I THINK THAT MODERN PHYSICS HAS DEFINITELY DECIDED IN FAVOR OF PLATO. IN FACT THE SMALLEST UNITS OF MATTER ARE NOT PHYSICAL OBJECTS IN THE ORDINARY SENSE; THEY ARE FORMS, IDEAS WHICH CAN BE EXPRESSED UNAMBIGUOUSLY ONLY IN MATHEMATICAL LANGUAGE.
WERNER HEISENBERG

THE ONLY SHIBBOLETH THE WEST HAS IS SCIENCE. IT IS THE PREMISE OF MODERNITY AND IT DEFINES ITSELF AS A RATIONALITY CAPABLE OF, INDEED REQUIRING SEPARATION FROM POLITICS, RELIGION AND REALLY, SOCIETY. MODERNISATION IS TO WORK TOWARDS THIS. . . IF ONE LOOKS AT THE WORKS OF NEWTON TO EINSTEIN, THEY WERE NEVER SCIENTISTS IN THE WAY MODERNITY UNDERSTANDS THE TERM.
BRUNO LATOUR

THE BOUNDARY BETWEEN SCIENCE FICTION AND SOCIAL REALITY IS AN OPTICAL ILLUSION.
DONNA HARAWAY

DR. DOEG

INTRODUCTION TO MECHANICS

THE INVISIBLE COLLEGE

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Dedicated to the ghosts of the college and

the spectral bodies of physics.

Note

This physics text is an OpenSource academic project developed in abstraction by the Invisible College. The manuscript is written in LaTeX and makes use of the `tufte-book` and `tufte-handout` document classes.

http://latex-project.org/ftp.html
https://git-scm.com/downloads

Introduction

The limits of my language means the limits of my world.
-Ludwig Wittgenstein

Physics is embodied through the process of measurement. Within physics any meaningful statement relates to some observable quantity. The process of observation requires measurement using physical instrumentation. In this way, physics is not only about the world but made of the world.

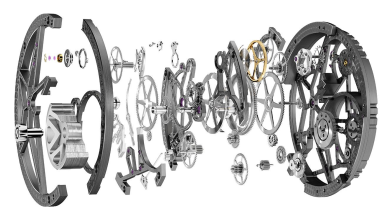


Figure 1: Exploded clock assembly

$$E = hf$$

Figure 2: Mathematical equation relating the energy and oscillatory frequency of a radiant particle.

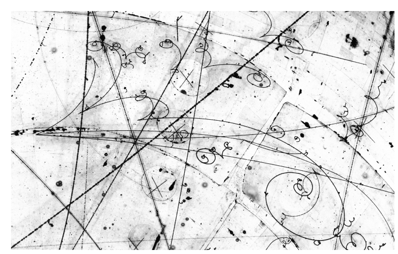


Figure 3: Bubble chamber image of a neutrino particle interaction event from the Fermi National Accelerator Laboratory.

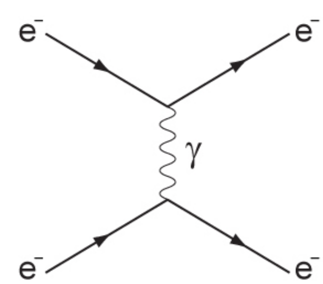


Figure 4: Feynman diagram representing photon mediated electron-electron scattering.

Physics is symbolic in the application of mathematics to the working world. Observable quantities are represented by algebraic variables and . Once quantities are abstracted as mathematical variables, physics becomes a language game.

Physics is visual in both the measurement and the representation.

Physics is social. In the end mathematics and physical observation are human activities. As is Soylent Green, physics too is made of people. People! This does not mean we can not gain insights beyond ourselves. Hopefully it means physicists will have jobs.

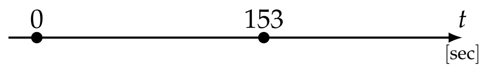
Quantities

A physical quantity is a physical property of a phenomenon, body, or substance, that can be quantified by measurement. We can represent a physical quantity using an algebraic variable. A physical quantity requires a value and unit of measure.

$$t = \overbrace{153}^{\textit{Value}} \ \overbrace{\text{seconds}}^{\textit{Unit}}$$

Consider the physical quantity, time. In the above equation we use t as the algebraic variable, seconds as the unit of measure and 153 as the value or magnitude of that unit.

So Simon Peter climbed back into the boat and dragged the net ashore. It was full of large fish, 153, but even with so many the net was not torn. **- John 21:11**



The same information in the above equation may be encoded graphically using an appropriately labeled number line. Here the quantity's magnitude may be represented a length measured by a point and the origin.

The **accuracy** of a measurement system is the degree of closeness of measurements of a quantity to that quantity's true value. The **precision** of a measurement system, related to reproducibility and repeatability, is the degree to which repeated measurements under unchanged conditions show the same results.

Scientific Notation

Scientific notation is a way of writing numbers that are too big or too small to be conveniently written in decimal form. In scientific notation the numerical value 153 would be written as 1.53×10^2.

$$153 = \overbrace{1.53}^{\text{significant figures}} \times \overbrace{10^2}^{\text{order of magnitude}}$$

In this format only one non-zero digit is to the left of the decimal place and to the right of the decimal place go the remaining significant figures. The order of magnitude is the exponent power of ten. In this example there are three significant figures and an order of magnitude of two. Scientific notation enables simpler order-of-magnitude comparisons.

Length	Meters
Observable Universe	10^{26}
Milky Way	10^{21}
Solar System	10^{13}
Earth to Sun	10^{11}
Earth	10^{7}
Football Field	10^{2}
Human	10^{0}
Cell	10^{-5}
Hydrogen Atom	10^{-10}
Proton	10^{-15}

Table 1: A list of length scales.

Approximation

MAKING ORDER OF MAGNITUDE APPROXIMATIONS involves using these 10^n representations. Consider for example a deep breathing yogi. How many breaths do they take per day? The time scale for a breath is on the order of 10 or (10^1) seconds while a day is on the order of 10^5 seconds. Therefore the number of breaths per day would be on the order of 10^4, or ten thousand breaths.

$$\frac{10^5}{10^1} = 10^{5-1} = 10^4$$

Said another way, the time for the rotation of the earth is four orders of magnitude greater than the cycle of the breath.

Time	Seconds
Age Universe	10^{17}
Age Earth	10^{17}
Life on Earth	10^{17}
Year	10^{7}
Month	10^{6}
Day	10^{5}
Sunlight to Earth	10^{2}
Heartbeat	10^{0}
Audible Latency	10^{-2}
Pion Lifetime	10^{-8}

Table 2: A list of time scales.

Units and Dimensions

meter The meter is the length of the path travelled by light in vacuum during a tim interval of $1/299,792,458$ of a second.

kilogram The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram. This international prototype is made of platinum-iridium and is kept at the International Bureau of Weights and Measures in France.

second The second is the duration of $9,192,631,770$ periods of the radiation corresponding to the transition between the two hyper fine levels of the ground state of the cesium-133 atom.

ampere The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed one meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length.

kelvin The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.

mole The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles or specified groups of such particle. In this definition, it is understood that the carbon 12 atoms are unbound, at rest and in their ground state.

candela The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of a frequency 540×10^{12} Hertz and has a radiant intensity in that direction of $1/683$ watt per steradian.

The *International System of Units* (**SI**) is the modern form of the metric system and is the world's most widely used system of measurement, used in both commerce and science. It comprises a coherent system of units of measurement built on seven base units. It defines twenty-two named units, and includes many more unnamed coherent derived units. The system also establishes a set of twenty prefixes to the unit names and unit symbols that may be used when specifying multiples and fractions of the units.

Prefix	Symbol	Value
yotta	Y	10^{24}
zetta	Z	10^{21}
exa	E	10^{18}
peta	P	10^{15}
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
hecto	h	10^{2}
deca	da	10^{1}
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}
zepto	z	10^{-21}
yocto	y	10^{-24}

Table 3: A list of metric prefixes.

Quantity	SI Unit	SI Symbol	Variable	Dimension
Fundamental				
distance	meter	m	l	$\mathbf{L}$
mass	kilogram	kg	m	$\mathbf{M}$
time	second	s	t	$\mathbf{T}$
electrical current	ampere	A	I	$\mathbf{I}$
temperature	kelvin	K	T	$\mathbf{\Theta}$
number particles	mole	mol	n	$\mathbf{N}$
luminous intensity	candela	cd	J	$\mathbf{J}$
Derived				
angle	radian	rad	θ	$\mathbf{1}$
frequency	hertz	Hz = $1/\mathrm{s}$	f	$\mathbf{T}^{-1}$
force	newton	N = $\mathrm{kg \cdot m/s^2}$	F	$\mathbf{MLT}^{-2}$
pressure	pascal	Pa = $\mathrm{N/m^2}$	P	$\mathbf{ML}^{-1}\mathbf{T}^{-2}$
energy	joule	J = N · m	E	$\mathbf{ML}^2\mathbf{T}^{-2}$
power	watt	W = $\mathrm{J/s}$	P	$\mathbf{ML}^2\mathbf{T}^{-3}$
electric charge	coulomb	C = A · s	q	$\mathbf{IT}$
electric potential	volt	V = $\mathrm{J/C}$	V	$\mathbf{ML}^2\mathbf{T}^{-3}\mathbf{I}^{-1}$
capacitance	farad	F = $\mathrm{C/V}$	C	$\mathbf{M}^{-1}\mathbf{L}^{-2}\mathbf{T}^4\mathbf{I}^2$
resistance	ohm	Ω = $\mathrm{V/A}$	R	$\mathbf{ML}^2\mathbf{T}^{-3}\mathbf{I}^{-2}$
magnetic field	tesla	T = $\mathrm{V \cdot s/m^2}$	B	$\mathbf{MT}^{-2}\mathbf{I}^{-1}$
inductance	henry	H = $\mathrm{V \cdot s/A}$	L	$\mathbf{ML}^2\mathbf{T}^{-2}\mathbf{I}^{-2}$
radioactivity	becquerel	Bq = $1/\mathrm{s}$	A	$\mathbf{T}^{-1}$

Table 4: A list of physical quantities with SI units and dimensions.

The concept of physical dimension was introduced by Joseph Fourier in 1822. Physical quantities that are commensurable have the same dimension; if they have different dimensions, they are incommensurable. For example, it is meaningless to ask whether a kilogram is less, the same, or more than an hour. Any physically meaningful equation will have the same dimensions on the left and right sides, a property known as "dimensional homogeneity".

ANY PHYSICAL QUANTITY Q is proportional to a product of fundamental quantities. The critical exponents always being integer values.

$$Q = C l^\alpha m^\beta t^\gamma I^\delta T^\epsilon n^\xi J^\eta$$

The dimension of the quantity, dim Q, is a product of the dimensions of the constituent quantity factors.

$$\dim Q = \mathbf{L}^\alpha \mathbf{M}^\beta \mathbf{T}^\gamma \mathbf{I}^\delta \mathbf{\Theta}^\epsilon \mathbf{N}^\xi \mathbf{J}^\eta$$

Here α, β, γ, δ, ϵ, ξ, η are all positive or negative integers

This constitutes what is called and Abelian group.

