Hausdorff Dimension of Sets Defined by P-adic Continued Fractions

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Abstract

In this document we summarise the work done during the Summer - Fall of 2021 under the supervision of Eyal Z. Goren. The main goal was to explore analogues of sets defined by p-adic continued fractions, their measure, and their dimension.

1 Introduction

We begin by recalling some concepts: p-adic numbers, Hausdorff Dimension for sets in $\mathbb{C}_p$ and Iterated function systems. Their introduction is quite quick and mostly serves as a reminder of the main ideas. For more details, references are given at the beginning of each section. After this we describe two different algorithms that produce continued fractions for p-adic numbers: Ruban's algorithm and Schneider's one. While we describe both algorithms, almost all of the results are for the Schneider one since it was our main focus. Most of the results related to measure have an analogue for Ruban's continued fractions. The principle of continuity is proved for Schneider's continued fractions.

2 P-adic Numbers [1]

Definition 1. *Fix a prime p, the **p-adic valuation** on $\mathbb{Z}$ is a function from $\mathbb{Z} \setminus \{0\}$ to $\mathbb{R}$ such that $v_p(n)$ is defined as the unique integer satisfying $n = p^{v_p(n)}a$ such that p does not divide a. We can also extend it uniquely to $\mathbb{Q} \setminus \{0\}$ such that if $x = \frac{a}{b}$, $v_p(x) = v_p(a) - v_p(b)$. By convention we set $v_p(0) = +\infty$.*

Definition 2. *Let p be a prime number. The **P-adic absolute value** is a non-Archimedian norm. It is defined as follows: for all $x \in \mathbb{Q}$, (i) $|0|_p = 0$ (ii) and $|x|_p = p^{-v_p(x)}$. This norm satisfies the strong triangle inequality $|x+y|_p \leq max(|x|_p, |y|_p)$.*

Consider the following diagram:

$$\begin{tikzcd}
 & \mathbb{R} \arrow[r, "\text{algebraic closure}"'] & \mathbb{C} & \\
\mathbb{Q} \arrow[ur, "\text{complete w.r.t. } |\ \ |"] \arrow[r, "\text{complete w.r.t. } |\ \ |_p"'] & \mathbb{Q}_p \arrow[r, "\text{algebraic closure}"'] & \overline{\mathbb{Q}_p} \arrow[r, "\text{completion}"'] & \mathbb{C}_p \\
\mathbb{Z} \arrow[u, phantom, "\cup|"] \arrow[r, "\text{complete w.r.t. } |\ \ |_p"'] & \mathbb{Z}_p \arrow[u, phantom, "\cup|"] & &
\end{tikzcd}$$

The set $\mathbb{Z}_p$ is called the set of p-adic integers, it is a subset of $\mathbb{Q}_p$. We can also write it as: $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. Furthermore, every element x of $\mathbb{Q}_p$ can be written as a unique formal series: $x = \sum_{i=m}^{\infty} b_i p^i$ with $0 \leq b_i \leq p-1$. We also have that $\mathbb{Z}_p$ is compact and $\mathbb{Q}_p$ is locally compact.

We will now talk a bit more about finite extensions of $\mathbb{Q}_p$. Let K be a finite extension of degree n of $\mathbb{Q}_p$. One can show that there is a unique way to extend the p-adic absolute value to K such that K is complete with respect to it, and it is equal to $|x|_p$, $\forall x \in \mathbb{Q}_p$.

Definition 3. *We call e the* ***ramification index*** *of K over $\mathbb{Q}_p$ if it is the unique integer dividing n, such that $v_p(K^\times) = \frac{1}{e}\mathbb{Z}$.*

Note that $v_p(K^\times) = \frac{1}{e}\mathbb{Z} \subseteq \frac{1}{n}\mathbb{Z}$

Definition 4. ***Ramified and unramified extensions.*** *Let K be a an algebraic extension of $\mathbb{Q}_p$ of degree n. We then have that $n = e \cdot f$, with e being the ramification index and f the degree of the residue field. An extension is* ***unramified*** *if $e = 1$,* ***totally ramified*** *if $e = n$ and* ***ramified*** *otherwise.*

$$\mathbb{Z}_p \xrightarrow{\text{completion in K}} \boxed{O_K = \{x \in K : |x|_p \leq 1\}}$$

$$p\mathbb{Z}_p \xrightarrow{\text{completion in K}} \boxed{P_K = \{x \in K : |x|_p < 1\}}$$

Definition 5. *The* ***residue field*** *is the quotient $O_K/P_K = k \cong \mathbb{F}_{p^f}$*

Definition 6. *An element $\pi \in K$ is called a* ***uniformizer*** *if $|\pi|_p = p^{\frac{-1}{e}}$. It is the smallest possible positive valuation. As before, every $x \in K$ can be written uniquely as $x = \sum_{i=m}^{\infty} a_i \pi^i$ with a special condition on the a_i's.*

We will now focus a bit more on totally ramified extension ($e = n$). Consider first the following criterion.

Theorem 1 (Eisenstein's irreducibility criterion)**.** *Say $f(x)$ is a polynomial with coefficients in $\mathbb{Z}_p[x]$, that is, $f(x) = a_n x^n + ... + a_1 x + a_0$ with $a_i \in \mathbb{Z}_p[x]$. If $f(x)$ satisfies the following 3 conditions:*

1. $|a_n| = 1$
2. $|a_i| < 1$ *for* $0 \leq i < n$ $(p|a_i,\ 0 \leq i < n)$
3. $|a_0| = \frac{1}{p}$ $(p^2 \not| a_0)$

Then $f(x)$ is irreducible over $\mathbb{Q}_p$.

