oplus \eta) = \sum_{i+j=n} c_i(\xi)c_j(\eta)$;
3. $c_1(L) = x \in \tilde{E}^2(\mathbb{C}P^\infty)$, where L denotes the tautological line bundle over $\mathbb{C}P^\infty$.

In singular cohomology one has
$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$
for line bundles L_1 and L_2 over the same base X. We can say similar thing about a complex oriented cohomology theory. It turns out that $c_1(L_1 \otimes L_2)$ can be written as
$$F(c_1(L_1), c_1(L_2))$$
for some $F(x, y) \in E^*[\![x, y]\!]$. If we write $x +_F y$ for $F(x, y)$, then this power series will have the following properties:

(1) $x +_F y = y +_F x$ (because $L_1 \otimes L_2 \cong L_2 \otimes L_1$);
(2) $x +_F 0 = x = 0 +_F x$ (because $L \otimes 1 \cong L$, where 1 denotes the trivial line bundle);
(3) $(x +_F y) +_F z = x +_F (y +_F z)$ (because tensor product of line bundles is associative).

Such an F is called a *formal group law* over the ring E^*. Here $E^* = \tilde{E}^*(\text{pt})$. To understand the Chern classes it is important to understand $E^*(\mathbb{C}P^\infty)$ as all the complex line bundles are classified by a map to $\mathbb{C}P^\infty$. To define the Chern class it's important to understand this cohomology ring.

Proposition 2.3. *Let E be a multiplicative cohomology theory and let $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ be an element restricting to 1. Then there is a isomorphism*
$$E^*(\mathbb{C}P^n) = E^*[x]/(x^{n+1})$$

Proof. The element x gives a map $E^*[x] \to E^*(\mathbb{C}P^n)$ (for each n). One can see that x^{n+1} must map to zero: First note that $\mathbb{C}P^n$ can be covered by $n+1$ contractible open sets U_i, and because x is a reduced cohomology class it must restrict to zero on each U_i. As a general rule one knows that if $a \in \tilde{E}^*(X, A)$ and $b \in E^*(X, B)$ then

$$a \smile b \in \tilde{E}^*(X, A \cup B).$$

But we can write $x \in E^*(\mathbb{C}P^n, U_i)$ for each i, and so x^{n+1} lies in $E^*(\mathbb{C}P^n, U_1 \cup \cdots \cup U_{n+1}) = E^*(\mathbb{C}P^n, \mathbb{C}P^n) = 0$.

Thus, we obtain a map :

$$E^*(\mathbb{C}P^n)[x]/(x^{n+1}) \to E^*(\mathbb{C}P^n).$$

Consider the Atiyah–Hirzebruch spectral sequence (AHSS) for the cohomology theory E^* applied to the CW–complex $\mathbb{C}P^n$ with the skeletal filtration:

$$E_2^{p,q} = H^p(\mathbb{C}P^n; E^q(*)) \Longrightarrow E^{p+q}(\mathbb{C}P^n).$$

Since $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[x]/(x^{n+1})$ with $|x| = 2$, the E_2–page may be identified (as an E^*–module) with

$$E_2^{*,*} \cong E^* \otimes H^*(\mathbb{C}P^n) \cong E^*[x]/(x^{n+1}),$$

where $x \in E_2^{2,0}$ is the class coming from the generator of $H^2(\mathbb{C}P^n)$.

By multiplicativity of the AHSS and the existence of the chosen orientation class/orientation-like element $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ which restricts to the generator on each 2–cell, both the classes coming from E^* (the coefficients) and the class x are permanent cycles. Degree reasons then force all differentials to vanish: any nonzero differential

$$d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$$

would change the total degree by 1, but the algebra generators lie in even total degrees and there is no room for differentials to hit the generators (more concretely, the E_2–page is concentrated in even total degrees and the target groups of possible differentials are zero). Hence the spectral sequence collapses at E_2.

Collapse of the AHSS gives an isomorphism of graded groups between the associated graded of the skeletal filtration on $E^*(\mathbb{C}P^n)$ and $E^*[x]/(x^{n+1})$:

$$\operatorname{Gr} E^*(\mathbb{C}P^n) \;\cong\; E^*[x]/(x^{n+1}).$$

It remains to check there are no nontrivial extension problems in passing from this associated graded ring to $E^*(\mathbb{C}P^n)$ itself. Multiplicativity of the spectral sequence provides a multiplicative identification of the associated graded algebra, and the class $x \in E^2(\mathbb{C}P^n)$ (the image of the degree-2 generator from $\mathbb{C}P^\infty$) lifts the degree–2 generator on the E_2–page. The powers $1, x, x^2, \ldots, x^n$ therefore give elements of $E^*(\mathbb{C}P^n)$ whose images in the associated graded are the corresponding monomials. This forces the multiplicative extensions to be trivial, so the natural map of E^*-algebras

$$E^*[x]/(x^{n+1}) \longrightarrow E^*(\mathbb{C}P^n)$$

sending the polynomial generator x to this element of $E^2(\mathbb{C}P^n)$ is an isomorphism.

Thus $E^*(\mathbb{C}P^n) \cong E^*[x]/(x^{n+1})$, as required □

Corollary 2.4. *Taking inverse limit we get*

$$E^*(\mathbb{C}P^\infty) \simeq E^*[\![x]\!]$$

For a line bundle $L \to X$ we have canonical map $f_L : X \to \mathbb{C}P^\infty$. We define, $C_1(L) = f_L^*(x)$. So, without loss of generality we may assume x is the chern class of the ‘tautological line bundle’ over $\mathbb{C}P^n$, for any n.

Corollary 2.5. *The most important corollary of the previous proposition is that*

$$E^*(BU(n)) \simeq E^*[\![c_1, \cdots, c_n]\!]$$

where c_i are in degree $2i$ and hence we can define n-th chern class for any complex vector bundle.

Proof. The proof of the corollary is exactly the same proof in ordinary cohomology theory. □

Remark 2.6. As we have mentioned earlier, the Chern classes gives us a formal group law over E^*. We already know for $E = H\mathbb{Z}$ it is just addition over $\mathbb{Z}[x, y]$. For complex K-theory, KU it turns out that the $c_1(L) = 1 - L$ and thus the formal group law is given by $F(x, y) = x + y - xy$.

The universal example of complex oriented cohomology theory is given by **complex cobordism**. Firstly, consider $BU(n)$ to the the cassifying space for complex n-plane bundle. Let, γ_n be the canonical n-plnae bundle over it often defined as $EU(n) \times_{U(n)} \mathbb{C}^n$. We define $MU(n)$ to be the Thom space of γ_n. More precisely

$$MU(n) = EU(n)^+ \wedge_{U(n)} \widehat{\mathbb{C}^n}$$

There is a map $BU(n) \to BU(n+1)$ that sends a bundle ξ to $\xi \oplus \epsilon$, ϵ is the trivial line bundle. Thus in Thom space we have a map

$$S^{\mathbb{C}} \wedge MU(n) \to MU(n+1)\ (\star)$$

If we consider the collection $MU = \{MU(n)\}_{n \geq 1}$ apparently this is not a spectra as there is no structure map between $MU(n)$ to $MU(n+1)$. Perhaps what we do is the following: consider $\widetilde{MU}_n = \mathrm{map}(S^n, MU(n))$. The collection $\widetilde{MU} = \left\{\widetilde{MU}_n\right\}$ forms a spectra (Exercise, or follow the paper of S.Schwede on *Symmetric spectra*). What one can note is that the stable homotopy groups of $\widetilde{MU}$ is given by

$$\pi_k(\widetilde{MU}) = \mathrm{colim}\ \pi_{k+2n}(MU(n))$$

here the colimit is taken from the map coming from the induced map in homotopy groups coming from $(\star)$. We shall abuse the notation and assume MU to be the spectra $\widetilde{MU}$. In the stable homotopy category we have

$$MU \simeq \mathrm{hocolim}\ S^{-2n} \wedge \Sigma^\infty MU(n)$$

This spectrum is called **complex cobordism.** From here, let us define the spectra $MU_n = S^{-2n} \wedge MU(n)$ in the stable homotopy category. So we can see MU is homotopy colimit of MU_n more precisely we have a filtration of MU using MU_n.

