IT IS IMPOSSIBLE TO ACHIEVE A COHERENT OBJECTIVE PICTURE OF THE WORLD ON THE BASIS OF CONCEPTS WHICH ARE TAKEN MORE OR LESS FROM INNER PSYCHOLOGICAL EXPERIENCE.

ALBERT EINSTEIN

DANCE IS A VOCABULARY. CAN YOU RECITE THE VOCABULARY FORWARDS AND BACKWARDS AND ADD NEW WORDS?

B-BOY VIETNAM

LIVE AS IF YOU WERE TO DIE TOMORROW. LEARN AS IF YOU WERE TO LIVE FOREVER.

MAHATMA GANDHI

DR. DOEG

ELECTROMAGNETISM & LIGHT

THE INVISIBLE COLLEGE

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Dedicated to the ghosts of the college and

the spectral bodies of physics.

Note

This physics text is an OpenSource academic project developed in abstraction by the Invisible College. The manuscript is written in LaTeX and makes use of the `tufte-book` and `tufte-handout` document classes.

http://latex-project.org/ftp.html
https://git-scm.com/downloads

Light & Optics

Music is the arithmetic of sounds as optics is the geometry of light.
- Claude Debussy

Most optical phenomena can be accounted for using the classical electromagnetic description of light. Complete electromagnetic descriptions of light are, however, often difficult to apply in practice. Geometric optics, treats light as a collection of rays that travel in straight lines and bend when they pass through or reflect from surfaces. Diffraction and interference require a wave model of light and cannot be accounted using geometric optics.

Figure 1: Depp's optics (*Sleepy Hollow*)

Light Rays

A light ray is an idealized model of light, obtained by choosing a line that is perpendicular to the wavefronts of the actual light, and that points in the direction of energy flow. Rays are used to model the propagation of light through an optical system, by dividing the real light field up into discrete rays and tracked through ray tracing.

Ray Angle

At an interface, ray angles are described relative to the surface normal.

Refractive Index

The refractive index or index of refraction n of a material is a dimensionless number that describes how light propagates through that medium. It is defined as the ratio between the speed of light through vacuum and the speed of light through that material.

$$n = \frac{c}{v}$$

Reflection, Refraction & Snell's Law

Reflection is the change in direction of a wavefront at an interface between two different media so that the wavefront returns into the medium from which it originated. For specular reflection the incident ray angle equals reflected ray angle.

$$\theta_{incidence} = \theta_{reflection}$$

Refraction is the change in direction of propagation of a wave due to a change in its transmission medium. **Snell's law** describes the relationship between the angle of the incident ray and that of the refracted ray.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

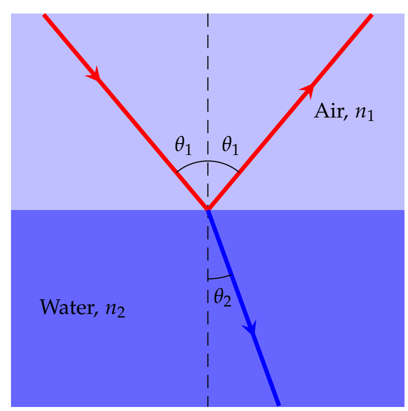

Figure 2: Reflection and refraction

Snell's law may be derived in various ways. One way is to match the frequency of the incident and refracted wave at the surface.

$$f_1 = f_2$$

Representing the frequency as a ratio of the velocity and the wavelength yields the following.

$$\frac{v_1}{\lambda_1} = \frac{v_2}{\lambda_2}$$

Substituting the velocity as the ratio of the speed of light and the index of refraction gives the following.

$$\frac{c}{n_1\lambda_1} = \frac{c}{n_2\lambda_2}$$

This simplifies further.

$$n_1\lambda_1 = n_2\lambda_2$$

The distance between wave peaks along the interface is d. It must be the same value on both sides of the interface. Representing the wavelength in terms of d gives the following.

$$n_1 d \sin\theta_1 = n_2 d \sin\theta_2$$

Finally cancelling d yields Snell's law.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Fermat's Principle

Fermat's principle states that when light travels between any two points its path is the one that requires the least time. Snell's law may also be derived from this principle. In truth it is less of a principle and more of a feature of light and refraction.

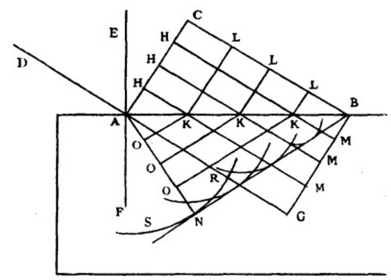

Figure 3: Huygens' construction

Total Internal Reflection

Total internal reflection is a limiting case of refraction. In the case of total internal reflection the refracted angle is 90 degrees, namely there is no refracted ray.

$$n_1 \sin\theta_c = n_2 \sin 90$$

The critical angle for total internal reflection is given as follows.

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

This is possible when light is passing from a medium of higher refractive index to lower refractive index.

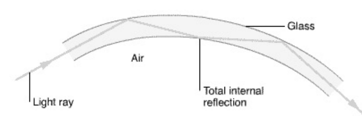

Figure 4: Total internal reflection in a fiber optic cable

Dispersion

In many wave-propagation media, wave velocity changes with frequency or wavelength of the waves; this is true of light propagation in most transparent substances other than a vacuum. These media are called dispersive. The result is that the angles determined by Snell's law also depend on frequency or wavelength, so that a ray of mixed wavelengths, such as white light, will spread or disperse. Such dispersion of light in glass or water underlies the origin of rainbows and other optical phenomena, in which different wavelengths appear as different colors. In short, indices of refraction are wavelength dependent.

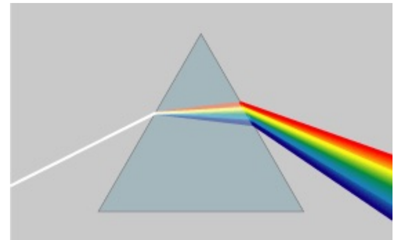

Figure 5: Dispersion of white light through a prism

Rainbows

A rainbow is a meteorological phenomenon that is caused by reflection, refraction and dispersion of light in water droplets resulting in a spectrum of light appearing in the sky. It takes the form of a multicoloured arc. Rainbows caused by sunlight always appear in the section of sky directly opposite the sun.

The following diagram shows how rainbows for through light refraction-reflection-refraction interaction with a water droplet.

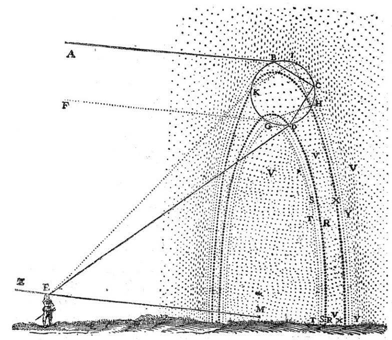

Figure 6: Descartes diagram of rainbow formation

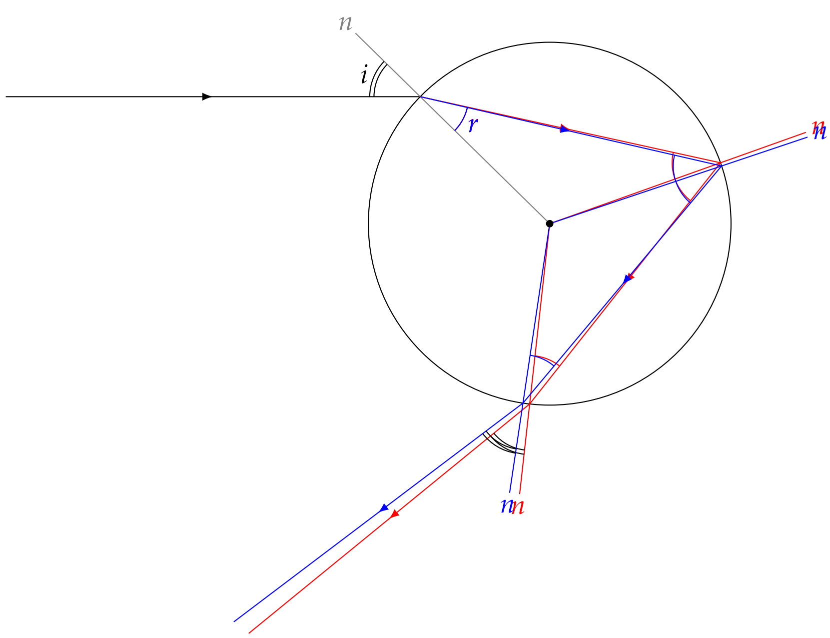

Figure 7: Raindrop producing a rainbow

Geometric Optics

Geometric optics uses ray tracing to model light.

Thin Lens Equation

Light's interaction with thin lenses can be modeled using the thin lens equation. Here f is the focal length, d_i is the distance from the lens to image and d_o is the distance from the lens to the object.

$$\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$$

Convergent/Convex Lenses (f>0)

As light rays parallel to the optical axis hit a convergent lens they are scattered to pass through the focus. The rays converge on the focal point. Convex lenses are convergent.

