Using Lie Algebras to find solutions of a simplified Navier-Stokes equation

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Abstract

In this paper we find a group of continuous symmetries of the equation $u_t + uu_x + u^2u_y = 0$. Furthermore, we create a lie algebra table and find a set of solutions.

1 Theory

Definition 1 *A partial differential equation is an equation* $\Delta(x_1, x_2, ..., x_p, u, u_{x_1}, u_{x_2}, ..., u_{x_p}, u_{x_1x_1}, u_{x_1,x_2}, ...) = 0$ *that includes a function* $u = u(x_1, x_2, x_3, ..., x_p)$ *and some of its partial derivatives up to a certain order. Here we use the notaion:* $u_{x_i} = \frac{\partial u}{\partial x_i}$, $u_{x_ix_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, *(where* $x_1, x_2, x_3, ..., x_p$ *are* p *different independent variables).*

Let $\Delta(x_1, x_2, ..., x_p, u, u_{x_1}, u_{x_2}, ..., u_{x_p}, u_{x_1x_1}, u_{x_1,x_2}, ...) = 0$ be a partial differential equation. A symmetry of the differential equation is defined as follows:

Definition 2 *A symmetry of a differential equation is a transformation on the space of independent and dependent coordinates* $(x_1, x_2, ..., x_p, u)$ *which maps each solution of the differential equation* $\Delta = 0$ *to another solution of the equation* $\Delta = 0$.

That is, if we suppose that $u(x_1, x_2, ...x_p)$ is a solution of $\Delta = 0$, and that the transformation ϕ_ε on the space of coordinates acting as:

$$\begin{aligned} x_1 \to \tilde{x}_1 &= X^1_\varepsilon(x_1, x_2, ..., x_p, u, \varepsilon) \\ x_2 \to \tilde{x}_2 &= X^2_\varepsilon(x_1, x_2, ..., x_p, u, \varepsilon) \\ &... \\ x_p \to \tilde{x}_p &= X^p_\varepsilon(x_1, x_2, ..., x_p, u, \varepsilon) \\ u \to \tilde{u} &= U_\varepsilon(x_1, x_2, ..., x_p, u, \varepsilon) \end{aligned} \tag{1}$$

is a symmetry of the differential equation (where ε is a continuous parameter) then the function $\tilde{u}(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_p)$ is also a solution of the differential equation $\Delta = 0$.

The set of all continuous symmetries of a differential equation forms a Lie group. Lie groups were originally introduced by Sophus Lie in order to model such symmetries. Before we define what a Lie group is, we first review the definitions of groups in general as well as differentiable manifolds.

A group is a set G with a binary operation $G \times G \to G$, defined as $(g_1, g_2) \mapsto g_1 \cdot g_2$ whose elements have the four following properties:

(1) If g_1 and g_2 are elements of G, then $g_1 \cdot g_2$ is also an element of G.

(2) There exists an element $e \in G$, called the identity of G, for which $g \cdot e = e \cdot g = g$ for any $g \in G$.

(3) For any element g of G, there exists an inverse element g^{-1} such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

(4) For any $g_1, g_2, g_3 \in G, \quad g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

In the case where G is the set of continuous symmetries of a differential equation, these become:

(1) If ϕ_1 and ϕ_2 are symmetries acting on the space of coordinates, then the transformation $\phi = \phi_2 \circ \phi_1$, where ϕ_1 and ϕ_2 act successively on the space of coordinates is also a symmetry.

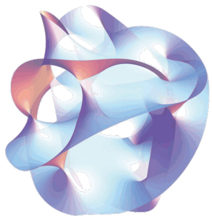
(2) There exists a transformation $\phi_{id} : \tilde{x_1} = x_1, \tilde{x_2} = x_2, ..., \tilde{x}_p = x_p, \tilde{u} = u$ called the identity transformation, for which $\phi \circ \phi_{id} = \phi_{id} \circ \phi = \phi$ for any symmetry ϕ.

(3) For any symmetry ϕ in G, there exists a symmetry ϕ^{-1} in G called the inverse of ϕ, such that $\phi \circ \phi^{-1} = \phi^{-1} \circ \phi = \phi_{id}$.

(4) For any three symmetries ϕ_1, ϕ_2 and ϕ_3 in G, $\phi_1 \circ (\phi_2 \circ \phi_3) = (\phi_1 \circ \phi_2) \circ \phi_3$

Definition 3 *A manifold is a topological space that locally resembles Euclidean space near each point.*

For example, a quite famous manifold is the Calabi-Yau manifold. It has a property called Ricci flatness, which is sought after in physics. This roughly means that small cubes in the Calabi-Yau manifold have the same volume as small cubes in $\mathbb{R}^3$. For this reason, it is used in superstring theory.