To understand why complex cobordism MU is the universal complex oriented cohomology theory, let us recall what a *complex orientation* is. A complex orientation of a multiplicative cohomology theory E consists of a functorial assignment of Thom classes to complex vector bundles, natural in the bundle and multiplicative under direct sum. Let E be a complex oriented cohomology theory represented by a spectrum E. For each n, consider the universal complex n-plane bundle

$$\gamma_n \longrightarrow BU(n),$$

where γ_n is the canonical bundle. A functorial choice of Thom classes for n-plane bundles is equivalent to choosing a Thom class for this universal bundle. Since the Thom space of γ_n is by definition $MU(n)$, such a Thom class is an element

$$u_n \in \widetilde{E}^{2n}(MU(n)).$$

giving a Thom class $u_n \in \widetilde{E}^{2n}(MU(n))$ is equivalent to giving a map of spectra (for each n)

$$MU_n \longrightarrow E.$$

It is known that MU is a ring spectrum, in fact an E_∞–ring spectrum. The multiplication comes from the block–sum (direct sum) operation on vector bundles:

$$BU(n) \times BU(m) \;\longrightarrow\; BU(n+m),$$

and this induces the structure maps

$$MU(n) \wedge MU(m) \;\longrightarrow\; MU(n+m)$$

making MU into a highly structured ring spectrum.We saw earlier that a complex orientation on E requires maps

$$\theta_n : MU_n \;\longrightarrow\; E$$

for each n. Equivalently, elements $\theta_n \in \widetilde{E}^*(MU_n)$. These must be compatible in the sense that the element $\theta_n\theta_m$ is the pullback of θ_{n+m} in $\widetilde{E}^*(MU_{n+m})$. Moreover, to be actual Thom classes, they must restrict to the unit map on

$$MU_0 = S^0.$$

Thus, giving a complex orientation on E is equivalent to giving maps of spectra

$$\theta_n : MU_n \longrightarrow E$$

for each n, such that the following diagram is always homotopy commutative:

$$\begin{tikzcd}
MU_n \wedge MU_m \arrow[r] \arrow[d, "\theta_n \wedge \theta_m"'] & MU_{n+m} \arrow[d, "\theta_{n+m}"] \\
E \wedge E \arrow[r] & E
\end{tikzcd}$$

and satisfy the restriction condition $\theta_{n+1}|_{MU_n} = \theta_n$.. These compatible maps assemble into a morphism of *ring spectra*

$$MU \;\longrightarrow\; E.$$

To justify this, observe that a complex orientation gives a family of maps

$$MU_n \longrightarrow E,$$

equivalently compatible elements of $\widetilde{E}^*(MU_n)$. Using the Thom isomorphism,

$$\widetilde{E}^*(MU_n) \;\cong\; \widetilde{E}^*(BU(n)) \;\cong\; (\pi_*E)[[c_1,\ldots,c_n]],$$

and the transition maps in n are surjections. Thus the Milnor exact sequence yields

$$\widetilde{E}^*(MU) \;\cong\; \varprojlim \widetilde{E}^*(MU_n).$$

So, given a complex orientation, we get a map of spectra $MU \to E$ that is unique to θ_n after restricting it to MU_n. Now we will check that the obtained map is a mp of ring spectra. This means that the multiplication on MU is sent to the multiplication on E, i.e. that the following diagram is homotopy commutative:

$$\begin{tikzcd}
MU \wedge MU \arrow[r] \arrow[d] & MU \arrow[d] \\
E \wedge E \arrow[r] & E
\end{tikzcd}$$

To see this, observe that the multiplication on MU arises from the direct-sum operation on complex vector bundles:

$$BU(n) \times BU(m) \longrightarrow BU(n+m),$$

which induces structure maps

$$MU_n \wedge MU_m \longrightarrow MU_{n+m}.$$

Given a complex orientation, the two composites

$$MU_n \wedge MU_m \longrightarrow MU_{n+m} \longrightarrow E \quad \text{and} \quad MU_n \wedge MU_m \longrightarrow E \wedge E \longrightarrow E$$

represent two elements of $\widetilde{E}^*(MU_n \wedge MU_m)$ that restrict to the same element in $\widetilde{E}^*(MU_{n+m})$. For every pair (n, m) these restrictions agree. Using the Milnor exact sequence and the fact that the transition maps

$$\widetilde{E}^*(MU_{n+m}) \longrightarrow \widetilde{E}^*(MU_n \wedge MU_m)$$

are surjective, we deduce that the two composites above must coincide in the inverse limit

$$\widetilde{E}^*(MU \wedge MU) \cong \varprojlim \widetilde{E}^*(MU_n \wedge MU_m).$$

Hence the multiplication diagram commutes, proving that $MU \longrightarrow E$ is indeed a *map of ring spectra* and this is unique. From this we obtain the result:

Theorem 2.7. *If E is a ring spectrum, then there is a natural bijection between complex orientations on E and maps of ring spectra*

$$MU \longrightarrow E.$$

Remark 2.8. We have previously seen that a complex orientation on cohomology theory leads to a formal group law. It will turn out that the ring $\pi_*(MU)$ or MU_* has universal formal group law in the same way we have MU is universal complex oriented cohomology theory. This remarkable result is due to Quillen. Later on this report we will explore that.

3. The complex cobordism ring MU_*

In this section we will compute the ring $\pi_*(MU)$ oe equivalently $MU_*(S^0)$. We will use Adams spectral sequence (the version discussed in appendix B) for this computation. The idea is to know about $\widehat{\pi_*(MU)}_p$ for all primes p from ASS and then use some local-global types argument to conclude things about MU_*. The first step is to compute $\mathcal{A}_p^*$-comodule structure of $H_*(MU; \mathbb{F}_p)$.

Proposition 3.1. *Since MU is a ring spectrum $H_*(MU)$ (integral homology) is a ring and as a ring it is isomorphic to $\mathbb{Z}[b_1, \cdots b_i, \cdots]$ where $\deg b_i = 2i$.*

Proof. Recall the fact $H^*(BU(n)) = (H^*(BU(1))^{\otimes n})^{\Sigma_n}$. Since $BU(n)$ has cohomology in even cells we can dualize this to get $H_*(BU(n)) = \mathrm{Sym}^n(H_*(BU(1)))$. Now we apply Thom isomorphism to get

$$H_*(MU(n)) = \mathrm{Sym}^n(H_*(MU(1)))$$

The multiplication structure of $H_*(MU(n))$ is similar to the multiplicative structure of $H_*(BU(n))$ as Thom isomorphism is multiplicative. Note that $H_*(MU(1)) = \mathbb{Z}\{\gamma_i\}$ for $i \geq 0$, and γ_i has degree $2i$. Taking limit with respect to n we get,

$$H_*(MU) = \varinjlim H_*(MU(n)) \simeq \mathbb{Z}[b_1, b_2, \cdots]$$

Here γ_0 goes to unit and γ_i goes to b_i. □

Corollary 3.2. *$H_*(MU; \mathbb{F}_p) \simeq \mathbb{F}_p[b_1, b_2, \cdots]$ where b_i has degree $2i$.*

Now we want to look at the $\mathcal{A}_p^*$-comodule structure on $H_*(MU; \mathbb{F}_p)$. Recall that $\mathcal{A}_p^*$ as an $\mathbb{F}_p$ algebra isomorphic to (for odd p) $\mathbb{F}_p[\xi_i] \otimes E(\tau_j)$ where $\deg \xi_i = 2(p^i - 1), \deg \tau_j = 2p^j - 1$. The τ_j's are coming from Bockstein. The action of Bockstein on $H^*(MU; \mathbb{F}_p)$ is trivial. Thus we can say there is a subcoalgbera $P \subset \mathcal{A}_p^*$ generated by ξ_i for odd primes p which coacts on $H_*(MU; \mathbb{F}_p)$. For $p = 2$ we have similar analogue of this. There P is generated by ξ_1^2 (Here $\mathcal{A}_2^* = \mathbb{F}_2[\xi_i]$, so that $\deg \xi_i = 2^i - 1$).

Lemma 3.3. *$H_*(MU; \mathbb{F}_p)$ is naturally a comodule over P.*

Proof. The key point is that $H_*(MU;\mathbb{F}_p)$ is concentrated in even degrees. Consider the comodule structure map

$$\varphi\colon H_*(MU;\mathbb{F}_p) \longrightarrow H_*(MU;\mathbb{F}_p)\otimes \mathcal{A}_p^*,$$

which is degree-preserving. This implies that only even-degree elements of $\mathcal{A}_p^*$ can appear in the image. Choose a homogeneous basis $\{m_i\}$ of $H_*(MU;\mathbb{F}_p)$. For any homogeneous class m, write

$$\varphi(m) = \sum_i m_i \otimes a_i,$$

where each $a_i \in \mathcal{A}_p^*$ is uniquely determined. From the degree considerations above, each a_i must lie in even degree. Now apply the comodule structure map again and use coassociativity. On one hand,

$$(\varphi\otimes 1)\,\varphi(m) = \sum_i \varphi(m_i)\otimes a_i.$$

On the other hand,

$$(1\otimes \phi^*)\,\varphi(m) = \sum_i m_i \otimes \phi^*(a_i).$$

here ϕ^* is the co-multiplication map as discussed in A.8. Comparing these two expressions forces each coproduct $\phi^*(a_i)$ to take the form

$$\phi^*(a_i) = \sum_j b_j' \otimes b_j'',$$

with b_j' and b_j'' still lying in even degrees (from the grading constraints). Consequently, the elements a_i must lie in the sub-Hopf algebra $P \subseteq \mathcal{A}_p^*$ generated by the even-degree primitives. A direct inspection of the coproduct formulas in $\mathcal{A}_p^*$ confirms this.