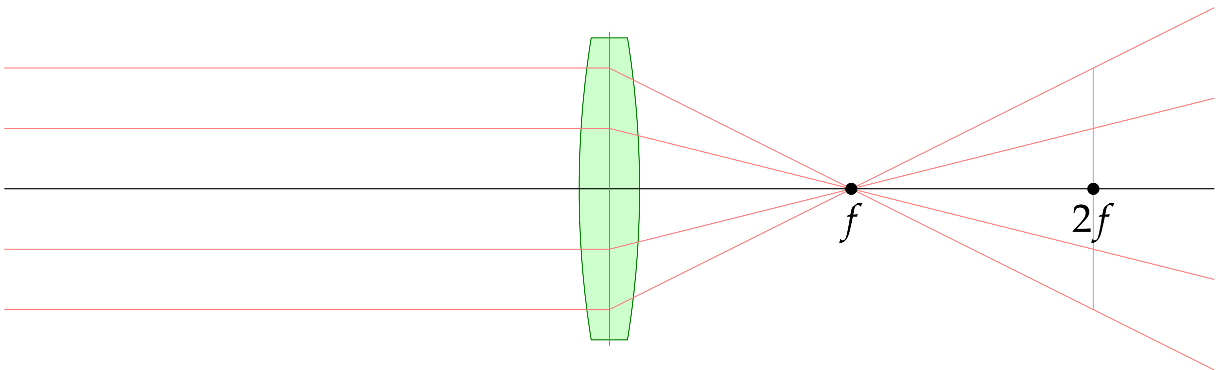

Figure 8: Convergent Lens

A light emitting object with a d_0 greater that $2f$ will produce an image with a d_i between f and $2f$. The magnification in this case is negative and has a magnitude which is less than one.

Magnification

Magnification is the ratio of the height of the focused image h_i to the height of the light emitting object h_o.

$$M = \frac{h_i}{h_o} = -\frac{d_i}{d_o}$$

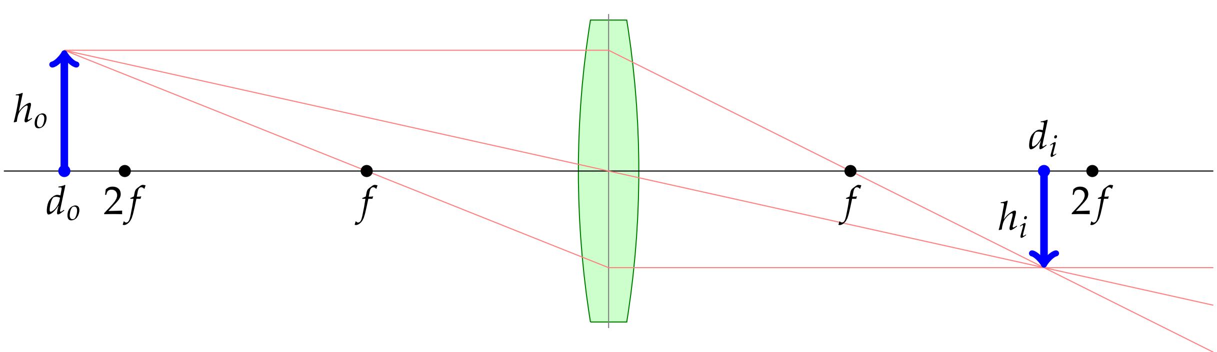

Figure 9: $d_0 > 2f$

A light emitting object with a d_0 equal $2f$ will produce an image with a d_i equal to $2f$. The magnification in this case is negative one.

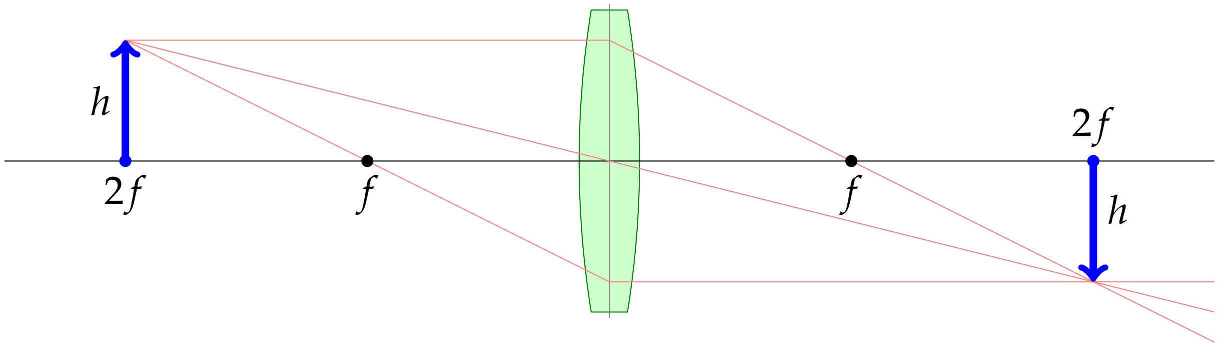

Figure 10: $d_0 = 2f$

A light emitting object with a d_0 between $2f$ and f will produce an image with a d_i greater than $2f$. The magnification in this case is negative and has a magnitude which is greater than one.

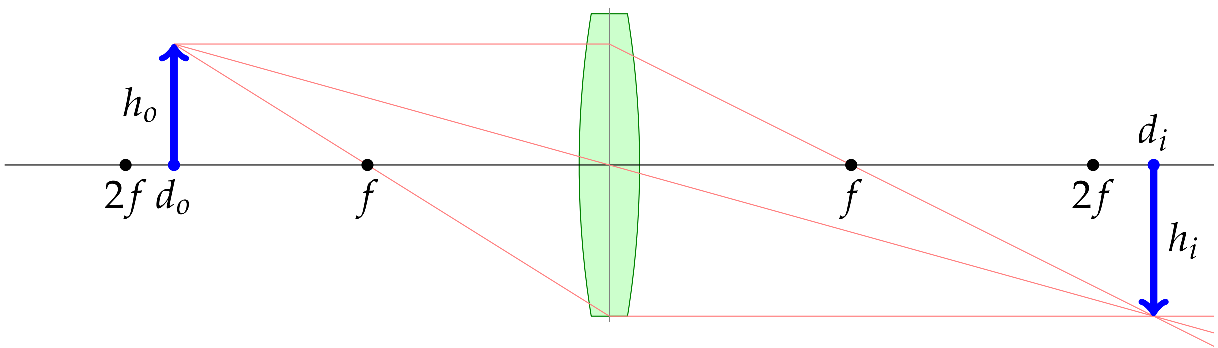

Figure 11: $2f > d_0 > f$

A light emitting object with a d_0 equal to f will not produce an image since the rays transmitted through the lens will be parallel. The image may be described as existing at infinity.

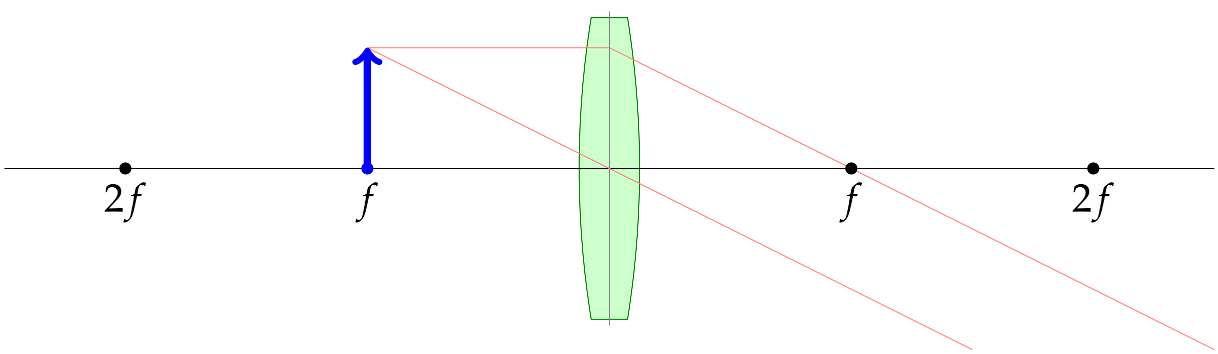

Figure 12: $d_0 = f$

A light emitting object with a d_0 less than f will not produce an image since the rays transmitted through the lens will diverge. The image may be described as existing with a d_i that is negative. The image does not exist. It is virtual. The magnification is positive and greater than one.

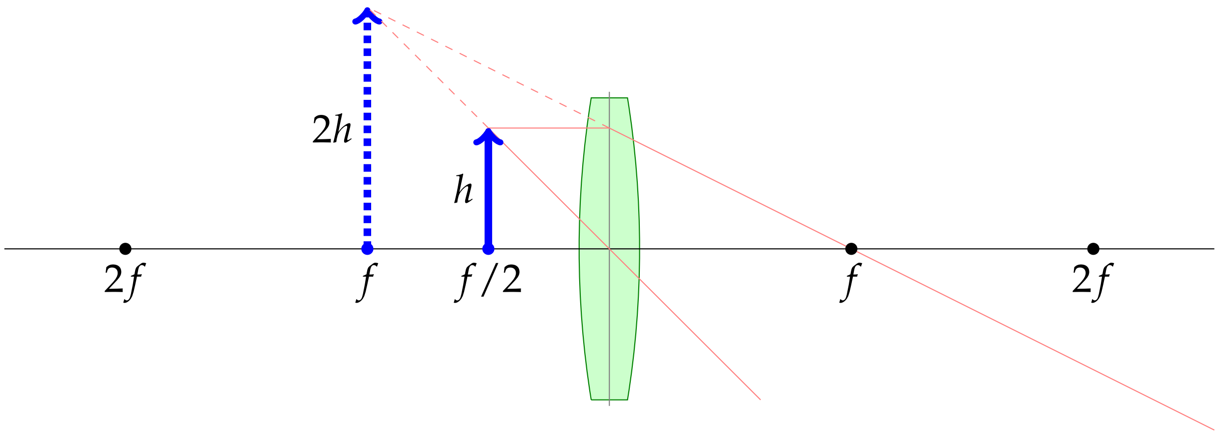

Figure 13: $d_0 < f$

Divergent/Concave Lenses (f<0)

As light rays parallel to the optical axis hit a divergent lens they are scattered away from the optical axis and do not converge at a point. Geometrically they diverge as if they come from a point on the other side of the lens. This point is the focal point of a divergent lens. The focal length of a divergent lens is negative. Concave lenses are divergent.