A 2D cross-section of a 6D Calabi-Yau manifold

Definition 4 *A differentiable manifold (also differential manifold) is a type of manifold that is locally similar enough to a linear space to allow one to do calculus.*

Definition 5 *A smooth function is a function which is infinitely differentiable on it's domain, also called a C^∞ function.*

We can now define what a Lie group is.

Definition 6 *A Lie group is a group that also has the structure of an n-dimensional differentiable manifold with the property that the group operations are smooth (by group operations we mean the group multiplication $(g_1, g_2) \mapsto g_1 \cdot g_2$ and the group inverse $g \mapsto g^{-1}$).*

For every point in a manifold there is a corresponding tangent space. In the case of Lie groups, the tangent space at the identity element is called a Lie algebra. In the case of a group of symmetry transformations, the Lie algebra consists of infinitesimal symmetries given in the form of vector fields:

$$\alpha = \xi_1(x_1, x_2, x_3..., x_p, u)\frac{\partial}{\partial x_1} + \xi_2(x_1, x_2, x_3..., x_p, u)\frac{\partial}{\partial x_2} + ... + \xi_p(x_1, x_2, x_3..., x_p, u)\frac{\partial}{\partial x_p} + \psi(x_1, x_2, x_3..., x_p, u)\frac{\partial}{\partial u} \tag{2}$$

involving $p+1$ functions, $\xi_1, \xi_2, ...\xi_p, \phi$, each of the p variables $x_1, ..., x_p$ and the function u.

The link between the transformation (1) and its corresponding generating infinitesimal transformation (2) is:

$$\xi_i(x_1,...x_p,u) = \tfrac{d}{d\varepsilon}X^i_\varepsilon(x_1,...x_p,u,\varepsilon)|_{\varepsilon=0} \text{ for } i=1,2,...,p,$$
$$\text{and } \phi(x_1,...x_p,u) = \tfrac{d}{d\varepsilon}U_\varepsilon(x_1,...x_p,u,\varepsilon)|_{\varepsilon=0}$$

Consider as an example the following rotation in the xy-plane:

$$\tilde{x} = cos(\varepsilon)x - sin(\varepsilon)y$$
$$\tilde{y} = sin(\varepsilon)x + cos(\varepsilon)y$$
$$\tilde{u} = u$$

$$\xi_1(x,y,u) = \tfrac{d}{d\varepsilon}\big[(cos(\varepsilon)x - (sin(\varepsilon)y\big]|_{\varepsilon=0} = \big[(-sin(\varepsilon)x - (cos(\varepsilon)y\big]|_{\varepsilon=0} = (0)x-(1)y = -y$$

$$\xi_2(x,y,u) = \tfrac{d}{d\varepsilon}(sin(\varepsilon)x + (cos(\varepsilon)y|_{\varepsilon=0} = (cos(\varepsilon)x - (sin(\varepsilon)y|_{\varepsilon=0} = (1)x-(0)y = x$$

$$\phi(x,y,u) = \tfrac{d}{d\varepsilon}[u]|_\varepsilon = [0]|_\varepsilon = 0$$

So the corresponding infinitesimal generator is:

$$\overrightarrow{V} = -y\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial y}$$

To conclude, the set of infinitesimal symmetries of a given differential equation $\Delta = 0$ constitutes a Lie algebra g which generates the Lie group G of symmetries.

In order to compute Lie groups and algebras of symmetries, we use the techniques described in the book "Applications of Lie Groups to Differential Equations" by P.J Olver. [1]

2 Problem

In this paper, we analyse a special case of the Navier-Stokes equation from the point of view of symmetries.

We begin by looking at the two-dimensional Navier-Stokes equation: [2]

$$\overrightarrow{u}_t + (\overrightarrow{u}\cdot\overrightarrow{\nabla})\overrightarrow{u} = \tfrac{-\overrightarrow{\nabla}P}{\rho} + \nu\nabla^2\overrightarrow{u}$$

where $\overrightarrow{u} = (u_1(t,x,y), u_2(t,x,y))$ is the velocity of a fluid, P is its pressure, ρ is its density and ν is its viscosity. We consider the special case where $\overrightarrow{u} = (u, u^2)$ for some scalar function, $u = u(t,x,y)$, $\overrightarrow{\nabla}P = 0$ (the fluid is isobaric) and $\nu = 0$ (the fluid is inviscid)

From this, we obtain the partial differential equation:

$$u_t + uu_x + u^2u_y = 0 \tag{3}$$

3 Results

In this paper, we find symmetries and solutions of equation (3). To find the symmetries of this equation, we begin by considering the most general form of the infinitesimal symmetry vector field:

$$\alpha = \xi(x,y,t,u)\frac{\partial}{\partial x} + \tau(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial t} + \psi(x,y,t,u)\frac{\partial}{\partial u} \tag{4}$$