Thus $H_*(MU;\mathbb{F}_p)$ is a comodule over P. $\square$

Proposition 3.4. *As an $\mathcal{A}_p^*$–comodule algebra, one has*

$$H_*(MU;\mathbb{F}_p) \cong P \otimes \mathbb{F}_p[y_i]_{\,i+1\neq p^k}.$$

Here $\deg y_i = 2i$.

Proof. We handle the case of odd primes p. We first recall the coaction on the homology of $\mathbb{C}P^\infty$. Let $\{\gamma_i\}_{i\geq 0}$ be the standard basis of $H_*(\mathbb{C}P^\infty;\mathbb{F}_p)$, where $|\gamma_i| = 2i$. The $\mathcal{A}_p^*$–coaction

$$H_*(\mathbb{C}P^\infty;\mathbb{F}_p) \longrightarrow H_*(\mathbb{C}P^\infty;\mathbb{F}_p)\otimes \mathcal{A}_p^*$$

is described by

$$\gamma_{p^f} \longmapsto \gamma_1\otimes \zeta_f + 1\otimes \gamma_{p^f} + \cdots,$$

where the dots indicate terms involving γ_i with $1 < i < p^f$. This expresses the fact that the class $x\in H^2(\mathbb{C}P^\infty;\mathbb{F}_p)$ dual to γ_1 satisfies x^{p^f} pairing only with ζ_f under the dual Steenrod operations. Now consider $H_*(MU;\mathbb{F}_p)\cong \mathbb{F}_p[b_i : i\geq 0]$. Using the coaction formulas on $H_*(\mathbb{C}P^\infty;\mathbb{F}_p)$ and applying the standard shifting that relates the b_i to the γ_i, one obtains in particular

$$b_{p^f-1} \longmapsto 1\otimes \zeta_f \;+\; b_{p^f-1}\otimes 1 \;+\cdots.$$

We now define the map

$$\Psi\colon H_*(MU;\mathbb{F}_p) \longrightarrow P\otimes \mathbb{F}_p[y_i]_{\,i+1\neq p^k}.$$

Its construction is as follows:

$$H_*(MU;\mathbb{F}_p) \longrightarrow P\otimes H_*(MU;\mathbb{F}_p) \longrightarrow P\otimes \mathbb{F}_p[y_i]_{\,i+1\neq p^k},$$

where the first arrow is the P–coaction and the second sends $b_i \mapsto y_i$, annihilating those b_i for which $i+1 = p^k$. The map Ψ is simultaneously a map of comodule algebras, and on indecomposable elements it satisfies

$$\Psi(b_i) = \begin{cases} y_i + \cdots, & i+1 \neq p^k, \\ \zeta_k + 1 + \cdots, & i+1 = p^k, \end{cases}$$

where the omitted terms are decomposables. Thus, on indecomposables of the target algebra, the elements y_i and the classes coming from P are all detected by Ψ. A comparison of dimension shows that the map is an isomorphism. □

The construction of the Adams spectral sequence for the complex cobordism spectrum MU requires an explicit computation of

$$\mathbf{Ext}^{s,t}_{\mathcal{A}_p^*}(\mathbb{F}_p,\ H_*(MU;\mathbb{F}_p)),$$

where $\mathcal{A}_p^*$ denotes the dual Steenrod algebra at the prime p, and $H_*(MU;\mathbb{F}_p)$ is regarded as a graded $\mathcal{A}_p^*$-comodule. The comodule structure identified previously shows that $H_*(MU;\mathbb{F}_p)$ is "almost" coinduced; consequently a systematic coalgebraic change–of–rings mechanism is required to compute the associated $\mathbf{Ext}$-groups. For this purpose we use change of rings theorem C.3. That would give us

$$\mathbf{Ext}^{s,t}_{\mathcal{A}_p^*}(\mathbb{F}_p, P \otimes \mathbb{F}_p[y_i : i \neq p^k - 1]) = \mathbf{Ext}^{s,t}_{\mathcal{A}_p^*//P}(\mathbb{F}_p, \mathbb{F}_p) \otimes \mathbb{F}_p[y_i : i \neq p^k - 1]$$

This gives us the entire E_2-page of the ASS. Now we use the following algebraic lemma to give a more concrete description of this E_2-page.

Lemma 3.5. *E be an exterior algebra over $\mathbb{F}_p$ on one primitive generator x in an odd degree n. Then $\mathbf{Ext}^{s,t}_E(\mathbb{F}_p, \mathbb{F}_p)$ is a polynomial algebra on a generator in bi-degree $(1, n)$.*

Using this we can give a complete description of the E_2-page of the ASS as follows

$$E_2^{*,*} = \mathbb{F}_p[y_i, z_j : i \neq p^k - 1]$$

Here $\deg y_i = (0, 2i), \deg z_j = (1, 2p^j - 1)$. In particular we get that the ASS degenerates at E_2-page.

We now analyse the p-adically completed homotopy groups $\widehat{\pi_* MU}$ using the Adams spectral sequence

$$E_2^{s,t} = \mathbf{Ext}^{s,t}_{\mathcal{A}_p^*}(\mathbb{F}_p, H_*(MU;\mathbb{F}_p)) \implies \pi_{t-s}\widehat{(MU)}.$$

As observed previously, the E_2-page contains an element

$$a_0 \in E_2^{1,1},$$

arising from the class z_0 in the computation of the $\mathbf{Ext}$-groups. This class corresponds to an element of Adams filtration one in the associated graded of $\widehat{\pi_0 MU} \cong \mathbb{Z}_p$. Since $\pi_0 MU \cong \mathbb{Z}$, this filtration-one class must detect p, up to multiplication by a unit in $\mathbb{Z}_p$. Next, for each generator y_i and z_j with $j > 0$ on the E_2-page, choose lifts

$$\widetilde{y}_i,\ \widetilde{z}_j \in \pi_* MU$$

representing their corresponding permanent cycles. These elements necessarily satisfy:

$$\deg(\widetilde{y}_i) = 2i \quad \text{for } i+1 \neq p^k, \qquad \deg(\widetilde{z}_j) = 2(p^j - 1) \quad (j > 0).$$

Proposition 3.6. *The p-adic completion $\widehat{\pi_* MU}$ is a polynomial algebra over $\mathbb{Z}_p$ on the classes $\{\widetilde{y}_i\}$ and $\{\widetilde{z}_j\}_{j>0}$:*

$$\widehat{\pi_* MU} \cong \mathbb{Z}_p[\widetilde{y}_i, \widetilde{z}_j \mid j > 0].$$

Proof. Fix k, and choose an element $x \in \widehat{\pi_k MU}$. In the associated graded object with respect to the Adams filtration, the class of x is represented by a polynomial expression in the y_i and z_j; denote such a polynomial by $Q(y_i, z_j)$. Since the class a_0 detects p, there exists an exponent r such that

$$x - a_0^r \, Q(y_i, z_j)$$

belongs to a strictly lower Adams filtration.

Lifting this relation to $\widehat{\pi_* MU}$, we may repeatedly subtract $p^r Q(\widetilde{y}_i, \widetilde{z}_j)$ from the element x, thereby lowering the filtration at each step. Iterating this process yields a sequence converging p-adically, and one eventually obtains that x is equal to a polynomial (with coefficients in $\mathbb{Z}_p$) in the elements $\widetilde{y}_i$ and $\widetilde{z}_j$.

