Overview Talk

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We will begin with 'cohomology of projective varieties' and we will see for smooth projective varieties many beautiful properties holds for de-Rahm cohomology, singular cohomology which don't get satiesfied for the case of 'singular projective varieties'. We will discuss those with examples in this talk.

Let $X \subseteq \mathbb{C}P^N$ be a projective variety of dimension n (in the sense of Krull dimension which will be same with the manifold dimension for the smooth case). X is given by zeroes of some homogeneous polynomial thus it a closed subspace of $\mathbb{C}P^N$ and hence it is compact. For the smooth case X is a 'smooth manifold'(complex manifold). Some properties of smooth X are described below,

- X is given by zeroes of $g_1, \cdots, g_{N-n}$ with the rank of the matrix $\left(\frac{\partial g_j}{\partial z_i}\right)_{ij}$ equal to $N-n$.
- Hermitian metric on Tangent space of X.
- X is an orientable manifold of dimension $2n$ admitting a Riemannian metric g and a 'complex structure' on it's Tangent space.
- There is also an alternating form ω (or Kähler differential).

1. Dualities

We can compute the singular(simplicial) homology(cohomology) for X with the coefficients in $\mathbb{R}$. Since X is compact orientable manifold we can talk about the cup product pairing as follows:

$$H^i(X;\mathbb{R}) \times H^{2n-i}(X;\mathbb{R}) \xrightarrow{\smile} H^{2n}(X;\mathbb{R}) \cong \mathbb{R}$$

is a 'non-degenerate' pairing. Thus we have **Poincare Duality**,

$$H^{2n-i}_{\text{Sing}}(X;\mathbb{R}) \cong H^i_{\text{Sing}}(X;\mathbb{R})^* \cong H_i^{\text{sing}}(X;\mathbb{R})$$

Since X is a smooth manifold we can talk about de-Rahm cohomology. In a shofesticated language 'de-Rahm cohomology is a cohomology of soft-resolution of constant sheaf'. In this case also we have the following as non-degenerate,

$$H^i_{DR}(X;\mathbb{R}) \times H^{2n-i}_{DR}(X;\mathbb{R}) \xrightarrow{\wedge} H^{2n}_{DR}(X;\mathbb{R}) \xrightarrow[\int-\text{vol-form}]{\sim} \mathbb{R}$$

Thus again we have the dualiy, $H^{2n-i}_{DR}(X;\mathbb{R}) \cong H^i_{DR}(X;\mathbb{R})^*$. Connecting de-Rahm cohomology and singular(simplicial) cohomology with coefficients in $\mathbb{R}$, there is a beautiful theorem by de-Rahm stated as follows,

> **Theorem 1.1 (De-Rahm's Theorem)** There is an isomorphism between the singular(simplicial) cohomology with coefficients in $\mathbb{R}$ and de-Rahm cohomology which is compatible with the product structure on both the V.S.