Following the methods described in the book by Olver [1], we calculate the first prolongation of the vector field (4):

$$pr^{(1)}(\alpha) = \alpha + \psi^x\frac{\partial}{\partial u_x} + \psi^y\frac{\partial}{\partial u_y} + \psi^t\frac{\partial}{\partial u_t}$$

This allows us to obtain the following determining equations, which describe the conditions that must be satisfied between the functions ξ, τ, ϕ and ψ in order for the vector field α given in (4) to be an infinitesimal symmetry of equation (3).

$$
\begin{aligned}
\phi_{xx} - \phi_t &= 0 \\
2\phi_{xu} - \xi_{xx} + \xi_t + 2\phi_x &= 0 \\
\phi_{uu} - 2\xi_{xu} + 2\phi_u - 2\xi_x &= 0 \\
-\xi_{uu} - 2\xi_u &= 0 \\
\phi_u - 2\xi_x &= 0 \\
-\tau_{xx} - \phi_u + \tau_t &= 0 \\
\tau_u &= 0 \\
\xi_u - 2\tau_x - 2\tau_{xu} &= 0 \\
-\tau_{uu} - 2\tau_u &= 0 \\
-2\tau_x &= 0 \\
-3_u &= 0 \\
-\tau_u &= 0 \\
-2\tau_u &= 0
\end{aligned} \tag{5}
$$

We did not solve the general case for equations (5), but a specific sub-case of solutions of (5) was found to be

$$
\begin{aligned}
&\phi = c_1 t + c_2 \\
&\xi = k_2 t + k_1 x + c_1 x + c_4 \\
&\tau = 2k_2 x + c_1 y + 2k_1 y + c_5 \\
&\psi = k_1 u + k_2
\end{aligned} \tag{6}
$$

where $c_1, c_2, c_3, c_4, c_5, k_1$ are arbitrary constants.

This gives us a six-dimensional set of solutions of (5) which corresponds to a six-dimensional Lie algebra of infinitesimal symmetries spanned by the following six vector fields

$$
\begin{aligned}
&\alpha_1 = t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} \\
&\alpha_2 = \frac{\partial}{\partial t} \\
&\alpha_3 = \frac{\partial}{\partial x} \\
&\alpha_4 = \frac{\partial}{\partial y} \\
&\alpha_5 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial y} \\
&\alpha_6 = t\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y} + \frac{\partial}{\partial u}
\end{aligned} \tag{7}
$$

Integrating, (7) we find that they generate the following one-parameter symmetry groups:

$$
\begin{aligned}
&G_1 : \tilde{t} = e^{\lambda} t, \quad \tilde{y} = e^{\lambda} y, \quad \tilde{x} = e^{\lambda} x, \quad \tilde{u} = u \\
&G_2 : \tilde{t} = t + \lambda, \quad \tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{u} = u \\
&G_3 : \tilde{x} = x + \lambda, \quad \tilde{u} = u, \quad \tilde{y} = y, \quad \tilde{t} = t \\
&G_4 : \tilde{y} = y + \lambda, \quad \tilde{u} = u, \quad \tilde{x} = x, \quad \tilde{t} = t \\
&G_5 : \tilde{x} = e^{\lambda} x, \quad \tilde{y} = e^{2\lambda} y, \quad \tilde{u} = e^{\lambda} u, \quad \tilde{t} = t \\
&G_6 : \tilde{x} = x + \lambda t, \quad \tilde{y} = y + 2\lambda x + \lambda^2 t, \quad \tilde{u} = u + \lambda, \quad \tilde{t} = t
\end{aligned} \tag{8}
$$

where λ is an arbitrary parameter

	α_1	α_2	α_3	α_4	α_5	α_6
α_1	0	-α_2	-α_3	-α_4	0	0
α_2	α_2	0	0	0	0	α_3
α_3	α_3	0	0	0	α_3	2α_4
α_4	α_4	0	0	0	2α_4	0
α_5	0	0	-α_3	-2α_4	0	-α_6
α_6	0	-α_3	-2α_4	0	α_6	0

Table 1: Lie Algebra commutation table giving [A,B] for each pair of generators A and B

4 Solutions

We begin by constructing the table of commutation relations between the generators (7). The commutation [A,B] of two elements A,B of a Lie algebra is defined as $[A, B] = AB - BA$. So we have $[A, B]f(x) = A(B(f(x))) - B(A(f(x)))$. So for example for the generators $\alpha_1 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $\alpha_2 = \frac{\partial}{\partial t}$, the commutator is:

$$\begin{aligned}
[\alpha_1, \alpha_2]f(x) &= \left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial t}\right)f(x) - \left(\frac{\partial}{\partial t}\right)\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f(x) \\
[\alpha_1, \alpha_2]f(x) &= \left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f_t - \left(\frac{\partial}{\partial t}\right)(tf_t + xf_x + yf_y) \\
[\alpha_1, \alpha_2]f(x) &= tf_{tt} + xf_{xt} + yf_{yt} - f_t - tf_{tt} - xf_{xt} - yf_{yt} \\
[\alpha_1, \alpha_2]f(x) &= -f_t \\
[\alpha_1, \alpha_2]f(x) &= -\alpha_2 f(x) \\
[\alpha_1, \alpha_2] &= -\alpha_2
\end{aligned} \tag{9}$$