Therefore the canonical morphism

$$\mathbb{Z}_p[\widetilde{y}_i, \widetilde{z}_j] \longrightarrow \widehat{\pi_* MU}$$

is surjective. Injectivity follows because algebraic relations among the generators would already appear in the associated graded, where the $\widetilde{y}_i$ and $\widetilde{z}_j$ correspond to algebraically independent generators. $\square$

From the proposition, for each prime p one may choose a sequence of homogeneous classes $\{x_i\} \subset \pi_* MU$ whose images in $\widehat{\pi_* MU}$ form polynomial generators. In particular,

Corollary 3.7. *For every prime p, the graded ring $\pi_* MU$ contains no p-torsion.*

Proof. Since $\widehat{\pi_* MU}$ is p-torsion free, any p-torsion element of $\pi_* MU$ would map to zero under the natural map $\pi_* MU \longrightarrow \widehat{\pi_* MU}$, and therefore also under the composite

$$\pi_* MU \longrightarrow \pi_* MU \otimes \mathbb{Q}.$$

However, the Hurewicz map

$$\pi_* MU \longrightarrow H_* MU$$

is known to become an isomorphism after tensoring with $\mathbb{Q}$; thus $\pi_* MU \to \pi_* MU \otimes \mathbb{Q}$ is injective. Consequently, no nonzero element of $\pi_* MU$ can be annihilated by p, proving the claim. $\square$

We now determine the integral structure of $\pi_* MU$, assembling the information obtained from all primes. Let $I \subset \pi_* MU$ denote the augmentation ideal, and consider its indecomposable quotient I/I^2. Since the Hurewicz homomorphism

$$\pi_* MU \longrightarrow H_* MU$$

is injective, we obtain induced maps on indecomposables

$$(I/I^2)_{2k} \longrightarrow (J/J^2)_{2k},$$

where $J \subset H_* MU$ is the augmentation ideal of the homology Hopf algebra. It is known that for each k, the graded component

$$(J/J^2)_{2k} \cong \mathbb{Z}.$$

From the p-adic analysis carried out earlier, the p-adic completion of I/I^2 satisfies

$$\left(\widehat{I/I^2}\right)_{2k} \cong \mathbb{Z}_p,$$

generated by the images of the polynomial generators of $\widehat{\pi_* MU}$. Since I/I^2 is a finitely generated $\mathbb{Z}$-module, it follows that $\left(I/I^2\right)_{2k} \cong \mathbb{Z}$, and the map

$$(I/I^2)_{2k} \longrightarrow (J/J^2)_{2k}$$

is injective, as it becomes an isomorphism modulo torsion.

For each k, choose an element $x_k \in \pi_{2k} MU$ whose image generates $(I/I^2)_k$. These elements will serve as homotopical polynomial generators.

Theorem 3.8 (Milnor)**.** *The graded ring $\pi_* MU$ is a polynomial algebra on the classes x_k:*

$$\pi_* MU \ \cong\ \mathbb{Z}[x_1, x_2, x_3, \dots], \qquad \deg(x_k) = 2k.$$

Proof. Consider the graded algebra homomorphism

$$\mathbb{Z}[x_1, x_2, \dots] \longrightarrow \pi_* MU.$$

By construction it is surjective, since the x_k map onto the indecomposable quotient I/I^2. After tensoring with $\mathbb{Q}$, it becomes a morphism between graded polynomial algebras of the same graded rank, hence an isomorphism. Because $\pi_* MU$ is torsion-free, the original map is therefore injective. □

4. The Universal Formal Group Law and MU_*

In section 2 we have seen that any complex oriented cohomology theory E, gives us a Formal Group Law over E_* or E^*. At the end of the section 2 we have indicated the formal group law corresponding to MU is universal. In this section we will explore that universality which is originally due to Quillen [Qui69b]. Before that we will start from the basics.

Definition 4.1 (Formal group Law)**.** Let R be a commutative ring. A (commutative, one-dimensional) *formal group law* over R is a power series

$$F(x, y) \in R[\![x, y]\!]$$

satisfying the identities

(1) (commutativity) $F(x, y) = F(y, x)$,
(2) (associativity) $F(x, F(y, z)) = F(F(x, y), z)$,
(3) (identity) $F(x, 0) = F(0, x) = x$.

standard inductive argument shows that any such F admits a unique inverse series $i(x) \in R[\![x]\!]$ such that $F(x, i(x)) = 0$.

In particular, F has the property that for any R-algebra S, the nilpotent elements of S form an abelian group under the addition law determined by F. To define a formal group law over R is equivalent to choosing coefficients $c_{i,j} \in R$ and forming the power series

$$F(x, y) = \sum_{i,j \geq 0} c_{i,j} x^i y^j,$$

subject to the polynomial relations among the $c_{i,j}$ imposed by the three axioms above. These relations are functorial in R, and they are finitely generated in each total degree. As a result, one obtains a universal algebraic object governing all such choices.

Theorem 4.2. *There exists a commutative ring L and a formal group law*

$$F_{\text{univ}}(x, y) \in L[\![x, y]\!]$$

with the following universal property: for every commutative ring R and every formal group law F over R, there is a unique ring homomorphism

$$L \longrightarrow R$$

carrying F_{univ} to F. In particular, the functor

$$R \longmapsto \mathrm{FGL}(R)$$

from commutative rings to sets is corepresented by L.

L is called the **Lazard ring**. A further structural feature, essential for topological considerations, is that L admits a natural grading. The coefficient $c_{i,j}$ occurring in

$$F_{\text{univ}}(x, y) = \sum_{i,j \geq 0} c_{i,j} x^i y^j$$

is assigned degree $\deg(c_{i,j}) = 2(i+j)-1$. These degrees are compatible with the polynomial relations defining L, and so they induce a well-defined grading on the Lazard ring. Under this grading, the universal map

$$L \longrightarrow \pi_* E$$

associated to a complex orientation is a morphism of graded rings. The motivation for this grading comes from the requirement that the formal group law respect the grading on the variables: if x and y are assigned degree -2, then the sum $F(x, y)$ must also lie in degree -2. This convention reflects the topological situation, where Chern classes live in cohomological degree 2, and the grading above is simply the homological reindexing appropriate for working with power series in x and y. Nevertheless, for $E = MU$ the above map will turn out to be an isomorphism [Qui69b]. Our objective is to determine the algebraic structure of the Lazard ring.

Theorem 4.3 (Lazard). *The Lazard ring L is a polynomial ring*

$$L \simeq \mathbb{Z}[x_1, x_2, \dots]$$

on countably many variables x_i, with $\deg(x_i) = 2i$.

The strategy is to construct an injective graded ring homomorphism

$$\phi\colon L \longrightarrow \mathbb{Z}[b_1, b_2, \dots]$$

into a sufficiently large polynomial ring. This map is the algebraic analogue of the Hurewicz homomorphism $\pi_* MU \longrightarrow H_*(MU)$.. To define ϕ, we place a formal group law on $\mathbb{Z}[b_1, b_2, \dots]$ as follows. Consider the formal power series

$$\exp(x) = \sum_{i=0}^{\infty} b_i x^{i+1}, \qquad b_0 = 1,$$

and define a formal group law by

$$F(x, y) = \exp\big(\exp^{-1}(x) + \exp^{-1}(y)\big),$$

where $\exp^{-1}(x)$ denotes the inverse power series. This generalizes the use of the ordinary exponential and logarithm to convert multiplication into addition. If each b_i is assigned degree $2i$, then ϕ becomes a graded ring map. To prove injectivity and describe the image, let $I \subset L$ and $J \subset \mathbb{Z}[b_1, b_2, \dots]$ denote the augmentation ideals. Their indecomposable quotients are I/I^2 and J/J^2. Our first goal is to understand the graded components $(I/I^2)_{2k}$.

For any abelian group M, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{graded}}(L, \mathbb{Z} \oplus M[2k]) \;\cong\; \mathrm{Hom}_{\mathrm{Ab}}\big((I/I^2)_{2k},\, M\big),$$

so that graded maps $L \longrightarrow \mathbb{Z} \oplus M[2k]$ (where M lies in degree k) correspond exactly to homomorphisms from $(I/I^2)_{2k}$ to M. A graded ring map $L \to \mathbb{Z} \oplus M[k]$ is equivalent to giving a formal group law over $\mathbb{Z} \oplus M[k]$. The grading forces such a law to take the form

$$F(x, y) = x + y + \sum_{i+j=k+1} m_{i,j}\, x^i y^j, \tag{1}$$

with coefficients $m_{i,j} \in M$. Thus, understanding the indecomposables in L amounts to understanding formal group laws over $\mathbb{Z} \oplus M[2k]$ of the above form (1).

We are interested not merely in the indecomposables of L, but in comparing them with those of $\mathbb{Z}[b_1, \dots]$. A graded homomorphism

$$\mathbb{Z}[b_1, \dots] \longrightarrow \mathbb{Z} \oplus M[2k]$$

is determined uniquely by the image of the generator b_k, which is an element of M. Via the resulting change of coordinates, such an element specifies a formal group law over $\mathbb{Z} \oplus M[2k]$. Thus, to understand the effect of the map $L \to \mathbb{Z}[b_1, \dots]$ on indecomposables, we must compare the formal group laws obtained in this way with *all* formal group laws over $\mathbb{Z} \oplus M[2k]$ of the form (1).

Let M be an abelian group, and consider a "polynomial"

$$P(x,y) = \sum_{i+j=2k+1} m_{i,j}\, x^i y^j$$

as in (1). One can ask *when does the expression $f(x,y) = x+y+P(x,y)$ define a formal group law over the graded ring $\mathbb{Z} \oplus M[2k]$* or equivalently, when does f satisfy the commutativity, unitality, and associativity identities for a one–dimensional commutative FGL.

i. Commutativity requires $P(x,y) = P(y,x)$.
ii. Unitality requires $P(x,0) = P(0,x) = 0$.
iii. Associativity requires $f(x, f(y,z)) = f(f(x,y),z)$, which, after substituting f, becomes

$$x+y+z+P(x,y)+P(x+y,z) \;=\; x+y+z+P(x,y+z)+P(y,z).$$

Thus the condition is

$$P(x,y+z)+P(y,z) = P(x,y)+P(x+y,z).$$

Polynomials satisfying this identity are called *symmetric* 2*-cocycles* with coefficients in M.