Algebra and Similitude A sum or difference of two commensurate quantities (having the same dimensions) is a physically meaningful expression.

$$402 \text{ meter} - 137 \text{ meter} = (402 - 137)\text{meter} = 265 \text{ meter}$$

The conversion of a quarter mile into meters.

$$0.25 \text{ mi} \underbrace{\left(\frac{1609 \text{ m}}{1 \text{ mi}}\right)}_{1} = \frac{0.25 \cdot 1609}{1} \frac{\cancel{\text{mi}}\ \text{m}}{\cancel{\text{mi}}}$$
$$0.25 \text{ mi} = 402 \text{ m}$$

A product or quotient of any quantities can be a physically meaningful expression.

$$\frac{265 \text{ meter}}{153 \text{ second}} = \frac{265}{153} \frac{\text{meter}}{\text{second}} = 1.73 \ \mathrm{m/s}$$

Unit Conversion The dimensions of a conversion factor are unity. Any physically meaningful equality can be turned into one of two conversion factors.

$$1 \text{ mile} = 1609 \text{ meters} \quad \Longleftrightarrow \quad \frac{1609 \text{ meters}}{1 \text{ mile}} = \frac{1 \text{ mile}}{1609 \text{ meters}} = 1$$

English $\Longleftrightarrow$ Metric
1 mile = 5280 feet = 1609 meters
1gallon = 3.785 liters = 4.000 quarts
1 cm^3 = 1 milliliter
1 hp = 746 Watts
1 lb = 4.45 Newtons

Table 5: English and Metric equivalences

Constants

A PHYSICAL CONSTANT IS a physical quantity that is generally believed to be both universal in nature and constant in time. It can be contrasted with a mathematical constant, which is a fixed numerical value, but does not directly involve any physical measurement.

Earth	Sun & Moon	Water
$g = 9.81 \text{m/s}^2$	$m_{sun} = 1.99 \times 10^{30}\text{kg}$	$c_{vapor} = 2.08 \times 10^3 \text{m/kg·K}$
$B_{earth} = 5.0 \times 10^{-5}\text{T}$	$R_{sun} = 6.96 \times 10^8\text{m}$	$c_{water} = 4.18 \times 10^3 \text{m/kg·K}$
$m_{earth} = 5.98 \times 10^{24}\text{kg}$	$m_{moon} = 7.36 \times 10^{22}\text{kg}$	$c_{ice} = 2.11 \times 10^3 \text{m/kg·K}$
$R_{earth} = 6.38 \times 10^6\text{m}$	$R_{moon} = 1.74 \times 10^6\text{m}$	$L_{fusion} = 3.3 \times 10^5 \text{J/kg}$
$r_{earth} = 1.50 \times 10^{11}\text{m}$	$r_{moon} = 3.84 \times 10^8\text{m}$	$L_{vapor} = 2.1 \times 10^6 \text{J/kg}$
$T_{earth} = 365.24$ days	$T_{moon} = 27.3$ days	$\rho_{water} = 1.00 \times 10^6 \text{kg/m}^3$

Table 6: A list of physical quantities with SI units and dimensions.

Two tables are given here with common constants. Above is a table of data solar system and thermodynamic properties of water.

Description	Symbol	Quantity
Gravitational Constant	G	$6.67 \times 10^{-11} \text{N·m}^2/\text{kg}^2$
Electrostatic Constant	k_e	$8.99 \times 10^9 \text{N·m}^2/\text{C}^2$
Boltzmann's Constant	k_B	$1.38 \times 10^{-23} \text{J/K}$
Avogado's Number	N_A	6.02×10^{23}
Plank's Constant	h	$6.63 \times 10^{-34} \text{J} \cdot \text{s}$
Speed of Light	c	$3.0 \times 10^8 \text{m/s}$
Fundamental Charge	e	$1.6 \times 10^{-19}\text{C}$
Mass of the Electron	m_e	$9.1 \times 10^{-31}\text{kg}$
Mass of Proton	m_p	$1.7 \times 10^{-27}\text{kg}$
Gas Constant	R	8.31J/mole·K
Vacuum Permativity	ε_0	$8.85 \times 10^{-12} \text{F/m}$
Vacuum Permeablity	μ_0	$4\pi \times 10^{-7} \text{T·m/A}$
Bohr Radius	a_0	$0.53 \times 10^{-10}\text{m}$
Fine Structure Constant	α	$1/137$

Table 7: A list of physical quantities with SI units and dimensions.

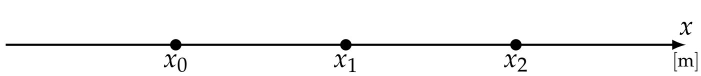
$$R = N_A k_B$$
$$k_e = 1/4\pi\varepsilon_0$$
$$\mu_0 c = 1/\varepsilon_0 c$$
$$\alpha = \frac{\mu_0 c e^2}{2h}$$
$$a_0 = \frac{\hbar^2}{m_e k_e e^2} = \frac{\hbar}{m_e\, c\, \alpha}$$

Sets, Sequences & Series

A set of N values of x may be ordered into a sequence using an index i running from 0 to $N-1$.

$$\{x_0, x_1, x_2 \ldots, x_{N-1}\} = \{x_i\}$$

They can represent various points on the number line.



A sum of values is known as a series. Series may be represented using sigma notation.

$$x_0 + x_1 + x_2 \ldots x_M = \sum_{i=0}^{M} x_i$$

A **set** is a collection without a sequential order while a **sequence** is an ordered set. An index subscript is used to order the sequence. A **series** is a summation of a sequence of numbers. Sigma notation is your friend in representing .

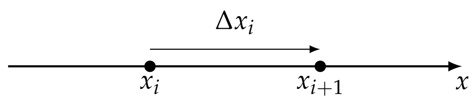
$\langle x \rangle$ is the mean, or average value of a set. σ is the uncertainty or standard deviation.

$$\langle x \rangle = \frac{x_0 + x_1 + x_2 \ldots}{N}$$
$$\langle x^2 \rangle = \frac{x_0^2 + x_1^2 + x_2^2 \ldots}{N}$$
$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

Difference

The delta operator Δ indicates difference. Δx is a change in x from $x = a$ to $x = b$. $\Delta x = b - a$, or indexed $\Delta x_i = x_{i+1} - x_i$.

$$\{\Delta x_i\} = \{x_1 - x_0, x_2 - x_1, \ldots, x_N - x_{N-1}\}$$



For any sequence x_i with N elements we can generate a sequence Δx_i of changes. This is a mapping of one sequence to another.

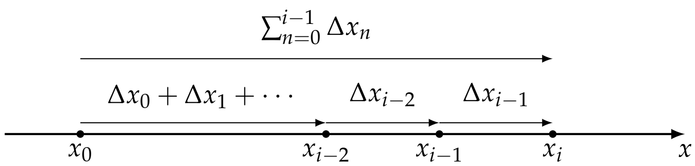
```
for (i=0,i<N-1,i=i+1)
    dx[i]=x[i+1]-x[i]
```

$$\{x_i\} \xrightarrow{\Delta} \{\Delta x_i\}$$

Accumulation

Accumulation is the inverse of difference. Rather than subtract successive terms we add them. It is the summation of a sequence as a series. The summation of Δx_i returns us to the sequence x_i.

$$\sum_{n=0}^{i-1} \Delta x_n = x_i - x_0$$



For any sequence Δx_i of changes we can generate a sequence x_i (up to a constant). This is the reverse mapping.

```
for (i=0,i<N-1,i=i+1)
    x[i+1]=x[i]+dx[i]
```

$$\{\Delta x_i\} \xrightarrow{\Sigma} \{x_i\}$$

Functions

Given a sequence of discrete ordered pairs $\{x_i, y_i\}$ we can graph points on a 2-dimensional graph using two number lines. We can model the relationship using a continuous function, $y = f(x)$.

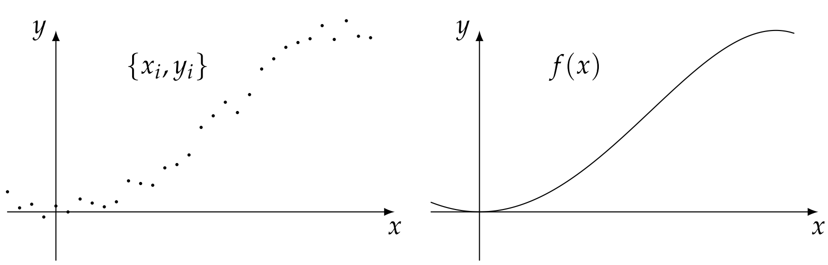
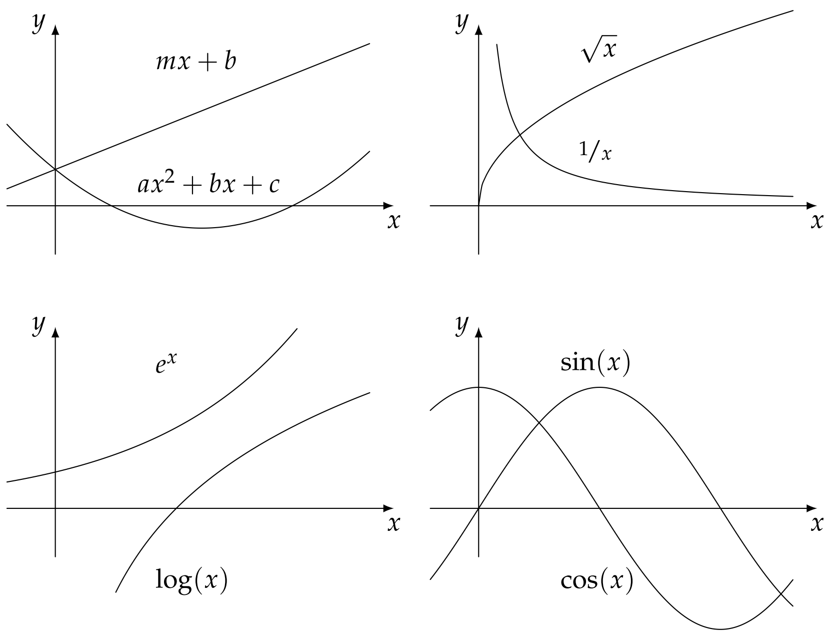


Figure 5: Ordered pairs $\{x_i, y_i\}$ and continuous function $f(x)$

Figure 6: Common functions



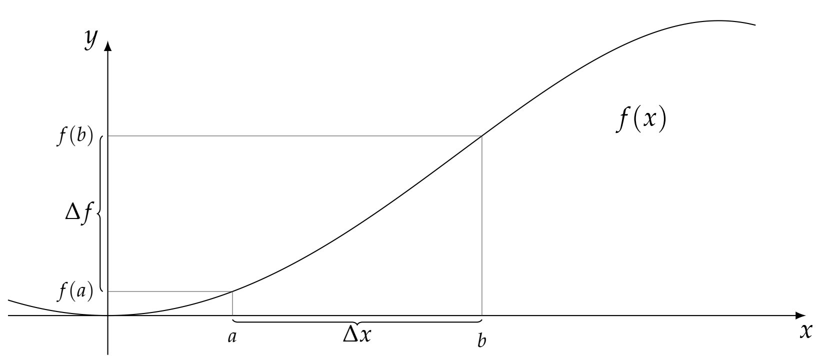
Function	Formula
Linear	$y = mx + b$
Quadratic	$y = ax^2 + bx + c$
Reciprocal	$y = \frac{1}{x}$
Power	$y = x^n$
Exponential	$y = a^x$
Logarithmic	$y = \log_a b$
Sine	$y = \sin x$
Cosine	$y = \cos x$
Tangent	$y = \tan x$
Arc sine	$y = \sin^{-1} x$
Arc cosine	$y = \cos^{-1} x$
Arc tangent	$y = \tan^{-1} x$

Table 8: A list of common functions

Changing Functions

Changes in the value of a function generate difference in $f(x)$. There is a relationship between Δf and Δx.

Figure 7: Difference on a continuous function $f(x)$



Difference is the object of a practical affirmation inseparable from essence and constitutive of existence. **- Gilles Deleuze**

$$\Delta f = f(b) - f(a)$$

$$\Delta f = f(a + \Delta x) - f(a)$$

Slope, Tangent Line & Slope Function

The **slope** is a ratio of difference between two points on the function graph. The construction of the right triangle with sides Δx and Δy yields a tangent which is equivalent to the slope. In the limit of small Δ we call the slope a **tangent line**.