This procedure is repeated for each pair of the vector fields given in (7). The results are given in Table 1.
We can use this commutation table to help us find invariant solutions of equation (3). For example we consider the sub-algebra generated by α_1 and α_2. We notice that $\xi = \frac{x}{y}$ is invariant under both sub-algebras.

$$\begin{aligned}
\alpha_1\xi &= t\partial t\left(\frac{x}{y}\right) + y\partial y\left(\frac{x}{y}\right) + x\partial x\left(\frac{x}{y}\right) \\
&= 0 + y\left(\frac{-x}{y^2}\right) + x\left(\frac{1}{y}\right) \\
&= \frac{-x}{y} + \frac{x}{y} = 0 \quad \text{and} \\
\alpha_2\xi &= \partial t\left(\frac{x}{y}\right) \\
&= 0
\end{aligned} \tag{10}$$

Since $\xi = \frac{x}{y}$is a symmetry variable for $\{\alpha_1, \alpha_2\}$ we look for a solution of the type $u = u(\xi)$. Then the derivatives u_x, u_y and u_t are given by:

$$\begin{aligned}
u_x &= u_\xi \xi_x = \frac{1}{y}u_\xi \\
u_y &= u_\xi \xi_y = \frac{-x}{y^2}u_\xi \\
u_t &= 0
\end{aligned} \tag{11}$$

Replacing these derivatives into equation (3), we get:

$$
\begin{aligned}
u_t + uu_x + u^2 u_y &= 0 \\
0 + u\left(\frac{1}{y}\right)u_\xi + u^2\left(\frac{-x}{y^2}\right)u_\xi &= 0 \\
\left(\frac{1}{y}\right)uu_\xi + \left(\frac{-x}{y^2}\right)u^2 u_\xi &= 0 \\
uu_\xi - \xi u^2 u_\xi &= 0
\end{aligned}
\tag{12}
$$

We then have that either $u = 0$, $u_\xi = 0$ or $1 - \xi_u = 0 \implies u = \frac{1}{\xi}$. Indeed, if $u = \frac{1}{\xi}, u^2 = \frac{1}{\xi^2}, u_\xi = \frac{-1}{\xi^2}$.

$$
uu_\xi - \xi u^2 u_\xi = \frac{1}{\xi}\left(\frac{-1}{\xi^2}\right) - \xi\left(\frac{1}{\xi^2}\right)\left(\frac{-1}{\xi^2}\right) = 0 \tag{13}
$$

Therefore, the solution $u(x, y, t) = \frac{y}{x}$ of the equation is invariant under α_1 and α_2.

As a second example, we consider the sub-algebras generated by α_1 and α_4. We notice that $\xi = \frac{t}{x}$ is invariant under both sub-algebras.

$$
\begin{aligned}
\alpha_1 \xi &= t\partial t\left(\frac{t}{x}\right) + y\partial y\left(\frac{t}{x}\right) + x\partial x\left(\frac{t}{x}\right) \\
&= \frac{t}{x} + 0 - x\frac{t}{x^2} \\
&= 0 \quad \text{and} \\
\alpha_2 \xi &= \partial y\left(\frac{t}{x}\right) \\
&= 0
\end{aligned}
\tag{14}
$$

Since $\xi = \frac{t}{x}$is a symmetry variable for $\{\alpha_1, \alpha_4\}$ we look for a solution of the type $u = u(\xi)$. We obtain the derivatives

$$
\begin{aligned}
u_x &= u_\xi \xi_x = \frac{-t}{x^2}u_\xi \\
u_y &= 0 \\
u_t &= u_\xi \xi_t = \frac{1}{x}u_\xi
\end{aligned}
\tag{15}
$$

Replacing these derivatives into equation (3), we obtain:

$$
\begin{aligned}
u_t + uu_x + u^2 u_y &= 0 \\
\frac{1}{x}u_\xi + u\frac{-t}{x^2}u_\xi &+ 0 \\
\frac{1}{x}(1 - u\xi) &= 0
\end{aligned}
\tag{16}
$$

We then have that $u_\xi = 0$ or $1 = u\xi$. Therefore, the solution $u(x, y, t) = \frac{x}{t}$ of the equation is invariant under α_1 and α_4.

References

[1] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer (1986).

[2] G.K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press (1967).