We obtain specific examples of such cocycles from graded homomorphisms $\mathbb{Z}[b_1,b_2,\dots] \longrightarrow \mathbb{Z} \oplus M[2k]$. Choosing the image $m \in M$ of the generator b_k determines, by change of coordinates, a formal group law over $\mathbb{Z} \oplus M[2k]$. Define

$$\exp(x) = x + mx^{k+1}, \qquad \exp^{-1}(x) = x - mx^{k+1}.$$

Conjugating the additive formal group law by exp yields

$$\exp\big(\exp^{-1}(x) + \exp^{-1}(y)\big) = x + y + m(x+y)^{k+1} - mx^{k+1} - my^{k+1}.$$

Thus the corresponding cocycle is

$$m\big((x+y)^{k+1} - x^{k+1} - y^{k+1}\big).$$

Our goal is to compute the map on indecomposables $(I/I^2)_{2k} \longrightarrow (J/J^2)_{2k}$ induced by $L \to \mathbb{Z}[b_1,b_2,\dots]$. Equivalently, we study the induced map

$$\mathbf{Hom}\big((J/J^2)_{2k}, M\big) \longrightarrow \mathbf{Hom}\big((I/I^2)_{2k}, M\big).$$

The first group is canonically identified with M, while the second is the group of symmetric 2-cocycles homogeneous of degree $k+1$. The map $M \longrightarrow \mathbf{Hom}\big((I/I^2)_{2k}, M\big)$ sends $m \in M$ to the cocycle $m\big((x+y)^{k+1} - x^{k+1} - y^{k+1}\big)$. Thus our task is to describe all symmetric 2-cocycles with coefficients in M.

Proposition 4.4. *Suppose $k+1$ is not a power of any prime p. Then every symmetric 2-cocycle with coefficients in M is uniquely of the form*

$$m\big((x+y)^{k+1} - x^{k+1} - y^{k+1}\big)$$

for a unique $m \in M$.

If $k+1 = p^r$ is a prime power, then every symmetric 2-cocycle is uniquely of the form

$$m \cdot \frac{1}{p}\big((x+y)^{k+1} - x^{k+1} - y^{k+1}\big)$$

for a unique $m \in M$.

Proof. We omit the proof. One cal look at [Hop99, Chapter 3] for proof. □

In particular, $\mathbf{Hom}\big((I/I^2)_{2k}, M\big) \;\simeq\; M$, and hence $(I/I^2)_{2k} \simeq \mathbb{Z}$. Moreover, the map $(I/I^2)_{2k} \longrightarrow (J/J^2)_{2k}$ is an isomorphism when $k+1$ is not a prime power, and has index p when $k+1 = p^r$.

The final step needed for the proof of Lazard's theorem is to justify that the map

$$L \longrightarrow \mathbb{Z}[b_1, b_2, \dots],$$

which classifies the formal group law obtained from the additive one by a change of coordinates using the exponential series

$$\exp(x) = \sum_{i \geq 0} b_i \, x^{i+1},$$

is an isomorphism modulo torsion. Equivalently,

$$L \otimes \mathbb{Q} \;\simeq\; \mathbb{Z}[b_1, b_2, \dots] \otimes \mathbb{Q}.$$

Although this step may be bypassed by proving the 2-cocycle lemma over arbitrary coefficient rings, the result in fact becomes elementary over $\mathbb{Q}$. Indeed, over a $\mathbb{Q}$-algebra, specifying a formal group law is equivalent to specifying an exponential series of the form $\exp(x) = x + b_1 x^2 + \cdots$.

Definition 4.5. Let $f(x, y)$ be a formal group law over a commutative ring. An *exponential* for f is a power series

$$\exp(x) = x + b_1 x^2 + b_2 x^3 + \cdots$$

such that

$$f(x, y) = \exp\bigl(\exp^{-1}(x) + \exp^{-1}(y)\bigr).$$

The inverse series $\exp^{-1}(x)$ is called the *logarithm* of f.

Thus a logarithm is precisely an isomorphism between f and the additive formal group law.

Proposition 4.6. *A formal group law over a $\mathbb{Q}$-algebra admits a unique logarithm; equivalently, it is uniquely isomorphic to the additive formal group law.*

This statement fails in characteristic p, where formal group laws need not be isomorphic and the additive formal group law admits nontrivial automorphisms.

Proof. Let $f(x, y)$ be a formal group law. We seek a formal power series

$$\log(x) = x + m_1 x^2 + m_2 x^3 + \cdots$$

such that

$$\log\bigl(f(x, y)\bigr) = \log(x) + \log(y).$$

Differentiating with respect to x yields the differential equation

$$\log'\bigl(f(x, y)\bigr) \, f_1(x, y) = \log'(x),$$

where f_1 denotes the partial derivative with respect to the first variable. Setting $x = 0$ gives

$$\log'(y) \, f_1(0, y) = 1,$$

since $f(0, y) = y$. Thus we must define

$$\log(y) = \int_0^y \frac{dt}{f_1(0, t)},$$

interpreted as a formal integral; this requires division by integers and hence the assumption of a $\mathbb{Q}$-algebra.

It remains to show that $\log(f(x, y)) = \log(x) + \log(y)$. Both sides are symmetric in x and y and have zero constant term, so it suffices to differentiate with respect to y. The derivatives are

$$\log'\bigl(f(x, y)\bigr) \, f_2(x, y) = \frac{1}{f_1(0, f(x, y))} \, f_2(x, y) \quad \text{and} \quad \frac{1}{f_1(0, x)}.$$

Thus we must verify the identity

$$f_1(0, f(x, y)) \;=\; f_1(x, y) \, f_1(0, x).$$

This is precisely the infinitesimal form of associativity: differentiate the associativity relation

$$f(z, f(x, y)) = f(f(z, x), y)$$

with respect to z at $z = 0$.

Finally, suppose two logarithms existed. Their difference would give an automorphism $g(x)$ of the additive formal group law in characteristic 0, i.e. a series satisfying

$$g(x + y) = g(x) + g(y).$$

Then $g(2x) = 2g(x)$, and iterating forces $g(x) = x$. Thus the logarithm is unique. □

Our objective is to return to topology and, in particular, to analyze the formal group law associated with the universal complex-oriented theory MU (complex cobordism). From the computation via the Adams spectral sequence, we know that

$$\pi_*\mathrm{MU} \;\simeq\; \mathbb{Z}[x_1, x_2, \dots], \qquad \deg(x_i) = 2i.$$

By the previous results, this ring is isomorphic to the Lazard ring L. However, it is not immediately evident that the map

$$L \longrightarrow \pi_*\mathrm{MU},$$

which classifies the formal group law arising from the canonical complex orientation of MU, is an isomorphism. Quillen's theorem asserts precisely this: the formal group law of MU is the universal one. We now begin laying the groundwork for its proof.

There is a Hurewicz homomorphism

$$\pi_*\mathrm{MU} \longrightarrow H_*(\mathrm{MU}) \;=\; \mathbb{Z}[b_1, b_2, \dots].$$

Consequently, the composite

$$L \longrightarrow \pi_*\mathrm{MU} \longrightarrow \mathbb{Z}[b_1, \dots]$$

classifies a formal group law over $\mathbb{Z}[b_1, \dots]$. This is exactly the same map considered earlier. Understanding the resulting formal group law therefore provides significant information about the formal group law on $\pi_*\mathrm{MU}$, since the rings $\pi_*\mathrm{MU}$ and $H_*(\mathrm{MU})$ are closely related.

Lemma 4.7. *The formal group law on* $H_*(\mathrm{MU}; \mathbb{Z})$ *classified by the composite*

$$L \to \pi_*\mathrm{MU} \to H_*(\mathrm{MU})$$

is given by

$$F(x, y) \;=\; \exp\!\big(\exp^{-1}(x) + \exp^{-1}(y)\big), \qquad \exp(t) = \sum_{i \geq 0} b_i\, t^{\,i+1}.$$

We prove this in a more general setting. Let E be a complex-oriented ring spectrum. Then the smash product $E \wedge \mathrm{MU}$ admits two natural complex orientations:

i. one obtained via the morphism $E \to E \wedge \mathrm{MU}$ from the orientation of E;
ii. one obtained via the morphism $\mathrm{MU} \to E \wedge \mathrm{MU}$ from the orientation of MU.

Let the corresponding formal group laws over $\pi_*E[b_1, b_2, \dots] = \pi_*(E \wedge \mathrm{MU})$ be denoted F_1 and F_2, respectively. The law F_1 is simply the formal group law associated with E, while F_2 is, a priori, different.