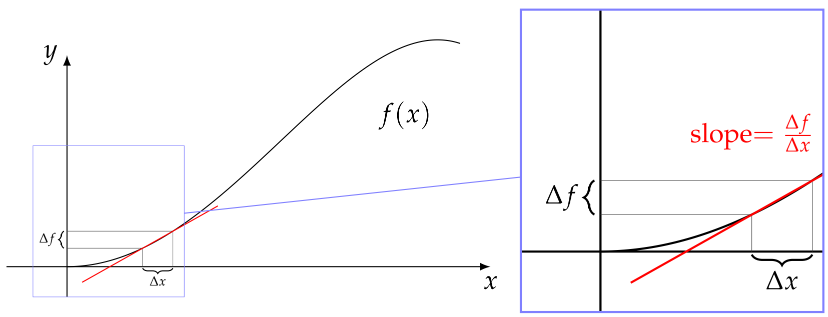
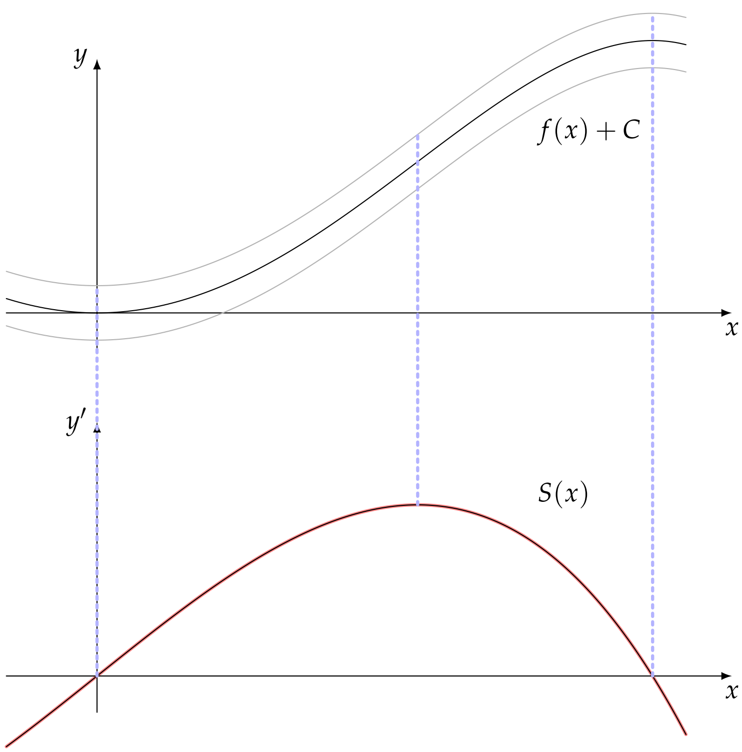


Figure 8: Tangent lines and the slope function S(x) for a function $f(x)$

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta f}{\Delta x}$$



This specific function $f(x)$ is

$$y = x^2 - \frac{x^4}{12}$$

$$S(x) = \text{slope of tangent line} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

This specific slope function $S(x)$ is

$$y = 2x - \frac{x^3}{3}$$

The **slope function** $S(x)$ represents the slope of the tangent line at any point x.

Accumulation of a Slope Function

Over some range Δx the area beneath the slope function represents some accumulation of $S(x)$. The total area represents Δf over the range Δx.

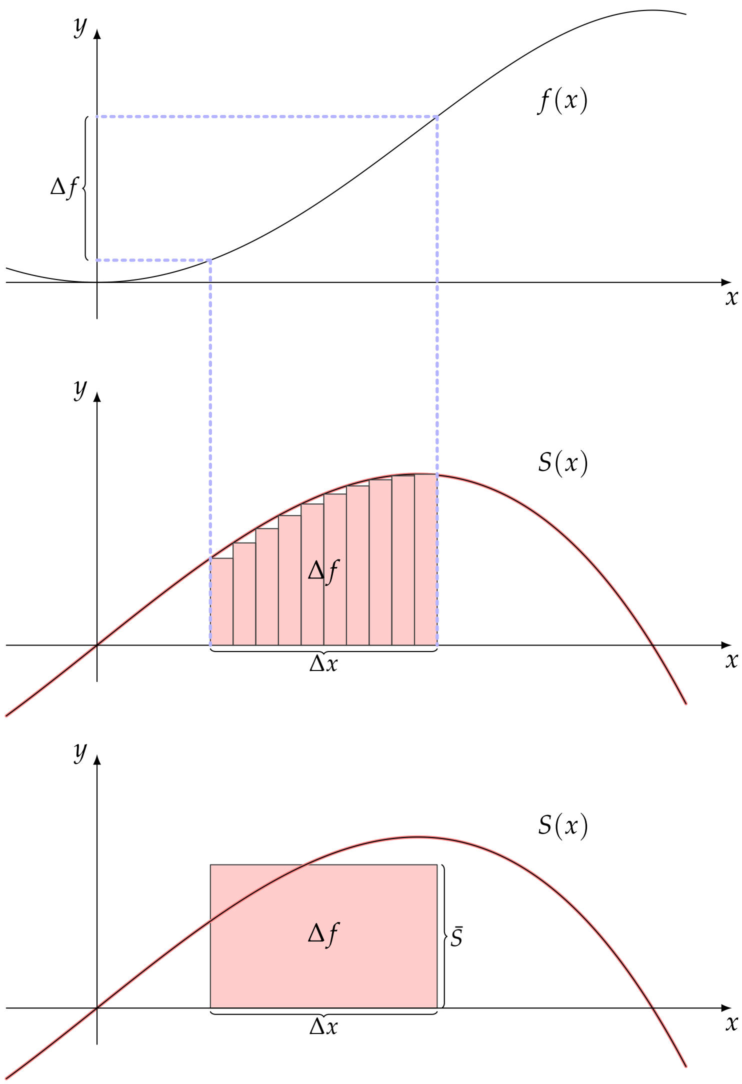


Figure 9: Area underneath a slope function $S(x)$

$$\bar{S} = \frac{\Delta f}{\Delta x}$$

$$\Delta f = \bar{S} \cdot \Delta x$$

If this area representing Δf is represented as a rectangle with base Δx the height of the rectangle is the average value $\bar{S}$.

Scalars and Vectors

A **scalar** is a one-dimensional physical quantity, i.e. one that can be described by a single real number (signed, with units). It is a physical quantity that only has magnitude but no direction. A **vector** is a geometric object that has magnitude (or length) and direction and can be added to other vectors according to vector algebra. A vector can be represented by a set of scalars.

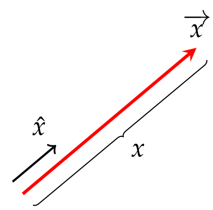
$\overrightarrow{x}$ is a vector. x is the magnitude (or length) of the vector $\overrightarrow{x}$. The vector $\hat{x}$ is the directional vector of $\overrightarrow{x}$. It has a magnitude of 1 rand points in the same direction as $\overrightarrow{x}$.

$$\overrightarrow{x} = x\hat{x}$$

$$|\overrightarrow{x}| = x$$

$$\hat{x} = \frac{\overrightarrow{x}}{x}$$

$$|\hat{x}| = 1$$



Spatial Coordinate Systems

2D Cartesian (x, y)

In two dimensions we use the vector representation $\overrightarrow{r}$ with components x and y. This can be represented as a column vector or using the unit vectors $\hat{x}$ and $\hat{y}$. The magnitude of the vector $\overrightarrow{r}$ is r. We can calculate r given the x and y components using the Pythagorean theorem.

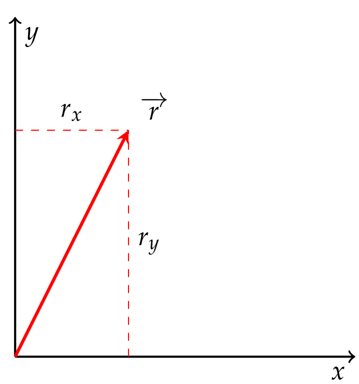
x (or r_x) and y (or r_y) are called orthogonal components of $\overrightarrow{r}$ meaning $\hat{x}$ and $\hat{y}$ do not overlap at all, namely they meet at 90 degrees.

$$\overrightarrow{r} = x\hat{x} + y\hat{y}$$

$$\overrightarrow{r} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$r = \sqrt{x^2 + y^2}$$

$$\hat{r} = \begin{pmatrix} x/\sqrt{x^2+y^2} \\ y/\sqrt{x^2+y^2} \end{pmatrix}$$



Polar Coordinates (r, θ)

In polar coordinates we parameterize the vector $\overrightarrow{r}$ in terms of its length and direction rather than in terms of the x and y components. The length of the vector $\overrightarrow{r}$ is r and the direction is expressed in terms of θ. The vector $\overrightarrow{r}$ makes an angle θ with the unit vector $\hat{x}$. We can easily convert between polar coordinates (r, θ) and 2-D cartesian coordinates (x, y).

$$\overrightarrow{r} = r\hat{r}$$

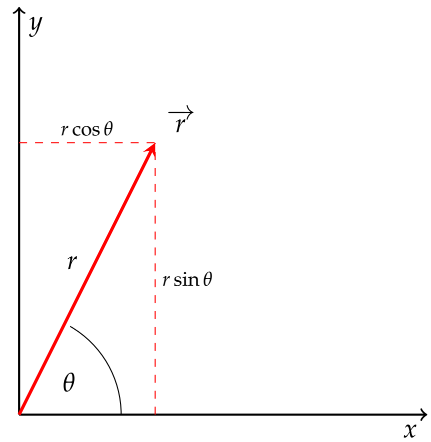
$$r = \sqrt{x^2 + y^2}$$

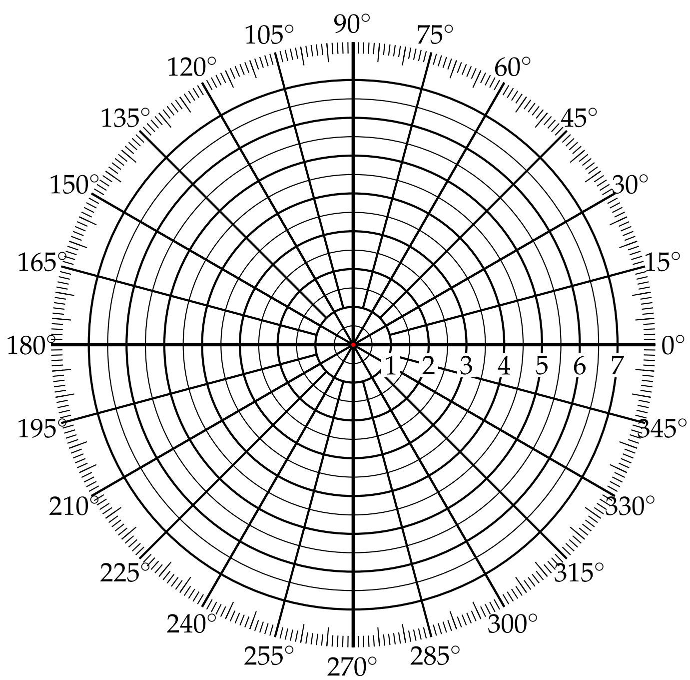
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\hat{r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$





In mathematics, the polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction.

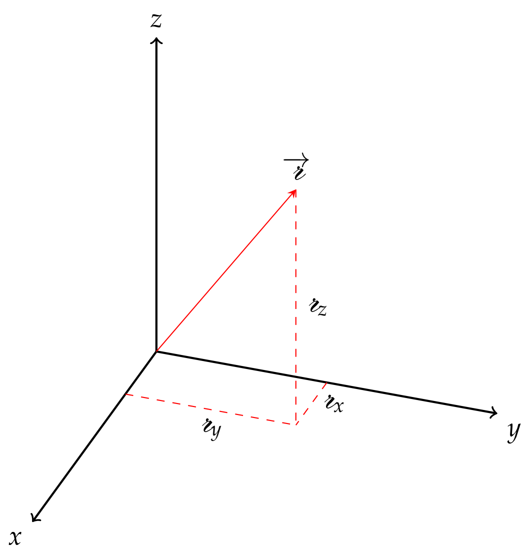
The reference point (analogous to the origin of a Cartesian system) is called the pole, and the ray from the pole in the reference direction is the polar axis. The distance from the pole is called the radial coordinate or radius, and the angle is the angular coordinate, polar angle, or azimuth.