Theorem 2. *Say $K/\mathbb{Q}_p$ is a totally ramified extension of $\mathbb{Q}_p$. Then $K = \mathbb{Q}_p(\pi)$. Furthermore, π is a root of $f(x) = x^n + ... + a_1 x + a_0 \in \mathbb{Q}_p[x]$ satisfying Eisenstein's criterion.*

Theorem 3. *For each f, $(f = \frac{n}{e})$, there is exactly one unramified $(e = 1)$ extension of degree f. It can be obtained by adjoining to $\mathbb{Q}_p$ a primite $(p^f - 1)$ roof of unity ζ. Meaning that ζ is a solution to $(\zeta^{p^f - 1} = 1)$.*

For further details on finite extensions, please consult chapter 5 of [1].

3 Hausdorff Dimensions [5]

Definition 7. *The* ***diameter*** *of a set $E \subset \mathbb{C}_p$, written as $|E|_p$ is defined as $sup\{|x-y|_p : \forall x, y \in E\}$.*

Definition 8. *An* ***r-cover*** *of a set $E \subset \mathbb{C}_p$ is a collection of sets $\{U_j\}$ such that:*
1. $U_j \subset \mathbb{C}_p$
2. $E \subset \cup_{j=1}^{\infty} U_j$
3. $|U_j|_p \leq r, \forall j$

Definition 9. *Let $E \subset \mathbb{C}_p$, $s \geq 0$, $r > 0$. An* ***s-dimensional approximate Hausdorff measure*** *is defined as follows:*

$$H_r^s(E) = inf\left\{ \sum_{j=1}^{\infty} |U_j|_p^s : \{U_j\} \text{ is a r-cover of } E \right\}$$

Definition 10. *An s-dimensional Hausdorff measure is defined as follows:*

$$\lim_{s \to \infty} H_r^s(E) = H^s(E)$$

Theorem 4. *Say $\lambda \in \mathbb{C}_p$, $E + \lambda = \{x + \lambda : x \in E\}$, $\lambda E = \{\lambda x : x \in E\}$*
1. $H^s(E + \lambda) = H^s(E)$
2. $H^s(\lambda E) = |\lambda|_p^s H^s(E)$

Note that if $s \leq t$ then we have that $H^t(E) \leq H^s(E)$

Theorem 5. *If $0 \leq s < t < \infty$ then: $H^s(E) < \infty$ implies $H^t(E) = 0$ and $H^t(E) > 0$ implies $H^s(E) = \infty$.*

Definition 11. *The* ***Hausdorff dimension*** *is defined as follows:*

$$\begin{aligned} dim_H E =& sup\{s : H^s(E) > 0\} \\ =& inf\{s : H^t(E) < \infty\} \end{aligned}$$

Theorem 6.

$$O = \{x \in \mathbb{C}_p : |x|_p \leq 1\}, \; dim_H O = \infty$$

For further details regarding the Hausdorff dimension of sets in $\mathbb{C}_p$, please see [5].

4 Haar measure [5]

Definition 12. *There is, up to a multiplicative constant a unique, countably additive, translation-invariant, measure μ on $\mathbb{Q}_p$. This measure is called the* ***Haar measure****. We usually choose μ such that $\mu(\mathbb{Z}_p) = 1$ and $\mu(p\mathbb{Z}_p) = \frac{1}{p}$. In this paper, we will mostly be working with $p\mathbb{Z}_p$. For that reason, we "renormalize" the Haar measure such that $\mu(\mathbb{Z}_p) = p$ and $\mu(p\mathbb{Z}_p) = 1$.*

Lemma 1. *For any measurable set U and $x \in \mathbb{Q}_p$ we have:*

(i) $\mu(x + U) = \mu(U)$

(ii) $\mu(p^k U) = p^{-k}\mu(U)$ when $(U \subseteq \mathbb{Z}_p, k \leq 0)$

Ex 1. *For example, for any $a \in \mathbb{Q}_p$ we have: $\mu(a + p^k\mathbb{Z}_p) = p^{1-k}$*

5 Iterated Function System for $\mathbb{R}^n$[2]

IFS are described in Falconer's book in detail.

Definition 13. *Let $D \subset \mathbb{R}^n$, a function $S : D \mapsto \mathbb{R}^n$ is called a* ***contraction*** *if $\exists\, 0 < c < 1$ s.t. $|S(x) - S(y)|_p \leq c|x - y| \; \forall x, y \in D$.*

Definition 14. *An* ***IFS*** *(Iterated Function System) is a finite family of contractions* $\{S_i\}_{i=1}^m$ $m \geq 2$.

Definition 15. *An* ***attractor*** *of the IFS is a non-empty, compact, subset of* D *s.t.* $E = \bigcup_{i=1}^{m} S_i(E)$.

Definition 16. *An IFS is said to satisfy the* ***open set condition*** *if* $\exists$ *a non-empty, bounded, open set* V *s.t.* $V \supset \bigsqcup_{i=1}^{m} S_i(V)$. *(Note that the* $S_i(V)$*'s are disjoint).*

Theorem 7. *Say we have an IFS satisfying the O.S.C. The following are true:*

(1) Assume that every contraction is exact, i.e. $\forall i$ $|S_i(x) - S_i(y)|_p = c_i|x - y|_p$. *Let* s *be defined by* $\sum_{i=1}^{m} c_i^s = 1$, *then*

$$s = dim_H(E).$$

(2) Assume that for every contraction S_i, $|S_i(x) - S_i(y)|_p \leq c_i|x - y|_p$. *With* s *defined as before. Then*

$$s \geq dim_H(E).$$

(3) Assume that for every contraction S_i, $|S_i(x) - S_i(y)|_p \geq c_i|x - y|_p$. *With* s *defined as before. Then*

$$s \leq dim_H(E).$$

5.1 Iterated Function System for $\mathbb{C}_p$ [7]

It turns out that IFS are very versatile, they may work for spaces other than $\mathbb{R}^n$. In fact, with a couple more conditions, we can compute the Hausdorff dimension of sets in complete metric spaces. In complete metric spaces the SOSC (Strong Open Set Condition) is enough to guarantee us that the theorem in the previous section is satisfied.