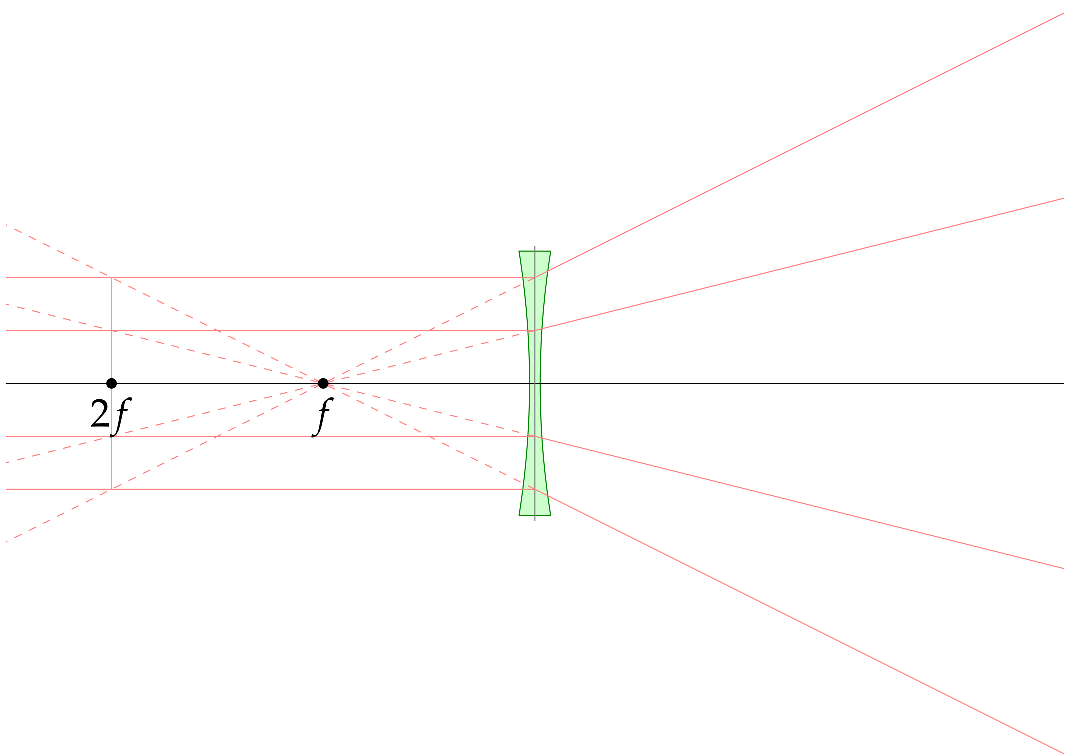

Figure 14: Divergent Lens: parallel lines diverge from focus

A light emitting object with a d_0 greater than $2f$ will not produce an image since the rays transmitted through the lens will diverge. The image may be described as existing with a d_i that is negative and less than f in absolute value. The image does not exist. It is virtual. The magnification is positive and less than one.

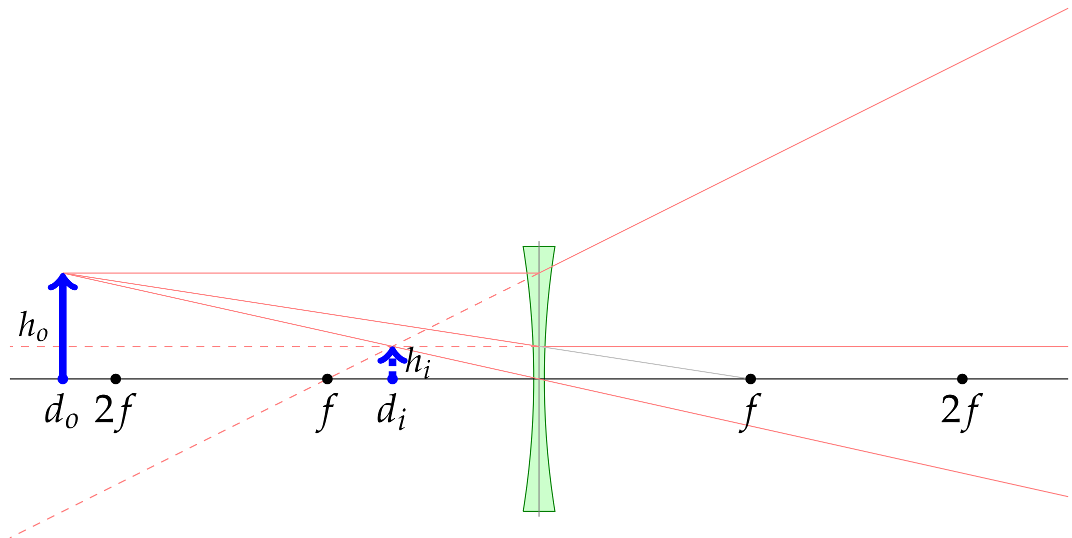

Figure 15: Convergent Lens: $d_o > 2f$

In the case where d_0 is equal to $2f$ no image is produced since the rays transmitted through the lens will diverge. The image may be described as existing with a d_i that is negative and two thirds of f. Again the image does not exist, it is virtual. The magnification is one third.

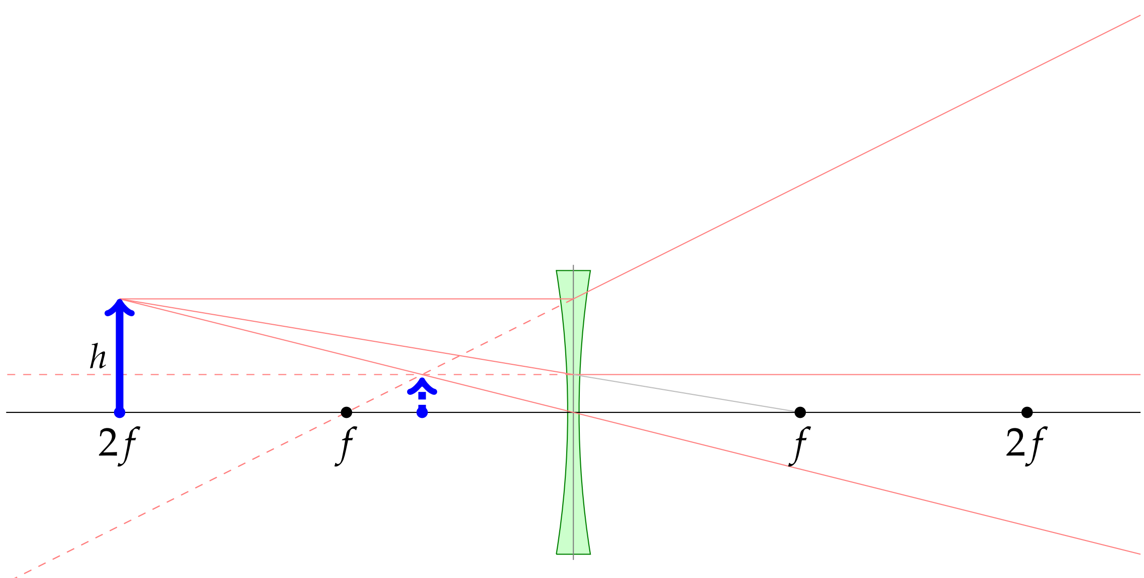

Figure 16: Convergent Lens: $d_o = 2f$

In the case where d_0 is equal to f no image is produced since the rays transmitted through the lens will diverge. The image may be described as existing with a d_i that is negative and two half of f. Again the image does not exist, it is virtual. The magnification is one half.