The concepts of angle and radius were already used by ancient peoples of the 1st millennium BC. The Greek astronomer and astrologer Hipparchus (190-120 BC) created a table of chord functions giving the length of the chord for each angle, and there are references to his using polar coordinates in establishing stellar positions. In On Spirals, Archimedes describes the Archimedean spiral, a function whose radius depends on the angle. The Greek work, however, did not extend to a full coordinate system.

3D Cartesian (x, y, z)

$$\vec{\mathscr{r}} = \vec{x} + \vec{y} + \vec{z} = x\hat{x} + y\hat{y} + z\hat{z} = \mathscr{r}_x\hat{x} + \mathscr{r}_y\hat{y} + \mathscr{r}_z\hat{z}$$

$$\vec{\mathscr{r}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathscr{r}_x \\ \mathscr{r}_y \\ \mathscr{r}_z \end{pmatrix}$$



In three dimensions a vector can be described by three orthogonal components (x, y, z). It is the 3D cartesian coordinate system. These form a right handed coordinate system so that if you look at the standard (x, y) plane the z-axis comes straight out of the page. The magnitude of $\vec{\mathscr{r}}$ is determined using the Pythagorean Theorem (in 3D).

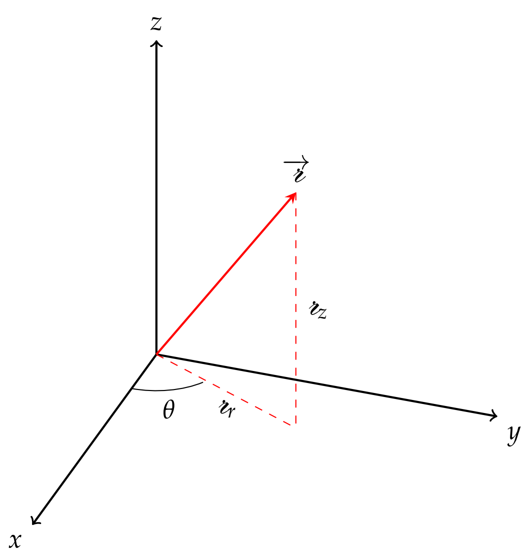
$$\mathscr{r} = \sqrt{x^2 + y^2 + z^2}$$

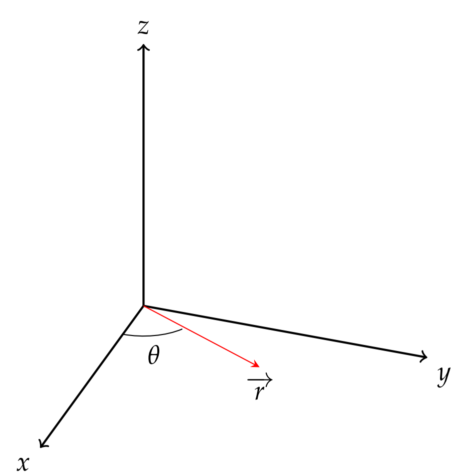
The concept of Cartesian coordinates generalizes to allow axes that are not perpendicular to each other, and/or different units along each axis. In that case, each coordinate is obtained by projecting the point onto one axis along a direction that is parallel to the other axis (or, in general, to the hyperplane defined by all the other axes). In such an oblique coordinate system the computations of distances and angles must be modified from that in standard Cartesian systems, and many standard formulas (such as the Pythagorean formula for the distance) do not hold.

We use the $\vec{\mathscr{r}}$ to notate the vector in 3D and reserve $\vec{r}$ to represent the component in the xy-plane. In cylindrical coordinates the vector $\vec{\mathscr{r}}$ is represented using polar coordinates for $\vec{r}$ and an orthogonal z-component. Converting between cylindrical and 3D cartesian coordinates is identical to the polar and 2D cartesian conversion.

Cylindrical (r, θ, z)

$$\vec{\mathscr{r}} = \vec{r} + \vec{z} = r\hat{r} + z\hat{z} = \mathscr{r}_r\hat{r} + \mathscr{r}_z\hat{z}$$





Spherical $(\mathscr{r}, \theta, \phi)$

In a spherical coordinate system we represent the 3D vector $\vec{\mathscr{r}}$ in terms of its magnitude (length) $\mathscr{r}$, azimuthal angle θ and polar angle ϕ. The conversion between 3D cartesian or cylindrical coordinates and spherical coordinates are given.

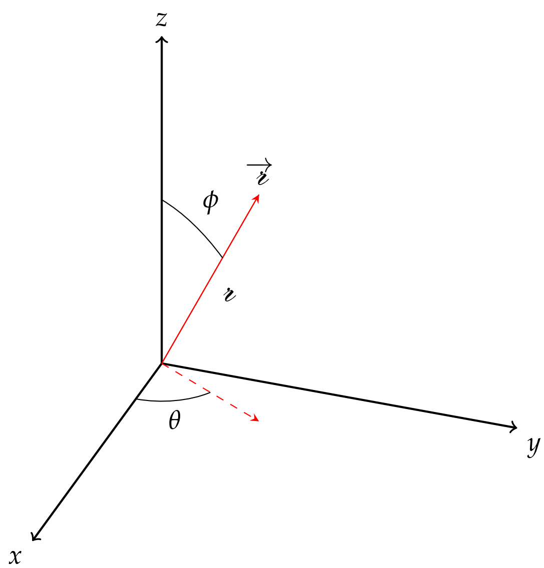
$$\vec{\mathscr{r}} = \mathscr{r}\,\hat{\mathscr{r}}$$

$$\hat{\mathscr{r}} = \begin{pmatrix} \sin\phi\ \cos\theta \\ \sin\phi\ \sin\theta \\ \cos\phi \end{pmatrix}$$

$$\mathscr{r} = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\phi = \tan^{-1}\left(\frac{r}{z}\right)$$



A spherical coordinate system is a coordinate system for three-dimensional space where the position of a point is specified by three numbers: the radial distance of that point from a fixed origin, its polar angle measured from a fixed zenith direction, and the azimuth angle of its orthogonal projection on a reference plane that passes through the origin and is orthogonal to the zenith, measured from a fixed reference direction on that plane.

The radial distance is also called the radius or radial coordinate. The polar angle may be called co-latitude, zenith angle, normal angle, or inclination angle.

The mathematics used to describe electron distributions around atoms uses spherical coordinates. The symmetry features of this mathematics gives rise to the structure of the periodic table.

$$r = \mathscr{r}\,\sin\phi$$

$$z = \mathscr{r}\,\cos\phi$$

$$x = \mathscr{r}\,\sin\phi\ \cos\theta$$

$$y = \mathscr{r}\,\sin\phi\ \sin\theta$$

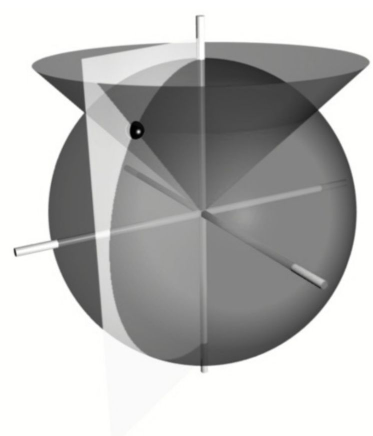


Figure 10: Spherical coordinates

Vector Operations

Vector Scaling

$$a\overrightarrow{p} = a\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} ap_x \\ ap_y \\ ap_z \end{pmatrix}$$

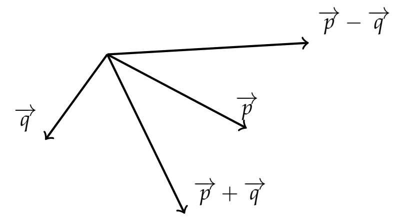


Figure 11: Vector Addition and Subtraction

Vector Addition

$$\overrightarrow{p} + \overrightarrow{q} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} + \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = \begin{pmatrix} p_x + q_x \\ p_y + q_y \\ p_z + q_z \end{pmatrix}$$

Dot Product

The dot product, or scalar product, or inner product, is an algebraic operation that takes two equal-length sequences of numbers (usually coordinate vectors) and returns a single number. This operation can be defined either algebraically or geometrically.

$$\overrightarrow{p} \cdot \overrightarrow{q} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \cdot \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = p_x q_x + p_y q_y + p_z q_z$$

$$|\overrightarrow{p}| = p = \sqrt{\overrightarrow{p} \cdot \overrightarrow{p}} = \sqrt{p_x^2 + p_y^2 + p_z^2}$$
$$\overrightarrow{p} \cdot \overrightarrow{q} = pq\cos\gamma$$

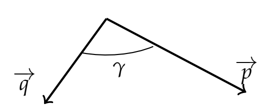


Figure 12: Dot product is proportional to the overlap of the two vectors

Cross Product

The cross product, or vector product, $\overrightarrow{p} \times \overrightarrow{q}$ is the very small displacement caused by rotation of the vector $\overrightarrow{p}$ around the vector $\overrightarrow{q}$ by an angle q. It yields a vector perpendicular to both $\overrightarrow{p}$ and $\overrightarrow{q}$.

$$\overrightarrow{p} \times \overrightarrow{q} = \begin{pmatrix} p_y q_z - p_z q_y \\ p_z q_x - p_x q_z \\ p_x q_y - p_y q_x \end{pmatrix}$$

$$|\overrightarrow{p} \times \overrightarrow{q}| = pq\sin\gamma$$
$$\overrightarrow{p} \times \overrightarrow{q} = -\overrightarrow{q} \times \overrightarrow{p}$$
$$\hat{x} \times \hat{x} = \hat{y} \times \hat{y} = \hat{y} \times \hat{y} = 0$$
$$\hat{x} \times \hat{y} = \hat{z} \qquad \hat{y} \times \hat{x} = \hat{x} \qquad \hat{z} \times \hat{x} = \hat{y}$$

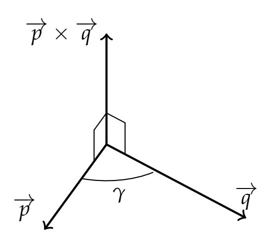


Figure 13: Cross product is proportional to the perpendicularity of the two vectors

Translational Kinematics

I can calculate the movement of the stars, but not the madness of men.
-Isaac Newton

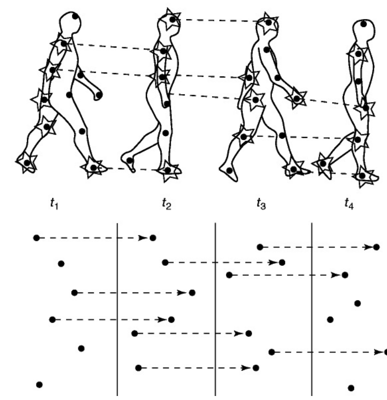


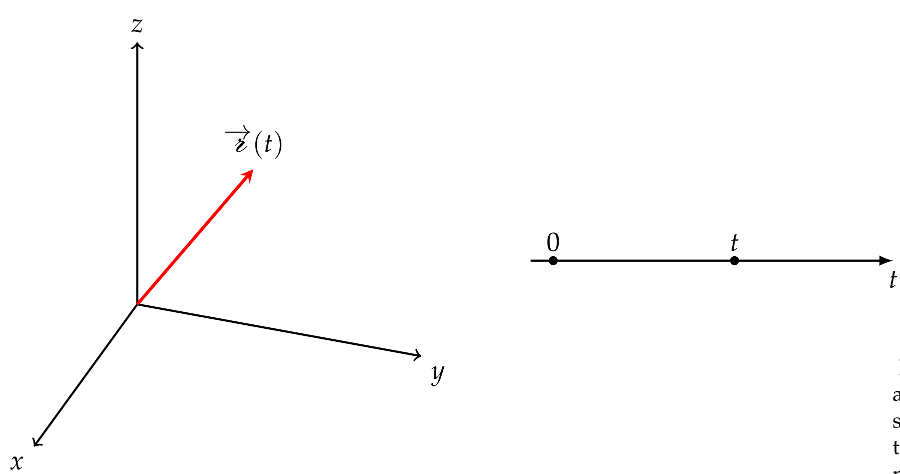
Figure 14: Time series of human motion

Translational Motion in Space and Time

Translation is the movement associated with a point particle. An extended body can have rotational motion and deformations. When a particle moves through space we describe its motion by collecting ordered pairs of position vectors and times. This is known as a time series. Algebraically this corresponds to and x,y,z and t value. Geometrically this is equivalent to a point in space and a point in time. These pairs can be considered part of a continuos function.

$$\{\vec{r}, t\} \ni \vec{r}(t)$$

Figure 15: Geometric position and time



In Euclidean geometry, a translation is a function that moves every point a constant distance in a specified direction. A translation can be described as a rigid motion: other rigid motions include rotations and reflections. A translation can also be interpreted as the addition of a constant vector to every point, or as shifting the origin of the coordinate system.