Definition 17. *The* ***strong open set condition*** *is the open set condition with the added requirement that the open set must have non-empty intersection with the attractor.*

Since $\mathbb{C}_p$ is a complete metric space, we are good. For applications of IFS on $\mathbb{C}_p$ please consult [5].

6 Ruban's Algorithm [3]

For any $x \in p\mathbb{Z}_p \setminus \{0\}$, do the following: Since $|x|_p < 1$, this implies that $|\frac{1}{x}|_p > 1$. Now recall that for any number in $\mathbb{Q}_p$ we can represent it uniquely as a formal series. In our case,

$$\frac{1}{x} = c_{-m}p^{-m} + ... + c_{-1}p^{-1} + c_0 + c_1 p + ... \text{ with } c_i \in \{0, 1, ..., p-1\}$$

where $m \geq 1$ and $c_{-m} \neq 0$.

Definition 18. *Let* $\left\langle \frac{1}{x} \right\rangle = c_{-m}p^{-m} + ... + c_{-1}p^{-1} + c_0$ *and* $\left(\frac{1}{x}\right) = c_1 p + c_2 p^2 ...$. *For each* x, *these quantities are unique. Therefore,* $\frac{1}{x} = \left\langle \frac{1}{x} \right\rangle + \left(\frac{1}{x}\right)$

Once the setup is done, let's begin the algorithm. Write $\frac{1}{x} = b_0 + x_1$ where $b_0 = \left\langle \frac{1}{x} \right\rangle$ and $x_1 = \left(\frac{1}{x}\right)$. Then $x = \frac{1}{b_0 + x_1}$. If $x_1 = 0$, then the algorithm terminates. Otherwise, since $x_1 \neq 0$ and $|x|_p < 1$, we repeat the process on x_1. We get

$$\frac{1}{x_1} = b_1 + x_2 \Leftrightarrow x_1 = \frac{1}{b_1 + x_2}$$

$$\text{And therefore } x = \frac{1}{b_0 + x_1} = \frac{1}{b_0 + \frac{1}{b_1 + x_2}}$$

Continue the process indefinitely. The algorithm will either terminate, in which case we have a finite continued fraction. If it does not terminate, we will have an infinite continued fraction. The final expression for x is then called the Ruban continued fraction. Note that since the b_i's are unique, every $x \in p\mathbb{Z}_p \setminus \{0\}$ has a unique Ruban continued fraction.

Definition 19.

$$J = \{b \in \mathbb{Q} : b = c_j p^{-j} + ... + c_0 \text{ for some } j \in \mathbb{Z}^+,\ c_i \in \{0, 1, ..., p-1\} \forall i \text{ and } c_{-j} \neq 0\}$$

Theorem 8 (Theorem 3.1 of [3])**.** *(⇒)For each $x \in p\mathbb{Z}_p \setminus \{0\}$, ∃! Ruban continued fraction $\frac{1}{b_0+x_1} = \frac{1}{b_0+\frac{1}{b_1+...}}$ with $b_i \in J$ that converges to x.*

(⇐) If $b_i \in J\ \forall i$, then the continued fraction $\frac{1}{b_0+x_1} = \frac{1}{b_0+\frac{1}{b_1+...}}$ is a Ruban continued fraction representing a unique $x \in p\mathbb{Z}_p \setminus \{0\}$.

Remark 1. *Contrary to the classical case, we might have rational numbers with periodic Ruban continued fractions. For example, consider the following Ruban fraction:*

$$x = \cfrac{1}{(p-1)p^{-1} + (p-1) + \cfrac{1}{(p-1)p^{-1} + (p-1) + ...}}$$

Solving for x, we get $px^2 + (p^2 - 1)x - p = 0$ with x being equal to p^{-1} or $-p$. Since $x \in p\mathbb{Z}_P$ it follows that $x = -p$.

7 Schneider's Algorithm [3, 4]

Without loss of generality, consider $x \in p\mathbb{Z}_p$ only. Expanding this to $\mathbb{Z}_p$ is then easy to do. Since $x \in p\mathbb{Z}_p$, its norm is < 1.

Definition 20. ***Schneider's map*** *(T_p) is a map from $p\mathbb{Z}_p \to p\mathbb{Z}_p$, defined as follows:*

$$T_p(x) = \frac{p^a}{x} - b \text{ with } a = v(x) \in \mathbb{N} \text{ and } \ b \in \{1, 2, ..., p-1\} \text{ uniquely chosen such that } |\frac{p^a}{x}|_p - b < 1$$

Remark that both the a and b are unique to our x. Shuffling the terms around in the previous expression, we get that

$$\begin{aligned} T_p(x) &= \frac{p^a}{x} - b \\ T_p(x) + b &= \frac{p^a}{x} \\ \frac{1}{T_p(x) + b} &= \frac{x}{p^a} \\ x &= \frac{p^a}{b + T_p(x)} \end{aligned}$$

Now note that by our choice of a and b, $|T_p(x)|_p < 1 \Rightarrow T_p(x) \in p\mathbb{Z}_p$. Therefore we can apply Schneider's map again on $T_p(x)$. Our new expression for x is then

$$x = \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + T_p^2(x)}}$$

Applying this map repeatedly, we get:

$$x = \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{b_3 + \frac{p^{a_4}}{b_4 + ...}}}}$$

The algorithm outputs $(\alpha_n)_{n=1}^{\infty}$ where $\alpha_n = (a_n, b_n) \in \mathbb{N} \times \{1, 2, ..., p-1\}$

For numerical examples please consult [4].