Lemma 4.8. *The formal group law* F_2 *is obtained from* F_1 *by the change of coordinates*

$$t \longmapsto \exp(t) \;:=\; \sum_{i \geq 0} b_i\, t^{\,i+1}.$$

Hence smashing with MU *has the effect of applying a universal change of coordinates to formal group laws.*

Let E be complex oriented. For any choice of coordinate $t \in (E \wedge \mathrm{MU})^2(\mathbb{CP}^\infty)$,

$$(E \wedge \mathrm{MU})^*(\mathbb{CP}^\infty) \;\cong\; \pi_* E[b_1, b_2, \dots][\![t]\!]$$

There are two natural choices of such a coordinate:

i. t_E, induced from the complex orientation of E;
ii. t_{MU}, induced from the complex orientation of MU.

The formal group laws F_1 and F_2 arise from the homomorphism

$$(E \wedge \mathrm{MU})^*(\mathbb{CP}^\infty) \longrightarrow (E \wedge \mathrm{MU})^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty),$$

expressed in terms of t_E and t_{MU}, respectively. Thus, to compare F_1 and F_2, it suffices to express t_{MU} in terms of t_E. We aim to establish the identity

$$t_{\mathrm{MU}} \;=\; \sum_{i \geq 0} b_i \, t_E^{i+1}, \qquad (b_0 = 1).$$

To determine the coefficients, we pair both sides with the standard basis $\{\beta_i\}_{i \geq 0} \subset (E \wedge \mathrm{MU})_*(\mathbb{CP}^\infty)$, defined so that β_i is the image of the element of $E_*(\mathbb{CP}^\infty)$ that is dual to t_E^i. Thus,

$$(t_E^i, \beta_j) = \delta_{ij}.$$

Pairing t_{MU} with β_i amounts to evaluating β_i under the map $\mathbb{CP}^\infty \to \mathrm{MU}$ in E-homology. By definition of the classes b_j, we obtain

$$(t_{\mathrm{MU}}, \beta_i) \;=\; b_{i-1}.$$

Since the β_i form the basis dual to the t_E^i, it follows that

$$t_{\mathrm{MU}} \;=\; \sum_{i \geq 1} b_{i-1} \, t_E^i \;=\; \sum_{i \geq 0} b_i \, t_E^{i+1}.$$

This proves that t_{MU} is obtained from t_E by the coordinate change $t \mapsto \exp(t)$, and therefore that F_2 is the corresponding transform of F_1.

Observe that the map

$$\pi_* MU \longrightarrow H_*(MU)$$

is injective, and its image contains the image of the Lazard ring L. Thus, to conclude that $L \to \pi_* MU$ is an isomorphism, it remains only to show that the image of $\pi_* MU \to H_*(MU)$ is contained in the image of L. It suffices to verify this on indecomposables.

In particular, we are reduced to the following statement: if $n + 1 = p^f$ for a prime p, then the induced map

$$\mathbb{Z} \;\simeq\; Q_{2n}(\pi_* MU) \;\longrightarrow\; Q_{2n}(H_*(MU)) \;\simeq\; \mathbb{Z}$$

has image contained in $p\mathbb{Z}$.

Proof. Equivalently, we must show that the mod–p Hurewicz homomorphism

$$Q_n(\pi_* MU) \longrightarrow Q_n(H_*(MU; \mathbb{Z}/p))$$

is zero. This follows directly from the Adams spectral sequence for MU computed earlier. Recall that in the ASS converging to $\pi_* MU$, the indecomposables in degree $p^f - 1$ appear in Adams filtration at least one. But any class of Adams filtration ≥ 1 is annihilated by the Hurewicz map; this is an immediate consequence of the construction of the spectral sequence using Adams resolutions. □

Thus we have proved:

Theorem 4.9 (Quillen). *The formal group law of MU is the universal formal group law.*

Remark 4.10. It is noteworthy that the proof of Quillen's theorem is predominantly computational, in contrast with the conceptual nature of the result itself—namely, that the universal complex-oriented cohomology theory corresponds to the universal formal group law. To the author's knowledge, no substantially non-computational proof is available. In particular, a construction of MU directly from the universal formal group law would be of considerable conceptual interest.

Appendix A. The Dual Steenrod Algebra $\mathcal{A}_p^*$

In this section we will compute the dual of Steenrod Algebra $\mathcal{A}_p$, as an algebra over $\mathbb{F}_p$. For most of the part we follow the outline of [Mil58]. Firstly let us recall Steenrod-operations on $H^*(X;\mathbb{F}_p)$ for a topological space X,

Theorem A.1. *We have natural homomorphisms $P^i : H^*(X;\mathbb{F}_p) \to H^*(X;\mathbb{F}_p)$ of degree $2i(p-1)$ for odd p and of degree i for $p = 2$ satisfying the following properties:*

(1) $P^0 = Id$ and $P^i(1) = 0$ for $i > 0$. Here, $1 \in H_0(X;\mathbb{F}_p)$ is the identity element.
(2) $P^i(x) = 0$ if $2i > \deg x$ (for p odd), $i > \deg x$ (for $p = 2$).
(3) $P^i(x) = x^p$ for $2i = \deg x$ (odd p), $i = \deg x$ (even p)
*(4) **Stability.** P^i commutes with the connecting morphism in cohomology. In other words it commutes with the suspension isomorphism $H^{*+1}(\Sigma X;\mathbb{F}_p) \to H^*(X'\mathbb{F}_p)$.*
*(5) **Cartan formula.** If $x \in H^q(X)$ and $y \in H^r(X)$ then*

$$P^s(xy) = \sum_{i+j=s} P^i(x)P^j(y)$$

*(6) **Adem relation.***

$$\text{If } p = 2 \text{ and } a < 2b, \quad P^aP^b = \sum_i (2i-a,\, a-b-i-1)\, P^{a+b-i}P^i,$$

$$\text{If } p > 2 \text{ and } a < pb, \quad P^aP^b = \sum_i (-1)^{a+i}(pi-a,\, a-(p-1)b-i-1)\, P^{a+b-i}P^i,$$

$$P^a\beta P^b = \sum_i (-1)^{a+i}(pi-a,\, a-(p-1)b-i)\, \beta P^{a+b-i}P^i - \sum_i (-1)^{a+i}(pi-a-1,\, a-(p-1)b-i)\, P^{a+b-i}\beta P^i.$$

Here $\beta P = \beta \circ P$ and β is the Bockstein corresponding to the short exact sequence

$$0 \to \mathbb{F}_p \to \mathbb{Z}/(p^2) \to \mathbb{F}_p \to 0$$

One can look at a construction of Steenrod operations in a general algebraic setting [Mon25, May70]. Using the Adem relations, we can multiply two Steenrod operations and thereby form an associative $\mathbb{F}_p$-algebra generated by these Steenrod operations and the Bocksteins (upto Adem relation), $\mathcal{A}_p$, which is known as the *Steenrod algebra*. This algebra carries a natural graded structure. From the point of view of stable homotopy theory, we know that

$$[H\mathbb{F}_p, H\mathbb{F}_p]_{\mathrm{Ho}(\mathbf{Sp})} \simeq \mathcal{A}_p,$$

which contains all stable cohomology operations. Hence, this algebra is of fundamental importance. However, the algebraic structure of $\mathcal{A}_p$ is quite intricate, mainly because it admits a basis over $\mathbb{F}_p$ that is rather complicated in nature (see, for instance, [Mon25]). The advantage of considering the dual algebra $\mathcal{A}_p^*$ is that, as an $\mathbb{F}_p$-algebra, it has a much simpler description. By the end of this section, we shall see this explicitly.

To begin this section we recall the Hopf algebra structure of $\mathcal{A}_p$. There is a obvious product:

$$\phi : \mathcal{A}_p \otimes \mathcal{A}_p \to \mathcal{A}_p$$

and the coproduct is given by the following lemma: (for proof one can look at [May70] or [Mil58, Lemma 3.1])

Lemma A.2. *For each element $\Theta \in \mathcal{A}_p$ there is unique element $\psi(\Theta) = \sum \Theta' \otimes \Theta'' \in \mathcal{A}_p \otimes \mathcal{A}_p$ such that the identity*

$$\Theta(a \smile b) = \sum (-1)^{\deg \Theta'' \deg a} \Theta'(a) \smile \Theta''(b)$$

is satisfied for all $a, b \in H^*(X; \mathbb{F}_p)$ for a topological space X. Further more the assignment $\Theta \mapsto \psi(\Theta)$ is a ring homomorphism.