Displacement and Time Lapse

Applying the delta operator Δ to spatial vector and time series yields the displacement vector and elapsed time. The vector difference between points in space is known as displacement. In the temporal dimension the difference between points represents time elapsed.

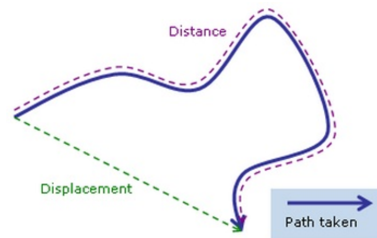


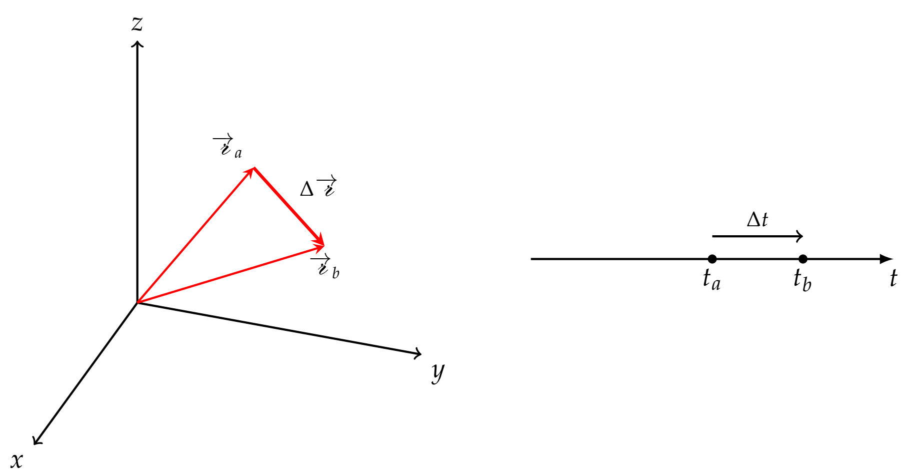
Figure 16: Displacement, path and distance traveled

$$\Delta \overrightarrow{\mathscr{r}} = \overrightarrow{\mathscr{r}}_b - \overrightarrow{\mathscr{r}}_a$$

$$\Delta t = t_b - t_a$$

A displacement is the shortest distance from the initial to the final position of a point. Thus, it is the length of an imaginary straight path, typically distinct from the path actually travelled. A displacement vector represents the length and direction of this imaginary straight path. A displacement may be also described as a ’relative position’: the final position of a point relative to its initial position, and a displacement vector can be mathematically defined as the difference between the final and initial position vectors.

Figure 17: Geometric displacement and elapsed time



Algebraically this corresponds to and $(\Delta x, \Delta y, \Delta z)$ and t value. Geometrically this is equivalent to a vector in space and a elapse of time.

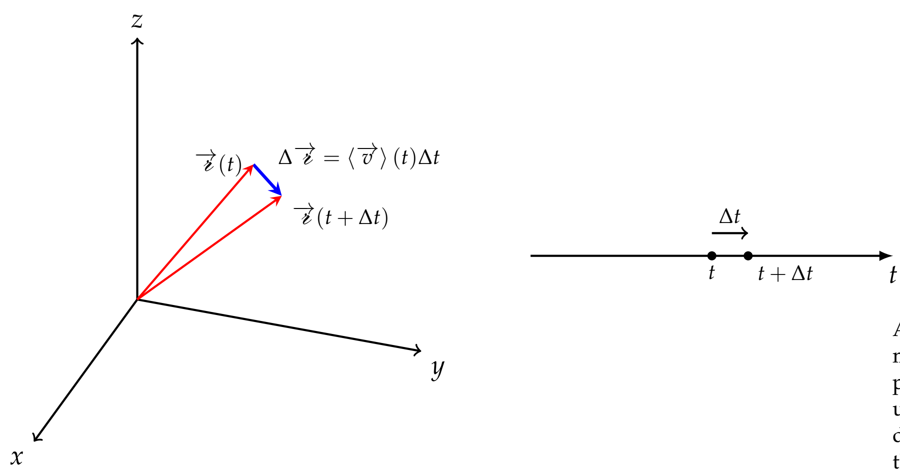
$$\Delta \overrightarrow{\mathscr{r}} = \Delta x \hat{x} + \Delta y \hat{y} + \Delta z \hat{z} = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

Average Velocity

Average velocity is defined as the displacement vector divided by the elapsed time. It is a time rate vector pointing in the direction of the displacement.

$$\bar{\vec{v}} = < \vec{v} > = \frac{\Delta \vec{r}}{\Delta t}$$

Figure 18: Geometric displacement as average velocity times elapsed time



An instant, $\lim \Delta t \to 0$, is an infinitesimal moment in time, a moment whose passage is instantaneous. The continuous nature of time and its infinite divisibility was addressed by Aristotle in his Physics, where he wrote on Zeno's paradoxes. Scientists, philosophers and artists still seek to define the exact nature of an instant thousands of years later.

In physics, a theoretical lower-bound unit of time called the Planck time has been proposed, that being the time required for light to travel a distance of 1 Planck length. The Planck time is theorized to be the smallest time measurement that will ever be possible, roughly 10^{-43} seconds. Within the framework of the laws of physics as they are understood today, for times less than one Planck time apart, we can neither measure nor detect any change. As of May 2010, the smallest time interval that was directly measured was on the order of 12 attoseconds (12×10^{-18} seconds), about 1024 times larger than the Planck time. It is therefore physically impossible, with current technology, to determine if any action exists that causes a reaction in "an instant", rather than a reaction occurring after an interval of time too short to observe or measure.

Position is described by a three dimensional vector space. Velocity too can be described by a three dimensional vector space. The dimensions of velocity space are distance over time. Since they are distant we should not technically draw a velocity vector on a position graph.

Instantaneous Velocity

Instantaneous velocity is defined as the average velocity in the limit as Δt and $\Delta\vec{r}$ go to zero. This is informally notated as $\Delta \to 0$.

$$\vec{v} = \lim_{\Delta \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

The instantaneous velocity vector $\vec{v}$ has a component in the x, y and z directions.

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

Change in Velocity

Applying the delta operator Δ to the velocity vector yields a change in velocity vector $\Delta \overrightarrow{v}$. The vector $\Delta \overrightarrow{v}$ is the difference between final velocity state and initial velocity state. Algebraically this is the final velocity vector $\overrightarrow{v}_b$ minus the initial velocity vector $\overrightarrow{v}_a$.

$$\Delta \overrightarrow{v} = \overrightarrow{v}_b - \overrightarrow{v}_a$$

Geometrically the change in velocity vector $\Delta \overrightarrow{v}$ represents the vector beginning at the point represented by $\overrightarrow{v}_a$ and ending at $\overrightarrow{v}_b$.

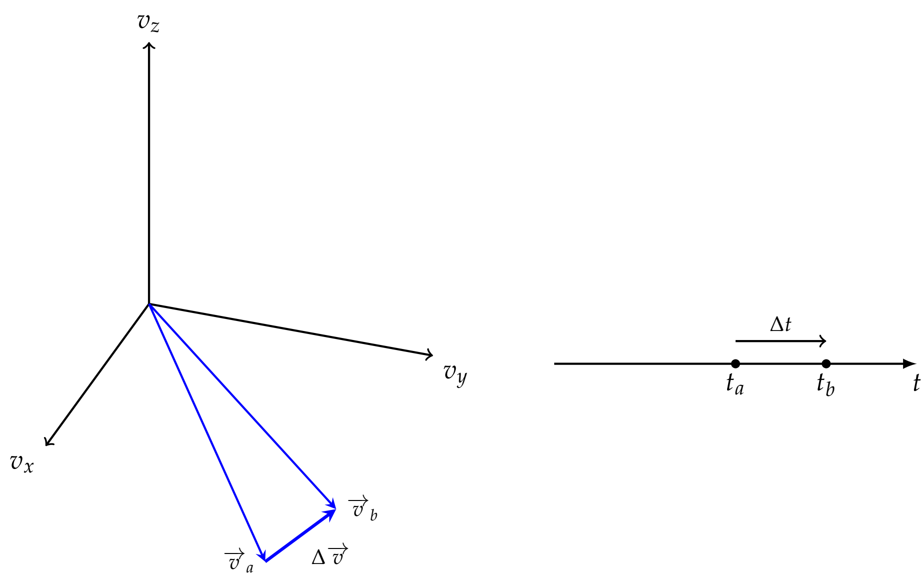


Figure 19: Geometric velocity and elapsed time

Average Acceleration

Average acceleration is defined as the change in velocity vector divided by the elapsed time. It is a time rate vector pointing in the direction of changing velocity.

$$\overline{\overrightarrow{a}} =< \overrightarrow{a} >= \frac{\Delta \overrightarrow{v}}{\Delta t}$$

Instantaneous Acceleration

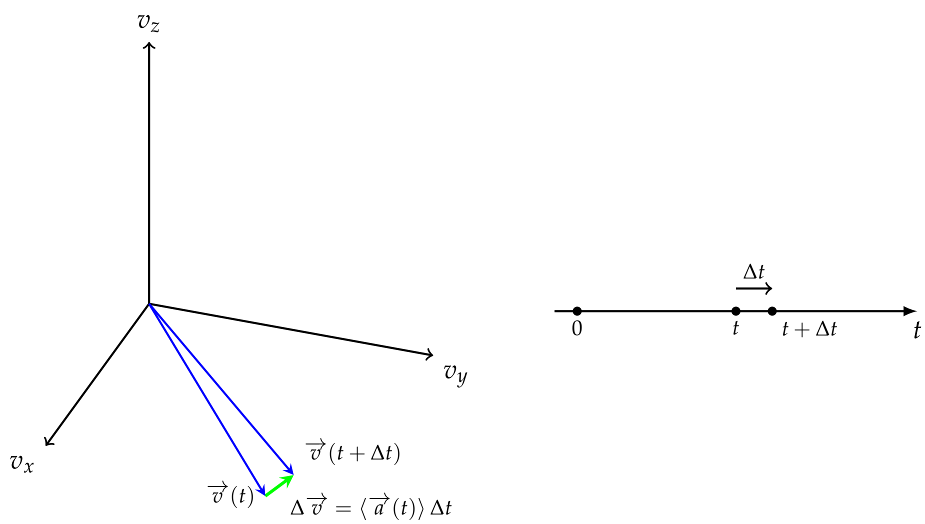
Instantaneous acceleration is defined as the average acceleration in the limit as Δt and $\Delta \overrightarrow{v}$ go to zero.

$$\overrightarrow{a} = \lim_{\Delta \to 0} \frac{\Delta \overrightarrow{v}}{\Delta t}$$

The instantaneous acceleration is a vector with an x, y, and z component.

$$\overrightarrow{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

Figure 20: Geometric position and time



In mathematics, infinitesimals are things so small that there is no way to measure them. The insight with exploiting infinitesimals was that entities could still retain certain specific properties, such as angle or slope, even though these entities were quantitatively small. The word infinitesimal comes from a 17th-century Modern Latin coinage infinitesimus, which originally referred to the "infinite-th" item in a sequence. It was originally introduced around 1670 by either Nicolaus Mercator or Gottfried Wilhelm Leibniz. Infinitesimals are a basic ingredient in the procedures of infinitesimal calculus as developed by Leibniz, including the law of continuity and the transcendental law of homogeneity. In common speech, an infinitesimal object is an object which is smaller than any feasible measurement, but not zero in size; or, so small that it cannot be distinguished from zero by any available means. Hence, when used as an adjective, "infinitesimal" means "extremely small". In order to give it a meaning it usually has to be compared to another infinitesimal object in the same context (as in a derivative). Infinitely many infinitesimals are summed to produce an integral.