Theorem 9 (Bundschuh). *If the p-adic continued fraction of a rational number is non-terminating, then the tail of the expansion has the form:*

$$p-1+\cfrac{p}{p-1+\cfrac{p}{p-1+...}}$$

Let $x =$ tail, then $x = -1$. Thus if the continued fraction does not terminate, then some step of the algorithm yields:

$$b_0+\cfrac{p^{a_1}}{b_1+\cfrac{p^{a_2}}{b_2+...\cfrac{p^{a_n}}{-1}}}$$

What follows is a list of basic formulas which follow closely the classical case.
Let $\frac{A_n}{B_n}$ be the nth partial quotient. That is:

$$\frac{A_n}{B_n}=b_0+\cfrac{p^{a_1}}{b_1+\cfrac{p^{a_2}}{b_2+...\cfrac{p^{a_n}}{b_n}}}$$

We then have the following formulas:

1. $A_n = p^{a_n}A_{n-2} + b_nA_{n-1}\ \forall n \geq 0$
2. $B_n = p^{a_n}B_{n-2} + b_nB_{n-1}\ \forall n \geq 0$
3. $|A_n|_p = |B_n|_p = 1\ \forall n \geq 0$
4. $A_{n-1}B_n - A_nB_{n-1} = (-1)^n p^{a_1+...+a_n}$ for $n \geq 1$
5. $gcd(A_n, B_n) = 1$ and hence the partial quotients are always in reduced form for $n \geq 1$

7.1 2-adic continued fractions

$$\cfrac{p^{a_1}}{b_1+\cfrac{p^{a_2}}{b_2+\cfrac{p^{a_3}}{b_3+\frac{p^{a_4}}{b_4+...}}}} \rightsquigarrow \cfrac{2^{a_1}}{1+\cfrac{2^{a_2}}{1+\cfrac{2^{a_3}}{1+\frac{2^{a_4}}{1+...}}}}$$

A lot of the general results for Schneider's continued fractions are much cleaner in the case $p = 2$. For more details consult [4].

8 Ergodicity [4, 5]

Although we didn't focus a lot on ergodicity during the Summer, during our literature review we found a lot of similar results to the classical case for both Ruban's and Schneider's continued fractions, results such as analogues of Khinchin's theorem for p-adic numbers. First recall the classical theorem.

Theorem 10 (Khinchin's theorem). *For all $x \in (0, 1)$ outside a set of measure 0, the continued fraction expansion $x = \frac{1}{b_1+\frac{1}{b_2+...}}$ satisfies the following:*

$$\lim_{n\to\infty}(b_1b_2...b_n)^{\frac{1}{n}} = \prod_{m=1}^{\infty}\left(\frac{m^2+2m+1}{m+2+2m}\right)^{log_2(m)}$$

Theorem 11 (Ergodic theorem). *If $f \in L^1(p\mathbb{Z}_p, \mu)$ and if δ is indecomposable and preserves μ, then:*

$$\lim_{n\to\infty}\frac{f(x)+f(\delta x)+...+f(\delta^{n-1}x)}{n} \to \int_X f\,d\mu$$

In [4] the conditions in the previous theorem are proven, giving us the following result:

Theorem 12. *Let p be a prime number. For all $x \in p\mathbb{Z}_p$ outside a set of measure 0, Schneider's p-adic continued fraction expansion with $b_0 = 0$ satisfies:*

$$\lim_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n} = \frac{p}{p-1}$$

9 Measure of Schneider's Continued fractions [3]

In this section we follow closely Laohakosol's master thesis. Sometimes giving proofs that were missing in the original manuscript. We will also change our notation a bit for our continued fractions similar to the one of Laohakosol's thesis. That is $\overline{a_i} = p^{a_i}$.

$$\cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{b_3 + \frac{p^{a_4}}{b_4 + ...}}}} \rightsquigarrow \cfrac{\overline{a_1}}{b_1 + \cfrac{\overline{a_2}}{b_2 + \cfrac{\overline{a_3}}{b_3 + \frac{\overline{a_4}}{b_4 + ...}}}}$$

Theorem 13.

1. $\frac{1}{1+p^r\mathbb{Z}_p} = 1 + p^r\mathbb{Z}_p \ \forall r \in \mathbb{Z}^+$
2. $\frac{1}{\delta+\beta\mathbb{Z}_p} = \frac{1}{\delta} + \frac{\beta\mathbb{Z}_p}{\delta^2}$ *if* $v_p(\frac{\beta}{\delta}) \geq 1$ *and* $\delta \neq 0$

Definition 21. *Let $[\overline{a_1}, ..., \overline{a_n}, b_1, ..., b_n]$ be the set* $\left\{ \frac{\overline{a_1}}{b_1 + ... + \frac{\overline{a_n}}{b_n}} \,\middle|\, x \in p\mathbb{Z}_p \right\}$

Theorem 14. *Let $I = p\mathbb{Z}_p \setminus \mathbb{Q}$ and $\Delta(\overline{a_1}, ..., \overline{a_n}, b_1, ..., b_n) = \{x \in I : \overline{a_1}(x) = \overline{a_1}, ..., b_n(x) = b_n\}$. Then we have that:*

$$\Delta(\overline{a_1}, ..., b_n) = [\overline{a_1}, ..., \overline{a_n}, b_1, ..., b_n] + \frac{\overline{a_1}...\overline{a_n}p\mathbb{Z}_p}{(b_1 + [\overline{a_1}, ..., b_n])^2...(b_{n-1} + [\overline{a_n}, b_n])^2 b_n^2}$$

- (1) Note that $\mathbb{Q}$ is countable, so we can safely ignore it in order to compute the measure.
- (2) Here when we talk about the functions $\overline{a_1(x)}, ..., b_n(x)$, we just mean to say that the first n terms in the Schneider expansion are just $\overline{a_1}, ..., b_n$.

Proof. We will prove it by induction.