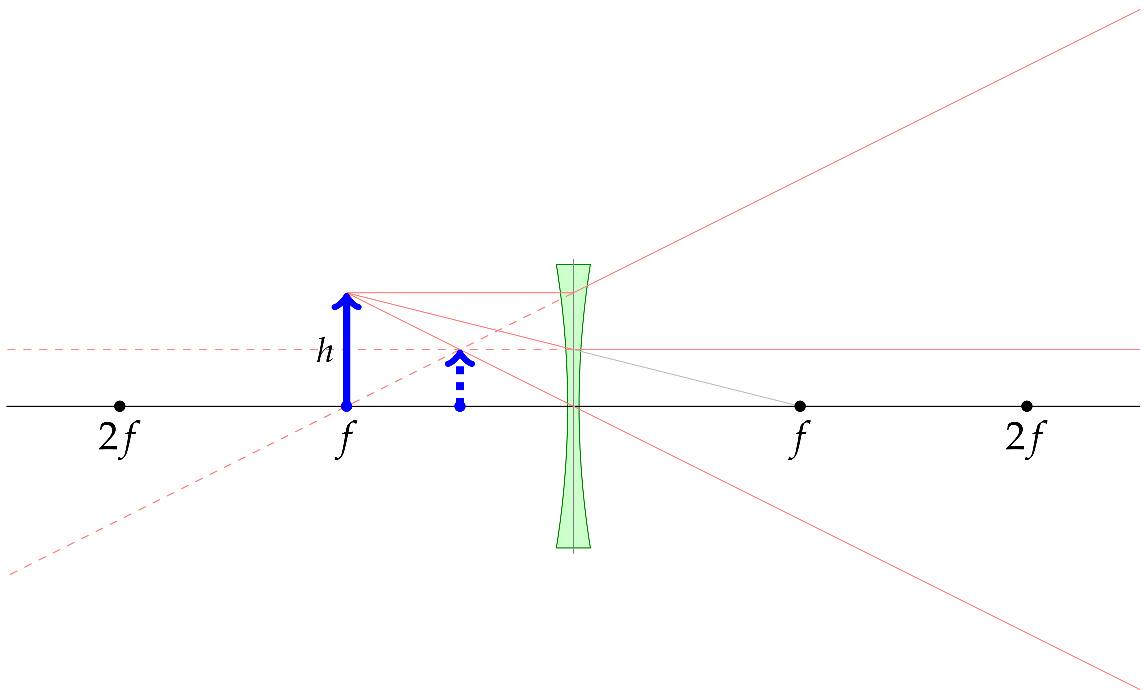

Figure 17: Convergent Lens: $d_o = f$

Mirrors

Mirrors work the same way lenses do except light is reflected rather than transmitted. Convergent mirrors are concave while divergent lenses are convex.

Characterizing Images

real/virtual Real images can be focused on a plane and occur when light converges. Virtual images cannot be focused on a plane and occur when light diverges.

upright/inverted Upright images are right-side-up versions of the object and occur for $M > 0$ while inverted images are up-side-down versions of the object and occur for $M < 0$.

enlarged/reduced Enlarged images are bigger versions of the object and occur for $|M| > 0$ while reduced images are smaller versions of the object and occur for $|M| < 0$.

Huygens Principle

Huygens principle states that every point which a luminous disturbance reaches becomes a source of a spherical wave. In other words, all points bombarded by waves become point sources for waves.

Figure 18: Christiaan Huygens had sausage fingers

Interference

Light wave interference requires the following criteria.

- Sources must be coherent, namely maintain a constant phase relative to one another.
- Sources must be monochromatic, namely be of a single wavelength.
- Superposition must apply

Double Slit Experiment

Conducted by Young 1801. Consider two slits separated by a distance d. Light interferes on a plane distance L away. For $L >> d$ we may use parallel ray approximation to determine path difference δ.

$$\delta = r_2 - r_1 = d \sin\theta$$

Constructive interference associated with bright spots on the plane.

$$d \sin\theta = N\lambda$$

Similar phenomena for a diffraction grating where d is the distance between slits.

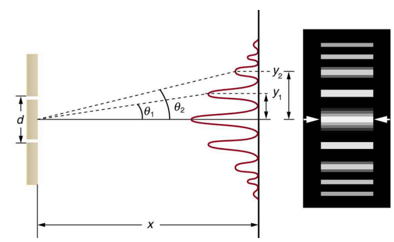

Figure 19: Two slit diffraction

Intensity

Phase difference is related to the path difference and wavelength.

$$\phi = \frac{2\pi}{\lambda}\delta = \frac{2\pi}{\lambda} d \sin\theta$$

The total electric field is the sum of the two components of electric field.

$$E_T = E_1 + E_2 = E_0 \cos(\omega t) + E_0 \cos(\omega t + \phi) = 2E_0 \cos(\phi/2)\cos(\omega t + \phi/2)$$

Then relating the average intensity to the average electric field squared removes the time dependence and yields the following.

$$I_{avg} = I_0 \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)$$

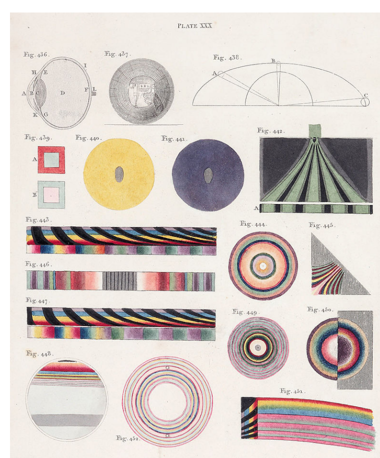

Figure 20: Plate from Thomas Young's 1807 *Lectures on Natural Philosophy and the Mechanical Arts*

Reflection Phase Change

An electromagnetic wave undergoes a phase change of 180 degrees upon reflection at an interface with a medium medium of higher refractive index than the one in which the wave is traveling. (reflection off a window)

Thin Film Interference

Consider a thin film of refractive index n and thickness l. An electromagnetic wave in air hits the interface, part of the wave is reflected with a 180 degree phase change and part is transmitted into the film. The transmitted component reflects off the other side of the film and is transmitted bad through the original interface to recombine with the reflected component.

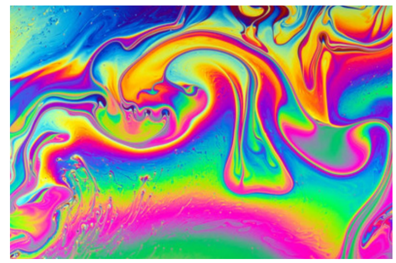

Figure 21: Thin film of oil

Constructive Interference

For constructive interference to occur the path difference must be equivalent an integer multiple of wavelengths plus half a wavelength. This is due to the fact that there is a reflection which takes place. This is equivalent to a half-phase shift.

$$2l = \left(N + \frac{1}{2}\right)\frac{\lambda}{n}$$

Constructive interference is desired for reflective coatings.

Destructive Interference

For destructive interference to occur the path difference must be an integer multiple of waves. The integer multiple of wave will produce destructive interference because there is an additional half-phase shift due to the reflection

$$2l = N\frac{\lambda}{n}$$

Destructive interference is desired for anti-glare coatings.

Iridescence

Iridescence caused by thin-film interference is a commonly observed phenomenon in nature, being found in a variety of plants and animals. One of the first known studies of this phenomenon was conducted by Robert Hooke in 1665. In *Micrographia*, Hooke postulated that the iridescence in peacock feathers was caused by thin, alternating layers of plate and air. In 1704, Isaac Newton stated in his book, *Opticks*, that the iridescence in a peacock feather was due to the fact that the transparent layers in the feather were so thin. In 1801, Thomas Young provided the first explanation of constructive and destructive interference.

Relativity

Imagination is more important than knowledge. Knowledge is limited. Imagination encircles the world.
- Albert Einstein

The theory of relativity usually encompasses two theories by Albert Einstein: special relativity and general relativity. Concepts introduced by the theories of relativity include spacetime as a unified entity of space and time, relativity of simultaneity, kinematic and gravitational time dilation, and length contraction.

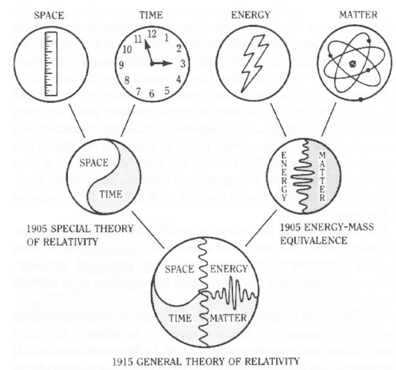

Figure 22: Special and general relativity

Galilean-Newtonian Relativity

Galilean invariance, Galilean relativity or Newtonian relativity states Newtonian physics operates the same in all inertial (non-accelerating) frames. Galileo Galilei first described this principle in 1632 in his *Dialogue Concerning the Two Chief World Systems*. He describes a ship traveling at constant velocity, without rocking, on a smooth sea. In this ship any observer doing experiments below the deck would not be able to tell whether the ship was moving or stationary.


Figure 23: Galilean boat

Unfortunately Galilean invariance did not agree with Maxwell's equations describing electromagnetism. The first problem is that electromagnetic theory describes electromagnetic waves, namely light, moving at a constant speed no matter the velocity of the frame of reference. In addition all experiments show light moving at a constant speed. The second problem with electromagnetic theory is that electromagnetic forces seem to be dependent on the velocity of the frame of reference. Namely, in a frame moving with a charged particle there are no magnetic forces however the same particle observed from a frame in which it is moving could see magnetic forces. Forces, however, should not be dependent on the velocity of the frame of reference.

Galilean Transformation

x transforms to x' in a frame moving in the x-direction at speed v.

$$\vec{v} = v\hat{x}$$

$$x' = x - vt \qquad y' = y$$

$$z' = z \qquad t' = t$$

$$\vec{u}' = \vec{u} - \vec{v}$$

$$\vec{E}' = \vec{E} - \vec{v} \times \vec{B}$$

$$\vec{B}' = \vec{B} + \frac{1}{c^2}\vec{v} \times \vec{E}$$

Einstein’s Relativity

Einstein’s relativity dictates the laws of mechanics, including electromagnetism, must be the same in all inertial frames and that the speed of light is constant.