Kinematic Difference: Slope and Curvature

Velocity

In one dimensional motion there is only one degree of freedom, represented by the variable x. When x is graphed against t in a position vs time graph the velocity is represented by the slope of the tangent line of $x(t)$.

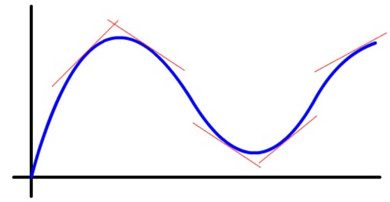


Figure 21: Tangent lines of a function

$$v = \text{Slope of tangent line on position vs time graph}$$

$$v = \lim_{\Delta \to 0} \frac{\Delta x}{\Delta t}$$

Acceleration

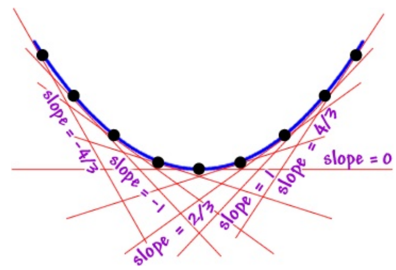


Figure 22: Curvature of a function

When v is graphed against t in a velocity vs time graph the acceleration is represented by the slope of the tangent line of $v(t)$. Acceleration is the time rate of change of the velocity.

$$a = \text{Slope of the tangent line on the velocity vs time graph}$$

$$a = \lim_{\Delta \to 0} \frac{\Delta v}{\Delta t}$$

The acceleration is the rate of change of the velocity and the velocity is the slope of $x(t)$. Therefore the acceleration is the rate of change of the slope, namely it is the curvature.

$$a = \text{Curvature on the position vs time graph}$$

$$a = \lim_{\Delta \to 0} \frac{\Delta \left({}^{\Delta x}/_{\Delta t} \right)}{\Delta t}$$

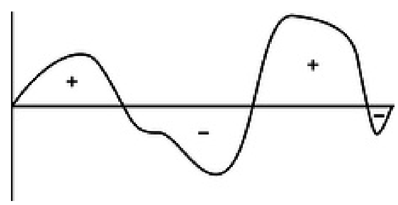


Figure 23: Area under a function

Kinematic Accumulation: Area Under Curve

Position

$$\Delta x = \text{Area under velocity vs time graph}$$

$$\Delta x = \langle v \rangle \Delta t$$

$$x(t) = x(0) + \Delta x$$

When v is graphed against t in a velocity vs time graph the displacement Δx is the area under the graph of $v(t)$. The displacement from the initial position $x(0)$ gives position as a function of time $x(t)$.

Velocity

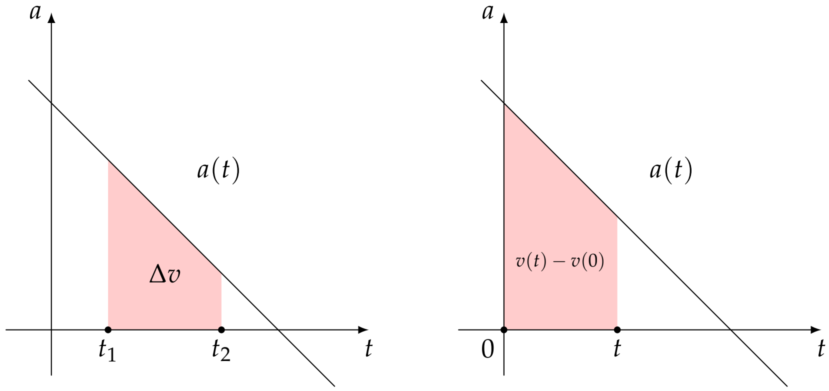
$$\Delta v = \text{Area under acceleration vs time graph}$$

$$\Delta v = \langle a \rangle \Delta t$$

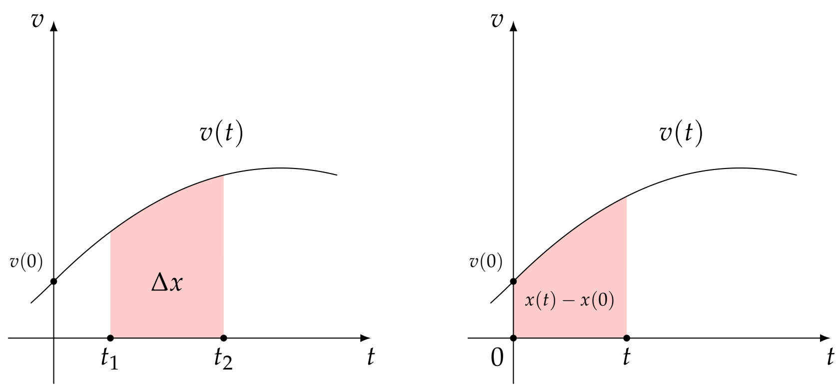
$$v(t) = v(0) + \Delta v$$

When a is graphed against t in an acceleration vs time graph the change in velocity Δv is the area under the graph of $a(t)$. The change from the initial velocity $v(0)$ gives velocity as a function of time $v(t)$.

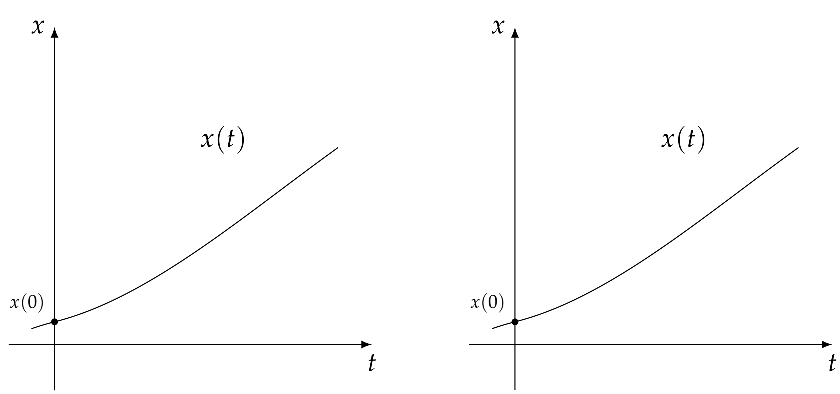
Graphical Representation



The slope of the tangent lines of the acceleration function is known as the jerk. The area under the acceleration function is the change in velocity. The change may be computed from the t=0 or an arbitrary time.



The slope of the tangent lines of the velocity function is the acceleration. The area under the velocity function is the change in position, or displacement. The change may be computed from the t=0 or an arbitrary time.



The slope of the tangent lines of the position function is the velocity. The area under the position function is not physically meaningful.

Simple 1-D Motion

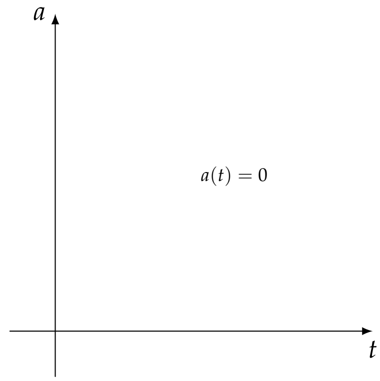
Constant Velocity

$$a(t) = \text{Slope}(v(t)) = 0$$

$$\Delta v = \text{Area under } a(t) = 0$$

$$v(t) = v(0) + \Delta v$$

$$v(t) = v(0)$$



The situation of constant velocity means it is not changing or its rate of change is zero. Constant velocity is zero acceleration. Zero acceleration yields no area under the curve. $\Delta v = 0$

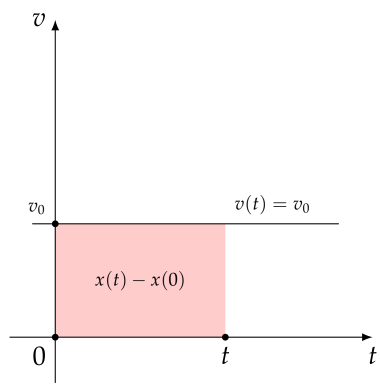
$$v(t) = v(0) = v_0 = \text{constant}$$

$$v(t) = \text{Slope}(x(t))$$

$$\Delta x = \text{Area under } v(t)$$

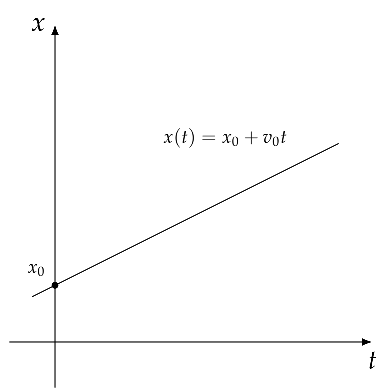
$$\Delta x = v_0 \Delta t$$

$$x(t) = x(0) + v_0 t$$



Since the velocity doesn't change $\Delta v = 0$ and the velocity stays as v_0, all day everyday.

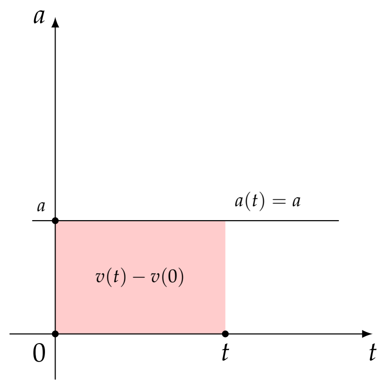
$$x(t) = x_0 + v_0 t$$



If the velocity is constant the slope of $x(t)$ is constant. It is a straight line with a slope v_0 and vertical intercept of x_0 . The curvature is zero.