Base Case: In the third equation, we apply the previous theorem with $\delta = b_1$ and $\beta = p$.

$$\Delta(\overline{a_1}, b_1) = \frac{\overline{a_1}}{b_1 + p\mathbb{Z}_p} = \frac{\overline{a_1}}{b_1} + \frac{\overline{a_1}p\mathbb{Z}_p}{b_1^2} = [\overline{a_1}, b_1] + \frac{\overline{a_1}p\mathbb{Z}_p}{b_1^2}$$

Inductive step: Suppose it holds for "r" $\overline{a_i}$'s and "r" b_i's. That is,

$$\Delta(\overline{a_1}, ..., b_r) = [\overline{a_1}, ..., b_r] + \frac{\overline{a_1}...\overline{a_r}p\mathbb{Z}_p}{(b_o + [\overline{a_1}, ..., b_r])^2...(b_{r-1} + [\overline{a_r}, b_r])^2 b_r^2}$$

Conclusion:

$$\Delta(\overline{a_1}, ..., b_r) = \frac{\overline{a_1}}{b_1 + \Delta(\overline{a_1}, ..., b_r)} = \overline{a_1}\frac{1}{(b_1 + [\overline{a_2}, ..., b_r])^2 + \frac{\overline{a_1}...\overline{a_r}p\mathbb{Z}_p}{(b_1 + [\overline{a_2}, ..., b_r])^2...(b_{r-1} + [\overline{a_r}, b_r])^2 b_r^2}}$$

Let $\delta = [\overline{a_1}, ..., b_r]$ and $\beta = \frac{\overline{a_1}...\overline{a_r}p\mathbb{Z}_p}{(b_1+[\overline{a_2},...,b_r])^2...(b_{r-1}+[\overline{a_r},b_r])^2 b_r^2}$. Then, $v_p(\delta) = 0$ and $v_p(\beta) = v_p(\overline{a_1}...\overline{a_r}p)$, thus $v_p(\frac{\beta}{\delta}) \geq 1$. Therefore we can apply our theorem again.

$$= \frac{\overline{a_1}}{b_1 + [\overline{a_1}, ..., b_r]} + \frac{\overline{a_1}\overline{a_1}...\overline{a_r}p\mathbb{Z}_p}{(b_1 + [\overline{a_2}, ..., b_r])^2...(b_{r-1} + [\overline{a_r}, b_r])^2 b_r^2}$$
$$= [\overline{a_1}, ..., \overline{a_r}, b_1, ..., b_r] + \frac{\overline{a_1}\overline{a_1}...\overline{a_r}p\mathbb{Z}_p}{(b_1 + [\overline{a_1}, ..., b_r])^2...(b_{r-1} + [\overline{a_r}, b_r])^2 b_r^2}$$

□

Computing the measure follows naturally by the use of the previous theorem.

Lemma 2. $\mu(\Delta(\overline{a_1},...,\overline{a_n},b_1,..,b_n)) = |\overline{a_1}...\overline{a_n}|_p$

Proof. $\mu(\Delta(\overline{a_1},...,\overline{a_n},b_1,..,b_n)) = \mu(\frac{\overline{a_1}...\overline{a_n}p\mathbb{Z}_p}{(b_1+[\overline{a_1},...,b_n])^2...b_r^2}) = |\frac{\overline{a_1}...\overline{a_n}}{(b_1+[\overline{a_1},..b_n])^2}\mu(p\mathbb{Z}_p)|_p = |\overline{a_1},...,\overline{a_n}|_p$ □

Lemma 3. *Therefore,* $\mu(\{x \in I : \overline{a_1}(x) = \overline{a_1},...,\overline{a_n}(x) = \overline{a_n}\}) = \sum_{b_1=1}^{p-1} ... \sum_{b_n=1}^{p-1} \mu(\Delta\overline{a_1},...,b_n)) = p^{-(a_1+...+a_n)}(p-1)^n$

Lemma 4. *So for a simpler set like:* $E_n = \{x \in \mathbb{Z}_p : a_1(x) = ... = a_n(x) = 1\}$*, the measure is:* $\mu(E_n) = \frac{(p-1)^n}{p^n}$.

Ex 2. *Let's now compute the measure of some more specific sets. Let's begin with the following set:* $A_n = \{x \in \mathbb{Z}_p \mid a_i \in \{1,2\}\ \forall\ 0 \le i \le n\}$*, using the theorem above we get the following:*

$$\begin{aligned}
\mu(A_n) =&(p-1)^n\mu(\{x \in \mathbb{Z}_p \mid a_i \in \{1,2\}\ \forall\ 0 \le i \le n \wedge b_i\text{'s are fixed}\}) \\
=&(p-1)^n\left(\binom{n}{0}p^{-n} + \binom{n}{1}p^{n-1} + ... + \binom{n}{n}p^{-2n}\right) \\
=&\frac{(p-1)^n}{p^n}\left(\binom{n}{0} + \binom{n}{1}p^{-1} + ... + \binom{n}{n}p^{-n}\right) \\
=&\frac{(p-1)^n}{p^n}\sum_{i=0}^{n}\binom{n}{i}p^{-i} \\
=&\frac{(p-1)^n}{p^n}\left(\frac{1}{p}+1\right)^n = \frac{(p-1)^n(p+1)^n}{p^{2n}}
\end{aligned}$$