Figure 24: Moving clocks run slowly

Time Dilation

Consider a light clock which consists of a light source next to a photocell opposite a mirror. They are a distance D apart. Light travels at speed c and travels a distance $2D$. The time it takes is $\Delta\tau$

$$\Delta\tau = \frac{2D}{c}$$

Now consider the light clock moving perpendicular to the length D at a speed v. In this frame the light takes a longer path but still travels at speed c. The time for the light to travel in this frame is $\Delta t'$.

$\Delta\tau$ is the proper time interval. This is defined as the time separating two events that take place at the same point in space.

$$\Delta t = \frac{2\sqrt{D^2 + (v\Delta t/2)^2}}{c}$$

Squaring each side of both equations.

$$(\Delta\tau)^2 = \frac{4D^2}{c^2} \qquad (\Delta t)^2 = \frac{4D^2 + v^2(\Delta t)^2}{c^2}$$

Combining these equations and solving for Δt yields the following.

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\Delta\tau$$

Note the factor $1/\sqrt{1-v^2/c^2}$ is always greater than one, $\gamma > 1$. Therefore $\Delta t > \Delta\tau$. The clock tics more slowly when it is observed moving.

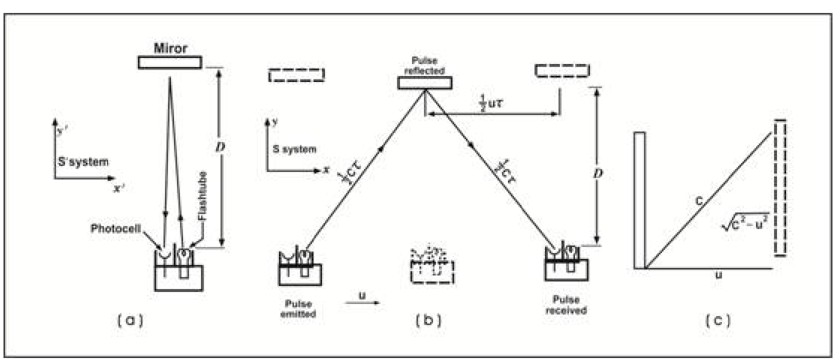

Figure 25: Time dilation

Length Contraction

Now consider measuring the velocity of an object. This always requires retrieving information from each frame of reference. One could view a moving clock and measure the distance it moved between flashes. The time measurement would be in the moving frame Δt while the length measurement would be in the rest frame L.

$$v = \frac{L}{\Delta t}$$

Alternately the velocity could be determined using a still clock with proper time interval $\Delta\tau$ and an object whose length is measured while moving l.

$$v = \frac{l}{\Delta\tau}$$

Equating the velocities gives the following.

$$\frac{L}{\Delta t} = \frac{l}{\Delta\tau}$$

Expressing $\Delta\tau$ in terms of Δt gives an expression for L in terms of l.

$$\frac{L}{\gamma\Delta\tau} = \frac{l}{\Delta\tau}$$

$$L = \gamma l = \frac{l}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This means the moving length l is shorter than the proper length L. This is known as length contraction. Moving objects appear squished.

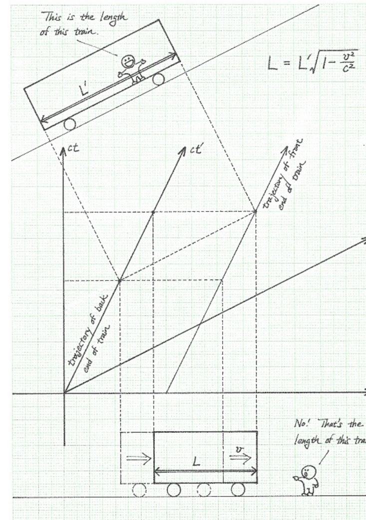

Figure 26: Length contraction

L is the proper length. The proper length or rest length refers to the length of an object in the object's rest frame.

Lorentz Transformation

The Lorentz transformation is a coordinate transformation between two coordinate frames that move at constant velocity relative to each other. Here x is transformed to x' by taking measurements in a frame moving at $\vec{v} = v\hat{x}$.

$$x' = \gamma(x - vt)$$

The y and z coordinates are left invariant in this transformation.

$$y' = y \qquad z' = z$$

Time is transformed in the moving frame as follows.

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

The velocity of an object in the rest frame is measured as $\vec{u}$. The x-component of the velocity is transformed as follows.

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

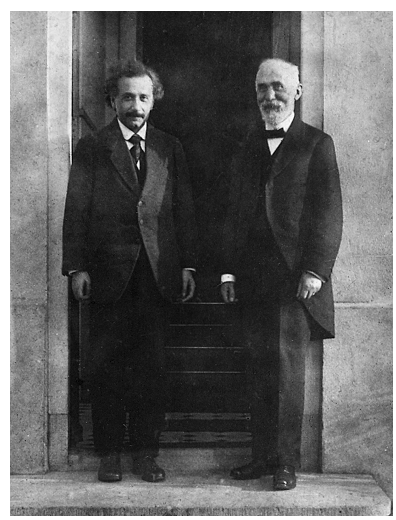

Figure 27: Albert Einstein and Hendrik Lorentz #biffles4life

Relativistic Mechanics

Special relativity gives new definitions to momentum and kinetic energy.

$$\overrightarrow{p} = m\overrightarrow{v} = \gamma m_0 \overrightarrow{v}$$

An interpretation relativistic momentum uses the concept of relativistic mass. At high velocity the mass transforms from the rest mass m_0 to m.

$$m = \gamma m_0$$

Once relativistic momentum is defined the relativistic force may be derived using Newton's second law.

$$\overrightarrow{F}_{net} = \frac{d\overrightarrow{p}}{dt}$$

Once the relativistic force is defined the relativistic kinetic energy may be derived using work-energy theorem.

$$\Delta KE = W_{net} = \int \overrightarrow{F}_{net} \cdot d\overrightarrow{r}$$

This yields the following expression for relativistic kinetic energy.

$$KE = \gamma m_0 c^2 - m_0 c^2 = mc^2 - m_0 c^2$$

The kinetic energy is the difference between two terms. The first term E changes with velocity. The second term, know as the rest energy E_0 is independent of velocity.

$$KE = E - E_0$$

The total energy E is defined as the sum of the kinetic energy and the rest energy.

$$E = KE + E_0$$

The total energy and rest energy are written as follows.

$$E = \gamma mc^2$$

$$E^2 = p^2c^2 + (mc^2)^2$$

$$E_0 = m_0 c^2$$

This expression for the rest energy describes a relationship between energy and mass. This is known as mass-energy equivalence.

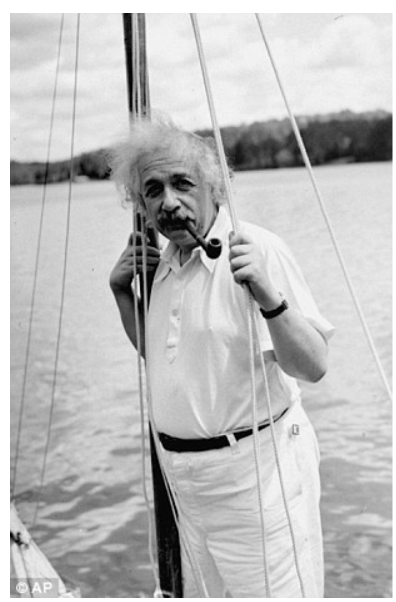

Figure 28: Einstein on a boat!

General Relativity

General relativity describes the effect of gravity on spacetime. Mass is an inertial quantity. Newton's second law relates to inertial mass.

$$F_{net} = m_i a$$

Gravity is different than the other forces in nature because mass gives rise to its field interaction.

$$F_g = m_g g$$

. Therefore the strength of the gravitational field g is equal to the acceleration a. The equivalence principle states that a gravitational field is indistinguishable from acceleration. This gives the theory the ability to analyze the behavior of time, space and light in proximity to mass.

The net time dilation in a spherically symmetric gravitational field can is considered. It is written below.

$$\Delta t = \frac{\Delta \tau}{\sqrt{1 - \frac{2GM}{rc^2}}} = \frac{\Delta \tau}{\sqrt{1 - \frac{v_e^2}{c^2}}}$$

The proper time is outside the gravitational field, out at $r = \infty$. Note that this is simply special relativity applied when the object is moving at the escape velocity from the field. Other effects include bending of light and curvature of space.

Figure 29: Albert Einstein at 14

Quantum Mechanics

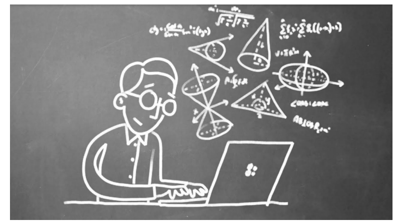

Figure 30: Snowden says NSA is building a quantum computer

The solution of the difficulty is that the two mental pictures which experiment lead us to form - the one of the particles, the other of the waves - are both incomplete and have only the validity of analogies which are accurate only in limiting cases.
- Werner Heisenberg

A quantum (plural: quanta) is the minimum amount of any physical quantity involved in an interaction. For example, angular momentum J is quantized into units $\hbar$.

$$J_n = n\hbar$$

Quantum mechanics describes systems with probability distributions in space and time. The probabilities describe discrete particle events. The structure of those distributions however have wavelike features. This is known as wave particle duality. Quantum mechanics gradually arose from Max Planck's solution in 1900 to the black-body radiation problem and Albert Einstein's 1905 paper which offered a quantum-based theory to explain the photoelectric effect (reported 1887).