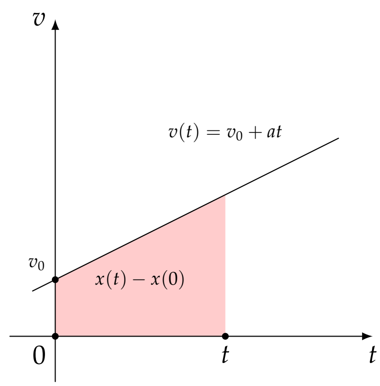
Constant Acceleration

$$\text{Slope}(a(t)) = 0$$
$$a(t) = a = \text{constant}$$
$$\Delta v = \text{Area under } a(t)$$
$$\Delta v = at$$
$$v(t) = v(0) + \Delta v$$
$$v(t) = v(0) + at$$



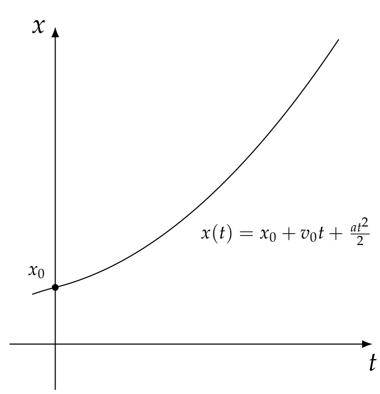
Constant acceleration is smooth, zero jerk. It generates a rectangular area beneath it representing the change in velocity. Area $=$ height $\times$ base so $\Delta v = at$. The acceleration is the average acceleration because it dent change. It stays constant a all day everyday.

$$v(t) = v_0 + at$$
$$v(t) = \text{Slope}(x(t))$$
$$\Delta x = \text{Area under } v(t)$$
$$\Delta x = v_0 t + \frac{1}{2}at^2$$
$$x(t) = x(0) + \Delta x$$
$$x(t) = x(0) + v_0 t + \frac{1}{2}at^2$$



Since the acceleration is constant the slope of the velocity versus time graph is constant. $v(t)$ is a straight line with slope a. The vertical intercept is v_0. The area underneath the function is the displacement. It may be computed using the triangular portion $^{at^2}/_2$ and rectangular portion $v_0 t$.

$$x(t) = x_0 + v_0 t + \frac{at^2}{2}$$



The slope of $x(t)$ is changing linearly so $x(t)$ is parabolic. It is a quadratic function with vertical intercept of x_0.

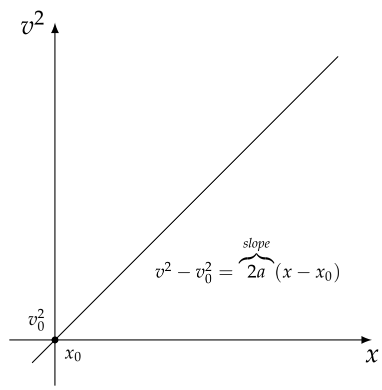
Removing Time Dependence

$$v = v_0 + at \implies t = \frac{v - v_0}{a}$$

$$x = x_0 + v_0 t + \frac{at^2}{2} = x_0 + v_0\left(\frac{v - v_0}{a}\right) + \frac{a\left(\frac{v-v_0}{a}\right)^2}{2} = x_0 + \frac{v^2 - v_0^2}{2a}$$

$$v^2 - v_0^2 = 2a(x - x_0)$$

Removing time dependence is to take the function $v(t)$ and determine $v(x)$. To determine the velocity of an object dependent on its position in space.



Graphically this corresponds to a linear graph with vertical axis v^2 and horizontal axis x.

Returning Time Dependence

$$\{v(x), \Delta x\} \longrightarrow t(x) \longrightarrow x(t)$$

Reversing this procedure is accomplished by determining Δt as the area beneath the function $1/v(x)$ graphed versus x.

$$\Delta t = \text{Area under } \frac{1}{v(x)}$$

Equations of Simple 1-D Translational Motion

Constant Velocity

$$v = v_0$$

$$x = x_0 + vt$$

Constant Acceleration

$$v = v_0 + at$$

$$\bar{v} = \frac{v_0 + v}{2}$$

$$x = x_0 + v_0 t + \frac{at^2}{2} = x_0 + \bar{v}t$$

$$v^2 - v_0^2 = 2a(x - x_0)$$

These are the kinematic equations of motion for a system of constant acceleration

Gravity at Earth's Surface

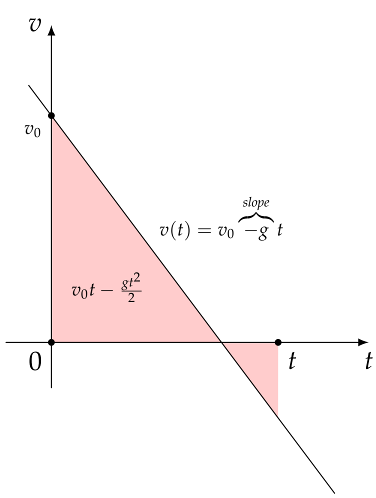
$$a = -g = -9.81\frac{\text{m}}{\text{s}^2}$$

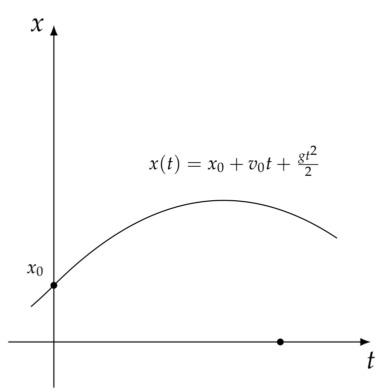
$$v = v_0 - gt$$

$$x = x_0 + v_0 t - \frac{gt^2}{2}$$

$$v^2 - v_0^2 = -2g(x - x_0)$$

Near the surface of the earth the acceleration due to gravity is fairly constant. Under a constant acceleration toward the ground. If up is taken as positive the acceleration is a negative constant.





Graphically this corresponds to a linear $v(t)$ function with a slope of $-g$ and a downwardly concave parabolic $x(t)$ function.

2-D Constant Velocity

$$\vec{a} = 0$$

In two dimensions position, velocity and acceleration are represented by 2D vectors. Acceleration is zero so velocity does not change from its initial state $\vec{v}_0$.

$$v_x = v_{0x}$$

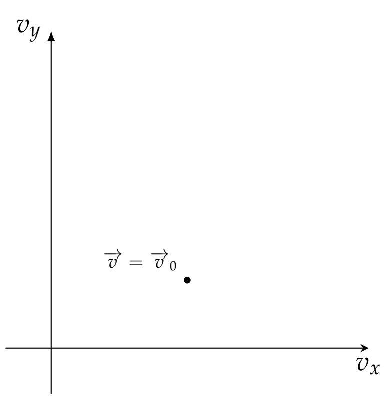
$$v_y = v_{0y}$$

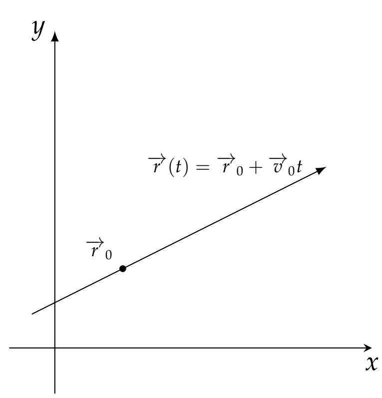
$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix} = \vec{v}_0$$

$$x = x_0 + v_{0x}t$$

$$y = y_0 + v_{0y}t$$

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 + v_{0x}t \\ y_0 + v_{0y}t \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix} t = \vec{r}_0 + \vec{v}_0 t$$





The velocity vector does not change. The position vector propagates linearly in the $\hat{v}$ direction from the initial position $\vec{r}_0$..

Projectile Motion

In projectile motion there is no acceleration in the horizontal direction and constant acceleration in the vertical direction. Under these conditions the initial position and velocity completely determines the evolution of the system. Take x and y as the horizontal and vertical components respectively. In the x direction there is no acceleration and constant velocity making $x(t)$ a linear function. In the y direction the acceleration is constant and velocity changing linearly as a function of time. This makes $y(t)$ a parabolic function. The trajectory is also a parabolic function.

$$\vec{a} = -g\hat{y} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$v_x = v_{0x}$$

$$v_y = v_{0y} + a_y t = v_{0y} - gt$$

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_{0x} \\ v_{0y} - gt \end{pmatrix} = \begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix} + \begin{pmatrix} 0 \\ -g \end{pmatrix} t = \vec{v}_0 + \vec{a}\, t$$

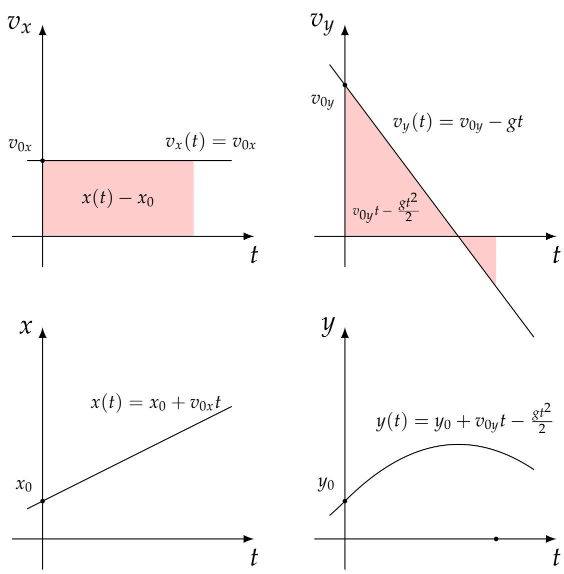
$$x = x_0 + v_{0x} t$$

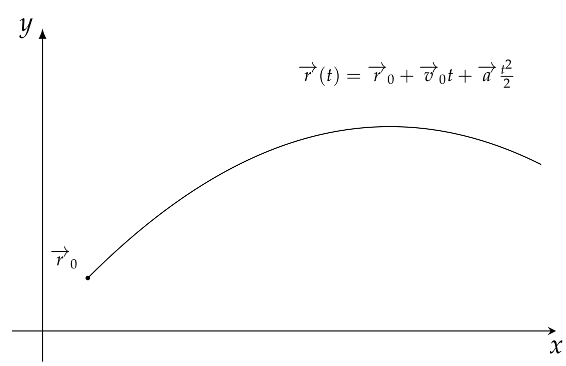
$$y = y_0 + v_{0y} t - \frac{gt^2}{2}$$

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 + v_{0x} t \\ y_0 + v_{0y} t - \frac{gt^2}{2} \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix} t + \begin{pmatrix} 0 \\ -g \end{pmatrix} \frac{t^2}{2} = \vec{r}_0 + \vec{v}_0 t + \vec{a}\frac{t^2}{2}$$

$$\vec{v} = \vec{v}_0 + \vec{a}\, t$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \vec{a}\frac{t^2}{2}$$

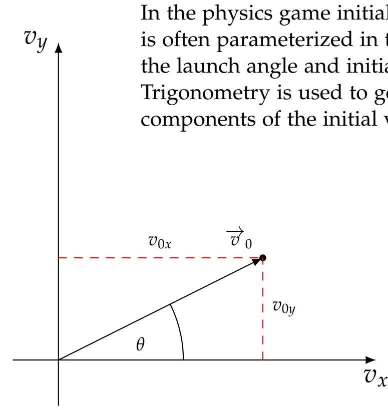




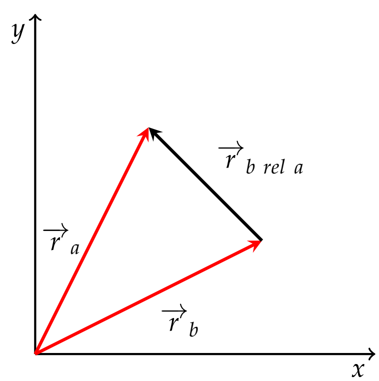
Components of Initial Velocity

In the physics game initial velocity is often parameterized in terms of the launch angle and initial speed. Trigonometry is used to get the x and y components of the initial velocity.

$$\vec{v}_0 = v_{0x}\hat{x} + v_{0y}\hat{y} = \begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix} = \begin{pmatrix} v_0 \ \cos\theta \\ v_0 \ \sin\theta \end{pmatrix} = v_0 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



Relative Position and Velocity



Relative position of "b" according to "a" is the position of "b" that would be measured if the origin of coordinates were positioned at "a". Relative velocity of "b" according to "a" is the velocity of "b" that would be measured if the origin of coordinates were moving at the velocity of "a".