The map $\psi : \mathcal{A}_p \to \mathcal{A}_p \otimes \mathcal{A}_p$ relates the cup-product structure in $H^*(X; \mathbb{F}_p)$ with the action of $\mathcal{A}_p$ on this cohomology ring. From the Cartan formula one obtain that for the power operation P^i,

$$\psi(P^i) = P^i \otimes 1 + P^{i-1} \otimes P^1 \cdots + 1 \otimes P^i$$

and for Bockstein β,

$$\psi(\beta) = 1 \otimes \beta + \beta \otimes 1$$

The homomorphisms ϕ, ψ gives $\mathcal{A}_p$ a *Hopf Algebra structure*. Only by looking at the effect of ψ, ϕ on the generators P^i, β one can conclude the following theorem:

Theorem A.3. *The homomorphisms*

$$\mathcal{A}_p \xrightarrow{\psi} \mathcal{A}_p \otimes \mathcal{A}_p \xrightarrow{\phi} \mathcal{A}_p$$

give $\mathcal{A}_p$ a Hopf algebra structure where ϕ is associative and ψ is both associative and commutative.

Dualizing the above theorem A.3, we get

Corollary A.4. *There is a dual Hopf algebra*

$$\mathcal{A}_p^* \xrightarrow{\phi^*} \mathcal{A}_p^* \otimes \mathcal{A}_p^* \xrightarrow{\psi^*} \mathcal{A}_p^*$$

with associative, commutative product operation.

Let X be a finite CW-complex. The Steenrod algebra $\mathcal{A}_p$ acts on cohomology

$$\mathcal{A}_p \otimes H^*(X; \mathbb{F}_p) \longrightarrow H^*(X; \mathbb{F}_p), \qquad \Theta \otimes a \mapsto \Theta(a),$$

it induces an action of $\mathcal{A}_p$ on $H_*(X; \mathbb{F}_p)$ by duality. Using the Kronecker pairing

$$\langle \ , \ \rangle : H^*(X; \mathbb{F}_p) \otimes H_*(X; \mathbb{F}_p) \longrightarrow \mathbb{F}_p,$$

the action is defined by the formula

$$\langle \Theta \cdot u, \ a \rangle \ = \ \langle u, \ \Theta(a) \rangle, \qquad u \in H_*(X; \mathbb{F}_p), \ a \in H^*(X; \mathbb{F}_p), \ \Theta \in \mathcal{A}_p. \tag{1}$$

In particular, this homology action corresponds to a unique homomorphism

$$\lambda_* : H_*(X; \mathbb{F}_p) \otimes \mathcal{A}_p^* \longrightarrow H_*(X; \mathbb{F}_p) \otimes \mathcal{A}_p^*$$

dual to the coaction $\lambda : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes \mathcal{A}_p^*$, in the sense that

$$\langle a, \ \lambda_*(u) \rangle = \langle \lambda(a), \ u \rangle.$$

Lemma A.5. *The coaction λ satisfies the identity*

$$(\lambda \otimes \mathrm{id}) \circ \lambda \ = \ (\mathrm{id} \otimes \phi^*) \circ \lambda,$$

where $\phi^* : \mathcal{A}_p^* \to \mathcal{A}_p^* \otimes \mathcal{A}_p^*$ is the coproduct dual to the product in $\mathcal{A}_p$.

Proof. For $a, b \in H^*(X; \mathbb{F}_p)$ and $\Theta \in \mathcal{A}_p$ we have, by Lemma A.2,

$$\Theta(ab) = \sum (-1)^{|\Theta''||a|}\, \Theta'(a)\, \Theta''(b), \qquad \psi(\Theta) = \sum \Theta' \otimes \Theta''.$$

Dualizing and using the defining property of λ, one obtains the commutative diagram

$$\begin{tikzcd} H^*(X) \otimes H^*(X) \arrow[r, "\lambda\otimes\lambda"] \arrow[d, "\smile"'] & H^*(X) \otimes \mathcal{A}_p^* \otimes H^*(X) \otimes \mathcal{A}_p^* \arrow[d, "\smile\otimes\phi^*"] \\ H^*(X) \arrow[r, "\lambda"] & H^*(X) \otimes \mathcal{A}_p^*, \end{tikzcd}$$

which expresses the desired identity. □

As an immediate consequence, the coaction is multiplicative.

Lemma A.6. *For all $a, b \in H^*(X; \mathbb{F}_p)$,*

$$\lambda(ab) = \lambda(a)\, \lambda(b).$$

Proof. Consider the commutative diagram of Lemma A.5 (applied to $X \times X$ and the cup product). Now apply this diagram to the diagonal $\Delta : X \to X \times X$. Writing $\Delta^*(a \otimes b) = ab$ and using naturality of λ with respect to Δ, the composition along the top then right hand side sends $a \otimes b$ to $\lambda(a)\lambda(b)$, while the left then bottom composition sends $a \otimes b$ to $\lambda(ab)$. Commutativity therefore gives

$$\lambda(ab) = \lambda(a)\lambda(b),$$

as required. □

Let $X = L^{2N+1}(p)$ denote the $(2N+1)$–skeleton of $K(\mathbb{Z}/p, 1)$. Fix classes

$$x \in H^1(X; \mathbb{F}_p), \qquad y \in H^2(X; \mathbb{F}_p),$$

that are the generators and are related by the Bockstein $y = \beta(x)$, where $\beta : H^*(X; \mathbb{F}_p) \to H^{*+1}(X; \mathbb{F}_p)$ denotes the Bockstein operator. The cohomology of X is generated (up to the stated truncation) by x and powers of y together with the mixed classes $x\, y^i$.

The coaction $\lambda : H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p) \otimes \mathcal{A}_p^*$ therefore has expansions of the following form:

$$\lambda(x) = x \otimes 1 \;+\; y \otimes \tau_0 \;+\; y^p \otimes \tau_1 \;+\; y^{p^2} \otimes \tau_2 \;+\; \cdots, \tag{2}$$

$$\lambda(y) = y \otimes \xi_1 \;+\; y^p \otimes \xi_2 \;+\; y^{p^2} \otimes \xi_3 \;+\; \cdots, \tag{3}$$

with uniquely determined coefficients $\tau_k, \xi_k \in \mathcal{A}_p^*$.

Define elements $\tau_k, \xi_k \in \mathcal{A}_p^*$ by the duality relations

$$\langle \tau_k,\ \beta P^{p^k} \rangle = 1, \qquad \langle \xi_k,\ P^{p^k} \rangle = 1,$$

and by requiring these functionals to vanish on all other admissible monomials of $\mathcal{A}_p$. Here β denotes the Bockstein and P^{p^k} denotes the reduced p-th power operation of appropriate index. The coefficients in (2) and (3) ensure that τ_k and ξ_k occur uniquely as the indicated coefficients of the powers y^{p^k}.

The algebra and coalgebra structures on $\mathcal{A}_p^*$ are then determined by the multiplicativity of λ and by the identity $(\lambda \otimes \mathrm{id}) \circ \lambda = (\mathrm{id} \otimes \phi^*) \circ \lambda$, from which the usual polynomial/exterior decomposition and the Milnor coproduct formulas follow.

The multiplicativity of the coaction λ implies that all monomials in the elements ξ_i and τ_j occur in the coaction of products of x and y. Linear independence follows from the duality with a known basis of $\mathcal{A}_p$. This yields the algebra structure of $\mathcal{A}_p^*$.

Theorem A.7 (Algebra structure of $\mathcal{A}_p^*$). *As a graded $\mathbb{F}_p$–algebra,*

$$\mathcal{A}_p^* \;\cong\; \mathbb{F}_p[\xi_1, \xi_2, \dots] \;\otimes\; E(\tau_0, \tau_1, \dots),$$

where ξ_i are polynomial generators and τ_j are exterior generators.