Since $(p-1)(p-1) \le p^2 \to p^2 - 1 \le p^2$. *Thus:*

$$\lim_{n\to\infty}\mu(A_n) = 0$$

Ex 3. *We can generalise this example. Consider the set* $B_n = \{x \in \mathbb{Z}_p \mid a_i \le M\ \forall\ 0 \le i \le n \text{ where } M \in\ \mathbb{N}\}$. *The cases* $M = 2,3$ *are fairly easy. Their measures are* $\left(\frac{p-1}{p}\right)^n\left(1+\frac{1}{p}\right)$ *and* $\left(\frac{p-1}{p}\right)^n\left(1+\frac{1}{p}+\frac{1}{p^2}\right)$ *respectively. We will solve the general case by the use of multinomial coefficients. Recall that:*

$$(x_1 + ... + x_m)^n = \sum_{k_1+...+k_m=n}\binom{n}{k_1,k_2,...,k_m}\prod_{i=1}^{m}x_t^{k_t}$$

Now let's apply this to our set.

$$\begin{aligned}
\mu(B_n) =&(p-1)^n\mu(\{x \in \mathbb{Z}_p \mid a_i \le M\ \forall\ 0 \le i \le n \wedge b_i\text{'s are fixed}\}) \\
=&(p-1)^n\sum_{k_1+...+k_m=n}\binom{n}{k_1,k_2,...,k_m}p^{-k_2}p^{-2k_3}...p^{-(m-1)k_m} \\
=&\left(\frac{p-1}{p}\right)^n\left(\sum_{j=0}^{M-1}\left(\frac{1}{p}\right)^j\right) \\
=&\left(\frac{p-1}{p}\right)^n\left(\frac{(\frac{1}{p})^M - 1}{(\frac{1}{p}-1)}\right)^n \\
=&\left(\frac{1-p^M}{p^M}\right)^n
\end{aligned}$$

Therefore even in the generalised case we have that:

$$\lim_{n\to\infty}\mu(B_n) = 0$$

10 Dimension of Schneider's Continued fractions [4]

Our main goal in this section is to apply IFS to different sets defined by Schneider's continued fraction. In the classical case, we have many examples of sets with measure 0 but with non-zero dimension. It is then natural to ask of the existence of such sets in $\mathbb{C}_p$.

The setup:

Let $E_r = \{[a_0, b_0, ...] : a_i's \in \{n_1, ..., n_r\}\}$, $D = p\mathbb{Z}_p$. We also define the following map:

$$S_j^i(x) = \frac{p^{n_i}}{j+x} \text{ where } i \in \{1, ..., r\} \text{ and } j \in \{1, ..., p-1\}.$$

We must first check that E_r is a compact set. Obviously it is bounded, which means we only have to show that it is closed.

Proof. We will show that the complement is open. Pick a point $x \notin E_r$. We know that we can write x as $[a_0, b_0, a_1, b_1, ...]$. Since $x \notin E_r$ $\exists j$ such that $a_j > r$. Therefore $\exists$ an open ball around x such that all $a_i, \quad i < j$ are fixed. Thus the set is open and E_r is closed. □

1. Exact contraction: $\frac{|S(x)-S(y)|_p}{x-y} = \frac{|\frac{p^i}{j+x} - \frac{p^i}{j+y}|_p}{x-y} = |p_i|_p \frac{|\frac{j+y-(j+x)}{(j+x)(j+y)}|_p}{|x-y|_p} = \frac{1}{p_i}|\frac{1}{(j+x)(j+y)}|_p = \frac{1}{p_i}$
2. Attractor: $E = \bigsqcup_{i,j} S_i^j(E)$
3. OSC: The open set condition is satisfied by setting $V = p\mathbb{Z}_|$ since that is an open set. The rest of the conditions follow.
4. SOSC: Obivously $E \cap V \neq \emptyset$

Let $E = \{[a_0, b_0, ...] : a_i \in \{1, 2\}$, $D = p\mathbb{Z}_p$. We also define the following map: $S_j^i(x) = \frac{p^i}{j+x}$ where $i \in \{1, 2\}$ and $j \in \{1, ..., p-1\}$.

Since all of our conditions are satisfied, and the contractive maps are exact, we can apply our big theorem from section 5.

$$(p-1)\left(\frac{1}{p}\right)^s + (p-1)\left(\frac{1}{p^2}\right)^s = 1$$

Substitute $\frac{1}{p-1}$ by α and $\frac{1}{p}$ by β.

$$\begin{aligned}
\alpha^s + \alpha^{2s} &= \beta \\
x + x^2 &= \beta \text{ (substitute } x = \alpha^s) \\
\text{Giving us: } x_1 &= \frac{-1+\sqrt{1+4\beta}}{2} \\
\alpha^s &= \frac{-1+\sqrt{1+4\beta}}{2} \\
\text{Thus: } s = H(E) &= \frac{\log(\frac{-1+\sqrt{1+4\beta}}{2})}{\log(\alpha)}
\end{aligned}$$

We will now show a couple of different sets and their dimensions for which the polynomial has an exact solution.

Ex 4. *Let* $E = \{[a_0, b_0, ...] \mid a_i \in \{a_i\},\ b_i \in \{1, 2, ..., p-1\}\}$*. Applying the previous method, we get the following:*

$$\alpha^{ns} = \beta$$
$$\left(\frac{1}{p}\right)^s = \left(\frac{1}{p-1}\right)^{\frac{1}{n}}$$
$$s = \frac{\frac{1}{n}log(\frac{1}{p-1})}{log(\frac{1}{p})} = \frac{\frac{1}{n}log(p-1)}{log(p)}$$

Ex 5. *Let $E = \{[a_0, b_0, ...] \mid a_i \leq M,\ b_i \in \{1\}\}$. Applying the previous method, we get the following:*

$$\begin{aligned}
\alpha^s + \alpha^{2s} + ... + \alpha^{Ms} &= 1 \\
x + x^2 + ... + x^M - 1 &= 0 \\
-x^{M+1} + 2x - 1 &= 0 \\
\text{Note that: } \lim_{M\to\infty} x_1 &= \frac{1}{2} \\
\frac{1}{2} + \epsilon &= \left(\frac{1}{p}\right)^s \\
s &= \frac{log(\frac{1+2\epsilon}{2})}{log(\frac{1}{p})} \\
s &\approx \frac{log(\frac{1}{2})}{log(\frac{1}{p})} = \frac{log(2)}{log(p)}
\end{aligned}$$

We can generalise this problem and boil it down to finding solutions to a specific polynomial. This is our main result.