For non-relativistic speeds we assume velocities much slower than the speed of light and can calculate as follows.

$$\vec{r}_{b\ rel\ a} = \vec{r}_a - \vec{r}_b$$

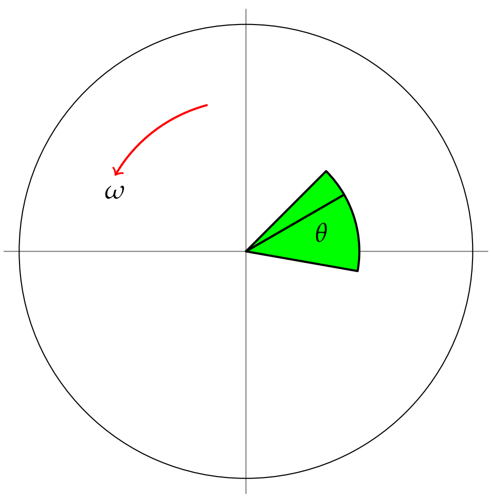
$$\vec{v}_{b\ rel\ a} = \vec{v}_a - \vec{v}_b$$

Rotational Kinematics

Joy is the mainspring... in Nature's calm rotation.
-Friedrich Schiller

Figure 24: Turntablism is largely an art of rotational kinematics

Pure Rotation



Translation	Rotation
Position (x)	Angle (θ)
Velocity (v)	Angular Velocity(ω)
Acceleration (a)	Angular Acceleration(α)

Table 9: Translational and rotational analogues

Angular Velocity

$$\omega = \lim_{\Delta \to 0} \frac{\Delta\theta}{\Delta t}$$

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\theta}{T}$$

The angular velocity is the rate of change of the angle value. The lowercase Greek letter omega is used to represent it as an algebraic variable.

Angular Acceleration

$$\alpha = \lim_{\Delta \to 0} \frac{\Delta\omega}{\Delta t}$$

Angular acceleration is the time rate of change of the angular velocity. The lowercase Greek letter alpha is used to represent it as an algebraic variable.

Kinematics Equations for Angular Motion

Constant Angular Velocity

$$\alpha = 0$$

$$\omega = \omega_0$$

$$\theta = \theta_0 + \omega t$$

Under zero acceleration the angular velocity is constant. The angle is a linear function of time with a slope equal to the angular velocity.

Constant Angular Acceleration

$$\omega = \omega_0 + \alpha t$$

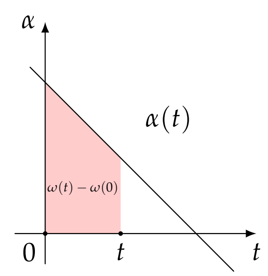
$$\bar{\omega} = \frac{\omega_0 + \omega}{2}$$

$$\theta = \theta_0 + \omega_0 t + \frac{\alpha t^2}{2} = \theta_0 + \bar{\omega} t$$

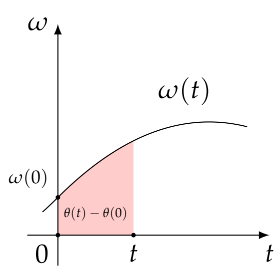
$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$$

Under constant acceleration the angular velocity is a linear function of time with a slope equal to the angular acceleration. The angle is a quadratic function of time. The angular velocity may also be parameterized as a function of angle.

Variable Angular Acceleration

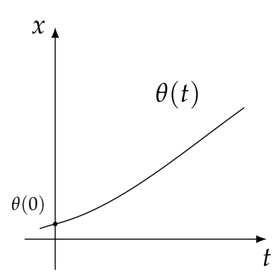


$\Delta\omega = \text{Area under } \alpha(t)$



$\omega(t) = \omega(0) + \text{Area}(\alpha(t))$

$\Delta\theta = \text{Area under } \omega(t)$



$\theta(t) = \theta(0) + \text{Area}(\omega(t))$

Angular and Radial Motion in Polar Coordinates

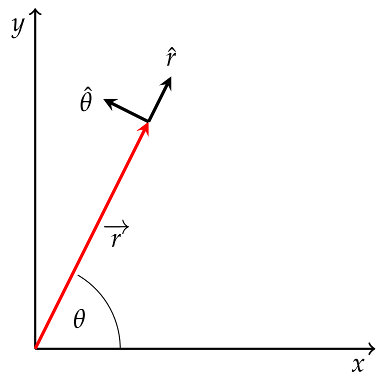


Figure 25: Position of a particle in polar coordintes

The unit vector in the radial direction is determined by normalizing the position vector. This is done by dividing the position vector by its magnitude.

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}} = \frac{r\cos\theta\hat{x} + r\sin\theta\hat{y}}{\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}} = \cos\theta\hat{x} + \sin\theta\hat{y} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Perpendicular to the radial unit vector is the unit angular vector. It points in the direction of increasing angle.

$$\hat{\theta} = -\sin\theta\hat{x} + \cos\theta\hat{y} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

The two unit vectors are orthogonal and together provide a handy coordinate system well suited for rotating systems.

$$\hat{r} \cdot \hat{\theta} = 0$$

Arc Length

$$\Delta S = r\Delta\theta$$

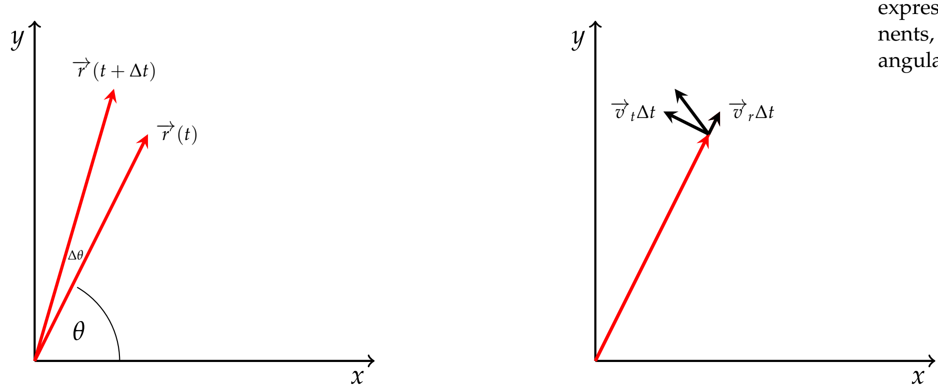
For a fixed radius the arc length of a curve is the product of the radius and the angular change. Angle is measured in radians.

Period

$$\text{time for a complete rotation} = T$$

The characteristic time for one whole rotation to occur is called the period.

Velocity



This figure shows change in position expressed in terms of polar components, in the radial direction and the angular (tangential) direction.

Tangential Velocity

The angular velocity and radius are the two factors contributing to the tangential velocity.

$$v_{tan} = r\omega$$

$$\vec{v}_{tan} = v_{tan}\hat{\theta}$$

Radial Velocity

The radial velocity is the time rate of change of the radius.

$$v_r = \lim_{\Delta\to 0}\frac{\Delta r}{\Delta t}$$

$$\vec{v}_{rad} = v_r\hat{r}$$

Total Velocity

The total velocity is the vector sum of these two orthogonal components.

$$\vec{v} = \lim_{\Delta\to 0}\frac{\Delta\vec{r}}{\Delta t}$$

$$\vec{v} = \underbrace{r\omega\hat{\theta}}_{tangential} + \overbrace{v_r\hat{r}}^{radial}$$

Consider a car driving around a circular racetrack. The magnitude of the tangential velocity determines how quickly the car circles the track. The magnitude of the radial velocity determines how quickly the car changes lanes.

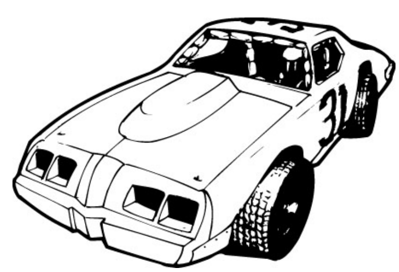


Figure 26: A rad car

Acceleration

Tangential Acceleration

$$a_{tan} = r\alpha$$
$$\overrightarrow{a}_{tan} = a_{tan}\hat{\theta}$$

The angular acceleration and radius are the two factors contributing to the tangential acceleration.

Centripetal Acceleration

$$a_c = r\omega^2 = \frac{v_{tan}^2}{r}$$
$$\overrightarrow{a}_c = -a_c\hat{r}$$

Centripetal acceleration is the acceleration associated with circular motion.

Coriolis Acceleration

$$a_C = 2v_r\omega = \frac{2v_r v_{tan}}{r}$$
$$\overrightarrow{a}_C = a_C\hat{\theta}$$

The coriolis acceleration is strange. It is dependent on the velocity in the tangential direction and the radial direction.

Radial Acceleration

$$a_r = \lim_{\Delta \to 0} \frac{\Delta v_r}{\Delta t}$$
$$\overrightarrow{a}_{rad} = a_r\hat{r}$$

The radial acceleration is simply the time rate of change of the radial velocity.

Total Acceleration

$$\overrightarrow{a} = \lim_{\Delta \to 0} \frac{\Delta \overrightarrow{v}}{\Delta t}$$
$$\overrightarrow{a} = \underbrace{r\alpha\hat{\theta}}_{tangential} - \overbrace{r\omega^2\hat{r}}^{centripetal} + \underbrace{2v_r\omega\hat{\theta}}_{Coriolis} + \overbrace{a_r\hat{r}}^{radial}$$

The total acceleration is the vector sum of these four components. The centripetal and radial acceleration are in the radial direction. The coriolis and tangential acceleration are in the tangential direction.

The Coriolis effect is a deflection of moving objects when the motion is described relative to a rotating reference frame. In a reference frame with clockwise rotation, the deflection is to the left of the motion of the object; in one with counter-clockwise rotation, the deflection is to the right. Although recognized previously by others, the mathematical expression for the Coriolis force appeared in an 1835 paper by French scientist Gaspard-Gustave Coriolis, in connection with the theory of water wheels.

Italian scientists Giovanni Battista Riccioli and his assistant Francesco Maria Grimaldi described the effect in connection with artillery in the 1651 Almagestum Novum, writing that rotation of the Earth should cause a cannonball fired to the north to deflect to the east. The effect was described in the tidal equations of Pierre-Simon Laplace in 1778.

Circular Motion

In circular motion the radius is fixed. There is no radial velocity.

$$v_r = 0$$

The angular motion may be variable over time with changing speed and acceleration of rotation.

$$\{\theta(t), \omega(t), \alpha(t)\}$$

There is only tangental velocity.

$$\overrightarrow{v} = \underbrace{r\omega\hat{\theta}}_{tangential} + \cancel{\overbrace{v_r\hat{r}}^{radial}} = \underbrace{r\omega\hat{\theta}}_{tangential}$$

The acceleration has only two terms, the tangential and the centripetal. The tangential is due to angular acceleration. The centrifugal is due to the constrained circularity of the motion.

$$\overrightarrow{a} = \underbrace{r\alpha\hat{\theta}}_{tangential} - \overbrace{r\omega^2\hat{r}}^{centrifugal} + \cancel{\underbrace{2v_r\omega\hat{\theta}}_{Coriolis}} + \cancel{\overbrace{a_r\hat{r}}^{radial}} = \underbrace{r\alpha\hat{\theta}}_{tangential} - \overbrace{r\omega^2\hat{r}}^{centrifugal} = r\alpha\hat{\theta} - \frac{v^2}{r}\hat{r}$$

Circular Motion: Constant Speed

For circular motion at constant angular velocity the tangential velocity is constant. There is no angular acceleration therefore and there is no tangential acceleration.

$$\omega(t) = \omega$$

$$\alpha = 0$$

$$\theta(t) = \omega t + \theta_0$$

$$\overrightarrow{v} = r\omega\hat{\theta} = v\hat{\theta}$$

The acceleration is centripetal, towards the center.

$$\overrightarrow{a} = \cancel{\underbrace{r\alpha\hat{\theta}}_{tangential}} - \overbrace{r\omega^2\hat{r}}^{centrifugal} = -\frac{v^2}{r}\hat{r}$$