Proof. By Lemma A.6, the coaction λ is multiplicative:

$$\lambda(x^a y^b) = \lambda(x)^a \, \lambda(y)^b,$$

for all non-negative integers a, b (within the truncated skeleton). Expanding $\lambda(x)$ and $\lambda(y)$ using (2)-(3) shows that all monomials in the elements ξ_i and τ_j appear as coefficients of linearly independent classes $x^a y^b$ in $H^*(X; \mathbb{F}_p)$. Thus, these monomials span $\mathcal{A}_p^*$ as an $\mathbb{F}_p$-vector space. Linear independence follows from the nondegenerate pairing

$$\langle \cdot \,, \cdot \rangle : \mathcal{A}_p^* \otimes \mathcal{A}_p \longrightarrow \mathbb{F}_p,$$

together with the fact that the corresponding admissible monomials in $\mathcal{A}_p$ form a basis. Therefore, the monomials in ξ_i and τ_j give an $\mathbb{F}_p$-basis of $\mathcal{A}_p^*$, and the algebra structure is exactly

$$\mathcal{A}_p^* \;\cong\; \mathbb{F}_p[\xi_1, \xi_2, \dots] \;\otimes\; E(\tau_0, \tau_1, \dots).$$

□

Finally, the coproduct on $\mathcal{A}_p^*$, dual to the multiplication in $\mathcal{A}_p$, is determined by the compatibility of the coaction:

$$(\lambda \otimes \mathrm{id}) \circ \lambda = (\mathrm{id} \otimes \phi^*) \circ \lambda.$$

Theorem A.8 (Coproduct formulas). *For each $k \geq 0$,*

$$\phi^*(\xi_k) = \sum_{i+j=k} \xi_i^{p^j} \otimes \xi_j, \qquad \phi^*(\tau_k) = \tau_k \otimes 1 + \sum_{i+j=k} \xi_i^{p^j} \otimes \tau_j.$$

Proof. Apply the identity $(\lambda \otimes \mathrm{id}) \circ \lambda = (\mathrm{id} \otimes \phi^*) \circ \lambda$ to the classes x and y and expand using the coaction formulas (2)–(3). Comparing coefficients of the linearly independent classes y^{p^k} in $H^*(X) \otimes \mathcal{A}_p^* \otimes \mathcal{A}_p^*$ gives the above formulas. □

Remark A.9. With the choice $x \in H^1$, $y = \beta(x) \in H^2$, the degrees of the dual Steenrod algebra generators are

$$|\tau_k| = 2p^k - 1, \qquad |\xi_k| = 2(p^k - 1),$$

compatible with the cohomological grading.

Appendix B. The Adams Spectral Sequence

The Adams spectral sequence is one of the central computational tools in stable homotopy theory. It allows one to approximate the stable homotopy groups of spectra by purely algebraic data derived from the Steenrod algebra (or dual steenrod algebra). In this section we give a brief introduction to its construction and state the main structural properties following [Ada74, Rav86] and [Mil58].

Definition B.1. A H_*- *Adams resolution* of X is a sequence of spectra

$$X = X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots$$

such that for each s the cofiber $K_s := \mathrm{cofib}(X_{s+1} \to X_s)$ is a retract of $H\mathbb{F}_p \wedge K_s$ and the induced map on mod-p homology $H_*(X_{s+1}; \mathbb{F}_p) \to H_*(X_s; \mathbb{F}_p)$ is zero.

Proposition B.2. *For any spectra X, H_* Adams resolution exists.*

Proof. Fix a prime p. Let $\eta : S^0 \to H\mathbb{F}_p$ denote the unit map of ring spectra; smashing with a spectrum X yields a natural map $f_0 : X \to H\mathbb{F}_p \wedge X$. Note that this map is injective in homology. Set, X_0 to be X and $K_0 = X \wedge H\mathbb{F}_p$, define $X_1 = \mathrm{hofib}(f_0)$. And by induction we have the following H_*-Adams resolution

$$\begin{tikzcd} X = X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_0"'] \arrow[d, "f_1"'] & X_2 \arrow[l, "g_1"'] \arrow[d, "f_2"] & \cdots \arrow[l] \\ K_0 & K_1 & K_2 & \end{tikzcd}$$

□

From the tower $\{X_s\}$ one obtains an exact couple of homotopy groups which produces a spectral sequence. The associated E_1-page is built from the groups $\pi_* K_s$, and the d_1-differential is the map induced by the connecting morphisms in the tower. This spectral sequence is obtained from an exact couple

$$\begin{tikzcd} \bigoplus \pi_*(X_i) \arrow[rr, "g"] & & \bigoplus \pi_*(X_i) \arrow[dl, "f"] \\ & \bigoplus \pi_*(K_i) \arrow[ul, "h"] & \end{tikzcd}$$

Where the map f and g comes from $\{f_i\}, \{g_i\}$ respectively and h comes from the maps $K_s \to \Sigma X_{s+1}$. Under some special hypotheses, the spectral sequence converges (under standard mild convergence hypotheses) to the p-adic completion of $\pi_* X$. The hypotheses in the definition of H_*-Adams resolution makes sure that the inverse limit

$$\varprojlim X_s$$

has trivial p-adic homology and consequently we can think of X as exhaustive when we are looking at the p-completion, which would give us convergence of the spectral sequence. This is described in [Rav86, Chapter 2].

Now we want to examine the E_2-page of this spectral sequence. The E_2 page is the homology of the differential fh on $\bigoplus \pi_* K_i$. Another way to say this is that the K_i form a complex in the homotopy category of spectra: that is, there are maps

$$X \to K_0 \to \Sigma^{-1} K_1 \to \Sigma^{-2} K_2 \to \cdots,$$

any two of which are nullhomotopic. These come from chasing around the cofiber sequences. Because we are working with an Adams resolution, we find that the sequence

$$0 \longrightarrow H_*(X; \mathbb{Z}/p) \longrightarrow H_*(K_0; \mathbb{Z}/p) \longrightarrow \cdots$$

is exact and is, in particular, a resolution of $H_*(X; \mathbb{Z}/p)$ by cofree $\mathcal{A}_p^*$-comodules.

Now

$$\pi_* K_i \;=\; \mathrm{Hom}_{\mathcal{A}_p^*}(\mathbb{Z}/p,\ H_*(K_i; \mathbb{Z}/p))$$

because maps of $\mathcal{A}_p^*$-comodules $\mathbb{Z}/p \to H_*(K_i; \mathbb{Z}/p)$ are precisely the primitive elements in $H_*(K_i; \mathbb{Z}/p)$. Consequently, if we want to take the homology of the bigraded group $\pi_* K_i$, this is the same as the homology of the bigraded group

$$\mathrm{Hom}_{\mathcal{A}_p^*}(\mathbb{Z}/p,\ H_*(K_i; \mathbb{Z}/p)).$$

But we have just seen that $H_*(K_i; \mathbb{Z}/p)$ is a resolution of $H_*(X; \mathbb{Z}/p)$ by cofree $\mathcal{A}_p^*$-comodules. The conclusion is that the homology of this bigraded group, or the E_2 term, is

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_p^*}(\mathbb{Z}/p,\ H_*(X; \mathbb{Z}/p)).$$

Theorem B.3. *(Adams Spectral Sequence) Let, X be a connective spectra whose homotopy groups are finitely generated. Then there is a spectral sequence whose E_2-page is*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p^*}(\mathbb{Z}/p,\ H_*(X; \mathbb{Z}/p)).$$

This spectral sequence converges to the p-adic completion $\widehat{\pi_{t-s}(X)}$, in the sense that there is a filtration on $\widehat{\pi_{t-s}(X)}$ whose successive quotient are the E_∞-terms.

In the above discussion , we have constructed this spectral sequence. The convergence is highly technical, we omit the proof. One can find it in [Rav86].

Appendix C. Change of rings

Let A be a coalgebra over a field k. A morphism of coalgebras $A \to B$ induces a corestriction functor

$$\mathrm{Res} : \mathbf{Comod}(A) \longrightarrow \mathbf{Comod}(B),$$

obtained by composing comodule structure maps with $A \to B$. In contrast to the algebra case, the restriction functor for coalgebras admits a *right* adjoint, formulated using the cotensor product.

Definition C.1. For a right A-comodule M and a left A-comodule N, the *cotensor product* is the equalizer

$$M \,\square_A\, N = \ker\big(M \otimes N \rightrightarrows M \otimes A \otimes N\big),$$

where the parallel maps are induced by the respective comodule structures.

If M carries both left and right A-comodule structures which commute, then $M \square_A N$ inherits a left comodule structure. In particular, A itself, with its canonical bicomodule structure, satisfies $A \square_A N \cong N$.

Suppose A is a connected graded Hopf algebra and $B \subset A$ a connected graded sub–Hopf algebra. Let $B^+ \subset B$ denote the augmentation ideal. If B is *normal*, i.e. $AB^+ = B^+A$, the quotient

$$A//B \; = \; A/AB^+$$

acquires a natural Hopf algebra structure.

A fundamental structural fact identifies the subalgebra B as a cotensor product over this quotient Hopf algebra:

Lemma C.2. *For a normal sub–Hopf algebra $B \subset A$,*

$$B \;\cong\; A \,\square_{A//B}\, k,$$

where k is regarded as the trivial $A//B$-comodule.

The adjunction between restriction and cotensor yields the following change–of–rings isomorphism.

Theorem C.3 (Change of Rings)**.** *Let M be an A-comodule and let V be a vector space endowed with the trivial $A//B$-comodule structure. Then for all $i \geq 0$,*

$$\mathbf{Ext}^i_A(M,\; B \otimes V) \;\cong\; \mathbf{Ext}^i_{A//B}(M,\; V).$$

This reduces **Ext**-computations over the larger Hopf algebra A to computations over the simpler quotient $A//B$. In the context of the Adams spectral sequence for MU, this allows the computation of $\mathbf{Ext}_{\mathcal{A}_p^*}(-,-)$ to be simplified by passing to a quotient of the dual Steenrod algebra where the resulting Hopf algebra has a tractable algebraic structure.

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