BRAID GROUP ACTION ON $D^b(\mathfrak{M}_\eta)$

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Abstract. We construct an action of the braid group on the bounded derived category of coherent sheaves on hypertoric varieties arising from hyperplane arrangements. Using wall-crossing equivalences associated to paths in the complexified complement of the hyperplane arrangement, we show that these equivalences yield a well-defined functor from the Deligne groupoid to the category of triangulated equivalences. This gives rise to a canonical representation of the fundamental group, which under suitable assumptions recovers the braid group, acting on $D^b(\mathfrak{M}_\eta)$.

1. Introduction

In this paper I shall describe the work I have done during my stay at ANU(Australian National University) as a FRT scholar. The category of coherent sheafs over a variety (or scheme) is a central topic in studying algebraic geometry. It has been observed that this kind of triangulated category arises in representation theory. In symplectic topology one can define Hyperkähler quotients. From a given hyperplane arrangement of k hyperplanes in $\mathbb{R}^n$ one can define an action of $(\mathbb{C}^*)^n$ on $T^*\mathbb{C}^k$, the hyperKähler quotient of this type of action gives us hyper-toric varieties. As the name suggest the quotient has a variety structure. If we define the hypertoric varieties by $\mathfrak{M}_\eta$, the category $D^b(\mathfrak{M}_\eta)$ makes sense. Whenever we talk about the derived category of a variety we mean this.

The derived category of $\mathfrak{M}_\eta$ is related to the derived category of modules over some algebra B (as in [1]), arises from a combinatorial setting. Also, the category of B-modules can be thought of as an analogue of Bernstein-Gelfand-Gelfand's category $\mathcal{O}$ in a combinatorial context. So, there is a natural connection of representation theory with this category. Thus categoryfication of braid groups via $D^b(\mathfrak{M}_\eta)$ can help to study deeper representations of the braid group.

Generally, if we have a braid group B_p acting on $D^b(X)$, we can construct some knot invariants in the following way: suppose $b \in B_p$ be a braid, if we take it's closure in S^3 we get a knot $\hat{b}$. If we denote T_b to the auto-equivalence related to b then for fixed object F in $D^b(X)$ (such as $\mathscr{O}_X$), we can compute $\mathbf{Ext}_{D^b(X)}(F, T_b(F))$ or catgorical traces. These actually gives some knot-invariant cohomology theories. This is one of the motivations behind braid group action on such categories.

2. Discussion of the Main work

To give braid group action on $D^b(\mathfrak{M}_\eta)$ one needs to define a group homomorphism

$$\Phi : B_p \to \mathbf{Auteq}(D^b(\mathfrak{M}_\eta))$$

here, $\mathbf{Auteq}(D^b(\mathfrak{M}_\eta))$ is the group of all derived auto-equivalences between $D^b(\mathfrak{M}_\eta)$. So, we need to find some autoequivalences that satisfy the braid group relation. The work in

[2] suggests that if we have $S_1, \cdots, S_p$ spherical objects in $D^b(X)$, the **spherical twists** are auto-equivalences and if the satisfy

$$\dim \mathbf{Ext}_{D^b(X)}(S_i, S_j) = 1_{|i-j|=1}$$

for $i \neq j$ then the corresponding twists T_{S_i} satisfy the braid relation. There is a core of $\mathfrak{M}_\eta$ which is union of smooth, projective, lagrangian subvarieties of $\mathfrak{M}_\eta$, call it $\mathcal{X}$. One can expect that the structure sheaf of these smooth projective complement can satisfy the braid relation. But it is not immediate from [2] as $\mathscr{O}_{\mathbb{P}^n}$ are not spherical for higher n. But this result can be used for some particular cases.

There is something called A_m-surfaces, their resolution are a hypertoric variety, the irreducible lagrangian subvarieties of this are copies of $\mathbb{P}^1$. Here, $\mathscr{O}_{\mathbb{P}^1}$ can be thought of a spherical object in the derived category of the A_m-surface. By [2], we can get a braid group action.

In order to tackle this dificulty we go down to the hyperplane arrangement corresponding to the hypertoric variety. When we define $\mathfrak{M}_\eta$, η comes from the character of the torus action. Suppose η' another such character differs from η by a discriminantal hper-plane crossing, we can define a **wall-crossing** functor

$$\Phi_\eta^{\eta'} : D^b(\mathfrak{M}_\eta) \to D^b(\mathfrak{M}_{\eta'})$$

which is an equivalence of triangulated derived categories. It will turn out that this wall-crossing functor is a Fourier-Mukai transform corresponding to the kernal $\mathscr{O}_Z$ here $Z = \mathfrak{M}_\eta \times_{\mathfrak{M}_\xi} \mathfrak{M}_{\eta'}$, $\mathfrak{M}_\xi$ comes from the stability of GIT or the hyperKähler quotient. Before proceeding towards the framework of the main theorem we would like to remark the following.

Remark. For the case of A_m-surfaces $\mathbb{C}^2/\mathbb{Z}_{m+1}$, the minimal resolution $\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}}$ is a hypertoric variety. It arises from a hyperplane arrangement with m points in $\mathbb{R}$. Clearly, there are $(m+1)$ chambers, and let η_i denote the corresponding characters of the torus for $i = 1, \cdots, m+1$. Moreover, there are m-spherical objects S_i, where the index $i = 1, \cdots, m$. There exists a map $\psi_i : \mathfrak{M}_{\eta_i} \to \mathfrak{M}_{\eta_{i+1}}$. There is an obvious relation

$$\Phi_{\eta_i}^{\eta_{i+1}} = \psi_{i*} T_{S_i}.$$

Thus, there is a way to use the wall-crossing functors to construct the braid group action in this case. We attempted to generalize the observations obtained here, which leads to the following discussion.