In 1900, the British physicist Lord Rayleigh derived the λ^{-4} dependence of the Rayleigh-Jeans law based on classical physical arguments and empirical facts. The proportionality constant was added by Rayleigh and Sir James Jeans in 1905. The Rayleigh-Jeans law revealed an important error in physics theory of the time as it predicted an energy output that diverges towards infinity as wavelength approaches zero and energy output at short wavelengths disagreed with this prediction. This was known as the ultraviolet catastrophe. The term "ultraviolet catastrophe" was first used in 1911 by Paul Ehrenfest, but the concept originated with the 1900 derivation of the Rayleigh-Jeans law. The phrase refers to the fact that the Rayleigh-Jeans law accurately predicts experimental results at radiative frequencies below 105 GHz, but begins to diverge with empirical observations as these frequencies reach the ultraviolet region of the electromagnetic spectrum

Blackbody Radiation

Rayleigh-Jeans Law (Classical)

The Rayleigh-Jeans law attempts to describe the spectral radiance of electromagnetic radiation at all wavelengths from a black body at a given temperature through classical arguments.

$$I(\lambda, T) = \frac{2\pi c k_B T}{\lambda^4}$$

In 1838, Michael Faraday discovered cathode rays. By 1859 the black-body radiation problem had been identified by Gustav Kirchhoff.

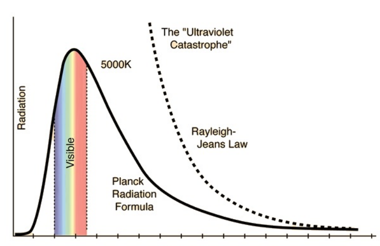

Figure 31: Ultraviolet catastrophe

Planck Spectrum (Modern)

Ludwig Boltzmann suggested that the energy states of a physical system can be discrete in 1877 and by 1900 the quantum hypothesis had ben made by Max Planck. The hypothesis states that energy is radiated and absorbed in discrete "quanta" (or energy elements). Statistically this models the intensity I as a particular function of wavelength and temperature. This distribution is known as the Planck spectrum.

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5\left(e^{\frac{hc}{\lambda k_B T}} - 1\right)}$$

The Planck spectrum precisely matched the observed patterns of black-body radiation. Specifically it matched Wien's displacement law.

Wien's displacement law states that the black body radiation curve for different temperatures peaks at a wavelength inversely proportional to the temperature.

$$\lambda_{peak} = \frac{2900\ \mu\text{m} \cdot \text{K}}{T}$$

The shift of that peak is a direct consequence of the Planck radiation law which describes the spectral brightness of black body radiation as a function of wavelength at any given temperature. However it had been discovered by Wilhelm Wien several years before Max Planck developed that more general equation, and describes the entire shift of the spectrum of black body radiation toward shorter wavelengths as temperature increases.

- Molecules have discrete energy states E_n
$$E_n = nhf$$
- Energy is released in discrete packets of electromagnetic radiation called "photons"
$$E_{photon} = hf$$

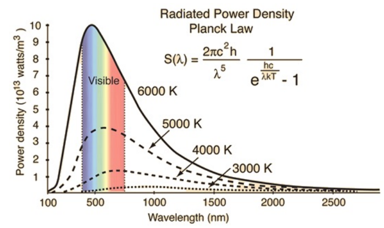

Figure 32: Planck spectrum

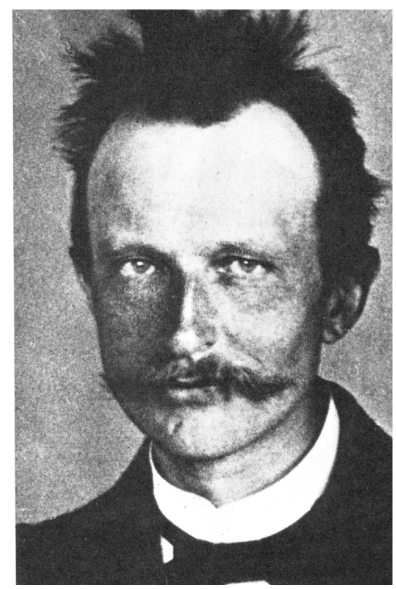

Figure 33: Young Max Planck

Photoelectric Effect

The photoelectric effect is the observation that many metals emit electrons when light shines upon them. Electrons emitted in this manner can be called photoelectrons.

According to classical electromagnetic theory, this effect can be attributed to the transfer of energy from the light to an electron in the metal. From this perspective, an alteration in either the intensity or wavelength of light would induce changes in the rate of emission of electrons from the metal. Furthermore, according to this theory, a sufficiently dim light would be expected to show a time lag between the initial shining of its light and the subsequent emission of an electron. However, the experimental results did not correlate with either of the two predictions made by classical theory.

Instead, electrons are only dislodged by the impingement of photons when those photons reach or exceed a threshold frequency.

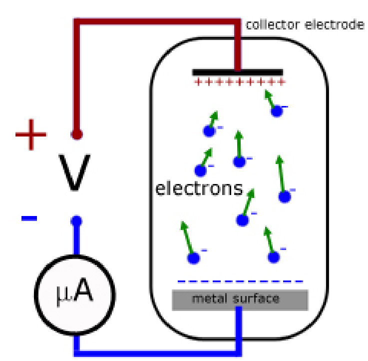

Figure 34: Photoelectric tube

Below that threshold, no electrons are emitted from the metal regardless of the light intensity or the length of time of exposure to the light. To make sense of the fact that light can eject electrons even if its intensity is low, Albert Einstein proposed that a beam of light is not a wave propagating through space, but rather a collection of discrete wave packets (photons), each with energy hf. This shed light on Max Planck's previous discovery of the Planck relation (E = hf) linking energy (E) and frequency (f) as arising from quantization of energy. The factor h is known as the Planck constant.

In 1887, Heinrich Hertz discovered that electrodes illuminated with ultraviolet light create electric sparks more easily. In 1905 Albert Einstein published a paper that explained experimental data from the photoelectric effect as the result of light energy being carried in discrete quantized packets. This discovery led to the quantum revolution. The work function W is the work required to free an electron from the metal.

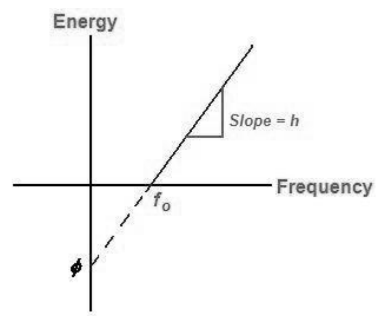

Figure 35: Photoelectric effect graph

$$KE_{max} = hf - W$$

$$KE_{max} = eV_{stop}$$

Electrons will not be emitted if incident light is below a certain cutoff frequency even if intensity of light is increased.

Compton Effect

Compton scattering, discovered by Arthur Holly Compton, is the inelastic scattering of a photon by a charged particle, usually an electron.

$$\lambda' - \lambda_0 = \frac{h}{mc}(1 - \cos\theta)$$

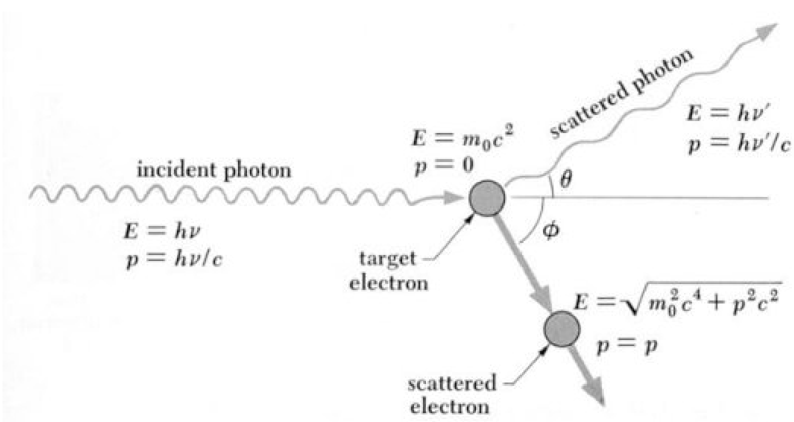

Figure 36: Compton scattering

Compton scattering results in a decrease in energy (increase in wavelength) of the photon. This is called the Compton effect. Part of the energy of the photon is transferred to the recoiling electron. The scattering supports particle nature of light.

De Broglie Relation

All matter can exhibit wave-like behavior. For example, a beam of electrons can be diffracted just like a beam of light or a water wave. Matter waves are a central part of the theory of quantum mechanics, being an example of wave-particle duality. The concept that matter behaves like a wave is also referred to as the de Broglie hypothesis due to having been proposed by Louis de Broglie in 1924. Matter waves are often referred to as de Broglie waves.