Theorem 15. *Let $E = \{[a_0, b_0, ...] \mid a_i \in \{n_1, n_2, .., n_r\},\ b_i \in S \mid S \subseteq \{1, ..., p-1\}\}$. Applying the previous method, we get the following polynomial, with x and β defined as before.*

$$x^{n_1} + x^{n_2} + ... + x^{n_r} - \beta = 0$$

We can go even further, restricting our b_i's based on the a_i's before it.

Theorem 16. *Let $E = \{[a_0, b_0, ...] \mid a_i \in \{n_1, n_2, .., n_r\},$ if $a_i = n_j$ for some $1 \leq j \leq r$ then $b_i \in B_j \mid B_j \subseteq \{1, ..., p-1\}\}$. Applying the previous method, we get the following polynomial, with x defined as before.*

$$|B_1|\, x^{n_1} + |B_2|\, x^{n_2} + ... + |B_r|\, x^{n_r} = 1 \text{ with } 1 \leq |B_j| \leq p-1$$

Finally for our last result, we will add some more liberties to our contractive maps, giving us much more freedom in what sets we can find the dimension of. Define the following map:

$$S_n(x) = S^{i_1,i_2,...,i_n}_{j_1,j_2,...,j_n}(x) = \cfrac{p^{i_1}}{j_1 + ... + \cfrac{p^{i_n}}{j_n + x}}$$

We will now both prove it's a contractive map and what the contractive constant is.

We have that $S_n(x) = \dfrac{A_n + xA_{n-1}}{B_n + xB_{n-1}}$, therefore:

$$\begin{aligned}
\frac{|S(x)-S(y)|_p}{|x-y|_p} &= \frac{|\frac{A_n+xA_{n-1}}{B_n+xB_{n-1}} - \frac{A_n+yA_{n-1}}{B_n+yB_{n-1}}|_p}{|x-y|_p} \\
&= \frac{|A_{n-1}B_n - A_nB_{n-1}|_p}{|(B_n+xB_{n-1})(B_n+yB_{n-1})|_p} \\
&= \frac{|(-1)^n p^{a_1 \dots a_n}|_p}{|(B_n+xB_{n-1})(B_n+yB_{n-1})|_p} \\
&= \frac{1}{p^{a_1+\dots+a_n}} \frac{1}{|(B_n+xB_n-1)|_p} \frac{1}{|(B_n+yB_n-1)|_p} \\
&= \frac{1}{p^{a_1+\dots+a_n}}
\end{aligned}$$

Let's say $n = 2$ in $S_n(x)$ and $E = \{[a_0, b_0, a_1, b_1, ...] \mid a_i = 1 \ \forall\ i,\ \ b_i \in \{1,2\}$ if i even, $\ b_i \in \{3,4\}$ if i odd$\}$ We have shown previously that S_n is contractive. The rest of the conditions also follow easily. Using the same method we get the following equation for s:

$$1 = 4\left(\frac{1}{p^2}\right)^s$$
$$s = \frac{log(4)}{log(p^2)}$$

11 Infinite Iterated Function Systems (IIFS) [6, 7]

We will now introduce IIFS as described in Fernau's paper [7] in order to find the dimension of sets that cannot be described by IFS. As an example of that consider:

$$E = \{[a_0, b_0, ...] \mid a_i \in S \subset \mathbb{N},\ b_i \in \{1, ..., p-1\} \land |S| = \infty\}$$

In the case of real continued fractions, we would be able to find the dimension of such sets by the principle of continuity. We were not able to recreate that argument for p-adic continued fractions. To prove the continuity principle for p-adics we required IIFS, which we will introduce now.

Definition 22. *An* ***IIFS*** *is a countable sequence of* ***contractive maps****.*

Definition 23. *Let $C = \{c_i\}$ be the sequence of contractive factors for the IIFS of the set E. This sequence is called a* ***zero sequence*** *if $lim_{i\to\infty}(c_i) = 0$.*

Theorem 17. *An IIFS of E satisfying the SOSC where C is a zero-sequence has Hausforff dimension s where $\inf\{a|\phi_C(s) \leq 1\}$ where $\phi_c(s) = \sum_{i\in\mathbb{N}} c_i^s$.*

Definition 24. *Let say $E_r\{[a_0, b_0, ...] \mid a_i \in S_r \subset \mathbb{N},\ b_i \in \{1, ..., p-1\}\}$, where $S_r \subset S$ containing the first r elements of S.*

Theorem 18. *This section concludes in the following result:* $\lim_{r\to\infty}(dim_H(E_r)) = dim_H(E)$

11.1 Using IIFS

Let's go back to our sets:

$$E = \{[a_0, b_0, ...] \mid a_i \in S \subset \mathbb{N},\ b_i \in \{1, ..., p-1\}\}$$
$$E_r = \{[a_0, b_0, ...] \mid a_i \in S_r,\ b_i \in \{1, ..., p-1\}\}$$

Our sequence in contractive factors for E in this case is: $C = \{\frac{1}{p^{a_1}}, ..., \frac{1}{p^{a_1}}, \frac{1}{p^{a_2}, ...}\}$. Obviously this sequence is contractive. Its also obvious that union of the sequence of IFS for $E_r, \forall r$ gives us an IIFS for E. We can apply our previous theorem directly.