We now recall the construction of the Deligne groupoid associated to a real hyperplane arrangement and prove that our assignment of wall-crossing functors to paths is well-defined. This will allow us to deduce a braid group action on derived categories of hypertoric varieties.

The Deligne groupoid. Let $\mathcal{A}$ be a real hyperplane arrangement in $\mathbb{R}^d$, and let $\mathcal{C}$ denote the set of chambers, i.e. the connected components of $\mathbb{R}^d \setminus \bigcup_{H \in \mathcal{A}} H$. For chambers $C, C' \in \mathcal{C}$ we define:

Definition 2.1. The *Deligne groupoid* $G = \Pi_1(\mathcal{A})$ is the groupoid whose

- objects are the chambers of $\mathcal{A}$,
- morphisms are generated by *elementary moves* $C \to C'$ whenever C and C' share a codimension-one wall, subject to the relations coming from minimal positive paths in the Salvetti complex of $\mathcal{A}$.

Concretely, a path γ in $\mathbb{C}^d \setminus \mathcal{A}_{\mathbb{C}}$ with endpoints in real chambers determines a morphism in G, and two such paths are equivalent if they are homotopic through such paths. The groupoid

G is equivalent to the fundamental groupoid of the complexified complement:

$$G \simeq \Pi_1(\mathbb{C}^d \setminus \mathcal{A}_{\mathbb{C}}, \mathcal{C}).$$

2.1. **Wall-crossing functors.** For each chamber $\eta \in \mathcal{C}$, we have the hypertoric variety M_η, and hence its bounded derived category $D^b(M_\eta)$. If two chambers η, η' are adjacent (separated by a single wall), we have constructed an equivalence

$$\Phi_\eta^{\eta'} : D^b(M_\eta) \xrightarrow{\sim} D^b(M_{\eta'}).$$

By composing such equivalences along a path $\gamma : \eta \to \eta'$ crossing successive walls, we obtain a functor

$$\Phi_\gamma : D^b(M_\eta) \xrightarrow{\sim} D^b(M_{\eta'}).$$

Well-definedness. The key issue is that a given pair of chambers η, η' may be connected by many distinct paths. We must show that the resulting functor Φ_γ depends only on the homotopy class of γ, i.e. is well-defined in the Deligne groupoid.

Theorem 2.2 (Well-definedness of wall-crossing functors)**.** *Let γ_1, γ_2 be two paths in $\mathbb{C}^d \setminus \mathcal{A}_{\mathbb{C}}$ connecting the same chambers η and η'. Then the associated wall-crossing functors*

$$\Phi_{\gamma_1}, \Phi_{\gamma_2} : D^b(M_\eta) \to D^b(M_{\eta'})$$

are canonically isomorphic. Equivalently, the assignment

$$\Phi : G \longrightarrow \mathbf{Cat}, \qquad \eta \mapsto D^b(M_\eta), \quad \gamma \mapsto \Phi_\gamma$$

is a well-defined functor from the Deligne groupoid G to the 2-category of triangulated categories and equivalences.

Sketch of proof. By construction, Φ_γ is defined by composition of elementary wall-crossing functors $\Phi_{\eta_i}^{\eta_{i+1}}$. The Deligne groupoid G imposes relations identifying any two minimal positive paths γ_1, γ_2 between the same pair of chambers. Thus it suffices to prove that the corresponding compositions of functors agree.

For each codimension-two intersection of hyperplanes, the Salvetti complex description of G implies that two distinct minimal paths around this intersection become homotopic in the complexified complement [4]. On the categorical side, this corresponds to the fact that the two possible compositions of elementary equivalences $\Phi_{\eta_i}^{\eta_{i+1}}$ yield naturally isomorphic functors: both functors realize the same wall-crossing bimodule, differing only by the order in which intermediate mutations are applied. This follows from the compatibility of wall-crossing functors with hyperplane relations, which was checked in the construction of $\Phi_\eta^{\eta'}$.

Therefore Φ_γ depends only on the morphism in G defined by γ, and the assignment Φ is functorial.

Consequence: braid group action. It is a theorem of Deligne [3] that if $\mathcal{A}$ is a real simplicial arrangement, then the fundamental group $\pi_1(\mathbb{C}^d \setminus \mathcal{A}_{\mathbb{C}})$ is isomorphic to an Artin braid group. Combining this with the previous theorem, we obtain:

Theorem 2.3 (Braid group action on derived categories)**.** *Let $\mathcal{A}$ be a simplicial real hyperplane arrangement with chambers $\mathcal{C}$. For each $\eta \in \mathcal{C}$, let M_η be the corresponding hypertoric variety. Then the braid group*

$$B = \pi_1(\mathbb{C}^d \setminus \mathcal{A}_{\mathbb{C}})$$

acts by equivalences on the derived categories $D^b(M_\eta)$. Explicitly, a loop γ based at η defines an autoequivalence

$$\Phi_\gamma : D^b(M_\eta) \xrightarrow{\sim} D^b(M_\eta).$$

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