The de Broglie wavelength is the wavelength, λ, associated with a massive particle and is related to its momentum, p, through the Planck constant, h.

$$p = \frac{h}{\lambda} = \hbar k$$

De Broglie, in his 1924 PhD thesis, proposed that just as light has both wave-like and particle-like properties, electrons also have wave-like properties.

Planck-Einstein Relation

$$E = \frac{h}{T} = hf = \hbar\omega$$

Figure 37: Louis De Broglie

Wave functions

In quantum mechanics a wave function is a mathematical object that represents a particular pure quantum state of a specific isolated system of one or more particles. It is a central entity in quantum mechanics and provides probability distribution of states.

$$\psi(x,t) = Ae^{-i(kx-\omega t)} = A\left(\cos(kx-\omega t) + i\sin(kx-\omega t)\right)$$

These distributions can be used to find average values of observable quantities as follows.

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x)x\ \psi(x)\ dx$$

The wave functions provide an inner product space for a set of observable operators. The DeBroglie relation is represented through the following momentum operator.

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x)\left(i\hbar\frac{d}{dx}\right)\ \psi(x)\ dx$$

The Planck-Einstein relation is represented through the following Hamiltonian (energy operator).

$$\langle H \rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\left(i\hbar\frac{d}{dt}\right)\ \psi(x,t)\ dx$$

The inner product of the wavefunction should be normalized.

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\ dx = 1$$

Probability distributions

For standard probability distributions average values are calculated as follows.

$$\langle x \rangle = \int_{-\infty}^{\infty} \rho(x)x\ dx$$

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} \rho(x)x^2\ dx$$

$$\int_{-\infty}^{\infty} \rho(x)\ dx = 1$$

$$i\hbar\frac{d}{dx}e^{-i(kx-\omega t)} = -i^2\hbar k e^{-i(kx-\omega t)}$$

$$i\hbar\frac{d}{dx}e^{-i(kx-\omega t)} = pe^{-i(kx-\omega t)}$$

$$i\hbar\frac{d}{dt}e^{-i(kx-\omega t)} = -i^2\hbar\omega e^{-i(kx-\omega t)}$$

$$i\hbar\frac{d}{dt}e^{-i(kx-\omega t)} = Ee^{-i(kx-\omega t)}$$

Schrodinger Equation

Figure 38: Erwin Schrodinger

The Schrödinger equation is the fundamental equation of physics for describing quantum mechanical behavior. It is also often called the Schrödinger wave equation, and is a partial differential equation that describes how the wavefunction of a physical system evolves over time. Viewing quantum mechanical systems as solutions to the Schrödinger equation is sometimes known as the Schrödinger picture, as distinguished from the matrix mechanical viewpoint, sometimes known as the Heisenberg picture.

$$\mathcal{H}\psi(x,t) = \frac{p^2}{2m}\psi(x,t) + V(x)\psi(x) = E\psi(x,t)$$

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x,t) + V(x)\psi(x) = E\psi(x,t)$$

$$\frac{d^2}{dx^2}\psi(x,t) = -\frac{2m}{\hbar^2}\left[E - V(x)\right]\psi(x,t)$$

Uncertainty Relations

Figure 39: Schrodinger's cat

The uncertainty principle, also known as Heisenberg's uncertainty principle, is any of a variety of mathematical inequalities asserting a fundamental limit to the precision with which certain pairs of physical properties of a particle, known as complementary variables, such as position x and momentum p, can be known simultaneously. Introduced first in 1927, by the German physicist Werner Heisenberg, it states that the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa

$$(\Delta x)^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2$$

$$\sigma_x = \Delta x$$
$$\sigma_p = \Delta p$$

The uncertainty of a quantity is calculated from the difference between the mean square and the square mean. The relationship between the uncertainty in position and momentum is as follows.

$$\Delta x \; \Delta p \geq \frac{\hbar}{2}$$

The relationship between the uncertainty in energy and time is as follows.

$$\Delta E \; \Delta t \geq \frac{\hbar}{2}$$

As a mathematical system for making statements about the world, quantum mechanics has limits of specificity built in. These uncertainties are not shortcomings of measurement. They are hard coded into quantum theory itself.

According to the Copenhagen interpretation, physical systems generally do not have definite properties prior to being measured, and quantum mechanics can only predict the probabilities that measurements will produce certain results. The act of measurement affects the system, causing the set of probabilities to reduce to only one of the possible values immediately after the measurement. This feature is known as wavefunction collapse.

Schrödinger's cat is a thought experiment, sometimes described as a paradox, devised by Austrian physicist Erwin Schrödinger in 1935. It illustrates what he saw as the problem of the Copenhagen interpretation of quantum mechanics applied to everyday objects.

Particle in a Box

the particle in a box model (also known as the infinite potential well or the infinite square well) describes a particle free to move in a small space surrounded by impenetrable barriers. The model is mainly used as a hypothetical example to illustrate the differences between classical and quantum systems. In classical systems, for example a ball trapped inside a large box, the particle can move at any speed within the box and it is no more likely to be found at one position than another. However, when the well becomes very narrow (on the scale of a few nanometers), quantum effects become important. The particle may only occupy certain positive energy levels. Likewise, it can never have zero energy, meaning that the particle can never "sit still". Additionally, it is more likely to be found at certain positions than at others, depending on its energy level. The particle may never be detected at certain positions, known as spatial nodes.

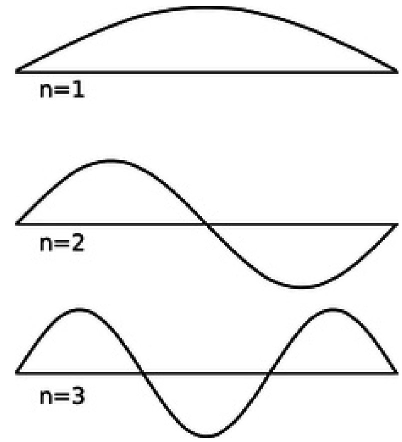

Figure 40: Particle in a box wavefunctions

$$\psi(x) = A \sin\left(\frac{n\pi x}{L}\right)$$

$$E_n = \frac{h^2 n^2}{8mL^2}$$

$$\sigma_x^2 = \frac{L^2}{12}\left(1 - \frac{6}{n^2\pi^2}\right)$$

$$\sigma_p^2 = \left(\frac{\hbar n\pi}{L}\right)^2$$

$$\sigma_x\sigma_p = \frac{\hbar}{2}\sqrt{\frac{n^2\pi^2}{3} - 2}$$

Hydrogen Atom and the Bohr Model

The 1913 Niels Bohr paper *On the Constitution of Atoms and Molecules* depicts the atom as a small, positively charged nucleus surrounded by electrons that travel in circular orbits around the nucleus. It is the solar system, but small and attraction provided by electrostatic forces rather than gravity.

The principal quantum number (symbolized n) is one of four quantum numbers which are assigned to each electron in an atom to describe that electron's state. As a discrete variable, the principal quantum number is always an integer. As n increases, the number of electronic shells increases and the electron spends more time farther from the nucleus. As n increases, the electron is also at a higher potential energy and is therefore less tightly bound to the nucleus.

The principal quantum number was first created for use in the semiclassical Bohr model of the atom, distinguishing between different energy levels. With the development of modern quantum mechanics, the simple Bohr model was replaced with a more complex theory of atomic orbitals. However, modern theory still requires the principal quantum number. Apart from the principal quantum number n, the other quantum numbers for bound electrons are the azimuthal quantum number l, the magnetic quantum number m, and the spin quantum number s.

Ryberg Formula

The model's key success lay in explaining the Rydberg formula for the spectral emission lines in of atomic hydrogen.

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

The Ryberg constant R fits the spectral lines of hydrogen with the following value.

$$R = \frac{1.097 \times 10^7}{\mathrm{m}}$$

While the Rydberg formula had been known experimentally, it did not gain a theoretical underpinning until the Bohr model was introduced.

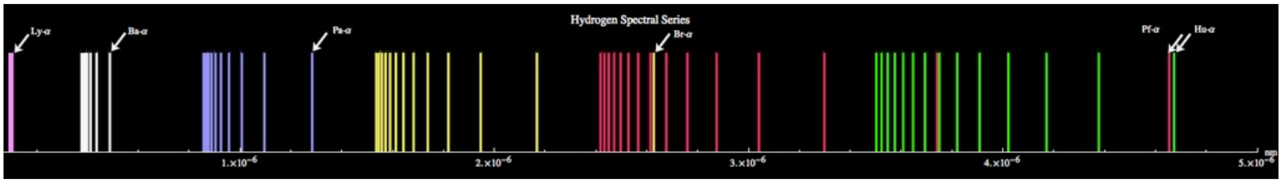

Figure 41: Hydrogen spectrum

Quantization of Angular Momentum

In order to derive the energy levels of the Bohr model begin with quantization of angular momentum.

$$L = n\hbar = mvr$$

This is the quantum fairy dust sprinkled on circular orbits of the electron in a Coulomb field.