Theorem 19. *Thus* $\lim_{r\to\infty}(dim_H(E_r)) = dim_H(E)$

12 Approximations to the Hausdorff dimension

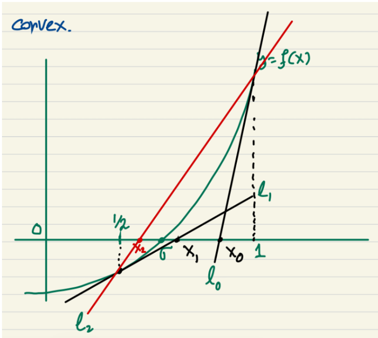
Although we have an exact description for the dimension, as a root to some polynomial, we were not able to find general solutions to these type of polynomials. For some specific cases it can and was done in earlier sections. In this part we will focus on approximations to this special root.

Our goal in this section will be to find approximations to the roots of the following polynomial:

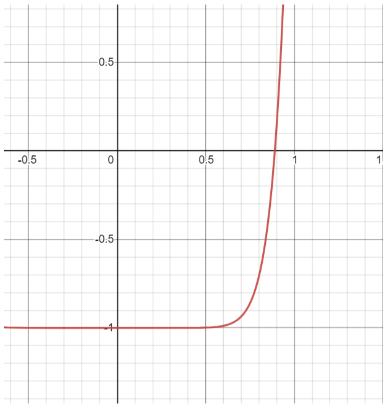
$$f(x) = x^{n_1} + x^{n_2} + ... + x^{n_r} - 1$$

First note that this function is increasing on $[0, +\infty]$
Also, $f(1/2) < 0$ and $f(1) > 0$, thus we have a unique root between $[1/2, 1]$.

It is natural to use the convexity of the function to try and get some simple bounds. In (a) we have 3 such bounds given by l_0, l_1, l_2.



(a) Plot of f(X)



(b) Plot of $x^{10} + x^{11} + x^{12} + x^{13} - 1$

- l_0 is a linear function passing trough $f(1/2)$ and $f(1)$, the point at wich it crosses the origin is an obvious lower bound. It turns out to be a quite week lower bound, we will present a better one in the future.
- Both l_1 and l_2 are approximations to $f(x)$ by taking first order derivatives. The problem with these type of approximations is that while for some polynomials they are good, if the exponents get too big, the function $f(x)$ (see (b))) tends to look like a 90 degree angle. The derivative then gives a useless upper bound.

We will now present the strongest bounds that we could find.

12.1 Upper Bounds

Since all of the x^{n_i} are positive on our interval, we can use the AM-GM inequality to bound the function from below.

$$\frac{x^{n_1} + x^{n_2} + ... + x^{n_r}}{r} \geq \sqrt[r]{x^{n_1+...+n_r}}$$
$$x^{n_1} + x^{n_2} + ... + x^{n_r} - 1 \geq r\sqrt[r]{x^{n_1+...+n_r}} - 1$$

Since the function is bounded from bellow and it is an increasing function, the root of the RHS is an upper bound for the root of the LHS. So if x^* is the root to the LHS, we have that:

$$x^* \leq \left(\frac{1}{r}\right)^{\frac{r}{n_1+...+n_r}}$$

Same method can be used for the most general case we have with the use of the weighted AM-GM inequality.

Theorem 20. *Let $w = w_1 + ... + w_k$ then the* ***weighted AM-GM*** *states that:*

$$\frac{w_1x_1 + ... + w_kx_k}{w} \geq \sqrt[w]{x_1^{w_1}...x_k^{w_k}}$$

Then if $f(x) = |B_1|x^{n_1} + ... + |B_r|x^{n_r} - 1$ and $1 \leq |B_i| \leq p-1$. We let $w = \sum_{i=1}^{r} |B_i|$ and $k = \sum_{i=1}^{r} n_i |B_i|$. Using the same method as before, this gives us an upper bound for the root.

$$x^* \leq \left(\frac{1}{w}\right)^{\frac{k}{w}}$$

Remark 2. *AM-GM is tight only when all terms are equal. In our case this means that the bigger the spread of our exponents, the more precision we lose. There is a way to mitigate this somewhat. Note that* $x^{n_1} + ... + x^{n_r} - 1 \geq x^{n_1} + ... + x^{n_{r-1}} - 1$*. Say our exponents are in increasing order. In that case if* n_r *is big enough, it turns out using the AM-GM upper bound on* $x^{n_1} + ... + x^{n_{r-1}} - 1$ *turns out to be a better upper bound then using AM-GM on* $x^{n_1} + ... + x^{n_r} - 1$ *directly. We have not yet found the exact point at which this happens.*

12.2 Lower Bounds

Define the following constant:

$$c := \inf\left\{\frac{n_j}{j \cdot n_1} : j = 1, 2, ..., r\right\} \to n_j \geq c \cdot n_1 \cdot j$$

$$x^{n_1} + x^{n_2} + ... + x^{n_r} - 1 \leq x^{cn_1} + (x^{cn_1})^2 + ... + (x^{cn_1})^r - 1 = y \cdot \left(\frac{1}{1-y}\right) - 1$$

Since $y = \frac{1}{2}$ is a solution to the R.H.S, we can use it as a lower bound for our root x^*.

$$x^* \geq \left(\frac{1}{2}\right)^{\frac{1}{cn_1}}$$

12.3 Lower Bound for $f(x) = \sum_{i=1}^{\infty} x^{n_i} - 1$

Consider $g(x) = n_1 \sum_{i=1}^{\infty} (x^\alpha)^i - 1$ where $\alpha = \inf\{\frac{n_{i+1} - n_1}{i} : i \in \mathbb{N}\}$. Then $f(x) \leq g(x)$.

$$g(x) = \frac{x^{n_1}}{1 - x^\alpha} - 1 = \frac{x^{n_1} + x^\alpha - 1}{1 - x^\alpha}$$

We are looking for the root of $x^{n_1} + x^\alpha - 1$. Here we must stop since we don't yet have a method of finding the root of polynomials of the following form: $x^a + x^b - 1$.

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