Orbital Radii

Applying the quantization of angular momentum yields the following.

$$r = \frac{k_e e^2}{mv^2} \qquad\qquad v = \frac{n\hbar}{mr}$$

This process identifies the quantized orbital radii of hydrogen. Remember n is tracking the angular momentum of the hydrogen atom.

$$r_n = \frac{n^2\hbar^2}{m_e k_e e^2} = n^2 a_0$$

From the quantized radii the energy levels may be expressed.

$$E_n = -\frac{k_e e^2}{2a_0 n^2} = \frac{E_1}{n^2}$$

The wavelength of photon emitted from an electron transition from m to n is written.

$$E_n - E_m = \frac{hc}{\lambda}$$

The Ryberg constant may be identified.

$$R = \frac{E_1}{hc}$$

Schrodinger's Hydrogen

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 - \frac{Ze^2}{4\pi\epsilon_0 r}\right)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

Above is the Schrodinger equation for 3-dimensions. Expanding in sperical coordinates leaves the following.

Coulomb Force

In this case the Coulomb force provides the centripetal force.

$$F_e = F_c$$

For an electron orbiting a proton the following applies.

$$\frac{k_e e^2}{r^2} = \frac{m_e v^2}{r}$$

This is useful to parameterize the kinetic energy as a function of radius.

Energy

Consider the total energy of the hydrogen atom.

$$E = KE + PE$$

$$E = \frac{m_e v^2}{2} - \frac{k_e e^2}{r}$$

Substituting yields a function for the total energy as a function of r or of v.

$$E = -\frac{k_e e^2}{2r} = -\frac{m_e v^2}{2}$$

This is used to derive an expression for the radius of orbit r as a function of v.

Quantum Stability

Experiments by Rutherford in 1909 showed the structure of the atom be a dense, positive nucleus with a light, negative charge orbiting around it. This immediately caused problems on how such a system could be stable. Classical electromagnetism had shown that any accelerating charge radiates energy described through the Larmor formula. If the electron is assumed to orbit in a perfect circle and radiates energy continuously, the electron would spiral into the nucleus.

$$P = \frac{2}{3}\frac{q^2 a^2}{4\pi\varepsilon_0 c^3} = \frac{q^2 a^2}{6\pi\varepsilon_0 c^3} \text{ (SI units)}$$

$$t_{\text{fall}} \approx \frac{a_0^3}{4r_0^2 c} \approx 1.6 \cdot 10^{-11}\text{s}$$

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]-\frac{Ze^2}{4\pi\epsilon_0 r}\psi=E\psi$$

The solution of the Schrödinger equation wave equation for the hydrogen atom is written as follows.

$$\psi_{n\ell m}(r,\vartheta,\varphi)=\sqrt{\left(\frac{2}{na_0}\right)^3\frac{(n-\ell-1)!}{2n[(n+\ell)!]}}e^{-\rho/2}\rho^\ell L_{n-\ell-1}^{2\ell+1}(\rho)Y_\ell^m(\vartheta,\varphi)$$

$$\rho=\frac{2r}{na_0}$$

$L_{n-\ell-1}^{2\ell+1}(\rho)$ is the generalized Laguerre polynomial of degree $n-\ell-1$. Here is a closed form formula.

$$L_n^\alpha(x)=\sum_{i=0}^{n}(-1)^i\binom{n+\alpha}{n-i}\frac{x^i}{i!}$$

$Y_\ell^m(\vartheta,\varphi)$ is the is a spherical harmonic function of degree ℓ and order m. Here is a closed form formula.

$$Y_\ell^m(\theta,\varphi)=\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\varphi}$$

Here $P_\ell^m(x)$ is the associated Legendre polynomial of degree ℓ and order m. Here is the Rodriguez formula.

$$P_\ell^m(x)=\frac{(-1)^m}{2^\ell\ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

The energy eigenstates may be classified by two angular momentum quantum numbers, ℓ and m (both are integers). The angular momentum quantum number $\ell=0,1,2,...$ determines the magnitude of the angular momentum. The magnetic quantum number $m=-\ell,...,+\ell$ determines the projection of the angular momentum on the (arbitrarily chosen) z-axis.

The Schrodinger hydrogen is a probability distribution. The ground state is spherically symmetric, in fact all $\ell=0$ states are symmetric. The average radial position of these probability distribution functions corresponds to the quantized radii of the Bohr atom.

$$\langle r\rangle_n=n^2a_0$$

Note the $\ell=0$ states correspond to zero angular momentum. Compare this to the Bohr model ground state.

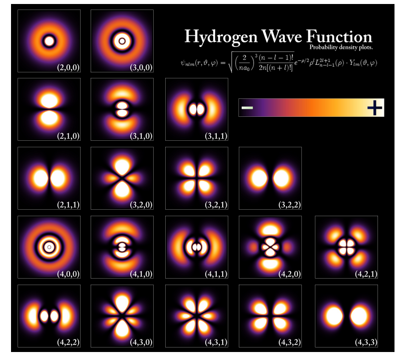

Figure 42: Hydrogen wave function

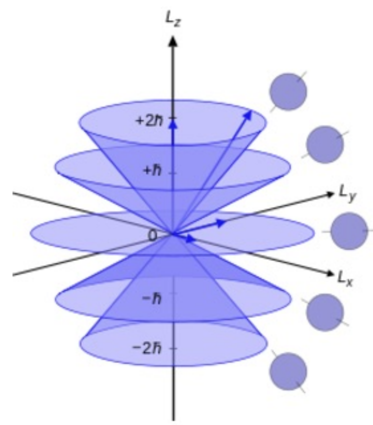

Figure 43: Quantum numbers ℓ and m

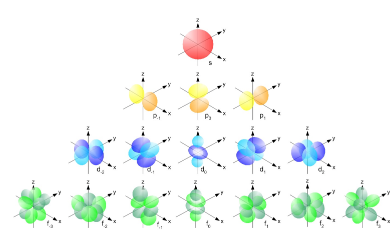

Figure 44: Spherical harmonics

Harmonic Oscillator

The quantum harmonic oscillator is the quantum-mechanical analog of the classical harmonic oscillator. Because an arbitrary potential can usually be approximated as a harmonic potential at the vicinity of a stable equilibrium point, it is one of the most important model systems in quantum mechanics. Furthermore, it is one of the few quantum-mechanical systems for which an exact, analytical solution is known.

$$\frac{d^2}{dx^2}\psi(x) = -\frac{2m}{\hbar^2}\left[E - \frac{m\omega^2 x^2}{2}\right]\psi(x) = -\left[\frac{2mE}{\hbar^2} - \left(\frac{m\omega}{\hbar}\right)^2 x^2\right]\psi(x)$$

Quantized Energy States

$$E_{n+1} - E_n = \hbar\omega$$

From the ground state it is possible to ladder up the states. This is the general solution of the wavefunction.

$$\psi_n(x) = \frac{1}{\sqrt{2^n\, n!}} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot e^{-\frac{m\omega x^2}{2\hbar}} \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \qquad n = 0, 1, 2, \ldots$$

Where H_n are Hermite polynomials. The Rodriquez formula is written below.

$$H_n(z) = (-1)^n\, e^{z^2} \frac{d^n}{dz^n}\left(e^{-z^2}\right)$$

The quantized energy states are written as follows.

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) = (2n+1)\frac{\hbar}{2}\omega$$

This energy spectrum is noteworthy for three reasons. First, the energies are quantized, meaning that only discrete energy values. Second, these discrete energy levels are equally spaced, unlike in the Bohr model of the atom, or the particle in a box. Third, the lowest achievable energy (the energy of the n = 0 state, called the ground state) is not equal to the minimum of the potential well, but $\hbar\omega/2$ above it; this is called zero-point energy. Because of the zero-point energy, the position and momentum of the oscillator in the ground state are not fixed (as they would be in a classical oscillator), but have a small range of variance, in accordance with the Heisenberg uncertainty principle. This zero-point energy further has important implications in quantum field theory and quantum gravity.

$$V(x) = \frac{kx^2}{2} = \frac{m\omega^2 x^2}{2}$$
$$E = \frac{m\omega^2 A^2}{2}$$

Ground State Wavefunction

$$\psi_0(x) = Be^{-Cx^2}$$

$$C = \frac{m\omega}{2\hbar} \qquad E_0 = \frac{\hbar\omega}{2}$$

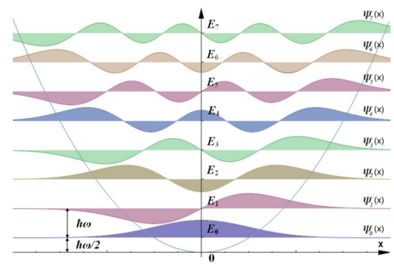

Figure 45: Quantized harmonic oscillator

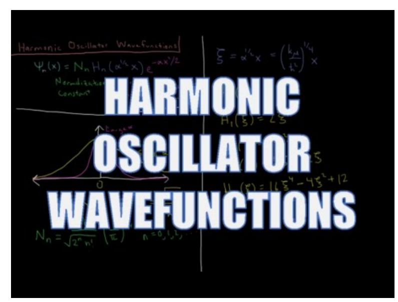

Figure 46: Harmonic oscillator wavefunctions

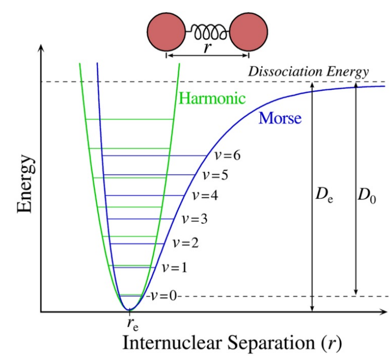

Figure 47: Harmonic approximation